CHAPTER 4: RANDOM VARIABLES

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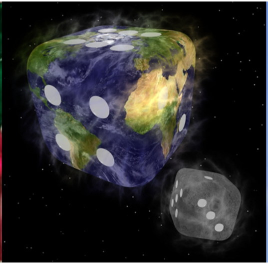



CONTENTS

Random variables [Ref. Chapter 4 in the textbook]

Example

Suppose that our experiment consists of tossing 3 fair coins. If we let Y denote the number of heads appearing, then Y is a random variable taking on one of the values $0, 1, 2, 3$ with respective probabilities.

State Ω? ω?

$$P(Y = 0) = P\{(T, T, T)\} = \frac{1}{8}$$

$$P(Y = 1) = P\{(T, T, H), (T, H, T), (H, T, T)\} = \frac{3}{8}$$

$$P(Y = 2) = P\{(T, H, H), (H, T, H), (H, H, T)\} = \frac{3}{8}$$

$$P(Y = 3) = P\{(H, H, H)\} = \frac{1}{8}$$

Random variables

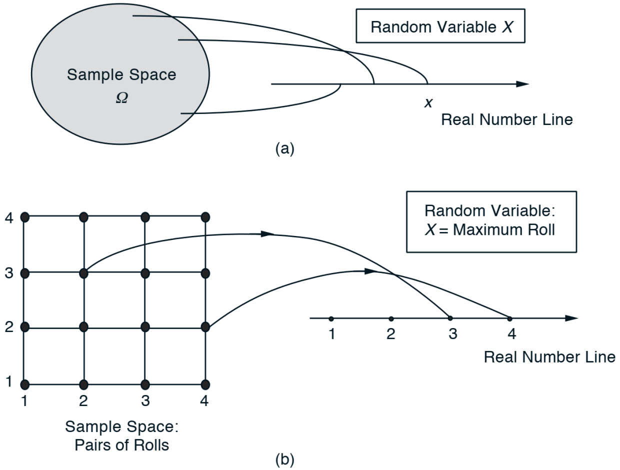


Figure 2.1: (a) Visualization of a random variable. It is a function that assigns a numerical value to each possible outcome of the experiment. (b) An example of a random variable. The experiment consists of two rolls of a 4-sided die, and the random variable is the maximum of the two rolls. If the outcome of the experiment is $(4, 2)$, the experimental value of this random variable is 4.

Probability space and random variable

Definition: Random variables

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. A random variable is a (measurable) function $X : \Omega \to \bar{\mathbb{R}}$ such that the set $\{\omega \in \Omega : X(\omega) \leqslant t\}$ is an element of $\mathcal{F}$ for all $t \in \mathbb{R}$.

Let A be a subset of $\bar{\mathbb{R}}$, and let X be a random variable. We use the notation $\{X \in A\}$ to denote the set

$$\{\omega \in \Omega : X(\omega) \in A\}.$$

[Reference for this chapter: Chapter 4 in the textbook by S. Ross; the first haft of Chapter 2 and Chapter 3 by D. Bertsekas, MIT]

Random variables

Example

Independent trials consisting of the flipping of a coin having probability p of coming up heads are continually performed until either a head occurs or a total of n flips is made. If we let X denote the number of times the coin is flipped, then X is a random variable taking on one of the values $1, 2, 3, \ldots, n$ with respective probabilities:

$$P(X = 1) = P(H) = p$$

$$P(X = 2) = P(T, H) = (1 - p)\, p, \quad \ldots$$

$$P(X = n - 1) = (1 - p)^{n-1} p$$

$$P(X = n) = (1 - p)^{n-1}$$

Random variables

- An event that may or may not occur is called random. Random variables (RV) is an uncertain quantity/number: **a function from a set S into $\mathbb{R}$.**

- An outcome is an observed value of a RV.

- Random variables whose set of possible values can be written either as finite sequence $x_1, \ldots, x_n$, or as an infinite sequence $x_1, x_2, \ldots$ are said to be discrete.

