EXERCISES FOR CHAPTER 1&2: LINEAR PROGRAMMING

Exercises for everyone: All exercises in parts A and B.

A. Non-assessed Exercises (corrected in class):

0.1.5; 0.1.8; 0.2.4; 0.3.4; 0.3.5; 0.4.1 (d); 0.5.4; 0.6.1 (e); 0.6.5; 0.7.1.

B. Assessed Assignments (to be submitted):

0.1.4; 0.1.6; 0.2.1; 0.3.1; 0.3.3; 0.3.6; 0.4.1 (a), (c); 0.5.1; 0.5.2; 0.5.3; 0.5.5; 0.6.1 (a), (c), (d); 0.6.4; 0.7.2.

C. Bonus Exercises: Remaining exercises.

0.1 PRELIMINARIES

0.1.1 Convex Sets

Exercise 0.1.1. Consider a region S defined by a set of linear constraints

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\},$$

where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Prove that S is convex.

Exercise 0.1.2. Let S be a convex set. Prove that $\mathbf{x} \in S$ is an extreme point of S if and only if $S \setminus \mathbf{x}$ is convex.

Exercise 0.1.3. Prove that if $\mathbf{x}$ and $\mathbf{y}$ are extreme points of a convex set S and $\mathbf{z} = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$ is a point of S, then $0 \leq t \leq 1$.

0.1.2 The Graphical Method

Solve linear programs 0.1.4–0.1.8 graphically.

Exercise 0.1.4. Maximize and minimize the objective function $z = x + 5y$ subject to the constraints

$$\begin{aligned} x + 4y &\leq 12 \\ x &\leq 8 \\ x + y &\geq 2 \\ x, y &\geq 0. \end{aligned}$$

ANS. The maximum value is 15, which occurs at $(0, 3)$; the minimum value is 2, which occurs at $(2, 0)$.

Exercise 0.1.5. •

$$\begin{array}{rr} \text{minimize} & x + 2y \\ \text{subject to} & x + y \geq 1 \\ & 2x + 4y \geq 3 \\ & x, y \geq 0. \end{array}$$

ANS. The minimum value is 3/2 which occurs at any point on the line segment joining $(0.5, 0.5)$ and $(1.5, 0)$.

Exercise 0.1.6. Nutt's Nuts has 75 pounds of cashews and 120 pounds of peanuts available. These are to be mixed in 1-pound packages as follows: a low-grade mixture that contains 4 ounces of cashews and 12 ounces of peanuts and a high-grade mixture that contains 8 ounces of cashews and 8 ounces of peanuts. The profit is \$0.25 on each package of the low-grade mixture and \$0.45 on each package of the high-grade mixture. How many packages of each type of mixture should be prepared to maximize the profit?

ANS. Maximize profit is \$69.75.

Exercise 0.1.7. A couple has up to $\$30,000$ to invest in mutual funds. Their broker recommends investing in two funds based on their average annual return for the 5 years ending on December 31, 2009: the Templeton Global Bond fund yielding 8% and the Franklin International Small Cap Growth fund yielding 12%. After some consideration the couple decides to invest at most $\$12,000$ in the Franklin International Small Cap Growth fund and at least \$6000 in the Templeton Global Bond fund. They also want the amount invested in the Templeton Global Bond fund to exceed or equal the amount invested in the Franklin International Small Cap Growth fund.What should the broker recommend if the couple (quite naturally) wants to maximize the return on their investment?

ANS. The maximum return on investment is \$2880, obtained by placing \$18,000 in the global Bond fund and \$12,000 in the Franklin International Small Cap growth fund.

Exercise 0.1.8. • A factory produces gasoline engines and diesel engines. Each week the factory is obligated to deliver at least 20 gasoline engines and at least 15 diesel engines. Due to physical limitations, however, the factory cannot make more than 60 gasoline engines nor more than 40 diesel engines in any given week. Finally, to prevent layoffs, a total of at least 50 engines must be produced.

(i) If gasoline engines cost \$450 each to produce and diesel engines cost \$550 each to produce, how many of each should be produced per week to minimize the cost?

(ii) What is the excess capacity of the factory? That is, how many of each kind of engine is being produced in excess of the number that the factory is obligated to deliver?

ANS. (i) min cost= \$24,000, with 35 gasoline engines and 15 diesel engines produced.

0.2 PIVOTS

Exercise 0.2.1. Solve the following using the pivot operation.

