Itô integral

July 24, 2021

Outline

- Textbook: Section 4.2, 4.3 Shreve II
- Content
  - Construction Itô integral
    - Itô integral for elementary Intergal
    - Itô integral for general Intergal
  - Simulation
  - Properties: continuity, adaptness, Martingale, Isometry, quadratic variation
  - Distribution of Itô integral with deterministic integrand

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Example of ODE

When we model some situations, we don't know a priori which function to use, since we only know the local behavior of our system.

- $f(t)$ represents a commodity price at time t
- Write
$$f(t+\Delta t) - f(t) = \mu \Delta t f(t)$$
in order to mean that the variation of $f(t+\Delta t) - f(t)$ of the commodity price over a time period is is proportional to the length Δt of the time period considered as well as the commodity price $f(t)$ at the start of the period, i.e. $\mu \Delta t f(t)$, μ being a constant.

- Divide both side by Δt

$$\frac{f(t+\Delta t)-f(t)}{\Delta t}=\mu f(t)$$

- Let $\Delta t \to 0$

$$\frac{df}{dt}=\mu f(t)$$

  or

$$df(t)=\mu f(t)dt$$

  It means that the derivative of the function is proportional to the function itself

- General solution $f(t) = ce^{\mu t}$

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- $c = f_0$
- Commodity price at time t is

$$f = f_0 e^{\mu t}$$

- $(S_t)_{t\geq 0}$: evolution of a risky asset price
- We don't know, in general, the law that governs such a process, but we may have an idea of its local behavior.

Example

- over a short time interval of length Δt, a price tends to vary proportionally to the period length and the asset price at the beginning of the period

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- we are not certain that the price varies proportionally to the period length and the asset price
- an unpredictable error needs to be incorporated into our equation.
- We can however control the magnitude of such a random error.

- Error term
  - the higher the price, the more the risky asset price can diverge from the trend.
  - the random error must also depend on the length of the time interval considered: the longer the interval, the greater the chance that the price diverges from the trend.

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  - the higher the price, the more the risky asset price can diverge from the trend.
  - the random error must also depend on the length of the time interval considered: the longer the interval, the greater the chance that the price diverges from the trend.
  - Add a stochastic term to the initial equation.
-
$$S_{t+\Delta t} - S_t = \mu \Delta t S_t + \sigma S_t \sqrt{\Delta t} \epsilon_t$$

  where
  - $\sigma > 0$
  - $\epsilon_t \hookrightarrow N(0,1)$ is indepedent of $(S_u)_{u \leq t}$

  The latter condition is important, since we must not be able to predict the error ϵ_t from observing the behavior of the risky asset price prior to date t.

- A candidate for error terms: Brownian motion B_t
- $B_{t+\Delta_t} - B_t \hookrightarrow N(0, \Delta t)$ and independent of $(B_u)_{u \le t}$
- $$S_{t+\Delta t} - S_t = \mu \Delta t S_t + \sigma S_t (B_{t+\Delta_t} - B_t)$$
- When length of time $\Delta t \to 0$, it leads to

  $$dS_t = \mu S_t dt + \sigma S_t dB_t$$

  which is called a **stochastic differential equation**

Questions

1. the term $\sigma S_t dB_t$ is not well defined, particularly we proved that the Brownian motion paths are nowhere differentiable!
2. does a solution to that equation exist?
3. if that solution exists, is it unique and how can it be found?

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Riemann integral

$$\int_0^t g(s)ds = \lim_{\|\Pi\|\to 0} \sum_{i=1}^{n} g(s_i)(\underbrace{s_{i+1} - s_i}_{\Delta s_i})$$

Lebesgue - Stieltjes integral

$$\int_0^t g(s)df(s) = \lim_{\|\Pi\|\to 0} \sum_{i=1}^{n} g(s_i)(\underbrace{f(s_{i+1}) - f(s_i)}_{\Delta f(s_i)})$$

where $\Pi : 0 = s_0 < s_1 < \cdots < s_n = t$ is a partition of $[0, t]$

Riemann integral

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where $\Pi : 0 = s_0 < s_1 < \cdots < s_n = t$ is a partition of $[0, t]$
Example: Left-end point rule $s_i = \frac{it}{n}$, $i = 0, 1, \ldots, n$