- Random variables that take on a continuum of possible values are called continuous random variables.

  Example: The random variable denoting the lifetime of a car. The car's lifetime is assumed to take on any value in some interval (a, b).

Random variables

Example

Three balls are to be randomly selected without replacement from an urn containing 20 balls numbered 1 through 20. If we bet that at least one of the balls that are drawn has a number as large as or larger than 17, what is the probability that we win the bet?

Random variables

- The cumulative distribution function (cdf) (or more simply the distribution function) is defined as

$$F(b) = P\{X \leqslant b\}$$

- $F(b)$ is a nondecreasing function of b and

$$\lim_{b \to \infty} F(b) = 1; \lim_{b \to -\infty} F(b) = 0$$

$$P(a < X \leqslant b) = F(b) - F(a)$$

Random variables

Example

Suppose the random variable X has distribution function

$$F(x) = \begin{cases} 0 \text{ if } x \leqslant 0 \\ 1 - e^{-x} \text{ if } x > 0 \end{cases}$$

What is the probability that X exceeds 1?

$$P(X > 1) = 1 - P(X \leqslant 1) = 1 - F(1) = \frac{1}{e}$$

Discrete Random variables

- The probability mass function (or distribution) $p(a) = P(X = a)$.
- Properties: $p(x_i) > 0, i = 1, 2, 3, ..., p(x) = 0$ all other values of x

$$\text{and } \sum_{i=1}^{\infty} p(x_i) = 1.$$

- The cumulative distribution function F can be expressed in terms of $p(x)$ by

$$F(a) = \sum_{all\ \mathrm{x \leqslant a}} p(x)$$

Discrete Random variables

Example

Consider a random variable X that is equal to $1, 2, 3$ or 4. If we know that $p(1) = \frac{1}{4}, p(2) = \frac{1}{2}, p(3) = \frac{1}{8}$.

(a) Find $p(X = 4)$.

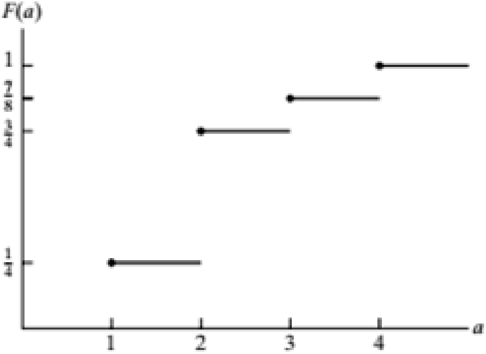
(b) Find and sketch the cumulative distribution function.

Hint (a) $p(X = 4) = p(4) = 1 - p(1) - p(2) - p(3) = 1/8$.

(b) The cumulative distribution function is

$$F(a) = \begin{cases} 0, a < 1 \\ \frac{1}{4}, 1 \leqslant a < 2 \\ \frac{3}{4}, 2 \leqslant a < 3 \\ \frac{7}{8}, 3 \leqslant a < 4 \\ 1, 4 \leqslant a \end{cases}$$

Discrete Random variables



The graph of $F(x)$

Note: The distribution function F of a discrete RV is a step function.

Discrete Random variables

Example

A random number generator produces an integer number X that is equally likely to be any element in the set

$$S = \{0, 1, 2, ..., M - 1\}$$

Find the probability mass function of X.

Answer:

$$P(X = k) = \frac{1}{M}, \quad k = 0, 1, 2.., M - 1.$$

Expected value

- If X is a (discrete) random variable having a probability mass function $p(x)$, the expectation or the expected value of X, denoted by $E(X)$, is defined by

$$E\left(X\right) = \sum_i p\left(x_i\right) x_i = p\left(x_1\right) x_1 + p\left(x_2\right) x_2 + ... + p\left(x_n\right) x_n.$$

The expected value of X is a weighted average of the possible values that X can take on ($p(x_i) > 0$).

Example

Toss a coin. If heads, then $X = 1$. If tail, then $X = -1$.

