VIETNAM NATIONAL UNIVERSITY-HCMC
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Chapter 3. Vector spaces

Linear Algebra

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Contents

Introduction

- Many physical quantities, such as area, length, mass, and temperature, are completely described once the magnitude of the quantity is given. Such quantities are called scalars.
- Other physical quantities are not completely determined until both a magnitude and a direction are specified. These quantities are called vectors.
- The goal of this section is to explain and use four ideas:

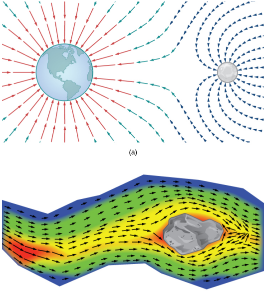
  1. Linear independence or dependence.
  2. Spanning a subspace.
  3. Basis for a subspace (a set of vectors).
  4. Dimension of a subspace (a number).

- Main references: Textbook by Kolman & Hill (Chapter: 4, 6); Textbook by D. Lay.

Recall: What is a Vector?

- A vector, represented by an arrow, has both a direction and a magnitude. Magnitude is shown as the length of a line segment. Direction is shown by the orientation of the line segment, and by an arrow at one end.

- Equal vectors have the same length and direction but may have different starting points.
- Examples of vectors in nature are velocity, force, electromagnetic fields,...

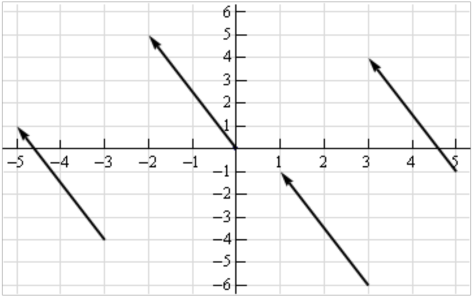


(a)

(b)

Recall: What is a Vector?

Consider the figure below:



Each of the directed line segments in the sketch represents the same vector. In each case the vector starts at a specific point then moves 2 units to the left and 5 units up.

Notation: $\vec{v} = \langle -2, 5 \rangle$ or $\vec{v} = (-2, 5)$.

Recall: Vectors

- Given the two points $A(a_1, a_2)$ and $B(b_1, b_2)$, the vector with the representation $\vec{AB}$ is $\vec{AB} = (b_1 - a_1, b_2 - a_2)$.
- The magnitude, or length, of the vector $\vec{v} = (a, b)$ is given by,

$$\|\vec{v}\| = \sqrt{a^2 + b^2}$$

- Example, if $\vec{v} = (-3, 5)$ then its magnitude

$$\|\vec{v}\| = \sqrt{9 + 16} = 5$$

- Any vector with magnitude of 1 is called a unit vector, e.g., $\vec{v}_1 = (0, 1)$, or $\vec{v}_2 = (1, 0)$ (standard basis vectors).
- Zero Vector, $\vec{0} = (0, 0)$, is a vector that has no magnitude or direction.

Vector Spaces

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below, for all $u, v, w \in V$ and for all c, d scalars:

1. The sum of u and v, denoted by $u + v$, is in V.
2. $u + v = v + u$.
3. $(u + v) + w = u + (v + w)$.
4. There is a zero vector 0 in V such that $u + 0 = u$.
5. For each u in V, there is a vector $-u$ in V such that $u + (-u) = 0$.
6. The scalar multiple of u by c, denoted by cu, is in V.
7. $c(u + v) = cu + cv$.
8. $(c + d)u = cu + du$.
9. $c(du) = (cd)u$.
10. $1u = u$.

Vector Spaces

- Technically, V is a real vector space. All of the theory in this chapter also holds for a complex vector space in which the scalars are complex numbers. From now on, all scalars are assumed to be real.
- The zero vector in Axiom 4 is unique. The vector $-u$ called the negative vector of u.

Properties

For each u in V and scalar c,
1. $0u = 0$
2. $c0 = 0$, where 0 is vector 0.
3. $-u = (-1)u$.

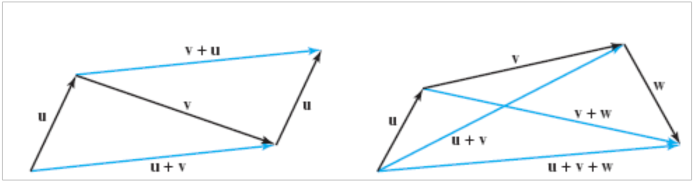
Vector Spaces

Example: Three-dimensional vector space

Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction.