Itô integral

Itô integral

$$I_t = \int_0^t \delta_s dB_s$$

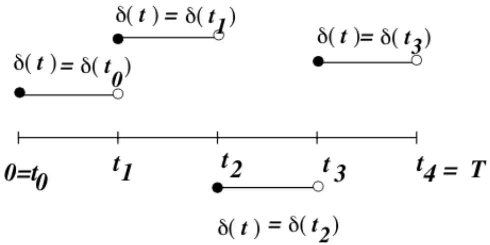
where

- **Integrator** $(B_t)_{t\geq 0}$ is a Brownian motion associated with filtration $(\mathcal{F}_t)_{t\geq 0}$
- **Integrand** $(\delta_t)_{t\geq 0}$ is $\mathcal{F}_t$ - adapted and square integrable

$$E\int_0^T \delta_s^2 ds < \infty, \ \forall T$$

Elementary Integrand

- $0 = t_0 < t_1 < ... < t_n = T$ is a partition of $[0, T]$
- δ_t is a constant on each subinterval $[t_k, t_{k+1}]$ then $(\delta_t)_{t \geq 0}$ is called an **elementary process**

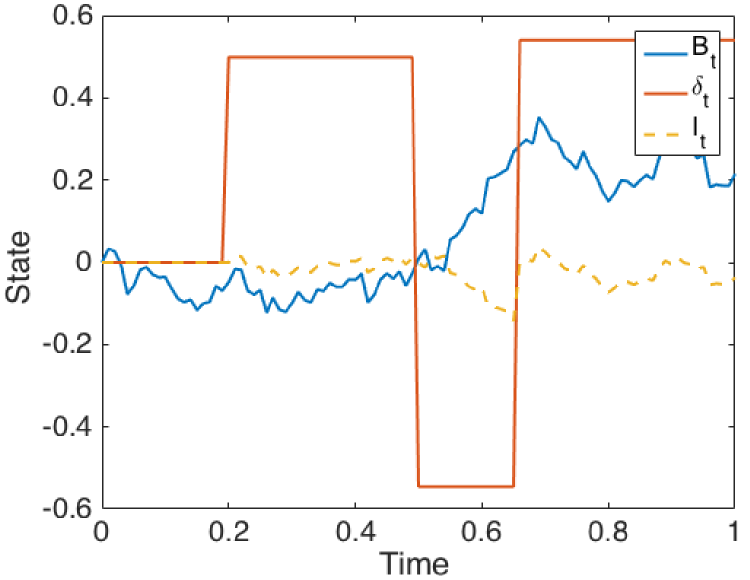


Construction of Itô integral of an Elementary Integrand

$$I_t = \sum_{i=0}^{k-1} \delta_{t_i}(B_{t_{i+1}} - B_{t_i}) + \delta_{t_k}(B_t - B_{t_k})$$

for $t_k \leq t \leq t_{k+1}$

A sample path of Itô integral of an Elementary Integrand



One interpretation

- B_t as the price per unit share of an asset at time t.
- $t_0, t_1, ..., t_n$ as the trading dates for the asset.
- δ_k as the number of shares of the asset acquired at trading date t_k and held until trading date t_{k+1}.
- the (accumulated) gain from trading at time t

$$I_t = \begin{cases} \delta_{t_0}(B_t - B_{t_0}), & t_0 \leq t \leq t_1 \\ \delta_{t_0}(B_{t_1} - B_{t_0}) + \delta_{t_1}(B_t - B_{t_1}), & t_1 \leq t \leq t_2 \\ \dots \end{cases}$$

In general, if $t_k \leq t \leq t_{k+1}$

$$I_t = \sum_{i=0}^{k-1} \delta_{t_i}(B_{t_{i+1}} - B_{t_i}) + \delta_{t_k}(B_t - B_{t_k})$$

