Introduction to Random Process

Outline

- Textbook: chapter 1 - Shreve I
- Introduction to Random process
- Classify random process
- Martingale property
- Stopping time

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Stochastic processes or Random process

- A stochastic process is a mathematical model of a probabilistic experiment that evolves in time and generates a sequence of numerical values.
- Each numerical value in the sequence is modeled by a random variable
- A collection of random variables

Example

- the sequence of daily prices of a stock;
- the sequence of scores in a football game;
- the sequence of failure times of a machine;
- the sequence of hourly traffic loads at a node of a communication network;
- the sequence of radar measurements of the position of an airplane

Random processes

Stochastic process is a collection of random variables $(X_t)_{t \in I}$ on a probability space $(\Omega, \mathcal{F}, P)$ taking values in $\mathcal{S}$

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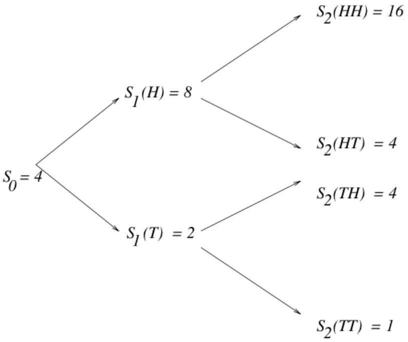
- t: time
- I: the index set of the process
- $\mathcal{S}$: the set of state of the stochastic process.
- For each fixed ω, $X_t(\omega)$ is a deterministic function of time, which is called the sample path (realization, trajectory, sample function)
- At each instant t, X_t is a random variable

Example

A collection (S_0, S_1, S_2) is a stochastic process.

S_0, S_1, S_2 are RV

Corresponding to an outcome $w = HH$, we have a sample path $(4, 8, 16)$



Binomial tree of stock prices with $S_0 = 4$, $u = 1/d = 2$.

Use random processes to

- model some phenomena which evolves over time
- take into account the dependence, e.g how knowledge about asset price up to today effect on the behavior of asset price tomorrow or in the future
- forecasting
- evaluate risk

Example

Consider a binomial asset pricing model

- $S_0 = 4$
- $p(H) = p(T) = \frac{1}{2}$
- $u = 2$, $d = \frac{1}{2}$

Suppose we know that $S_1 = 2$, $S_2 = 4$, $S_3 = 8$.
Then $E(S_4 | S_1 = 2, S_2 = 4, S_3 = 8)$ is used to forecast the asset price at period 4.

Example - Auto regressive model AR(1)

Let S_n be asset price at period n and $r_n = \frac{S_n - S_{n-1}}{S_{n-1}}$ be percentage return at period n

Return at period n depends on the return at period $n-1$ and random noise ϵ_n

$$r_n = c + \phi r_{n-1} + \epsilon_n$$

$\epsilon_1, \epsilon_2, \ldots$ are independent (unpredictable term effects on return)

- Assume that $c = 3$, $\phi = 1$ and $\epsilon_n \sim \mathcal{N}(0, 1)$
- Given that $r_0 = 3$, $r_1 = 1$, $r_2 = 4$, $r_3 = -1$
- the conditional distribution of r_5

$$r_5 | (r_0 = 3, r_1 = 1, r_2 = 4, r_3 = -1) \sim \mathcal{N}(2, 1)$$

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- Forecast return at period 5

$$E(r_5|(r_0 = 3, r_1 = 1, r_2 = 4, r_3 = -1) = 2$$

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- Forecast return at period 5

$$E(r_5|(r_0 = 3, r_1 = 1, r_2 = 4, r_3 = -1) = 2$$

- Risk that the return at period 5 is negative

$$P(r_5 < 0|(r_0 = 3, r_1 = 1, r_2 = 4, r_3 = -1) = P(X < 0)$$

where $X \sim \mathcal{N}(2, 1)$

Exercise - AR(2)

Return at period n depends on the two last returns at period $n-1$ and $n-2$ and random noise ϵ_n

$$r_n = 1 + 0.5S_{n-2} + 2S_{n-1} + \epsilon_n$$

$\epsilon_1, \epsilon_2, \ldots$ are independent and normally distributed $\mathcal{N}(0,1)$.
Given that $r_0 = 3$, $r_1 = 1$, $r_2 = 4$, $r_3 = -1$

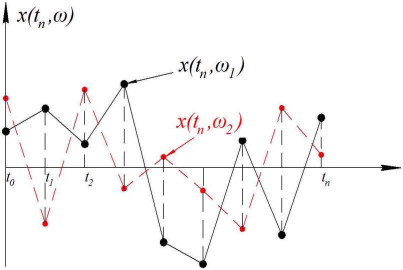
1. Forecast return at period 5
2. Evaluate the risk that the return at period 5 is less than -1.