Answer

$$E(X) = p(1) \times (1) + p(-1) \times (-1) = (0.5) \times (1) + (0.5) \times (-1) = 0.$$

Expected value

Example

Find $E[X]$, where X is the outcome when we roll a fair die.

Answer

$$E(X) = p(1) \times 1 + p(2) \times 2 + p(3) \times 3 + p(4) \times 4 + p(5) \times 5 + p(6) \times 6$$

$$= \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6 = 7/2$$

Expected value

Example

We say that I is an indicator variable for the event A if

$$I = \begin{cases} 1, if \quad A \quad occurs \\ 0, if \quad A^c \quad occurs \end{cases}$$

Find E(I).

Answer

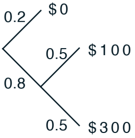
$$E(I) = p(A) \times 1 + p(A^c) \times 0 = p(A).$$

That is, the expected value of the indicator variable for the event A is equal to the probability that A occurs.

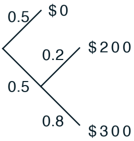
Expected value

Example

Consider a quiz game where a person is given two questions and must decide which question to answer first. Question 1 will be answered correctly with probability 0.8, and the person will then receive as prize \$100, while question 2 will be answered correctly with probability 0.5, and the person will then receive as prize \$200. If the first question attempted is answered incorrectly, the quiz terminates, i.e., the person is not allowed to attempt the second question. If the first question is answered correctly, the person is allowed to attempt the second question. Which question should be answered first to maximize the expected value of the total prize money received?



Question 1
Answered 1st



Question 2
Answered 1st

Expected value

Example

A school class of 120 students is driven in 3 buses to a symphonic performance. There are 36 students in one of the buses, 40 in another, and 44 in the third bus. When the buses arrive, one of the 120 students is randomly chosen. Let X denote the number of students on the bus of that randomly chosen student, and find E[X].

Expectation of a function of a RV

Example

Let X denote a random variable that takes on any of the values -1, 0, and 1 with respective probabilities

$$P\left(X=-1\right)=0.2, P\left(X=0\right)=0.5, P\left(X=1\right)=0.3$$

Compute $E[X^2]$.

Solution

Let $Y=X^2$. Then the probability mass function of Y is given by

$$P\left(Y=1\right)=P\left(X=-1\right)+P\left(X=1\right)=0.5$$

$$P\left(Y=0\right)=P\left(X=0\right)=0.5$$

$$E\left(X^2\right)=E\left(Y\right)=1\left(0.5\right)+0\left(0.5\right)=0.5$$

Expectation of a function of a RV

Preposition

If X is a discrete random variable that takes on one of the values x_i, $i \geq 1$, with respective probabilities $p(x_i)$, then, for any real-valued function g,

$$E\left(g\left(X\right)\right) = \sum_i g\left(x_i\right) p\left(x_i\right)$$

Therefore, apply the preposition for the previous example, we get:

$$\begin{aligned} E\left(X^2\right) &= (-1)^2\left(0.2\right) + 0^2\left(0.5\right) + 1^2\left(0.3\right) \\ &= 1\left(0.2 + 0.3\right) + 0\left(0.5\right) = 0.5 \end{aligned}$$

Expectation of a function of a RV

Corollary

If a and b are constants, then

$$E\left(aX+b\right)=aE\left(X\right)+b$$

Remark: $E\left(X^n\right)=\sum_{x:p(x)>0} x^n p\left(x\right)$ is called the nth moment of X.

Expectation of a function of a RV

Definition

If X is a random variable with mean μ, then the variance of X, denoted by $Var(X)$,

$$Var\left(X\right) = E\left[\left(X - \mu\right)^2\right]$$

is defined by

Note that the variance of X can be re-written as

$$Var\left(X\right) = E\left(X^2\right) - \left(E\left(X\right)\right)^2$$

Expectation of a function of a RV

Example

Calculate $Var(X)$ if X represents the outcome when a fair die is rolled.