Example - non random elementary integrand

- $\delta_t = 1$ for all t then

$$I_t = B_t - B_0 = B_t \sim \mathcal{N}(0, t)$$

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- $$\delta_t = \begin{cases} -1 & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } 1 < t \leq 2 \\ 2 & \text{if } 2 < t \leq 3 \end{cases}$$

  then

  $$I_3 = -1(B_1 - B_0) + 1(B_2 - B_1) + 2(B_3 - B_2) = 2B_3 - B_2 - 2B_1$$

  with distribution $I_3 \sim \mathcal{N}(0, 6)$

Example - non random elementary integrand

- $\delta_t = 1$ for all t then

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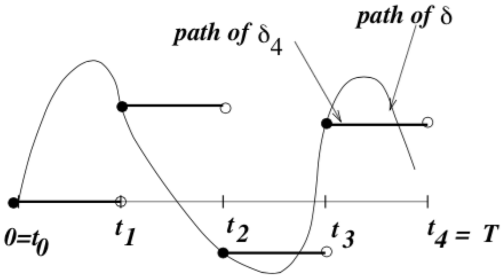
$$I_{2.5} = ?$$

Approximating a general process by an elementary process

Let (δ_t) be an integral then there exists a sequence of elementary processes $(\delta_t^{(n)})_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} E \int_0^T |\delta_s^{(n)} - \delta_s|^2 ds = 0$$



Using "left - end point approximation"

Construction of Itô's integrals for general integrands

$$I_t = \int_0^t \delta_s dB_s = \lim_{n\to\infty} \int_0^t \delta_s^{(n)} dB_s$$

Example

In order to compute

$$I_1 = \int_0^1 \underbrace{s}_{\delta_s} dB_s$$

- Partition for $[0,1]$

$$\Pi = \{s_0 = 0, s_1 = \frac{1}{n}, s_2 = \frac{2}{n}, \dots, s_n = \frac{n}{n} = 1\}$$

  $(s_i = \frac{i}{n})$
- Approximate general process δ_s by elementary process

$$\delta_s^{(n)} = \begin{cases} 0 & \text{if } 0 \leq s < \frac{1}{n} \\ \frac{1}{n} & \text{if } \frac{1}{n} \leq s < \frac{2}{n} \\ \dots & \\ \frac{n-1}{n} & \text{if } \frac{n-1}{n} \leq s < \frac{n}{n} \end{cases}$$

- Compute Itô integral for elementary process

$$
\begin{aligned}
I_1^{(n)} &= \int_0^1 \delta_s^{(n)} dB_s \\
&= 0(B_{\frac{1}{n}} - B_0) + \frac{1}{n}(B_{\frac{2}{n}} - B_{\frac{1}{n}}) + \cdots + \frac{n-1}{n}(B_{\frac{n}{n}} - B_{\frac{n-1}{n}}) \\
&= \sum_{i=0}^{n-1} \frac{i}{n}(B_{\frac{i+1}{n}} - B_{\frac{i}{n}})
\end{aligned}
$$

- Let $n \to \infty$ to obtain the Itô integral

$$
I_1 = \lim_{n\to\infty} I_1^{(n)} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{i}{n}(B_{\frac{i+1}{n}} - B_{\frac{i}{n}})
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$$\begin{aligned} I_1^{(n)} &= \int_0^1 \delta_s^{(n)} dB_s \\ &= 0(B_{\frac{1}{n}} - B_0) + \frac{1}{n}(B_{\frac{2}{n}} - B_{\frac{1}{n}}) + \cdots + \frac{n-1}{n}(B_{\frac{n}{n}} - B_{\frac{n-1}{n}}) \\ &= \sum_{i=0}^{n-1} \frac{i}{n}(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}) \end{aligned}$$

- Let $n \to \infty$ to obtain the Itô integral

$$I_1 = \lim_{n\to\infty} I_1^{(n)} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{i}{n}(B_{\frac{i+1}{n}} - B_{\frac{i}{n}})$$

- Distribution of I_1?