Classification of stochastic processes

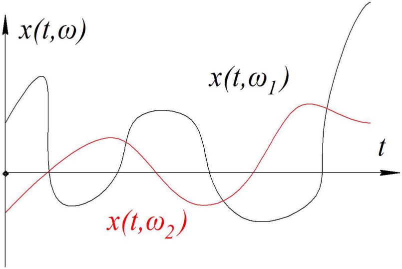
$$X: \quad I \times \Omega \to \mathcal{S}$$
$$(t, \omega) \to X_t(\omega)$$

- Based on time observation I
  - Discrete: I is countable, e.g, $\{0, 1, 2, \dots\}$
  - Continuous: I is uncountable, e.g $[]0, \infty)$, $[0, 1]$
- Based on state observation S
  - Discrete S is countable, e.g, $\{0, 1, 2, \dots\}$, $\{2^0, 2^{\pm 1}, 2^{\pm 2}, \dots\}$
  - Continuous: S is uncountable, e.g $[0, \infty)$, $[0, 1]$

Discrete time vs Continuous time

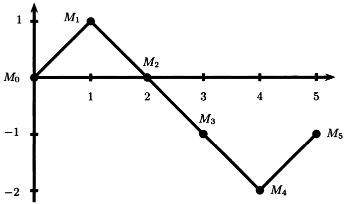


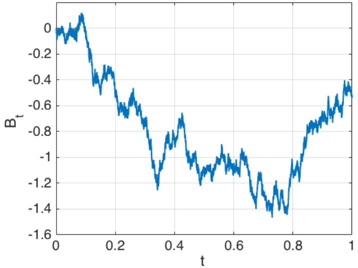
Discrete time process.

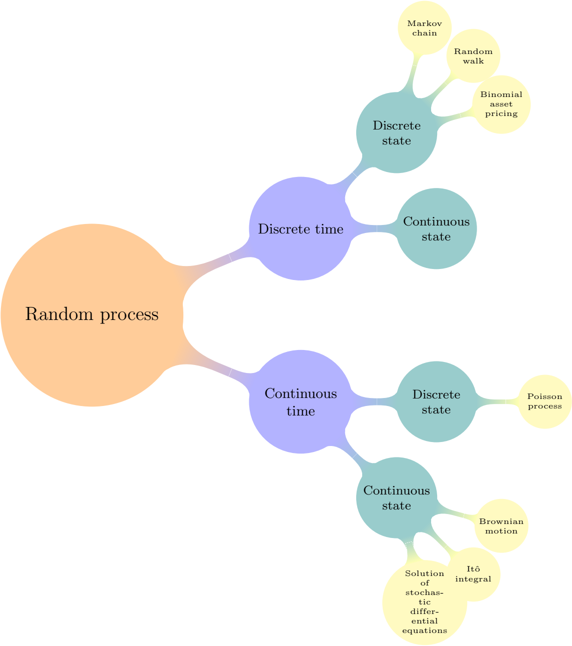


Continuos time process.

Discrete state vs Continuous state






Random process
Discrete time
Discrete state
Markov chain
Random walk
Binomial asset pricing
Continuous state
Continuous time
Discrete state
Poisson process
Continuous state
Brownian motion
Itô integral
Solution of stochas- tic differ- ential equations

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- X_i: the concentration level at i-th hour.
- $X_0, X_1, ...$: is discrete-time continuous-state stochastic process.

Continuous-time discrete-state stochastic process

- Dany receives text messages at random times day and night.

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Continuous-time discrete-state stochastic process

- Dany receives text messages at random times day and night.
- X_t: the number of texts he receives up to time t
- $\{X_t, t \in [0, \infty]\}$ is a continuous-time stochastic process with discrete state space $\{0, 1, 2, ...\}$

Discrete-time discrete-state stochastic process

- Flip a coin infinitely many times, X_k: the number of heads in the first k flips
- Stock price in binomial stock model

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Modeling stock prices as stochastic processes

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- If we only observe the closed price at the end of each trading day, then we have a discrete collection $(S_t)_{t=0,1,2,\ldots}$: discrete-time continuous state stochastic process.
- If we observe all intra-day prices (all prices during a trading day), we obtain a continuous collection $(S_t)_{0\leq t\leq T}$:a continuous-time continuous-state stochastic process.

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- Supply increases: low expectation of the company's profitability.
- Demand increases: high expectation of the company's profitability.
- News changes the expectation of the company's profitability
- Filtration: time-evolving information structure to study random process of a stock price.

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- $\mathcal{F}_t$ represents the information available up to time t.
- The condition $\mathcal{F}_s \subset \mathcal{F}_t, \forall 0 \leq s \leq t$ ensures that the amount of information grows as time evolves and that no information is lost with increasing time.

