VIETNAM NATIONAL UNIVERSITY-HO CHI MINH CITY
INTERNATIONAL UNIVERSITY

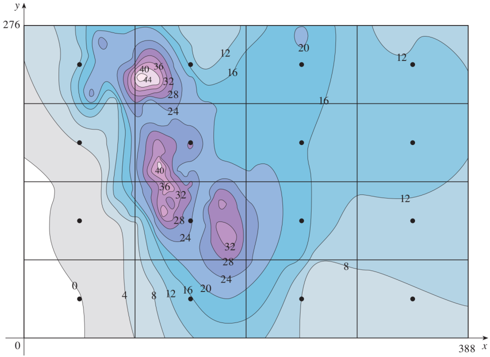
Chapter 2. Multiple Integrals

Analysis 3

Lecturer: Nguyen Minh Quan, PhD
quannm@hcmiu.edu.vn

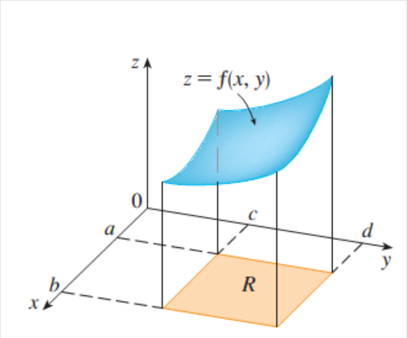
Contents

Introduction



In this chapter, we will learn how to approximate the snowfall in Colorado (2006) given by the figure above by double integrals.
Integrals of functions of several variables, called multiple integrals, are a natural extension of the single-variable integrals. They are used to compute many quantities that appear in applications, such as volumes, surface areas, centers of mass, probabilities, and average values.

Volumes and Double Integrals

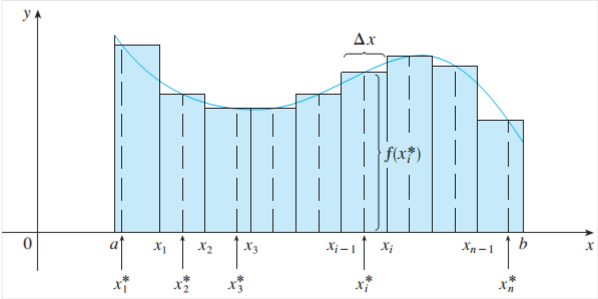


The integral of a function of two variables $f(x,y)$, called a double integral, is denoted $\iint\limits_R f(x,y)\,dA$.

It represents the signed volume of the solid region between the graph of $f(x,y)$ and a domain R in the $xy-$plane, where the volume is positive for regions above the xy-plane and negative for regions below.

Review Partition for function of one variable

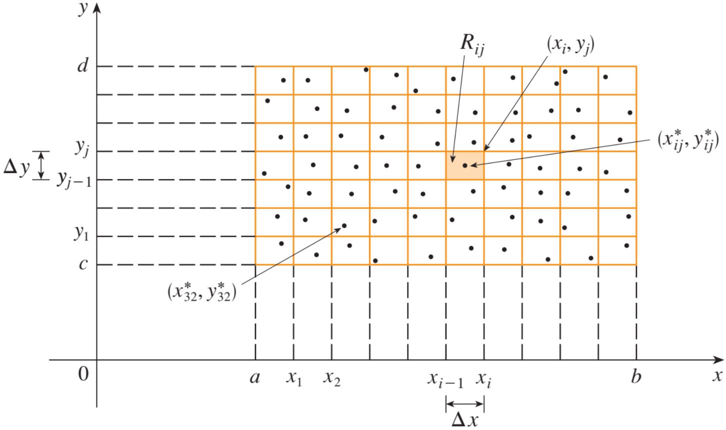
- Like integrals in one variable, double integrals are defined through a three-step process: subdivision, summation, and passage to the limit
- Recall the definite integrals of functions of a single variable:



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Partition for function of two variables

- Suppose $P_1 = \{x_0, x_1, \ldots, x_n; x_1^*, \ldots x_n^*\}$, and $P_2 = \{y_0, y_1, \ldots, y_m; y_1^*, \ldots y_m^*\}$ are partitions of $[a, b]$ and $[c, d]$. Then $P = P_1 \times P_2$ is called a *partition* of $R = [a, b] \times [c, d]$

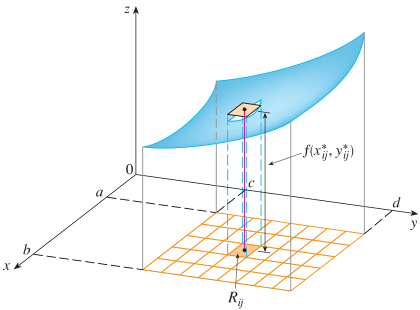


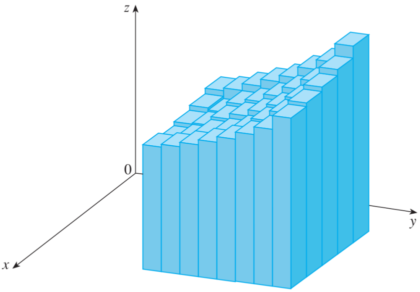
Riemann sum

Riemann sum of $f(x, y)$ corresponding to the partition P is

$$S(f, P) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

where $\Delta A = \Delta x \Delta y$, $\Delta x = x_i - x_{i-1}$, and $\Delta y = y_j - y_{j-1}$.





Definition of the double integral

Let $\mathcal{P}(R)$ be the set of all partition of $R = [a, b] \times [c, d]$. For $P \in \mathcal{P}$, let:

$$|P| = \max\{(x_i - x_{i-1})(y_j - y_{j-1}) : 1 \leq i \leq n, 1 \leq j \leq m\}$$

Definition

We say that $f(x, y)$ is Riemann integrable over a rectangle R if there exists $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ satisfying:

$$|S(f, P) - \alpha| \leq \varepsilon, \ \ \forall P \in \mathcal{P}(R), \ \ |P| < \delta$$

The value α is called the double integral of $f(x, y)$ over R:

$$\iint_R f(x, y)\, dA = \alpha$$

In other words,

$$\iint_R f(x, y)\, dA = \lim_{|P| \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

Midpoint Rule for Double Integrals

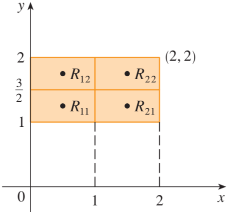
$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example

Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)dA$, where $R = [0, 2] \times [0, 2]$.

Answer: $f(\frac{1}{2}, \frac{5}{4})\Delta A + f(\frac{1}{2}, \frac{7}{4})\Delta A + f(\frac{3}{2}, \frac{5}{4})\Delta A + f(\frac{3}{2}, \frac{7}{4})\Delta A = -\frac{95}{8}$.



Properties

1. $$\iint_R [f(x,y)+g(x,y)]\,\mathrm{d}x\mathrm{d}y = \iint_R f(x,y)\mathrm{d}x\mathrm{d}y + \iint_R g(x,y)\mathrm{d}x\mathrm{d}y$$
2. $$\iint_R cf(x,y)\mathrm{d}x\mathrm{d}y = c\iint_R f(x,y)\mathrm{d}x\mathrm{d}y$$
3. If $f(x,y) \leq g(x,y)$ for all $(x,y) \in R$ then:
$$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y \leq \iint_R g(x,y)\mathrm{d}x\mathrm{d}y$$

Iterated integrals

Suppose that is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. We use the notation $\int_a^b f(x, y)\, dx$ to mean that y is held fixed and $f(x, y)$ is integrated with respect to x from $x = a$ to $x = b$.

