1 Symmetric Random walk

We toss a fair coin infinitely many times. The probability for landing head up or landing tail up is 0.5. Let ω_i be the outcome of the ith toss then $\omega_i = H$ if the coin lands head up or $\omega_i = T$ if the coin lands tail up. Let X_i be a random variable such that $X_i(\omega_i) = \begin{cases} 1 & \text{if} \quad \omega_i = H \\ -1 & \text{if} \quad \omega_i = T \end{cases}$. As X_i can only receive two values, it can be considered as a Bernoulli random variable. The outcomes of different tosses are independent to each other, thus X_i and X_j are independent random variables for any $i \neq j$.

For the whole process of infinitely tossing coin, each outcome of the experiment would be a sequence as $\omega = \omega_1\omega_2\cdots$, where ω_i denotes the outcome of the ith toss. Let Ω be a sample space of the experiment, which collects all possible outcomes $\omega = \omega_1\omega_2\cdots$. For each $k \in \mathbb{N}$, we denote $M_k = \sum_{i=1}^{k} X_i$, a sum of Bernoulli random variables on Ω. Thus, M_k is a binomial random variable. The collection $(M_k)_{k\in\mathbb{N}}$ forms a process on the probability space and is called a symmetric random walk.

Example 1.1. Consider the following game. For each toss of a fair coin, you will win \$1 if it lands up heads and lose \$1 if it lands up tails. Suppose that you start the game with \$0 and denote M_k is the amount of money you obtain after k tosses. Then $(M_k)_{k\in\mathbb{N}}$ is a random walk.

1.1 Martingale processes

A continuous-time stochastic process $(X_t)_{t\geq 0}$ is said to be a martingale with respect to the filtration $\{\mathcal{F}_t\}$ if it is adapted to $\{\mathcal{F}_t\}$ and satisfies the condition $E(|X_t|) < \infty, \quad \forall t \geq 0$ and $E(X_t|\mathcal{F}_s) = X_s, \ \forall 0 \leq s < t$ (the best prediction of the value of X_t given the information available at time $s < t$ is the value of X_s). If $E[X_t|\mathcal{F}_s] \geq X_S, \ \forall 0 \leq s \leq t$ then the process is called submartingale (no tendency to fall, might have a tendency to rise). If $E[X_t|\mathcal{F}_s] < X_S, \ \ \forall 0 \leq s \leq t$ then the process is called supermartingale (no tendency to rise, might have a tendency to fall).

Taking the unconditional expectation to both side of $E(X_t|\mathcal{F}_s) = X_s$, we have $E(X_s) = E(E(X_t|\mathcal{F}_s)) = E(X_t)$. Thus the expected future gain is zero and a martingale process models a fair game. The importance of martingale processes in modern finance is self-explanatory since the security evaluation is the determination of the fair price of a security.

DISCRETE VERSION. A discrete-time stochastic process $(X_n)_{n\in\mathbb{N}}$ on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is a martingale process with respect to the filtration $(F_n)_{n\in\mathbb{N}}$ if $(X_n)_{n\in\mathbb{N}}$ satisfies the condition $E(|X_n|) < \infty, \ \forall n \geq 0$ and $E(X_{n+1}|\mathcal{F}_n) = X_n$.

According to Fama, a market in which market prices always fully reflect available information is called efficient. Efficient market hypothesis is closely related to the martingale condition.

Important note 1.1. Sometimes, a stochastic process $(X_t)_{t\geq 0}$ is said to be a martingale process, without specifying the corresponding filtration. In that case, the corresponding filtration is "quietly" understood as the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the process, i.e., $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ (the σ-algebra generated from all random variables $X_s, 0 \leq s \leq t$)

In other cases, a stochastic process $(X_t)_{t\geq 0}$ is said to be a martingale process with respect to another stochastic process $(Y_t)_{t\geq 0}$. In these cases, the corresponding filtration is the natural filtration of the process $(Y_t)_{t\geq 0}$.

1.2 SOME PROPERTIES OF A SYMMETRIC RANDOM WALK

1. A random walk has independent increments. Increments over non-overlapping time intervals are independents as they depend on different coin tosses. Mathematically, $M_k - M_l$ and $M_t - M_s$ are independent random variables for any $0 \le k \le l \le s \le t$.

2. Each increment has expected value 0. More precisely, $E[M_k - M_l] = E[\sum_{i=l+1}^{k} X_i] = \sum_{i=l+1}^{k} E[X_i] = 0$ (note that $(X_i)_{i \ge 1}$ are independent random variables).

3. Each increment $M_k - M_l$ has variance $k - l$, for any $l \le k$.

4. $(M_k)_{k \in \mathbb{N}}$ is a martingale process. More precisely,

$$E[M_t|\mathcal{F}_s] = E[(M_t - M_s) + M_s|\mathcal{F}_s] = E[M_t - M_s|\mathcal{F}_s] + M_s = E[M_t - M_s] + M_s = M_s. \tag{1}$$

5. Quadratic variation up to time t, defined as $[M, M]_k = \sum_{j=1}^{k}(M_j - M_{j-1})^2$ is equal to k. This is because that $(M_j - M_{j-1})^2 = X_j^2 = 1, \quad \forall j \in \mathbb{N}$.

1.3 The limit of a symmetric random walk

It can be proved that the limit of the symmetric random walks is a continuous stochastic process, called Brownian motion(see [**?**]).

We now simulate symmetric random walk to illustrate this graphically.
R codes to illustrate this convergence is

```
RandWalk=function(n)
{set.seed (123)
T <- 1
t <- seq (0,T, length =1000) ### divide [0,T] into 100 interval
### We now simulate a sequene of random variables $X_i$ taking values $1$ and $-1$ with
equal probability via the uniform distribution. The R command runif(n)
generates n random numbers from the distribution in (0,1),
and runif(n)>0.5 transforms these into a sequence of zeros and ones.
Now, if x is either 0 and 1, the function 2*x-1 maps 0 to -1 and 1 to 1.
Thus we now have a sequence of n equally distributed random numbers -1 and 1
and command cumsum claculates $S_n$ for us.
S <- cumsum (2*( runif (n ) >0.5) -1)
W <- sapply (t, function (x) ifelse (n*x >0,S[n*x] ,0))
W <- as.numeric (W)/ sqrt (n)
plot (t,W, type ="l",ylim =c( -1.2 ,1.2))
}
```

Running the command RandWalk(1000) in R, we obtain Figure 1.

2 Brownian motion

Brownian motion is a key thing in building stochastic processes. Understanding Brownian motion is a very first step in studying stochastic processes.