Define addition by the parallelogram rule and for each v in V , define cv to be the arrow whose length is $|c|$ times the length of v, pointing in the same direction as v if $c > 0$ and otherwise pointing in the opposite direction.

Show that V is a vector space. This space is a common model in physical problems for various forces



Vector Spaces

Spaces of Matrices

The set of all $m \times n$ matrices with matrix addition and multiplication of a matrix by a real number (scalar multiplication), is a vector space (verify). We denote this vector space by M_{mn}.

Vector Spaces

Example: Vector space of Matrices with zero trace

Let V be the set of all 2×2 matrices with trace equal to zero, that is,

$$V = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : Tr\left(A\right) = a + d = 0 \right\}$$

V is a vector space with the standard matrix addition, and the standard scalar multiplication of matrices.

Vector Spaces

Example: n-dimensional vector space

Let R_n be the set of all vector in the following form

$$u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

This is the set of all matrices of size $n \times 1$, a specific case of the previous example. So R_n is a vector space.

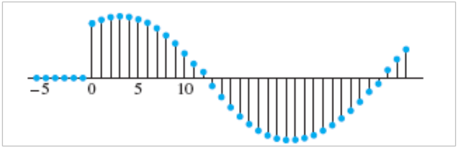
Vector Spaces

Example: discrete-time signals

Let S be the space of all doubly infinite sequences of numbers (usually written in a row rather than a column) with operations

$$\{y_k\} + \{z_k\} = \{y_k + z_k\}\,;\, c\,\{y_k\} = \{cy_k\}$$

Elements of S arise in engineering, for example, whenever a signal is measured (or sampled) at discrete times. A signal might be electrical, mechanical, optical, and so on. For convenience, we will call S the space of (discrete-time) signals.



Vector Spaces

Example: The vector spaces of polynomials of degree nth

For $n > 0$, the set P_n of polynomials of degree at most n consists of all polynomials of the form

$$p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n,$$

where the coefficients $a_0, a_1, ..., a_n$ and the variable x are real numbers. If all the coefficients are zero, p is called the *zero polynomial*.

If $q(x) = b_0 + b_1x + ... + b_nx^n$, then we define

$$(p + q)(x) = p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + ... + (a_n + b_n)x^n$$

$$(cp)(x) = cp(x) = ca_0 + (ca_1)x + (ca_2)x^2 + ... + (ca_n)x^n$$

Then P_n is a vector space.

Vector Spaces

Example: The vector space of all real-valued functions

Let V be the set of all real-valued functions defined on a set D. (Typically, D is the set of real numbers or some interval on the real line.) Functions are added in the usual way

$$(f+g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x)$$

Two functions in V are equal if and only if their values are equal for every t in D.

Hence the zero vector in V is the function that is identically zero, $f(t) = 0$ for all t, and the negative of f is $(-1)f$. Axioms 1 and 6 are obviously true, and the other axioms follow from properties of the real numbers, so V is a vector space.

Subspaces

Definition

A subspace of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

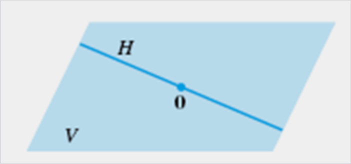
b. H is closed under vector addition. That is, for each u and v in H, the sum $u + v$ is in H.

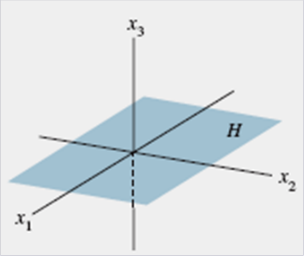
c. H is closed under multiplication by scalars. That is, for each u in H and each scalar c, the vector cu is in H.

Properties $(a), (b)$, and (c) guarantee that a subspace H of V is itself a vector space, under the vector space operations already defined in V.

Subspaces

Example





(a) A line H through 0 is a subspace of V=R^2.
(b)The xy-plane defined as:

$$H = \{(a, b, 0) : a, b \in \mathbb{R}\}$$

as a subspace of $\mathbb{R}^3$.

Subspaces

Example

The set consisting of only the zero vector in a vector space V is a subspace of V, called the zero subspace and written as $\{0\}$.

Example

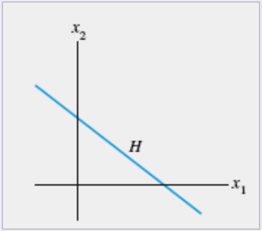
Let P_2 be the set of all polynomials of egree ≤ 2 . Then P_2 is a subspace of P, the space of all polynomials. Also, for each $n > 0$, P_n is a subspace of P.