Example

$$\int_0^1 \left(x^2 + xy + x^2y^2\right) dx = \frac{x^3}{3} + \frac{x^2y}{2} + \frac{x^3y^2}{3}\Bigg|_{x=0}^{x=1} = \frac{1}{3} + \frac{y}{2} + \frac{y^2}{3}$$

In general,

$$\int_a^b f(x, y) dx = g(y), \quad \int_c^d f(x, y) dy = h(x)$$

The results are functions of one variable. Therefore, they can also be integrated!

Iterated integrals

Recall the previous example,

$$\int_0^1 \left(x^2 + xy + x^2y^2\right) dx = \frac{x^3}{3} + \frac{x^2y}{2} + \frac{x^3y^2}{3}\bigg|_{x=0}^{x=1} = \frac{1}{3} + \frac{y}{2} + \frac{y^2}{3}$$

Example

Evaluate the integral

$$I_1 = \int_0^1 \left[\int_0^1 \left(x^2 + xy + x^2y^2\right) dx\right] dy$$

$$I_1 = \int_0^1 \left[\frac{1}{3} + \frac{y}{2} + \frac{y^2}{3}\right] dy = \frac{y}{3} + \frac{y^2}{4} + \frac{y^3}{9}\bigg|_0^1 = \frac{25}{36}$$

Iterated integrals

Example

Evaluate the integral

$$I_2 = \int_0^1 \left[\int_0^1 \left(x^2 + xy + x^2y^2 \right) dy \right] dx$$

We first integrate w.r.t. y

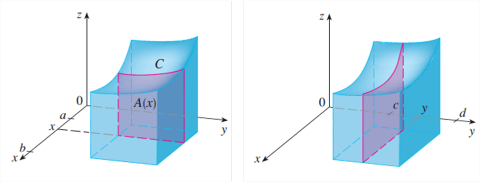
$$\int_0^1 \left(x^2 + xy + x^2y^2 \right) dy = x^2y + \frac{xy^2}{2} + \frac{x^2y^3}{3} \Big|_{y=0}^{y=1} = x^2 + \frac{x}{2} + \frac{x^2}{3}$$

Thus

$$I_2 = \int_0^1 \left[x^2 + \frac{x}{2} + \frac{x^2}{3} \right] dx = \frac{x^3}{3} + \frac{x^2}{4} + \frac{x^3}{9} \Big|_0^1 = \frac{25}{36} = I_1$$

Iterated integrals

Suppose f is continuous and positive on $R = [a,b] \times [c,d]$. The volume V of the solid that lies above R and under the surface $z = f(x,y)$ can be calculated by two ways.



$$V = \int_a^b A(x)\,dx = \int_a^b dx \int_c^d f(x,y)\,dy$$

and

$$V = \int_a^b C(y)\,dy = \int_c^d dy \int_a^b f(x,y)\,dx$$

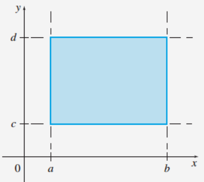
Iterated integrals

Funibi's Theorem

If f is continuous on the rectangle $R = [a, b] \times [c, d]$

$$\iint\limits_R f(x, y)\, dA = \int\limits_c^d \left(\int\limits_a^b f(x, y) dx \right) dy = \int\limits_a^b \left(\int\limits_c^d f(x, y) dy \right) dx$$

Either of these integrals is called an iterated integral since it is evaluated by integrating twice, first using one variable and then using the other.



Rectangle R is given by $a \le x \le b$, $c \le y \le d$.

Double integrals over a rectangular region

Example

Find $\iint\limits_R x^2y\,dxdy$ over a rectangular region R is defined by $0 \leqslant x \leqslant 3, 1 \leqslant y \leqslant 2$.

Solution

One can integrate first with respect to y, then with respect to x.

$$\int_0^3 \int_1^2 x^2 ydydx = \int_0^3 \left[x^2 \left. \frac{y^2}{2} \right|_{y=1}^{y=2} \right] dx$$

$$= \int_0^3 \frac{3}{2}x^2 dx = \left. \frac{1}{2}x^3 \right|_0^3 = \frac{27}{2}$$

Or, integrate first w.r.t. x then w.r.t y

$$\int_1^2 \int_0^3 x^2 ydxdy = \int_1^2 \left[y \left. \frac{x^3}{3} \right|_{x=0}^{x=3} \right] dy = \int_1^2 9ydy = \frac{27}{2}$$

Double integrals over a rectangular region

Example

Find $\iint\limits_R 6xy^2 + 12x^2y + 4y\, dxdy$ over a rectangular region R is defined by $3 \leqslant x \leqslant 5, 1 \leqslant y \leqslant 2$

Answer: 712

Double integrals over a rectangular region

Example

Find $\iint\limits_R \frac{3\sqrt{x}y}{y^2+1}\,dxdy$ over a rectangular region R is defined by $0 \leqslant x \leqslant 4, 0 \leqslant y \leqslant 2$

Answer: $8\ln 5$

Iterated integrals

A special case

If $f(x,y) = g(x)h(y)$ and $R = [a,b] \times [c,d]$, then

$$\iint\limits_R g(x)\,h(y)\,dA = \left(\int\limits_a^b g(x)\,dx\right) \times \left(\int\limits_c^d h(y)\,dy\right)$$

Example

Evaluate

$$\int\limits_0^2 \int\limits_0^{\pi/2} e^x \cos y \; dydx$$

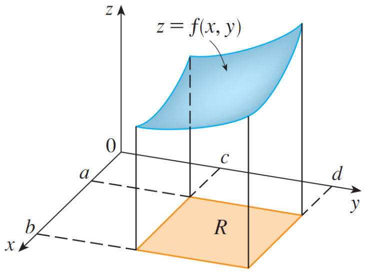
Answer: $e^2 - 1$.

Volume

Definition

Let be a function that is non-negative on the rectangular region R defined by $a \leqslant x \leqslant b, c \leqslant y \leqslant d$. The volume of the solid under the graph of $f(x, y)$ and over the region R is

$$V = \iint\limits_R f(x, y)\, dA$$



Volume

Example

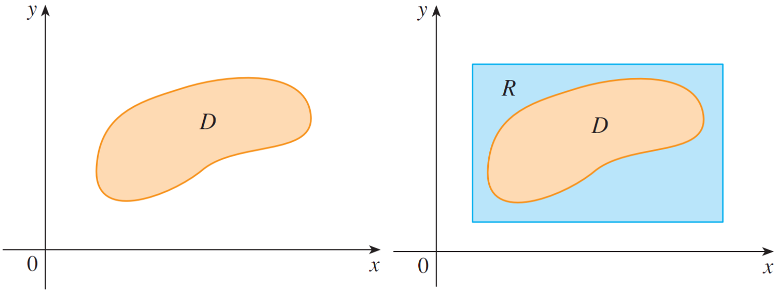
Find the volume under the surface $z = f(x, y) = x^2 + y^2$ over the region $0 \leqslant x \leqslant 4, 0 \leqslant y \leqslant 4$.

Answer:

$$V = \iint_R x^2 + y^2 dxdy = \frac{512}{3}$$

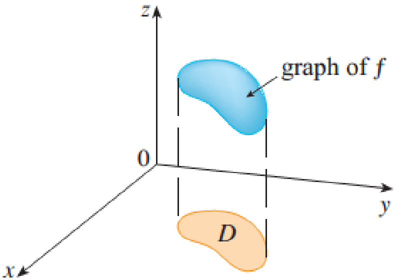
Double integrals over other regions

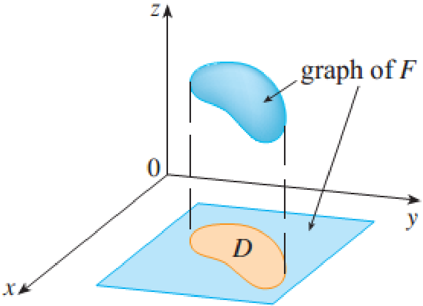
We suppose that D is *a bounded region*, which means that can be enclosed in a rectangular R.



