Chapter 1: Solutions of Nonlinear Equations

Lecture 3: Open Methods

- **Newton-Raphson Method**
- **Secant Method**

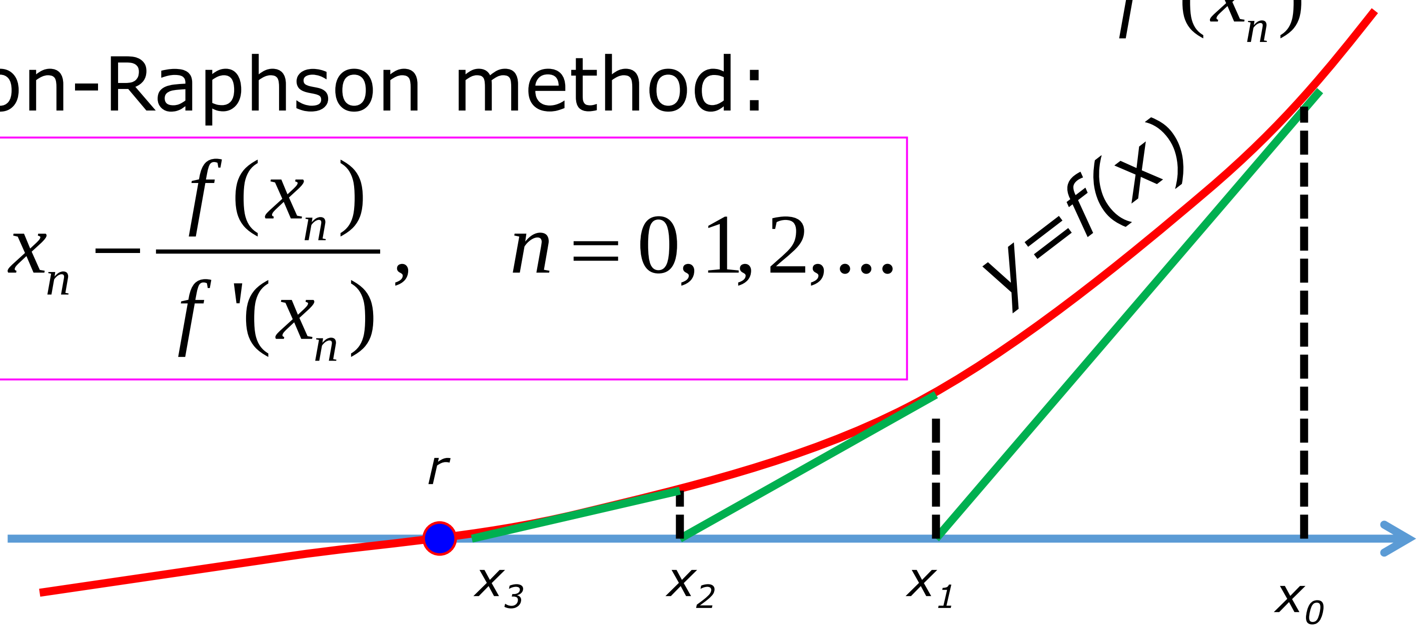
Newton-Raphson Method

Given an approximation x_n of the root r. The x-intercept of tangent at x_n can give a better approximation of root r

$$y = f(x_n) + f'(x_n)(x - x_n) = 0 \Rightarrow x = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton-Raphson method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots$$



Example

Using Newton-Raphson method to determine the root of equation

$$f(x) = \frac{x^3}{2} + x - 6 = 0$$

with $$x_0 = 3 \text{ until } |\varepsilon_a| \le \varepsilon = 0.1\%$$

Solution

- We have

$$f(x)=\frac{x^3}{2}+x-6 \quad \Rightarrow f'(x)=\frac{3}{2}x^2+1$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{\frac{x_n^{\;3}}{2}+x_n-6}{\frac{3}{2}x_n^{\;2}+1}, \quad n=0,1,2,\ldots$$

$$x_0=3, x_1=x_0-\frac{\frac{x_0^{\;3}}{2}+x_0-6}{\frac{3}{2}x_0^{\;2}+1}= \ 2.2759,$$

$$x_1 = 2.2759, \quad x_2 = x_1 - \frac{\frac{x_1{}^3}{2} + x_1 - 6}{\frac{3}{2} x_1{}^2 + 1} = \text{ 2.0284}, \varepsilon_a = \frac{x_2 - x_1}{x_2} = \text{ -0.1087}$$

