The Welfare Effects of Social Security in a Model with Aggregate and Idiosyncratic Risk*

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Abstract

We study the welfare effects of social security in an overlapping generations general equilibrium model with aggregate and idiosyncratic risk. Prior research on social security has only considered either aggregate or idiosyncratic risk. We show analytically that the aggregate and idiosyncratic risks interact due to the life-cycle structure of the economy. This interaction increases the welfare gains of a marginal introduction of an unfunded social security system. Adding a second interaction by making the variance of the idiosyncratic risk countercyclical further increases the welfare gains. In our quantitative experiment, raising the contribution rate from zero to two percent leads to long-run welfare gains of 3.5% of life-time consumption on average, even though the economy experiences substantial crowding out of capital. Approximately one third of these gains can be attributed to the interactions between idiosyncratic and aggregate risk.

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1 Introduction

Many countries operate large social security systems. One reason is that social security can provide insurance against risks for which there are no private markets. However, these systems also impose costs by distorting prices and decisions. The question arises whether the benefits of social security outweigh the costs.

We address this question in a model which features both aggregate and idiosyncratic risk. We follow the literature and assume that insurance markets for both forms of risk are incomplete. In such a setting social security can increase economic efficiency by partially substituting for missing markets. The analysis is embedded in a general equilibrium framework to account for the costs of crowding out. The difference to the previous literature is that, so far, only models with one kind of risk were examined. One strand of the literature has looked at social security when only aggregate risk is present (e.g. Krueger and Kubler (2006)). There, social security can improve the intergenerational sharing of aggregate risks. The other strand included only idiosyncratic risk (e.g. Imrohoroglu, Imrohoroglu, and Joines (1995, 1998)). There, social security is valuable because of intragenerational insurance. However, households face both kinds of risk over their lifetime. To get a more complete picture of how much insurance social security can provide, the different risks need to be included in one model. By doing that, we can assess the contribution of each risk to total insurance. More importantly, we can analyze the role played by interactions between the two types of risk.

The first interaction is an interaction over the life-cycle and accordingly we call it the life-cycle interaction (LCI). To better understand this new effect, consider a standard model in which idiosyncratic wage risk is statistically independent of aggregate risk. Due to the nature of a life-cycle economy, aggregate and idiosyncratic risks directly interact despite their statistical independence. The reason is that when retired, consumption is mainly financed out of private savings. The level of private savings depends on the realizations of idiosyncratic wage risk and aggregate return risk during working life. As a consequence, the variance of private savings contains an interaction term between idiosyncratic and aggregate risk. Because households face these risks for many years before they go into retirement, this interaction term becomes large.

The second interaction operates via the so-called counter-cyclical cross-sectional variance of idiosyncratic productivity shocks (CCV). This means that the variance of idiosyncratic shocks is higher in a downturn than in a boom. The CCV has been documented in the data (Storesletten, Telmer, and Yaron (2004)), and has been analyzed with respect to asset pricing (Mankiw

(1986), Constantinides and Duffie (1996), Storesletten, Telmer, and Yaron (2007)). We want to understand whether social security can provide insurance against this interaction.

In order to evaluate how much these interactions matter quantitatively, we build a large-scale overlapping generations (OLG) model in the tradition of Auerbach and Kotlikoff (1987), extended by various forms of risk. Aggregate wage risk is introduced through a standard shock to total factor productivity (TFP). Aggregate return risk is introduced through a depreciation shock. The two shocks enable us to calibrate the model in such a way that it produces realistic fluctuations of wages and returns, both of which are central to the welfare implications of social security.

The social security system is a pure pay-as-you-go (PAYG) system. Every period, all the contributions are paid out as a lump-sum to all the retirees. Households can also save privately by investing in a risk-free bond and risky stock. Having this portfolio choice in the quantitative model is important, because social security can be seen as an asset with a low risk and a low return. Therefore, the risk-return structure of the bond and the stock directly affect the value of social security. In order to match a high expected risky return and a low risk-free rate at the same time we need Epstein-Zin-preferences. Finally, households also face survival risk. Therefore, they value social security because it partially substitutes for missing annuity markets.

Our experiment consists of a marginal introduction of social security. We use a two-period model to expose the new life-cycle interaction LCI. We show analytically that social security provides insurance against both LCI and against the countercyclical variance CCV. We also show analytically that the benefit of the insurance against CCV becomes larger when aggregate risk in the economy increases.

When we calibrate the model to the U.S. economy, we find that the introduction of social security leads to a strong welfare gain. This stands in contrast to the previous literature, because social security in our model provides insurance against both idiosyncratic and aggregate risk, as well as their interactions. To be precise, increasing the contribution rate from zero to two percent leads to welfare gains of 3.5% in terms of consumption equivalent variation. This welfare improvement is obtained even though we observe substantial crowding out of capital. About one third of the welfare gains is attributed to the two interactions LCI and CCV.

The welfare gains are not caused by reducing an inefficient overaccumulation of capital in the sense of Samuelson (1958) or Diamond (1965). To control for that, we ensure in our calibration that the economy is dynamically efficient. The welfare numbers do not hinge on the specific experiment: when we increase the contribution rate from 12% to 14%, the welfare gains are still positive, though smaller. When we follow different calibration strategies, the welfare numbers also retain the same sign and relative magnitude.

The idea that social security can insure against aggregate risks goes back to Diamond (1977) and Merton (1983). They show how it can partially complete financial markets and thereby increase economic efficiency. Building on these insights, Shiller (1999) and Bohn (2001, 2009) show that social security can reduce consumption risk of all generations by pooling labor income and capital income risks across generations if labor income and capital returns are imperfectly correlated.

Gordon and Varian (1988), Ball and Mankiw (2001), Matsen and Thogersen (2004) and Krueger and Kubler (2006) use a two-period partial equilibrium model where households consume only in the second period of life, i.e. during retirement. For our analytical results, we extend the model by adding idiosyncratic risk. In our companion paper Harenberg and Ludwig (2011) we relax the assumption of zero first-period consumption and conclude that one of the results breaks down: a smaller or negative covariance between wages and risky returns does not necessarily improve intergenerational risk-sharing. In the present paper, we address this insight by analyzing two calibrations which differ with respect to this covariance.

Quantitative papers with aggregate uncertainty and social security are scarce. Krueger and Kubler (2006) is the closest to us.[1] They also look at a marginal introduction of a PAYG system and find that it does not constitute a Pareto-improvement. The concept of a Pareto-improvement requires that they take an ex-interim welfare perspective, whereas we calculate welfare from an ex-ante perspective. Our paper differs in that it adds idiosyncratic risks and analyzes the interactions.

Quantitative papers with idiosyncratic uncertainty and social security, on the other hand, are plenty (e.g. Conesa and Krüger (1999), Imrohoroglu, Imrohoroglu, and Joines (1995, 1998), Huggett and Ventura (1999) and Storesletten, Telmer, and Yaron (1999)). On a general level, a conclusion from this literature is that welfare in a stationary economy without social security is higher than in one with a PAYG system. That is, the losses from crowding out dominate the gains from completing insurance markets. The more recent work by Nishiyama and Smetters (2007) and Fehr and Habermann (2008) are examples of papers which focus on modeling institutional features of existing social security systems in detail. Our approach is less policy oriented than

[1]Ludwig and Reiter (2010) ask how pension systems should optimally adjust to demographic shocks. Olovsson (2010) claims that pension payments should be very risky because this increases precautionary savings and thereby welfare improving capital formation.

theirs and we abstract from such details. Our results show the benefits of a flat pension scheme without additionally optimizing over the exact design of the pension benefit formula.

Huggett and Parra (2010) argue that it is important to look at a simultaneous reform of both the social security system and of the general tax system. They report strong welfare gains from joint reforms of both systems. We instead follow the more standard approach and take the general income tax system as given. Consequently, we calibrate our model to income processes after taxation.

The remainder of this paper is structured as follows. We derive our analytical results in section 2. Section 3 develops the quantitative model and section 4 presents the calibration. The main results of our quantitative analysis are presented in section 5, where we make much use of our analytical results. We conclude in section 6. Proofs, computational details, and robustness checks are relegated to separate appendices.

2 A Two-Generations Model

We first develop an analytical model that provides useful insights for our quantitative analysis. We develop our model in several steps. We start by adopting the partial equilibrium framework of Gordon and Varian (1988), Ball and Mankiw (2001), Matsen and Thogersen (2004), Krueger and Kubler (2006) and others who assume that members of each generation consume only in the second period of life. We show that the aforementioned literature—which focuses on aggregate risk only—misses important interaction mechanisms between idiosyncratic and aggregate risk. Furthermore, as shown in Harenberg and Ludwig (2011) a two period model misses an important aspect of the inter-temporal nature of the savings problem which biases results against social security if wages and returns are positively correlated. To avoid this discussion here—which would in any case lead us on a sidetrack—, we simply shut down the correlation between wages and returns.

We argue that this simple setup can only provide a partial characterization of the total welfare effects of social security. It misses the effects of taxation on reallocation of consumption and savings as well as the welfare losses induced by crowding out. To accommodate both channels at once we extend our model to a standard Diamond (1965) model with risk. Hence, consumption and savings decisions take place in the first period and wages and returns are determined in general equilibrium.[2]

[2]Without explicitly acknowledging how prices are determined in general equilibrium

2.1 Households

A household born in period t has preferences over consumption in two periods whereby second period consumption is discounted with the raw time discount factor β. In the first period of life, the household experiences an idiosyncratic productivity shock which we denote by η. This shock induces heterogeneity by household type which we denote by i. In addition, we index age by $j = 1, 2$. Consequently, all variables at the individual level carry indices i, j, t. The expected utility function of a household born in period t is given by

$$E_t U_t = E_t \left[v(c_{i,1,t}) + \beta u(c_{i,2,t+1}) \right],$$

where the per period Bernoulli utility functions are (weakly) increasing and concave, i.e., $v' \geq 0, u' > 0, v'' \leq 0, u'' < 0$.

Consumption in the two periods is given by

$$c_{i,1,t} + s_{i,1,t} = (1 - \tau)\eta_{i,1,t} w_t \tag{1a}$$

$$c_{i,2,t+1} = s_{i,1,t}(1 + r_{t+1}) + b_{t+1} \tag{1b}$$

where $\eta_{i,1,t}$ is an idiosyncratic shock to wages in the first period of life. We assume that $E\eta_{i,1,t} = 1$ for all i, t. Furthermore, a law of large numbers applies so that $\int \eta_{i,1,t} d\eta = 1$ for all t. b_{t+1} are social security benefits to be specified next and τ is the contribution rate to social security.

2.2 Government

The government organizes a PAYG financed social security system. Assuming perfect insurance against idiosyncratic wage risk through social security, the social security budget constraint writes as

$$b_t N_{2,t} = \tau w_{t+1} N_{1,t}$$

where $N_{j,t}$ is the population in period t of age j, i.e., $N_{j,t} = \int N_{i,j,t} di$. We ignore population growth, hence

$$b_t = \tau w_t.$$

We can therefore rewrite consumption in the second period as

$$c_{i,2,t+1} = s_{i,1,t}(1 + r_{t+1}) + w_{t+1}\tau.$$

we cannot derive the effect of taxation on first period consumption. The reason is that a human capital wealth effect—caused by the discounted value of future pension income—inhibits closed form solutions in the partial equilibrium setup. Hence, moving to general equilibrium "kills two birds with one stone".

2.3 Welfare

We take an ex-ante Rawlsian perspective and hence specify the social welfare function (SWF) of a cohort born in period t as the expected utility of a generation from the perspective of period $t-1$:

$$SWF_t \equiv E_{t-1}U_t = E_{t-1}\left[v(c_{i,1,t}) + \beta u(c_{i,2,t+1})\right].$$

2.4 Partial Equilibrium Analysis

We start by looking at a degenerate version of our model where first-period utility is zero. We assume that utility from consumption in the second period is CCRA with a coefficient of relative risk aversion of θ:

Assumption 1. *Let* $v(c_{i,1,t}) = 0$ *and* $u(c_{i,2,t+1}) = \frac{c_{i,2,t+1}^{1-\theta}-1}{1-\theta}$.

Stochastic Processes

Wages and interest rates are stochastic. We denote by ζ_t the shock on wages and by $\tilde{\varrho}_t$ the shock on returns. We further assume that wages grow deterministically at rate g. We therefore have:

$$\begin{aligned} w_t &= \bar{w}_t\zeta_t = \bar{w}_{t-1}(1+g)\zeta_t \\ R_t &= \bar{R}\tilde{\varrho}_t \end{aligned}$$

To simplify the analysis we assume that both ζ_t and $\tilde{\varrho}_t$ are not serially correlated. Despite the observed positive serial correlation of wages and asset returns in annual data, this assumption can be justified on the grounds of the long factual periodicity of each period in a two-period OLG model which is about 30 to 40 years. We also assume that ζ_t and $\tilde{\varrho}_t$ are statistically independent. We do so because, as we point out in Harenberg and Ludwig (2011), any conclusion from such a simple and inherently a-temporal model on the effects of the correlation structure between wages and returns is misleading. The idiosyncratic shock $\eta_{i,1,t}$ is not correlated with either of the two aggregate shocks. We relax this assumption once we introduce the CCV mechanism below. All shocks are assumed to have bounded support. We now summarize these assumptions:

Assumption 2. *a) Bounded support:* $\zeta_t > 0$, $\tilde{\varrho}_t > 0$ *for all* t, $\eta_{i,1,t} > 0$ *for all* i, t.

b) Means: $E\zeta_t = E\tilde{\varrho}_t = E\eta_{i,1,t} = 1$, *for all* i, t.

c) *Statistical independence of* (ζ_{t+1}, ζ_t) *and* $(\tilde{\varrho}_{t+1}, \tilde{\varrho}_t)$. *Therefore:* $E(\zeta_{t+1}\zeta_t) = E\zeta_{t+1}E\zeta_t$ *for all* t *and, correspondingly,* $E(\tilde{\varrho}_{t+1}\tilde{\varrho}_t) = E\tilde{\varrho}_{t+1}E\tilde{\varrho}_t$ *for all* t.

d) *Statistical independence of* $(\zeta_t, \tilde{\varrho}_t)$. *Therefore:* $E(\zeta_t\tilde{\varrho}_t) = E\zeta_t E\tilde{\varrho}_t$ *for all* t.

e) *Statistical independence of* $(\zeta_t, \eta_{i,1,t})$. *Therefore:* $E(\eta_{i,1,t}\zeta_t) = E\eta_{i,1,t}E\zeta_t$ *for all* i, t.

f) *Statistical independence of* $(\tilde{\varrho}_t, \eta_{i,1,t})$. *Therefore:* $E(\eta_{i,1,t}\varrho_t) = E\eta_{i,1,t}E\varrho_t$ *for all* i, t.

Life-Cycle Interaction

Under assumption 1, utility maximization implies that $c_{i,1,t} = 0$ and $s_{i,1,t} = (1-\tau)\eta_{i,1,t}\bar{w}\zeta_t$. Consumption in the second period can accordingly be rewritten as

$$c_{i,2,t+1} = \bar{w}\left(\eta_{i,1,t}\zeta_t\bar{R}\tilde{\varrho}_{t+1} + \tau\left((1+g)\zeta_{t+1} - \eta_{i,1,t}\zeta_t\bar{R}\tilde{\varrho}_{t+1}\right)\right). \tag{3}$$

We then have:

Proposition 1. *Under assumptions 1 and 2, a marginal introduction of social security increases ex-ante expected utility if*

$$(1+g)\frac{E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}^{\theta}_{t+1}}\right]E_{t-1}\left[\frac{1}{\zeta^{\theta}_t}\right]E_{t-1}\left[\frac{1}{\eta^{\theta}_{i,1,t}}\right]}{E_{t-1}\left[\tilde{\varrho}^{1-\theta}_{t+1}\right]E_{t-1}\left[\zeta^{1-\theta}_t\right]E_{t-1}\left[\eta^{1-\theta}_{i,1,t}\right]} > \bar{R}. \tag{4}$$

The RHS of equation (4) reflects the costs of introducing social security represented here by the ex-risk return on savings. We speak of the LHS of equation (4) as the risk-adjusted implicit return of social security which reflects the value (or benefit) of introducing social security. Obviously, the implicit return increases if g increases. This is the standard Aaron condition.

To interpret the risk adjustment, we next assume that all stochastic variables are jointly distributed as log-normal.

Assumption 3. *Joint log-normality:* $\eta_{i,1,t}, \zeta_t, \zeta_{t+1}, \tilde{\varrho}_{t+1}$ *are jointly distributed as log-normal with parameters* $\mu_{\ln\eta}$, $\mu_{\ln\zeta}$, $\mu_{\ln\tilde{\varrho}}$, $\sigma^2_{\ln(\eta)}$, $\sigma^2_{\ln(\zeta)}$, $\sigma^2_{\ln(\tilde{\varrho})}$ *for means and variances, respectively.*

We then have:

Proposition 2. *Under assumptions 1 through 3, a marginal introduction of social security increases ex-ante expected utility if*

$$(1+g)\cdot(1+TR)^{\theta} > \bar{R}, \tag{5}$$

where

$$TR \equiv var(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}) = \underbrace{\sigma_\eta^2}_{IR} + \underbrace{\sigma_\zeta^2 + \sigma_{\tilde{\varrho}}^2 + \sigma_\zeta^2\sigma_{\tilde{\varrho}}^2}_{AR} + \underbrace{\sigma_\eta^2\left(\sigma_\zeta^2 + \sigma_{\tilde{\varrho}}^2 + \sigma_\zeta^2\sigma_{\tilde{\varrho}}^2\right)}_{LCI=IR\cdot AR}. \quad (6)$$

To interpret this condition, observe that, according to equation (6), term TR—standing in for "total risk"—consists of three components, reflecting the effect of idiosyncratic risk in term IR, total aggregate risk in term AR and a mechanical interaction between idiosyncratic and aggregate risk in term LCI. To understand the nature of these terms notice that, in absence of social security, savings cum interest in our simple model is given by $s_{i,1,t}R_{t+1} = \bar{w}_t\bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}$. Hence, from the ex-ante perspective, the product of three sources of risk are relevant, idiosyncratic wage risk, $\eta_{i,1,t}$, aggregate wage risk, ζ_t, and aggregate return risk, $\tilde{\varrho}_{t+1}$. Term TR is the variance of the product of these stochastic elements. It can be derived by applying the product formula of variances presented in Goodman (1960).