Then we define a new function with domain R by

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \setminus D \end{cases}$$





Definition

If F is integrable over R, then we define the double integral of over D by

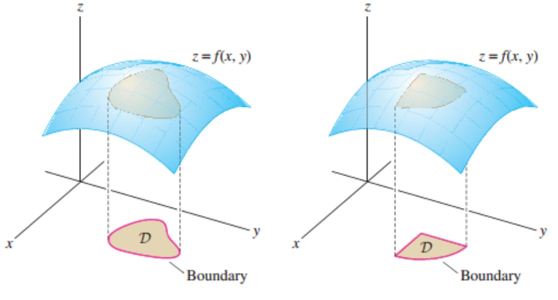
$$\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_R F(x,y)\mathrm{d}x\mathrm{d}y$$

Note: F is discontinuous because its values jump suddenly to zero beyond the boundary. When does $\iint\limits_R F(x,y)\mathrm{d}x\mathrm{d}y$ exist?

Double integrals over other regions

1. A curve is simple if it does not intersect itself.

2. We assume that D is closed, and that the boundary is a simple closed curve which is either smooth or has at most infinitely many points of nondifferentiability. A boundary curve of this type is called piecewise smooth.



(A) $\mathcal{D}$ has a smooth boundary.

(B) $\mathcal{D}$ has a piecewise smooth boundary, consisting of three smooth curves joined at the corners.

Double integrals over other regions

The following theorem guarantees that the integral of F over R exists if our original function f is continuous.

Theorem

If $f(x, y)$ is continuous on a closed domain D whose boundary is a closed, simple, piecewise smooth curve, then $\iint\limits_D f(x, y)\mathrm{d}x\mathrm{d}y$ does exist.

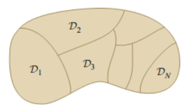
Properties of Double Integrals

1. $$\iint_D [f(x,y)+g(x,y)]\,\mathrm{d}x\mathrm{d}y$$
   $$= \iint_D f(x,y)\mathrm{d}x\mathrm{d}y + \iint_D g(x,y)\mathrm{d}x\mathrm{d}y$$
2. $$\iint_D cf(x,y)\mathrm{d}x\mathrm{d}y = c\iint_D f(x,y)\mathrm{d}x\mathrm{d}y$$
3. If $f(x,y) \leq g(x,y)$ for all $(x,y) \in D$, then:
   $$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y \leq \iint_D g(x,y)\mathrm{d}x\mathrm{d}y$$

Properties of Double Integrals

4. If D is the union of domains D_1, D_2,...,D_N that do not overlap except possibly on boundary curves, then

$$\iint\limits_{D} f(x,y)\,dA = \iint\limits_{D_1} f(x,y)\,dA + \ldots + \iint\limits_{D_N} f(x,y)\,dA$$



5. The area of D is:

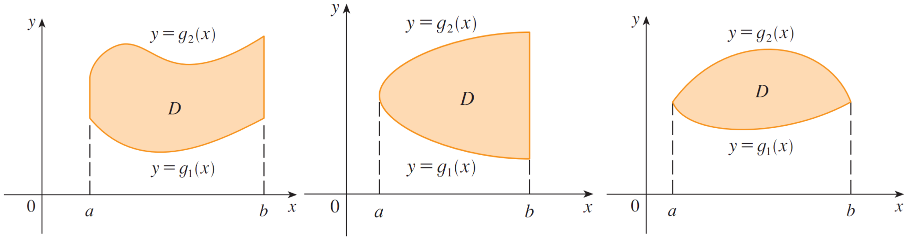
$$A(D) = \iint_D \mathrm{d}x\mathrm{d}y$$

6. The signed volume of the solid region between the graph of $f(x,y)$ and a domain D in the xy-plane is

$$V = \iint_D f(x,y)\mathrm{d}x\mathrm{d}y$$

Double integrals over type I region

$$D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$



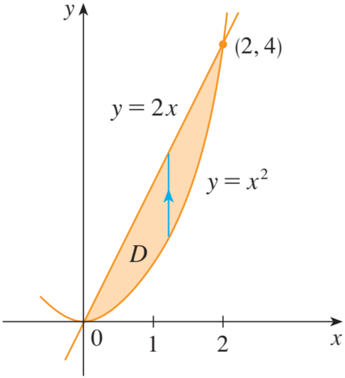
$$\iint_D f(x, y)dxdy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

Integrate first with respect to y, then with respect to x.

Double integrals over type I region

Example

Evaluate $I = \iint_D (x + 2y)dxdy$ where D is the region enclosed by $y = 2x$ and $y = x^2$.



Double integrals over type I region

Solution

$$D = \left\{(x, y) : 0 \leqslant x \leqslant 2, x^2 \leqslant y \leqslant 2x\right\}$$

Integrate first with respect to y, then with respect to x.

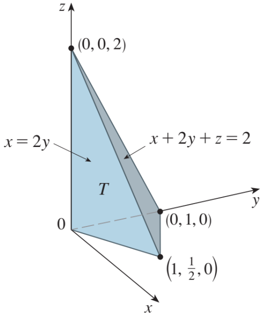
$$I = \iint\limits_R (x + 2y)\, dxdy = \int\limits_0^2 \int\limits_{x^2}^{2x} (x + 2y)\, dydx = \int\limits_0^2 \left(xy + y^2 \Big|_{y=x^2}^{y=2x} \right) dx$$

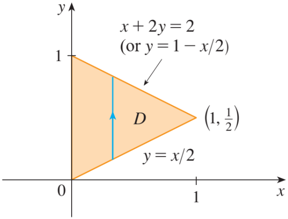
$$I = \int\limits_0^2 \left(6x^2 - x^3 - x^4\right) dx = \frac{28}{5}$$

Double integrals over type I region

example

Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, $z = 0$.



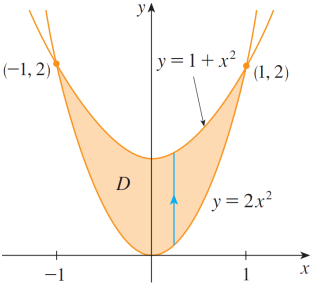


$$V = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dydx = \int_0^1 (x^2 - 2x + 1)dx = \frac{1}{3}$$

Double integrals over type I region

Example

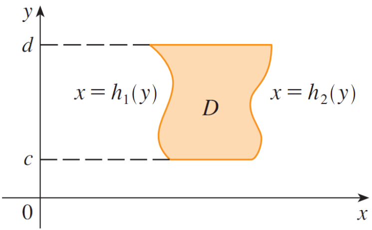
Evaluate $I = \iint_D (x + 2y)dxdy$ where D is the region enclosed by $y = 2x^2$ and $y = 1 + x^2$.

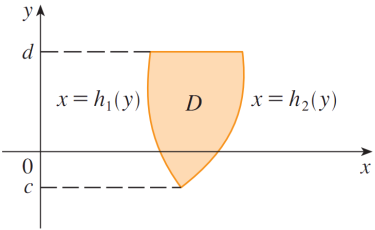


$$V = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dydx = \frac{32}{15}$$

Double integrals over type II region

$$D = \{(x, y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$$



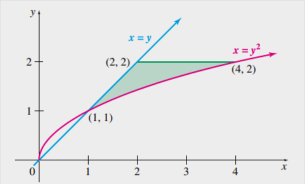


$$\iint_D f(x, y)dxdy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$$

Double integrals over type II region

Example

Evaluate $\iint_D xy\,dxdy$ where D is the domain enclosed by $y = x, y = 2$, and $x = y^2$.



