$\mathbb{MA}1505$ Cheat Sheet

Functions

$(f \pm g)(x) = f(x) \pm g(x)$	$\lim_{x\to a}(f \pm g)(x) = L \pm L'$	
$(fg)(x) = f(x)g(x)$	$\lim_{x\to a}(fg)(x) = LL'$	All
$(f/g)(x) = f(x)/g(x)$	$\lim_{x\to a} \frac{f}{g}(x) = \frac{L}{L'}$	
	$\lim_{x\to a} kf(x) = kL, k \in \mathbb{R}$	

polynomials are continuous at every point in $\mathbb{R}$. All rational functions $\frac{p(x)}{q(x)}$ where p and q are polynomials are continuous at every point such that $q(x) \neq 0$.
Composition is given by $\circ$, e.g. $(f \circ g)(x) = f(g(x))$

Differentiation

Product Rule $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient Rule $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
Chain Rule $(f \circ g)'(x) = f'(g(x))g'(x) = (f' \circ g)(x)g'(x)$
$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

Maxima and Minima

A function f has a local/relative maximum value at a point c in its domain if $f(x) \leq f(c)$ for all x in the neighborhood of c.
The function has an absolute maximum value at c if $f(x) \leq f(c)$ for all x in the domain.
Reverse signs for minimum.

Extreme and Critical Points

Points where f can have an extreme value are: interior points where $f'(x) = 0$, interior points where $f'(x)$ does not exist and end points of the domain of f.
An interior point of the domain of a function f where f' is zero or does not exist is a *critical point* of f.

Increasing or Decreasing Functions

f is increasing on an interval I when $f'(x) > 0$ for all $x \in I$.
f is decreasing on an interval I when $f'(x) < 0$ for all $x \in I$.

Concavity

The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$ and concave up on any interval where $y'' > 0$.
A point c is a point of inflection of the function f if f is continuous at c and there is an open interval containing c such that the graph of f changes from concave up (or down) to concave down (or up). The function need not be differentiable at c.

Derivative Tests for Maxima and Minima

First derivative test: Suppose $c \in (a, b)$ is a critical point of f.
If $f'(x) > 0$ for $x \in (a, c)$ and $f'(x) < 0$ for $x \in (c, b)$ then $f(c)$ is a local maximum.
If $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$ then $f(c)$ is a local minimum.
Second derivative test: If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at $x = c$.
If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at $x = c$.

Indeterminacy and L'Hopital's Rule

If the functions f and g are continuous at $x = a$ but $f(a) = g(a) = 0$, then the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ cannot be directly evaluated.
Suppose that f and g are differentiable in a neighborhood of a, $f(a) = g(a) = 0$ and $g'(x) \neq 0$ except possibly at a.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Integration

If f is continuous on $[a, b]$, then

$$F(x) = \int_a^x f(t)dt$$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Integration by parts.

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Area between two curves $f_2(x)$ and $f_1(x)$ where $f_1(x) \leq f_2(x)$ in $[a, b]$

$$\int_a^b f_2(x) - f_1(x)dx$$

Partial Fraction Decomposition

$$\frac{px + q}{(ax + b)(cx + d)} = \frac{A}{ax + b} + \frac{B}{cx + d}$$

$$\frac{px^2 + qx + r}{(ax + b)(cx + d)^2} = \frac{A}{ax + b} + \frac{B}{cx + d} + \frac{C}{(cx + d)^2}$$

$$\frac{px^2 + qx + r}{(ax + b)(x^2 + c^2)} = \frac{A}{ax + b} + \frac{Bx + C}{x^2 + c^2}$$

Series

Arithmetic Series

$$\sum_1^n a_n = \frac{n}{2}(a_1 + a_n)$$

Geometric Series

$$\sum_1^n ar^{n-1} = a\frac{1 - r^n}{1 - r}$$

If $|r| < 1$ then as $n \to \infty$,

$$\sum_1^n ar^{n-1} \to \frac{a}{1 - r}$$

Ratio Test

For a series $\sum a_n$, let

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$$

Series is convergent if $\rho < 1$, divergent if $\rho > 1$ and no conclusion reached if $\rho = 1$.

Power Series

A power series has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n + ...$$

If the series is centered about $x = a$,

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_n(x-a)^n + ...$$

Standard Series

$$(1 + x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + ... + \frac{r(r-1)...(r-n+1)}{n!}x^n$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + ... \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + ...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + \frac{(-1)^n x^{2n}}{(2r)!}$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + \frac{(-1)^{n+1}x^n}{n} + ...$$

Radius of Convergence

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

Taylor Series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

Taylor's Theorem

The nth order Taylor polynomial of f at a is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Then $f(x) = P_n(x) + R_n(x)$ where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x, where $R_n(x)$ is the remainder of order n or the error term for the approximation of $f(x)$ by $P_n(x)$.