For standard random variables, an interaction term involving products of variances—such as LCI in our context—would be small. However, we here deal with long horizons so that the single variance terms might be quite large. Illustration 1 in the appendix gives a simple numerical example which uses parameters of our calibrated income processes. Based on this example we conclude that LCI adds about 40 percent times AR. Whatever the exact size of AR is, this interaction is clearly a non-negligible increase in overall income risk.

We next show how LCI translates into utility. Take a second order Taylor series approximation of the utility function around expected consumption, which, after some transformations given in the appendix, gives

$$E_{t-1}u_t \approx \frac{1}{1-\theta}\left(\bar{w}_t\bar{R}\right)^{1-\theta} - \frac{\theta}{2}\left(\bar{w}_t\bar{R}\right)^{1-\theta}(IR + AR + IR\cdot AR) \quad (7)$$

whereby the second term reflects the utility losses from fluctuations. Observe that

$$\frac{\partial E_{t-1}u_t}{\partial AR} = -\frac{\theta}{2}\left(\bar{w}_t\bar{R}\right)^{1-\theta}(1 + IR).$$

Hence, the aforementioned 40% that the interaction term LCI adds to the variance of consumption translates into roughly the same proportional increase of the utility loss from fluctuations when the variance of aggregate shocks—term AR—increases.

Modification: Counter-Cyclical Conditional Variance

We now return to condition (4) and modify assumption 3 slightly in order to reflect the CCV mechanism. Observe that CCV, by definition, does away with assumption 2e.

Assumption 4. *$\zeta_t \in [\zeta_l, \zeta_h]$ for all t where $\zeta_h > \zeta_l > 0$. We let $\zeta_h = 1 + \Delta_\zeta$ and $\zeta_l = 1 - \Delta_\zeta$ where $\Delta_\zeta < 1$. Notice that $\frac{1}{2}(\zeta_l + \zeta_h) = 1$. $\eta_{i,1,t}$ is distributed as log-normal whereby*

$$\eta_{i,1,t} = \begin{cases} \eta_{i,1,l} & \text{for } \zeta_t = \zeta_l \\ \eta_{i,1,h} & \text{for } \zeta_t = \zeta_h. \end{cases}$$

and $E \ln \eta_{i,1,l} = E \ln \eta_{i,j,h} = E \ln \eta_{i,j,t} = E \ln \eta$ and

$$\sigma^2_{\ln \eta} = \begin{cases} \sigma^2_{\ln \eta h} = \sigma^2_{\ln \eta} + \Delta & \text{for } \zeta_t = \zeta_l \\ \sigma^2_{\ln \eta l} = \sigma^2_{\ln \eta} - \Delta & \text{for } \zeta_t = \zeta_h. \end{cases}$$

For simplicity, we focus only at the log-uility case, hence $\theta = 1$. The RHS of equation (4) then rewrites as

$$(1+g)E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}}\right] E_{t-1}\left[\frac{1}{\zeta_t}\right] E_{t-1}\left[\frac{1}{\eta_{i,1,t}}\right]$$

Under assumption 4, the expression rewrites as

$$(1+g)E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}}\right] \frac{1}{2}\left(\frac{1}{\zeta_l}E_{t-1}\left[\frac{1}{\eta_{i,1,l}}\right] + \frac{1}{\zeta_h}E_{t-1}\left[\frac{1}{\eta_{i,1,h}}\right]\right) \tag{8}$$

and, without CCV, the corresponding expression is

$$(1+g)E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}}\right] \frac{1}{2}\left(\frac{1}{\zeta_l} + \frac{1}{\zeta_h}\right) E_{t-1}\left[\frac{1}{\eta_{i,j,t}}\right]. \tag{9}$$

We can then show the following:

Proposition 3. *a) the LHS of equation (8) is larger than the LHS of equation (9).*

b) the difference between the LHS of equation (8) and the LHS of equation (9) increases in the variance of aggregate shocks.

We can therefore conclude that, on top of the previously illustrated mechanical interaction between idiosyncratic and aggregate risk, the direct interaction via the CCV mechanism will further increase the beneficial effects of social security. Importantly, finding 3b establishes that the effect of CCV is larger when the variance of aggregate risk is higher.

2.5 General Equilibrium Analysis

In general equilibrium, we relax assumption 1, thereby modeling utility from consumption in the first period. To derive analytical solutions we have to restrict attention to log-utility in both periods. Furthermore, again for analytical reasons, we assume absence of idiosyncratic shocks:[3]

Assumption 5. *a)* $v(\cdot) = u(\cdot) = \ln(\cdot)$.

b) $\eta_{i,1,t} = E\eta_{i,1,t} = 1$ *for all* i, t.

As a consequence of discarding idiosyncratic risk, our analysis in this subsection does not contribute anything particularly new to the literature on social security so that the quick reader may wish to proceed with section 3. We nevertheless regard the general equilibrium extension as very useful to provide guidance for interpretation of our quantitative results in section 5 where we will occasionally refer back to our analytical expressions.

Firms

To close the model in general equilibrium, we add a firm sector. We take a static optimization problem. Firms maximize profits operating a neo-classical production function. Let profits of the firm be

$$\Pi = \zeta_t F(K_t, \Upsilon_t L_t) - (\bar{\delta} + r_t)\varrho_t^{-1} K_t - w_t L_t$$

where ζ_t is a technology shock with mean $E\zeta_t = 1$. Υ_t is the technology level growing at the exogenous rate g, hence $\Upsilon_{t+1} = (1+g)\Upsilon_t$. ϱ_t is an exogenous shock to the unit user costs of capital with mean $E\varrho_t = 1$. We add this non-standard element in order to model additional shocks to the rate of return to capital. These shocks are multiplicative in the user costs to capital for analytical reasons. In our full-blown quantitative model, these shocks will be replaced by shocks to the depreciation rate.[4]

[3] Our proof of equilibrium dynamics requires that all households are ex-ante identical which is not the case if idiosyncratic is present in the first period of life. We reintroduce idiosyncratic risk by slightly altering our model in a separate appendix. In this appendix we also shed more light on the analytical reasons for discarding idiosyncratic risk in the first place. Our extension features a subperiod structure where households also work a constant fraction in the second period. This allows us to reintroduce idiosyncratic risk in the second period while otherwise preserving the structure of the model. We briefly summarize our findings from this extension once we complete the discussion of the simpler version.

[4] Let shocks to the depreciation rate be δ_t. Our formulation with shocks to the user costs to capital can be translated into shocks to the depreciation rate to capital. In our quantitative model, we have that $R_t = \zeta_t \bar{R}_t - \delta_t$. Let $\zeta_t = 1$. We then have $\delta_t = \bar{R}_t(1 - \varrho_t)$.

Throughout we assume full depreciation and Cobb-Douglas production, hence $\bar{\delta} = 1$ and

$$F(K_t, \Upsilon_t L_t) = K_t^{\alpha} \left(\Upsilon_t L_t\right)^{1-\alpha},$$

where α is the capital elasticity of production. The firm first-order conditions then give:

$$1 + r_t = \alpha k_t^{\alpha-1} \zeta_t \varrho_t = \bar{R}_t \zeta_t \varrho_t \tag{10a}$$

$$w_t = (1-\alpha)\Upsilon_t k_t^{\alpha} \zeta_t = \bar{w}_t \zeta_t. \tag{10b}$$

Hence, ϱ_t is a shock to the gross return on savings. We also define by $\bar{R}_t \equiv \alpha k_t^{\alpha-1}$ the "ex-shock" component of the gross return and, correspondingly, by $\bar{w}_t = (1-\alpha)\Upsilon_t k_t^{\alpha}$ the "ex-shock" component of per capita wages.

Stochastic Processes in General Equilibrium

Observe that, relative to the partial equilibrium model of subsection 2.4, instead of shocks to wages and returns, shocks to productivity and the user costs of capital take over as stochastic primitives of the model. This also implies an explicit linkage the deterministic and stochastic components of wages and asset returns.[5] Consequently, assumption 2d is dropped. Also notice that assumptions 2e–f are irrelevant because of the absence of idiosyncratic risk. Replacing $\tilde{\varrho}_t$ by ϱ_t, assumptions 2a–c remain unaltered.

General Equilibrium Dynamics

Proposition 4. *Equilibrium dynamics in the economy are given by*

$$k_{t+1} = \frac{1}{1+g} \chi (1-\tau)(1-\alpha)\zeta_t k_t^{\alpha} \tag{11a}$$

where the savings rate χ is given by

$$\chi \equiv \frac{1}{1 + \left(\beta \alpha \bar{E}\right)^{-1}} \tag{11b}$$

and

$$\bar{E} \equiv E_t \left[\frac{1}{\alpha + (1-\alpha)\varrho_{t+1}^{-1}\tau} \right]. \tag{11c}$$

Analyzing the system of equations in (11) we find that, for $\tau > 0$, increasing the variance of return shocks, ϱ_t, reduces $\bar{E}$ and therefore decreases

[5] Notice that, in the notation of the partial equilibrium version of the model, $\tilde{\varrho}_t = \zeta_t \varrho_t$.

the saving rate, χ if $\tau > 0$. This is the human capital wealth effect of discounted pension income.[6] Notice that the discounted value of pension income is given by $\frac{b_{t+1}}{R_{t+1}}$. An increase in the variance of R_{t+1} hence increases discounted pension income in expectation. This increase the (expected) human capital wealth which, as in a standard deterministic model, increases first-period consumption and thereby decreases the saving rate. Increasing τ decreases $\bar{E}$ and therefore decreases the saving rate, χ. This is the crowding-out of private capital formation. Increasing α or β increases the savings rate.

A useful concept are mean shock equilibria which we will refer to below to evaluate welfare. Mean shock equilibria will also play an important role below in our computational analysis of our more elaborate quantitative model.

Definition 1. *In a mean shock equilibrium the realizations of all aggregate shocks are at their respective unconditional means, hence $\zeta_t = E\zeta_t = 1$, $\varrho_t = E\varrho_t = 1$ for all t. The corresponding equilibrium dynamics follow from (11a) as*

$$k_{t+1,ms} = \frac{1}{1+g}\chi(1-\tau)(1-\alpha)k_{t,ms}^{\alpha} \tag{12}$$

where χ is defined in (11b).

A stationary mean shock equilibrium is equivalent to a stochastic steady state:

Definition 2. *In a stationary mean shock equilibrium (=stochastic steady state) all variables grow at constant rates. In particular, we have that $k_{t,ms} = k_{ms}$ for all t which is given by*

$$k_{ms} = \left(\frac{1}{1+g}\chi(1-\tau)(1-\alpha)\right)^{\frac{1}{1-\alpha}}. \tag{13}$$

We then have:

Proposition 5. *A no social security ($\tau = 0$) stationary mean-shock equilibrium is dynamically efficient, i.e., $\bar{R}^{ms} > 1+g$, iff $\frac{\alpha}{1-\alpha} > \frac{\beta}{1+\beta}$.*

The proof is trivial and therefore omitted in the appendix. It immediately follows from the definition of $\bar{R}^{ms}$ which is given by

$$\bar{R}_{ms} = \alpha k_{ms}^{\alpha-1} = (1+g)\frac{\alpha}{(1-\alpha)}\frac{(1+\beta)}{\beta}. \tag{14}$$

[6] Since we assume log utility, income and substitution effects of stochastic interest rates cancel.

and the fact that the golden rule capital stock is $k^\star = \left(\frac{\alpha}{1+g}\right)^{\frac{1}{1-\alpha}}$.

In what follows, we assume a dynamically efficient economy. This implies that households loose in terms of welfare from decreasing the capital stock. Discounting is sufficiently strong such that induced gains from increasing returns (caused by decreasing the capital stock) are offset by decreasing wages.

Welfare Analysis in General Equilibrium

To simplify the analysis, we compare two long-run mean shock equilibria and thereby (again) ignore endogenous fluctuations of k_t and all transitional dynamics:

Proposition 6. *A marginal introduction of social security increases ex-ante expected utility in the long-run mean shock stationary equilibrium if*

$$\underbrace{\frac{\beta(1-\alpha)}{\alpha} E_{t-1}\left[\frac{1}{\zeta_t}\right] E_{t-1}\left[\frac{1}{\varrho_{t+1}}\right] - (1+\beta)}_{=A} + \underbrace{\frac{\beta(1-\alpha)}{\alpha(1+\beta)} E_{t-1}\left[\frac{1}{\varrho_{t+1}}\right]}_{=B} \\ - \underbrace{(\alpha(1+\beta) - \beta(1-\alpha)) \frac{1}{1-\alpha}\left(1 + \frac{1}{1+\beta}\frac{1-\alpha}{\alpha} E_t\left[\frac{1}{\varrho_{t+1}}\right]\right)}_{=C} > 0 \quad (15)$$

In equation (6), term A reflects the implicit rate of return condition. Combining equation (6) with equation (14) we get that $A > 0$ iff

$$(1+g)\left(E_{t-1}\left[\frac{1}{\zeta_t}\right] E_{t-1}\left[\frac{1}{\varrho_{t+1}}\right]\right) > \bar{R}_{ms}.$$

Observe that this condition immediately follows from condition (4) under assumption 5, equation (14) and acknowledging that $\tilde{\varrho}_t = \zeta_t \varrho_t$. It is therefore the general equilibrium equivalent to the partial characterization we gave above. Term B reflects the effects of introducing social security on the marginal propensity to consume, $(1-\chi)$, which was not present in the partial equilibrium setup. By assumption 2a term B is always positive. Finally, by the assumption of dynamic efficiency, term C is positive. As it enters with a negative sign, the term reflects the welfare losses incurred by crowding out of capital.

3 The Quantitative Model

Our quantitative model extends our simple model along several dimensions. First, rather than considering a stylized setup with two generations we take a periodicity of one calendar year and consider J overlapping generations. Second, we introduce one period ahead risk-free bonds. The primary reason for this extension is to impose discipline on calibration. Having a bond in the model means that our model entails predictions about general equilibrium asset prices. Any model on the welfare effects of social security should have realistic asset pricing implications. By providing a bond, we give households an additional asset to self-insure against idiosyncratic and aggregate risk. Ceteris paribus, this reduces the beneficial effects of social security. However, the presence of the bond also reduces the effect of decreasing savings on the crowding out of productive capital because part of the reduced savings is absorbed by the bond market.

3.1 Risk and Time

Time is discrete and runs from $t = 0, \ldots, \infty$. Risk is represented by an event tree. The economy starts with some fixed event z_0, and each node of the tree is a history of exogenous shocks $z^t = (z_0, z_1, ..., z_t)$. The shocks are assumed to follow a Markov chain with finite support Z and strictly positive transition matrix $\pi^z(z' \mid z)$. Let Π^z denote the invariant distribution associated with π^z. In our notation, we will make all aggregate and idiosyncratic shocks contingent on z_t. For notational convenience, we will suppress the dependency of all other variables on z^t but history dependence of all choice variables is understood.

3.2 Demographics

In each period t, the economy is populated by J overlapping generations of agents indexed by $j = 1, \ldots, J$, with a continuum of agents in each generation. Population grows at the exogenous rate of n. Households face an idiosyncratic (conditional) probability to survive from age j to age $j+1$ which we denote by ς_{j+1}, hence $\varsigma_1 = 1$ and $\varsigma_{J+1} = 0$. Consequently, given an initial population distribution $\{N_{0,j}\}_{j=1}^{J}$ which is consistent with constant population growth for all periods $t = 0, 1, \ldots$ and normalized such that $N_0 = \sum_{j=1}^{J} N_{t,j} = 1$, the exogenous law of motion of population in our model is given by

$$N_{t+1,1} = (1+n)N_{t,1}$$
$$N_{t+1,j+1} = \varsigma_{j+1} \cdot N_{t,j} \quad \text{for } j = 1, \ldots, J.$$

by $n = f^{\frac{1}{1+j_f}} - 1$. Households retire at the fixed age j_r. Labor supply is exogenous in our model and during the working period $j = 1, \ldots, j_r - 1$ each household supplies one unit of labor. Observe that constant population growth implies that population shares, e.g., the working age to population ratio, are constant.

3.3 Firms

Production of the final good takes place with a standard Cobb-Douglas production function with total output at time t given by

$$Y_t = F(\zeta(z_t), K_t, L_t) = \zeta(z_t)K_t^{\alpha}(\Upsilon_t L_t)^{1-\alpha} \tag{16}$$

where K_t is the aggregate stock of physical capital, L_t is labor, $\zeta(z_t)$ is a stochastic shock to productivity and Υ_t is the deterministic level of technology which grows at the exogenous rate of g.

The economy is closed. The consumption good can either be consumed in the period when it is produced or can be used as an input into a production technology producing capital. We ignore capital adjustment costs. Accordingly, the production technology for capital is

$$\begin{aligned} K_{t+1} &= I_t + K_t(1-\delta(z_t)) \\ &= Y_t - C_t + K_t(1-\delta(z_t)) \end{aligned} \tag{17}$$

where $\delta(z_t)$ is the stochastic depreciation rate of physical capital.