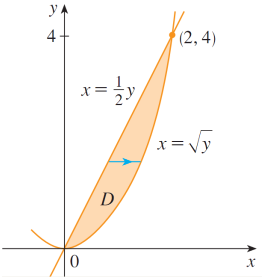
$$D = \left\{(x, y) : 1 \leq y \leq 2, y \leq x \leq y^2\right\}$$

$$I = \int_1^2 \int_y^{y^2} xydxdy = \int_1^2 \left(\int_y^{y^2} xydx \right) dy = \int_1^2 \left(\frac{y^5}{2} - \frac{y^3}{2} \right) dy = \frac{27}{8}$$

Interchanging Limits of Integration

- Sometimes it is easier to integrate first with respect to x, and then y, while with other integrals the reverse process is easier.
- The limits of integration can be reversed whenever the region D can be re-expressed in the above two types.
- Back to the example in slide 29, we integrated first with respect to y, then with respect to x and found $I = 28/5$. We now interchange limits of integration: Integrate first with respect to x, then with respect to y.

Solve the example in slide 29 using type II



Re-write $D = \left\{(x, y) : 0 \leqslant y \leqslant 4, y/2 \leqslant x \leqslant \sqrt{y}\right\}$.

$$I = \iint\limits_R (x + 2y)\, dxdy = \int\limits_0^4 \int\limits_{y/2}^{\sqrt{y}} (x + 2y)\, dxdy$$

$$I = \int\limits_0^4 \left(\frac{\left(\sqrt{y}\right)^2}{2} - \frac{\left(\frac{y}{2}\right)^2}{8} + 2y\left(\sqrt{y} - \frac{y}{2}\right) \right) = \frac{28}{5}$$

Interchanging Limits of Integration

Example

Evaluate

$$\int_0^{16}\int_{\sqrt{y}}^{4}\sqrt{x^3+4}\,dxdy$$

Comments: It is very difficult to integrate first with respect to x, because we can NOT evaluate

$$\int \sqrt{x^3+4}\,dx$$

We therefore interchange limits of integration by integrating first with respect to y!

What should we do?

Interchanging Limits of Integration

Solution

The region is given by

$$D = \{(x, y) : \sqrt{y} \leqslant x \leqslant 4, 0 \leqslant y \leqslant 16\}$$

We re-write D as

$$D = \left\{(x, y) : 0 \leqslant y \leqslant x^2, 0 \leqslant x \leqslant 4\right\}$$

$$I = \int_0^4 \int_0^{x^2} \sqrt{x^3 + 4} dydx = \int_0^4 y\sqrt{x^3 + 4}\Big|_0^{x^2} dx$$

$$= \int_0^4 x^2\sqrt{x^3 + 4} dx = \frac{2}{9}(x^3 + 4)^{3/2}\Big|_0^4 = 122.83$$

Interchanging Limits of Integration

Exercise

Evaluate

$$I = \int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3} dy$$

Hint:

Re-write D as

$$D = \left\{(x, y) : 0 \leqslant y \leqslant 1, 0 \leqslant x \leqslant y^2\right\}$$

$$I = \frac{e-1}{3}.$$

Average value

Definition

The average value of the function $z = f(x, y)$ over a region D is defined as

$$\overline{f} = \frac{1}{A} \iint_R f(x, y) dxdy$$

where A is the area of the region D.

Example

Find the average value for the function $f(x, y)$ over the given region D

$$f(x, y) = 3x^2 + 6y^2,$$

where D: $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2$.

Solution

We have A $= 1 \times 2 = 2$.

$$\iint\limits_D (3x^2 + 6y^2)dxdy = \int\limits_0^1 \int\limits_0^2 (3x^2 + 6y^2)dydx = \int\limits_0^1 (3x^2 y + 2y^3)\Big|_0^2 dx =$$

$$= \int\limits_0^1 (6x^2 + 16)dx = (2x^3 + 16x)\Big|_0^1 = 2 + 16 = 18$$

Thus, the average value of $f(x, y)$ is $\overline{f} = \frac{1}{A}(18) = 9$.

Average value

Example: Production function

A production function is given by

$$P\left(x,y\right)=500x^{0.2}y^{0.8}$$

where x is the number of units of labor and y is the number of units of capital. Find the average production level if x varies from 10 to 50 and y from 20 to 40.

Connected domain

Definition

A set D in the plane is said to be connected if any two points in D can be joined by a continuous parametric curve

$$x = x(t), \; y = y(t), \quad a \leqslant t \leqslant b$$

lying entirely in D.

(A) Connected domain: Any two points can be joined by a curve lying entirely in $\mathcal{D}$.

(B) Nonconnected domain.

Mean Value Theorem for Double Integrals

Theorem

If $f(x, y)$ is continuous and D is closed, bounded, and connected, then there exists a point $P \in D$ such that

$$\frac{1}{Area\,(D)} \iint_D f\,(x, y)\,dA = f\,(P)$$

Equivalently, $f(P) = \bar{f}$, where $\bar{f}$ is the average value of f on D.

Example

Let $\bar{f}$ be the average of $f(x, y) = xy^2$ on $D = [0, 1] \times [0, 4]$. Find a point $P \in D$ such that $f(P) = \bar{f}$ (the existence of such a point is guaranteed by the Mean Value Theorem for Double Integrals).

Double Integrals in Polar Coordinates

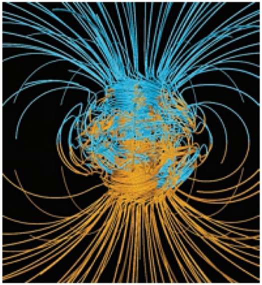
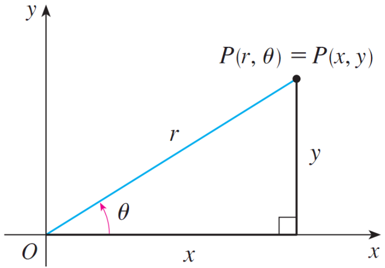


FIGURE 1 Spherical coordinates are used in mathematical models of the earth's magnetic field. This computer simulation, based on the Glatzmaier–Roberts model, shows the magnetic lines of force, representing inward and outward directed field lines in blue and yellow, respectively.

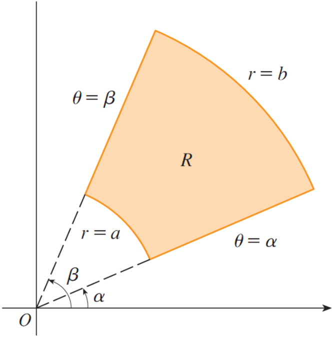
Like single-variable functions, change of variables is also useful in multivariable functions, but the emphasis is different. In the multivariable case, we are usually interested in simplifying not just the integrand, but also the domain of integration.