(a)

$$\begin{aligned} 3x_2 - 3x_3 &= 15 \\ x_1 + x_2 + x_3 &= 0 \\ 3x_1 + 5x_2 + 3x_3 &= 4. \end{aligned}$$

(b)

$$\begin{aligned} 3x_1 + 2x_2 - 7x_3 &= 1 \\ x_1 - 5x_2 - 6x_3 &= -4. \end{aligned}$$

(c)

$$\begin{aligned} x_1 + 2x_2 \qquad - 2x_4 &= 5 \\ -3x_2 + x_3 + 4x_4 &= 2. \end{aligned}$$

Exercise 0.2.2. A system of equations is said to be **redundant** if one of the equations in the system is a linear combination of the other equations. Show by using the pivot operation that the following system is redundant. Is this system equivalent to a system of equations in canonical form?

$$\begin{aligned} x_1 + x_2 - 3x_3 &= 7 \\ -2x_1 + x_2 + 5x_3 &= 2 \\ 3x_2 - x_3 &= 16. \end{aligned}$$

ANS. The system is equivalent to various systems of equations in canonical form.

Exercise 0.2.3. A system of equations is said to be **inconsistent** if the system has no solution. Show by using the pivot operation that the following system is inconsistent. Is this system equivalent to a system in canonical form?

$$\begin{aligned} x_1 + x_2 - 3x_3 &= 7 \\ -2x_1 + x_2 + 5x_3 &= 2 \\ 3x_2 - x_2 &= 15. \end{aligned}$$

Exercise 0.2.4. •

(i) Solve the following system of equations by finding an equivalent system in canonical form with basic variables x_1 and x_2.

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 17 \\ x_1 \qquad - \ x_3 &= 4. \end{aligned}$$

(ii) Is this system equivalent to a system in canonical form with basic variables x_1 and x_3?

(iii) Interpret these results geometrically.

ANS. (i) $x_2 = 9$, $x_1 - x_3 = 4$; (ii) No; (iii) $\mathbf{b} = (17, 4)$ can be expressed as a linear combination of $\mathbf{a}_1 = (2, 1)$ and $\mathbf{a}_2 = (1, 0)$, but not as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_3 = (-2, -1)$.

Exercise 0.2.5. For the linear programming problem of

$$\begin{aligned} \text{minimize} \quad & 5x_1 + 2x_2 + 3x_3 + x_4 \\ \text{subject to} \quad & x_1 + x_2 - 2x_3 + 3x_4 = 2 \\ & -2x_1 \qquad + \ x_3 \qquad = 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

(i) Show geometrically that there can be only two basic feasible solutions to the problem.

(ii) Compute these two basic feasible solutions.

(iii) Assume that the minimal value of the objective function is attained at a basic feasible solution and determine this minimal value.

ANS. (ii) $(0, 6, 2, 0)$ and $(0, 0, 2, 2)$; (iv) The minimum value of the objective function is 8, attained at $(0, 0, 2, 2)$.

Exercise 0.2.6. Prove that although there may be different ways of driving a system of equations into canonical form with a specified set of basic variables, there is a unique basic solution associated with this specified set of basic variables.

0.3 RELATIONS TO CONVEXITY

Exercise 0.3.1. Suppose S is the set of those feasible solutions to a linear programming problem at which the objective function of the problem attains its optimal value. Show that S is a convex set.

Exercise 0.3.2. Suppose the canonical form of a liner programming problem is given by the constraint matrix $\mathbf{A}$ and resource vector $\mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 6 \end{bmatrix}.$$

Determine which of the following points is

(i) a feasible solution to the linear programming problem.

(ii) an extreme point of the set of feasible solutions.

(iii) a basic solution.

(iv) a basic feasible solution.

For each basic feasible solution given below, list the basic variables.

$$\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 0 \\ -9 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Exercise 0.3.3. Consider the system of equations $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 & 0 & 4 \\ 1 & 0 & 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Determine whether each of the following 5-tuples is a basic solution to the system.

(a) $(1, 0, 1, 0, 0)$, (b) $(0, 2, -1, 0, 0)$
(c) $(2, -2, 3, 0, -2)$ (d) $\left(0, 0, \frac{1}{2}, 0, 0\right)$.

Exercise 0.3.4. • Consider the system of equations $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 0 & 6 & 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

Determine whether each of the following 5-tuples is a basic solution to the system.

(a) $(0, 2, -5, 0, -1)$, (b) $(0, 0, 1, 0, 1)$, (c) $(1, 0, -1, 1, 0)$.

ANS. (a) Not basic.

