# Chapter 6
## Numerical Methods for Partial Differential Equations

# Lecture 1: Finite-difference method for elliptic equations

# Introduction

A partial differential equation (**PDE**) is an equation that involves an unknown function and its partial derivatives.

## Example: Heat equation

$$\boxed{\frac{\partial u(x,y,z,t)}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}$$

$\alpha = \dfrac{K}{\rho C}$ : thermal diffusivity $(\mathrm{cm}^2 / s)$

$\rho$: density of the material $(\mathrm{g/cm}^3)$

$C$: heat capacity of the material $[\mathrm{cal/(g \cdot {}^{O}}C)]$

$K$: thermal conductivity $[\mathrm{cal/(s \cdot cm \cdot {}^{O}}C)]$

$u =$ temperature $({}^{O}C)$ in a body at point $(x, y, z)$, time $t$ (s)

# Laplace Equation

$$\boxed{\frac{\partial^2 u(x,y,z)}{\partial x^2}+\frac{\partial^2 u(x,y,z)}{\partial y^2}+\frac{\partial^2 u(x,y,z)}{\partial z^2}=0}$$

Used to describe the steady state distribution of heat in a body.

Also used to describe the steady state distribution of electrical charge in a body.

# Wave Equation

$$\boxed{\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}$$

It describes the displacement at time *t* of a particle having coordinates (*x*,*y*,*z*) .

The constant *c* represents the propagation speed of the wave.

# The Black–Scholes equation

In mathematical finance, **Black–Scholes equation** is

$$\boxed{\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0}$$

where $V$ is the price of the option as a function of stock price $S$ and time $t$, $r$ is the risk-free interest rate, and $\sigma$ is the volatility of the stock

Finite-difference methods in this lecture can be applied to this model

# Linear Second Order PDEs

A PDE is said to be *linear* if it is linear in the unknown function and all its derivatives

Because of their widespread application, our treatment of PDEs will focus on linear, second-order equations of two-independent variables

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D(x, y, u, u_x, u_y) = 0,$$

where $A$, $B$, and $C$ are functions of $x$ and $y$

# Linear Second Order PDEs
## Classification

A second order linear PDE (2-independent variables)

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D(x, y, u, u_x, u_y) = 0,$$

is classified based on $(B^2 - 4AC)$ as follows:

$$B^2 - 4AC < 0 \qquad \text{Elliptic}$$

$$B^2 - 4AC = 0 \qquad \text{Parabolic}$$

$$B^2 - 4AC > 0 \qquad \text{Hyperbolic}$$

# Poisson Equation Boundary Conditions

Let $\Omega \subset R^2$ be a planar domain, and denote its boundary by $\partial\Omega$. The boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in \Omega$$

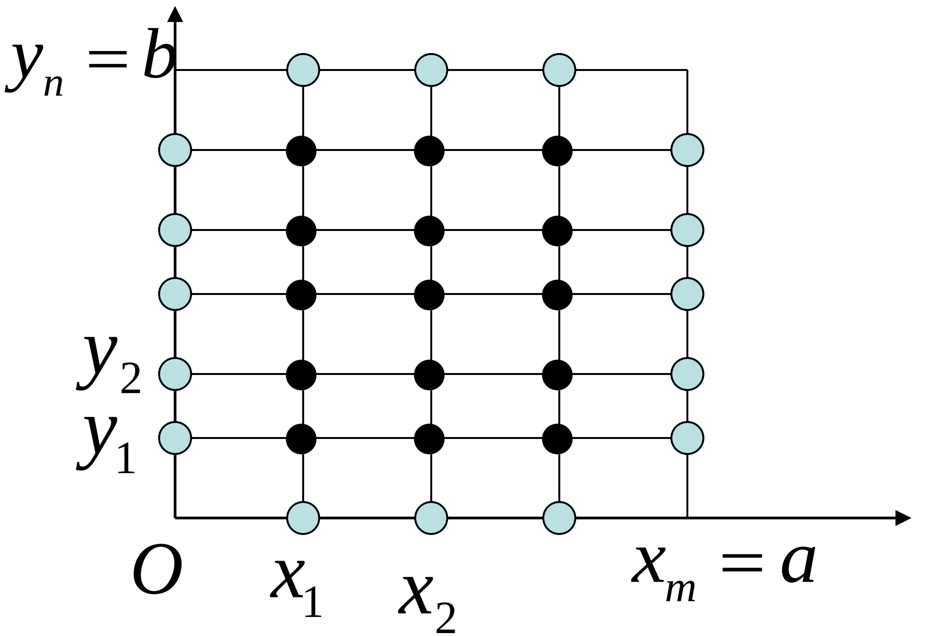
$$u = g(x,y), \quad (x,y) \in \partial\Omega$$

is called a Dirichlet problem because the value of u is specified on the boundary. For simplicity we will assume that $\Omega = (0,a)\times(0,b)$ and therefore

$$\partial\Omega = \{\,(x,y): 0 \le x \le a, y = 0, b \text{ or } 0 \le y \le b, x = 0, a\,\}$$