Subspaces

Example

A linear line not through the origin is not a subspace of $\mathbb{R}^2$.

A plane in $\mathbb{R}^3$ not through the origin is not a subspace of $\mathbb{R}^3$.



Linear combination

Definition

Let $v_1, v_2, ..., v_k$ be vectors in a vector space V. A vector v in V is called a linear combination of $v_1, v_2, ..., v_k$ if

$$v = a_1 v_1 + a_2 v_2 + + ... + a_k v_k = \sum_{j=1}^{k} a_j v_j$$

Linear combination

Example

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

then every $w = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$ is a linear combination of v_1, v_2 since $w = av_1 + bv_2$.

Subspace

Example

Which of the given subsets of the vector space P_2 are subspace?

(a) $a_2t^2 + a_1t + a_0, \quad \text{where } a_1 = 0, a_0 = 0$

(b) $a_2t^2 + a_1t + a_0, \quad \text{where } a_1 = 2a_0$

(c) $a_2t^2 + a_1t + a_0, \quad \text{where } a_2 + a_1 + a_0 = 2$

Answers: (a) and (b).

Null space

Example

if A is an $m \times n$ matrix, then the homogeneous system of m equations in n unknowns with coefficient matrix A can be written as

$$Ax = 0$$

where x is a vector in $\mathbb{R}^n$ and 0 is the zero vector. Show that set W of all solutions is a subspace of $\mathbb{R}^n$.

Hint: Let x and y be solutions. Then $x + y \in W$ and $tx \in W$ for any scalar t.

W is called the solution space of the homogeneous system, or the null space of the matrix A.

Subspace

Exercises

Let W be the set of all 3×3 matrices of the form

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$$

Show that W is a subspace of M_{33}.

Subspace

Exercises

Which of the following subsets of the vector space M_{mn} are subspaces?
(a) The set of all $n \times n$ symmetric matrices.
(b) The set of all $n \times n$ diagonal matrices.
(c) The set of all $n \times n$ nonsingular matrices.

Answer: (a) and (b).
(c) Find the nonsingular matrices A and B such that $A + B$ is singular.

A Subspace Spanned by a Set

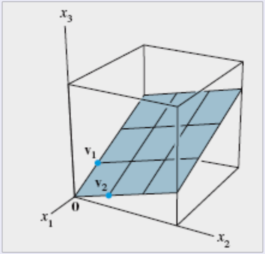
The next example illustrates one of the most common ways of describing a subspace.

Example

Given v_1 and v_2 in a vector space V, let

$$H = span\{v_1, v_2\} = \{av_1 + bv_2 : a, b \in \mathbb{R}\}$$

Show that H is a subspace of V.



A Subspace Spanned by a Set

SOLUTION The zero vector is in H, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$. To show that H is closed under vector addition, take two arbitrary vectors in H, say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

By Axioms 2, 3, and 8 for the vector space V,

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \\ &= (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2 \end{aligned}$$

So $\mathbf{u} + \mathbf{w}$ is in H. Furthermore, if c is any scalar, then by Axioms 7 and 9,

$$c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$$

which shows that $c\mathbf{u}$ is in H and H is closed under scalar multiplication. Thus H is a subspace of V. ∎

A Subspace Spanned by a Set

Theorem

If $v_1, v_2, ..., v_p$ are in a vector space V , then

$$span\{v_1, v_2, ..., v_p\} = \{c_1 v_1 + c_2 v_2 + ... + c_p v_p : c_j \in \mathbb{R}\}$$

is a subspace of V.

Proof...

We call $span\{v_1, v_2, ..., v_p\}$ the subspace spanned (or generated) by $\{v_1, v_2, ..., v_p\}$

A Subspace Spanned by a Set

Example

Consider the set S of 2×3 matrices given by

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right\}$$

Then span S is the set in M_{23} consisting of all vectors of the form

$$\begin{bmatrix} a_1 & a_2 & 0 \\ 0 & a_3 & a_4 \end{bmatrix}$$

where $a_j \in \mathbb{R}$.

A Subspace Spanned by a Set

Example

Let $S = \{t^2, t, 1\}$, then we have span S be a subset of P_2.
Then span S is the subspace of all polynomials of the form $a_2t^2 + a_1t + a_0$.

Example

Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then span S is the subspace of all all 2×2 diagonal matrices .