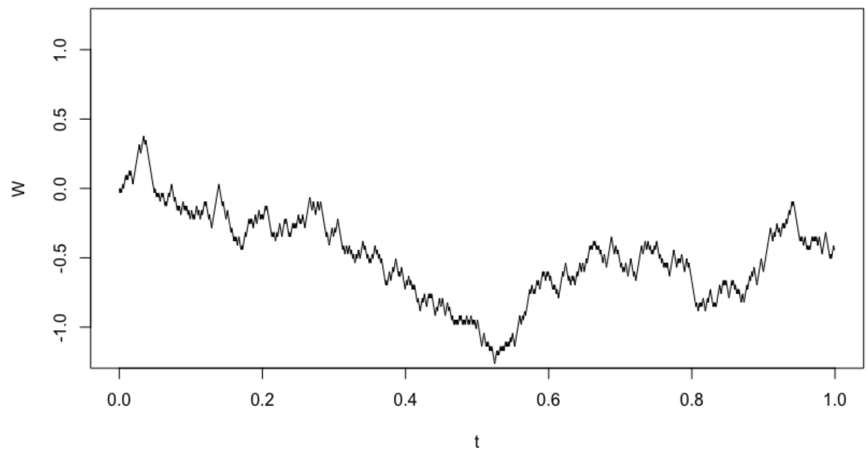

Figure 1: Random walk S_{1000}

Brownian motion is a stochastic process, which is rooted in a physical phenomenon discovered almost 200 years ago. In 1827, the botanist Robert Brown, observing pollen grains suspended in water, noted the erratic and continuous movement of tiny particles ejected from the grains. He studied the phenomenon for many years, ruled out the belief that it emanated from some "life force" within the pollen, but could not explain the motion. Neither could any other scientist of the 19th century.

In 1905, Albert Einstein solved the riddle in his paper "On the movement of small particles suspended in a stationary liquid demanded by the molecular-kinetic theory of heat. Einstein explained the movement by the continual bombardment of the immersed particles by the molecules in the liquid, resulting in "motions of such magnitude that these motions can easily be detected by a microscope." Einstein's theoretical explanation was confirmed 3 years later by empirical experiment, which led to the acceptance of the atomic nature of matter.

Einstein showed that the position x of a particle at time t was described by the heat equation:

$$\frac{\partial}{\partial t} f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(x,t),$$

where $f(x,t)$ represents the density (the number of particles per unit volume) at position x and t. The solution to that equation is $f(x,t) = \dfrac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$, which is the probability density function of the normal distribution with mean 0 and variance t.

The mathematical process known as Brownian motion arises as the limiting process of a discrete-time random walk. This is obtained by speeding up the walk, letting the interval time between discrete steps tend to 0. The process is used as a model for many phenomena that exhibit "erratic, zigzag motion", such as stock prices, the growth of crystals, and signal noise.

Definition 2.1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{P})$ be a filtered probability space and let $(B_t)_{0\leq t<\infty}$ be an adapted process of this space. The process $(B_t)_{0\leq t<\infty}$ is called a standard Brownian motion if it satisfies the following properties:

1. $B_0 = 0$ (the process starts at 0 at $t = 0$)

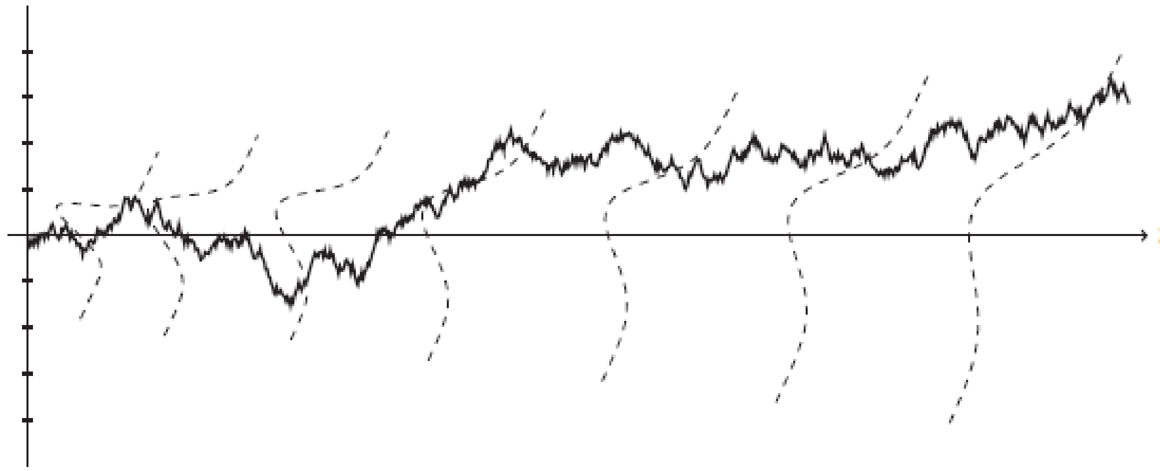

Figure 2: An illustration of Brownian path

2. Independent increments: $B_t - B_s$ is independent of $\mathcal{F}_s$, $0 \le s < t$. That means

$$P(B_t - B_s \le k|\mathcal{F}_s) = P(B_t - B_s \le k).$$

3. Stationary increments: $B_t - B_s$ is a Gaussian (normal) random variable with mean 0, variance $t - s$, i.e.,

$$B_t - B_s \stackrel{d}{=} B_{t-s} \sim N(0, \sqrt{t-s}), \ \forall \ 0 < s < t.$$

The density function of the Normal random variable is $f(x) = \dfrac{1}{\sqrt{2\pi t}} e^{-\dfrac{x^2}{2t}}$. Thus

$$P(B_{t+s} - B_s \le z) = P(B_t \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

4. Continuous paths: all sample paths of process $(B_t)_{t \ge 0}$ are almost surely continuous, i.e.

$$P(\omega \in \Omega | B_t(\omega) \text{ is a continuous sample path}) = 1.$$

Brownian motion can be thought of as the motion of a particle that diffuses randomly along a line. At each point t, the particle's position is normally distributed about the line with variance t. As t increases, the particle's position is more diffuse as illustrated in Figure 2.

It is not obvious that a stochastic process with the properties of Brownian motion actually exists. The American mathematician Nobert Wiener rigorously proved the existence of Brownian motions. Brownian motion is also called the Wiener process, named after him.

Computation involving Brownian motion are often tackled by exploiting stationary and independent increments.

Example 2.1. For $0 < s < t$, find the distribution of $B_s + B_t$, i.e., compute the expectation and variance of the random variable.