Example

Compute the Itô integral

$$I_T = \int_0^T B_s dB_s$$

Example

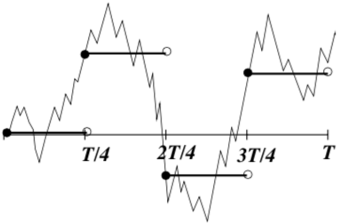
Compute the Itô integral

$$I_T = \int_0^T B_s dB_s$$

Solution

Approximate integrand B_s by the simple process

$$\delta_u^{(n)} = \begin{cases} B_0 = 0 \text{ if } 0 \leq s < T/n \\ B_{T/n} \text{ if } T/n \leq s < 2T/n \\ B_{2T/n} \text{ if } 2T/n \leq s < 3T/n \\ \dots \\ B_{(n-1T)/n} \text{ if } (n-1)T/n \leq s < T \end{cases}$$



Then

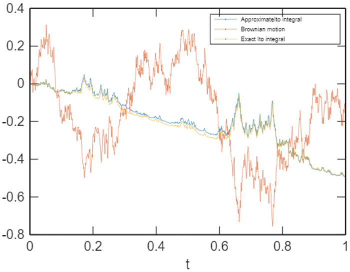
$$I_T = \lim_{n\to\infty} \sum_{k=0}^{n-1} B_{kT/n}(B_{(k+1)T/n} - B_{kT/n})$$

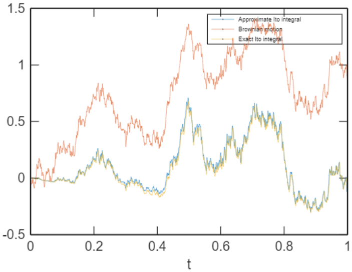
- Prove that

$$\sum_{k=0}^{n-1} B_{kT/n}(B_{(k+1)T/n} - B_{kT/n}) = \frac{1}{2}B_T^2 - \frac{1}{2}\sum_{k=0}^{n-1}(B_{(k+1)T/n} - B_{kT/n})^2$$

- let $n \to \infty$, we have

$$I_T = \frac{1}{2}B_T^2 - \frac{1}{2}\langle B\rangle(T) = \frac{1}{2}B_T^2 - \frac{1}{2}T$$


0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
0.2
0.4
0.6
0.8
1
t
ApproximateIto integral
Brownian motion
Exact Ito integral


1.5
1
0.5
0
-0.5
0
0.2
0.4
0.6
0.8
1
t
Approximate Ito integral
Brownian motion
Exact Ito integral

Practice

Write the following limits in form of Itô integral

1. $$\lim_{n\to\infty} \sum_{i=0}^{n-1} \sin\left(\frac{i}{n}\right)\left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right)$$

2. $$\lim_{n\to\infty} \sum_{i=0}^{n-1} e^{\left(\frac{i}{n}\right)}\left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right)$$

Practice

Write the following Itô integrals in form of limits by definition

1. $\int_0^1 s^2 dB_s$
2. $\int_0^1 B_s^2 dB_s$

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Example - deterministic integrand

Consider Itô integral

$$I_1 = \int_0^t sB_s$$

Let's try to find some possible value of I_t by simulation first

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Consider Itô integral

$$I_1 = \int_0^t sB_s$$

Let's try to find some possible value of I_t by simulation first

- Choose step size $h = 10^{-5}$ (close to 0)
- Simulate a sequence value of standard Brownian motion at time $0, h, \ldots, nh = 1$

$$B = \begin{pmatrix} B_0 & B_h & B_{2h} & \ldots & B_{nh} \end{pmatrix}$$

- Compute the value of integrand $\delta_s = s$ at the left - end point of each subinterval

$$\begin{aligned}
\delta_0 &= 0, && [0, h) \\
\delta_h &= h, && [h, 2h) \\
\ldots \\
\delta_{(n-1)h} &= (n-1)h, && [(n-1)h, nh)
\end{aligned}$$

- Stock values of integrand in a vector

$$\delta = \begin{pmatrix} \delta_0 & \delta_h & \delta_{2h} & \dots & \delta_{(n-1)h} \end{pmatrix}$$

- Evaluate change of Brownian motion in each subinterval

$$dB = \begin{pmatrix} B_h - B_0 & B_{2h} - B_h & \dots & B_{nh} - B_{(n-1)h} \end{pmatrix}$$

- Compute

$$\begin{aligned} I_1 =& \delta_0(B_h - B_0) + \delta_h(B_{2h} - B_h) + \delta_{2h}(B_{3h} - B_{2h}) \\ &+ \cdots + \delta_{(n-1)h}(B_{nh} - B_{(n-1)h}) \end{aligned}$$