A Subspace Spanned by a Set

Example

Let

$$H = \left\{ (a-3b, b-a, a, b)^T : a, b \in \mathbb{R} \right\}$$

Show that H is a subspace of $\mathbb{R}^4$.

Proof:

An arbitrary vector in H has the form

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $H = span\{v_1, v_2\}$, where

$$v_1 = (1, -1, 1, 0)^T, \; v_2 = (-3, 1, 0, 1)^T.$$

Hence H is a subspace of $\mathbb{R}^4$.

Span and Linear combination

Remarks:

$$v \in span\{v_1, v_2, ..., v_k\} \Leftrightarrow \text{ There exist } a_j \text{ such that } v = \sum_{j=1}^{k} a_j v_j.$$

Or, $v \in span\{v_1, v_2, ..., v_k\}$ iff v is a linear combination of $v_1, v_2, ..., v_k$.

A Subspace Spanned by a Set

Example

Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Is $v \in span\{v_1, v_2, ..., v_p\}$?

Outline solution: Find a_1, a_2, a_3 such that

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Solve this linear system to obtain $a_1 = 1, a_2 = 2, a_3 = -1$.
Thus, $v = v_1 + 2v_2 - v_3$ so $v \in span\{v_1, v_2, ..., v_p\}$

A Subspace Spanned by a Set

Example

For what value(s) of a will y be in the subspace of $\mathbb{R}^3$ spanned by v_1, v_2, v_3, if

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \text{ and } y = \begin{bmatrix} -4 \\ 3 \\ a \end{bmatrix}$$

Answer: $a = 5$.

A Subspace Spanned by a Set

Example

In P_2 let

$$v_1 = 2t^2 + t + 2, v_2 = t^2 - 2t, v_3 = 5t^2 - 5t + 2, v_4 = -t^2 - 3t - 2$$

Determine whether the vector

$$v = t^2 + t + 2$$

belongs to span $\{v_1, v_2, v_3, v_4\}$.

A Subspace Spanned by a Set

Solution

Find scalars a_1, a_2, a_3, a_4 such that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = v$$

$$(2a_1 + a_2 + 5a_3 - a_4)\, t^2 + (a_1 - 2a_2 - 5a_3 - 3a_4)\, t + (a_1 + 2a_3 - 2a_4)$$
$$= t^2 + t + 2$$

Thus we get the linear system:

$$2a_1 + a_2 + 5a_3 - a_4 = 1$$

$$a_1 - 2a_2 - 5a_3 - 3a_4 = 1$$

$$a_1 + 2a_3 - 2a_4 = 2$$

A Subspace Spanned by a Set

Solution

To determine whether this system of linear equations is consistent. We form the augmented matrix and transform it to reduced row echelon form, obtaining (verify)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

which indicates that the system is inconsistent; that is, it has no solution. Hence v does not belong to span $\{v_1, v_2, v_3, v_4\}$.

A Subspace Spanned by a Set

Example

Let V be the vector space R^3. Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Show that $span\{v_1, v_2, v_3\} = \mathbb{R}^3$.

Solution:

Pick up any

$$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in V$$

A Subspace Spanned by a Set

Solution (Cont.):
This leads to the linear system

$$a_1 + a_2 + a_3 = a$$

$$2a_1 + a_3 = b$$

$$a_1 + 2a_2 = c$$

A solution is (verify)

$$a_1 = \frac{-2a + 2b + c}{3}, a_2 = \frac{a - b + c}{3}, a_3 = \frac{4a - b - 2c}{3}$$

A Subspace Spanned by a Set

Example

Explain why the set S is not a spanning set for the vector space V.

(a) $S = \left\{t^3, t^2, t\right\}, V = P_3$

(b)

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, V = \mathbb{R}^2$$

(c)

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}, V = M_{22}$$

Linear Independence

Definition

The vectors $v_1, v_2, v_3, ..., v_k$ in a vector space V are said to be linearly dependent if there exist constants $a_1, a_2, ..., a_k$, not all zero, such that

$$\sum_{j=1}^{k} a_j v_j = 0$$

Otherwise, $v_1, v_2, v_3, ..., v_k$ are called linearly independent.

That is, $v_1, v_2, v_3, ..., v_k$ are linearly independent if $\sum\limits_{j=1}^{k} a_j v_j = 0 \Leftrightarrow a_j = 0, \forall j = 1, .., k.$

Linear Independence

Example

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

are linearly dependent since $v_1 + v_2 - v_3 = 0$.

Example

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

are linearly independent since $av_1 + bv_2 = 0$ iff $a = b = 0$.