Answer 2.1. We express $B_s + B_t = 2B_s + (B_t - B_s)$. The reason for this expression is that we know the distributions of B_s and $B_t - B_s$ as $B_s \sim N(0, s)$ and $B_t - B_s \sim B_{t-s} \sim N(0, t-s)$. In addition, using the property of independent increment, we deduce that B_s and $B_t - B_s$ are independent random variables. As the expectation (the variance) of the sum of independent random variables is the sum of expectations (variances) of individual random variables, we obtain: $E(B_s + B_t) = E(2B_s + B_t - B_s) = E(2B_s) + E(B_t - B_s) = 0$ and

$$\mathrm{Var}(B_s + B_t) = \mathrm{Var}(2B_s) + \mathrm{Var}(B_t - B_s) = 4s + t - s = 3s + t.$$

Example 2.2. A particle's position is modeled by a standard Brownian motion. If the particle is at position 1 at time $t = 2$, then find the probability that its position is at most 3 at time $t = 5$.

Answer 2.2. As the particle position is modeled by a standard Brownian motion $(B_t)_{t\geq 0}$, the position of the particle at time t is B_t. In particular, the position of the particle at time $t = 2$ and $t = 5$ are B_2 and B_5. The desired probability is

$$\begin{aligned} P(B_5 \leq 3|B_2 = 1) &= P(B_5 - B_2 + B_2 \leq 3|B_2 = 1)P(B_5 - B_2 \leq 2|B_2 = 1) = P(B_5 - B_2 \leq 2) \\ &= P(B_3 \leq 2) = \int_{-\infty}^{2} \frac{1}{\sqrt{3.2\pi}} e^{-x^2/(2.3)} dx \approx 0.875893. \end{aligned}$$

To obtain the result 0.875893, we can use either the "wolframealpha" website in the search command) or using software R with the command "pnorm(2,0,sqrt(3))".

Exercise 2.1. Find the covariance of B_s and B_t.

Hint 2.1. Proved that $\mathrm{cov}(B_s, B_t) = s, \ \ \forall\ 0 \leq s < t$.

Theorem 1. Brownian motion is a Markov process. Let $(B_t)_{t\geq 0}$ be a Brownian motion under the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$. Then B_t is also a Markov process.

Proof. We now show that whenever $0 \leq s \leq t$ and f is a Borel-measurable function, there is another Borel-measurable function g such that $E[f(B_t)|\mathcal{F}_s] = g(B_s)$. In fact, $E[f(B_t)|\mathcal{F}_s] = E[f(B_t - B_s + B_s)|\mathcal{F}_s]$. From the definition of Brownian motion, we have B_s is $\mathcal{G}$-measurable and $B_t - B_s$ is independent of $\mathcal{G}$. According to the independence property of conditional expectation, we have: $E[f(B_t - B_s + B_s)|\mathcal{F}_s] = g(B_s)$, where $g(x) = E[f(B_t - B_s + x)] = \dfrac{1}{\sqrt{2\pi(t-s)}} \displaystyle\int_{-\infty}^{\infty} f(x+s) e^{-\frac{s^2}{2(t-s)}} ds$, because $B_t - B_s$ is a normal random variable with mean 0 and variance $t - s$. ☐

Theorem 2. Brownian motion is a martingale process.

Proof. The proof is very similar to that of the property of the symmetric random walk. ☐

Example 2.3. Let $Y_t = B_t^2 - t$, for $t \geq 0$. Show that $(Y_t)_{t\geq 0}$ is a martingale with respect to Brownian motion. This is called the quadratic martingale.

Answer 2.3. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of the Brownian motion. For $0 \leq s \leq t$, we have:

$$\begin{aligned} E(Y_t|\mathcal{F}_s) &= E(B_t^2 - t|\mathcal{F}_s) = E((B_t - B_s + B_s)^2|\mathcal{F}_s) - t = E((B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2|\mathcal{F}_s) - t \\ &= E((B_t - B_s)^2|\mathcal{F}_s) + 2B_s E(B_t - B_s|\mathcal{F}_s) + B_s^2 - t = E((B_t - B_s)^2) + 2B_s E(B_t - B_s) + B_s^2 - t \\ &= \mathrm{Var}(B_t - B_s) + B_s^2 - t = (t - s) + B_s^2 - t = B_s^2 - s = Y_s. \end{aligned}$$

We conclude that $(Y_t)_{t\geq 0}$ is a martingale process with respect to the Brownian motion.

Theorem 3. EXPONENTIAL MARTINGALE. Let $(B_t)_{t\geq 0}$ be the standard Brownian motion on the filter probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$, with $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of $(B_t)_{t\geq 0}$. Then $(Z_t)_{t\geq 0}$, with $Z_t = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$ (σ is a positive constant), is a martingale with respect to the standard Brownian motion.

Proof. Because B_t is completely determined under the information available in σ-algebra $\mathcal{F}_t$, is Z_t so. In other words, $(Z_t)_{t\geq 0}$ is an adapted process with the filtration $(\mathcal{F}_t)_{t\leq 0}$.

For $0 \leq s \leq t$, we have

$$\begin{aligned} E[Z_t|\mathcal{F}_s] &= E[\exp(\sigma B_t - \frac{1}{2}\sigma^2 t)|\mathcal{F}_s] = \exp(-\frac{1}{2}\sigma^2 t)E[\exp(\sigma(B_t - B_s + B_s))|\mathcal{F}_s] && (2) \\ &= \exp(\sigma B_s - \frac{1}{2}\sigma^2 t)E[\exp(\sigma(B_t - B_s))|\mathcal{F}_s] = \exp(\sigma B_s - \frac{1}{2}\sigma^2 t)E[\exp(\sigma(B_t - B_s))] && (3) \end{aligned}$$

Recall that if $X \sim N(\mu, \sigma)$ (σ is the standard deviation of the normal random variable) then $E(\exp(X)) = \exp(\mu + \frac{1}{2}\sigma^2)$. Here $\sigma(B_t - B_s)$ is a normal random variable with mean 0 and variance $\sigma^2(t-s)$. Thus, $E[\exp(\sigma(B_t - B_s))] = \exp(\frac{1}{2}\sigma^2(t-s))$. As a result, we have

$$E[Z_t|\mathcal{F}_s] = \exp(\sigma B_s - \frac{1}{2}\sigma^2 t)\exp(\frac{1}{2}\sigma^2(t-s)) = \exp(\sigma B_s - \frac{1}{2}\sigma^2 s) = Z_s.$$

□

Example 2.4. Let $S_t = S_0 e^{X_t}$, with $X_t = \mu t + \sigma B_t$, and $(B_t)_{t\geq 0}$ is the standard Brownian motion. Let $r = \mu + \sigma^2/2$. Show that $e^{-rt}S_t$ is a martingale with respect to standard Brownian motion.