Three Dimensional Space $\mathbb{R}^3$

Dot Product

$$\vec{v_1} \cdot \vec{v_2} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2 = |\vec{v_1}|\,|\vec{v_2}|\cos\theta$$

Unit Vector

For some vector $\mathbf{u}$, its unit vector $\hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|}\mathbf{u}$

Cross Product

$$\vec{v_1} \times \vec{v_2} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{pmatrix} = |\vec{v_1}|\,|\vec{v_2}| \sin\theta$$

The distance dist from a point $P(x_0, y_0, z_0)$ to a plane $\Pi : ax + by + cz = d$ is given by

$$\text{dist } = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \text{proj}_{\mathbf{n}} \vec{OP}$$

Space Curves

For some curve with the vector equation
$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,
its arc length (if the curve is traversed once)

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt = \int_a^b |\mathbf{r}'(t)|$$

Fourier Series

A function is said to be odd if $-f(x) = f(-x)$ and even if $f(x) = f(-x)$. Examples are $\sin x$ for the former and $\cos x$ for the latter.
A periodic function of period T can be represented by a Fourier series $f(x)$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Let $2L = T$.

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

For the mth term where $m \in \mathbb{Z}^+$,

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{\pi m x}{L} dx$$

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{\pi m x}{L} dx$$

If the function is even, we only need to consider cosine terms. Similarly, if the function is odd, we only need to consider sine terms.

Multivariate Functions

Partial Derivatives

Let $z = f(x, y)$ be a function of two variables.
The *partial derivative* of a function $f(x, y)$ w.r.t. x is denoted by $f_x(x, y)$ or $\frac{\partial f}{\partial x}$ where the y term is taken as a constant.

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} \text{ and } f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2} \text{ and } f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x \partial y}$$

For most functions in practice, $f_{xy}(a, b) = f_{yx}(a, b)$.

Chained Derivatives

Suppose $z = f(x, y, z)$ where $x = x(t)$, $y = y(t)$ and $z = z(t)$. Thus, $z = f(x(t), y(t), z(t))$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Suppose $w = f(x, y, z)$ and $x = x(s, t)$, $y = y(s, t)$, and $z = z(s, t)$, giving $w = f(x(s, t), y(s, t), z(s, t))$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Directional Derivatives

Note that $D_{\mathbf{i}} f(a, b) = f_x(a, b)$ and $D_{\mathbf{j}} f(a, b) = f_y(a, b)$ for the standard unit vectors of the x and y direction.
For some <u>unit vector</u> $\hat{\mathbf{u}} = u_1\mathbf{i} + u_2\mathbf{j}$,

$$D_{\mathbf{u}} f(a, b) = f_x(a, b) \cdot u_1 + f_y(a, b) \cdot u_2 = \nabla f(a, b) \cdot \hat{\mathbf{u}}$$

The directional derivative $D_{\mathbf{u}} f(a, b)$ measures the change in the value df of a function f when moved a distance dt from the point (a, b) in the direction of the vector $\mathbf{u}$, where $df = D_{\mathbf{u}} f(a, b) \cdot dt$.

Gradient Vector

The gradient vector ∇f is given by

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j}$$

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u} = |\nabla f(a, b)| \cos\theta$$

The function f increases most rapidly in the direction $\nabla f(a, b)$ and decreases most rapidly in the direction $-\nabla f(a, b)$.

Maxima and Minima

$f(x, y)$ has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) near (a, b)
$f(x, y)$ has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) near (a, b)
A function f may have a local maximum or minimum at (a, b) if: $f_x(a, b) = 0$ and $f_y(a, b) = 0$ <u>or</u> $f_x(a, b)$ or $f_y(a, b)$ is not defined. A point that satisfies either condition is known as a *critical point*.
Suppose that (a, b) is a critical point of $f(x, y)$. Let us define D as

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

$D > 0$ and $f_{xx}(a, b) > 0$	relative minimum at (a, b)
$D > 0$ and $f_{xx}(a, b) < 0$	relative maximum at (a, b)
$D < 0$	saddle point at (a, b)
$D = 0$	no conclusion reached

Saddle Point

At a point (a, b) of f where $f_x(a, b) = 0$ and $f_y(a, b) = 0$, the point (a, b) is known as a *saddle point* of f if there are some directions along which f has a local maximum at (a, b) and some directions along f which has a local minimum at (a, b).

