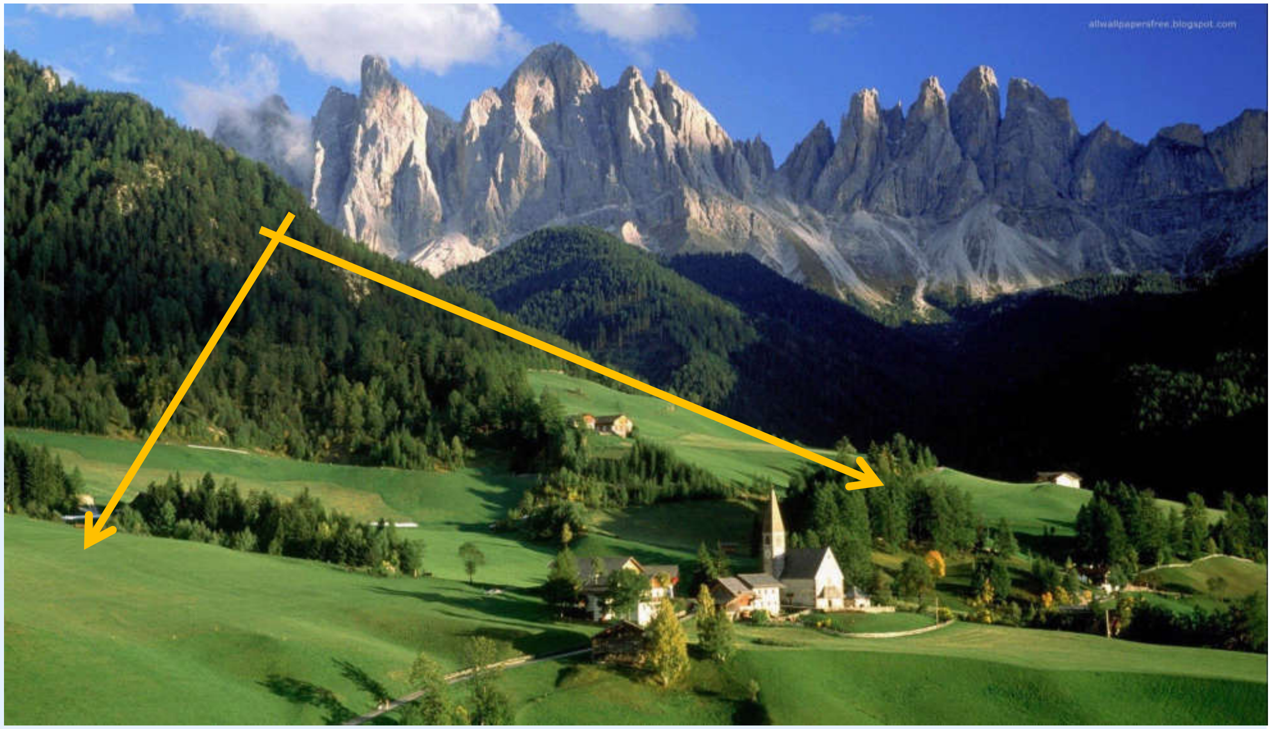
Chapter 2: Partial derivatives

Lecture 4

Partial Derivatives
Tangent Line and Linear Approximations

1. Partial Derivatives



- Rate of change of a function *f(x,y)* depends on the *direction*
- Begin by measuring the rate of change if we move parallel to the x or y axes
- These are called the *partial derivatives* of the function

Definitions

Partial derivative of $f(x, y)$ with respect to x:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Partial derivative of $f(x, y)$ with respect to y:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Partial derivatives can also be interpreted as *rates of change*