Linear Independence

Example

Determine whether the vectors

$$v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

are linearly independent.

Solution

Forming equation:

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$$

$$a_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Linear Independence

We obtain the homogeneous system

$$3a_1 + a_2 - a_3 = 0$$

$$2a_1 + 2a_2 + 2a_3 = 0$$

$$a_1 - a_3 = 0$$

Doing the row operations

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{Row\ \ operations} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The nontrivial solution is

$$k\,(1, -2, 1)^T, k \neq 0$$

so the vectors are linearly dependent!

Linear Independence

Example

Determine whether the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

are linearly independent.

Solution

Forming equation:

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Linear Independence

Doing the row operations

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus the only solution is the trivial solution $a_1 = a_2 = a_3 = 0$, so the vectors are linearly independent.

Linear Independence

Example

Are the vectors

$$v_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$$

in M_{22} linearly independent?

Solution

Forming equation:

$$a_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Linear Independence

$$\begin{bmatrix} 2a_1 + a_2 & a_1 + 2a_2 + a_3 \\ a_2 - 2a_3 & a_1 + a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving the linear system to find a_j:

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{Row\ \ operations} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The nontrivial solution is

$$k\,(-1,2,1)^T\,, k \neq 0$$

so the vectors are linearly dependent.

Linear Independence

Example

Are the vectors

$$v_1 = t^2 + t + 2, v_2 = 2t^2 + t, v_3 = 3t^2 + 2t + 2$$

in P_2 linearly independent?

Answer The given vectors are linearly dependent

Linear Independence

Theorem

Let $S = \{v_1, v_2, ..., v_n\}$ be a set of n vectors in $\mathbb{R}^n$ (R_n). Let A be the matrix whose columns (rows) are the elements of S. Then S is linearly independent if and only if $\det(A) \neq 0$.

Proof

We will prove the result for column-vectors.

Suppose that S is linearly independent. Then it follows that the reduced row echelon form of A is I_n. Thus, A is row equivalent to I_n, and hence $\det(A) \neq 0$.

Conversely, if $\det(A) \neq 0$, then A is row equivalent to I_n. Hence, the rows of A are linearly independent.

Linear Independence

Example

Is $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$ a linearly independent set of vector in $\mathbf{R}^3$?

Solution

We form the matrix A whose columns are the vectors in S:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix}$$

$$\det(A) = 2$$

So S is linearly independent.

Linear Independence

Theorem

Let S_1 and S_2 be finite subsets of a vector space and let S_1 be a subset of S_2. Then the following statements are true:

(a) If S_1 is linearly dependent, so is S_2.

(b) If S_2 is linearly independent, so is S_1.

Proof Let

$$S_1 = \{v_1, v_2, ..., v_k\}, S_2 = \{v_1, v_2, ..., v_k, v_{k+1}, .., v_m\}$$

(a) Since S_1 is linearly dependent, there exist constants $a_1, a_2, ..., a_k$, not all zero, such that

$$\sum_{j=1}^{k} a_j v_j = 0$$

Linear Independence

Proof (Cont.) Therefore,

$$a_1v_1 + a_2v_2 + \ldots + a_kv_k + 0v_{k+1} + \ldots + 0v_m = 0$$

Since not all the coefficients in the equations above are zero, we conclude that S_2 is linearly dependent.

Statement (b) is the contrapositive of statement (a), so it is logically equivalent to statement (a).

Linear Independence

Remarks

- The set $S = \{0\}$ consisting only of 0 is linearly dependent.

  From this it follows that if S is any set of vectors that contains 0, then S must be linearly dependent.

- A set of vectors consisting of a single nonzero vector is linearly independent.

- If $v_1, v_2, ..., v_k$ are vectors in a vector space V and any two of them are equal, then $v_1, v_2, ..., v_k$ are linearly dependent.

Linear Independence

Theorem

The nonzero vectors $v_1, v_2, \ldots, v_k$ in a vector space V are linearly dependent if and only if one of the vectors $v_j (j \geq 2)$ is a linear combination of the preceding vectors $v_1, v_2, \ldots, v_{j-1}$.

Example

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 + 0v_3 - v_4 = 0$$

so v_1, v_2, v_3, and v_4 are linearly dependent. We then have

$$v_4 = v_1 + v_2 + 0v_3.$$

Basis and dimensions

Definition

The vectors $v_1, v_2, ..., v_k$ in a vector space V are said to form a basis for V if

(a) $v_1, v_2, ..., v_k$ span V and

(b) $v_1, v_2, ..., v_k$ are linearly independent.