Firms maximize profits and operate in perfectly competitive markets. Accordingly, the rate of return to capital and the wage rate are given by

$$w_t = (1-\alpha)\Upsilon_t \zeta(z_t)k_t^{\alpha} \tag{18a}$$
$$r_t = \alpha\zeta(z_t)k_t^{\alpha-1} - \delta(z_t) \tag{18b}$$

where $k_t = \frac{K_t}{\Upsilon_t L_t}$ is the capital stock per unit of efficient labor which we refer to as "capital intensity".

3.4 Endowments

Agents are endowed with one unit of labor which is supplied inelastically for ages $j = 1, \ldots, j_r - 1$. After retirement, labor supply is zero. Households have access to two savings storage technologies. Either they save in the risky technology at rate of return r_t or in a one-period risk-free bond at return r_t^f which is in zero net supply. Households are subject to idiosyncratic shocks to their labor productivity. In our notation, we suppress an index to denote household types. We denote total assets by A_j, and the share invested in the risky asset by κ_j.

Additional elements of the dynamic budget constraint are income, Υ_j, to be specified below and consumption, C_j. The dynamic budget constraint of a household at age j then reads as

$$A_{j+1} = A_j(1 + r_t^f + \kappa_{j-1}(r_t - r_t^f)) + \Upsilon_j - C_j \tag{19}$$

where $\kappa_{i,j-1} \in [-\underline{\kappa}, \overline{\kappa}]$, for all j. This restricts the leverage in stocks in our model.[7]

Income is given by

$$\Upsilon_j = \begin{cases} (1-\tau)\epsilon_j w_t \eta & \text{for } j < jr \\ B_j & \text{for } j \geq jr \end{cases} \tag{20}$$

where ϵ_j is age-specific productivity and η is an idiosyncratic stochastic component.

We assume that η follows a time and age-independent Markov chain whereby the states of the Markov chain are contingent on aggregate states z. Accordingly, let the states be denoted by $\mathcal{E}_z = \{\eta_{z1}, \ldots, \eta_{zM}\}$ and the transition matrices be $\pi^\eta(\eta' \mid \eta) > 0$. Let Π^η denote the invariant distribution associated with π^η.

As for pension income, we assume that pension payments are lump-sum, hence

$$B_j = b_t \Upsilon_t \tag{21}$$

where b_t is some normalized pension benefit level which only depends on t. Accordingly, the pension system fully redistributes across household types. This is an approximation to the U.S. pension system.[8]

[7] In a model without a constraint of the form $\kappa_{j-1} \in [-\underline{\kappa}, \overline{\kappa}]$ we have a singularity at $X_j - C_j = 0$ so that, for $X_j - C_j \to +0$, $\kappa_j \to +\infty$ and for $X_j - C_j \to -0$, $\kappa_j \to -\infty$. The presence of the singularity has consequences for aggregation because the set for κ will not be compact. We set the constraint in order to rule out this technicality, but we set the bounds so high that the constraint will rarely be binding in equilibrium.

[8] The U.S. pension system links contributions to AIME, the average indexed monthly

3.5 Preferences

We take Epstein-Zin preferences. Let θ be the coefficient of relative risk-aversion and φ denote the inter-temporal elasticity of substitution. Then

$$U_j = \left[C_j^{\frac{1-\theta}{\gamma}} + \beta \varsigma_{j+1} \left(\mathbb{E}_j \left[U_{j+1}^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \tag{22}$$

where $\gamma = \frac{1-\theta}{1-\frac{1}{\varphi}}$, and $\beta > 0$ is the standard discount factor. For $\theta = \frac{1}{\varphi}$ we have $\gamma = 1$ and are back to CRRA preferences. $\mathbb{E}_j$ is the expectations operator and expectations, conditional on information at age j, are taken with respect to idiosyncratic wage shocks and aggregate productivity and depreciation shocks. As $\varsigma_{J+1} = 0$, equation (22) implies that $U_J = C_J$.

3.6 The Government

The government organizes a PAYG financed social security system. We take the position that social security payments are not subject to political risk. We assume that the budget of the social security system is balanced in all periods. We describe various social security scenarios below. We further assume that the government collects all accidental bequests and uses them up for government consumption which is otherwise neutral.

3.7 Equilibrium

To define equilibrium we adopt a de-trended version of the household model. We therefore first describe transformations of the household problem and then proceed with the equilibrium definition.

[To do: move transformations to the appendix. Define equilibrium in terms of assets, not cash-on-hand.]

Transformations

Following Deaton (1991), define cash-on-hand by $X_j = A_j(1 + r_t^f + \kappa_{j-1}(r_t - r_t^f)) + \mathrm{Y}_j$. The dynamic budget constraint (19) then rewrites as

$$X_{j+1} = (X_j - C_j)(1 + r_{t+1}^f + \kappa_j(r_{t+1} - r_{t+1}^f)) + \mathrm{Y}_{j+1} \tag{23}$$

earnings and has an additional distributional component by the so-called bend point formula. From an ex-ante perspective, given this distributional component and provided that income shocks are non-permanent, an approximation with lump-sum pension benefits is a good first-order approximation.

We next transform the problem to de-trend the model and work with stationary variables throughout. That is, we de-trend with the deterministic trend component induced by technological progress. Along this line, define by $x_j = \frac{X_j}{\Upsilon_t}$ transformed cash-on-hand and all other variables accordingly. Using $\omega_t = \frac{w_t}{\Upsilon_t}$ to denote wages per efficiency unit we have

$$\mathrm{y}_j = \begin{cases} (1-\tau)\epsilon_j \omega_t \eta & \text{for } j < jr \\ b & \text{for } j \geq jr. \end{cases}$$

Now divide the dynamic budget constraint (23) by Υ_t and rewrite to get

$$x_{j+1} = (x_j - c_j)\tilde{R}_{j+1} + \mathrm{y}_{j+1}. \tag{24}$$

where $\tilde{R}_{j+1} = \frac{(1+r^f_{t+1}+\kappa_j(r_{t+1}-r^f_{t+1}))}{1+g}$.

Transform the per period utility function accordingly and take an additional monotone transformation to get

$$u_j = \left[c_j^{\frac{1-\theta}{\gamma}} + \beta_{j+1} \left(\mathbb{E}_j \left[(u_{j+1})^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \tag{25}$$

where $\tilde{\beta}_{j+1} = \beta\varsigma_{j+1} (1+g)^{\frac{1-\theta}{\gamma}}$.

Definition of Equilibrium

Individual households, at the beginning of period t are indexed by their age j, their idiosyncratic productivity state η, their cash on hand holdings x, and a measure $\Phi(j,x,\eta)$ which describes the beginning of period wealth distribution in the economy, i.e., the share of agents at time t with characteristics (j,x,η). We normalize such that $\int d\Phi = 1$. Existence of aggregate shocks implies that Φ evolves stochastically over time. We use H to denote the law of motion of Φ which is given by

$$\Phi' = H(\Phi, z, z') \tag{26}$$

Notice that z' is a determinant of Φ' because it determines $\tilde{R}_{j+1}$ and therefore the distribution over $x\prime$.

A change in policy induces a transition of the economy from an initial stationary equilibrium to another. In analogy to models without aggregate risk, the aggregate law of motion will be time dependent, see Ludwig (2010) and the appendix for further details.

The de-trended version of the household problem writes as

$$u(j,x,\eta;z,\Phi) = \max_{c,\kappa,x'} \left\{ \left[c^{\frac{1-\theta}{\gamma}} + \tilde{\beta} \left(\mathbb{E}\left[\left(u(j+1,x',\eta';z',\Phi')\right)^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \right\} \tag{27a}$$

$$\text{s.t. } x' = (x-c)\tilde{R}' + \mathrm{y}' \tag{27b}$$

$$\tilde{R}' = \frac{(1 + r^{f'} + \kappa(r' - r^{f'}))}{1+g} \tag{27c}$$

$$\Phi' = H(\Phi, z, z'). \tag{27d}$$

We therefore have the following definition of the recursive equilibrium of our economy:[9]

Definition 3. *A recursive competitive equilibrium is a value function u, policy functions for the household, $x'(\cdot)$, $a'(\cdot)$, $c(\cdot)$, $\kappa(\cdot)$, policy functions for the firm, $K(\cdot)$, $L(\cdot)$, pricing functions $r(\cdot)$, $q(\cdot)$, $w(\cdot)$, policies, τ, b, aggregate measures $\Phi(\cdot)$ and an aggregate law of motion, H_t such that*

1. *$u(\cdot), x'(\cdot), a'(\cdot), c(\cdot), \kappa(\cdot)$ are measureable, $u(\cdot)$ satisfies the household's recursive problem and $x'(\cdot), a'(\cdot), c(\cdot), \kappa(\cdot)$ are the associated policy functions, given r, q, ω, τ and b.*

2. *K, L satisfy, given $r(\Phi, z)$ and $w(\Phi, z)$,*

$$\omega(\Phi,z) = (1-\alpha)\zeta(z)k(\Phi,z)^{\alpha} \tag{28a}$$

$$r(\Phi,z) = \alpha\zeta(z)k(\Phi,z)^{\alpha-1} - \delta(z). \tag{28b}$$

where $k(\Phi,z) = \frac{K(\Phi,z)}{\Upsilon L}$ is the capital stock per efficiency unit (or "capital intensity") and $\Upsilon = (1+g)\Upsilon_{-1}$ is the technology level in period t.

3. *neutral government consumption financed by bequests is given by*

$$gc' = \frac{\int (1-\varsigma_{j+1}) a'(j,x,\eta;z,\Phi)\, R'(\kappa(\cdot)) d\Phi}{\ell(1+n)(1+g)} \tag{29}$$

where

$$R'(\kappa(\cdot)) = (1 + r^{f'} + \kappa(j,x,\eta;z,\Phi)(r' - r^{f'})).$$

[9] We will add the measure theoretic stuff later. We use the integration operator $\int$ as a short-cut notation for all sums and integrals involved but discreteness of the characteristics (j,z) is understood. When integrating out with respect to all characteristics of the distribution, we simply write $d\Phi$, hence $\int \cdot d\Phi = \int \cdot \Phi(dj \times dx \times d\eta)$. When we integrate only with respect to a subset of characteristics, we make this explicit by, e.g., writing $\int \cdot \Phi(j, dx \times d\eta)$.

4. *the pension system budget constraint holds, i.e.*

$$\tau(\Phi, z)\omega(\Phi, z) = b(\Phi, z)p \tag{30}$$

where p is the economic dependency ratio which is stationary in our model.[10]

5. *For all Φ and all z*

$$k(H(\Phi, z, z'), z')(1+g)(1+n) = \frac{1}{\ell}\int \kappa(j, x, \eta; z, \Phi)a'(j, x, \eta; z, \Phi)d\Phi \tag{31a}$$

$$0 = \int (1 - \kappa(j, x, \eta; z, \Phi))a'(j, x, \eta; z, \Phi)d\Phi \tag{31b}$$

$$i(\Phi, z) = f(k(\Phi, z)) - gc - \frac{1}{\ell}\int c(j, x, \eta; z, \Phi)d\Phi \tag{31c}$$

$$k(H(\Phi, z, z'), z')(1+g)(1+n) = k(\Phi, z)(1 - \delta(z)) + i(\Phi, z) \tag{31d}$$

where ℓ is the working age to population ratio[11]*, equation (31b) is the bond market clearing condition and the bond price q is determined such that it clears the bond market in each period t and $i(\cdot) = \frac{I(\cdot)}{\Upsilon L}$ is investment per efficiency unit.*

6. *The aggregate law of motion H is generated by the exogenous population dynamics, the exogenous stochastic processes and the endogenous asset accumulation decisions as captured by the policy functions x'.*

Definition 4. *A stationary recursive competitive equilibrium is as described above but with time constant individual policy functions $x'(\cdot)$, $a'(\cdot)$, $c(\cdot)$, $\kappa(\cdot)$ and a time constant aggregate law of motion H.*

3.8 Welfare Criteria

At this stage, we only compare two stationary equilibria and do not take into account transitional dynamics. Our welfare concept is the consumption-equivalent variation for a newborn before any shocks are realized. It is an ex-ante perspective where the agent does not know the aggregate state nor the level of capital that he will be born into. A positive number then states

[10] It is given by $p = \frac{\sum_{j=j_r}^{J}(1+n)^{J-j}\prod_{i=1}^{j-1}\varsigma_i}{\sum_{j=1}^{j_r-1}(1+n)^{J-j}\epsilon_j\prod_{i=1}^{j-1}\varsigma_i}$.

[11] It is given by $\ell = \frac{\sum_{j=1}^{j_r-1}(1+n)^{J-j}\epsilon_j\prod_{i=1}^{j-1}\varsigma_i}{\sum_{j=1}^{J}(1+n)^{J-j}\prod_{i=1}^{j-1}\varsigma_i}$.

the amount an agent would be willing to give up in order to be born into the second long-run equilibrium (i.e. into an economy with some social security).

Note that this comparison between the long-run equilibria provides a lower bound on the expected welfare gains for the newborns along the transition, because they are spared some of the negative effects of crowding out, and because they get to save less and consume more as the level of capital moves toward its new, lower level.

3.9 Thought Experiment

At this stage, we only compare two stationary equilibria. In our initial equilibrium a social security system does not exist. In the second equilibrium, the economy features a social security system with a contribution rate of 2 percent. One can think of this as the introduction of a 'marginal' social security system as described in Krueger and Kubler (2006). We use their proposition 1 to ensure that the initial economy is dynamically efficient so as to rule out any welfare gains that would come from curing dynamic inefficiency.

We then use the exact same economy to conduct partial equilibrium (PE) experiments that enable us to disentangle the welfare gains due to insurance from the welfare losses due to crowding out and its associated price changes. In this partial equilibrium, we feed in the sequence of shocks and prices $\{z_t, r_t, r_t^f, w_t\}_{t=1}^T$ obtained from the associated general equilibrium (GE). It is like a small open economy, where aggregate prices are determined by the world and fluctuate over time and are not influenced by domestic policy changes. If we do not change any other parameter, then the results are naturally exactly the same as in the associated GE. To isolate the total insurance effects, we let agents optimize under the new policy, i.e. $\tau = 0.2$, but with the 'old' approximate laws of motion that still hold for the evolution of aggregate prices. Then we simulate by feeding in the old sequence of shocks and prices, but with the new policy functions and the new social security system.

In a very similar fashion, we isolate the insurance against aggregate risk, idiosyncratic risk, CCV, and survival risk. In order to isolate the interaction effect LCI, we conduct a difference-in-difference calculation, which works as follows: we keep idiosyncratic wage risk IR constant and compare two economies 1 and 2 with different levels of aggregate risk. This provides information on the interaction term. To understand this, recall from equation (7) that, under the level of aggregate risk in economy i, we have the second-order approximation

$$E_{t-1}u_{t,i} \approx \frac{1}{1-\theta}\left(\bar{w}_t\bar{R}\right)^{1-\theta} - \theta\left(\bar{w}_t\bar{R}\right)^{1-\theta}\left(IR_i + AR_i + LCI_i\right).$$

for $i = 1, 2$. Since $IR_1 = IR_2$ by assumption we then get

$$\Delta E_{t-1}u_t = E_{t-1}[u_{t,1} - u_{t,2}] = -\theta\left(\bar{w}_t\bar{R}\right)^{1-\theta}\left(\Delta AR + \Delta LCI\right).$$

Hence, given that we can measure ΔAR we can compute ΔLCI as

$$\Delta LCI = \frac{\Delta E_{t-1}u_t}{\theta\left(\bar{w}_t\bar{R}\right)^{1-\theta}} - \Delta AR$$

Recall that the interaction was defined as $LCI \equiv IR \cdot AR$, so that, by the simple product rule of differentiation,

$$\frac{\partial LCI}{\partial \sigma_{AR}} = \frac{\partial IR}{\partial \sigma_{AR}}AR + \frac{\partial AR}{\partial \sigma_{AR}}IR.$$

By definition $\frac{\partial IR}{\partial \sigma_{AR}} = 0$, so that we can isolate IR as

$$\frac{\frac{\partial LCI}{\partial \sigma_{AR}}}{\frac{\partial AR}{\partial \sigma_{AR}}} = \frac{\Delta LCI}{\Delta AR} = IR$$

Note that this is the level of pure idiosyncratic risk in both economies $i = 1, 2$. The level of the interaction term in the first economy is then

$$LCI_1 = \frac{E_{t-1}u_{t,1} - \frac{1}{1-\theta}\left(\bar{w}_t\bar{R}\right)^{1-\theta}}{-\theta\left(\bar{w}_t\bar{R}\right)^{1-\theta}} - AR_1 - IR_1$$

3.10 Computational Details

Following Gomes and Michaelides (2006) and Storesletten, Telmer, and Yaron (2007) we compute an approximate equilibrium of our model by applying the Krusell and Smith (1998) method. We approximate the solution by considering forecast functions of the average capital stock in the economy and the ex-ante equity premium. In the general equilibrium version of our model, we loop on the postulated laws of motion until convergence. We do so by simulating the economy for $T = 5000$ periods and discard the first 500 initialization periods. In each period, we compute the market clearing bond price. The goodness of fit of the approximate laws of motion is $R^2 = 0.99$.

We compute solution to the household model by adopting Carroll's endogenous grid method, which reduces computational time strongly. Written in Fortran 2003, the model takes about one hour to converge to a solution, given a decent initial guess for the laws of motion.

A more detailed description of our computational methods can be found in our appendix.

4 Calibration

4.1 Overview

Part of our parameters are exogenously calibrated either by reference to other studies or directly from the data. We refer to these parameters as first stage parameters. A second set of parameters is calibrated by informally matching simulated moments to respective moments in the data. Accordingly, we refer to those parameters as second stage parameters.