Recall that $E[X] = 7/2$.

$$E\left(X^2\right) = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{6}\right) = \frac{91}{6}$$

$$Var\left(X\right) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Variance and standard deviation

- Variance measure the possible variation of X around $E(X)$.
- Standard deviation: $SD(X) = \sqrt{Var(X)}$ [or, another notation $SD(X) \equiv \sigma_X$].

Theorem

Let X be a random variable and k a real number. Then

$$\text{(i) } Var\left(X + k\right) = Var\left(X\right)$$

$$\text{(ii) } Var\left(kX\right) = k^2 Var\left(X\right)$$

We now consider some of important discrete random variables.

The Bernoulli random variables

Suppose that a trial, or an experiment, whose outcome can be classified as either a success or as a failure is performed. If we let $X = 1$ when the outcome is a success and $X = 0$ when it is a failure, then the probability mass function of X is given by

$$P(X = 1) = p$$

$$P(X = 0) = 1 - p$$

where p, $0 \leq p \leq 1$, is the probability that the trial is a "success."

A random variable X such that is said to be a Bernoulli random variable.

The Binomial Random Variables

Suppose that n independent trials, each of which results in a success with probability p and in a failure with probability $1 - p$, are to be performed. If X represents the number of successes that occur in the n trials, then X is said to be a binomial random variable with parameters (n, p).

The probability mass function of a binomial random variable having parameters (n, p) is given by

$$p(X = i) = p\,(i) = \binom{n}{i} p^i \,(1 - p)^{n-i}$$

Remark: If X is a binomial random variable with parameters (n, p), then we say that X has a binomial distribution with parameters (n, p).

Q: Verify $\sum_{i=0}^{\infty} p\,(i) = 1$?

Hint: Use the expansion $(x + y)^n$ for $x = p$ and $y = 1 - p$.

The Binomial Random Variables

Example

Four fair coins are flipped. If the outcomes are assumed independent, what is the probability that two heads and two tails are obtained?

Solution

Letting X equal the number of heads ("successes") that appear, then X is a binomial random variable with parameters $(n = 4, p = 1/2)$. Hence,

$$p(X = 2) = \binom{4}{2} p^2 (1 - p)^2 = \frac{3}{8}$$

The Binomial Random Variables

Example

It is known that any item produced by a certain machine will be defective with probability 0.1, independently of any other item. What is the probability that in a sample of three items, at most one will be defective?

Solution

If X is the number of defective items in the sample, then X is a binomial random variable with parameters $(3, 0.1)$. Hence, the desired probability is given by

$$p(X=0)+p(X=1)=\begin{pmatrix} 3 \\ 0 \end{pmatrix}(0.1)^0(0.9)^3+\begin{pmatrix} 3 \\ 1 \end{pmatrix}(0.1)^1(0.9)^2=0.972$$

The Binomial Random Variables

Example

Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.

Hint

$$P(X = k) = \binom{5}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{5-k}, k = 0, 1, 2, .., 5$$

The Binomial Random Variables

Exercise

Nine percent of undergraduate students carry credit card balances greater than \$7000 (Reader's Digest, July 2002). Suppose 10 undergraduate students are selected randomly to be interviewed about credit card usage.
(a) Is the selection of 10 students a binomial experiment? Explain.
(b) What is the probability that two of the students will have a credit card balance greater than \$7000?
(c) What is the probability that none will have a credit card balance greater than \$7000?

The Binomial Random Variables

Exercise

A binary communications channel introduces a bit error in a transmission with probability p. Let X be the number of errors in n independent transmissions.

a. Find the probability mass function of X.

b. Find the probability of one or fewer errors.

Answer b. $(1-p)^n + np\,(1-p)^{n-1}$.

The Binomial Random Variables

Exercise

Suppose that an airplane engine will fail, when in fight, with probability $1 - p$ independently from engine to engine; suppose that the airplane will make a successful fight if at least 50 percent of its engines remain operative. For what values of p is a four-engine plane preferable to a two-engine plane?