Interpretation of Partial Derivatives

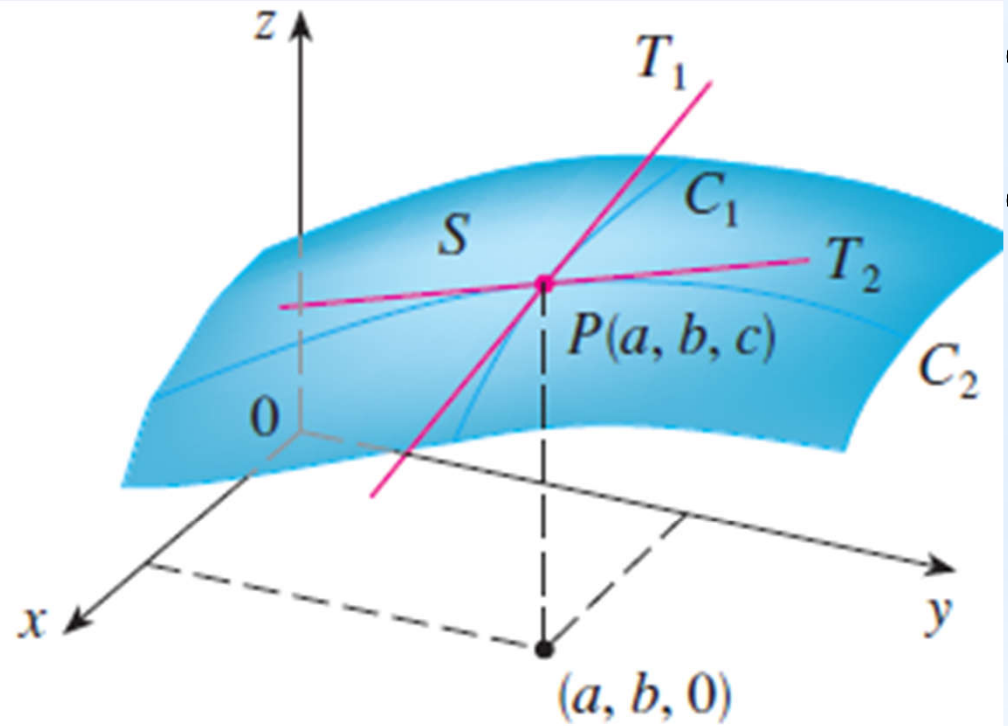


FIGURE 1

The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

C_1: graph $z=f(x,y)$ intersects $y=b$

C_2: graph $z=f(x,y)$ intersects $x=a$

Computing partial derivatives

- To compute $f_x(x, y)$, treat y as a constant, and calculate the derivative of the function f with respect to x
- To compute $f_y(x, y)$, treat x as a constant, and calculate the derivative of the function f with respect to y.
- ***Notations:***

$$\boxed{f_x(x,y) = \frac{\partial f(x,y)}{\partial x}, \qquad f_y(x,y) = \frac{\partial f(x,y)}{\partial y}}$$

Example

Find partial derivatives of the function

$$f(x, y) = 2x^2 y + y^3$$

Solution.

$$f_x(x, y) = 4xy + 0 = 4xy$$

$$f_y(x, y) = 2x^2 + 3y^2$$

Functions of more than two variables

We have similar definitions of partial derivatives for functions of more than two variables.

Function of three variables:

$$f_x(x,y,z)=\frac{\partial f(x,y,z)}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h}$$

Function of n variables:

$$f_{x_k}(x_1,\cdots,x_k,\cdots,x_n)=\lim_{h\to 0}\frac{f(x_1,\cdots,x_k+h,\cdots,x_n)-f(x_1,\cdots,x_k,\cdots,x_n)}{h}$$

$$=\frac{\partial f(x_1,\cdots,x_k,\cdots,x_n)}{\partial x_k}$$

Example $f(x, y, z) = 3x^2yz + z^3y$

To find partial derivative of *f(x,y,z)* with respect to *x*, we treat *y* and *z* as constants

$$\frac{\partial f(x, y, z)}{\partial x} = f_x(x, y, z) = 6xyz$$

To find partial derivative of *f(x,y,z)* with respect to *y*, we treat *x* and *z* as constants

$$\frac{\partial f(x, y, z)}{\partial y} = f_y(x, y, z) = 3x^2z + z^3$$

To find partial derivative of *f(x,y,z)* with respect to *z*, we treat *x* and *y* as constants

$$\frac{\partial f(x, y, z)}{\partial z} = f_z(x, y, z) = 3x^2y + 3z^2y$$

Higher Derivatives

- If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables.
- So we can consider their partial derivatives

$$(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y$$

which are called **second partial derivatives** of the function $f(x,y)$

- If $z=f(x,y)$, we use notations

$$\left(f_x\right)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$\left(f_x\right)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$\left(f_y\right)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$\left(f_y\right)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Example

- Find all second partial derivatives of the function $$f(x, y) = 2x^2 y + y^3$$
- Solution. We have
$$f_x(x, y) = 4xy, \quad f_y(x, y) = 2x^2 + 3y^2$$
- So
$$f_{xx}(x, y) = 4y, \quad f_{xy}(x, y) = 4x$$
$$f_{yx}(x, y) = 4x, \quad f_{yy}(x, y) = 6y$$