# Finite Difference Grid

$\Omega$ is divided into m equal parts along x-axis and n parts along y-axis




boundary points
interior points

Coordinates of mesh (grid) points: $(x_i, y_j)$,

where $x_i = i\Delta x,\ y_j = j\Delta y,\ 0 \le i \le m, 0 \le j \le n$

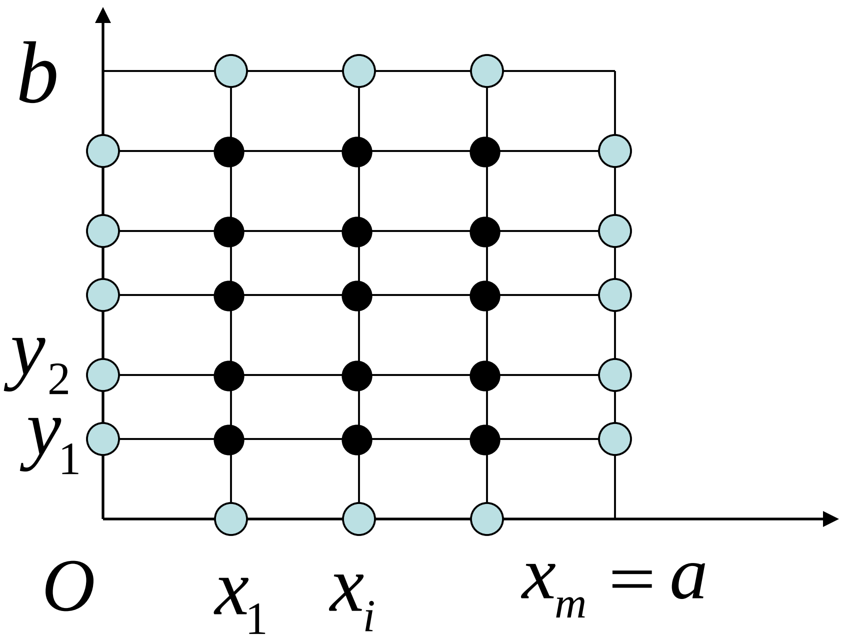
Step sizes: $\Delta x = \dfrac{a}{m},\quad \Delta y = \dfrac{b}{n}$

# Finite Difference Approximation

We **approximate the PDE by CDA of 2nd derivatives**:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) = (x_i, y_j)$$

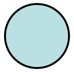
$$\frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{(\Delta x)^2} + \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{(\Delta y)^2} \approx f_{ij}$$




Notations:

$$f_{ij} = f(x_i, y_j)$$

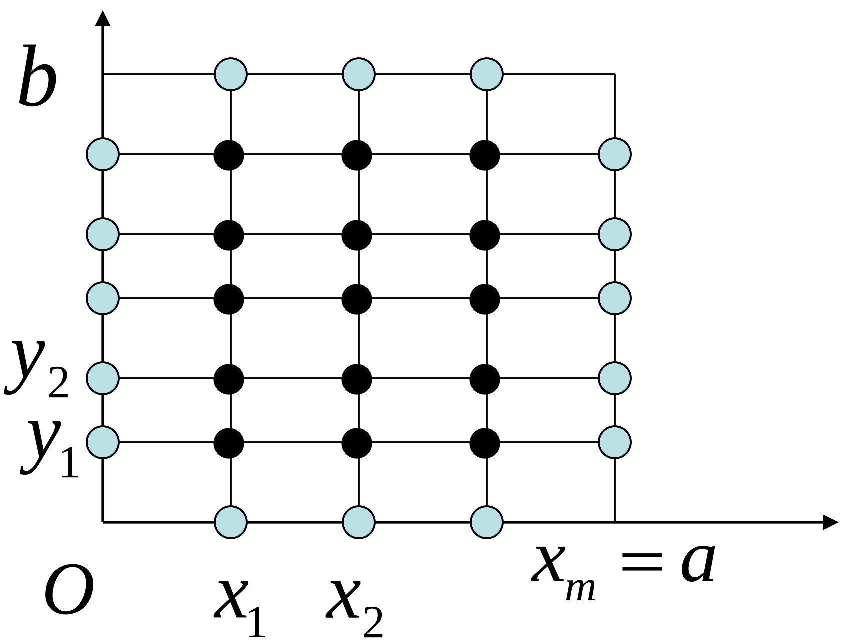
$$u_{i,j} \approx u(x_i, y_j)$$

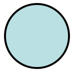
For boundary points  $u(x_i, y_j) = g(x_i, y_j)$

# Finite Difference Approximation

We approximate $u(x_i, y_j)$ by the solutions $u_{i,j}$ of

$$\boxed{\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = f_{i,j} \qquad 1 \le i \le m-1, \;\; 1 \le j \le n-1}$$




Where, at every boundary point,  we define

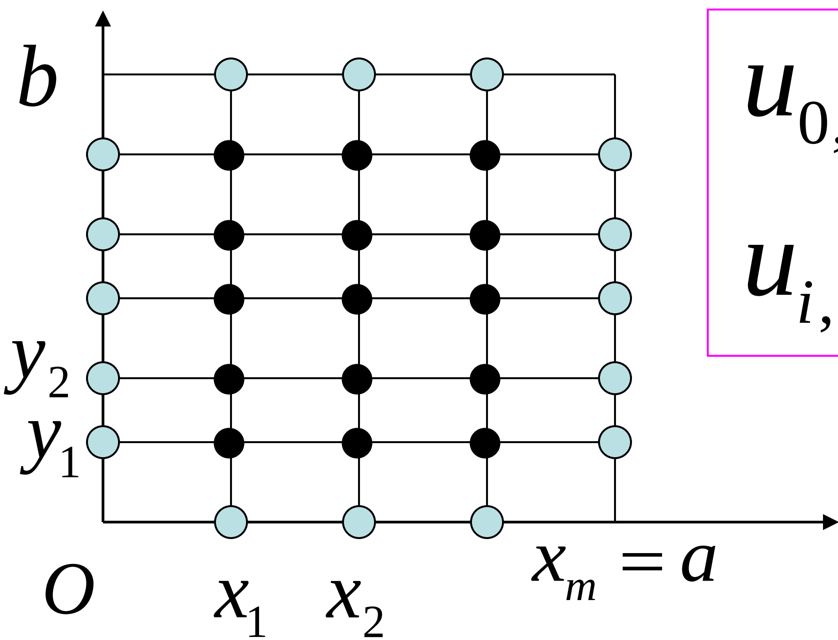
$$u_{i,j} = g(x_i, y_j)$$

# Finite Difference Approximation

If we choose $\Delta x = \Delta y = h$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j}$$