Example

Some important $\sigma - algebra$ of subsets of Ω in Example 2

1. $\mathcal{F}_0 = \{\emptyset, \Omega\}$: trial $\sigma - algebra$ - contains **no information**. Knowing whether the outcome w of the three tosses is in $\emptyset$ and whether it is in Ω tells you nothing about w.

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Some important $\sigma - algebra$ of subsets of Ω in Example 2

1. $\mathcal{F}_0 = \{\emptyset, \Omega\}$: trial $\sigma - algebra$ - contains **no information**. Knowing whether the outcome w of the three tosses is in $\emptyset$ and whether it is in Ω tells you nothing about w.
2.

$$
\begin{aligned}
\mathcal{F}_1 &= \{0, \Omega, \{HHH, HHT, HTH, HTT\}, \{THH, THT, TTH, TTT \\
&= \{0, \Omega, A_H, A_T\}
\end{aligned}
$$

where

$$A_H = \{HHH, HHT, HTH, HTT\} = \{ \text{ H on first toss } \}$$

$$A_T = \{THH, THT, TTH, TTT\} = \{ \text{ H on first toss } \}$$

$\mathcal{F}_1$**: information of the first coin or "information up to time 1"**. For example, you are told that the first coin is H and no more.

Example

3.

$$\mathcal{F}_2 = \{\emptyset, \Omega, A_{HH}, A_{HT}, A_{TH}, A_{TT}\}$$

and all sets which can be built by taking unions of these }

where

$$A_{HH} = \{HHH, HHT\} = \{\text{HH on the first two tosses}\}$$

$$A_{HT} = \{HTH, HTT\} = \{\text{HT on the first two tosses}\}$$

$$A_{TH} = \{THH, THT\} = \{\text{TH on the first two tosses}\}$$

$$A_{TT} = \{TTH, TTT\} = \{\text{TT on the first two tosses}\}$$

$\mathcal{F}_2$: **information of the first two tosses or "information up to time 2"**

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$\mathcal{F}_2$: information of the first two tosses or "information up to time 2"

4. $\mathcal{F}_3 = \mathcal{G}$ set of all subsets of Ω: **"full information"** about the outcome of all three tosses

Natural filtration

Consider a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$.

- The filtration $(\mathcal{F}_t)_{t \geq 0}$ is called a natural filtration of the process $(X_t)_{t \geq 0}$ if $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t), t \geq 0$.

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- $\mathcal{F}_t$ is the σ-algebra generated by random variables $X_s, 0 \leq s \leq t$

Adapted process

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- X_t can be completely determined from the information available in $\mathcal{F}_t$.
- For each $t > s$, X_t may not be $\mathcal{F}_s$-measurable, i.e., at time s, X_t is considered unknown
- The notion of adaptedness can be interpreted as inability to have knowledge about future events.

Example

- Binomial asset pricing
- $S = (S_0, S_1, S_2, S_3)$: stock price up to time period 3
- Filtration $(\mathcal{F}_t) = \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ with

$$
\begin{aligned}
\mathcal{F}_0 &= \{\emptyset, \Omega\} \\
\mathcal{F}_1 &= \{\emptyset, \Omega, A_T, A_H\} = \sigma(A_H, A_T) \\
\mathcal{F}_2 &= \sigma(A_{HH}, A_{HT}, A_{TH}, A_{TT}) \\
\mathcal{F}_3 &= \sigma(A_{HHH}, A_{HHT}, A_{THH}, A_{THT}, \\
&\quad A_{THH}, A_{THT}, A_{TTH}, A_{TTT})
\end{aligned}
$$

- S is adapted to the filtration $(\mathcal{F}_t)$

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Martingale

1. **(Discrete case)**A (discrete time) stochastic process $(X_n)_{n\in\mathbb{N}}$ is a martingale with respect to the filtration $(\mathcal{F}_n)$ if
   - X_n is $\mathcal{F}_n$ - measurable.
   - $E(X_{n+1}\mathcal{F}_n) = X_n$ for all n.
2. **(Continuous case)** A (continuous time) stochastic process $(X_t)_{t\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)$ if
   - X_t is $\mathcal{F}_t$ - measurable.
   - $E(X_t\mathcal{F}_s) = X_s$ for all $t \geq s$.

Property

Martingale tends to neither go up nor go down.
The expectation of a martingale is constant over time

$$E(X_t) = E(E(X_t|\mathcal{F}_0)) = E(X_0)$$

for all t

Remark

- In order to verify
$$E(X_{n+1}|\mathcal{F}_n) = X_n$$
for a discrete time stochastic process
- Represent X_{n+1} as
$$X_{n+1} = U_n + Y_{n+1}$$
or
$$X_{n+1} = U_n Y_{n+1}$$
where U_n is F_n - measurable and Y_{n+1} is independent of $\mathcal{F}_n$

Example

Binomial asset pricing model $(S_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$ where $\mathcal{F}_n$ is σ - algebra generated by the first n tossing results.