(b) Basic if x_2 and x_4 are taken as nonbasic variables; basic if x_1 and x_2 are taken as nonbasic variables; and not basic if x_1 and x_4 are taken as nonbasic variables.

(c) Not basic.

Exercise 0.3.5. • Consider the linear programming problem

$$\begin{aligned} \text{maximize} \quad & 3x + 2y \\ \text{subject to} \quad & 2x - y \leq 6 \\ & 2x + y \leq 10 \\ & x, y \geq 0. \end{aligned}$$

(i) Transform this problem to a problem in standard form.

(ii) For each extreme point of the new problem, identify the basic variables.

(iii) Solve the problem geometrically.

Exercise 0.3.6. Consider the linear programming problem

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 2x_2 + 7x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 4x_3 \leq 18 \\ & 4x_1 + 2x_2 + 5x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(i) Transform this problem to a problem in standard form.

(ii) For each extreme point of the new problem, identify the basic variables.

(iii) Which of the extreme points are optimal solutions to the problem?

ANS.

x_1	x_2	x_3	x_4	x_5	Basic variables	Optimal
0	0	0	18	10	$x_4,\ x_5$	No
0	0	2	10	0	$x_3,\ x_4$	Yes
0	5	0	23	0	$x_2,\ x_4$	No
$\frac{5}{2}$	0	0	13	0	$x_1,\ x_4$	No

0.4 THE SIMPLEX ALGORITHM

Exercise 0.4.1. For each of the following, put the problem into canonical form, set up the initial tableau, and solve by hand using the simplex method. At most, two pivots should be required for each. Along the way, objective functions requiring some initial adjustments and unbounded objective functions should be encountered.

(a)

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 4x_2 - 4x_3 + 7x_4 \\ \text{subject to} \quad & 8x_1 - 2x_2 + x_3 - x_4 \leq 50 \\ & 3x_1 + 5x_2 \qquad + 2x_4 \leq 150 \\ & x_1 - x_2 + 2x_3 - 4x_4 \leq 100 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

(b)

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 - x_3 \\ \text{subject to} \quad & x_2 + 4x_3 \leq 36 \\ & 5x_1 - 4x_2 + 2x_3 \leq 60 \\ & 3x_1 - 2x_2 + x_3 \leq 24 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(c)

$$\begin{aligned} \text{minimize} \quad & -5x_1 + 4x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 - 3x_3 \leq 8 \\ & 2x_2 - 2x_3 \leq 7 \\ & -x_1 - 2x_2 + 4x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(d) •

$$\begin{aligned}
\text{maximize} \quad & 9x_2 + 2x_3 - x_5 \\
\text{subject to} \quad & x_1 - 3x_2 - 4x_4 + 2x_6 = 60 \\
& 2x_2 - x_4 - x_5 + 4x_6 = -20 \\
& x_2 + x_3 + 3x_6 = 10 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{aligned}$$

(e)

$$\begin{aligned}
\text{maximize} \quad & x_1 + 12x_2 + 9x_3 \\
\text{subject to} \quad & 3x_1 + 2x_2 - 6x_3 \leq 20 \\
& 2x_1 + 6x_2 + 3x_3 \leq 30 \\
& 6x_1 + 2x_3 \leq 16 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}$$

(f)

$$\begin{aligned}
\text{minimize} \quad & x_3 - x_4 \\
\text{subject to} \quad & x_1 - 3x_4 + x_5 = 1 \\
& x_2 + 6x_4 - 5x_5 = 6 \\
& x_3 - 3x_4 + 2x_5 = 5 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}$$

ANS. (a) $\min z = -200$ attained at $(0, 0, 50, 0)$; (c) Unbounded objective function. (d) $\max z = 90$ attained at $(250, 10, 0, 40, 0, 0)$.

Exercise 0.4.2. For each of the following, determine two distinct basic feasible solutions at which the optimal value of the objective function is attained.

(a)

$$\begin{aligned}
\text{minimize} \quad & 4x_1 + 12x_2 + 8x_3 \\
\text{subject to} \quad & 3x_1 + 2x_2 - 6x_3 \leq 20 \\
& 3x_1 + 6x_2 + 4x_3 \leq 30 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}$$

(b)

$$\begin{aligned}
\text{minimize} \quad & x_1 - 3x_2 - 6x_3 \\
\text{subject to} \quad & 2x_1 - x_2 + x_3 + x_4 \leq 60 \\
& 3x_1 + 4x_2 + 2x_3 - 2x_4 \leq 150 \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}$$

0.5 ARTIFICIAL VARIABLES

In Exercises 0.5.1–0.5.2 set up the initial simplex tableau for solving the problem using the two-phase method.