$$1 \le i \le m-1, \;\; 1 \le j \le n-1$$




$$u_{0,j} = g_{0,j}, \quad u_{m,j} = g_{m,j}$$

$$u_{i,0} = g_{i,0}, \quad u_{i,n} = g_{i,n}$$

# 2D Laplace Equation

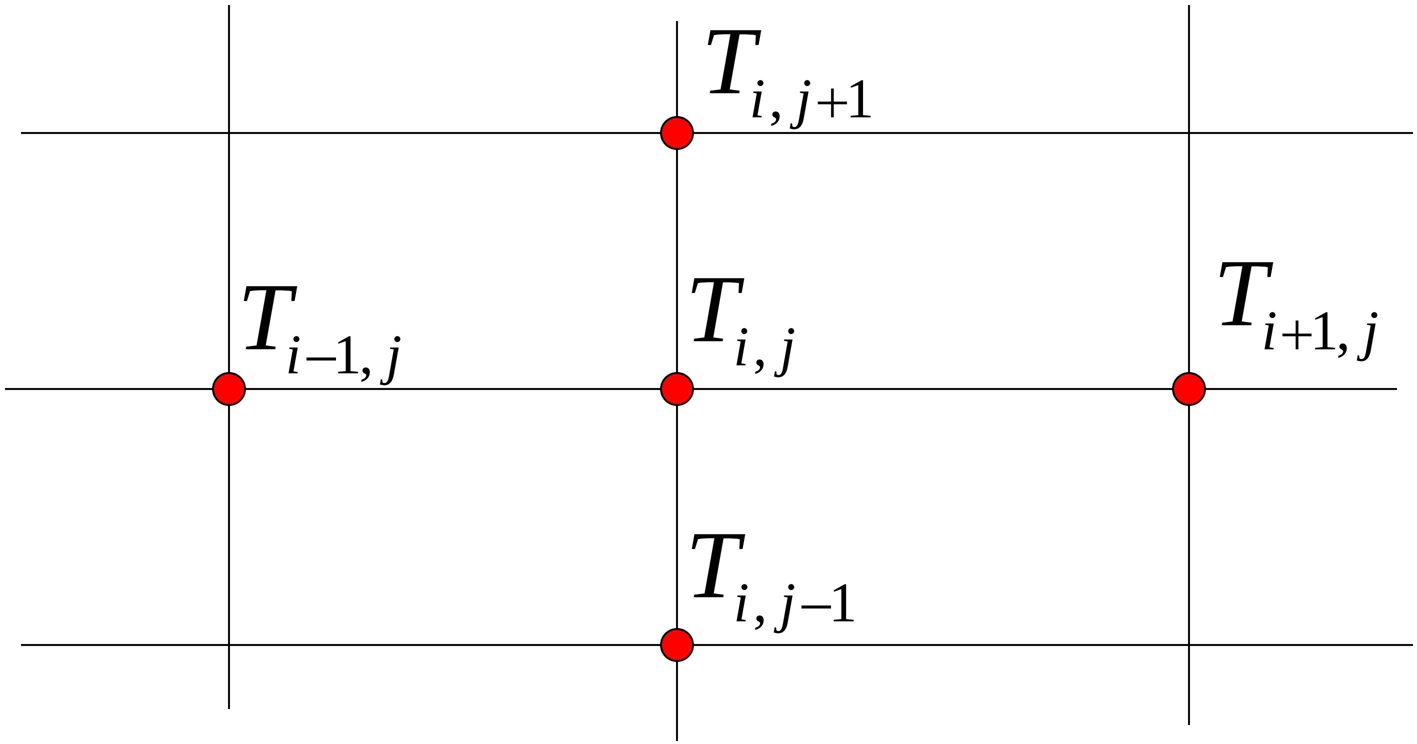
$$\boxed{\frac{\partial^2 \ T(x,y)}{\partial x^2} + \frac{\partial^2 \ T(x,y)}{\partial y^2} = 0}$$

It represents heat distribution over a body occupying a planar region without heat source