We continue this process to get:

$$x_3 = 2.0003, \varepsilon_a = \frac{x_3 - x_2}{x_3} = \text{-0.0138}$$

$$x_4 = 2.000, \varepsilon_a = \frac{x_4 - x_3}{x_4} = \text{-}1.7057 \times 10^{-4}$$

$$|\,\varepsilon_a\,| = 1.7057 \times 10^{-4} < 0.1\%$$

So the approximate root is $r = x_4 = 2.000$

Convergence Analysis

Theorem:

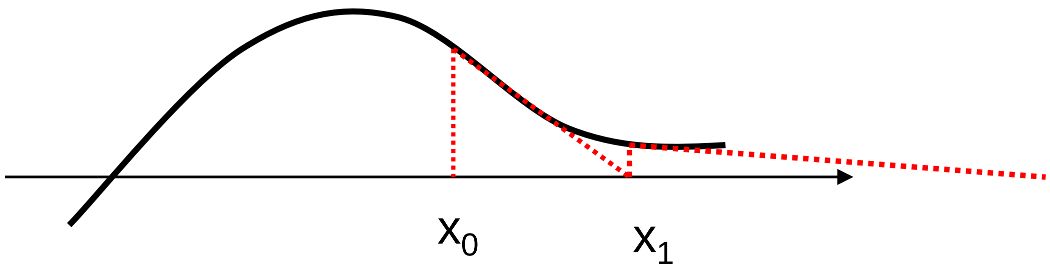
Let $f(x), f'(x)$ *and* $f''(x)$ be continuous at $x \approx r$

where $f(r) = 0$. *If* $f'(r) \neq 0$ then there exists $\delta > 0$

such that $|x_0 - r| \leq \delta \Rightarrow \dfrac{|x_{k+1} - r|}{|x_k - r|^2} \leq C$

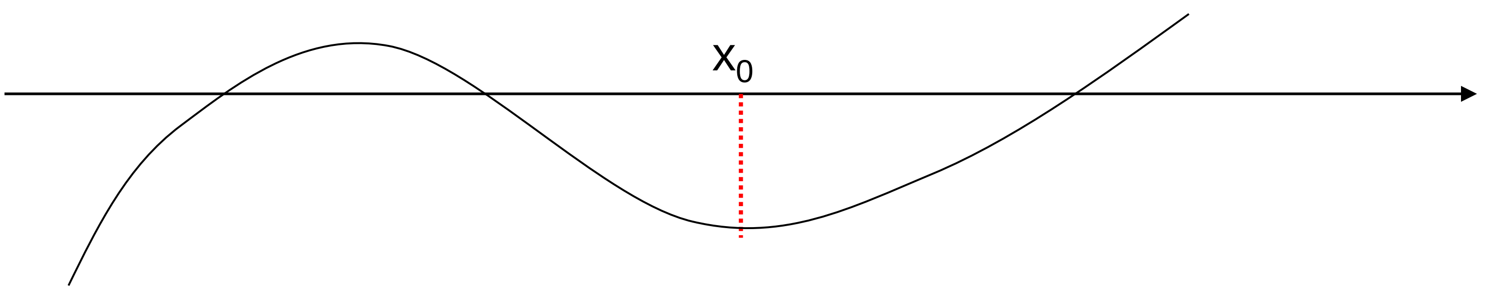
$$C = \frac{1}{2} \frac{\displaystyle\max_{|x_0 - r| \leq \delta} |f''(x)|}{\displaystyle\min_{|x_0 - r| \leq \delta} |f'(x)|}$$

Problems with Newton's Method
- Runaway -



The estimates of the root is going away from the root.

Problems with Newton's Method
- Flat Spot -



The value of $f'(x)$ is zero, the algorithm fails.

If $f'(x)$ is very small then x_1 will be very far from x_0.