The theoretical discussion in section 2.4 emphasized that the correlation between TFP and returns will play a crucial role when evaluating social security benefits. To address this, we take two views with regard to the data generating process of observed TFP (or wage) fluctuations. First, we detrend the data with a linear trend, thereby following the approach of Krueger and Kubler (2006), and -like them- find that the correlation is negative. Second, we assume a unit root process for (the log of) TFP and detrend by first differences, which yields a highly significant positive correlation. We will argue that this is a more appropriate approach. In our discussion of robustness in section 5.3 we show that when a Hodrick-Prescott filter is used, the correlation is again positive and significant.

Table 1 summarizes the calibration. Table 2 contains the information on the stochastic processes of log TFP and the corresponding approximation according to our two approaches: NC stands for the negative correlation between TFP and returns which results from the linear trend specification, while PC stands for the positive correlation which results from the de-trending with first differences. Since all other targets in the calibration remain the same, we need to slightly adjust some endogenous parameters, which are displayed in table 3 for both specifications. The next subsections contain a detailed description of our methodology.

Table 1: Calibration: Summary

Parameter	Value	Target (source)	Stage
Preferences			
Discount factor, β	cf. table 3	Capital output ratio, 2.65 (NIPA)	2
Coefficient of relative risk aversion, θ	cf. table 3	Average equity premium, 0.056 (Shiller)	2
Intertemporal elasticity of substitution, φ	1.5	Consumption Profile	2
Technology			
Capital share, α	0.32	Wage share (NIPA)	1
Leverage, b	0.66	Croce (2010)	1
Technology growth, g	0.018	TFP growth (NIPA)	1
Mean depreciation rate of capital, δ_0	0.0418	Risky return, 0.079 (Shiller)	1
Std. of depreciation $\bar{\delta}$	cf. table3	Std. of risky return, 0.168 (Shiller)	2
Aggregate productivity states, $1 \pm \bar{\zeta}$	$\{1.029, 0.971\}$	Std. of TFP, 0.029 (NIPA)	1
Transition probabilities of productivity, π^{ζ}	0.941	Autocorrelation of TFP, 0.88 (NIPA)	1
Conditional prob. of depreciation shocks, π^{δ}	cf. table 3	Corr.(TFP, returns), 0.36 (NIPA, Shiller)	2
Idiosyncratic Productivity			
Age productivity, $\{\epsilon_j\}$	-	Earnings profiles (PSID)	1
CCV $\sigma_{\nu(z)}$	$\{0.21, 0.13\}$	Storesletten, et al. (2007)	1
Autocorrelation ρ	0.952	Storesletten, et al. (2007)	1
Demographics: Exogenous parameters			
Biological age at birth	1		1
Retirement age, j_r	45		1
End of life, J	70		1
Survival rates, $\{s_j\}$	-	Population data (HMD)	1
Population growth, n	0.011	U.S. Social Sec. Admin. (SSA)	1

Table 2: Calibration: Estimates of aggregate risk

	NC	PC
Corr. (TFP, returns), $cor(\zeta_t, r_t)$	-0.08 (0.57)	0.50 (0.00)
Corr. (wages, returns), $cor(w_t, r_t)$	-0.33 (0.016)	0.306 (0.025)

Notes: NC: Negative correlation between TFP shocks and returns (linear trend estimation), PC: Positive correlation between TFP shocks and returns (first differences estimation).
p-values are reported in brackets.

Table 3: Calibration: Endogenous parameters

	NC	PC
Preferences		
Discount factor, β	0.96	0.97
Relative risk aversion, θ	12	8
Technology		
Std. of depreciation $\bar{\delta}$	0.10	0.11
Cond. prob. depr. shocks, π^{δ}	0.435	0.86

Notes: NC: Negative correlation between TFP shocks and returns (linear trend estimation), PC: Positive correlation between TFP shocks and returns (first differences estimation).

4.2 Production Sector

We set the value of the capital share parameter, a first stage parameter, to $\alpha = 0.32$. This is directly estimated from NIPA data (1960-2005) on total compensation as a fraction of (adjusted) GDP. Our estimated value is in the range of values considered as reasonable in the literature. It is close to the preferred value of 0.3 as used by Krueger and Kubler (2006). To estimate α, we take data on total compensation of employees (NIPA Table 1.12) and deflate it with the GDP inflator (NIPA Table 1.1.4). In the numerator, we adjust GDP (NIPA Table 1.1.5), again deflated by the GDP deflator, by nonfarm proprietors' income and other factors that should not be directly related to wage. Without these adjustments, our estimate of α would be considerably higher, i.e., at $\alpha = 0.43$.

To determine the mean depreciation rate of capital, a first stage parameter in our model, we proceed as follows. We first estimate the capital output ratio in the economy. To measure capital, we take the stock of fixed assets (NIPA Table 1.1), appropriately deflated. We relate this to total GDP. This gives an estimate of the capital output ratio of $K/Y = 2.65$, in line with the estimates by, e.g., Fernandez-Villaverde and Krüger (2005), or of the ratio of output to capital of 0.38. This implies an average marginal product of capital $E[mpk] = \alpha E[Y/K] = 0.12$. Given this estimate for the marginal product of capital and our estimate for the average risky return on capital of 0.079 based on data since 1950 provided by Rob Shiller, we set $E[\delta] = E[mpk] - E[r] = 0.042$.[12]

Our estimate of the deterministic trend growth rate, also a first stage parameter, is $g = 0.018$ which is in line with other studies. We determine it by estimating the Solow residual from the production function, given our estimate of α, our measure for capital, and a measure of labor supply determined by multiplying all full- and part-time employees in domestic employment (NIPA Table 6.4A) with an index for aggregate hours (NIPA Table 6.4A).[13] We then fit a linear trend specification to the Solow residual. Acknowledging the labor augmenting technological progress specification chosen, this gives the aforementioned point estimate.

[12] The data was downloaded from Rob Shillers webpage, http://www.econ.yale.edu/ shiller/data.htm.

[13] Notice that we thereby ignore age-specific productivity which should augment our measure of employment.

4.3 Aggregate states and shocks

We assume that aggregate risk is driven by a four state Markov chain with support $Z = \{z_1, \dots, z_4\}$ and transition matrix $\pi = (\pi_{ij})$. Each aggregate state maps into a combination of low or high technology shocks and low or high physical capital depreciation. To be concrete, we let

$$\zeta(z) = \begin{cases} 1 - \bar{\zeta} \text{ for } z \in z_1, z_2 \\ 1 + \bar{\zeta} \text{ for } z \in z_3, z_4 \end{cases} \quad \text{and } \delta(z) = \begin{cases} \delta_0 + \bar{\delta} \text{ for } z \in z_1, z_3 \\ \delta_0 - \bar{\delta} \text{ for } z \in z_2, z_4. \end{cases} \tag{32}$$

With this setup, z_1 corresponds to a low wage and a low return, while z_4 corresponds to a high wage and a high return.

To calibrate the entries of the transition matrix, denote by $\pi^\zeta = \pi(\zeta' = 1-\bar{\zeta} \mid \zeta = 1-\bar{\zeta})$ the transition probability of remaining in the low technology state. Assuming that the transition of technology shocks is symmetric, we then also that $\pi(\zeta' = 1 + \bar{\zeta} \mid \zeta = 1 + \bar{\zeta}) = \pi^\zeta$ and, accordingly $1 - \pi^\zeta = \pi(\zeta' = 1 - \bar{\zeta} \mid \zeta = 1 + \bar{\zeta}) = \pi(\zeta' = 1 + \bar{\zeta} \mid \zeta = 1 - \bar{\zeta})$.

To govern the correlation between technology and depreciation shocks, let the probability of being in the high (low) depreciation state conditional on being in the low (high) technology state, assuming symmetry, be $\pi^\delta = \pi(\delta' = \delta_0 + \bar{\delta} \mid \zeta' = 1 - \bar{\zeta}) = \pi(\delta' = \delta_0 - \bar{\delta} \mid \zeta' = 1 + \bar{\zeta})$. We then have that the transition matrix of aggregate states follows from the corresponding assignment of states in (32) as

$$\pi^z = \begin{bmatrix} \pi^\zeta \cdot \pi^\delta & \pi^\zeta \cdot (1-\pi^\delta) & (1-\pi^\zeta) \cdot (1-\pi^\delta) & (1-\pi^\zeta) \cdot \pi^\delta \\ \pi^\zeta \cdot \pi^\delta & \pi^\zeta \cdot (1-\pi^\delta) & (1-\pi^\zeta) \cdot (1-\pi^\delta) & (1-\pi^\zeta) \cdot \pi^\delta \\ (1-\pi^\zeta) \cdot \pi^\delta & (1-\pi^\zeta) \cdot (1-\pi^\delta) & \pi^\zeta \cdot (1-\pi^\delta) & \pi^\zeta \cdot \pi^\delta \\ (1-\pi^\zeta) \cdot \pi^\delta & (1-\pi^\zeta) \cdot (1-\pi^\delta) & \pi^\zeta \cdot (1-\pi^\delta) & \pi^\zeta \cdot \pi^\delta \end{bmatrix}$$

In sum, the Markov chain process of aggregate shocks is characterized by four parameters, $(\bar{\zeta}, \bar{\delta}, \pi^\zeta, \pi^\delta)$. All of these parameters are second stage parameters which we calibrate jointly to match the following targets: (i) an average variance of the cyclical component of TFP, again estimated from NIPA data, (ii) the average fluctuation of the risky return which features a standard deviation in the data of 0.16, (iii) the autocorrelation of the cyclical component of TFP in the data and (iv) the estimated correlation of the cyclical component of TFP with risky returns.

As to the latter targets we calibrate two versions which reflect different views on the nature of the data generating process of observed TFP fluctuations.

First, we adopt the Krueger and Kubler (2006) approach to the data by assuming a linear trend as a filter to get the stochastic components in the data. Such a linear trend specification can be justified on the grounds that the model features such a trend, and that the underlying covariance structures should remain unaffected.[14] The results are in line with Krueger and Kubler (2006) and are shown in the first column (labeled NC) of table 2.[15] The correlation between wages and returns is estimated to be negative, while the correlation between TFP and returns is negative but statistically not different from zero.

In our second approach, we assume a unit root process for the log of TFP. Applying a first difference filter to the data, we find that the correlation between TFP and returns as well as wages and returns is positive, the former being larger in magnitude than the latter. This finding coincides with our economic intuition as we would expect these variables to co-move over the cycle.

For sake of consistency, we then transform the numbers to an equivalent deterministic trend specifications in the following way. We stick to the Krueger and Kubler (2006) calibration and only adopt the new correlation structure between TFP innovations and returns. This means that we implicitly compute the average horizon h in the unit root model such that the unconditional variance over h periods coincides with the KK calibration. This gives an average horizon of $h = 19.2751$ years.[16]

In order to check the robustness of our findings, we also adopt a standard RBC view of the data and de-trend with the Hodrick-Prescott filter. This yields a highly significant, positive correlation which is comparable in magnitude to our preferred PC (finite difference) calibration. Details can be found in the robustness section 5.3.

[14] Rather than to TFP fluctuations, Krueger and Kubler (2006) refer to stochastic processes of aggregate wages. By the static first order conditions of the firm problem, these two are highly related, cf. equation (18).

[15] The differences between their and our estimates arise because they have to aggregate the data to 6-year intervals to match the period length of their model, whereas we use the yearly data since we have 1-year intervals.

[16] Observe that the unit root estimates in fact imply even stronger aggregate fluctuations. Adjusting the variance in the linear trend specification such that the average horizon equals the average horizon of households in our model, appropriately adjusted to account for the correlation of TFP innovations, gives an average horizon of 34.88 years. This implies a standard deviation of 0.039. Relative to the PC calibration this means that the standard deviation of innovations increases by roughly 76 percent. However, the overall effects of this additional increase in risk are small. Results are available upon request.

4.4 Population data

We assume that agents start working at the biological age of 21, which therefore corresponds to $j = 1$. We set $J = 70$, implying that agents die with certainty at biological age 90, and $jr = 45$, corresponding to a statutory retirement age of 65. Population grows at the rate of 1.1% which reflects the current trend growth of the US population. The conditional survival rates ς_j are imputed from mortality data retrieved from the Human Mortality Database (HMD).

4.5 Household sector

The value of household's raw time discount factor, β, and the coefficient of relative risk aversion θ are calibrated endogenously (second stage parameters) such that our model produces a capital output ratio of 2.65 and an average equity premium of 0.056.

We determine the intertemporal substitution elasticity as a second-stage parameter such that our model generates a hump-shaped consumption profile. This is achieved via a relatively high value of $\varphi = 1.5$. It is consistent with the range discussed in Bansal and Yaron (2004) and lower than their benchmark value of 2. We document the sensitivity of our results with respect to this parameter in section 5.3.2.

The age-specific productivity profile ϵ_j is calibrated to match PSID data applying the method of Huggett, Ventrua, and Yaron (2011).

Our calibration of states $\mathcal{E}_z$ and transition probabilities π^η of the idiosyncratic Markov chain income processes is based on estimates of Storesletten, Telmer, and Yaron (2004), henceforth STY, for individual wage income processes. STY postulate that the permanent shocks obey an $AR(1)$ process given as

$$\ln(\eta)_{j,t} = \rho \ln(\eta)_{j-1,t-1} + \epsilon_{j,t} \tag{33}$$

where

$$\epsilon_{j,t} \sim \mathcal{N}(0, \sigma_t^2) \tag{34}$$

Building on Constantinides and Duffie (1996), STY assume a countercyclical, cross-sectional variance of the innovations (CCV). Their estimates are $\rho = 0.952$ and

$$\sigma_t^2 = \begin{cases} \sigma_c^2 = 0.0445 \text{ for } z \in z_1, z_2 \\ \sigma_e^2 = 0.0156 \text{ for } z \in z_3, z_4 \end{cases} \tag{35}$$

where e stands for expansion and c for contraction.

We approximate the above process by discrete two-state Markov process. Denoting state contingency of the innovations by $\sigma^2(z)$, observe that $\sigma(z)^2_{\ln\eta} = \frac{\sigma^2(z)}{1-\rho^2}$. We then approximate the underlying η_t by the following symmetric Markov process:

$$\mathcal{E}_z = [\eta_1(z), \eta_2(z)] = [\eta_-(z), \eta_+(z)] \tag{36}$$

$$\pi^\eta = \begin{bmatrix} \bar{\pi}^\eta & 1-\bar{\pi}^\eta \\ 1-\bar{\pi}^\eta & \bar{\pi}^\eta \end{bmatrix} \tag{37}$$

$$\Pi = [0.5, 0.5]$$

so that the unconditional mean of the state vector is equal to 1.

Our approximation is different from standard approximations of log income processes in two respects. First, standard approximations do not condition on aggregate states. Second, standard approximations ignore a bias term which gets large when the variance of the estimates increases. We describe the details of our procedure in appendix B.3. Resulting estimates are

$$\eta_1 = \eta_- = \begin{cases} 0.4225 & \text{for } z = z_1, z_2 \\ 0.6196 & \text{for } z = z_3, z_4 \end{cases} \qquad \eta_2 = \eta_+ = \begin{cases} 1.5775 & \text{for } z = z_1, z_2 \\ 1.3804 & \text{for } z = z_3, z_4 \end{cases}$$

and $\bar{\pi}^\eta = 0.9741$.

5 Results

In the discussion of the main results of our quantitative analysis we will refer to the insights derived from the simple model of section 2. In particular, we will highlight the insurance effects against idiosyncratic risk (IR), aggregate risk (AR), and their interaction (LCI) as defined in equations (6) and (7), and oppose them with the costs of crowding out denoted by term C in equation (15).

In order to expose the commonalities and differences to Krueger and Kubler (2006), we will first analyze the results of the calibration with a negative correlation between TFP and returns. Then we compare that to the calibration with a positive correlation. As argued by KK, a negative correlation should increase the intergenerational insurance provided by social security. However, our companion paper Harenberg and Ludwig (2011) exposes this to be a fallacy resulting from the essentially atemporal structure of the simple toy model, and shows that a positive correlation should increase the insurance effects of social security. The next two subsections will give some quantitative answers on how much these effects matter.

In the third subsection, we discuss the robustness of our results when we either conduct a different thought experiment, or change the elasticity of intertemporal substitution φ, or use a calibration with a nonlinear trend.

5.1 Calibration with negative correlation between TFP and returns (NC)

This calibration matches the aggregate statistics in the column labeled NC in table 2. The corresponding endogenous parameters are displayed in the first column of table 3. As expected, the negative correlation between TFP and returns, $cor(\zeta, r)$, leads to a negative correlation between wages and returns, $cor(w, r)$, which is documented in table 4. While we target $cor(\zeta, r)$, KK target a negative $cor(w, r)$ directly. The table also shows that the standard deviation of aggregate consumption growth, $std(\Delta C/C)$, is counterfactually high, which is a direct consequence of the depreciation shocks that we employ to match the variance of risky returns.[17]

Table 4: Model-generated moments (NC)

$cor(w, r)$	$std(\Delta C/C)$
-0.079	0.073

The effects of our social security experiment on welfare, capital, and prices are documented in table 5. In the first column (labeled 'GE' for general equilibrium), we compare the two long-run equilibria without any transition. We see that the increase of the contribution rate from $\tau = 0.0$ to $\tau = 0.02$ leads to welfare gains of +0.51%. This number represents the percent of lifetime consumption the agent would be willing to give up to be born into the economy with some social security. There is substantial crowding out of capital of -5.91%, which leads to the displayed price changes, but this adverse effect is not strong enough to overturn the benefits from insurance.