Answer 2.4. We have $e^{-rt}S_t = e^{-(\mu+\sigma^2/2)t}S_0 e^{\mu t + \sigma B_t} = S_0 e^{\sigma B_t - (\sigma^2/2)t}$. As $e^{\sigma B_t - (\sigma^2/2)t}$ is a martingale with respect to the standard Brownian motion, so is $e^{-rt}S_t$.

The paths of Brownian motion are unusual in that their quadratic variation is not zero. This makes stochastic calculus different from ordinary calculus. Most functions have continuous derivatives and hence their quadratic variations are zero. For this reason, one never considers quadratic variation in ordinary calculus. The paths of Brownian motion, on the other hand, cannot be differentiated with respect to the time variable. Thus the quadratic variation of Brownian motion might not be zero. In fact, it can be proved that $[W,W](T) = \lim_{\Pi \to 0} \sum_{j=0}^{n-1}[W(t_{j+1}) - W(t_j)]^2 = T, \forall T \geq 0$ almost surely (there might be some paths of the Brownian motion for which the assertion $[W,W](T) = T$ is not true, but the set of such paths has zero probability). Here $\Pi = \{t_0, t_1, ...t_n\}$ and $0 = t_0 < t_1 < ... < t_n = T$. From the quadratic variation property, it can be formally written: $dB_t^2 = dt$ (see **?** , p. 104]). This fact can be expressed as Brownian motion accumulates quadratic variation at rate one per unit time.

FREQUENTLY ASKED QUESTIONS ABOUT BROWNIAN MOTIONS

The first condition is that stochastic process starts at 0 when $t = 0$. What happens if we consider Brownian motion from time $t > 0$ on ward, i.e., the process is at B_t, not at 0? From time t onward, the process is still a Brownian motion (we change the coordinate system so that the position B_t is moved to 0).

Can we predict the value of B_t from time $s < t$? We cannot predict exactly the value of the B_t, but we know the possible range of values B_t. In particular, we have $B_t = B_s + \sqrt{t-s}N(0,1)$ or $B_t \sim N(B_s, \sqrt{t-s})$. In other words, B_t is a normal random variable, with mean B_s and variance $t - s$. We therefore can guess the possible ranges of values of B_t, with corresponding probability as follows:

- $P(B_s - \frac{\sqrt{t-s}}{2} < B_t < B_s + \frac{\sqrt{t-s}}{2}) = 38.2\%$
- $P(B_s - \sqrt{t-s} < B_t < B_s + \sqrt{t-s}) = 68.2\%$
- $P(B_s - \frac{3\sqrt{t-s}}{2} < B_t < B_s + \frac{3\sqrt{t-s}}{2}) = 86.6\%$
- $P(B_s - 2\sqrt{t-s} < B_t < B_s + 2\sqrt{t-s}) = 95.4\%$
- $P(B_s - \frac{5\sqrt{t-s}}{2} < B_t < B_s + \frac{5\sqrt{t-s}}{2}) = 98.8\%$
- $P(B_s - 3\sqrt{t-s} < B_t < B_s + 3\sqrt{t-s}) = 99.8\%$

As we can clearly see, when t is close to s, the value of B_t is close to B_s, with high probability. On the other hand, when $t - s$ is large, the possible range values of B_t become large, and thus it is less likely to predict exactly the value of B_t. In other words, the farther the time distance, the more fluctuate the value of B_t could be. Fortunately, we almost surely that the value of B_t is in the range $[B_s - 3\sqrt{t-s}, < B_s + 3\sqrt{t-s}]$, with 99.8%

How do we model a standard Brownian process? Suppose we want to model a Brownian motion in a time interval $[0, T]$. We divide the interval into n equal subintervals by discrete time points $0 = t_0 < t_1 < \cdots < t_n = T$, with the time step $h = t_i - t_{i-1}$. The value of B_{t_i} is determined by a recursive formula: $B_{t_i} = B_{t_{i-1}} + \sqrt{h} * X$, where $X \sim N(0, 1)$. All we need to simulate the Brownian motion is to simulate a standard normal random variable, which can be done by using command "rnorm(1)" in software R. The recursive (iteration) process occurs as follows:

- $B_0 = 0$;
- $B_1 = B_0 + \sqrt{h} * X$
- $B_2 = B_1 + \sqrt{h} * X$
- $\cdots$
- $B_n = B_{n-1} + \sqrt{h} * X$

For each iteration, as X is a random variable, the value of X may change.

```
## this is R code for simulating standard Brownian motion.
BM=function(T,N) {
### we need to specify the input of the Brownian process
# T=1  expiry time
# N=100  number of simulation points
h=T/N  # the timestep of the simulation
t=(0:T*N)/N # discrete time points  #may be not needed in this code
X=rep(0, (N+1)) # generate Brownian vector, #with length N+1
X[1]=0
for(i in 1:N) { X[i+1]=X[i] +sqrt(h)*rnorm(1)}
return(X)
}
```

We now try to vectorize the above code using the fact that $B_t - B_s$ is a normal random variable with mean 0 and variance (t-s) and

$$B_{t_i} = (B_{t_i} - Wt_{i-1}) + (B_{t_{i-1}} - Wt_{i-2}) + \cdots (B_1 - B_0).$$

In other words, each B_{t_i} is a sum of i i.i.d random variables with mean 0 and variance h, which is the time step. This idea is the key for producing a vectorized code. We first generate n i.i.d normal random variable by using command in R, "rnorm(n,0,$\sqrt{h}$)", then we use "cumsum" command to cumulatively sum up the resulted vector. The code is provided as follows:

```
### an alternative code for Brownian motion
## this is R code for simulating standard Brownian motion.
BM=function(T,N) {
### we need to specify the input of the Brownian process
# T=1  expiry time
# N=100  number of simulation points
ptm <- proc.time() #start clock
X<-c(0,cumsum(rnorm(N,0,sqrt(t/N)))) ## a method of vetorized code
## we use the fact that B_t - B_s
##is a normal random variable with mean 0 and variance (t-s)
return(X)
#stop clock
proc.time() - ptm}


## code used to plot a sample Brownian motion if we want to
steps <- seq(0,t,length=n+1)
plot(steps,X,type="l")
```

Brownian motion can be very difficult to visualize; in fact, in various respects it's impossible. Brownian motion has some "surprising" features that make it seem strange and somewhat intimidating.

Brownian motion has continuous sample paths, and continuous functions are quite nice already, aren't they? Here's one slightly strange feature, just to get started. Recall that $B_0 = 0$. It turns out that for almost all sample paths of Brownian motion, the path has infinitely many zeros in the interval $(0, \epsilon)$. That is, the path changes sign infinitely many times, cutting through the horizontal axis infinitely many times, within the interval $(0, \epsilon)$. Another rather mind-boggling property is that with probability 1, a sample path of Brownian motion does not have a derivative at any time! It's easy to imagine functions, like $f(t) = |t|$, for example that fail to be differentiable at isolated points. But try to imagine a function that everywhere fails to be differentiable, so that there is not even one time point at which the function has a well-defined slope.