Example

Let $V = \mathbb{R}^3$. The vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$, called the natural basis or standard basis for $\mathbb{R}^3$.

Basis and dimensions

Example

Generally, the natural basis or standard basis for $\mathbb{R}^n$ is denoted by

$$\{e_1, e_2, ..., e_n\}$$

where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Basis and dimensions

Example

Show that the set

$$S = \left\{t^2 + 1, t - 1, 2t + 2\right\}$$

is a basis for the vector space P_2.

Solution We must show that S spans V and is linearly independent.

To show that it spans V, we take any vector in V, that is a polynomial $at^2 + bt + c$ and find constants a_1, a_2 and a_3 such that

$$at^2 + bt + c = a_1\left(t^2 + 1\right) + a_2\left(t - 1\right) + a_3\left(2t + 2\right)$$

Basis and dimensions

Solution (cont.)

$$a_1 = a$$
$$a_2 = \frac{a+b-c}{2}$$
$$a_3 = \frac{c+b-a}{4}$$

Hence S spans V.
To show that S is linearly independent, we form

$$a_1\left(t^2+1\right) + a_2\left(t-1\right) + a_3\left(2t+2\right) = 0$$
$$a_1 t^2 + \left(a_2 + 2a_3\right)t + \left(a_1 - a_2 + 2a_3\right) = 0$$

This can hold for all values of t only if

$$a_1 = a_2 + 2a_3 = a_1 - a_2 + 2a_3 = 0$$

Thus $a_1 = a_2 = a_3 = 0$.
Remark: The set of vectors $S = \left\{t^n, t^{n-1}, .., t, 1\right\}$ forms a basis for the vector space P_n called the natural, or standard basis, for P_n.

Basis and dimensions

Example

Show that the set

$$S = \{v_1, v_2, v_3, v_4\}$$

where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is a basis for the vector space $\mathbb{R}^4$.

Basis and dimensions

Hint

We need to show that S spans $\mathbb{R}^4$ and S is linearly independent.
(a) To show that S spans $\mathbb{R}^4$, we let

$$v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

in $\mathbb{R}^4$ and find a_1, a_2, a_3 and a_4 such that $v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$.
(b) S is linearly independent since $\det(A) = 1$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

Basis and dimensions

Example

Find a basis for the subspace V of P_2, consisting of all vectors of the form at $at^2 + bt + c$ where $c = a - b$.

Hint:

$$S = \left\{ t^2 + 1, t - 1 \right\}$$

Basis and dimensions

Remarks

A vector space V is called finite-dimensional if there is a finite subset of V that is a basis for V. If there is no such finite subset of V, then V is called infinite-dimensional.

Basis and dimensions

Theorem

If $S = \{v_1, v_2, .., v_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of the vectors in S.

Proof

Suppose

$$v = a_1 v_1 + a_2 v_2 + ... + a_n v_n, \text{ and } v = b_1 v_1 + b_2 v_2 + ... + b_n v_n$$

This implies

$$(a_1 - b_1) v_1 + (a_2 - b_2) v_2 + ... + (a_n - b_n) v_n = 0$$

Since S is linearly independent, we have $a_j = b_j$ for $j = 1, 2, ..., n$.

Basis and dimensions

Theorem

Let $S = \{v_1, v_2, .., v_n\}$ be a set of nonzero vectors in a vector space V and let $W = span\ S$. Then some subset of S is a basis for W.

Proof

Finding a basis

Let $S = \{v_1, v_2, .., v_n\}$ be a set of nonzero vectors in a vector space V. We find a subset T of S that is a basis for $W = span\ S$.

Procedure for finding a basis from a subset

- Form the equation

$$a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0$$

- Construct the augmented matrix and transform it to reduced row echelon form.
- The vectors corresponding to the columns containing the leading 1's form a basis T for $W = span\ S$.

Finding a basis

Example

Find a basis for $S = \{v_1, v_2, .., v_5\} := Col\ A$, where

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding a basis

Solution

Each nonpivot column of A is a linear combination of the pivot columns. In fact, $v_2 = 4v_1$, $v_4 = 2v_1 - v_3$.

Let

$$S = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Since no vector in S is a linear combination of the vectors that precede it, S is linearly independent.

Thus S is a basis for Col A.

Finding a basis

Example

Let

$$v_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, v_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}.$$

Find a basis for the subspace W spanned by $\{v_1, v_2, v_3, v_4\}$.