In order to isolate those insurance benefits, we conduct the partial equilibrium (PE) experiment described in section 3.9. One can think of it as a small open economy, where aggregate prices are determined in the world, and social security is introduced in the small home country. As the second column in table 5 shows, the net welfare gains attributable to the total insurance provided by social security amount to +5.10%. Aggregate prices

[17] We reduced $std(\Delta C/C)$ significantly by introducing a capital structure of the firm, but we haven't yet included that model element in the description of the firm.

in this world do not change by construction, and that is why we can isolate the insurance effects. Therefore, the difference between the two welfare numbers $0.51\% - 5.10\% = -4.61\%$ can be attributed to the crowding out of capital, which corresponds to term C in equation (15). Finally, the $\Delta K/K = -24.90\%$ in PE should be interpreted as 'less capital being invested abroad': of course agents save much less for old-age retirement, and this effect is much smaller in GE because of the mitigating price adjustments.

Table 5: The social security experiment (NC)

	GE	PE
ΔWelf/Welf	+0.51%	+5.10%
$\Delta K/K$	-5.91%	-24.90%
$\Delta E(r)$	+0.29%	*0.00%*
Δr_f	+0.59%	*0.00%*
$\Delta w/w$	-3.88%	*0.00%*

But where do these +5.10% of total net insurance come from, how much can be attributed to insurance against aggregate risk, how much to idiosyncratic wage risk, how much to CCV, and to survival risk? That is answered in table 6, where we start with an economy with only aggregate risk, then add idiosynratic wage risk on top, then add CVV, and finally also include survival risk. For each economy, we look at the welfare gains from the experiment in PE, so that in the last column, we end up with the same +5.10% that we just saw. The first column looks at an economy with only aggregate risk, which therefore is comparable to the partial equilibrium of KK. The welfare gains of +0.18% represent the intergenerational insurance against aggregate risk. This number is close to zero, because the additional insurance only just outweighs the cost of the higher contributions, which are painful in particular for those agents that are close to the natural borrowing limit. So the actual insurance itself is larger, and we will quantify it below.[18]

The second column of table 6 looks at an economy with both aggregate and idiosyncratic risk. Introducing social security in this economy leads to substantially larger welfare gains of +1.99%, which -since this is still PE- are attributable to the intergenerational insurance against aggregate risk plus the intergenerational insurance against idiosyncratic risk. When we add CCV

[18]Optimally, we would compute welfare gains/losses in a model without aggregate risk, but this would require such drastic recalibration that any comparison would very hard to interpret.

risk, insurance gains go up by another 0.47% (calculated as 2.46% − 1.99%), and looking at the last column we see that adding survival risk adds another 2.64%. Summing up, insurance against idiosyncratic wage risk and survival risk is very large, that against aggregate risk and CCV only moderate.

Table 6: Insurance against sources of risk (NC)

	aggr. risk	+ idios. wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	+0.18%	+1.99%	+ 2.46%	+5.10%

Table 7: Identification of the direct interaction term LCI (NC)

$\sigma\zeta, \sigma_\delta$	aggr. risk (AR)	+ idios. wage risk	$IR + LCI$	ΔLCI
benchmark	+0.18%	+1.99%	+1.81	-
−10%	+0.09%	+1.83%	+1.74	-0.072

Now let's turn to the interaction effect LCI. To isolate it, we conduct the difference-in-difference calculation described in section 3.9. The logic of this calculation becomes transparent in table 7. The first row looks at our benchmark economy, and the second row at the same economy with aggregate risk reduced by −10%.[19] The first two columns simply show the insurance against aggregate risk (AR) and against aggregate risk + idiosynratic wage risk in the same way we just discussed. Indeed the two numbers in the first row are simply copied from table 6.

By taking the difference between column two and column one we are left with $IR + LCI$, since AR drops out, and this number is displayed in the third column. We now take the difference of the numbers in the third column and thereby get ΔLCI, because IR remains constant by construction. So the term ΔLCI represents by how much LCI changes if we reduce AR by −10%. Relating the change in insurance against LCI to the change in insurance against AR, we get $\frac{-0.072}{0.09-0.18} = 0.8$. In other words, LCI increases insurance by 80% of AR.

[19]Reducing aggregate risk by −10% is achieved by reducing the standard deviations of the TFP and depreciation shocks by −10% each.

With these numbers at hand, we can now easily recoup the levels of insurance against IR and LCI for the benchmark economy. We already know from the first number in table 7 that insurance against AR is 0.18%. The other two numbers are calculated as $LCI \approx 0.8 \cdot 0.18\% = 0.144\%$ and $IR = 1.81\% - 0.144\% = 1.036\%$. The total interaction between aggregate and idiosyncratic risk is the sum of CCV and LCI which is $0.47\% + 0.144\% = 0.61\%$. the last is an order

5.2 Calibration with positive correlation between TFP and returns (PC)

The discussion of this calibration will be much more concise than the previous one, because all components were already explained there, and the exposition is structured in exactly the same way. The value of the targeted correlation $cor(\zeta, r)$ is shown in column two (labeled PC) of table 2, and in contrast to before it is now positive. In order to match it, we now need the conditional probability of depreciation shocks to be $\pi^\delta = 0.86$. In comparison to the NC calibration, we need to adjust the discount factor β and the standard deviation of depreciation $\bar{\delta}$ slightly, and considerably reduce the coefficient of relative risk aversion θ so as to match all statistics, see table 3 column two. The large, positive $cor(\zeta, r)$ induces a large, positive $cor(w, r) = 0.236$, and also drives up a bit the standard deviation of consumption growth, cf. table 8.

Table 8: Model-generated moments ($corr(TFP, returns) > 0$)

$cor(w,r)$	$std(\Delta C/C)$
0.236	0.076

Introducing social security into this economy leads to welfare gains of +3.52% when comparing the long-run general equilibria (table 9). Since we are talking about consumption-equivalent variations, this is a very large number. Note that it is much larger than in the NC calibration although we reduced θ substantially. The crowding out of capital and its associated price changes take virtually the same values as in the NC calibration. This surprising similarity is probably due to the fact that we still have the same elasticity of inter-temporal substitution, see section 5.3 for a sensitivity analysis with respect to this parameter.

When we repeat the PE experiment by again keeping prices fixed, we see that the net benefits attributable to the total insurance amount to +9.37%, so that the welfare costs of crowding out can be calculated as $3.52\% - 9.37\% = -5.85\%$. As before, the $\Delta K/K = -29.39\%$ in this PE should be interpreted as 'less capital being invested abroad' in a small open economy.

Table 9: The social security experiment (PC)

	GE	PE
ΔWelf/Welf	+3.52%	+9.37%
$\Delta K/K$	-5.90%	-29.39%
$\Delta E(r)$	+0.30%	*0.00%*
Δr_f	+0.67%	*0.00%*
$\Delta w/w$	-3.87%	*0.00%*

Table 10: Insurance against sources of risk (PC)

	aggr. risk	+ idios. wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	+1.26%	+3.92%	+5.69%	+9.37%

Table 10 decomposes the total insurance into its four sources. Insurance against aggregate risk is a lot higher than before. This suggests that the mechanism described in our companion paper Harenberg and Ludwig (2011) obtains in our model. To put it in a nutshell, a positive $cor(\zeta, r)$ (and correspondingly positive $cor(w, r)$) increases the value of social security, because it increases the variance of lifetime income. This effect quantitatively dominates the effect from a negative $cor(\zeta, r)$, which would increase the value of social security as a hedge against volatile savings income at old age.

The additional insurance when idiosyncratic wage risk is included amounts to $+3.92\% - 1.26\% = 2.66\%$, which is larger than in the NC calibration and which already indicates that LCI will be larger than before, because IR remained the same. Similarly, including CCV has a much larger impact at $5.69\% - 3.92\% = 1.77\%$ as opposed to 0.47%.[20] Finally, the impact of survival risk is also larger.

[20] The impact of CCV on both the equity premium and insurance is larger, the larger $cor(\zeta, r)$.

Table 11: Identification of the direct interaction term LCI (PC)

$\sigma\zeta, \sigma_\delta$	aggr. risk (AR)	+ wage risk	$IR + LCI$	ΔLCI
benchmark	+1.26%	+3.92%	+2.66	-
-10%	+0.83%	+3.10%	+2.27	-0.392

In order to isolate the interaction term LCI and the pure idiosyncratic risk term IR, we again proceed with the difference-in-difference calculation that is explained both in the previous subsection and in section 3.9. The results in table 11 show that ΔLCI is much larger than before. Also in relation to the change in AR it is larger than before: $\frac{\Delta LCI}{\Delta AR} = \frac{-0.392\%}{0.83\% - 1.26\%} \approx 0.91$, i.e. LCI adds 90% on top of AR in terms of insurance. From this, we can again compute the levels of insurance against IR and LCI in our benchmark economy. Since insurance against AR is +1.26% in the table, we get that the welfare gains from the other two sources amount to $LCI \approx 0.91 \times 1.26\% = 1.15\%$ and $IR = 2.66\% - 1.15\% = 1.51\%$. The total interaction between aggregate and idiosyncratic risk as the sum of CCV and LCI is hence $1.77\% + 1.15\% = 2.92\%$.

5.3 Robustness

This section discusses the sensitivity of our results with respect to the three most crucial model elements. First, we show that when the thought experiment is not a marginal introduction of a contribution rate, but instead a marginal increase from the current level in the U.S., all results remain qualitatively unchanged. Next, we document the sensitivity of the welfare and crowding out numbers when a much smaller elasticity of intertemporal substitution is used. Finally, we show that using a standard Hodrick-Prescott filter yields both empirical estimates as well as computational results that support the findings and conclusions from our preferred PC calibration.

5.3.1 Robustness of the thought experiment

We chose to perform the thought experiment of a marginal introduction of social security mainly because it conforms to both the approach taken in our theory section and the approach taken by KK. However, it is crucial to understand that the positive welfare results do not hinge on this specific example. Since it is beyond this paper to check all possible experiments, we chose one that is sufficiently different and that still seems relevant from an empirical perspective. For that, we take the current U.S. value of social security contributions $\tau = 0.12$ and increase it by 2% to keep it comparable to the previous results.

We perform this new experiment for both the NC and PC calibrations, and document all findings in the same way as before. Here, we will summarize the key insights, and relegate the tables to appendix C.1.

Since now we start from a situation with substantial old age income, we need to adjust the coefficient of relative risk aversion θ and the discount factor β considerably in order to match the same aggregate statistics. Specifically, θ has to be increased by five in both calibrations, because higher social security income puts downward pressure on the equity premium, as agents have more safe income and thus demand more of the risky stock. At the same time, β needs to be increased by approximately 0.02 in both calibrations in order to match the same capital-output-ratio, because agents' savings are of course reduced by the higher social security income. The model-generated, not-targeted moments $cor(w, r)$ and $std(\Delta C/C)$ remain basically unchanged.

Despite the fact that the changes in θ and β should, ceteris paribus, both increase the welfare gains of social security, we find these to be substantially smaller throughout. Still, welfare gains are positive in GE, with +0.17% for NC and +1.94% for PC, while at the same time, the crowding out of capital is half as large as it was before. The relative ordering of the various sources of insurance remains the same with the exception of survival risk, which now is much less important than before. The most relevant difference is that LCI now only adds approximately $0.3 * AR$ in terms of insurance in both cases, which -while being a notable reduction to before - still is a substantial amount. Finally, for the NC calibration, we find that insurance against aggregate risk is negative at -0.31%, meaning that the intergenerational insurance in this case does not outweigh the utility costs of having to pay the contributions, which is painful in particular for agents close to the natural borrowing limit.

5.3.2 Sensitivity with respect to the elasticity of intertemporal substitution

This section discusses the effects of a reduction in the elasticity of intertemporal substitution φ from 1.5 to 0.5. The first value resulted from our calibration strategy as described in section 4, where this parameter was set so as to get a more hump-shaped life-cycle consumption profile.[21] The value of

[21] A higher IES implies that agents are more willing to accept higher growth rates of consumption in return for higher mean consumption. In our model this generates an

0.5 is substantially smaller and does not deliver a clear hump in the life-cycle profile, but we use this value for the sensitivity analysis because it is the value chosen by Krueger and Kubler (2006). Both values can be defended on empirical grounds, as the recent estimates in the literature range from close to zero to a value of two.

One word of caution before we present the results. With a high EIS, agents react to the changes in interest rates to a stronger degree and thereby mitigate the crowding out of capital, whereas with a low EIS, the crowding out will be larger.[22] However, we only asses the negative welfare impact of crowding out since we ignore the transition to the new long-run equilibrium. If crowding out of capital between the long-run equilibria is larger, then agents will save less and consume more along the transition, which would enhance their welfare. So the welfare numbers we discuss now should be seen as a lower bound, and we expect that once we include the transition, welfare will react less sensitively to changes in this parameter.

The tables are shown in appendix C.2. Note that due to the nature of the experiment, they need to be compared the tables in appendix C.1. As shown in the last section, that experiment is much more unfavorable for social security, and welfare gains for our original experiment should be larger. As before, we document the numbers for both the calibration with the positive and the negative correlation between TFP and returns (PC and NC, respectively).

The reduction in the EIS necessitates a recalibration, which can be effectuated by adjusting only two parameters. The coefficient of relative risk aversion has to be reduced by two (three for the PC calibration) so that the model generates the same equity premium. The discount factor β needs to be reduced only very slightly. The endogenously generated $cor(w, r)$ is smaller than before, and more importantly, the standard deviation of aggregate consumption growth diminishes substantially to $std(\Delta C/C) = 0.043$ ($std(\Delta C/C) = 0.049$ for PC), which is much closer to the data. The reason for this decrease is that agents now prefer a smooth consumption path to high consumption growth.[23]

We find that the welfare numbers in general equilibrium are reduced by approximately 3.5%. This is a lot, and for the experiment conducted means that overall welfare gains are clearly negative, as opposed to the positive

empirically plausible peak at the age of 55.

[22] Higher social security contributions crowd out capital, which increases interest rates, which incentivizes agents to save more, and agents will respond to this incentive more strongly when their EIS is higher.

[23] Note that we only achieve such small values for $std(\Delta C/C)$ because we have introduced a capital structure of the firm, which is not yet included in the description of the model.

numbers for $\varphi = 1.5$. When looking at the GE, it seems that crowding out causes much of these losses, as the percentage of capital lost more than doubles. However, the PE experiment reveals that also the insurance gains are reduced by approx. 2.5%, which means that the capital loss accounts only for approximately 1% of the welfare losses. The fact that agents seem to value social security less with a smaller EIS even when prices are kept fixed seems an interesting finding. Our hypothesis is that agents still value the reduction in the variance of old-age consumption, but they dislike that their consumption profile becomes somewhat steeper, even though that means higher average consumption. On the contrary, when agents have a high EIS, they like this second effect. To verify this, we will look at mean consumption and the variance of consumption over the life-cycle for the GE and PE experiments.

Turning to the sources of risk, insurance effects drop for all of them. Insurance against aggregate risk drops the most, insurance against survival risk the least. The interaction term, on the other hand, is of about the same order of magnitude as before: it adds about 30% on top of aggregate risk, the same number as in the previous section (this number is so small because we look at the less favorable experiment).

5.3.3 Nonlinear trend calibration

We now take take a standard RBC or Hodrick and Prescott (1997) perspective according to which, while the model is stationary, the data are rather non stationary and driven by some deterministic trend component of unknown functional form. This view implies that we merge a large proportion of observed fluctuations into the deterministic component of the data. The autocorrelation of the cyclical component of log TFP is 0.43 with an unconditional standard deviation of 0.0125, cf. table 12. We also find that the correlation between stock returns and TFP is strongly positive and highly significant, whereas the correlation between returns and wages is not significantly different from zero. This gives strong support to our view that the PC calibration is the more relevant. While the value of $corr(\zeta_t, R_t)$ lies between the NC and PC calibrations, it is clearly closer to the latter.

The results from the experiments confirm our findings. All welfare numbers are between those found in the NC and PC calibrations, and slightly closer to the latter. Of course, insurance against aggregate risk decreases substantially when compared to the PC calibration, but the interaction effect LCI still amounts to about $0.4 * AR$, our prediction from the toy model. The tables are relegated to appendix C.3.

Table 12: Aggregate Risk, HP-filter

Point Estimates	
Autocorrelation of TFP	0.43 (0.00)
Standard deviation of TFP	0.012
Corr. TFP, R, $corr(\zeta_t, R_t)$	0.35 (0.01)
Corr. w, R, $corr(w_t, R_t)$	-0.03 (0.82)
Markov Approximation	
Aggregate states, $1 \pm \bar{\zeta}$	$[1.012, 0.988]$
Transition probabilities, π^{ζ}	0.7152

Notes: p-values are reported in brackets.

Table 13: Endogenous Parameters, HP-filter

θ	φ	β	$\bar{\delta}$	π^{δ}
14	1.5	0.98	0.1	0.7

6 Conclusion

In a life-cycle model, idiosyncratic and aggregate risk interact despite the fact that they are statistically independent. This interaction increases the value of social security. In our general equilibrium analysis, the introduction of a PAYG system leads to strong welfare gains. This stands in contrast to the related literature. The reason for this difference is that in our model, social security provides partial insurance against both idiosyncratic and aggregate risk, as well as their interactions. In fact, the interactions account for one third of the total welfare gains.

In our analysis, we abstracted from endogenous labor supply. This biases the results in favor of social security, because a higher contribution rate would distort the households' labor supply decision. In addition to the crowding out of capital, we would also see less labor being supplied. While it would be interesting to see this extension, one would probably have to restrict the model in some other way in order to clearly expose the mechanisms.