Such functions are not easy to imagine. In fact, before around the middle of the 19th century mathematicians generally believed that such functions did not exist, that is, they believed that every continuous function must be differentiable somewhere. Thus, it came as quite a shock around 1870 when Karl Weierstrass produced an example of a nowhere-differentiable function. Some in the mathematical establishment reacted negatively to this work, as if it represented an undesirable preoccupation with ugly, monstrous functions. Perhaps it was not unlike the way adults might look upon the ugly, noisy music of the next generation. It is interesting to reflect on the observation that, in a sense, the same sort of thing happened in mathematics much earlier in a different context with which we are all familiar. Pythagorus discovered that $\sqrt{2}$ which he knew to be a perfectly legitimate number, being the length of the hypotenuse of a right triangle having legs of length one is irrational. Such numbers were initially viewed with great distrust and embarrassment. They were to be shunned;

notice how even the name "irrational" still carries a negative connotation. Apparently some Pythagoreans even tried to hide their regretable discovery. Anyway, now we know that in a sense "almost all" numbers are of this "undesirable" type, in the sense that the natural measures that we like to put on the real numbers [[like Lebesgue measure (ordinary length)]] place all of their "mass" on the set of irrational numbers and no mass on the set of rational numbers. Thus, the proof of existence of irrational numbers by producing an example of a particular irrational number was dwarfed by the realization that if one chooses a real number at random under the most natural probability measures, the result will be an irrational number with probability 1. The same sort of turnabout has occurred in connection with these horrible nowhere-differentiable functions. Weierstrass constructed a particular function and showed that it was nowhere differentiable. The strange nature of this discovery was transformed in the same sense by Brownian motion, which puts probability 0 on "nice" functions and probability 1 on nowhere differentiable functions.

3 Gaussian process

Definition 3.1. Random variables $X_1, ..., X_k$ have a multivariate normal distribution if for all real numbers $a_1, ..., a_k$, the linear combination $a_1X_1 + ... + a_kX_k$ has a univariate normal distribution. A multivariate normal distribution is completely determined by its mean vector $\mu = (\mu_1, ..., \mu_k)$ and covariance matrix V, where $V_{ij} = \text{Cov}(X_i, X_j),\ \forall 1 \le i, j \le k$. The joint density function of the multivariate normal distribution is:

$$f(x) = \frac{1}{(2\pi)^{k/2}|V|^{1/2}} \exp\big(\frac{1}{2}(x-\mu)^T V^{-1}(x-\mu)\big),$$

where $x = (x_1, x_2, ..., x_k)$ and $|V|$ is the determinant of V.

The multivariate normal distribution has the remarkable property that all marginal and conditional distributions are normal. If $X_1, ..., X_k$ have a multivariate normal distribution, then X_i are normally distributed, joint distributions of subsets of X_i have multivariate normal distributions, and conditional distributions given subsets of the X_i are normal.

If $X_1, ..., X_k$ are independent normal random variables, then their joint distribution is multivariate normal. For jointly distributed normal random variables, independent is equivalent to being uncorrelated. That is, if X and Y are jointly distributed normal random variables, then X and Y are independent if and only if $E(XY) = E(X)E(Y)$.

A Gaussian process extends the multivariate normal distribution to stochastic processes.

Definition 3.2. A Gaussian process $(X_t)_{t\ge 0}$ is a continuous-time stochastic process with the property that for all $n = 1, 2, ...$ and $0 \le t_1 < ... < t_n$, the random variables $X_{t_1}, X_{t_2}, ..., X_{t_n}$ have a multivariate normal distribution.

A Gaussian process is completely determined by its mean function $E(X_t)$ for $t \ge 0$, and covariance function $\text{Cov}(X_s, X_t),\ \forall s, t \ge 0$.

Important note 3.1. A stochastic process $(B_t)_{t\ge 0}$ is a standard Brownian motion if and only if it is a Gaussian process with the following properties:

1. $B_0 = 0$.
2. $E(B_t) = 0$.
3. $\text{Cov}(B_s, B_t) = \min(s, t),\ \forall s, t$.

4. The function $t \to B_t$ is continuous, with probability 1.

Proof. The detailed proof is presented in pages in [**?** , p. 332-333].

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. We now prove it is a Gaussian process satisfying the above four properties.

To prove $(B_t)_{t\geq 0}$ is a Gaussian process, we need to prove that for $0 \leq t_1 < t_2 < ... < t_n$, the random variables $B_{t_1}, B_{t_2}, ..., B_{t_n}$ have a multivariate normal distribution. In other words, for constant $a_1, a_2, ..., a_k$, we need to show that $a_1 B_{t_1} + a_2 B_{t_2} + ... + a_k B_{t_k}$ is a normal random variable (having a univariate normal distribution). We can express $a_1 B_{t_1} + a_2 B_{t_2} + ... + a_k B_{t_k}$ as follows:

$$\begin{aligned} & a_1 B_{t_1} + a_2 B_{t_2} + ... + a_k B_{t_k} \\ = & a_1 B_{t_1} + a_2(B_{t_1} + B_{t_2} - B_{t_1}) + ... + a_k(B_{t_1} + B_{t_2} - B_{t_1} + ... + B_{t_k} - B_{t_{k-1}}) \\ = & (a_1 + ... + a_k)B_{t_1} + (a_2 + ... + a_k)(B_{t_2} - B_{t_1}) + ... + a_k(B_{t_k} - B_{t_{k-1}}). \end{aligned}$$

Thus $a_1 B_{t_1} + a_2 B_{t_2} + ... + a_k B_{t_k}$ is a linear combination of independent normal random variables $B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_k} - B_{t_{k-1}}$. As a result, it is also a normal random variable. The Brownian has the four required properties deduced from its definition.

Conversely assume that $(B_t)_{t\geq 0}$ is a Gaussian process that satisfies the stated properties. We need to show the process has stationary and independent increments. Since the process is Gaussian, for $s, t \geq 0, B_{t+s} - B_s$ is normally distributed with mean $E(B_{t+s} - B_s) = E(B_{t+s}) - E(B_s) = 0$ and variance $\text{Var}(B_{t+s} - B_s) = \text{Var}(B_{t+s}) + \text{Var}(B_s) - 2\text{Cov}(B_{t+s}, B_s) = t + s + s - 2s = t$. Thus $B_{t+s} - B_s \overset{d}{=} B_t$. This deduces that $(B_t)_{t\geq 0}$ has stationary increments.