Lagrange Multiplier

Suppose a function $f(x, y)$ subject to the constraint $g(x, y)$.

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

Solve for $F_x = 0$, $F_y = 0$ and $F_\lambda = 0$ to solve for λ.

Multiple Integrals

For $R = R_1 \cup R_2$ where R_1 and R_2 do not overlap except maybe at their boundary,

$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$$

Suppose a rectangular region R in the xy-plane where $a \leq x \leq b$ and $c \leq y \leq d$, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

Type A Regions

Bottom and top boundaries are curves given by $y = g_1(x)$ and $y = g_2(x)$ respectively, while left and right boundaries are $x = a$ and $x = b$ respectively.

$$R : g_1(x) \leq y \leq g_2(x),\ a \leq x \leq b$$

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Type B Regions

Left and right boundaries are curves given by $x = h_1(y)$ and $x = h_2(y)$ and bottom and top boundaries are straight lines $y = c$ and $y = d$ respectively.

$$R : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)$$

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Polar Coordinates

Circular regions/sectors can be described with polar coordinates r and θ.
In general, a region R in polar coordinates is described by

$$R : a \leq r \leq b,\ \alpha \leq \theta \leq \beta$$

When transforming from Cartesian to polar coordinates, (x, y) is transformed to (r, θ) where

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

and dA is changed from $dx\, dy$ to $r\, dr\, d\theta$.

Application of Double Integrals

Suppose D is a solid region under a surface defined by $f(x, y)$ over a plane region R.

$$\text{Volume of } D = \iint_R f(x, y)\, dA$$

If f has continuous first partial derivatives on a closed region R of the xy-plane, then the area S of that portion of the surface $z = f(x, y)$ that projects onto R is given by

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Line Integrals

Vector Fields

A vector field on R is a vector function $\mathbf{F}$ that assigns to each point a vector $\mathbf{F}(x, y, z)$.

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

Gradient Fields

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

Conservative Fields

A vector field $\mathbf{F}$ is called a conservative vector field if it is the gradient of some scalar function f such that $\mathbf{F} = \nabla f$, where f is known as the potential function for $\mathbf{F}$.
Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on the xy-plane.

$$\text{If } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ then } \mathbf{F} \text{ is conservative.}$$

Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field on xyz-space.

$$\text{If } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \text{ then } \mathbf{F} \text{ is conservative.}$$

If $\mathbf{F}$ is a conservative vector field, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path taken.
If $\mathbf{F}$ is a conservative vector field, then $\oint_l \mathbf{F} \cdot d\mathbf{r} = 0$ for any <u>closed</u> curve l, i.e. a curve with a terminal point that coincides with its initial point.

Line Integrals of Scalar Functions

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) ||\mathbf{r}'(t)|| dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Line Integrals of Vector Fields

$$\int_C \mathbf{F} \cdot\, d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

Geometrically, the line integral of $\mathbf{F}$ over C is summing up the tangential components of $\mathbf{F}$ with respect to the arc length of C.

$$\int_{-C} f(x, y, z) ds = \int_C f(x, y, z) ds$$

The vector equation of a curve C determines the orientation or direction of C.
For some $\mathbf{F}(x, y, z) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} + R(x, y)\mathbf{k}$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz = \int_a^b P(\mathbf{r}(t)) \frac{dx}{dt} + Q(\mathbf{r}(t)) \frac{dy}{dt} dt$$

Fundamental Theorem for Line Integrals

If f is a function of 2 or 3 variables whose gradient ∇f is continuous,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Green's Theorem

Let D be a bounded region in the xy-plane and ∂D the boundary of D. Suppose $P(x, y)$ and $Q(x, y)$ has continuous partial derivatives on D. Thus,

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

The orientation of ∂D is such that, as one traverses along the boundary in this direction, the region D is always on the left-hand side, i.e. the positive orientation of the boundary.

Surface Integrals

A parametric representation of a surface is given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

Standard Parametric Representation: Sphere

For a sphere of radius a: $x^2 + y^2 + z^2 = a^2$

$$\mathbf{r}(u, v) = (a \sin u \cos v)\mathbf{i} + (a \sin u \sin v)\mathbf{j} + (a \cos u)\mathbf{k}$$

When $0 \le u \le \pi$ and $0 \le v \le 2\pi$, the representation gives a full sphere.
When $0 \le u \le \frac{\pi}{2}$ and $0 \le v \le 2\pi$, the representation gives the upper hemisphere.