Take $\Delta x = \Delta y = h$ for discretizing the PDE and the region

# 2D Laplace equation Solution Technique

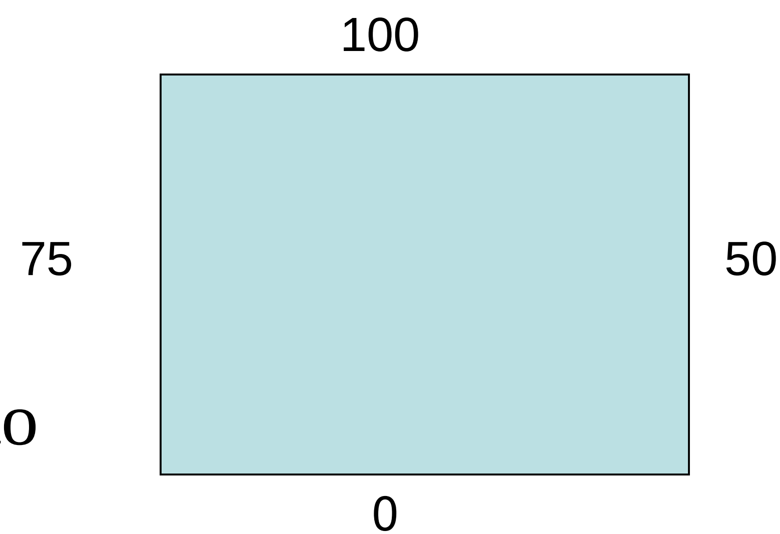
$$\Delta x = \Delta y = h$$




$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

# Example 1

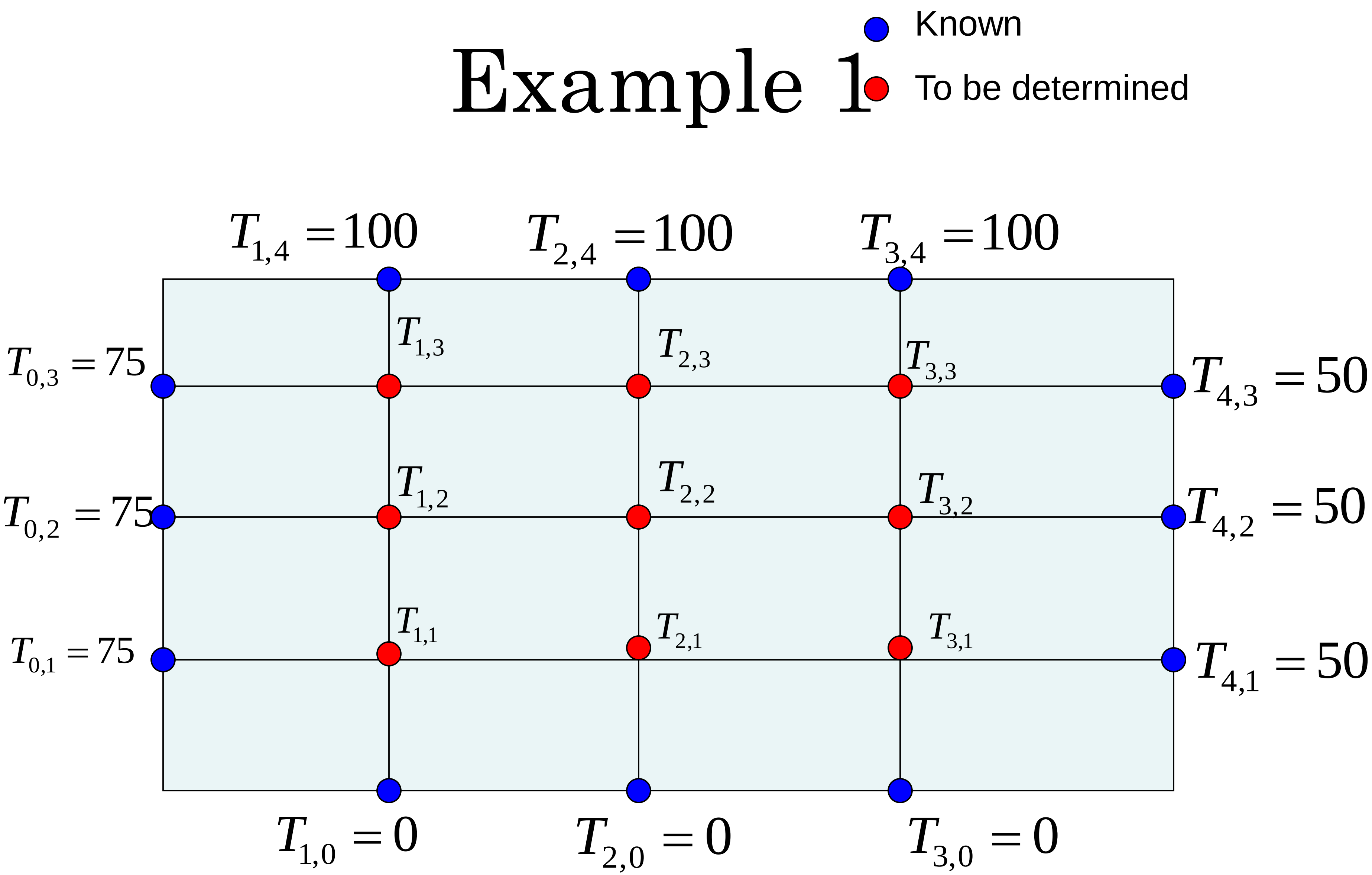