Example - random integrand

Simulate a value of Itô integral

$$I_1 = \int_0^1 B_s^2 dB_s$$

- Choose step size $h = 10^{-5}$ (close to 0)
- Simulate a sequence value of standard Brownian motion at time $0, h, \dots, nh = 1$

$$B = \begin{pmatrix} B_0 & B_h & B_{2h} & \dots & B_{nh} \end{pmatrix}$$

- Compute the value of integrand $\delta_s = s$ at the left - end point of each subinterval

$$\begin{aligned} \delta_0 &= B_0^2, & [0, h) \\ \delta_h &= B_h^2, & [h, 2h) \\ \dots & & \\ \delta_{(n-1)h} &= B_{(n-1)h}^2, & [(n-1)h, nh) \end{aligned}$$

- Stock values of integrand in a vector

$$\delta = \begin{pmatrix} \delta_0 & \delta_h & \delta_{2h} & \dots & \delta_{(n-1)h} \end{pmatrix}$$

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- Compute

$$I_1 = \overbrace{\overbrace{\overbrace{\delta_0(B_h - B_0)}^{I_h} + \delta_h(B_{2h} - B_h)}^{I_{2h}} + \delta_{2h}(B_{3h} - B_{2h})}^{I_{3h}} \\ + \cdots + \delta_{(n-1)h}(B_{nh} - B_{(n-1)h})$$

- Stock values of integrand in a vector

$$\delta = \begin{pmatrix} \delta_0 & \delta_h & \delta_{2h} & \dots & \delta_{(n-1)h} \end{pmatrix}$$

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- Compute

$$\begin{aligned} I_1 = {} & \overbrace{\overbrace{\overbrace{\delta_0(B_h - B_0)}^{I_h} + \delta_h(B_{2h} - B_h)}^{I_{2h}} + \delta_{2h}(B_{3h} - B_{2h})}^{I_{3h}} \\ & + \cdots + \delta_{(n-1)h}(B_{nh} - B_{(n-1)h}) \end{aligned}$$

$(I_t)_{t \geq 0}$ is a random (stochastic) process

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Properties of general Itô integral

$$I_t = \int_0^t \delta_s dB_s \qquad\qquad I_t = \int_0^t \gamma_s dB_s$$

- **Continuity** I_t is a continuous function of the upper limit of integration
- **Adaptness** For each t, I_t is $\mathcal{F}_t$ - measurable
- **Linearity**

$$I_t \pm J_t = \int_0^t (\delta_s \pm \gamma_u) dB_s \qquad c I_t = \int_0^t c\delta_s dB_s$$

- **Martingale** (I_t) is a $(\mathcal{F}_t)$ martingale
- **Isometry**

$$E I_t^2 = E \int_0^t \delta_s^2 ds$$

- **Quadratic variation**

$$\langle I, I\rangle(t) = \int_0^t \delta_s^2 ds$$

Proof

- Provide detail proof for elementary integrand
- Taking limit to get the corresponding result for general integrand

Proof for Martingale property

Case 1 $t_k \le s < t \le t_{k+1}$

$$E(I_t|\mathcal{F}_s) = E\left(\sum_{i=0}^{k-1} \underbrace{\delta_{t_i}}_{\mathcal{F}_s-\text{measurable}} \underbrace{(B_{t_{i+1}} - B_{t_i})}_{\mathcal{F}_s-\text{measurable}} + \delta_{t_k}(B_t - B_{t_k})|\mathcal{F}_s\right)$$

$$= \underbrace{\sum_{i=0}^{k-1} \delta_{t_i}(B_{t_{i+1}} - B_{t_i})}_{\text{take out of what is known}} + E(\underbrace{\delta_{t_k}}_{\mathcal{F}_s-\text{measurable}} (B_t - B_{t_k})|\mathcal{F}_s)$$

$$= \sum_{i=0}^{k-1} \delta_{t_i}(B_{t_{i+1}} - B_{t_i}) + \underbrace{\delta_{t_k}}_{\text{take out of what is known}} E(B_t - B_{t_k}|\mathcal{F}_s)$$