Clairaut's Theorem

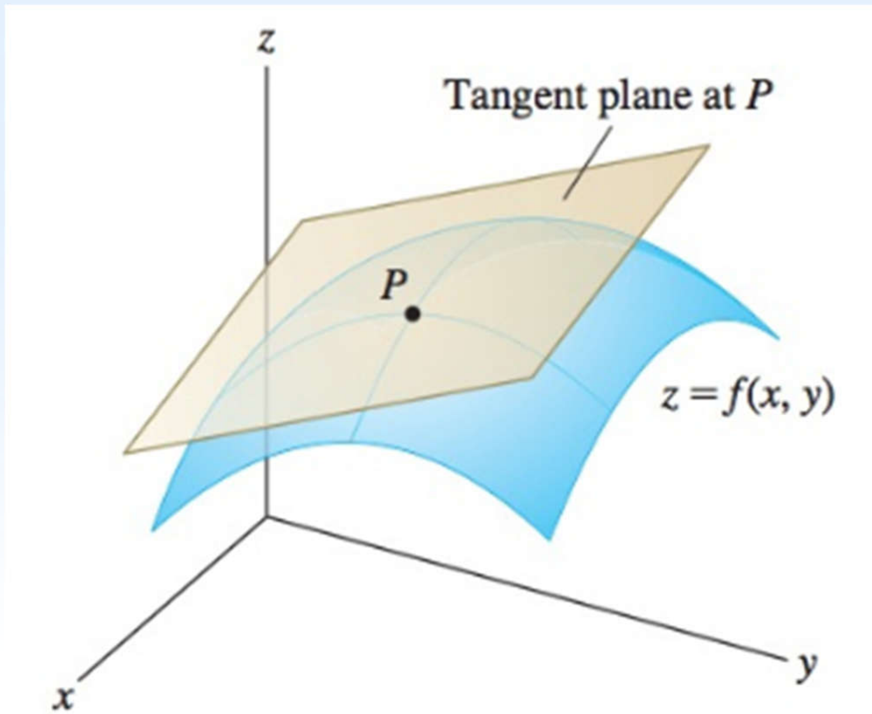
If (a,b) is in a disk D, and f_{xy} and f_{yx} are both continuous on D, then:

$$f_{xy}(a,b) = f_{yx}(a,b)$$

So the order of partial derivatives usually doesn't matter.

2. Tangent Planes & Linear Approximations

Find an equation of ***tangent plane***to a graph of *z = f(x,y)* at the point $P(x_0, y_0, z_0), z_0 = f(x_0, y_0)$