Answer $p \geq 2/3$

Properties of Binomial Random Variables

Theorem

If X is a binomial random variable with parameters n and p, then

$$E(X) = np, \quad Var(X) = np(1-p)$$

A recursive formula for computing the binomial distribution function

$$P(X = k+1) = \frac{p(n-k)}{(1-p)(k+1)} P(X = k)$$

The Poisson Random Variables

Definition

A random variable X, taking on one of the values $0, 1, 2, \ldots$, is said to be a Poisson random variable with parameter λ, if for some $\lambda > 0$

$$p\left(i\right) = P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!},\ \text{i} = 0,1,2,..$$

Verify $\sum\limits_{i=0}^{\infty} p\left(i\right) = 1$?

$$\sum_{i=0}^{\infty} p\left(i\right) = \sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1$$

The Poisson Random Variables

The Poisson random variable has a tremendous range of applications in diverse areas because it may be used as an approximation for a binomial random variable with parameters (n, p) when n is large and p is small enough.

Preposition

Suppose that X is a binomial random variable with parameters (n, p), and let $\lambda = np$. Then

$$P(X = i) \approx e^{-\lambda} \frac{\lambda^i}{i!}$$

The Poisson Random Variables

Some examples of random variables that generally obey the Poisson probability law are as follows:

- The number of misprints on a page (or a group of pages) of a book.
- The number of people in a community who survive to age 100.
- The number of wrong telephone numbers that are dialed in a day.
- The number of customers entering a post office on a given day.
- Thenumberof α-particles discharged in a fixed period of time from some radioactive material.

  ...

The Poisson Random Variables

Example

Suppose that the number of typographical errors on a single page of this book has a Poisson distribution with parameter $\lambda = 1$. Calculate the probability that there is at least one error on this page.

Solution

$$P(X \geqslant 1) = 1 - P(X = 0) = 1 - e^{-1} \approx 0.633$$

The Poisson Random Variables

Example

If the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$, what is the probability that no accidents occur today?

Solution

$$P(X = 0) = e^{-3} \approx 0.05$$

The Poisson Random Variables

Example

Suppose that the probability that an item produced by a certain machine will be defective is .1. Find the probability that a sample of 10 items will contain at most 1 defective item.

Solution The desired probability is

$$\binom{10}{0}(0.1)^0(0.9)^{10} + \binom{10}{1}(0.1)^1(0.9)^9 = 0.7361$$

whereas the Poisson approximation yields the value $e^{-1} + e^{-1} \approx .7358$.

The Poisson Random Variables

Theorem

If X a Poisson random variable, then

$$E(X) = \lambda; \quad Var(X) = \lambda$$

The Poisson Random Variables

Example of Poisson RV

Every week the average number of wrong-number phone calls received by a certain mail-order house is seven. Assume that the number of wrong numbers received tomorrow is approximately a Poisson RV. What is the probability that they will receive
(a) two wrong call tomorrow; (b) at least one wrong call tomorrow?

Solution

Let X be the number of wrong-number phone calls per day.

$$\lambda = E(X) = 1 \Rightarrow P(X = 2) = e^{-1}\frac{\lambda^2}{2!} = \frac{1}{2e} = 0.18$$

$$P(X \geqslant 1) = 1 - P(X = 0) = 1 - \frac{1}{e} = 0.63.$$

The Poisson Random Variables

Example of Poisson RV

More than 50 million guests stay at bed and breakfasts (B&Bs) each year. The website for the Bed and Breakfast Inns of North America, which averages seven visitors per minute, enables many B&Bs to attract guests (Time, September 2001).

(a) Compute the probability of no website visitors in a one-minute period.
(b) Compute the probability of two or more website visitors in a one-minute period.
(c) Compute the probability of one or more website visitors in a 30-second period

The Poisson Random variables

Example

The probability mass function of a random variable X is given by $p(i) = c\frac{\lambda^i}{i!}, i = 0, 1, 2, ...,$ where λ is a positive constant.
Find (a) $P(X = 0)$ and (b) $P(X > 1)$.

