INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

FINAL EXAMINATION
January 2016
Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Deputy Head of Dept. of Mathematics:	Lecturer:
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INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 *(25 marks)* Let $(X, \mathcal{M})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$ a measurable function. Let $a, b \in \mathbb{R}$, $a < b$. Show that the following sets are measurable:

$$E = \{x \in X : a < f(x) \leq b\},$$
$$F = \{x \in X : f(x) = +\infty\}.$$

Question 2

(a) *(15 marks)* Let

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine the following Lebesgue integrals

$$\int_{\frac{1}{n}}^{1} f(x)dx \quad \text{and} \quad \int_0^1 f(x)dx, \quad n = 1, 2, \ldots.$$

(b) *(10 marks)* Let $f : [a, b] \to \mathbb{R}$ be decreasing. Show that

$$V_a^b(f) = f(a) - f(b).$$

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Question 3

(a) *(10 marks)* Show that if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu$, then $\mu_1 \ll \mu$.

(b) *(15 marks)* Let $\int_X f d\mu$ be defined and let

$$\nu(E) = \int_E f d\mu \quad \text{if } E \text{ is measurable.}$$

Show that if S is a negative set for ν, then $\mu\big(S \cap \{x \in X : f(x) > 0\}\big) = 0$.

Question 4

(a) *(15 marks)* Let f be integrable on X with respect to the measure μ. Let $E_n = \{x \in X : |f(x)| \leq n\}$, $n = 1, 2, \ldots$. Show that $\chi_{E_n} f \to f$ a.e. Apply the Dominated Convergence Theorem to the sequence $\{\chi_{E_n} f\}$ to show that

$$\lim_{n\to\infty} \int_{E_n} f d\mu = \int_X f d\mu.$$

(*Hint:* Show that $\lim_{n\to\infty} \big(\chi_{E_n} f\big)(x) = f(x)$ if $f(x)$ is finite.)

(b) *(10 marks)* Show that if f is Lebesgue integrable on $\mathbb{R}$, then for any $a, b \in \mathbb{R}$, $a < b$, the function $F(x) = \int_{-\infty}^{x} f(t)dt$ is absolutely continuous on $[a, b]$ and $F'(x) = f(x)$ a.e. on $[a, b]$.

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SOLUTIONS

Question 1 Since f is measurable on X, the sets $\{x : f(x) \leq a\}$ and $\{x : f(x) \leq b\}$ are measurable. It follows that

$$E = \{x \in X : a < f(x) \leq b\} = \{x : f(x) \leq a\} \cap \{x : f(x) \leq b\}$$

is measurable.

For each n, the set $\{x \in X : f(x) > n\}$ is measurable. Thus

$$F = \{x \in X : f(x) = +\infty\} = \bigcap_{n=1}^{\infty} \{x \in X : f(x) > n\}$$

is measurable.

Question 2

(a) Since f is continuous on $(0, 1]$, it is Riemann integrable on each subinterval $[1/n, 1]$, hence the Riemann integral and Lebesgue integral of f on this subinterval are both equal to

$$\int_{1/n}^{1} \frac{1}{\sqrt{x}} dx = 2\sqrt{x}\Big|_{1/n}^{1} = 2\left(1 - \frac{1}{\sqrt{n}}\right), \quad n = 1, 2, \ldots .$$

Moreover, f is Lebesgue measurable on $(0, 1]$ so that

$$\nu(E) = \int_E f(x)dx$$

is a measure. Thus

$$\int_0^1 f(x)dx = \int_{(0,1]} f(x)dx = \lim_{n\to\infty} \int_{1/n}^{1} f(x)dx = \lim_{n\to\infty} 2\left(1 - \frac{1}{\sqrt{n}}\right) = 2.$$

The first equality holds because the set $\{0\}$ has Lebesgue measure zero.