For $0 \leq q < r \leq s < t$, we have:

$$\begin{aligned} E((B_r - B_q)(B_t - B_s)) &= E(B_r B_t) - E(B_q B_t) - E(B_r B_s) + E(B_q B_s) \\ &= \text{Cov}(B_r, B_t) - \text{Cov}(B_q, B_t) - \text{Cov}(B_r, B_s) + \text{Cov}(B_q, B_s) = r - q - r + q = 0. \end{aligned}$$

Thus $B_r - B_q$ and $B_t - B_s$ are uncorrelated. Since $B_r - B_q$ and $B_t - B_s$ are normal random variables, it follows that they are independent. □

Example 3.1. For $a > 0$, let $X_t = B_{at}/\sqrt{a}$, for $t \geq 0$. Show that $(X_t)_{t\geq 0}$ is a standard Brownian motion.

Answer 3.1. We need to prove that $(X_t)_{t\geq 0}$ is a Gaussian process with the four properties.

For real numbers $a_1, a_2, ..., a_k$ and for $0 \leq t_1 < t_2 < ... < t_k$, we have: $\sum_{i=1}^{k} a_i X_{t_i} = \sum_{i=1}^{k} \frac{a_i}{\sqrt{a}} B_{at_i}$. In other words, $\sum_{i=1}^{k} a_i X_{t_i}$ is a linear combination of random variables B_{at_i}, $i = 1, ..., k$. As $(B_t)_{t\geq 0}$ is a Gaussian process, $\sum_{i=1}^{k} a_i X_{t_i}$ is a normal random variable. This deduces that $(X_t)_{t\geq 0}$ is a Gaussian process. We now check whether $(X_t)_{t\geq 0}$ satisfies the four properties.

- $X_0 = B_0/\sqrt{a} = 0/\sqrt{a} = 0$.
- $E(X_t) = E(B_{at}/\sqrt{a}) = E(B_{at})/\sqrt{a} = 0$.

- $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(B_{as}/\sqrt{a}, B_{at}/\sqrt{a}) = \dfrac{1}{a}\mathrm{Cov}(B_{as}, B_{at}) = \dfrac{1}{a}as = s, \quad \forall 0 < s \leq t.$
- The path continuity of $(X_t)_{t\geq 0}$ follows from the path continuity of standard Brownian motion, as the function $t \to B_{at}/\sqrt{a}$ is continuous for all $a > 0$, with probability 1.

—

Important note 3.2. Brownian motion paths are nowhere differentiable

The above property shows that Brownian motion preserves its character after rescaling. For instance, given a standard Brownian motion on $[0, 1]$, if we look at the process on an interval of length one-trillionth ($= 10^{-12}$) then after resizing by a factor of $1/\sqrt{10^{-12}} = 10^6$, what we see is indistinguishable from the original Brownian motion.

This highlights the invariance, or fractal, structure of Brownian motion sample paths. It means that the jagged character of these paths remains jagged at all time scales. This leads to the remarkable fact that Brownian motion sample paths are nowhere differentiable. It is hard to even contemplate a function that is continuous at every point on its domain, but not differentiable at any point. Indeed, for many years, mathematicians believed that such a function was impossible, until Karl Weierstrass considered the founder of modern analysis, demonstrated their existence in 1872.

The proof that Brownian motion is nowhere differentiable requires advanced analysis. Here is a heuristic argument. Consider the formal derivative: $\dfrac{d}{dt}B_t = \lim\limits_{h\to 0} \dfrac{B_{t+h} - B_t}{h}$. By stationary increments, $B_{t+h} - B_t$ has the same distribution as B_h, which is normal with mean 0 and variance h. Thus the difference quotient $\dfrac{B_{t+h} - B_t}{h}$ is normally distributed with mean 0 and variance $1/h$. As h tends to 0, the variance tends to infinity. Since the difference quotient takes arbitrarily large values, the limit, and hence the derivative, does not exist.

Example 3.2. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then each of the following transformation is a standard Brownian motion.

- Reflection $(-B_t)_{t\geq 0}$.
- Translation $(X_t)_{t\geq 0}, \quad X_t = B_{t+s} - B_s \quad \forall s \geq 0$.
- Rescaling $B_{at}/\sqrt{a}, \quad \forall a > 0$.
- Inversion: the process $(X_t)_{t\geq 0}$ defined by $X_0 = 0$ and $X_t = tB_{1/t}, \quad \forall t > 0$.

Answer 3.2. It is straightforward to prove $(-B_t)_{t\geq 0}$ is a Brownian motion. The third claim is already proved in the above example. We now focus on the second and fourth claims.

We first prove the second claim that $(X_t)_{t\geq 0}$ is a Gaussian process with $X_t = B_{t+s} - B_s$. For real numbers $a_1, a_2, ..., a_k$ and for $0 \leq t_1 < t_2 < ... < t_k$, we have: $\sum\limits_{i=1}^{k} a_i X_{t_i} = \sum\limits_{i=1}^{k} a_i(B_{t_i+s} - B_{t_i})$, which could always be expressed a linear combination of independent normal random variables. Thus $\sum\limits_{i=1}^{k} a_i X_{t_i}$ is a normal random variable. This deduces that $(X_t)_{t\geq 0}$ is a Gaussian process. We now check whether $(X_t)_{t\geq 0}$ satisfies the four properties.

- $X_0 = B_s - B_s = 0$.

- $E(X_t) = E(B_t - B_s) = E(B_t) - E(B_s) = 0$.
- $\text{Cov}(X_u, X_v) = \text{Cov}(B_{u+s} - B_s, B_{v+s} - B_s) = \text{Cov}(B_{u+s}, B_{v+s}) - \text{Cov}(B_s, B_{v+s}) - \text{Cov}(B_s, B_{v+s}) + \text{Cov}(B_s, B_s) = v + s - s - s + s = v, \quad \forall 0 < v \leq u$.
- The path continuity of $(X_t)_{t\geq 0}$ follows from the path continuity of standard Brownian motion, as the function $t \to B_{t+s} - B_s$ is continuous, with probability 1.

We now prove the fourth claim that $(X_t)_{t\geq 0}$ is a Gaussian process with $X_0 = 0,\ X_t = tB_{1/t}$. For real numbers $a_1, a_2, ..., a_k$ and for $0 \leq t_1 < t_2 < ... < t_k$, we have: $\sum_{i=1}^{k} a_i X_{t_i} = \sum_{i=1}^{k} t_i B_{1/t_i}$, which is a linear combination of normal random variables $B_{1/t_i},\ i = 1, 2, ..., k$. As $(B_t)_{t\geq 0}$ is a Gaussian process, $\sum_{i=1}^{k} a_i X_{t_i}$ is a normal random variable. Thus $(X_t)_{t\geq 0}$ is a Gaussian process. We now check whether $(X_t)_{t\geq 0}$ satisfies the four properties.