Double Integrals in Polar Coordinates



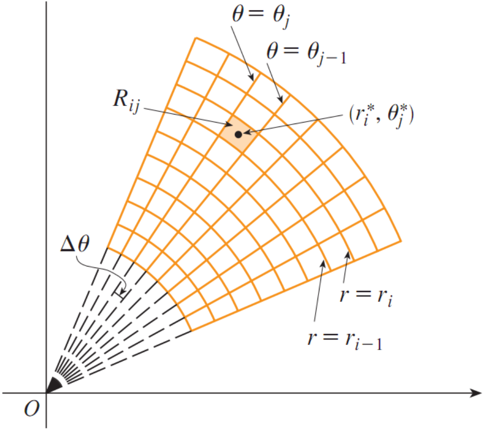
$$r = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta, \;\; y = r\sin\theta$$



A polar rectangle is

$R = \{(r, \theta) | \; a \leq r \leq b, \alpha \leq \theta \leq \beta\}$



The area of R_{ij} is:

$$
\begin{aligned}
\Delta A_i &= \tfrac{1}{2} r_i^2 \Delta\theta - \tfrac{1}{2} r_{i-1}^2 \Delta\theta = \tfrac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta \\
&= \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^* \Delta r \Delta\theta,
\end{aligned}
$$

where $r_i^* = (r_{i-1} + r_i)/2$.

Polar Coordinates (1)

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j, r_i^* \sin\theta_j)\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j, r_i^* \sin\theta_j) r_i^* \Delta r \Delta\theta$$

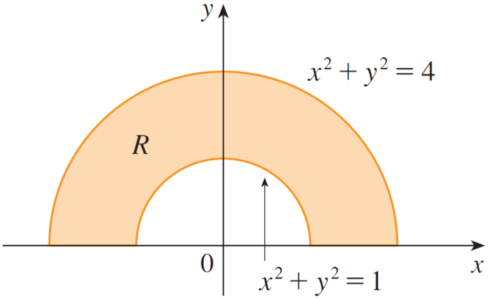
If *f is continuous* on a polar rectangle $R : 0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

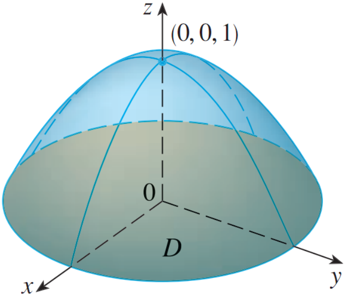
Polar Coordinates (1)

$$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}r\mathrm{d}\theta$$

Examples

1. Evaluate $\iint_R (3x + 4y^2)\mathrm{d}x\mathrm{d}y$, where R s the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
2. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$

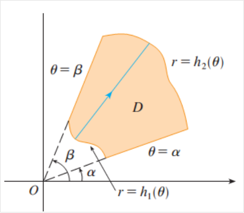




Hint: 1. $I_1 = \int\limits_0^{\pi} d\theta \int\limits_1^{2} \left(3r\cos\theta + 4r^2\sin^2\theta\right) rdr = \frac{15\pi}{2}$

2. $I_2 = \int\limits_0^{2\pi} d\theta \int\limits_0^{1} \left(1 - r^2\right) rdr = \dfrac{\pi}{2}.$

Polar Coordinates (2)



If f is continuous on a polar region of the form:

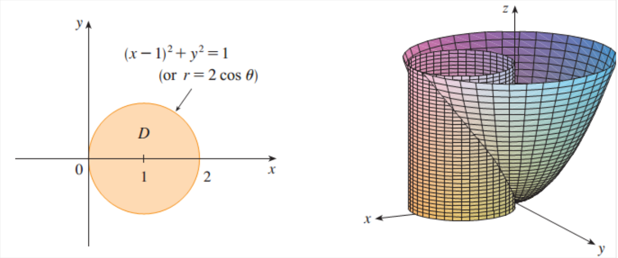
$$D = \left\{(x,y) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\right\}$$

Then:

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\mathrm{d}r\mathrm{d}\theta$$

Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.



$$D = \{(x, y) : -\pi/2 \le \theta \le \pi/2, 0 \le r \le 2\cos\theta\}$$

$$V = \iint_D (x^2 + y^2)\mathrm{d}x\mathrm{d}y = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 r \mathrm{d}r \mathrm{d}\theta = \frac{3\pi}{2}$$

Exercises.

Evaluate

1. $\iint_D (x+y)\mathrm{d}x\mathrm{d}y$, D is the region bounded by $y=\sqrt{x}, y=x^2$.
2. $\iint_D xy\mathrm{d}x\mathrm{d}y$, D is the region bounded by Oy, $x+y=1$ and $x-2y=4$
3. $\iint_D y^3\mathrm{d}x\mathrm{d}y$, D is the triangle defined by $(0,2),(1,1),(3,2)$.
4. $\iint_D \sqrt{4-x^2-y^2}\mathrm{d}x\mathrm{d}y$, $D: x^2+y^2 \le 4, y \ge x$
5. $\int_0^1 \int_x^1 e^{x/y}\mathrm{d}y\mathrm{d}x$
6. (a) Show that $\int\limits_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

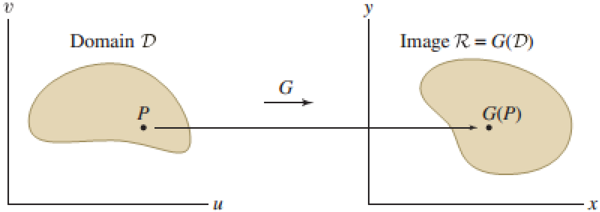
   (b) Show that $\dfrac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-x^2/2}dx = 1$.

   Note for Q.6: This is a fundamental result for probability and statistics (for normal distributions).

Change of Variables

We consider maps $G : D \to \mathbb{R}^2$ defined on a domain D in $\mathbb{R}^2$. We will often use u, v as our domain variables and x, y for the range. Thus, we will write $G(u, v) = (x(u, v), y(u, v))$, where the components x and y are functions of u and v:

$$x = x(u, v), y = y(u, v).$$

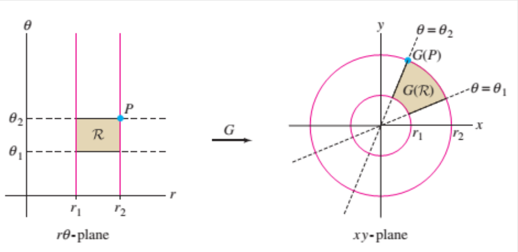


The polar coordinates map $G : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $G(r, \theta) = (r\cos\theta, r\sin\theta)$.

Change of Variables

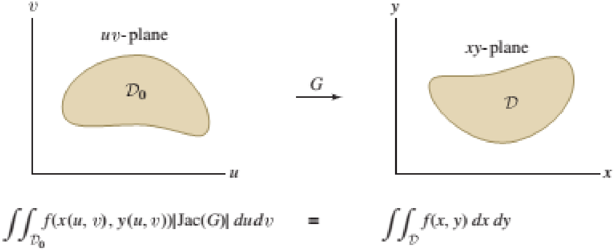
Example: Polar Coordinates Map

Describe the image of a polar rectangle $R = [r_1, r_2] \times [\theta_1, \theta_2]$ under the polar coordinates map.



The image of $R = [r_1, r_2] \times [\theta_1, \theta_2]$ under the polar coordinates map $G(r, \theta) = (r\cos\theta, r\sin\theta)$ is the polar rectangle in the xy-plane defined by $r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2$.

Change of Variables Formula



Theorem: Change of Variables Formula

Let $G : D_0 \to D$ be a C^1 mapping that is one-to-one on the interior of D_0. If $f(x, y)$ is continuous, then

$$\iint_D f(x, y)\, dxdy = \iint_{D_0} f(x(u, v), y(u, v)) \left| \frac{\partial (x, y)}{\partial (u, v)} \right| dudv$$

Change of Variables Formula

Example: Polar Coordinates Revisited

Use the Change of Variables Formula to derive the formula for integration in polar coordinates.