It is required to determine the steady state temperature at all points of a heated square sheet of metal. The edges of the sheet are kept at a constant temperature: 100, 50, 0, and 75 degrees.




The sheet is divided to 5x5 grids.

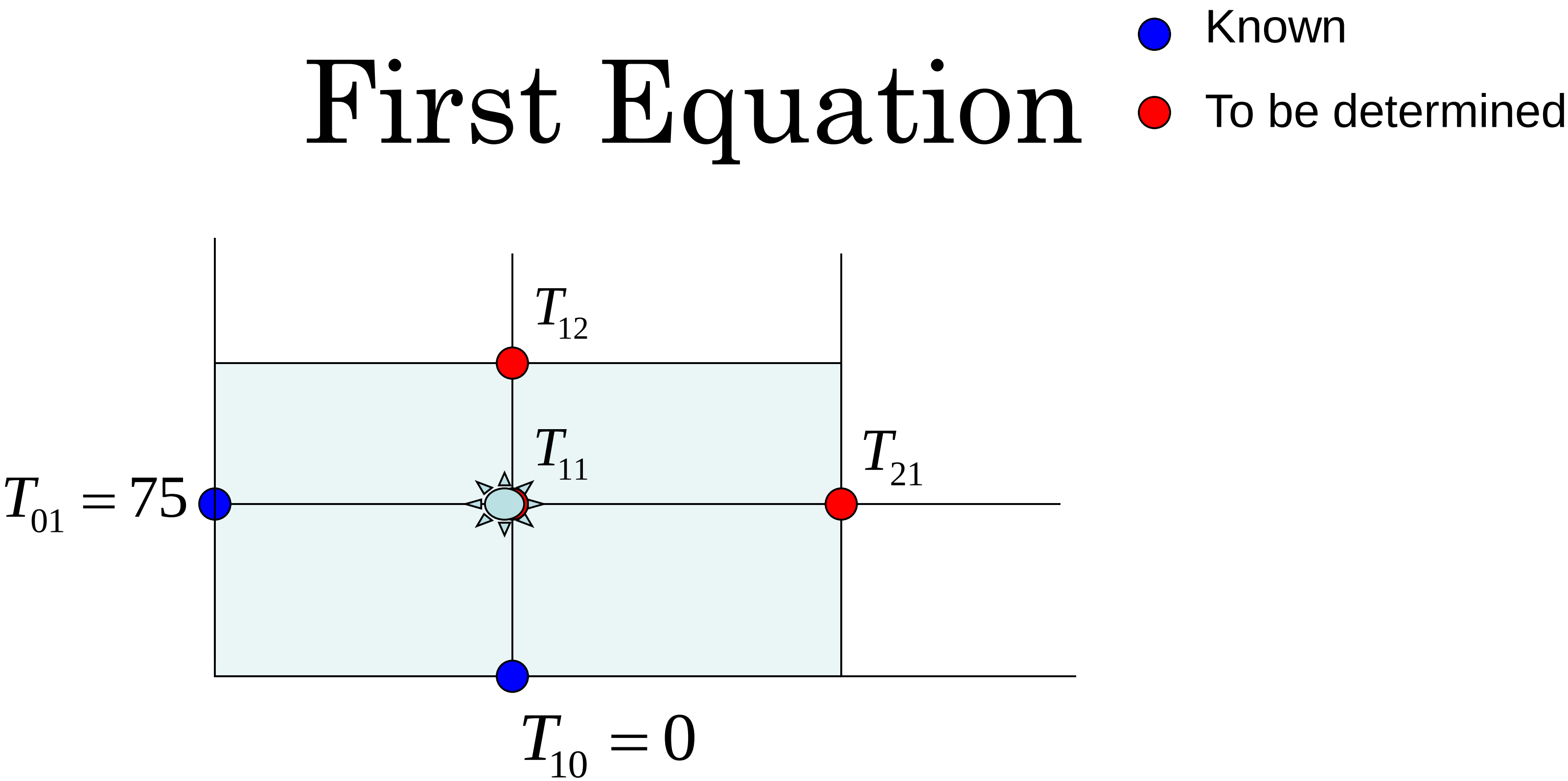
$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