While our results do not depend on the calibration, we have seen that the covariance between wages and risky returns plays a crucial role. Interestingly, a positive correlation leads to substantially larger welfare gains. Previous analyses suggested that a negative correlation should increase the welfare gains, because then social security income is a better hedge against volatile asset income at old age. It became apparent that this mechanism is opposed by other forces. We elaborate on this in our companion paper (Harenberg and Ludwig (2011)).

In our economy, the intergenerational sharing of aggregate risks is limited to those generations alive at the same point in time. From a social planner's point of view, it would be desirable to share the risk also with future, unborn generations. This could be achieved by allowing the government to take up debt to smooth shocks over time. That would open up an additional insurance channel, which would increase the welfare gains of introducing social security.

Finally, we document in our robustness section that increasing the contribution rate from the current level in the U.S. of 12 percent to 14 percent also leads to welfare gains. While the welfare gains are still large, they are smaller than when the contribution rate is increased from zero to two percent. It seems that the higher the current level of contributions, the smaller the welfare gains are for a fixed percentage point increase in contributions. From these results it seems that there is an optimal level of social security, and that it lies somewhere above the current level observed in the U.S. today. We leave this and the other extensions to future research.

A Proofs

Proof of proposition 1. Maximize

$$E_{t-1}u(c_{i,2,t+1}) = \frac{1}{1-\theta}E_{t-1}\left(\bar{w}_t\left(\bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1} + \tau\left((1+g)\zeta_{t+1} - \bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)\right)\right)^{1-\theta}.$$

This is equivalent to maximizing

$$\max E_{t-1}R_{p,t,t+1}^{1-\theta}$$

where $R_{p,t,t+1} \equiv \eta_{i,1,t}\zeta_t\bar{R}\tilde{\varrho}_{t+1} + \tau\left((1+g)\zeta_{t+1} - \bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)$ is a consumption (or portfolio) return. Increasing ex-ante utility for a marginal introduction of social security requires the first-order condition w.r.t. τ to exeed zero, hence:

$$E_{t-1}\left[R_{p,t,t+1}^{-\theta}\frac{\partial R_{p,t,t+1}}{\partial\tau}\right]|_{\tau=0} > 0 \tag{38}$$

Evaluated at $\tau = 0$ we have

$$\begin{aligned} R_{p,t,t+1}^{-\theta}|_{\tau=0} &= \left(\eta_{i,1,t}\zeta_t\bar{R}\varrho_{t+1}\right)^{-\theta} \\ \frac{\partial R_{p,t,t+1}}{\partial\tau}|_{\tau=0} &= (1+g) - \eta_{i,1,t}\zeta_t\bar{R}\varrho_{t+1} \end{aligned}$$

Equation (38) therefore rewrites as

$$(1+g)E_{t-1}\left[\left(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)^{-\theta}\zeta_{t+1}\right] > \bar{R}E_{t-1}\left[\left(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)^{1-\theta}\right]. \tag{39}$$

Rewriting the above and imposing assumption 2 we get equation (4). □

Proof of proposition 2. Define

$$\begin{aligned} Z_1 &\equiv \left(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)^{-\theta}\zeta_{t+1} \\ Z_2 &\equiv \left(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)^{1-\theta}. \end{aligned}$$

By log-normality we have that $EZ_i = \exp(E\ln Z_i + \frac{1}{2}\sigma^2_{\ln Z_i})$, $i = 1, 2$. Observe that

$$\begin{aligned} E\ln Z_1 &= -\theta\left(E\ln\eta_{i,1,t} + E\ln\tilde{\varrho}\right) + (1-\theta)E\ln\zeta \\ \sigma^2_{\ln Z_1} &= \theta^2\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}}\right) + (1+\theta^2)\sigma^2_{\ln\zeta} \end{aligned}$$

Therefore

$$
\begin{aligned}
E_{t-1}[Z_1] = & \exp\left(-\theta\left(E\ln\eta_{i,1,t} + \frac{\sigma^2_{\ln\eta}}{2}\right)\right)\cdot\exp\left(-\theta\left(E\ln\tilde{\varrho} + \frac{\sigma^2_{\ln\tilde{\varrho}}}{2}\right)\right)\cdot \\
& \cdot\exp\left((1-\theta)\left(E\ln\zeta + \frac{\sigma^2_{\ln\zeta}}{2}\right)\right)\cdot\exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}} + \sigma^2_{\ln\zeta}\right)\right) \\
= & \left(E[\eta_{i,1,t}]\right)^{-\theta}\left(E[\tilde{\varrho}]\right)^{-\theta}\left(E[\zeta]\right)^{1-\theta}\cdot\exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}} + \sigma^2_{\ln\zeta}\right)\right) \\
= & \exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}} + \sigma^2_{\ln\zeta}\right)\right)
\end{aligned}
$$

whereby the last line follows from assumption 2b.

Next, observe that log-normality implies that

$$
\begin{aligned}
\sigma^2_\eta = var_{t-1}(\eta_{i,1,t}) &= \exp\left(2E\ln\eta_{i,1,t} + \sigma^2_{\ln\eta}\right)\left(\exp\left(\sigma^2_{\ln\eta}\right) - 1\right) \\
&= \left(E\eta_{i,1,t}\right)^2\left(\exp\left(\sigma^2_{\ln\eta}\right) - 1\right) \\
&= \left(\exp\left(\sigma^2_{\ln\eta}\right) - 1\right)
\end{aligned}
$$

whereby the last line again follows from assumption 2b. Hence:

$$\sigma^2_{\ln\eta} = 1 + \sigma^2_\eta$$

with corresponding expressions for $\sigma^2_{\ln\zeta}$ and $\sigma^2_{\ln\tilde{\varrho}}$. Therefore:

$$\exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}} + \sigma^2_{\ln\zeta}\right)\right) = \left((1+\sigma^2_\eta)(1+\sigma^2_\zeta)(1+\sigma^2_{\tilde{\varrho}})\right)^{\frac{1}{2}\theta(1+\theta)}$$

We consequently have

$$E_{t-1}[Z_1] = \left((1+\sigma^2_\eta)(1+\sigma^2_\zeta)(1+\sigma^2_{\tilde{\varrho}})\right)^{\frac{1}{2}\theta(1+\theta)}$$

As to $E_{t-1}[Z_2]$ observe that

$$
\begin{aligned}
E\ln Z_2 &= (1-\theta)\left(E\ln\eta_{i,1,t} + E\ln\zeta + E\ln\tilde{\varrho}\right) \\
\sigma^2_{\ln Z_2} &= (1-\theta)^2\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\zeta} + \sigma^2_{\ln\tilde{\varrho}}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E_{t-1}[Z_2] = & \exp\left((1-\theta)\left(E\ln\eta_{i,1,t} + \frac{\sigma^2_{\ln\eta}}{2}\right)\left(E\ln\tilde{\varrho} + \frac{\sigma^2_{\ln\tilde{\varrho}}}{2}\right)\left(E\ln\zeta + \frac{\sigma^2_{\ln\zeta}}{2}\right)\right) \\
& \cdot\exp\left(\frac{1}{2}\theta(\theta-1)\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}} + \sigma^2_{\ln\zeta}\right)\right) \\
= & \left((1+\sigma^2_\eta)(1+\sigma^2_\zeta)(1+\sigma^2_{\tilde{\varrho}})\right)^{\frac{1}{2}\theta(\theta-1)}
\end{aligned}
$$

Hence

$$\frac{E_{t-1}[Z_1]}{E_{t-1}[Z_2]} = \left((1+\sigma_\eta^2)(1+\sigma_\zeta^2)(1+\sigma_{\tilde{\varrho}}^2)\right)^\theta$$

$\square$

Illustration 1. *Let us provide a simplified numerical illustration. Below, we calibrate our model with an annual income processes given by* $\ln(\eta_{i,j,t}) = \rho\ln(\eta_{i,j-1,t-1}) + \epsilon_{i,j,t}$ *where* j *is actual age of a working household,* t *is time,* $\epsilon_{i,j,t} \sim \mathcal{N}(0, \sigma_t^2)$*, hence* $\eta_{i,j,t}$ *is distributed as log-normal for all* i, j, t *and* ρ *is the autocorrelation coefficient. While we consider time variation in variances below, let us assume constant variances for now. Our calibration has an average variance of* $\sigma^2 \approx 0.03$*. We also calibrate* $\rho = 0.952$*. Consider the overall variance of income risk at retirement, that is, after a period in the work force of about* 45 *years. For AR(1) processes with such a long horizon, the approximate infinite horizon*[24] *formula to compute the variance of* $\ln\eta_{i,1,t}$ *at retirement is given by* $\frac{1}{1-\rho^2}\sigma_\epsilon^2$*. Using our numbers we accordingly have that the variance of* $\ln\eta_{i,1,t}$ *at retirement is given by* $\frac{1}{1-0.952^2}\cdot 0.03 = 10.67\cdot 0.03$*. By the formula for log-normal random variables, the variance of* $\eta_{i,1,t}$ *at retirement is therefore* $var(\eta_{i,1,t}) = (E[\eta_{i,1,t}])^2\left(\exp\left(\sigma^2_{\ln\eta}\right)-1\right) = \exp(10.67\cdot 0.03) - 1 = 0.37$*.*[25]

[24]The exact formula is $\frac{1-\rho^{2(j_r-1)}}{1-\rho^2}$ where j_r is the retirement age but the term $\rho^{2(j_r-1)}$ is negligible.

[25]As we describe in our main text, our estimates are based on Storesletten, Telmer, and Yaron (2004) who use after tax earnings data and control for aggregate fluctuations. Observe that these numbers are a conservative estimate of the overall dispersion of earnings inequalities at retirement because we ignore the dispersion of skills and learning abilities at the beginning of the life-cycle. The more recent work by Huggett, Ventrua, and Yaron (2011) attributes about 60 of the overall variation in life-time income to variations in initial conditions. However, it is rather education policies than pension policies and social insurance that should target such differences. Huggett et al. (2011)'s specification for income shocks is a unit root process. Their estimate of the standard deviation of the innovation of this process is 0.111. This would roughly double the relevance of the interaction term at retirement to 0.74. However, the estimates of Huggett et al. (2011) are based on pre-tax earnings data and the authors do not control for the business cycle. This may explain these substantial differences.

Derivation of equation 7. We have

$$\begin{aligned}
E_{t-1}u_t \approx E_{t-1}&\left[u(Ec_{i,2,t+1}) + u'(Ec_{i,2,t+1})\cdot(c_{i,2,t+1} - Ec_{i,2,t+1}) + \right.\\
&\left. + \frac{1}{2}u''(Ec_{i,2,t+1})\cdot(c_{i,2,t+1} - Ec_{i,2,t+1})^2\right]\\
&= u(Ec_{i,2,t+1}) + \frac{1}{2}u''(Ec_{i,2,t+1})\cdot var(c_{i,2,t+1})\\
&= \frac{1}{1-\theta}\left(\bar{w}_t\bar{R}\right)^{1-\theta} - \frac{\theta}{2}\left(\bar{w}_t\bar{R}\right)^{-(1+\theta)}\cdot\left(\bar{w}_t\bar{R}\right)^2\cdot(IR + AR + IR\cdot AR)
\end{aligned}$$

from which equation (7) immediately follows. □

Proof of proposition 3. To establish proposition 3a we have to show that

$$\begin{aligned}
&\frac{1}{\zeta_l}E_{t-1}\frac{1}{\eta_{i,1,l}} + \frac{1}{\zeta_h}E_{t-1}\frac{1}{\eta_{i,1,h}} > \left(\frac{1}{\zeta_l} + \frac{1}{\zeta_h}\right)E_{t-1}\frac{1}{\eta_{i,1,t}}\\
\Leftrightarrow\quad &\frac{1}{\zeta_l}\left(E_{t-1}\frac{1}{\eta_{i,1,l}} - E_{t-1}\frac{1}{\eta_{i,1,t}}\right) + \frac{1}{\zeta_h}\left(E_{t-1}\frac{1}{\eta_{i,1,h}} - E_{t-1}\frac{1}{\eta_{i,1,t}}\right) > 0. \quad (40)
\end{aligned}$$

Under assumption 4 we have that

$$\begin{aligned}
E_{t-1}\frac{1}{\eta_{i,1,t}} &= \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma^2_{\ln\eta}\right)\right)\\
E_{t-1}\frac{1}{\eta_{i,1,l}} &= \exp\left(-\left(E\ln\eta + \frac{1}{2}\left(\sigma^2_{\ln\eta} - \Delta\right)\right)\right)\\
E_{t-1}\frac{1}{\eta_{i,1,h}} &= \exp\left(-\left(E\ln\eta + \frac{1}{2}\left(\sigma^2_{\ln\eta} + \Delta\right)\right)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
E_{t-1}\frac{1}{\eta_{i,1,l}} - E_{t-1}\frac{1}{\eta_{i,1,t}} &= \exp\left(-\left(E\ln\eta + \frac{1}{2}\left(\sigma^2_{\ln\eta} - \Delta\right)\right)\right) - \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma^2_{\ln\eta}\right)\right)\\
&= \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma^2_{\ln\eta}\right)\right)\exp\left(\frac{1}{2}\Delta\right) - \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma^2_{\ln\eta}\right)\right)\\
&= \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma^2_{\ln\eta}\right)\right)\left(\exp\left(\frac{1}{2}\Delta\right) - 1\right)\\
E_{t-1}\frac{1}{\eta_{i,1,h}} - E_{t-1}\frac{1}{\eta_{i,1,t}} &= \exp\left(-\left(E\ln\eta + \frac{1}{2}\left(\sigma^2_{\ln\eta} + \Delta\right)\right)\right) - \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma^2_{\ln\eta}\right)\right)\\
&= \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma^2_{\ln\eta}\right)\right)\exp\left(-\frac{1}{2}\Delta\right) - \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma^2_{\ln\eta}\right)\right)\\
&= \exp\left(-\left(E\ln\eta + \frac{1}{2}\sigma^2_{\ln\eta}\right)\right)\left(\exp\left(-\frac{1}{2}\Delta\right) - 1\right)
\end{aligned}$$

Equation (40) therefore rewrites as

$$\frac{1}{\zeta_l}\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)+\frac{1}{\zeta_h}\left(\exp\left(-\frac{1}{2}\Delta\right)-1\right)>0. \tag{41}$$

Observe that $\exp\left(-\frac{1}{2}\Delta\right)-1<0,\ \ \exp\left(\frac{1}{2}\Delta\right)-1>0$ and convexity of the exponential function implies that

$$|\exp\left(-\frac{1}{2}\Delta\right)-1|<|\exp\left(\frac{1}{2}\Delta\right)-1|.$$

Therefore

$$\begin{aligned}
&\frac{1}{\zeta_l}\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)+\frac{1}{\zeta_h}\left(\exp\left(-\frac{1}{2}\Delta\right)-1\right)\\
&>\frac{1}{\zeta_l}\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)-\frac{1}{\zeta_h}\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)\\
&=\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)\left(\frac{1}{\zeta_l}-\frac{1}{\zeta_h}\right)\\
&=\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)\left(\frac{1}{\zeta_l}-\frac{1}{2-\zeta_l}\right)\\
&=\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)\frac{2(1-\zeta_l)}{\zeta_l(2-\zeta_l)}>0.
\end{aligned}$$

To establish proposition 3b use assumption 4 to rewrite equation (41) as

$$f(\Delta_\zeta)\equiv\frac{1}{1-\Delta_\zeta}\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)+\frac{1}{1+\Delta_\zeta}\left(\exp\left(-\frac{1}{2}\Delta\right)-1\right)>0.$$

Observe that

$$\begin{aligned}
\frac{\partial f(\Delta_\zeta)}{\partial\Delta_\zeta}&=\left(\frac{1}{1-\Delta_\zeta}\right)^2\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)-\left(\frac{1}{1+\Delta_\zeta}\right)^2\left(\exp\left(-\frac{1}{2}\Delta\right)-1\right)\\
&=\left(\frac{1}{1-\Delta_\zeta}\right)^2\underbrace{\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)}_{>0}+\left(\frac{1}{1+\Delta_\zeta}\right)^2\underbrace{\left(1-\exp\left(-\frac{1}{2}\Delta\right)\right)}_{>0}>0
\end{aligned}$$

Hence, a mean preserving spread of ζ increases the effect of CCV. $\square$

Proof of Proposition 4. The proof is by guessing and verifying. We guess that

$$\begin{aligned}
s_{1,t}&=\chi(1-\tau)w_t\\
&=\chi(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha.
\end{aligned}$$

If this is correct, then the equilibrium dynamics are given by

$$\begin{aligned}
K_{t+1} &= N_{1,t} s_{1,t} \\
&= N_{1,t}\chi(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha \\
\Leftrightarrow \qquad k_{t+1} &= \frac{N_{1,t}\chi(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha}{\Upsilon_{t+1}N_{1,t+1}} \\
&= \frac{1}{1+g}\chi(1-\tau)(1-\alpha)\zeta_t k_t^\alpha
\end{aligned}$$

To verify this, notice that our assumptions on savings implies that

$$c_{1,t} = (1-\chi)(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha \tag{42}$$

and, by the budget constraint, we have

$$c_{2,t+1} = \chi(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha \alpha\zeta_{t+1}\varrho_{t+1}k_{t+1}^{\alpha-1} + (1-\alpha)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha\tau. \tag{43}$$

Using (11a) in (43) we get

$$\begin{aligned}
c_{2,t+1} &= k_{t+1}(1+g)\Upsilon_t\alpha\zeta_{t+1}\varrho_{t+1}k_{t+1}^{\alpha-1} + (1-\alpha)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha\tau \\
&= \Upsilon_{t+1}\alpha\zeta_{t+1}\varrho_{t+1}k_{t+1}^\alpha + (1-\alpha)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha\tau \\
&= \left(\alpha\varrho_{t+1} + (1-\alpha)\tau\right)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha
\end{aligned}$$