Solution

We have

$$A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two column are the pivot columns. Hence $S = \{v_1, v_2\}$ is a basis for the subspace W spanned by $\{v_1, v_2, v_3, v_4\}$.

Basis

Theorem

If $S = \{v_1, v_2, .., v_n\}$ is a basis for a vector space V and $T = \{w_1, w_2, .., w_m\}$ is a linearly independent set of vectors in V, then $m \leq n$.

Corollary

If $S = \{v_1, v_2, .., v_n\}$ and $T = \{w_1, w_2, .., w_m\}$ are bases for a vector V, then $n = m$.

Q: Does numbers of vectors in basis tell us the dimension of V?

A: Yes, it does!

Dimensions

Definition

The dimension of a nonzero vector space V is the number of vectors in a basis for V. We often write *dim* V for the dimension of V.

We also define the dimension of the trivial vector space $\{0\}$ to be zero.

Example

If $S = \left\{t^2, t, 1\right\}$ is a basis for P_2, so $dim\ P_2 = 3$.

Basis

Definition

Let S be a set of vectors in a vector space V. A subset T of S is called a maximal independent subset of S if T is a linearly independent set of vectors that is not properly contained in any other linearly independent subset of S.

Example

Let

$$S = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Then maximal independent subsets of S are $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_1, v_3\}$.

Basis

Theorem

Let V be an n-dimensional vector space.

- If $S = \{v_1, v_2, ..., v_n\}$ is a linearly independent set of vectors in V. then S is a basis for V.
- If $S = \{v_1, v_2, ..., v_n\}$ spans V, then S is a basis for V.

Theorem

Let S be a finite subset of the vector space V that spans V. A maximal independent subset T of S is a basis for V.

Row Space and column space

Definition

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

be an $m \times n$ matrix.

The rows of A span a subspace of $\mathbb{R}^n$ called the row space of A, denoted by $Row\ A$.

The columns of A span a subspace of $\mathbb{R}^m$ called the column space of A, denoted by $Col\ A$.

Row Space and column space

Theorem

If A and B are two $m \times n$ row (column) equivalent matrices, then the row (column) spaces of A and B are equal.

Example

Find the row space, the null space, and the column space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -13 \end{bmatrix}$$

Row Space and column space

Solution We have

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three rows of B form a basis for the row space of A (as well as the row space of B).
Thus, a basis for *Row A* is

$$\{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$$

Row Space and column space

Solution (Cont.) For the column space, observe from B that the pivots are columns $1, 2, 4$. So a basis for $Col\ A$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

For the $Nul\ A$, we can do further row operations on B, this yields

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$$x_1 = -x_3 - x_5, x_2 = 2x_3 - 3x_5, x_4 = 5x_5,$$

where x_3 and x_5 are free variables

Rank-Nullity

Definition

The dimension of the row (column) space of A is called the row (column) rank of A.

Thus, if A and B are row equivalent, then row rank $A =$ row rank B; and if A and B are column equivalent, then column rank $A =$ column rank B.

Definition

The nullity of A is the dimension of the null space of A, that is, the dimension of the solution space of $Ax = 0$.

Rank-Nullity

Example

Let

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -13 \end{bmatrix}$$

We have

$$A \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, nullity $A = 2$, row rank A=column rank $A = 3$.

Rank-Nullity

Theorem

The row rank and column rank of the $m \times n$ matrix A are equal.

Theorem

If A is an $m \times n$ matrix, then rank A + nullity $A = n$.

Rank-Nullity

Example

Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 7 & 1 & 8 \end{bmatrix}$$

We have

$$A \sim B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, Nullity A=1, rank A = 2 and nullity A + rank A = 3 = numbers of column of A.

Rank-Nullity

Example

Let

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

We have

$$A \sim B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, nullity A=2, rank A = 3 and nullity A + rank A = 5 = numbers of column of A.

Rank and Singularity

The rank of a square matrix can be used to determine whether the matrix is singular or nonsingular.

Theorem

If A is an $n \times n$ matrix, then rank $A = n$ if and only if A is row equivalent to I_n.

Thus,

$$rank\ \ A = n \Leftrightarrow \det(A) \neq 0 \Leftrightarrow nonsingular(invertible)$$

Rank and Singularity

Corollary

(a) A is nonsingular if and only if rank $A = n$.

(b) If A is an $n \times n$ matrix, then *rank* $A = n$ if and only if $\det(A) \neq 0$.

(c) The homogeneous system $Ax = 0$, where A is $n \times n$, has a nontrivial solution if and only if *rank* $A < n$.