Hint:

$$\sum_{i=0}^{\infty} p(i) = 1 \rightarrow c = e^{-\lambda}$$

$$P(X = 0) = e^{-\lambda}$$

$$P(X > 1) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-\lambda} - \lambda e^{-\lambda}$$

Computing the Poisson Distribution Function

If X is Poisson with parameter λ, then

$$\frac{P(X=i+1)}{P(X=i)}=\frac{e^{-\lambda}\lambda^{i+1}/(i+1)!}{e^{-\lambda}\lambda^{i}/i!}=\frac{\lambda}{i+1}$$

Starting with

$$P(X=0)=e^{-\lambda}$$

Then,

$$P(X=1)=\lambda P(X=0)$$

$$P(X=2)=\frac{\lambda}{2}P(X=1)$$

....

Poisson Processes

Definition of the counting processes

A **counting process** is a process $X(t)$ in discrete or continuous time for which the possible values of $X(t)$ are the natural numbers $(0, 1, 2, ...)$ with the property that $X(t)$ is a non-decreasing function of t. Often, $X(t)$ can be thought of as counting the number of 'events' of some type that have occurred by time t. The basic example of a counting process is the Poisson process.

Definition of Poisson process

Suppose that, starting from a time point labeled $t = 0$, we begin counting the number of events that have occurred during $[0, t]$. For each value of t, we obtain a number denoted by $N(t)$, which is the number of events that have occurred during $[0, t]$.

Poisson Processes

Assumptions: Suppose that $N(t)$ satisfied the following properties

- **Stationarity**: For all $n \geq 0$, and for any two equal time intervals Δ_1 and Δ_2, the probability of n events in Δ_1 is equal to the probability of n events in Δ_2.
- **Independent Increments**: For all $n \geq 0$, and for any time interval $(t, t+s)$, the probability of n events in $(t, t+s)$ is independent of how many events have occurred earlier or how they have occurred.
- **Orderliness**: The occurrence of two or more events in a very small time interval is practically impossible.
- $0 < P(N(t) = 0) < 1$, for all $t > 0$.

If $\{N(t) : t \geqslant 0\}$ satisfies the four properties above then $\{N(t) : t \geqslant 0\}$ is a Poisson process and $N(t)$ is a Poisson RV with parameter λt.

Poisson Processes

Theorem

If random events occur in time in a way that the preceding conditions stationarity, independent increment, and orderliness are always satisfied, $N(0) = 0$, and $0 < P(N(t) = 0) < 1$ for all $t > 0$, then there exists a positive number λ such that

$$P\left(N\left(t\right) = n\right) = \frac{\left(\lambda t\right)^n e^{-\lambda t}}{n!}.$$

That is, for all $t > 0$, $N(t)$ is a Poisson random variable with parameter λt. Hence $E[N(t)] = \lambda t$ and therefore $\lambda = E[N(1)]$.

Poisson Processes

Example

Suppose that children are born at a Poisson rate of five per day in a certain hospital. What is the probability that (a) at least two babies are born during the next six hours; (b) no babies are born during the next two days?

Solution

Let $N(t)$ denote the number of babies born at prior to t. If we choose one day as time unit, then

$$\lambda = E\left[N\left(1\right)\right] = 5$$

Therefore,

$$P\left(N\left(t\right) = n\right) = \frac{\left(5t\right)^n e^{-5t}}{n!}$$

Poisson Processes

Example

Solution (Cont.)

The probability that at least two babies are born during the next 6 hours (1/4 of a day) is

$$P(N(1/4) \geq 2) = 1 - P(N(1/4) = 0) - P(N(1/4) = 1)$$

$$= 1 - \frac{(5/4)^0 e^{-5/4}}{0!} - \frac{(5/4)^1 e^{-5/4}}{1!} \approx 0.36.$$

The probability that no babies are born during the next 2 days is

$$P(N(2) = 0) = \frac{(10)^0 e^{-10}}{0!} \approx 4.54 \times 10^{-5}.$$

Poisson Processes

Example

Suppose that earthquakes occur in a certain region, in accordance with a Poisson process, at a rate of seven per year.
(a) What is the probability of no earthquakes in one year?
(b) What is the probability that in exactly three of next eight years no earthquakes will occur?