$$= \sum_{i=0}^{k-1} \delta_{t_i}(B_{t_{i+1}} - B_{t_i}) + \delta_{t_k}\underbrace{(E(B_t|\mathcal{F}_s) - E(\overbrace{B_{t_k}}^{\mathcal{F}_s-\text{measurable}}|\mathcal{F}_s))}_{\text{linear property}}$$

$$= \sum_{i=0}^{k-1} \delta_{t_i}(B_{t_{i+1}} - B_{t_i}) + \delta_{t_k}(\underbrace{B_s}_{B \text{ is a martingale}} - \underbrace{B_{t_k}}_{\text{take out of what is known}}) = I_s$$

Case 2 $t_l \le s \le t_{l+1} \le t_k \le t \le t_{k+1}$

- We have

$$I_t = \sum_{i=0}^{k-1} \delta_{t_i}(B_{t_{i+1}} - B_{t_i}) + \delta_{t_k}(B_t - B_{t_k})$$
$$= \sum_{i=0}^{l} \delta_{t_i}(B_{t_{i+1}} - B_{t_i}) + \sum_{i=l}^{k-1} \delta_{t_i}(B_{t_{i+1}} - B_{t_i}) + \delta_{t_k}(B_t - B_{t_k})$$
$$= I_{t_{l+1}} + \sum_{i=l}^{k-1} \delta_{t_i}(B_{t_{i+1}} - B_{t_i}) + \delta_{t_k}(B_t - B_{t_k})$$

So

$$E(I_t|\mathcal{F}_s) = \underbrace{E(I_{t_{l+1}}|\mathcal{F}_s)}_{I} + \underbrace{\sum_{i=l}^{k-1} E(\delta_{t_i}(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_s)}_{II}$$
$$+ \underbrace{E(\delta_{t_k}(B_t - B_{t_k})|\mathcal{F}_s)}_{III}$$

- $I = E(I_{t_{l+1}}|\mathcal{F}_s) = I_s$ (case 1)
- Need to prove $E(II|\mathcal{F}_s) = 0$ and $E(III|\mathcal{F}_s) = 0$

- **Each terms in II is equal to 0**

For $l \leq i \leq k-1$, we have

$$II_i = E(\delta_{t_i}(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_s) = \underbrace{E(E(\delta_{t_i}(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_{t_i})|\mathcal{F}_s))}_{\text{iteration } \mathcal{F}_s \subset \mathcal{F}_{t_i} \text{ since } s \leq t_i}$$

Compute the inner part

$$E(\underbrace{\delta_{t_i}}_{\mathcal{F}_{t_i}-measurable}(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_{t_i}) = \underbrace{\delta_{t_i}}_{\text{take out of what is known}} E(B_{t_{i+1}} - B_{t_i}|\mathcal{F}_{t_i})$$

$$= \delta_{t_i}\left(\underbrace{\overbrace{E(B_{t_{i+1}}|\mathcal{F}_{t_i})}^{B_{t_i} \text{ since } B \text{ is a martingale}} - E(\overbrace{B_{t_i}}^{\mathcal{F}_{t_i}-measurable}|\mathcal{F}_{t_i})}_{\text{linear property}}\right)$$

$$= \delta_{t_i}(B_{t_i} - B_{t_i}) = 0$$

Hence

$$II_i = E(0|\mathcal{F}_s) = 0$$

- Prove that $III = 0$

$$III = E(\delta_{t_k}(B_t - B_{t_k})|\mathcal{F}_s) = \underbrace{E(E(\delta_{t_k}(B_t - B_{t_k})|\mathcal{F}_{t_k})|\mathcal{F}_s))}_{\text{iteration } \mathcal{F}_s \subset \mathcal{F}_{t_k} \text{ since } t_k \geq s}$$