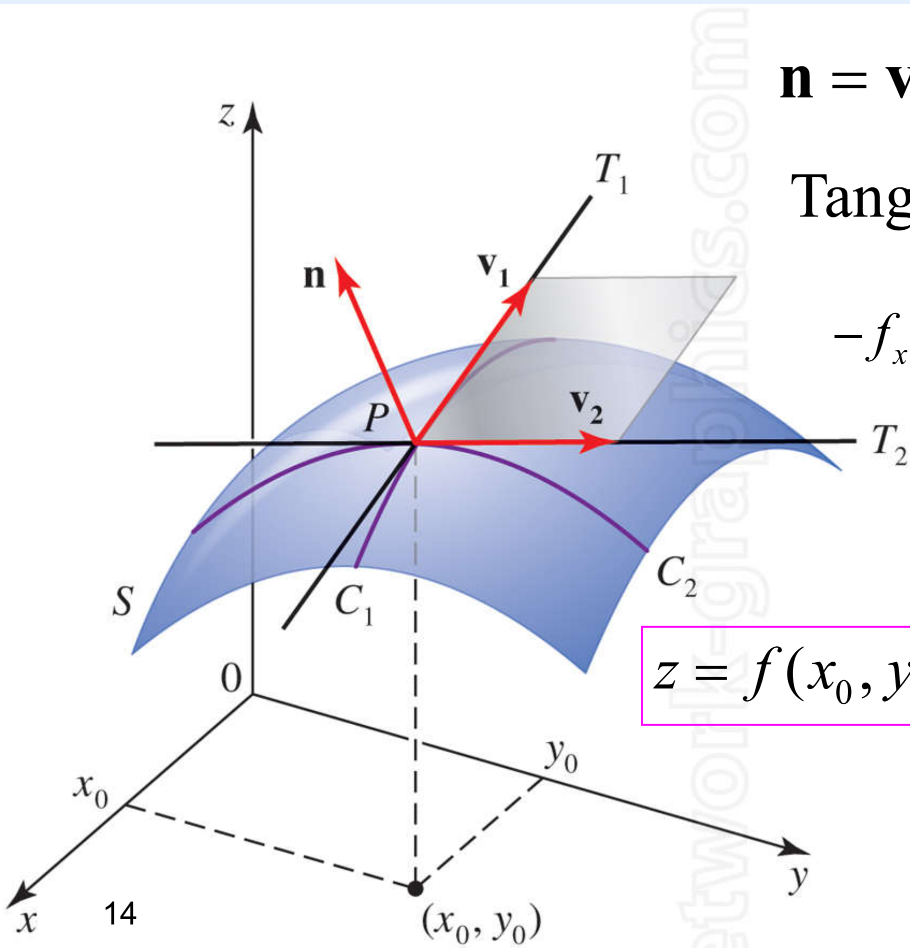


$$C_1 = S \cap \{y = y_0\} : z = f(x, y_0), \mathbf{r}(x) = < x, y_0, f(x, y_0) >$$

$$\mathbf{v}_1 = \mathbf{r}'(x_0) = < 1, 0, f_x(x_0, y_0) > // T_1 : \text{ tangent line to } C_1 \text{ at } P$$

$$C_2 = S \cap \{x = x_0\} : z = f(x_0, y), \mathbf{s}(y) = < x_0, y, f(x_0, y) >$$

$$\mathbf{v}_2 = \mathbf{s}'(y_0) = < 0, 1, f_y(x_0, y_0) > // T_2 : \text{ tangent line to } C_2 \text{ at } P$$



$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \left\langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1 \right\rangle$$

Tangent plane has equation:

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + z - z_0 = 0$$

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Example

- Find tangent plane to the elliptic paraboloid

  $z = 2x^2 + y^2$ at the point (1,1,3)

- Solution:

Let $f(x, y) = 2x^2 + y^2$. Then

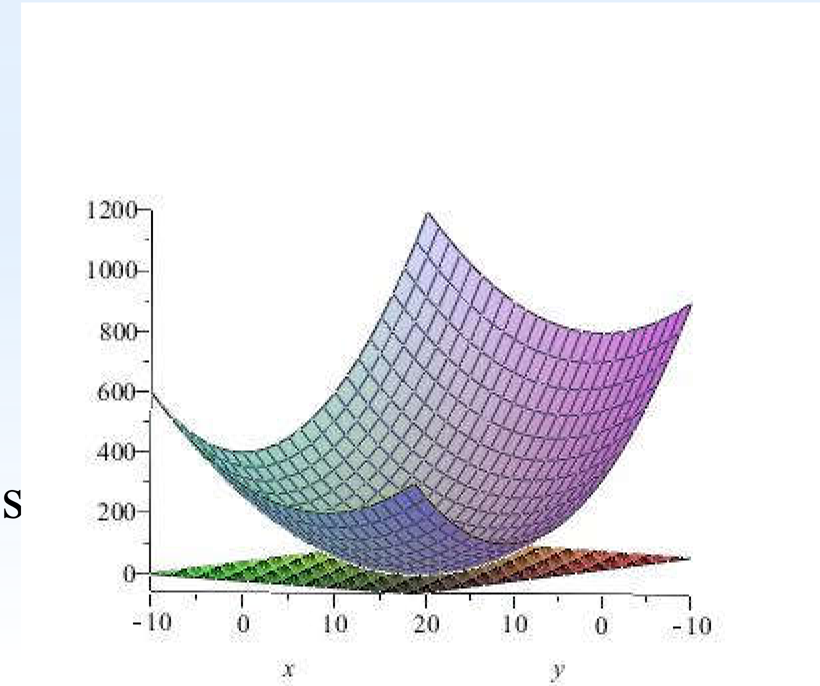
$f_x(x, y) = 4x, f_y(x, y) = 2y$

$f_x(1,1) = 4, f_y(1,1) = 2$

Equation of tangent plane at (1,1,3) is

$z = 3 + 4(x - 1) + 2(y - 1)$

or $z = 4x + 2y - 3$



Linear Approximations

Functions of two variables: Linear approximation at point (a,b) is given by

$$\boxed{f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

Linearization of f(x,y): Find

$$\boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

Functions of three variables: Linear approximation at point (a,b,c) is given by

$$\boxed{f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)}$$

Differentials

It is sometimes convenient to use the language of *differentials*:

$$\Delta z = f(x,y) - f(a,b)$$

$$dz = f_x(a,b)(x - a) + f_y(a,b)(y - b)$$

$$= f_x(a,b)dx + f_y(a,b)dy$$

$$\Delta z \approx dz$$

Differentiability

We say that a function $f(x,y)$ is *differentiable* at (a,b) if the linear approximation is good as (x,y) approaches (a,b) (i.e. Δz approaches dz).

This is guaranteed if both f_x and f_y exist near (a,b) and are continuous at (a,b).