Exercise 0.5.1.

$$\begin{aligned}
\text{maximize} \quad & z = x_1 + 2x_2 + x_4 \\
\text{subject to} \quad & x_1 + 3x_2 - x_3 + x_4 \leq 5 \\
& x_1 + 7x_2 + x_3 \geq 4 \\
& 4x_1 + 2x_2 + x_4 = 3 \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}$$

Exercise 0.5.2.

$$\begin{aligned}
\text{minimize} \quad & z = x_1 + 2x_2 + 7x_3 - x_4 \\
\text{subject to} \quad & 3x_1 + x_2 - 2x_3 - x_4 = 2 \\
& 2x_1 + 4x_2 + 7x_3 \geq 3 \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}$$

In Exercises 0.5.3–0.5.7 solve the indicated linear programming problem using the two-phase method.

Exercise 0.5.3.

$$\begin{aligned}
\text{maximize} \quad & z = 2x_1 + 5x_2 - x_3 \\
\text{subject to} \quad & -4x_1 + 2x_2 + 6x_3 = 4 \\
& 6x_1 + 9x_2 + 12x_3 = 3 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}$$

Exercise 0.5.4. •

$$\begin{aligned}
\text{maximize} \quad & z = 3x_1 - x_2 + 2x_3 + 4x_4 \\
\text{subject to} \quad & x_2 + 7x_3 + 2x_4 \geq 3 \\
& x_1 + 2x_2 + x_3 = 9 \\
& 2x_1 + 3x_2 + x_3 - 4x_4 \leq 7 \\
& x_4 \text{ free}, \quad x_1, x_2, x_3 \geq 0.
\end{aligned}$$

ANS. No finite optimal solution

Exercise 0.5.5.

$$\begin{aligned} \text{maximize} \quad & z = 2x_1 - x_2 + x_3 - x_4 + x_5 \\ \text{subject to} \quad & x_1 + x_2 - x_3 + x_4 + x_5 = 3 \\ & 2x_1 - x_2 + x_3 - 2x_4 = 2 \\ & 3x_1 - x_3 + 3x_4 \geq 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

Exercise 0.5.6.

$$\begin{aligned} \text{maximize} \quad & z = 3x_1 + x_2 - x_3 + 2x_4 - x_5 + 2x_6 \\ \text{subject to} \quad & 2x_1 + x_2 - x_3 + x_6 = 3 \\ & 3x_1 + 2x_3 + x_4 + 2x_5 = 4 \\ & x_2 - 3x_3 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

ANS. $\mathbf{x}^* = (0, 2, 0, 4, 0, 1),\ z_{\max} = 12.$

Exercise 0.5.7.

$$\begin{aligned} \text{maximize} \quad & 10x_1 + 10x_2 + 20x_3 + 30x_4 \\ \text{subject to} \quad & x_1 + x_3 + x_4 = 1 \\ & x_2 + x_3 + x_4 = 2 \\ & 3x_1 + 2x_2 + 2x_3 + x_4 = 7. \\ & x_i \geq 0, \quad i = 1, 2, 3, 4. \end{aligned}$$

Replace the last constraint by $3x_1+2x_2+5x_3+5x_4 = 7$ and solve the problem again.

0.6 DUAL LINEAR PROGRAMS

Exercise 0.6.1. • Determine the dual of each of the following linear programming problems.

(a)

$$\begin{aligned} \text{maximize} \quad & 20x_1 + 30x_2 \\ \text{subject to} \quad & 5x_1 - 4x_2 \leq 100 \\ & -x_1 + 12x_2 \leq 90 \\ & x_2 \leq 50 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(b)

$$\begin{aligned} \text{minimize} \quad & 4x_1 - 3x_2 \\ \text{subject to} \quad & 6x_1 + 11x_2 \geq -30 \\ & 2x_1 - \ \ 7x_2 \leq 50 \\ & x_2 \leq 80 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(c)

$$\begin{aligned} \text{maximize} \quad & -x_1 + 2x_2 \\ \text{subject to} \quad & 5x_1 + \ \ x_2 \leq 60 \\ & 3x_1 - 8x_2 \geq 10 \\ & x_1 + 7x_2 = 20 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(d)

$$\begin{aligned} \text{minimize} \quad & 6x_1 + 12_2 - 18x_3 \\ \text{subject to} \quad & x_1 - 3x_2 + \ \ 6x_3 = 30 \\ & 2x_1 + 8x_2 - 16x_3 = 70 \\ & x_1, x_2 \geq 0, \ x_3 \text{ unrestricted.} \end{aligned}$$