# Example 1



Known
To be determined
$T_{1,4} = 100$
$T_{2,4} = 100$
$T_{3,4} = 100$
$T_{0,3} = 75$
$T_{0,2} = 75$
$T_{0,1} = 75$
$T_{4,3} = 50$
$T_{4,2} = 50$
$T_{4,1} = 50$
$T_{1,3}$
$T_{2,3}$
$T_{3,3}$
$T_{1,2}$
$T_{2,2}$
$T_{3,2}$
$T_{1,1}$
$T_{2,1}$
$T_{3,1}$
$T_{1,0} = 0$
$T_{2,0} = 0$
$T_{3,0} = 0$

# First Equation

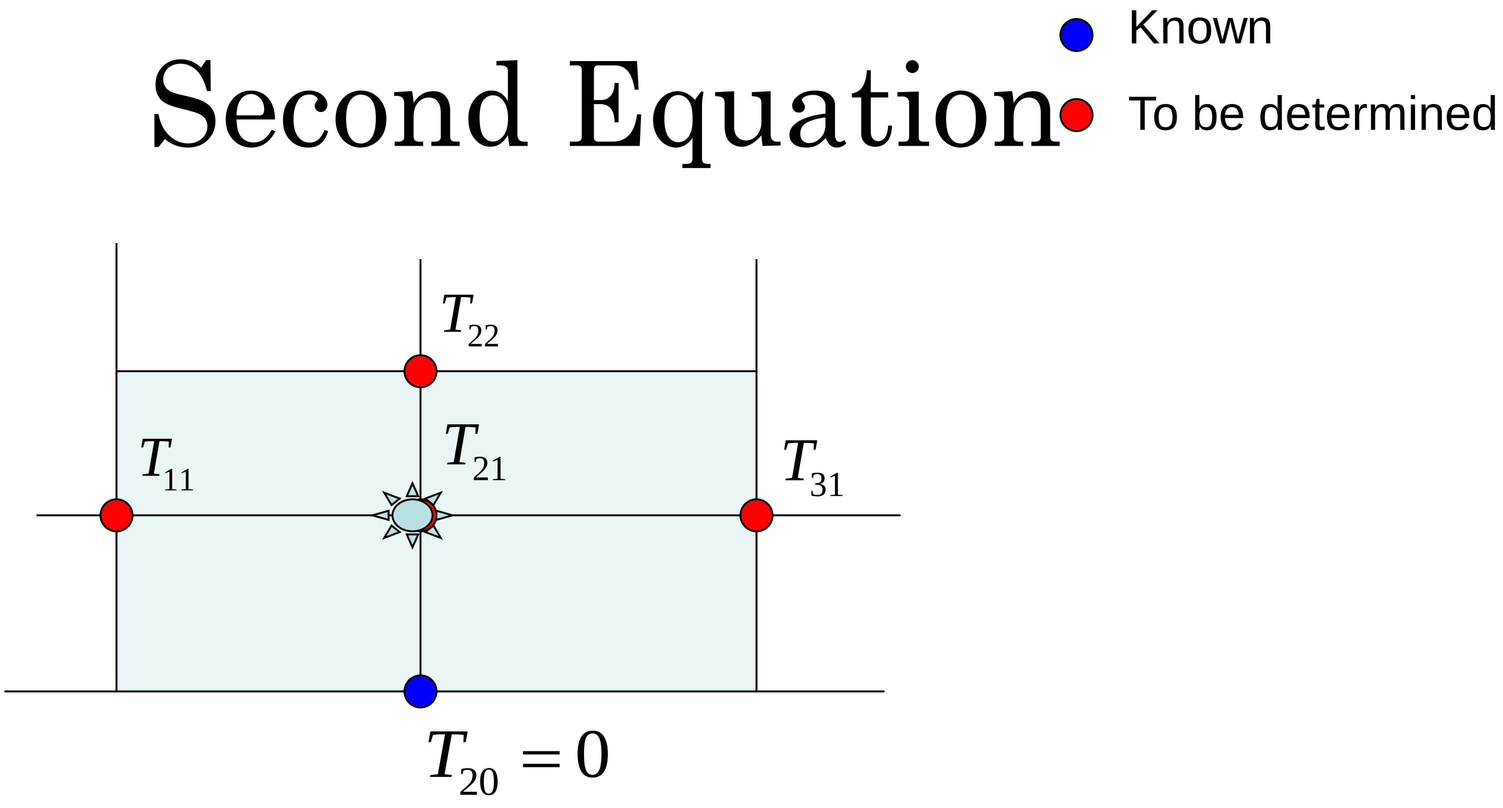



$$T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0$$

$$T_{21} + 75 + T_{12} + 0 - 4T_{11} = 0$$

$$4T_{11} - T_{21} - T_{12} = 75$$

# Second Equation




$$T_{31} + T_{11} + T_{22} + T_{20} - 4T_{21} = 0$$

$$T_{31} + T_{11} + T_{22} + 0 - 4T_{21} = 0$$

$$-T_{11} + 4T_{21} - T_{31} - T_{22} = 0$$

Full system:

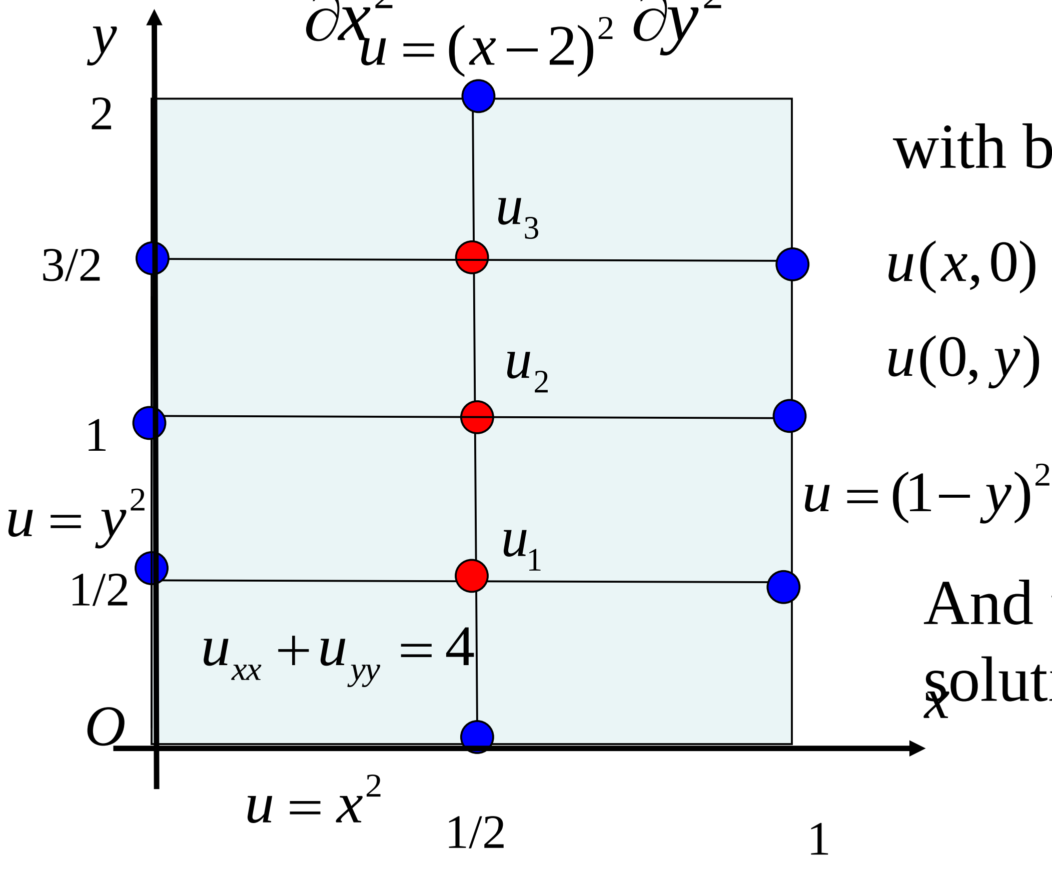
$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{pmatrix} = \begin{pmatrix} 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150 \end{pmatrix}$$

$$\boxed{\begin{array}{lll} T_{11} = 43.00061 & T_{21} = 33.29755 & T_{31} = 33.88506 \\ T_{12} = 63.21152 & T_{22} = 56.11238 & T_{32} = 52.33999 \\ T_{13} = 78.58718 & T_{23} = 76.06402 & T_{33} = 69.71050 \end{array}}$$

# Example 2

Use a finite-difference method with $\Delta x = \Delta y = 1/2$ to compute Approximate values of solution of the boundary-value problem

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 4, \qquad 0 < x < 1,\ 0 < y < 2$$




with boundary conditions

$$u(x,0) = x^2, \quad u(x,2) = (x-2)^2, \quad 0 \le x \le 1$$

$$u(0,y) = y^2, \quad u(1,y) = (y-1)^2, \quad 0 \le y \le 2$$

And find the errors, given the exact solution:

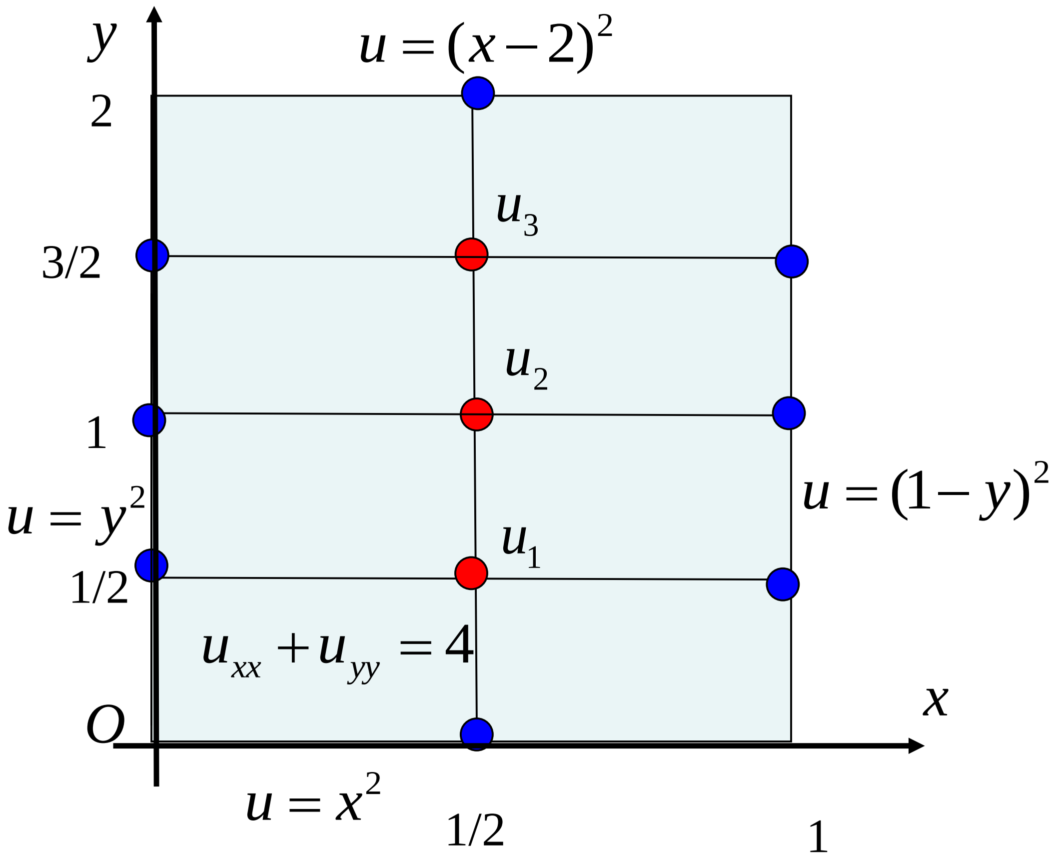
$$u(x,y) = (x-y)^2$$

Solution

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j} = 1$$

$$x_i = 1/2, y_j = j/2, \ 1 \le i \le 2, \ \ 1 \le j \le 4$$

Set $u_1 = u_{11}, u_2 = u_{12}, u_3 = u_{13}$




$$1/4 + 1/4 + 1/4 + u_2 - 4u_1 = 1$$

$$-4u_1 + u_2 = 1/4 \qquad (1)$$

$$0 + 1 + u_3 + u_1 - 4u_2 = 1$$

$$u_1 - 4u_2 + u_3 = 0 \qquad (2)$$

$$1/4 + 9/4 + 9/4 + u_2 - 4u_3 = 1$$

$$u_2 - 4u_3 = -15/4 \qquad (3)$$

From (1), (2) and (3): Au=b

$$(u_1, u_2, u_3) = (0, 1/4, 1)$$

Exact values $u = (0, 1/4, 1)$

errors $= 0$

# Exercise

Approximate solution of the elliptic equation

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=(x^2+y^2)e^{xy}, \qquad 0<x<2,\ 0<y<1$$

with boundary conditions

$$u(0,y)=1, \quad u(2,y)=e^{2y}, \quad 0\le y\le 1$$

$$u(x,0)=1, \quad u(x,1)=e^{x}, \quad 0\le x\le 2$$

with $\Delta x = \Delta y = 1/2$ and find the error. Exact solution:

$$u(x,y)=e^{xy}$$

# Exercise 2

Using $\Delta x = \Delta y = 1/3$, find approximate solution of the boundary value problem

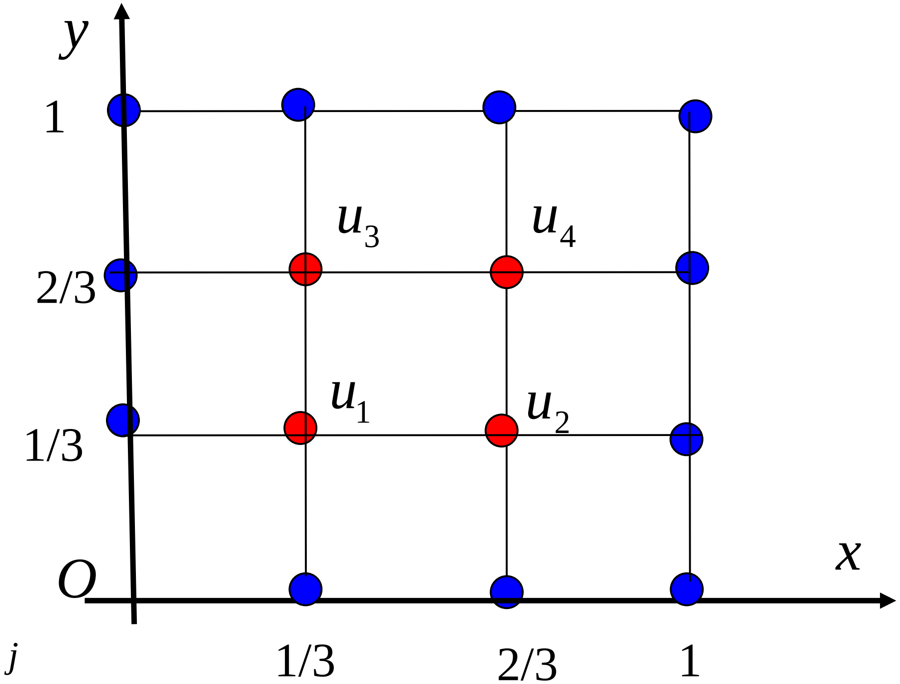
$$u_{xx} + u_{yy} = 6 + e^{x}, \qquad 0 < x < 1,\ 0 < y < 1$$

$$u(0, y) = 2y^2 + 1, \quad u(1, y) = 2y^2 + e + 1, \quad 0 \le y \le 1$$

$$u(x, 0) = x^2 + e^{x}, \quad u(x, 1) = x^2 + e^{x} + 2, \quad 0 \le x \le 1$$

Find the error given exact solution:

$$u(x, y) = x^2 + 2y^2 + e^{x}$$




$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j}$$

# Exercise 3

Using $\Delta x = \Delta y = 1/3$, find approximate solution of the boundary value problem