Exercise

The height (in m) at the time t (in seconds) of a ball projected vertically into the air is given by

$$f(t) = 1180(1 - e^{-t/10}) - 98t$$

Find the time at which the ball hits the ground using Newton-Raphson method initial guess $t_0=3$ and stopping criterion

$$|\varepsilon_a| < 0.1\%$$

Exercise 2

Using Newton-Raphson method, find the approximate root of equation $f(x) = x^{4/3} + x^2 - 2 = 0$

with x_0 =2 stopping condition $|\varepsilon_a| < 0.2\%$

Quiz 1

The height (in m) at the time t (in seconds) of a ball projected vertically into the air is given by

$$h(t) = 1200(1 - e^{-t/10}) - 98t$$

(a) Find the first time the ball goes up and reaches the height of 8m using Secant Method

(b) Find the time the ball comes down and reaches the height of 8m using False Position Method

(c) Find the time the ball comes down and reaches the height of 7m using Newton-Raphson Method

Stopping criterion: $|\varepsilon_a| < 0.2\%$

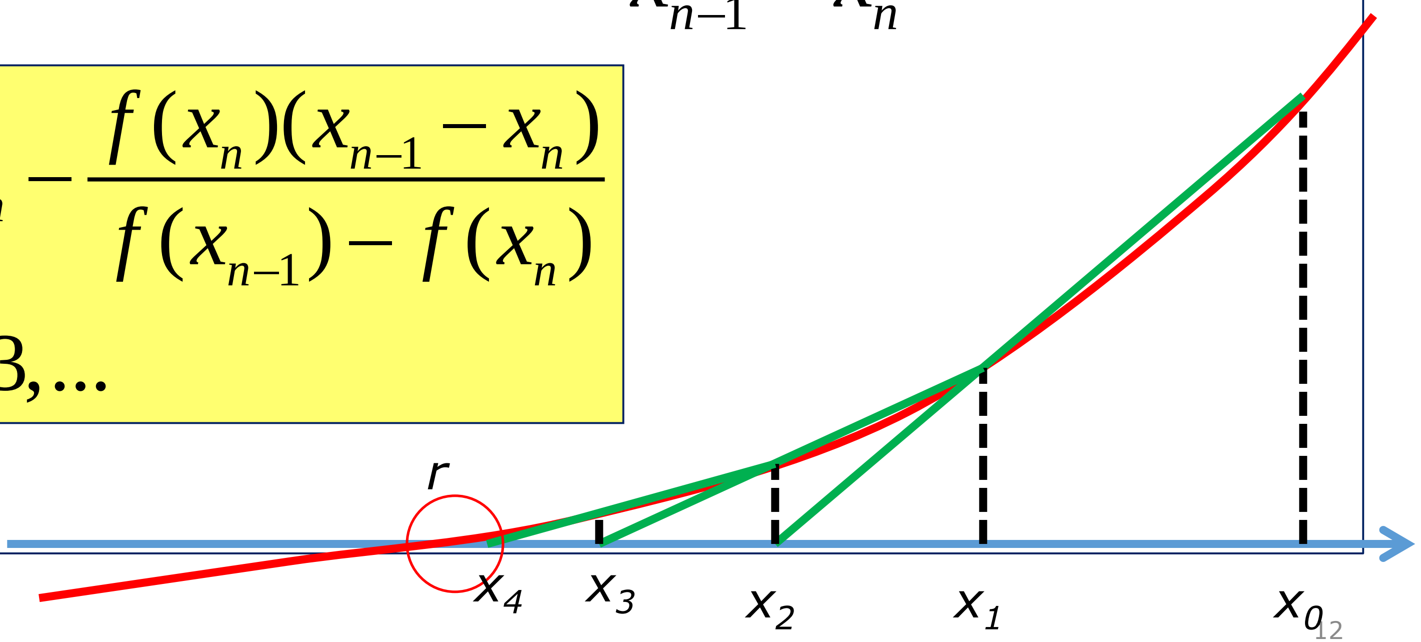
2. Secant Method

- x-intercept of the secant line near the root can approximate the root.

$$y = f(x_n) + \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}(x - x_n) = 0$$

$$x_{n+1} = x_n - \frac{f(x_n)(x_{n-1} - x_n)}{f(x_{n-1}) - f(x_n)}$$

$$n = 1, 2, 3, \ldots$$



Convergence Analysis

- The rate of convergence of the Secant method is super linear:

$$\frac{|x_{i+1} - r|}{|x_i - r|^{\alpha}} \le C, \qquad \alpha \approx 1.62$$

$$r : root \qquad x_i : \text{estimate of the root at the i}^{\text{th}} \text{ iteration.}$$

- It is better than Bisection method but not as good as Newton's method.