(e) •

$$\begin{aligned} \text{maximize} \quad & x_1 - 7x_2 + 3x_3 \\ \text{subject to} \quad & \qquad 2x_2 + 5x_3 = 20 \\ & 8x_1 \qquad - 3x_3 = 40 \\ & \qquad x_2 + 4x_3 \geq 60 \\ & x_1, x_3 \geq 0, \ x_2 \text{ unrestricted.} \end{aligned}$$

(f)

$$\begin{aligned} \text{minimize} \quad & 2y_1 - 3y_2 + 4y_3 \\ \text{subject to} \quad & 8y_1 \qquad - y_3 = 50 \\ & \qquad 6y_2 + y_3 \leq 60 \\ & y_1, y_2 \geq 0, \ -15 \leq y_3 \leq 0. \end{aligned}$$

Exercise 0.6.2. (i) Determine the dual to the problem of

$$\begin{aligned} \text{maximize} \quad & x_1 - 2x_2 \\ \text{subject to} \quad & x_2 \geq 1 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(ii) Rewrite your answer to part (i) as an equivalent maximization problem.

(iii) Compare your response in part (ii) to the original problem of part (i). Observation?

(iv) Show that the following problem is also its own dual.

$$\begin{array}{ll} \text{maximize} & x_1 - 2x_2 - 3x_3 \\ \text{subject to} & \quad\;\; x_2 + 2x_3 \geq 1 \\ & x_1 \qquad\; + 3x_3 \leq 2 \\ & 2x_1 - 3x_2 \qquad = 3 \\ & x_1, x_2 \geq 0, x_3 \text{ unrestricted.} \end{array}$$

Exercise 0.6.3. Consider the following linear programming problem

$$\begin{array}{ll} \text{maximize} & 6x_1 + x_2 + 4x_3 \\ \text{subject to} & 3x_1 + 7x_2 + x_3 \leq 15 \\ & x_1 - 2x_2 + 3x_3 \leq 20 \\ & \qquad x_1, x_2, x_3 \geq 0. \end{array}$$

(i) Show that the objective function of the dual problem is bounded below.

(ii) Solve the dual problem graphically.

(iii) Solve the maximization problem using the simplex method. Note that the optimal values of the objective functions are equal.

(iv) Compare the bottom two entries in the slack variable columns of the last simplex tableau of part (iii) with the point in part (ii) that yielded the minimal value.

ANS. (b) $\min \mathbf{b}^T\mathbf{y}$ is $41\frac{1}{4}$, attained at $(7/4, 3/4)$.
(c) $\max \mathbf{c}^T\mathbf{x}$ is $41\frac{1}{4}$, attained at $(25/8, 0, 45/8)$.

Exercise 0.6.4. Consider the linear programming problem of

$$\begin{array}{ll} \text{maximize} & 4x_1 + 10x_2 - 3x_3 + 2x_4 \\ \text{subject to} & 3x_1 - 2x_2 + 7x_3 + x_4 \leq 26 \\ & x_1 + 6x_2 - x_3 + 5x_4 \leq 30 \\ & -4x_1 + 8x_2 - 2x_3 - x_4 \leq 10 \\ & \qquad x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

(i) Show that $\left(\frac{54}{5}, \frac{16}{5}, 0, 0\right)$ is a feasible solution to this problem. Compute the value of the objective function at this point.

(ii) Write out the dual problem. Show that $\left(\frac{7}{10}, \frac{19}{10}, 0\right)$ is a feasible solution to this problem. What is the value of the objective function of the dual at this point?

Exercise 0.6.5. • Verify that $\left(0, 5\frac{2}{3}, 8\frac{1}{3}, \frac{1}{3}\right)$ is an optimal solution point to the problem of

$$\begin{aligned} \text{minimize} \quad & = 7x_1 + 11x_2 - 3x_3 - x_4 \\ \text{subject to} \quad & 2x_1 + 2x_2 - \ \ x_3 - 3x_4 \geq 2 \\ & -x_1 + 5x_2 - 2x_3 + \ \ x_4 \geq 12 \\ & x_1 - 4x_2 + 3x_3 + 5x_4 \geq 4 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

and $\left(3\frac{1}{2}, 2, 1\frac{1}{2}\right)$ is an optimal solution point to the dual.