Next, notice that the first-order-condition of household maximization gives:

$$1 = \beta E_t\left[\frac{c_{1,t}(1+r_{t+1})}{c_{2,t+1}}\right]. \tag{44}$$

Using the above equations for consumption in the two periods, we can rewrite (44) as:

$$\begin{aligned}
1 &= \beta E_t\left[\frac{c_{1,t}\alpha\zeta_{t+1}\varrho_{t+1}k_{t+1}^{\alpha-1}}{\left(\alpha\varrho_{t+1}+(1-\alpha)\tau\right)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha}\right] \\
&= \beta E_t\left[\frac{c_{1,t}\alpha\varrho_{t+1}}{\left(\alpha\varrho_{t+1}+(1-\alpha)\tau\right)\Upsilon_{t+1}k_{t+1}}\right] \\
&= \beta E_t\left[\frac{(1-\chi)(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha\alpha\varrho_{t+1}}{\left(\alpha\varrho_{t+1}+(1-\alpha)\tau\right)\Upsilon_{t+1}\frac{1}{1+g}\chi(1-\tau)(1-\alpha)\zeta_t k_t^\alpha}\right] \\
&= \beta E_t\left[\frac{(1-\chi)\alpha\varrho_{t+1}}{\left(\alpha\varrho_{t+1}+(1-\alpha)\tau\right)\chi}\right] \\
&= \frac{\beta(1-\chi)\alpha}{\chi}E_t\left[\frac{\varrho_{t+1}}{\alpha\varrho_{t+1}+(1-\alpha)\tau}\right] \\
&= \frac{\alpha\beta(1-\chi)}{\chi}\bar{E}
\end{aligned}$$

where

$$\bar{E} \equiv E_t \left[\frac{1}{\alpha + (1-\alpha)\varrho_{t+1}^{-1}\tau} \right].$$

It follows that

$$\chi = \frac{1}{1 + \left(\alpha\beta\bar{E}\right)^{-1}}$$

Uniqueness is established by convexity of the problem. Given that the solution is unique and given that we have characterized one solution, this is the solution to the problem. □

Proof of proposition 6. Rewrite the above consumption equations to get

$$c_{1,t,ms} = (1-\chi)(1-\tau)\Upsilon_t\zeta_t(1-\alpha)k_{ms}^{\alpha}$$

and

$$c_{2,t+1,ms} = \left(\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \tau\left((1+g) - \chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}\right)\right)\Upsilon_t\zeta_{t+1}(1-\alpha)k_{ms}^{\alpha}$$

We now look at ex-ante utility:

$$\begin{aligned} E_{t-1}u_t = {} & E_{t-1}\mathbf{c} + E_{t-1}\left[\ln(1-\chi) + \ln(1-\tau)\right] + \alpha\ln k_{ms} + \\ & \beta E_{t-1}\left[\ln\left(\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \tau\left((1+g) - \chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}\right)\right)\right] + \beta\alpha\ln k_{ms} \\ = {} & E_{t-1}\mathbf{c} + E_{t-1}\left[\ln(1-\chi) + \ln(1-\tau)\right] + \alpha(1+\beta)\ln k_{ms} + \\ & \beta E_{t-1}\left[\ln\left(\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \tau\left((1+g) - \chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}\right)\right)\right] \end{aligned} \tag{45}$$

where $\mathbf{c}$ encompasses all elements that are not affected by τ.

We evaluate the derivative of the above at $\tau = 0$. To this end, we look separately at the different terms. Notice that

$$E_{t-1}\frac{\partial\ln(1-\chi)}{\partial\tau} = -E_{t-1}\frac{1}{1-\chi}\frac{\partial\chi}{\partial\bar{E}}\frac{\partial\bar{E}}{\partial\tau}$$

We have

$$\frac{\partial\chi}{\partial\bar{E}} = \alpha\beta\left(\frac{1}{1+(\alpha\beta\bar{E})^{-1}}\right)^2\left(\frac{1}{\alpha\beta\bar{E}}\right)^2.$$

and, since

$$\begin{aligned} & \chi = \frac{1}{1+(\alpha\beta\bar{E})^{-1}} \\ \Leftrightarrow \qquad & \alpha\beta\bar{E} = \frac{\chi}{1-\chi} \end{aligned}$$

we get

$$\begin{aligned}
\frac{\partial \chi}{\partial \bar{E}} &= \alpha\beta \left(\frac{1}{1 + \frac{1-\chi}{\chi}} \right)^2 \left(\frac{1-\chi}{\chi} \right)^2 \\
&= \alpha\beta\chi^2 \left(\frac{1-\chi}{\chi} \right)^2 \\
&= \alpha\beta(1-\chi)^2.
\end{aligned}$$

Therefore:

$$\frac{1}{1-\chi}\frac{\partial \chi}{\partial \bar{E}} = \alpha\beta(1-\chi).$$

Next, observe that

$$1 - \chi = \frac{1}{1 + \alpha\beta\bar{E}}$$

and $\bar{E}$ evaluated at $\tau = 0$ gives

$$\bar{E} = \frac{1}{\alpha}.$$

Hence

$$1 - \chi = \frac{1}{1+\beta}$$

and we therefore have

$$\frac{1}{1-\chi}\frac{\partial \chi}{\partial \bar{E}} = \alpha\beta\frac{1}{1+\beta}.$$

Finally, evaluated at $\tau = 0$ we get

$$\frac{\partial \bar{E}}{\partial \tau} = -\frac{1-\alpha}{\alpha^2} E_t\left[\varrho_{t+1}^{-1}\right]$$

hence

$$\begin{aligned}
\frac{1}{1-\chi}\frac{\partial \chi}{\partial \bar{E}}\frac{\partial \bar{E}}{\partial \tau} &= -\alpha\beta\frac{1}{1+\beta}\frac{1-\alpha}{\alpha^2} E_t\left[\varrho_{t+1}^{-1}\right] \\
&= -\frac{\beta(1-\alpha)}{\alpha(1+\beta)} E_t\left[\varrho_{t+1}^{-1}\right]
\end{aligned}$$

Hence:

$$
\begin{aligned}
E_{t-1}\frac{\partial \ln(1-\chi)}{\partial \tau} &= -E_{t-1}\frac{1}{1-\chi}\frac{\partial \chi}{\partial \bar{E}}\frac{\partial \bar{E}}{\partial \tau} \\
&= \frac{\beta(1-\alpha)}{\alpha(1+\beta)}E_{t-1}\left[\frac{1}{\varrho_{t+1}}\right] > 0
\end{aligned}
$$

This first term captures the effect that increasing social security contributions decreases the need to provide private savings. This has positive welfare effects.

The second term captures taxation (evaluated again at $\tau = 0$):

$$
E_{t-1}\frac{\partial \ln(1-\tau)}{\partial \tau} = -\frac{1}{1-\tau} = -1
$$

We next investigate the implicit return equation for social security. We get, evaluated at $\tau = 0$, that:

$$
\begin{aligned}
&\frac{\partial \ln\left(\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \tau\left((1+g) - \chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}\right)\right)}{\partial \tau} \\
&= \frac{1}{\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}}\left((1+g) - \chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1} - (1-\tau)\chi\zeta_t\varrho_{t+1}(1-\alpha)\alpha k_{ms}^{\alpha-2}\frac{\partial k_{ms}}{\partial \tau}\right) \\
&|_{\tau=0}= \frac{1+g}{\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}} - 1 - (1-\alpha)k_{ms}^{-1}\frac{\partial k_{ms}}{\partial \tau} \\
&= \frac{1+g}{\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}} - 1 - (1-\alpha)\frac{\partial \ln k_{ms}}{\partial \tau} \\
&= \frac{1+g}{\chi\zeta_t\varrho_{t+1}\alpha\frac{1+g}{\chi(1-\tau)(1-\alpha)}} - 1 - (1-\alpha)\frac{\partial \ln k_{ms}}{\partial \tau} \\
&|_{\tau=0}= \frac{1-\alpha}{\alpha\zeta_t\varrho_{t+1}} - 1 - (1-\alpha)\frac{\partial \ln k_{ms}}{\partial \tau}
\end{aligned}
\tag{46}
$$

From (45) and (46) and subtracting -1 (the above effect of taxation on income), term A accordingly writes as

$$
A \equiv \frac{\beta(1-\alpha)}{\alpha}E_{t-1}\left[\frac{1}{\zeta_t}\right]E_{t-1}\left[\frac{1}{\varrho_{t+1}}\right] - (1+\beta)
$$

Finally, from (45) and (46) all terms incorporating $\frac{\partial \ln k_{ms}}{\partial \tau}$ are given by

$$
(\alpha(1+\beta) - \beta(1-\alpha))\frac{\partial \ln k_{ms}}{\partial \tau} \tag{47}
$$

Turning to $\frac{\partial \ln k_{ms}}{\partial \tau}$ we find that, at $\tau = 0$, we have

$$\begin{aligned}
\frac{\partial \ln k_{ms}}{\partial \tau} &= \frac{1}{1-\alpha}\left(\frac{\partial \ln \chi}{\partial \tau} + \frac{\partial \ln(1-\tau)}{\partial \tau}\right) \\
&= \frac{1}{1-\alpha}\left(\frac{1}{\chi}\frac{\partial \chi}{\partial \bar{E}}\frac{\partial \bar{E}}{\partial \tau} - 1\right) \\
&= \frac{1}{1-\alpha}\left(\frac{\alpha}{1+\beta}\frac{\partial \bar{E}}{\partial \tau} - 1\right) \\
&= \frac{1}{1-\alpha}\left(-\frac{\alpha}{1+\beta}\frac{1-\alpha}{\alpha^2}E_t\left[\frac{1}{\varrho_{t+1}}\right] - 1\right) \\
&= -\frac{1}{1-\alpha}\left(1 + \frac{1}{1+\beta}\frac{1-\alpha}{\alpha}E_t\left[\frac{1}{\varrho_{t+1}}\right]\right).
\end{aligned}$$

□

B Computational Appendix

B.1 Aggregate Problem

In order to compute the stationary competitive equilibrium of our model, we apply the Krusell and Smith (1997) method. Specifically, we follow Storesletten, Telmer, and Yaron (2007) (STY) and approximate the aggregate law of motion as

$$(k', \mu') = \hat{H}(t; k, \mu, z, z') \tag{48}$$

where k is the capital stock per efficiency unit and $\mu = \mathbb{E}r' - r^{f'}$ is the equity premium. That is, we approximate the distribution Φ by two "moments" where the equity premium captures information about equity and bond holding moments. Our approach differs from STY in three ways: (i) we plan to explicitly compute transitional dynamics between two stationary competitive equilibria (which fluctuate in two ergodic sets), (ii) we do not use simulation techniques to aggregate on the idiosyncratic states of the distribution and (iii) we compute an approximate equilibrium, referred to as a "mean shock equilibrium", which serves three purposes: first, it enables us to initialize the cross-sectional distribution of agents second, we use it in order to calibrate our model in the initial competitive equilibrium in all periods $t \leq 0$ (for $\tau_t = \tau_0$) and third, it determines the means of the aggregate grids which we employ in the stochastic solution of our model. Computation of the "mean shock equilibrium" is by standard methods to solve OLG models without any aggregate risk. But in contrast to fully deterministic models, the mean shock equilibrium gives rise to an equity premium.

B.1.1 Mean Shock Equilibrium

As an initialization step, we solve for a degenerate path of the economy where the realizations of all aggregate shocks are at their respective means. We accordingly set $z = \bar{z} = \mathbb{E}z$ and $\delta = \bar{\delta} = \mathbb{E}\delta$. We assume that households accurately solve their forecasting problem for each realization of the aggregate state. This means that we approximate the above approximate law of motion as

$$(k', \mu') = \hat{\hat{H}}(k, \mu, \bar{z}, \bar{z}') \tag{49}$$

Observe that in the two stationary equilibria of our model, we have that fixed point relation

$$(k', \mu') = \hat{\hat{H}}(k, \mu, \bar{z}, \bar{z}') = (k, \mu) \tag{50}$$

With these assumptions, we can solve the mean shock path by standard Gauss-Seidel iterations as, e.g., described in Auerbach and Kotlikoff (1987). We adopt the modifications described in Ludwig (2006). While the numerical methods are the same as in the solution to a deterministic economy, the actual behavior of households fully takes into account the stochastic nature of the model. This also means that we solve the household problem using recursive methods and store the solutions to the household problem on grids of the idiosyncratic state x. The fixed-point computed in this auxiliary equilibrium gives k^{ms} and μ^{ms} as aggregate moments and cross-sectional distributions of agents as induced by the mean shock path. We denote these distributions by Φ^{ms}.

B.1.2 Grids

To construct the grids for the the aggregate states k and μ, $\mathcal{G}^k$, $\mathcal{G}^\mu$, define scaling factors s^k and s^μ and the number of grid points, n. We set $s^k = 0.8$ $s^\mu = 0.6$, and $n = 7$. Using these factors, we construct symmetric grids around k^{ms}, μ^{ms}.

B.1.3 Stochastic Solution

In order to solve for the stochastic recursive equilibria of our model, we use simulation methods. To this end, we specify the approximate law of motion in (48) as:

$$\ln(k_{t+1}) = \psi_0^k(z) + \psi_1^k(z)\ln(k_t) + \psi_2^k(z)\ln(\mu_t) \tag{51a}$$

$$\ln(\mu_{t+1}) = \psi_0^\mu(z) + \psi_1^\mu(z)\ln(k_{t+1}) + \psi_2^\mu(z)\ln(\mu_t) \tag{51b}$$

Like in Krusell and Smith (1997), the forecast for k_{t+1} is used to forecast μ_{t+1}. Intuitively, k_{t+1} contains a lot of information on the savings choice of the agent and therefore on the returns next period. Note that, in each period, μ_t is an "endogenous state", the realization of which has to be pinned down in that particular period (in contrast to k_t which is given in period t from decisions $t-1$). As in the standard application of the Krusell and Smith (1998) method, the coefficients also depend on the realization of the aggregate state, z.

Stationary Equilibria

Define a number M of stochastic simulations and a number of $s < M$ of simulations to be discarded. We follow GM and set $M = 5500$ and $s = 500$. Also define a tolerance ζ. Further, draw a sequence for z for periods $t = -M, \ldots, 0$ and denote these realizations by $z_{-M}, \ldots, z_0$. Notice that we thereby use the same sequence of aggregate shocks (as given by a random number generator) in each iteration. Collecting coefficients as $\Psi = [\psi_0^k, \psi_1^k, \psi_2^k, \psi_0^\mu, \psi_1^\mu, \psi_2^\mu]'$, the iteration is as follows:

1. Initialization: Guess Ψ.

2. In each iteration i do the following:

   (a) Solution of household problem. We store the solutions of the household problem on the $\mathcal{G}^{hh} = \mathcal{G}^J \times \mathcal{G}^x \times \mathcal{G}^z \times \mathcal{G}^k \times \mathcal{G}^\mu$. This gives us policy functions for all households, e.g., $c(j, x; z, k, \mu)$, $\kappa(j, x; z, k, \mu)$, $a'(j, x; z, k, \mu)$.

   (b) Simulation and aggregation. We simulate the model economy for the M realizations of aggregate shocks, z. To aggregate on the idiosyncratic states, we start in period $t = -M$ with the initial distribution generated by the mean shock path, Φ^{ms}. We then loop forward using the transition functions Q as defined in (**??**) to update distributions, cf. equation (**??**). Notice that, conditional on the realization of z, this aggregation is by standard methods that are used in OLG models with idiosyncratic risk. Simulation and aggregation then gives us M realizations of k_t and μ_t for $t = -M, \ldots, 0$. Observe that, in order to compute the realizations for μ_t, we have to solve for the bond market clearing equilibrium in each t. We do so by using a univariate function solver (Brent's method). We are thereby more accurate than Gomes and Michaelides (2006) who simply interpolate on $\mathcal{G}^\mu$.

(c) From the stochastic simulations, discard the first s observations and, for the remaining periods $t = s, \ldots, 0$ run regressions on:

$$\ln(k_{t+1}) = \tilde{\psi}_0^k(z) + \tilde{\psi}_1^k(z)\ln(k_t) + \tilde{\psi}_2^k(z)\ln(\mu_t) + \vartheta_{t+1}^k \tag{52a}$$

$$\ln(\mu_{t+1}) = \tilde{\psi}_0^\mu(z) + \tilde{\psi}_1^\mu(z)\ln(k_t) + \tilde{\psi}_2^\mu(z)\ln(\mu_t) + \vartheta_{t+1}^\mu \tag{52b}$$

and collect the resulting coefficient estimates in the vector $\tilde{\Psi}$.

(d) `IF` $\|\Psi_i - \tilde{\Psi}_i\| < \zeta$ then `STOP`, `ELSE` define

$$g(\Psi) = \Psi - \tilde{\Psi}(\Psi) \tag{53}$$

as the distance function (=root finding problem) and update Ψ_{i+1} as

$$\Psi_{i+1} = \Psi_i - sJ(\Psi)^{-1}g(\Psi) \tag{54}$$

where $J(\Psi)$ is the Jacobi matrix of the system of equations in (53) and s is a scaling factor. Continue with step 2a. We sole the root finding problem using Broyden's method, see Ludwig (2006).