- $X_0 = 0$ as defined.
- $E(X_t) = tE(B_{1/t}) = 0$.
- $\text{Cov}(X_t, X_s) = \text{Cov}(tB_{1/t}, sB_{1/s}) = ts\text{Cov}(B_{1/t}, B_{1/s}) = ts\frac{1}{t} = s, \quad \forall 0 < s \leq t$.
- Continuity, for all $t > 0$, is inherited from $(B_t)_{t\geq 0}$. What remains is to show the process is continuous at $t = 0$. We do not prove this rigorously. Suffice it to show that

$$\lim_{t\to 0} X_t = \lim_{t\to 0} tB_{1/t} = \lim_{s\to\infty} \frac{B_s}{s} = 0.$$

This is because B_s is a normal random variable with mean 0 and variance $1/s$, which tends to 0 as $s \to \infty$.

Definition 3.3. For real value x, the process defined by $X_t = x + B_t$, for $t \geq 0$ is called Brownian motion started at x. For such a process, $X_0 = x$ and X_t is normally distributed with mean function $E(X_t) = x$ and variance t, for all t. The process retains all other defining properties of standard Brownian motion: stationary and independent increments, and contiuous sample paths. The probability density function of X_t is given by $K_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/(2t)}$.

Example 3.3. Let $(X_t)_{t\geq 0}$ be a Brownian motion started at $x = 3$. Find $P(X_2 > 0)$.

Answer 3.3. The process X_t can be expressed as $X_t = B_t + 3$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Then, $P(X_2 > 0) = P(B_t + 3 > 0) = P(B_t > -3) = \int_{-3}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-x^2/4} dx = 0.983$.
In R, type 1-pnorm(-3,0,sqrt(2)) will help us compute the integral.

4 Conditional distribution for Brownian motion

Suppose $t > s$, we have:

$$E[B_t|B_s] = E[B_t - B_s + B_s|B_s] = E[B_t - B_s|B_s] + E[B_s|B_s] = E[B_t - B_s] + B_s = B_s$$

and

$$Var[B_t|B_s] = Var[B_t - B_s + B_s|B_s] = Var[B_t - B_s|B_s] + Var[B_s|B_s] = Var[B_t - B_s] + 0 = t - s.$$

Things becomes more complicated if we want compute $E[B_s|B_t]$ and $Var[B_s|B_t]$, with $s < t$.

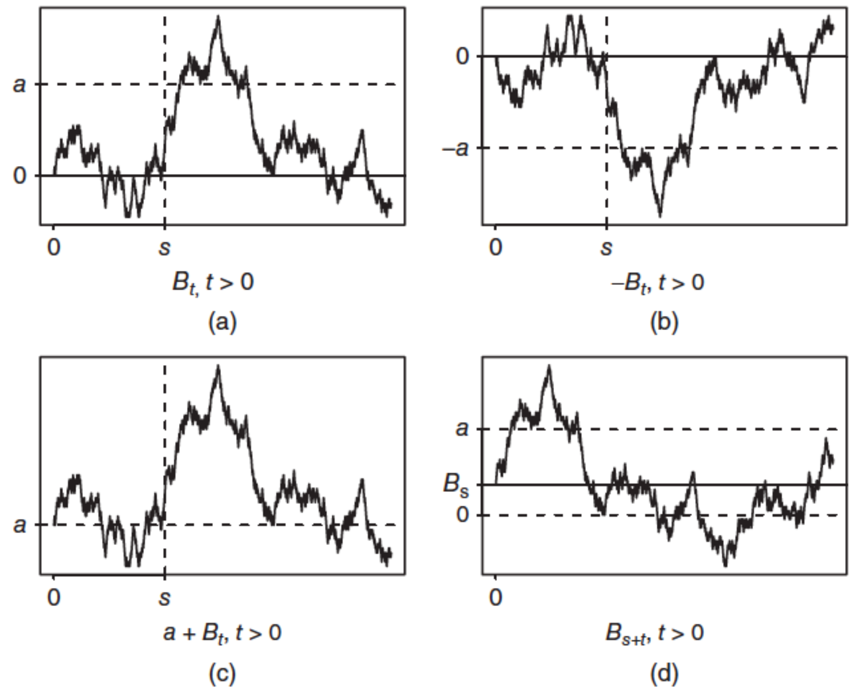

Figure 3: Brownian motion started at time s

We have $E[B_s|B_t] = E[B_s|B_s + B_t - B_s]$. Let $X = B_s$, $U = B_t - B_s$, and $Y = X + U = B_t$, then $E[B_s|B_t] = E[X|Y]$, where $X \sim N(0, s)$ and $U \sim N(0, t-s)$, $Y \sim N(0, t)$. The density distributions of X, U, Y are $f_X(x) = \frac{1}{\sqrt{2\pi s}} e^{-x^2/(2s)}$, $f_U(u) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-u^2/(2(t-s))}$, $f_Y(y) = \frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)}$. It is clear that X and U are independent, while Y is the sum of X and U.

To find $E[X|Y]$, we first find the conditional distribution of X given Y. We have:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{X,U}(x, y-x)}{f_Y(y)} = \frac{f_X(x) f_U(y-x)}{f_Y(y)} = \frac{\frac{1}{\sqrt{2\pi s}} e^{-x^2/(2s)} \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/(2(t-s))}}{\frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)}}$$

$$= \frac{\sqrt{t}}{\sqrt{2\pi s(t-s)}} e^{-x^2/(2s) - (y-x)^2/(2(t-s)) + y^2/(2t)} = \frac{1}{\sqrt{2\pi \frac{s}{t}(t-s)}} e^{-\frac{(x - \frac{sy}{t})^2}{2\frac{s}{t}(t-s)}}.$$

This is the density of a normal random variable with mean $\frac{sy}{t}$ and variance $\frac{s}{t}(t-s)$. In other words, $X|Y = y \sim N(\frac{sy}{t}, \frac{s}{t}(t-s))$. Thus, for all y, $E[B_s|B_t = y] = \frac{sy}{t}$ and $Var[B_s|B_t = y] = \frac{s}{t}(t-s)$. As a result, $E[B_s|B_t] = \frac{sB_t}{t}$, and $Var[B_s|B_t] = \frac{s}{t}(t-s)$.