(b) Since f is decreasing, $|f(\beta) - f(\alpha)| = f(\alpha) - f(\beta)$ whenever $\alpha, \beta \in [a, b]$, $\alpha < \beta$. If $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$, then

$$\begin{aligned} V_a^b(f, P) &= \sum_{i=1}^{n} \left|f(x_n) - f(x_{n-1})\right| = \sum_{i=1}^{n} \left[f(x_{n-1}) - f(x_n)\right] \\ &= f(x_0) - f(x_n) = f(a) - f(b). \end{aligned}$$

Thus $V_a^b(f) = \sup_P V(f; P) = f(a) - f(b)$.

Question 3

(a) Assume that $\mu(A) = 0$. Since $\mu_2 \ll \mu$, $\mu_2(A) = 0$. Furthermore, $\mu_1 \ll \mu_2$ implies that $\mu_1(A) = 0$. Consequently, $\mu_1 \ll \mu$.

(b) Let $A = S \cap \{x \in X : f(x) > 0\}$. For each n, set $A_n = \{x \in S : f(x) > \frac{1}{n}\}$. We have $A_n \subset A_{n+1}$ and $A_n \subset A$ for all n. Hence $\bigcup_{n=1}^{\infty} A_n \subset A$. Conversely, if $x \in A$, $f(x) > 0$, so there is an n satisfying $f(x) > \frac{1}{n}$, that is $x \in A_n$. Thus $A \subset \bigcup_{n=1}^{\infty} A_n$ and therefore, $A = \bigcup_{n=1}^{\infty} A_n$. Since S is a negative set for ν and $A_n \subset S$, we obtain

$$0 = \nu(A_n) = \int_{A_n} f d\mu \geq \int_{A_n} \frac{1}{n} d\mu = \frac{1}{n}\mu(A_n),$$

implying $\mu(A_n) = 0$. Thus $\mu(A) \leq \sum_{i=1}^{n} \mu(A_n) = 0$, that is $\mu(A) = 0$.

Alternative solution: Let $A = S \cap \{x \in X : f(x) > 0\}$. Since S is a negative set for ν,

$$\nu(A) = \int_A f d\mu \leq 0. \tag{0.0.1}$$

On the other hand, $f \geq 0$ on A so

$$\int_A f d\mu \geq 0. \tag{0.0.2}$$

(0.0.1) and (0.0.2) yield $\int_A f d\mu = 0$. Again, since $f \geq 0$ on A, $f = 0$ a.e., on A, that is $\mu\big(\{x \in A : f(x) \neq 0\}\big) = 0$. However, as $f(x) > 0$ for all $x \in A$, $\{x \in A : f(x) \neq 0\} = A$. Consequently, $\mu(A) = 0$.

Question 4

(a) Since f is integrable on X, it is finite a.e., that is the set $A := \{x \in X : |f(x)| = \infty\}$ has μ-measure zero. Fix $x_0 \notin A$. Since $f(x_0)$ is finite, there exists n_0 satisfying $n_0 > |f(x_0)|$. This implies that $x_0 \in E_n$ for all $n \geq n_0$. Thus $\big(\chi_{E_n} f\big)(x_0) = \chi_{E_n}(x_0) f(x_0) = f(x_0)$ for $n \geq n_0$ and therefore $\lim_{n\to\infty} \big(\chi_{E_n} f\big)(x_0) = \lim_{n\to\infty} f(x_0) = f(x_0)$. This holds for all $x_0 \in A^c$, and so $\chi_{E_n} f \to f$ a.e. Moreover, $|\chi_{E_n} f| \leq |f|$ for all n and f is integrable on X, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{E_n} f d\mu = \lim_{n\to\infty} \int_X \chi_{E_n} f d\mu = \int_X f d\mu.$$

(b) Since f is Lebesgue integrable on $\mathbb{R}$, it is integrable on $(\infty, a]$ and $[a, x]$ for $a \leq x \leq b$. Let $C = \int_{-\infty}^{a} f(t)dt$ and $G(x) = \int_a^x f(t)dt$. By additivity, we have $F(x) = G(x) + C$ for $a \leq x \leq b$. Since G is absolutely continuous on $[a, b]$ and $G'(x) = f(x)$ a.e. on $[a, b]$, it follows that F is absolutely continuous on $[a, b]$ and $F'(x) = G'(x) = f(x)$ a.e.