Proof

Denote

$$X_n = \begin{cases} u & \text{if the } n\text{th toss is H} \\ d & \text{if the } n\text{th toss is T} \end{cases}$$

then $(X_n)_n$ are independent and identically distributed

$$P(X_n = u) = p \qquad\qquad P(X_n = d) = q$$

and

$$S_{n+1} = S_n X_{n+1}$$

$$\Rightarrow E(S_{n+1}|\mathcal{F}_n) = E(\underbrace{S_n}_{\text{depends on } \mathcal{F}_n} \quad \underbrace{X_{n+1}}_{\text{independent of } \mathcal{F}_n} |\mathcal{F}_n)$$

$$= \underbrace{S_n}_{\text{take out of what is known}} E(X_{n+1}|S_n) = S_n \underbrace{E(X_{n+1})}_{\text{independent property}}$$

The price process $(S_k)_k$ is a martingale if $E(X_{n+1}) = 1$ or $pu + qd = 1$.

Example - Gambler

Consider a fair game in which the chances of winning and losing equal amounts are the same, i.e. if we denote X_k the outcome of k-th trial at the game, then it is known that $E[X_k] = 0$. Suppose that the initial wealth of a gambler is 0 and he is allowed to borrow as much as possible at no extra cost to play. Then his total wealth after k trials is determined by

$$M_k = X_1 + \cdots + X_k = \sum_{n=1}^{k} X_n$$

Denote $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$: information sets generated by the first k-trial. Then

$$E(M_{k+1}|\mathcal{F}_k) = E\left(\sum_{n=1}^{k+1} X_n|\mathcal{F}_k\right)$$

$$\sum_{n=1}^{k+1} X_n = \underbrace{\sum_{n=1}^{k} X_n}_{\mathcal{F}_k-\text{measurable}} + \underbrace{X_{k+1}}_{\text{independent of } \mathcal{F}_k}$$

Hence

$$E(M_{k+1}|\mathcal{F}_k) = E\left(\sum_{n=1}^{k+1} X_n|\mathcal{F}_k\right) = \sum_{n=1}^{k} X_n + E(X_{k+1}|\mathcal{F}_k)$$

$$= \sum_{n=1}^{k} X_n + \underbrace{E(X_{k+1})}_{=0}$$

$$= \sum_{n=1}^{k} X_n = M_k$$

So the wealth proces $(M_k)_{k\geq 0}$ is a martingale

Practice - Double or nothing

The gambler starts with an initial wealth of 1 dollar and she always bets all of her wealth on the head of a fair coin. If the coin lands on a head, she doubles her wealth. Otherwise, she goes broke. Let

$$X_n = \begin{cases} 2 \text{ if the } n^{th} \text{ tossing is head} \\ 0 \text{ if the } n^{th} \text{ tossing is tail} \end{cases}$$

1. Determine a formula for her wealth process $(M_k)_k \geq 0$
2. Is her wealth process a martingale?

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Stopping time

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on a probability space $(\Omega, \mathcal{F}, P)$. A nonnegative random variable T is called to be a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$ if $(\tau \leq t) \in \mathcal{F}_t)$ for all $t \geq 0$

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Example

In gambler problem, the first time that a gambler gains \$8 is a stopping time.

Stopped process

Let $((X_t)_{t\geq 0}$ be a random process and T is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ then the stopped process X_t^T is given by

$$X_t^T(\omega) = X_{\min(t,T(\omega))}(\omega) = \begin{cases} X_t(\omega) & \text{if } t < T(\omega) \\ X_{T(\omega)}(\omega) & \text{if } t \geq T(\omega) \end{cases}$$

Optimal stopping theorem

Let $(X_t)_{t\geq 0}$ be a martingale and T is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.
If T is bounded ($T < c$ a.s for some c) then the stopped process X_t^T is also a martingale.
Consequently

$$E(X_t^T) = E(X_0)$$

Example

Consider a fair game in which the change of winning or losing \$1 each round are equal . The gambler starts with an initial wealth of \$3. He plays until he broke or his wealth reaches \$10. What is the probability that he wins the game.

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where X_k are i.i.d with $P(X_k = 1) = P(X_k = -1) = \frac{1}{2}$
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- Let τ be the time that the game stops then τ is a stopping time
- Apply optimal stopping time, we have

$$E(M_\tau) = E(M_0) = 3$$

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- $$M_\tau \begin{cases} 10, & \text{with prob} p \\ 0, & \text{with prob} 1-p \end{cases}$$
- $E(M_\tau) = 10p + 0(1-p) = 10p$
- But $E(M_\tau) = 3$. So

$$10p = 3 \Rightarrow p = 0.3$$