Solution

Let $N(t)$ denote the number of earthquakes in this region at prior to t. If we choose one year as time unit, then

$$\lambda = E\left[N\left(1\right)\right] = 7$$

Therefore,

$$P\left(N\left(t\right) = n\right) = \frac{\left(7t\right)^n e^{-7t}}{n!}.$$

Poisson Processes

Example

Solution (Cont.)

(a) The probability of no earthquakes in one year is

$$p = P\left(N\left(1\right) = 0\right) = \frac{\left(7\right)^0 e^{-7}}{0!} \approx 0.00091$$

(b) Suppose that a year is called success if during its course no earthquakes occur. Of the next eight years, let X be the number of years in which no earthquakes will occur. Then X is a binomial random variable with parameters $(8, p)$. Thus

$$P\left(X = 3\right) \approx \binom{8}{3} \left(0.00091\right)^3 \left(1 - 0.00091\right)^5 \approx 4.2 \times 10^{-8}.$$

Poisson Processes

Example

Suppose that X is a Poisson random variable with $P(X=1)=P(X=3)$. Find $P(X=5)$.

Solution

$P(X=1)=P(X=3)$ implies that

$$e^{-\lambda}\lambda=\frac{e^{-\lambda}\lambda^3}{3!}.$$

This leads to $\lambda=\sqrt{6}$. The answer is

$$\frac{e^{-\sqrt{6}}\left(\sqrt{6}\right)^5}{5!}.$$

The Geometric Random Variables

Definition

Suppose that independent trials, each having probability p of being a success, are performed until a success occurs. If we let X be the number of trials required until the first success, then X is said to be a geometric random variable with parameter p.

$$p(n) = P(X = n) = (1-p)^{n-1} p, \;\; n = 1, 2, ...$$

Q:

$$\sum_{n=0}^{\infty} p(n) = 1?$$

The Geometric Random Variables

Example

An urn contains N white and M black balls. Balls are randomly selected, one at a time, until a black one is obtained. If we assume that each ball selected is replaced before the next one is drawn, what is the probability that

(a) exactly n draws are needed?

(b) at least k draws are needed?

$$P(X = n) = \left(\frac{N}{M+N}\right)^{n-1} \frac{M}{M+N}$$

$$P(X = n) = \left(\frac{N}{M+N}\right)^{n-1} \frac{M}{M+N}$$

$$P(X \geqslant k) = \frac{M}{M+N} \sum_{n=k}^{\infty} \left(\frac{N}{M+N}\right)^{n-1} = \left(\frac{N}{M+N}\right)^{k-1}$$

The Hypergeometric Random Variable

Suppose that a sample of size n is to be chosen randomly (without replacement) from an urn containing N balls, of which m are white and N - m are black. If we let X denote the number of white balls selected, then

$$P(X = i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$$

The Hypergeometric Random Variable

Example

A purchaser of electrical components buys them in lots of size 10. It is his policy to inspect 3 components randomly from a lot and to accept the lot only if all 3 are nondefective. If 30 percent of the lots have 4 defective components and 70 percent have only 1, what proportion of lots does the purchaser reject?

Let A denote the event that the purchaser accepts a lot.

$$P(A) = P(A|lot\ has\ 4\ defectives)\frac{3}{10} + P(A|lot\ has\ 1\ defective)\frac{7}{10}$$

$$= \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}}\frac{3}{10} + \frac{\binom{1}{0}\binom{9}{3}}{\binom{10}{3}}\frac{7}{10} = \frac{54}{100}$$

Hence, 46 percent of the lots are rejected.

–END OF CHAPTER 4–