The Jacobian of the polar coordinate map $G(r, \theta) = (r\cos\theta, r\sin\theta)$ is

$$Jac\,(G) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Let $D = G(D_0)$ be the image under the polar coordinates map G, where

$$D_0 = \{(r, \theta) : r_0 \le r \le r_1, \theta_0 \le \theta \le \theta_1\}$$

Then the change of variables formula gives

$$\iint_D f\,(x, y)\,dxdy = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} f\,(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Change of Variables Formula

Example

Use an appropriate change of variables to find the area of the elliptic disk E given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1$$

Under the transformation $x = au, y = bv$, the elliptic disk E is the one-to-one image of the circular disk D given by $u^2 + v^2 \leqslant 1$. Assuming $a > 0$ and $b > 0$, we have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

Using the change of variables formula:

$$\iint_D 1 dxdy = \int_0^{2\pi} \int_0^1 abr dr d\theta = \pi ab$$

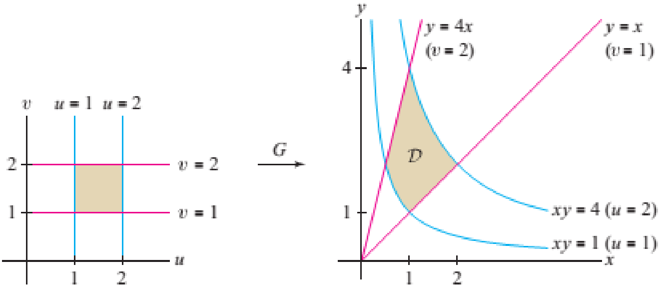
Change of Variables Formula

Example

Use the Change of Variables Formula with $x = uv^{-1}, y = uv$ to compute

$$I = \iint\limits_D \left(x^2 + y^2\right) dxdy$$

where D is the domain $1 \leq xy \leq 4, 1 \leq y/x \leq 4$.



Change of Variables Formula

Hint:

$$D_0 : 1 \leqslant u \leqslant 2, 1 \leqslant v \leqslant 2$$

$$Jac\left(G\right) = \frac{\partial\left(x,y\right)}{\partial\left(u,v\right)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = \frac{2u}{v}$$

$$I = \iint\limits_{D_0} u^2\left(v^{-2} + v^2\right)\left|\frac{2u}{v}\right| dudv = 2\int\limits_1^2 u^3 du \int\limits_1^2 \left(v^{-3} + v\right) dv = 225/16.$$

Additional reading: The general change of variables theorem

Let f be a continuous function of compact support. Then

$$\int_K f(\varphi(x))\, J(x)\, dx = \int_{\varphi(K)} f(y)\, dy$$

where J is the Jacobian determinant of the mapping

$$J(x) = \det \frac{\partial \varphi_j}{\partial x_i}$$

here φ_j is the j^{th} component of φ.
Reference: Peter D. Lax, *Change of Variables in Multiple Integrals*, The American Mathematical Monthly, Vol. 106, No. 6 (Jun. - Jul., 1999), pp. 497-501.
For detail proof, click on the following link to the paper by P. Lax

Triple Integrals

Fubini's Theorem for Triple Integrals

The triple integral of a continuous function $f(x, y, z)$ over a box $B = [a, b] \times [c, d] \times [p, q]$ is equal to the iterated integral:

$$\iiint\limits_B f(x, y, z)\, dV = \int\limits_{x=a}^{b} \int\limits_{y=c}^{d} \int\limits_{z=p}^{q} f(x, y, z)\, dzdydx$$

Or,

$$\iiint\limits_B f(x, y, z)\, dV = \int\limits_a^b dx \int\limits_c^d dy \int\limits_p^q f(x, y, z)\, dz$$

Remark: There are five other possible orders in which we can integrate, all of which give the same value.

Triple Integrals

Example

Calculate the integral

$$\iiint\limits_B x^2 e^{y+3z} dV$$

where $B = [1,4] \times [0,3] \times [2,6]$

Solution

$$I := \iiint\limits_B x^2 e^{y+3z} dV = \int_1^4 \int_0^3 \int_2^6 x^2 e^{y+3z} dz dy dx$$

First, evaluate the inner integral with respect to z:

$$\int_2^6 x^2 e^{y+3z} dz = \frac{1}{3} x^2 e^{y+3z} \Big|_2^6 = \frac{1}{3} \left(e^{18} - e^{6}\right) x^2 e^{y}$$

Triple Integrals

Second, evaluate the middle integral with respect to y:

$$\int_{y=0}^{3} \frac{1}{3}\left(e^{18}-e^{6}\right)x^2 e^y dy = \frac{1}{3}\left(e^{18}-e^{6}\right)\left(e^{3}-1\right)x^2$$

Finally, evaluate the outer integral with respect to x:

$$I = \int_{x=1}^{4} \frac{1}{3}\left(e^{18}-e^{6}\right)\left(e^{3}-1\right)x^2 dx = 7\left(e^{18}-e^{6}\right)\left(e^{3}-1\right)$$

Triple Integrals

Example

Calculate the integral

$$I = \iiint\limits_B xy^2 + z^3 dV$$

where $B = [0, a] \times [0, b] \times [0, c]$

Hint:

$$I = \int_0^c dz \int_0^b dy \int_0^a \left(xy^2 + z^3\right) dx = \int_0^c dz \int_0^b \left(\frac{a^2y^2}{2} + az^3\right) dy$$

$$I = \int_0^c \left(\frac{a^2b^3}{6} + abz^3\right) dz = \frac{a^2b^3c}{6} + \frac{abc^4}{4}$$

Triple Integrals

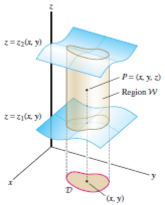
Theorem

The triple integral of a continuous function $f(x,y,z)$ over the region

$$W = \{(x,y,z) : (x,y) \in D \text{ and } z_1(x,y) \leqslant z \leqslant z_2(x,y)\}$$

is equal to the iterated integral

$$\iiint\limits_W f(x,y,z)\,dV = \iint\limits_D \left(\int\limits_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right) dA$$



Triple Integrals

Furthermore, the volume V of a region W is defined as the triple integral of the constant function $f(x, y, z) = 1$:

$$V = \iiint_W 1 dV$$

Example

Evaluate

$$I = \iiint_W z dV$$

where W is the region between the planes $z = x + y$ and $z = 3x + 5y$ lying W over the rectangle $D = [0, 3] \times [0, 2]$ in the xy-plane.

Solution

Apply previous theorem with $z_1(x,y) = x + y$ and $z_2(x,y) = 3x + 5y$:

$$\iiint\limits_W z dV = \iint\limits_D \left(\int\limits_{z=x+y}^{3x+5y} z dz \right) dA = \iint\limits_D \left(\frac{1}{2}(3x+5y)^2 - \frac{1}{2}(x+y)^2 \right) dA$$

$$= \int\limits_0^3 dx \int\limits_0^2 \left(4x^2 + 14xy + 12y^2\right) dy = \int\limits_0^3 \left(\frac{8}{3}x^2 + 28x + 32 \right) dx = 294$$

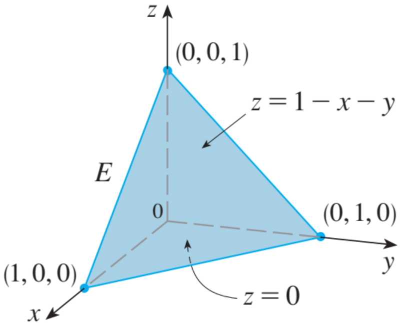
Triple Integrals

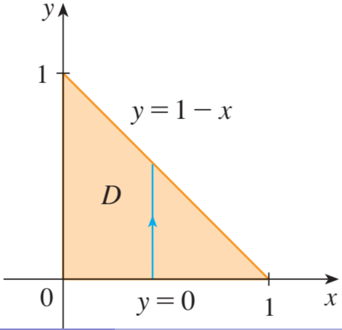
Example

Evaluate

$$I = \iiint\limits_T y dV$$

where T is the tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$, and $(0,0,1)$.