Compute the inner part

$$E(\underbrace{\delta_{t_k}}_{\mathcal{F}_{t_k}-measurable}(B_t - B_{t_k})|\mathcal{F}_{t_k}) = \underbrace{\delta_{t_k}}_{\text{take out of what is known}} E(B_t - B_{t_k}|\mathcal{F}_{t_k})$$

$$=\delta_{t_k}\left(\underbrace{\overbrace{E(B_t|\mathcal{F}_{t_k})}^{B_{t_k} \text{ since } B \text{ is a martingale}} - E(\overbrace{B_{t_k}}^{\mathcal{F}_{t_k}-measurable}|\mathcal{F}_{t_k})}_{\text{linear property}}\right)$$

$$=\delta_{t_k}(B_{t_k} - B_{t_k}) = 0$$

Hence

$$III = E(0|\mathcal{F}_s) = 0$$

Proof for Isometry property $EI_t^2 = E\int_0^t \delta_s^2 ds$

- To simplification, assume $t = t_k$ then

$$I_t = \sum_{i=0}^{k-1} \delta_{t_i} D_i$$

  where $D_i = B_{t_{i+1}} - B_{t_i} \sim \mathcal{N}(0, t_{i_1} - t_i)$ are independent

- $$I_t^2 = \left(\sum_{i=0}^{k-1} \delta_{t_i} D_i \right)^2 = \underbrace{\sum_{i=0}^{k-1} \delta_{t_i}^2 D_i^2}_{I} + \underbrace{2 \sum_{i<j} \delta_{t_i} D_i \delta_{t_j} D_j}_{II}$$

- Each term in II has expectation zero and then $E(II) = 0$
  For $i < j$

$$E(\underbrace{\delta_{t_i} D_i \delta_{t_j}}_{\mathcal{F}_{t_j}-measurable} \underbrace{D_j}_{\text{independent of } \mathcal{F}_{t_j}}) \overbrace{=}^{iteration} E(E(\underbrace{\delta_{t_i}\delta_{t_j} D_i}_{\mathcal{F}_{t_j}-measurable} D_j|\mathcal{F}_{t_j}))$$

The inner conditional expectation is

$$E(\underbrace{\delta_{t_i}\delta_{t_j} D_i}_{\mathcal{F}_{t_j}-measurable} D_j|\mathcal{F}_{t_j}) = \delta_{t_i}\delta_{t_j} D_i E(\underbrace{D_j}_{\text{independent of } \mathcal{F}_{t_j}} |\mathcal{F}_{t_j})$$

$$= \delta_{t_i}\delta_{t_j} D_i \underbrace{E(D_j)}_{=0} = 0$$

- Expectation of each term in I

  For $0 \leq i \leq k-1$

$$I_i = E\left(\underbrace{\delta_{t_i}^2}_{\mathcal{F}_{t_i}-measurable} \underbrace{D_i^2}_{\text{independent of } \mathcal{F}_{t_i}}\right) = E\left(E\left(\delta_{t_i}^2 D_i^2 | \mathcal{F}_{t_i}\right)\right)$$

  Compute the inner conditional expectation

$$E\left(\delta_{t_i}^2 D_i^2 | \mathcal{F}_{t_i}\right) = \underbrace{\delta_{t_i}^2}_{\text{take out of what is known}} E\left(D_i^2 | \mathcal{F}_{t_i}\right)$$

$$= \delta_{t_i}^2 \underbrace{E(D_i^2)}_{Var(D_i)+(E(D_i))^2=t_{i+1}} = \delta_{t_i}^2 (t_{i+1} - t_i)$$

- So

$$E(I_t^2) = E(I) = \sum_{i=0}^{k-1} I_i = \sum_{i=0}^{k-1} (t_{i+1} - t_i) E(\delta_{t_i}^2) = E\left(\underbrace{\sum_{i=0}^{k-1} (t_{i+1} - t_i) \delta_{t_i}^2}_{\int_0^t \delta_s ds}\right)$$

Proof for Quadratic variation $\langle I, I\rangle(t) = \int_0^t \delta_s^2 ds$

- Let $\Pi = \{t_0, t_1, ..., t_n\}$ be the partition for δ, i.e $\delta_t = \delta_{t_k}$ for $t_k \leq t < t_{k+1}$. To simplify notation, we assume that $t = t_n$. So

$$\langle I, I\rangle(t) = \sum_{k=1}^{n-1} \langle I\rangle(t_{k+1}) - \langle I\rangle(t_k)$$

