INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

MIDTERM EXAMINATION

April 2022

Duration: 100 minutes

SUBJECT: REAL ANALYSIS	
Head of Department of Mathematics:	Lecturer:
Prof. Pham Huu Anh Ngoc	Assoc. Prof. Nguyen Ngoc Hai

INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 For $x = (x_1, x_2) \in \mathbb{R}^2$, define

$$\varphi(x) = \max\{|x_1|, |x_1 + x_2|\}, \qquad \xi(x) = |x_2 - x_1|.$$

(a) *(35 marks)* Show that φ is a norm on $\mathbb{R}^2$. Sketch the closed unit ball $B = \{x \in \mathbb{R}^2 : \varphi(x) \leq 1\}$.

(b) *(5 marks)* Is ξ a norm on $\mathbb{R}^2$? Explain.

Question 2 *(30 marks)* Which of the following subsets of $\mathbb{R}^2$ are compact? Explain.

$$A = (-\infty, 1] \times [0, 1], \qquad B = [0, 1] \times [0, 1), \qquad C = [0, 1] \times [0, 1].$$

Question 3 *(20 marks)* Let $X = C([0, 1])$ be equipped with the norm

$$\|x\| = \max_{0 \leq t \leq 1} |x(t)|.$$

Show that the function $\Phi : X \to \mathbb{R}$ defined by

$$\Phi(x) = x\left(\tfrac{1}{2}\right) + \int_0^1 |x(t)|dt$$

is Lipschitz continuous.

(*Hint:* Show that $|\Phi(x) - \Phi(y)| \leq 2\|x - y\|$ for all $x, y \in X$.)

Question 4 *(10 marks)* Let A be an infinite subset of a metric space (X, d) such that $d(x, y) \geq 1$ when x and y are different points in A. Prove that A is not compact.

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