Solution

Apply previous theorem with $z_1(x,y) = 0$ and $z_2(x,y) = 1 - x - y$:

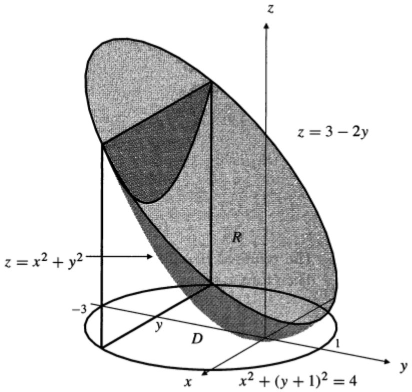
$$I = \iiint\limits_T z dV = \iint\limits_D \left(\int\limits_{z=0}^{1-x-y} y dz \right) dA = \iint\limits_D \left(y \left(1 - x - y \right) \right) dA$$

$$I = \int\limits_0^1 dx \int\limits_0^{1-x} y \left(1 - x - y \right) dy = \int\limits_0^3 \frac{1}{6} (1-x)^3 dx = \frac{1}{24}$$

Triple Integrals

Exercise

Find the volume of the region R lying below the plane $z = 3 - 2y$ and above the paraboloid $z = x^2 + y^2$.



Triple Integrals

Hint:

$$V = \iiint\limits_{W} 1dV = \iint\limits_{D} \left(\int\limits_{z=x^2+y^2}^{3-2y} 1dz \right) dA$$

$$V = \iint\limits_{D} \left(3 - 2y - x^2 - y^2\right) dA, D : x^2 + (y+1)^2 \leq 4$$

$$V = \int\limits_{0}^{2\pi} d\theta \int\limits_{0}^{2} \left(4 - r^2\right) rdr = 8\pi$$

Change of Variables in Triple Integrals

Change of Variables: Double integral extends to triple integrals

Consider the transformation:

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w),$$

where x, y, and z have continuous first partial derivatives with respect to u, v, and w. Then:

$$dV = dxdydz = \left|\frac{\partial\left(x, y, z\right)}{\partial\left(u, v, w\right)}\right| dudvdw$$

Or,

$$\iiint\limits_{D} f\left(x, y, z\right) dV =$$

$$\iiint\limits_{S} f\left(x\left(u, v, w\right), y\left(u, v, w\right), z\left(u, v, w\right)\right) \left|\frac{\partial\left(x, y, z\right)}{\partial\left(u, v, w\right)}\right| dudvdw$$

Change of Variables in Triple Integrals

Example

Evaluate the volume of the solid ellipsoid E given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1$$

Solution:

Under the change of variables $x=au, y=bv, z=cw$, where $a, b, c>0$, the solid ellipsoid E becomes the ball B given by $u^2+v^2+w^2\leq 1$. The Jacobian of this transformation is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}=abc$$

So the volume of the ellipsoid is

$$V=\iiint\limits_E 1dxdydz=abc\iiint\limits_B dudvdw=abcV_B=\frac{4}{3}\pi abc$$

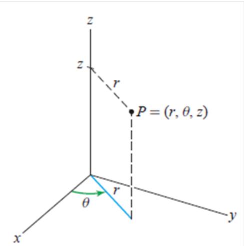
Cylindrical coordinates

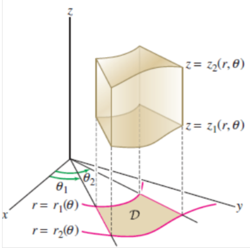
Cylindrical coordinates

Cylindrical coordinates are useful when the domain has axial symmetry, that is, symmetry with respect to an axis. In cylindrical coordinates (r, θ, z), the axis of symmetry is the z-axis:

$$x = r\cos\theta, y = r\sin\theta, z = z$$

.





Cylindrical coordinates

Note that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

so $dV = rdrd\theta dz$, we thus have following theorem

Theorem

For a continuous function f on the region

$$\theta_1 \le \theta \le \theta_2, r_1(\theta) \le r \le r_2(\theta), z_1(r,\theta) \le z \le z_2(r,\theta)$$

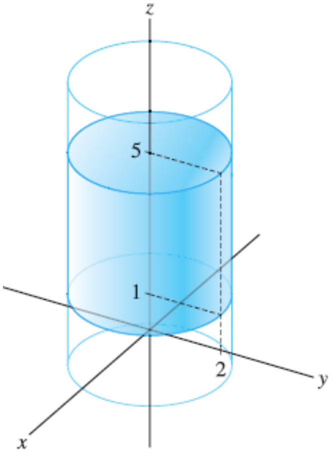
the triple integral $\iiint\limits_W f(x,y,z)dV$ is equal to

$$\int\limits_{\theta_1}^{\theta_2} d\theta \int\limits_{r=r_1(\theta)}^{r_2(\theta)} dr \int\limits_{z=z_1(r,\theta)}^{z_2(r,\theta)} f\left(r\cos\theta, r\sin\theta, z\right) rdz$$

Cylindrical coordinates

Example

Integrate $f(x, y, z) = z\sqrt{x^2 + y^2}$ over the cylinder $x^2 + y^2 \leq 4$ for $1 \leq z \leq 5$.



Cylindrical coordinates

Outline solution:

$$W : 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r \leqslant 2, 1 \leqslant z \leqslant 5.$$

$$I = \iiint_W z\sqrt{x^2+y^2}dV = \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{2} dr \int_{z=1}^{5} (zr)\, rdz$$

$$I = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{2} r^2 dr\right)\left(\int_1^{5} zdz\right) = 64\pi$$

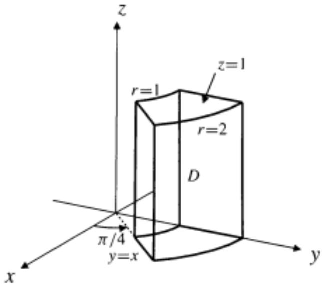
Cylindrical coordinates

Example

Evaluate

$$\iiint_D \left(x^2 + y^2\right) dV$$

over the first octant region bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and the planes $z = 0, z = 1, x = 0$, and $x = y$.



Cylindrical coordinates

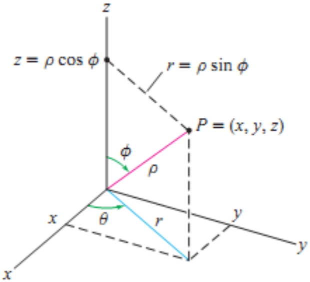
Solution:

In terms of cylindrical coordinates the region is bounded by $r = 1$, $r = 2$, $\theta = \pi/4$, $0 = \pi/2, z = 0$, and $z = 1$.