- Compute $\langle I\rangle(t_{k+1}) - \langle I\rangle(t_k)$
Let $\Xi = \{s_0, s_1, \ldots, s_m\}$ be a partition of $[t_k, t_{k+1}]$ then

$$I_{s_{j+1}} - I_{s_j} = \int_{s_j}^{s_{j+1}} \delta_{t_k} dB_u = \delta_{t_k}(B_{s_{j+1}} - B_{s_j})$$

So

$$\langle I, I\rangle(t_{k+1}) - \langle I, I\rangle(t_k) = \lim_{\|\Xi\|\to 0} \sum_{j=1}^{m-1} (I_{s_{j+1}} - I_{s_j})^2$$

$$= \lim_{\|\Xi\|\to 0} \sum_{j=1}^{m-1} \delta_{t_k}^2 (B_{s_{j+1}} - B_{s_j})^2 = \sum_{j=1}^{m-1} \delta_{t_k}^2 (s_{j+1} - s_j) = \delta_{t_k}^2 (t_{k+1} - t_k)$$

- $$\langle I, I\rangle(t) = \delta_{t_k}^2 (t_{k+1} - t_k) = \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \delta_u^2 du = \int_0^t \delta_u^2 du$$

Integral by parts theorem for deterministic integrand

Suppose the $f(s)$ is a continuous function on $[0, t]$ with bounded variation then

$$\underbrace{\int_0^t f(s)dB_s}_{\text{Itô integral}} = f(t)B_t - \underbrace{\int_0^t B_s df(s)}_{\text{Stieltjes integral}}$$

Example

$$\int_0^t s dB_s = tB_t - \int_0^t B_s ds$$

Analyze the LHS

- Choose $s_k = \frac{kt}{n}$, $k = 0, 1, \ldots, n$
- Left - end point rule gives an approximation for the LHS

$$\underbrace{\overbrace{f(0)\underbrace{\left(B_{\frac{t}{n}} - B_0\right)}_{\mathcal{N}(0,t/n)}}^{\mathcal{N}(0,f^2(0)\frac{t}{n})} + \overbrace{f(\frac{t}{n})\underbrace{\left(B_{\frac{2t}{n}} - B_{\frac{t}{n}}\right)}_{\mathcal{N}(0,t/n)}}^{\mathcal{N}(0,f^2(\frac{t}{n}))\frac{t}{n}} + \ldots \overbrace{f(\frac{(n-1)t}{n})\underbrace{\left(B_{\frac{nt}{n}} - B_{\frac{(n-1)t}{n}}\right)}_{\mathcal{N}(0,t/n)}}^{\mathcal{N}(0,f^2(\frac{(n-1)t}{n})\frac{t}{n})}}_{independent}$$

$$\sim \mathcal{N}\left(0, \underbrace{f^2(0)\frac{t}{n} + f^2(\frac{t}{n})\frac{t}{n} + \cdots + f^2(\frac{(n-1)t}{n})\frac{t}{n}}_{\to \int_0^t f^2(s)ds}\right)$$

Distribution of Itô integral with deterministic integrand

Suppose the $f(s)$ is a nonrandom function on $[0,t]$ then

$$I_t = \int_0^t f(s)dB_s$$

is normally distributed with

- mean

$$EI_t \underbrace{=}_{Martingale} EI_0 = \int_0^0 f(s)dB_s = 0$$

- variance

$$Var(I_t) \underbrace{=}_{E(I_t=0)} E(I_t^2) \underbrace{=}_{isometry} E\left(\int_0^t f^2(s)ds\right) = \int_0^t f^2(s)ds$$

.

Example

Find the distribution of

$$I_t = \int_0^t s dB_s$$

Solution

- The integrand $\delta_s = s$ is a deterministic function (only depends on time s)
- I_t is normally distributed
- Mean 0
- Variance

$$Var(I_t) = EI_t^2 = \int_0^t u^2 du = \frac{t^3}{3}$$

- $I_t \sim \mathcal{N}(0, \frac{t^3}{3})$

Practice

Find the distribution of

$$I_t = \int_0^t \sin(s) dB_s$$

Exercise

1. Show that
$$\int_0^t (t-s)dB_s = \int_0^t B_s ds$$
2. Find the distribution of
$$I_t = \int_0^t (t-s)dB_s$$