Standard Parametric Representation: Cylinder

For a circular cylinder of radius a: $x^2 + y^2 = a^2$

$$\mathbf{r}(u, v) = (a \cos u)\mathbf{i} + (a \sin u)\mathbf{j} + v\mathbf{k}$$

Here, u measures the angle from the positive x-axis about the z-axis while v measures the height from the xy-plane along the cylinder.
The same applies for $x^2 + z^2 = a^2$ (cylinder about y-axis)

$$\mathbf{r}(u, v) = (a \cos u)\mathbf{i} + v\mathbf{j} + (a \sin u)\mathbf{k}$$

and $y^2 + z^2 = a^2$ (cylinder about x-axis)

$$\mathbf{r}(u, v) = v\mathbf{i} + (a \cos u)\mathbf{j} + (a \sin u)\mathbf{k}$$

Tangent Planes

Let S be a surface given by the parametric representation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

For some position vector $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$ at a point P_0,
Fixing $v = v_0$ for a resulting curve C_1, the tangent vector of the space curve C_1 is given by

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

Fixing $u = u_0$ for a resulting curve C_2, the tangent vector of the space curve C_2 is given by

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$

Both vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ lie in the tangent plane to S at P_0.
Thus, the cross product $\mathbf{r}_u \times \mathbf{r}_v$, assuming it is non-zero, provides a normal vector to the tangent plane to S at P_0.
The equation of the tangent plane is described by

$$(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0$$

Surface Integrals of Scalar Functions

Suppose $f(x, y, z)$ be a function defined on a surface S, where we can find $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, k)\mathbf{k}$ of S over a domain D.

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

Surface Integrals of Vector Fields

Let $\mathbf{F}$ be a continuous vector fields defined on a surface S with a unit normal vector $\mathbf{n}$. The surface integral of $\mathbf{F}$ over S is given as

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS \text{ or more simply } \iint_S \mathbf{F} \cdot\, dS$$

This integral is also known as the flux of $\mathbf{F}$ over S.
If S is given by a parametric representation $\mathbf{r} = \mathbf{r}(u, v)$ with domain D,

$$\iint_S \mathbf{F} \cdot\, dS = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Orientation of Surfaces

By convention, a curve has a positive orientation if it progresses counter-clockwise.

$$\iint_{-S} \mathbf{F} \cdot\, dS = -\iint_S \mathbf{F} \cdot\, dS$$

The Little Man Analogy

Imagine a little man traversing a curve C. His head is pointed in the direction of the normal vector of C. C thus has a positive orientation if the region bounded by the curve is always on his left-hand side.

Curl and Divergence

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in xyz-space.

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

The curl of a vector field is also itself a vector field.

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence of a vector field is a scalar function.

Del Operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

The curl and divergence operations can be expressed in terms of the del operator.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} \text{ and div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

Conservative Fields*

Let $\mathbf{F}$ be a vector field in xyz-space.
If curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative field. The converse is also true.

Stokes' Theorem

Let S be an oriented, piecewise-smooth surface that is bounded by a closed, piecewise-smooth boundary curve C. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on S. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot dS$$

Stokes' Theorem can also be expressed as

$$\oint_C P\,dx + Q\,dy + R\,dz =$$

$$\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

The orientation of C must be consistent with that of S.

Gauss' Theorem

Let E be a solid region, S be the boundary of E given with outward orientation (where the normal vector on the surface always points away from E). Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives in E. Then,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\, dV$$

Trigonometric Identities

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1$$

$$1 + \tan^2 u = \sec^2 u$$

$$1 + \cot^2 u = \csc^2 u$$

Sum and Difference Formulae

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Double Angle Formulae

$$\sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

Half Angle Identities

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Sum → Product Identities

$$\sin u + \sin v = 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2}$$

$$\sin u - \sin v = 2 \cos \frac{u+v}{2} \sin \frac{u-v}{2}$$

$$\cos u + \cos v = 2 \cos \frac{u+v}{2} \cos \frac{u-v}{2}$$

$$\cos u - \cos v = -2 \sin \frac{u+v}{2} \sin \frac{u-v}{2}$$

Product → Sum Identities

$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$$

Parity Identities

$$\sin(-u) = -\sin u$$

$$\cos(-u) = \cos u$$

$$\tan(-u) = -\tan u$$

$$\cot(-u) = -\cot u$$

$$\csc(-u) = -\csc u$$

$$\sec(-u) = \sec u$$

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