(d) Let A be an $n \times n$ matrix. The linear system $Ax = b$ has a unique solution for every $n \times 1$ matrix b if and only if *rank* $A = n$.

Coordinates

Definition: Coordinates

If V is an n-dimensional vector space, we know that V has a basis S with n vectors in it,

$$S = \{v_1, v_2, ..., v_n\}.$$

Every vector $\in V$ can be uniquely expressed in the form:

$$v = a_1 v_1 + a_2 v_2 + ... + a_n v_n.$$

We will refer

$$[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

as the coordinate vector of v with respect to the (ordered) basis S.

Coordinates

Example

Consider the vector space $\mathbb{R}^3$ and let $S = \{v_1, v_2, v_3\}$ be an ordered basis for $\mathbb{R}^3$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

If

$$v = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$$

compute $[v]_S$

Coordinates

Solution

To find $[v]_S$, we need to find the constants a_1, a_2, a_3 such that $a_1 v_1 + a_2 v_2 + a_3 v_3 = v$. Solve the linear system

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -5 \end{array}\right]$$

We get $a_1 = 3, a_2 = -1, a_3 = -2$. Thus,

$$[v]_S = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Isomorphism

Definition

Let V and W be real vector spaces. A one-to-one function L mapping V onto W is called an isomorphism of V onto W if
(a) $L(v + w) = L(v) + L(w)$,
(b) $L(cv) = cL(v)$. In this case we say that V is isomorphic to W.

Theorem

(a) If V is an $n-$dimensional real vector space, then V is isomorphic to $\mathbb{R}^n$.

(b) Two finite-dimensional vector spaces are isomorphic if and only if their dimensions are equal.

Changes of bases-Transition matrices

Thus, let $S = \{v_1, v_2, ..., v_n\}$ and $T = \{w_1, w_2, ..., w_n\}$ be two ordered bases for the $n-$dimensional vector space V. Let v be a vector in V and let

$$[v]_T = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Question: Can we compute $[v]_S$ via $[v]_T$?

We have

$$[v]_S = [c_1 w_1 + c_2 w_2 + ... + c_n w_n]_S$$

$$[v]_S = [c_1 w_1]_S + [c_2 w_2]_S + ... + [c_n w_n]_S$$

$$[v]_S = c_1 [w_1]_S + c_2 [w_2]_S + ... + c_n [w_n]_S$$

Changes of bases-Transition matrices

Let the coordinate vector of w_j with respect to S be denoted by

$$[w_j]_S = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

The $n \times n$ matrix whose jth column is $[w_j]_S$ is called the transition matrix (or the change-of-coordinates matrix) from the T-basis to the S-basis and is denoted by $P_{S \leftarrow T}$. That is,

$$P_{S \leftarrow T} = ([w_1]_S, [w_2]_S, ..., [w_n]_S)$$

Therefore,

$$[v]_S = P_{S \leftarrow T} [v]_T$$

Changes of bases-Transition matrices

Example

Let

$$T = \{w_1, w_2\}, S = \{v_1, v_2\},$$

where

$$w_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

Find the transition matrix $P_{S \leftarrow T}$ from the T-basis to the S-basis.

Solution:

Let

$$[w_1]_S = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, [w_2]_S = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We need to solve the following linear systems

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = w_1, \quad \text{and} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = w_2$$

Changes of bases-Transition matrices

Example

We can solve both systems simultaneously.

$$\left[\begin{array}{cc|cc} v_1 & v_2 & w_1 & w_2 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

$$[w_1]_S = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, [w_2]_S = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Thus,

$$P_{S\leftarrow T} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

Changes of bases-Transition matrices

Example

Let

$$T = \{w_1, w_2, w_3\}, S = \{v_1, v_2, v_3\},$$

where

$$w_1 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the transition matrix $P_{S \leftarrow T}$ from the T-basis to the S-basis.

Changes of bases-Transition matrices

Hint

To find $[w_j]_S$, $j = 1, 2, 3$, we can solve three systems simultaneously

$$\left[\begin{array}{ccc|ccc} v_1 & v_2 & v_3 & w_1 & w_2 & w_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 6 & 4 & 5 \\ 0 & 2 & 1 & 3 & -1 & 5 \\ 1 & 0 & 1 & 3 & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} v_1 & v_2 & v_3 & w_1 & w_2 & w_3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$P_{S \leftarrow T} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Q: Verify $[v]_S = P_{S \leftarrow T} [v]_T$?