Since the integrand is $x^2 + y^2 = r^2$, the integral is

$$\iiint_D \left(x^2 + y^2\right) dV = \int\limits_{\theta=\pi/4}^{\pi/2} d\theta \int\limits_{r=1}^{2} dr \int\limits_{z=0}^{1} r^2 r dz = \frac{15}{16}\pi$$

Triple Integrals in Spherical Coordinates



$$x = \rho\cos\theta\sin\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\phi$$

In spherical coordinates, we have the analog for changing of variables:

$$dV = \rho^2 \sin\phi d\rho d\phi d\theta$$

Triple Integrals in Spherical Coordinates

Theorem

For a region W defined by

$$\theta_1 \leq \theta \leq \theta_2, \phi_1(\theta) \leq \phi \leq \phi_2(\theta), \rho_1(\theta, \phi) \leq \rho \leq \rho_2(\theta, \phi)$$

the triple integral $\iiint\limits_W f(x, y, z)dV$ is equal to

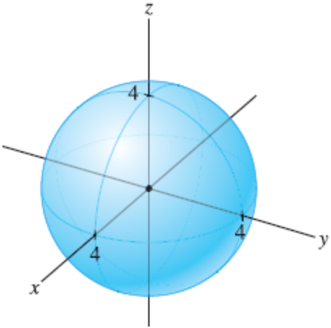
$$\int\limits_{\theta_1}^{\theta_2} d\theta \int\limits_{\phi=\phi_1}^{\phi_2} d\phi \int\limits_{\rho=\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f\left(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi\right)\rho^2\sin\phi d\rho$$

Note that $\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2, r = \sqrt{x^2 + y^2} = \rho\sin\phi$.

Triple Integrals in Spherical Coordinates

Example

Compute the integral of $f(x, y, z) = x^2 + y^2$ over the sphere S of radius 4 centered at the origin.



Triple Integrals in Spherical Coordinates

Solution:

First, write $f(x, y, z)$ in spherical coordinates:

$$f(x, y, z) = x^2 + y^2 = (\rho \cos\theta \sin\phi)^2 + (\rho \sin\theta \sin\phi)^2 = \rho^2 \sin^2\phi$$

We are integrating over the entire sphere S of radius 4, ρ varies from 0 to 4, θ from 0 to π:

$$I = \iiint\limits_S \left(x^2 + y^2\right) dV = \int\limits_0^{2\pi} d\theta \int\limits_{\phi=0}^{\pi} d\phi \int\limits_{\rho=0}^{4} \left(\rho^2 \sin^2\phi\right) \rho^2 \sin\phi \, d\rho$$

$$I = 2\pi \int\limits_0^{\pi} d\phi \int\limits_0^{4} \rho^4 \sin^3\phi \, d\rho = 2\pi \left(\int\limits_0^{\pi} \sin^3\phi \, d\phi \right) \left(\left. \frac{\rho^5}{5} \right|_0^4 \right)$$

$$I = \frac{8192\pi}{15}$$

Applications in Economics and Engineering

This section discusses some applications of multiple integrals. First, we consider quantities (such as mass, charge, and population) that are distributed with a given density ρ in $\mathbb{R}^2$ or $\mathbb{R}^3$.

$$\text{Total amount} = \iint_D \rho(x, y)\, dA$$

$$\text{Total amount} = \iiint_W \rho(x, y, z)\, dV$$

The density function ρ has units of "amount per unit area" (or per unit volume).

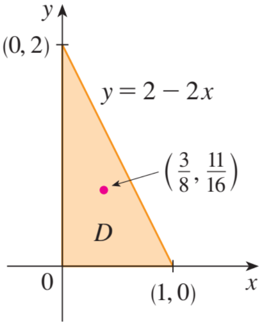
Center of mass in $\mathbb{R}^2$: $(\bar{x}, \bar{y})$, where $\bar{x} = \frac{1}{m} \iint_D x\rho(x, y)\, dA$ and $\bar{y} = \frac{1}{m} \iint_D y\rho(x, y)\, dA$.

Applications in Engineering and Economics

Example

Find the mass and center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, $(0, 2)$, if the density function $\rho(x, y) = 1 + 3x + y$.

Answer: $m = \dfrac{8}{3}$; $(\bar{x}, \bar{y}) = (\dfrac{3}{8}, \dfrac{11}{16})$.



Applications in Engineering and Economics

Example

A solid half-ball H of radius R has density depending on the distance ρ from the centre of the base disk. The density is given by $k(2R-\rho)$, where k is a constant. Find the mass of the half-ball.

Solution:

Choosing coordinates with origin at the centre of the base, and so that the half-ball lies above the xy-plane. The mass of the solid half-ball is

$$m = \underset{H}{\iiint} k\,(2R-\rho)dV = \iiint_H k\,(2R-\rho)\rho^2 \sin\phi d\rho d\phi d\rho$$

$$m = k\int_0^{2\pi} d\theta \int_{\phi=0}^{\pi/2} \sin\phi d\phi \int_{\rho=0}^{R} \rho^2\,(2R-\rho)\,d\rho$$

$$m = 2k\pi \times 1 \times \left(\frac{2R}{3}\rho^3 - \frac{\rho^4}{4}\right)\Bigg|_0^R = \frac{5}{6}\pi k R^4 \ \text{ units}$$

Applications in Engineering and Economics

Example

The population in a rural area near a river has density

$$\rho(x, y) = 40xe^{0.1y} \text{ people per km}^2$$

How many people live in the region $R : 2 \leq x \leq 6, 1 \leq y \leq 3$?

Solution:

The total population is the integral of population density:

$$\iint_R 40xe^{0.1y} dA = \left(\int_2^6 40x dx\right)\left(\int_1^3 e^{0.1y} dy\right) \approx 1566(\text{ people})$$

Applications in Engineering and Economics

Example

Find the total population within a 4-kilometer radius of the city center (located at the origin) assuming a population density of $\rho(x, y) = 2000(x^2 + y^2)^{-0.2}$ people per square kilometer.

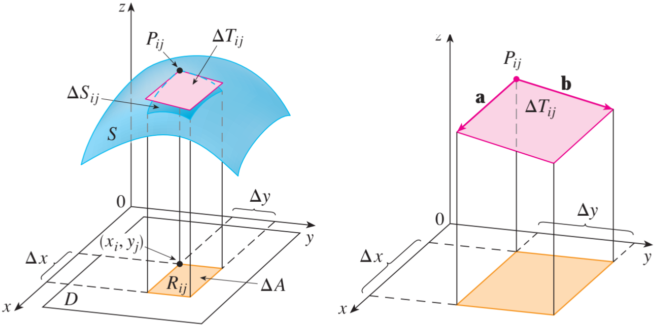
Applications in Engineering and Economics

The Surface Area of a Graph

The total surface area of the surface S with equation $z = f(x, y)$ defined for (x, y) in D is

$$S = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$

[Or, $dS = \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$].



Applications in Engineering and Economics

Example

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution:
We have $f_x = 2x, f_y = 2y$, thus

$$dS = \sqrt{1 + 4x^2 + 4y^2}dA = \sqrt{1 + 4r^2}rdrd\theta$$

The surface area is

$$S = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}dA = \int\limits_0^{2\pi} d\theta \int\limits_0^{3} \sqrt{1 + 4r^2}r\,dr$$

$$S = (2\pi)\,\frac{1}{8}\int\limits_1^{36} \sqrt{u}du = \frac{\pi}{6}(37\sqrt{37} - 1)$$

Applications in Engineering and Economics

Exercise

Find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Hint:

$$S = \int_0^2 dx \int_0^{-\frac{3}{2}x+3} \sqrt{(-3)^2 + (-2)^2 + 1}\, dy = 3\sqrt{14}$$

Applications in Engineering and Economics

Example in Probability

Without proper maintenance, the time to failure (in months) of two sensors in an aircraft are random variables X and Y with joint density

$$\rho(x,y) = \begin{cases} \frac{1}{864} e^{-x/24 - y/36} \text{ for } x \geqslant 0, y \geqslant 0 \\ 0 \text{ otherwise} \end{cases}$$

What is the probability that neither sensor functions after two years?

Hint:

$$P(0 \leqslant X \leqslant 24, 0 \leqslant Y \leqslant 24) = \int_{x=0}^{24} \int_{y=0}^{24} \rho(x,y)\, dy \approx 0.31$$

-END OF CHAPTER 2. THANK YOU!-