Example

Using Secant method to approximate the root of equation

$$f(x) = x^3 + 2x - 3 = 0$$

with

$$x_0 = 2,\ x_1 = 1.5 \ \text{ until } \ |E_a| \le 0.001$$

Solution

x_n	$f(x_n)$	E_a
2.0000	9.0000	--
1.5000	3.3750	-0.5000
1.2000	3.3750	-0.3000
1.0494	1.1280	-0.1506
1.0055	0.2544	-0.0439
1.0002	0.0278	-0.0054
1.0000	0.0008	-0.0002

Exercise

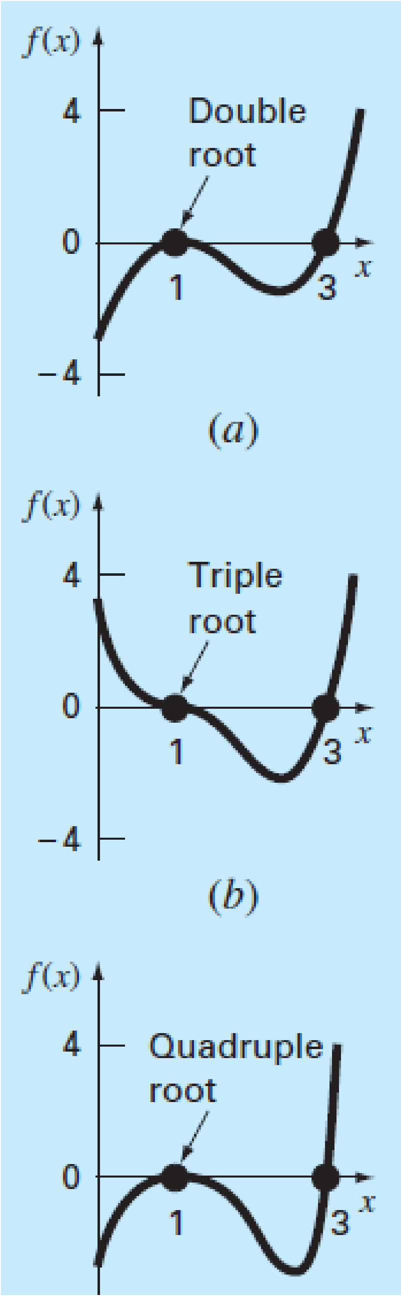
The height (in m) at the time t (in seconds) of a ball projected vertically into the air is given by

$$f(t) = 1180(1 - e^{-t/10}) - 98t$$

Find the time at which the ball hits the ground using Secant method with initial guesses t_0=3, t_1=3.5 and stopping criterion

$$|\varepsilon_a| < 0.1\%$$

Multiple Roots



Problems:

- Function does not change sign at even multiple roots

- Not only $f(x)$ but also $f'(x)$ goes to zero at the root

$Definition$: $A\ root\ p\ of\ f(x) = 0\ has\ multiplicity\ k$ $if\ for\ x \neq p, we\ can\ write$

$$f(x) = (x-p)^k g(x), \qquad where\ \lim_{x\to p} g(x) \neq 0$$

Multiple Roots

Substitute *f(x)* by $u(x) = \frac{f(x)}{f'(x)}$

$$u(x) = \frac{(x-p)^k g(x)}{k(x-p)^{k-1} g(x) + (x-p)^k g'(x)} = \frac{(x-p) g(x)}{k g(x) + (x-p) g'(x)}$$