Exercise 0.6.6. Suppose for the linear programming problem

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

we know that $\mathbf{b} = (12, 21, 8, 2, 5)$. Assume that $\mathbf{w} = (0, 4, 5, 0, 3)$ is an optimal solution to the dual of the given problem. Calculate the optimal value of the objective function for the given problem.

ANS. $z = 139$.

0.7 COMPLEMENTARY SLACKNESS

Exercise 0.7.1. • Consider the linear programming problem of

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + \ \ x_2 \leq 3 \\ & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(i) Determine the dual problem.

(ii) Show that $\mathbf{x}^* = (1, 1)$ and $\mathbf{y}^* = (0, 1)$ are optimal solutions for the original and dual problems, respectively, by using the Complementary Slackness Theorem.

(iii) Note that at these solution points, both y_1^* and the slack in the corresponding first constraint of the max problem are zero.

Exercise 0.7.2. Consider the linear programming problem of

$$
\begin{array}{rl}
\text{maximize} & 2x_1 + 2x_2 \\
\text{subject to} & x_1 \phantom{{}+x_2} + \ x_3 + x_4 \leq 1 \\
& \phantom{x_1 +{}} x_2 + \ x_3 - x_4 \leq 1 \\
& x_1 + x_2 + 2x_3 \phantom{{}-x_4} \leq 3 \\
& \phantom{x_1+{}} x_1, x_2, x_3, x_4 \geq 0.
\end{array}
$$

(i) Determine the dual problem.

(ii) Show that $\mathbf{x}^* = (1, 1, 0, 0)$ and $\mathbf{y}^* = (1, 1, 1)$ are feasible solutions to the original and dual problems, respectively.

(iii) Show that for this pair of solutions, for each j, $x_j^* > 0$ implies that the slack in the corresponding dual constraint is zero.

(iv) Show that $\mathbf{y}^*$ is not an optimal solution to the dual.

(v) Does this contradict the Complementary Slackness Theorem?

Exercise 0.7.3. Prove or disprove each of the following, using complementary slackness.

(i) $(3, 0, 1, 0, 5)$ is an optimal solution point to the problem of

$$
\begin{array}{rl}
\text{maximize} & 5x_1 + 16x_2 - 4x_3 - x_4 + 7x_5 \\
\text{subject to} & 8x_1 - 2x_2 + 3x_3 \phantom{{}+3x_4} - 2x_5 \leq 18 \\
& 2x_1 + 4x_2 - 7x_3 + 3x_4 + \ x_5 \leq 4 \\
& x_1 + 3x_2 + \ x_3 \ - \ x_4 + 2x_5 \leq 14 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{array}
$$

(ii) $(1,0,1,0)$ is an optimal solution point to the problem of

$$\begin{array}{rl} \text{minimize} & 5x_1 + 8x_2 + 4x_3 + 2x_4 \\ \text{subject to} & x_1 + 2x_2 - x_3 + x_4 \geq 0 \\ & 2x_1 + 3x_2 + x_3 - x_4 \geq 3 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

(iii) $(0,3,12)$ is an optimal solution point to the problem of

$$\begin{array}{rl} \text{minimize} & 2y_1 - 5y_2 - 3y_3 \\ \text{subject to} & -3y_1 - 6y_2 + 2y_3 \geq 6 \\ & y_1 + 3y_2 + y_3 \geq 20 \\ & 4y_1 + 7y_2 - 3y_3 \geq -15 \\ & y_1, y_2, y_3 \geq 0. \end{array}$$

ANS. (i) $(1,1,0,0)$ optimal; complementary slackness generates $(2,2,0)$, a feasible solution to the dual.

(ii) $(0,4,0,2)$ optimal; complementary slackness generates $(3,2,0)$, a feasible solution to the dual.

(iii) $(3,0,1,0,5)$ not optimal; complementary slackness generates $(0,1,3)$, but this point is not a feasible solution to the dual.

Exercise 0.7.4. Suppose that $x_1 = 2$, $x_2 = 0$, $x_3 = 4$ is an optimal solution to the linear programming problem

$$\begin{array}{rl} \text{maximize} & z = 4x_1 + 2x_2 + 3x_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 \leq 12 \\ & x_1 + 4x_2 + 2x_3 \leq 10 \\ & 3x_1 + x_2 + x_3 \leq 10 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{array}$$

Using the principle of complementary slackness and the duality theorem, find an optimal solution to the dual problem. What value will the objective function of the dual problem have at this optimal solution?