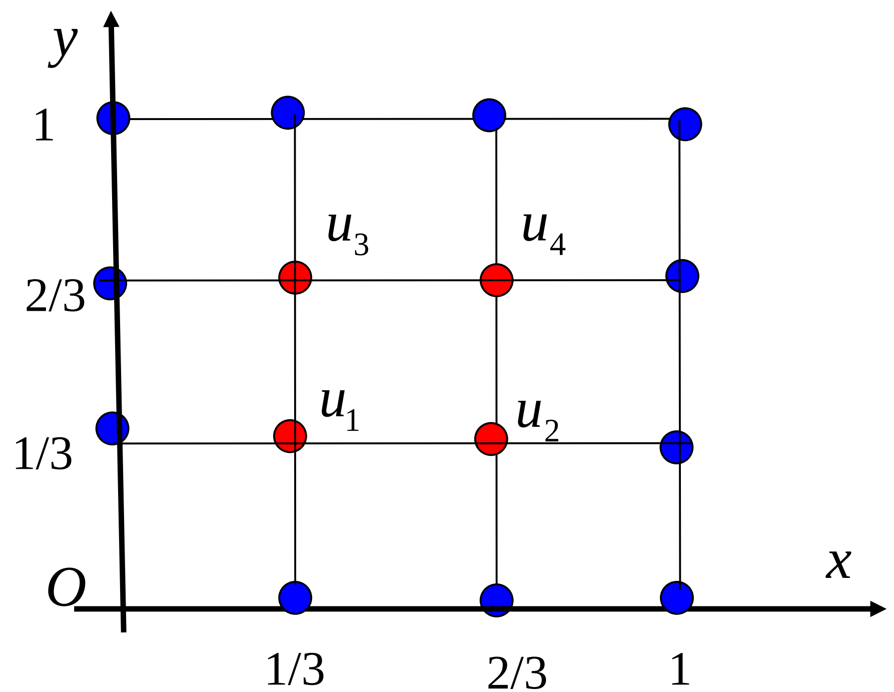
$$u_{xx} + u_{yy} - u_x = 6 - 2x, \qquad 0 < x < 1,\ 0 < y < 1$$

$$u(0, y) = 2y^2 + 1, \quad u(1, y) = 2y^2 + e + 1, \quad 0 \le y \le 1$$

$$u(x, 0) = x^2 + e^x, \quad u(x, 1) = x^2 + e^x + 2, \quad 0 \le x \le 1$$

Find the error given exact solution:

$$u(x, y) = x^2 + 2y^2 + e^x$$



# Homework Chapter 6

$\overline{(m-2)(n-2)}$ is the two last digits of your student ID number

**Problem 1**: Approximate solution of the elliptic equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = (mx^2 + ny^2)e^{xy}, \qquad 0 < x < 2,\ 0 < y < 1$$

with boundary conditions

$$u(0,y) = m, \quad u(2,y) = e^{2y}, \quad 0 \le y \le 1$$

$$u(x,0) = n, \quad u(x,1) = e^{x}, \quad 0 \le x \le 2$$

with $\Delta x=2/3$, $\Delta y=1/3$ and find the error.

# Homework Chapter 6

**Problem 2:** Approximate solution of the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{m}{5n} \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < 1, \quad t > 0$$

with boundary conditions $$u(0,t) = u(1,t) = 0, \quad t > 0$$

and initial condition $$u(x,0) = 2\sin(2\pi x), \qquad 0 \le x \le 1$$

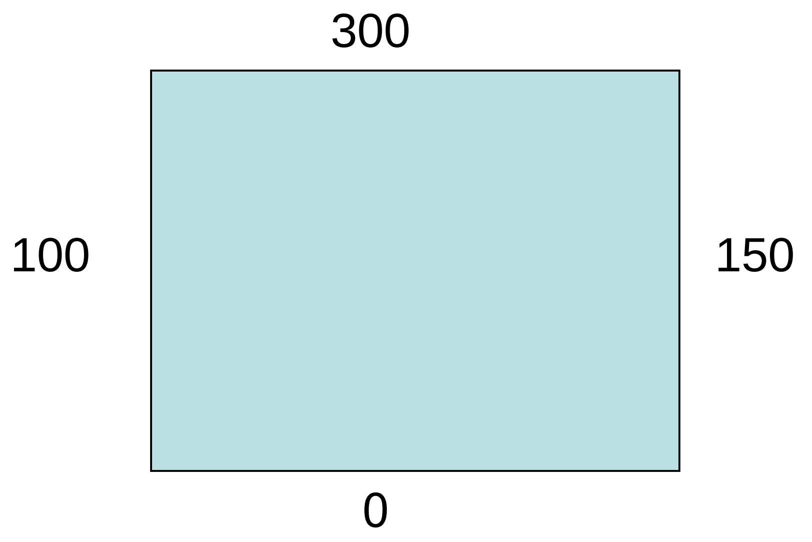
at the time t=0.1 and t=0.2 with $\Delta$x=0.2, $\Delta$t=0.1 using

a) explicit method and
b) implicit Crank-Nicholson method

# Homework Chapter 6: Problem 3

Determine the steady state temperature at all points of a heated sheet of metal. The edges of the sheet are kept at a constant temperature: 300, 150, 100, and 0 degrees as in Figure below.

The sheet is divided to 5X5 grids.




**Deadline: 2 weeks**