$p \text{ is a simple root of } u(x) = 0$

$$u'(x) = \frac{[f'(x)]^2 - f(x) f''(x)}{[f'(x)]^2}$$

- Modified Newton-Raphson method:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n) f'(x_n)}{[f'(x_n)]^2 - f(x_n) f''(x_n)}, \quad n = 0, 1, 2, \ldots}$$

- Modified Secant method:

$$\boxed{x_{n+1} = x_n - \frac{u(x_n)(x_{n-1} - x_n)}{u(x_{n-1}) - u(x_n)}, \quad n = 1, 2, 3, \ldots}$$

Systems of Nonlinear Equations: Newton's Method

Find approximate root of $F(X)=0$ with given initial guess X_0

$$\boxed{X_{k+1}=X_k-\left[DF(X_k)\right]^{-1}F(X_k),\quad k=0,1,2,....}$$

$$X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\quad F(X)=\begin{bmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_n(X) \end{bmatrix},\quad DF(X)=\begin{bmatrix} \frac{\partial f_1(X)}{\partial x_1} & \frac{\partial f_1(X)}{\partial x_2} & \cdots & \frac{\partial f_1(X)}{\partial x_n} \\ \frac{\partial f_2(X)}{\partial x_1} & \frac{\partial f_2(X)}{\partial x_2} & \cdots & \frac{\partial f_2(X)}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n(X)}{\partial x_1} & \frac{\partial f_n(X)}{\partial x_2} & \cdots & \frac{\partial f_n(X)}{\partial x_n} \end{bmatrix}$$

Example Find approximate solution of equations

$$y + x^2 - 0.5 - x = 0$$

$$x^2 - 5xy - y = 0$$

$$\text{Initial guess } \ x_0 = 1, \ \ y_0 = 0$$

Solution

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad F(X) = \begin{bmatrix} y + x^2 - 0.5 - x \\ x^2 - 5xy - y \end{bmatrix}, \ DF(X) = \begin{bmatrix} 2x - 1 & 1 \\ 2x - 5y & -5x - 1 \end{bmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad ad - bc \neq 0$$

$$\text{Iteration 1:}\quad X_0=\begin{bmatrix}1\\0\end{bmatrix},F(X_0)=\begin{bmatrix}y_0+x_0^2-0.5-x_0\\x_0^2-5x_0y_0-y_0\end{bmatrix}=\begin{bmatrix}-0.5\\1\end{bmatrix},$$

$$DF(X_0)=\begin{bmatrix}2x_0-1 & 1\\2x_0-5y_0 & -5x_0-1\end{bmatrix}=\begin{bmatrix}1 & 1\\2 & -6\end{bmatrix},\qquad \begin{bmatrix}1 & 1\\2 & -6\end{bmatrix}^{-1}=\begin{bmatrix}3/4 & 1/8\\1/4 & -1/8\end{bmatrix}$$

$$X_1=X_0-\left[DF(X_0)\right]^{-1}F(X_0)$$

$$=\begin{bmatrix}1\\0\end{bmatrix}-\begin{bmatrix}1 & 1\\2 & -6\end{bmatrix}^{-1}\begin{bmatrix}-0.5\\1\end{bmatrix}=\begin{bmatrix}1.25\\0.25\end{bmatrix}$$

$$\text{Iteration 2:}$$

$$F(X_1)=\begin{bmatrix}0.0625\\-0.25\end{bmatrix},\ DF(X_1)=\begin{bmatrix}1.5 & 1\\1.25 & -7.25\end{bmatrix}$$

$$X_2=X_1-\left[DF(X_1)\right]^{-1}F(X_1)$$

$$=\begin{bmatrix}1.25\\0.25\end{bmatrix}-\begin{bmatrix}1.5 & 1\\1.25 & -7.25\end{bmatrix}^{-1}\begin{bmatrix}0.0625\\-0.25\end{bmatrix}=\begin{bmatrix}1.2332\\0.2126\end{bmatrix}$$

Exercise:

Use Newton's method to find approximate roots of the system of nonlinear equations

$$x^2 + y^3 - 20 - xy = 0$$

$$(x-4)^2 + (y-4)^2 - 5 = 0$$

Stopping condition:

$$|E_a| \leq 0.01$$

for each variable