B.2 Household Problem

We iterate on the Euler equation, using ideas developed in Carroll (2002). As derived in section 3.7, the transformed dynamic programming problem of the household reads as

$$u(t,j,x,\eta;z,k,\mu) = \max_{c,\kappa,a'} \left\{ \left[c^{\frac{1-\theta}{\gamma}} + \tilde{\beta}\left(\mathbb{E}\left[u\left(t+1,j+1,x',\eta';z',k',\mu'\right)^{1-\theta}\right]\right)^{\frac{1}{\gamma}}\right]^{\frac{\gamma}{1-\theta}} \right\}$$

$$\text{s.t.} \quad x = a' + c,$$

where $x' = a'\tilde{R}' + \mathrm{y}'$, with $\tilde{R}' = \frac{(1+r^{f\prime}+\kappa(r'-r^{f\prime}))}{\exp(v'(z'))(1+g)}$, and $\tilde{\beta} = \beta\varsigma_{j+1}\left((1+g)\right)^{\frac{1-\theta}{\gamma}}$.

Dropping the time index to simplify notation and using the dynamic budget constraint in the continuation value we get

$$u(j,\cdot) = \max_{c,\kappa} \left\{ \left[c^{\frac{1-\theta}{\gamma}} + \tilde{\beta}\left(\mathbb{E}\left[u(j+1,(x-c)\tilde{R}'+\mathrm{y}',\cdot)^{1-\theta}\right]\right)^{\frac{1}{\gamma}}\right]^{\frac{\gamma}{1-\theta}} \right\} \tag{55}$$

The first-order conditions are given by:

$$c: \quad c^{\frac{1-\theta-\gamma}{\gamma}} - \tilde{\beta}\left(\mathbb{E}\left[u(j+1,\cdot)^{1-\theta}\right]\right)^{\frac{1-\gamma}{\gamma}} \cdot \mathbb{E}\left[u(j+1,\cdot)^{-\theta}u_{x'}(j+1,\cdot)\tilde{R}'\right] = 0 \tag{56a}$$

$$\kappa: \quad \mathbb{E}\left[u(j+1,\cdot)^{-\theta}u_{x'}(j+1,\cdot)\left(r' - r^{f\prime}\right)\right] = 0 \tag{56b}$$

and the envelope condition reads as:

$$
\begin{aligned}
u_x = & \left(c^{\frac{1-\theta}{\gamma}} + \tilde{\beta}\left(\mathbb{E}\left[u(j+1,\cdot)^{1-\theta}\right]\right)^{\frac{1}{\gamma}}\right)^{\frac{\gamma-1+\theta}{1-\theta}} \cdot \ldots \\
& \cdot \tilde{\beta}\left(\mathbb{E}\left[u(j+1,\cdot)^{1-\theta}\right]\right)^{\frac{1-\gamma}{\gamma}} \mathbb{E}\left[\tilde{R}' u(j+1,\cdot)^{-\theta} u_{x'}(j+1,\cdot)\right] \\
= & u(j,\cdot)^{\frac{\gamma-1+\theta}{\gamma}} \tilde{\beta}\left(\mathbb{E}\left[u(j+1,\cdot)^{1-\theta}\right]\right)^{\frac{1-\gamma}{\gamma}} \cdot \ldots \\
& \cdot \mathbb{E}\left[\tilde{R}' u(j+1,\cdot)^{-\theta} u_{x'}(j+1,\cdot)\right] \\
= & \left(\frac{c}{u(j,\cdot)}\right)^{\frac{1-\theta-\gamma}{\gamma}}, \qquad (57)
\end{aligned}
$$

where the last line follows from equation (56a) and is exactly the result one would get from direct application of the Benveniste/Scheinkman theorem to recursive preferences, namely $v_x = u_1(c, Ev)$ (see Weil (1989)). Plugging this into the FOCs we get

$$
\begin{aligned}
c: \quad & c^{\frac{1-\theta-\gamma}{\gamma}} - \tilde{\beta}\left(\mathbb{E}\left[u(j+1,\cdot)^{1-\theta}\right]\right)^{\frac{1-\gamma}{\gamma}} \ldots \\
& \cdot \mathbb{E}\left[u(j+1,\cdot)^{\frac{(1-\theta)(\gamma-1)}{\gamma}} (c')^{\frac{1-\theta-\gamma}{\gamma}} \tilde{R}'\right] = 0 \qquad (58a) \\
\kappa: \quad & \mathbb{E}\left[u(j+1,\cdot)^{\frac{(1-\theta)(\gamma-1)}{\gamma}} (c')^{\frac{1-\theta-\gamma}{\gamma}} \left(r' - r^{f'}\right)\right] = 0 \qquad (58b)
\end{aligned}
$$

With respect to our numerical solution, we will interpolate the functions $u(j,\cdot)$ and $c(j,\cdot)$. Note that we can expect $u(j,\cdot)$ to be approximately linear, since in period J it is simply given by $u(J) = c_J = x_J$.

Next, notice that $u(j+1,\cdot)$ and c' are functions of $(x-c)$ so that c shows up on both sides of the equation in (58a). This would require calling a non-linear solver whenever we solve optimal consumption and portfolio shares. To alleviate this computational burden we employ the endogenous grid method of Carroll (2002). Accordingly, instead of working with an *exogenous* grid for x (and thereby an *endogenous* grid for savings, $s = x - c$) we revert the order and work with an *exogenous* grid for $s = x - c$ and an *endogenous* grid for x.

So, roughly speaking, for each age j and each grid point in the savings grid $\mathcal{G}^s$, our procedure is the following:

1. Solve equation (58b) for κ using a univariate equation solver (Brent's method).

2. Given the solution to (58b) invert (58a) to compute

$$c = \left(\tilde{\beta} \left(\mathbb{E}\left[(\exp(\upsilon'(z'))u(j+1,\cdot))^{1-\theta} \right] \right)^{\frac{1-\gamma}{\gamma}} \mathbb{E}\left[(\exp(\upsilon'(z')))^{1-\theta} v(j+1,\cdot)^{-\theta} \tilde{R}' \right] \right)^{\frac{\gamma}{1-\theta-\gamma}}. \tag{59}$$

3. Update u, u_x and v.

More precisely, our procedure is as follows:

1. Loop on the grids of the aggregate states, $\mathcal{G}^z$, $\mathcal{G}^k$, $\mathcal{G}^\mu$.
2. For each (z, k, μ) use (51) to compute the associated k', μ'.
3. Initialize the loop on age for $j = J$ by setting $c_J = x_J$ and compute $u(x_J) = c_J$, $u_x(x_J) = 1$ and $v(x_J) = u(x_J) = c_J$.
4. Loop backwards in age from $j = J - 1, \ldots, 0$ as follows:
   (a) As $k' \notin \mathcal{G}^k$, $\mu' \notin \mathcal{G}^\mu$, interpolate on the aggregate states and store the interpolated objects $u(k', \mu', x', z', j+1)$, $v(k', \mu', x', z', j+1)$ as a projection on $\mathcal{G}^x$. Do so for each $z' \in \mathcal{G}^z$. Denote the interpolated objects as $\bar{u}(x', z', j+1)$, $\bar{v}(x', z', j+1)$.
   (b) For each s in $\mathcal{G}^s$ first solve (58b) for κ. To so, we have to loop on $z' \in \mathcal{G}^z$ (as well as the idiosyncratic shock) to evaluate the expectation taking into account the Markov transition matrix $\pi(z'|z)$. In this step, we also use the law of motion of the idiosyncratic state x:
   $$x' = s\tilde{R}' + \mathrm{y}' \tag{60}$$
   As, generally, $x' \notin \mathcal{G}^x$ we have to interpolate on $\bar{u}(x', z', j+1)$ and $\bar{v}(x', z', j+1)$ before evaluating the expectation.
   (c) Taking the optimal κ as given, next compute c from (59). Again we interpolate on $\bar{u}(x', z', j+1)$ and $\bar{v}(x', z', j+1)$ before evaluating the expectation.

B.3 Calibration of Income Process

We determine $\eta\mp(z)$ and $\bar{\pi}^\eta$ such that we match the unconditional variance of the STY estimates, i.e.,

$$E[(\ln \eta')^2 \mid z'] = \sigma(z')^2_{\ln \eta} \tag{61}$$

and the unconditional autocorrelation, i.e.,

$$\frac{E[\ln \eta' \ln \eta]}{E[(\ln \eta')^2]} = \rho. \tag{62}$$

To match the variance we specify the states of the Markov process as

$$\eta_{\mp}(z) = \frac{2 \exp(1 \mp \tilde{\sigma}(z))}{\exp(1 - \tilde{\sigma}(z)) + \exp(1 + \tilde{\sigma}(z))}$$

so that the unconditional mean equals one.

We pick $\tilde{\sigma}(z)$ such that the unconditional variance—which of course preserves its contingency on z—satisfies (61). To achieve this, observe that

$$\ln \eta_{\mp} = \ln\left(\frac{2}{\exp(1 - \tilde{\sigma}(z)) + \exp(1 + \tilde{\sigma}(z))}\right) + 1 \mp \tilde{\sigma}(z) \equiv \phi(\tilde{\sigma}(z)) \mp \tilde{\sigma}(z).$$

Hence

$$\begin{aligned} E[(\ln \eta')^2 \mid \eta = \eta_-, z'] &= \bar{\pi}^{\eta}(\phi(\tilde{\sigma}(z')) - \tilde{\sigma}(z'))^2 + \\ &\qquad (1 - \bar{\pi}^{\eta})(\phi(\tilde{\sigma}(z')) + \tilde{\sigma}(z'))^2 \\ E[(\ln \eta')^2 \mid \eta = \eta_+, z'] &= \bar{\pi}^{\eta}(\phi(\tilde{\sigma}(z')) + \tilde{\sigma}(z'))^2 + \\ &\qquad (1 - \bar{\pi}^{\eta})(\phi(\tilde{\sigma}(z')) - \tilde{\sigma}(z'))^2. \end{aligned}$$

The unconditional mean of the above—conditional on z'—is

$$E[(\ln \eta')^2 \mid z'] = \phi(\tilde{\sigma}(z'))^2 + \tilde{\sigma}(z')^2.$$

To determine $\tilde{\sigma}(z)$, we then numerically solve the distance function

$$f(\tilde{\sigma}(z)) = \phi(\tilde{\sigma}(z))^2 + \tilde{\sigma}(z)^2 - \sigma(z)^2_{\ln \eta} = 0$$

for all z. Standard procedures ignore the bias term $\phi(\tilde{\sigma}(z))^2$.

To determine $\bar{\pi}^{\eta}$ observe that, in the stationary invariant distribution, we have

$$\begin{aligned} E[\ln \eta' \ln \eta \mid \eta = \eta_-] &= \sum_z \Pi^z(z) \left\{\bar{\pi}^{\eta}(\phi(\tilde{\sigma}(z)) - \tilde{\sigma}(z))^2 + \right. \\ &\qquad \left. (1 - \bar{\pi}^{\eta})(\phi(\tilde{\sigma}(z)) - \tilde{\sigma}(z))(\phi(\tilde{\sigma}(z)) + \tilde{\sigma}(z))\right\} \\ E[\ln \eta' \ln \eta \mid \eta = \eta_+] &= \sum_z \Pi^z(z) \left\{\bar{\pi}^{\eta}(\phi(\tilde{\sigma}(z)) + \tilde{\sigma}(z))^2 + \right. \\ &\qquad \left. (1 - \bar{\pi}^{\eta})(\phi(\tilde{\sigma}(z)) + \tilde{\sigma}(z))(\phi(\tilde{\sigma}(z)) - \tilde{\sigma}(z))\right\} \end{aligned}$$

and

$$E[\ln \eta' \ln \eta] = \sum_{\eta} \Pi^{\eta}(\eta) E[\ln \eta' \ln \eta \mid \eta]$$

as well as

$$E[(\ln \eta')^2] = \sum_{z'} \Pi^{z'} E[(\ln \eta')^2 \mid z'].$$

Noticing that

$$\frac{E[\ln \eta' \ln \eta]}{E[(\ln \eta')^2]} = \rho$$

we use the above relationships to determine $\bar{\pi}^{\eta}$.

C Robustness

The tables shown below are all discussed in section 5.3 of the main text.

C.1 Robustness of the thought experiment

Here we report the results of the experiment $\tau = 0.12 \rightarrow \tau = 0.14$ ('HL experiment'). The value $\tau = 0.12$ is the average contribution rate in the U.S. today. As in the main text, we first report the NC calibration (negative correlation between TFP and returns), then the PC calibration (positive correlation between TFP and returns).

C.1.1 Calibration with negative correlation between TFP and returns (NC)

Parameters					Moments	
θ	φ	β	$\bar{\delta}$	π^{δ}	$cor(r, w)$	$std(\Delta C/C)$
17	1.5	0.975	0.1	0.435	-8.27 E-2	6.84 E-2

	GE	PE
ΔWelf/Welf	+0.17%	+2.62%
$\Delta K/K$	-2.88%	-21.27%
$\Delta E(r)$	+0.16%	*0.00%*
Δr_f	+0.28%	*0.00%*
$\Delta w/w$	-3.18%	*0.00%*

	aggr. risk	+ wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	-0.31%	+1.01%	+ 1.26%	+2.62%

Welfare gains					
	aggr. risk	+ wage risk	Δ	IAR	$\widehat{IAR}$
orig	-0.31%	+1.01%	+1.32	-	-
-10%	-0.39%	+0.90%	+1.28	-0.028	-0.031

C.1.2 Calibration with positive correlation between TFP and returns (PC)

Parameters					Moments	
θ	φ	β	$\bar{\delta}$	π^{δ}	$cor(r,w)$	$std(\Delta C/C)$
13	1.5	0.99	0.11	0.86	25.1 E-2	7.40 E-2

	GE	PE
ΔWelf/Welf	+1.94%	+4.18%
$\Delta K/K$	-2.21%	-17.41%
$\Delta E(r)$	+0.12%	*0.00%*
Δr_f	+0.26%	*0.00%*
$\Delta w/w$	-2.98%	*0.00%*

	aggr. risk	+ wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	+0.87%	+2.65%	+3.33%	+4.18%

Welfare gains					
	aggr. risk	+ wage risk	Δ	IAR	$\widehat{IAR}$
orig	+0.87%	+2.65%	+1.78	-	-
-10%	+0.35%	+1.98%	+1.63	-0.146	-0.207

C.2 Sensitivity with respect to the elasticity of intertemporal substitution

We now set the value of the elasticity of intertemporal substitution to $\varphi = 0.5$, the value used by Krueger and Kubler (2006). Of course we have to recalibrate the model, but the changes are restricted to a few parameters, which we document. First we report the results for the NC calibration, then for the PC calibration.

NOTE: The following results are for the HL experiment ($\tau = 0.12 \rightarrow \tau = 0.14$). This should be changed to the KK experiment ($\tau = 0.00 \rightarrow \tau = 0.02$). Nonetheless, the results are still valid, and can be compared to the values in the previous section (section C.1). The corresponding discussion in the main text gives insight on how the results would change with the KK experiment.

C.2.1 Calibration with negative correlation between TFP and returns (NC)

Parameters					Moments	
θ	φ	β	$\bar{\delta}$	π^{δ}	$cor(r,w)$	$std(\Delta C/C)$
15	0.5	0.975	0.1	0.435	-10.27 E-2	4.32 E-2

	GE	PE
ΔWelf/Welf	-3.10%	–
$\Delta K/K$	-5.91%	–
$\Delta E(r)$	+0.44%	*0.00%*
Δr_f	+0.47%	*0.00%*
$\Delta w/w$	-4.14%	*0.00%*

	aggr. risk	+ wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	-0.89%	+0.04%	–	–

Welfare gains					
	aggr. risk	+ wage risk	Δ	IAR	$\widehat{IAR}$
orig	-0.89%	+0.04%	+0.92	-	-
+**10**%	-0.80%	+0.15%	+0.96	+0.032	+0.034

C.2.2 Calibration with positive correlation between TFP and returns (PC)

This is actually the PC2 calibration, but that doesnt change anything.

Parameters					Moments	
θ	φ	β	$\bar{\delta}$	π^{δ}	$cor(r,w)$	$std(\Delta C/C)$
10	0.5	0.98	0.11	0.86	19.1 E-2	4.85 E-2

	GE	PE
ΔWelf/Welf	-2.51%	+1.74%
$\Delta K/K$	-5.87%	-20.55%
$\Delta E(r)$	+0.47%	*0.00%*
Δr_f	+0.48%	*0.00%*
$\Delta w/w$	-4.14%	*0.00%*

	aggr. risk	+ wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	-0.41%	+0.60%	+ 1.07%	+1.74%

Welfare gains					
	aggr. risk	+ wage risk	Δ	IAR	$\widehat{IAR}$
orig	-0.41%	+0.60%	+1.01	-	-
-10%	-0.35%	+1.98%	+0.94	-0.075	-0.142

C.3 Nonlinear trend calibration

Insurance larger with our standard KK experiment ($\tau = 0.00 \rightarrow \tau = 0.02$), which we still need to report.

Parameters					Moments	
θ	φ	β	$\bar{\delta}$	π^{δ}	$cor(r, w)$	$std(\Delta C/C)$
14	1.5	0.98	0.1	0.7	6.45 E-2	6.53 E-2

	GE	PE
ΔWelf/Welf	+0.8%	+3.0%
$\Delta K/K$	-2.7%	-22.1%
$\Delta E(r)$	+0.2%	*0.0%*
Δr_f	+0.3%	*0.0%*
$\Delta w/w$	-3.2%	*0.0%*

	aggr. risk	+ wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	-0.2%	+1.4%	+ 1.7%	+3.0%

Welfare gains					
	aggr. risk	+ wage risk	Δ	IAR	$\widehat{IAR}$
orig	-0.21%	+1.35%	+1.56	-	-
-10%	-0.33%	+1.18%	+1.51	-0.054	-0.045

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