Question 1.

(a) *(15 marks)* Let $(X, \mathcal{M})$ be a measurable space and let μ and ν be measures defined on $\mathcal{M}$. Show that the set function $\lambda : \mathcal{M} \to \overline{\mathbb{R}}$ defined by $\lambda(E) = \mu(E) + \nu(E)$ is a measure.

(b) *(10 marks)* Let $(X, \mathcal{M}, \mu)$ be a measure space and let $A, B \in \mathcal{M}$ and $A \subset B$. Show that if $\mu(A) = \mu(B) < \infty$, then for every measurable subset E of B we have $\mu(E \setminus A) = 0$ and $\mu(A \cap E) = \mu(E)$.

Question 2. Let $\{f_n\}$ be a sequence of measurable functions on X. Let

$$A = \{x : \lim_{n\to\infty} f_n(x) \text{ exists}\},$$
$$B = \{x : \lim_{n\to\infty} f_n(x) \text{ exists and finite}\}.$$

(a) *(15 marks)* Show that A and B are measurable. (*Hint:* $\lim_n f_n(x)$ exists if and only if $\liminf_n f_n(x) = \limsup_n f_n(x)$.)

(b) *(10 marks)* Show that the function g defined by

$$g(x) = \begin{cases} \lim f_n(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is measurable.

Question 3. *(25 marks)* Let $(X, \mathcal{M}, \mu)$ be a measure space.

Let $f, f_n : X \to \overline{\mathbb{R}}$ be measurable functions. Suppose that $f_n \geq 0$ and $f_n \to f$ a.e. Suppose also that $\lim_{n\to\infty} \int_X f_n d\mu = c$ where $0 < c < \infty$. Show that $\int_X f d\mu$ is defined and $0 \leq \int_X f d\mu \leq c$. Show by an example that $\int_X f d\mu$ may be equal to 0.

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Question 4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) *(10 marks)* Suppose that f and g are nonnegative measurable functions on X such that $\{f \neq 0\} \cap \{g \neq 0\} = \emptyset$. Show that the two measures $\nu_1, \nu_2 : \mathcal{M} \to \overline{\mathbb{R}}$ defined by $\nu_1(E) = \int_E f d\mu$ and $\nu_2(E) = \int_E g d\mu$ are mutually singular.

(b) *(15 marks)* Let h be a nonnegative measurable function on X. Defined $\lambda : \mathcal{M} \to \overline{\mathbb{R}}$ by

$$\lambda(E) = \int_E h d\mu, \qquad E \in \mathcal{M}.$$

Show that if $\mu(\{h = 0\}) = 0$, then $\mu \ll \lambda$.

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