5 First passage time

Let $(B_t)_{t\geq 0}$ be a Brownian motion and let m be a real number. We define the first passage time to level m is $\tau_m = \min\{t \geq 0 | B_t = m\}$. This is the first time the Brownian motion B_t reaches the level m. If B_t never

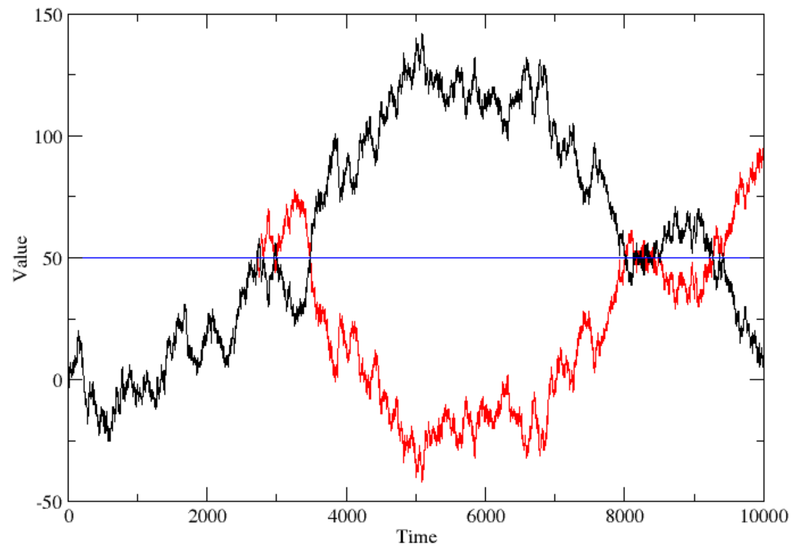

Figure 4: Reflection principle to standard Brownian motion

reaches the level m, we set $\tau_m = \infty$.

We fix a positive level m and a positive time t. We wish to count the Brownian paths that reach level m before time t (i.e., those paths for which the first passage time τ_m to level m is less than or equal to t). We can classify such paths into two types: those that reach level m prior to t but at time t are at some level $w < m$, and those that exceed level m at time t. There are also Brownian motion paths that are exactly at level m at time t, but the probability for these paths occur is zero. We may thus ignore this possibility.

As can be seen from Figure 4 that for each Brownian motion path that reaches level m before time t but is at a level $w < m$ at time t, there is a "reflected path" that is at level $2m - w$ at time t. This reflected path is constructed by switching the up and down moves of the Brownian motion from time τ_m onward. That observation leads to the key reflection equality

$$P(\tau_m \le t, B_t \le w) = P(B_t \ge 2m - w), \quad \forall w \le m, m > 0$$

or

$$P(\tau_m \le t, B_t \ge w) = P(B_t \le 2m - w), \quad \forall w \ge m, m < 0$$

FIRST PASSAGE TIME DISTRIBUTION. For all $m \ne 0$, the random variable τ_m has cumulative distribution function

$$P(\tau_m \le t) = \frac{2}{\sqrt{2\pi}} \int_{|m|/\sqrt{t}}^{\infty} e^{-y^2/2} dy$$

and the density function:

$$f_{\tau_m}(t) = \frac{d}{dt} P(\tau_m \le t) = \frac{|m|}{t\sqrt{2\pi t}} e^{-m^2/(2t)}, t \ge 0.$$

PROOF. We first consider the case $m > 0$. We have $P(\tau_m \le t) = P(\tau_m \le t, B_t \le m) + P(\tau_m \le t, B_t \ge m)$. As $B_t \ge m$ implies that $\tau_m \le t$, we have $P(\tau_m \le t, B_t \ge m) = P(B_t \ge m)$. Also, applying the Refection principle, we have $P(\tau_m \le t, B_t \le m) = P(B_t \ge 2m - m) = P(B_t \ge m)$. Thus,

$$P(\tau_m \le t) = 2P(B_t \ge m) = 2\frac{1}{\sqrt{2\pi t}}\int_m^\infty e^{-x^2/(2t)}dx = \frac{2}{\sqrt{2\pi}}\int_{m/\sqrt{t}}^\infty e^{-y^2/2}dy, \text{ with } y = x/\sqrt{t}.$$

We now consider the case $m < 0$. We have $P(\tau_m \le t) = P(\tau_m \le t, B_t \le m) + P(\tau_m \le t, B_t \ge m)$. As $B_t \le m$ implies that $\tau_m \le t$, we have $P(\tau_m \le t, B_t \le m) = P(B_t \le m)$. Also, applying the Refection principle, we have $P(\tau_m \le t, B_t \ge m) = P(B_t \le 2m - m) = P(B_t \le m)$. Thus,

$$P(\tau_m \le t) = 2P(B_t \le m) = 2\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^m e^{-x^2/(2t)}dx = 2\frac{1}{\sqrt{2\pi t}}\int_{-m}^\infty e^{-x^2/(2t)}dx = \frac{2}{\sqrt{2\pi}}\int_{-m/\sqrt{t}}^\infty e^{-y^2/2}dy, \text{ with } y = x/\sqrt{t}.$$

In short, we have for $m \ne 0$,

$$P(\tau_m \le t) = \frac{2}{\sqrt{2\pi}}\int_{|m|/\sqrt{t}}^\infty e^{-y^2/2}dy.$$

By differentiating the cumulative distribution, we obtain the density function:

$$f_{\tau_m}(t) = \frac{d}{dt}P(\tau_m \le t) = \frac{|m|}{t\sqrt{2\pi t}}e^{-m^2/(2t)}, t \ge 0.$$

The probability that a standard Brownian motion reaches level m sometime in the interval $[0, T]$ is

$$P(\tau_m \le T) = \int_0^T \frac{|m|}{\sqrt{2\pi t^3}}e^{-m^2/(2t)}dt$$

Example 5.1. A particle moves according to Brownian motion started at $x = 1$. After $t = 3$ hours, the particle is at level 1.5. Find the probability that the particle reaches level 2 sometime in the next hour.

Answer 5.1. After $t = 3$, the particle moves according to Brownian motion started at $x = 1.5$. The probability of the event that the particle reaches level 2 sometime in the next hour (before $t = 4$) is identical with the probability of the event the standard Brownian motion first hits level $m = 2 - 1.5 = 0.5$ in the time interval $[0, 1]$. The desired probability is

$$P(\tau_{0.5} \le 1) = \int_0^1 \frac{0.5}{\sqrt{2\pi t^3}}e^{-0.5^2/(2t)}dt = 0.617.$$

Important note 5.1. The first passage time distribution has some surprising properties. Consider

$$P(\tau_m < \infty) = \lim_{t\to\infty} P(\tau_m < t) = \lim_{t\to\infty} 2\int_{|m|/\sqrt{t}}^{+\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = 2\int_0^{+\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = 1. \tag{4}$$

This means the Brownian motion will hit the level a in a finite period of time, with probability 1, for all a, no matter how large the value of a. On the contrary, $E(\tau_m) = \displaystyle\int_0^{+\infty} \frac{t|m|}{\sqrt{2\pi t^3}}e^{-m^2/(2t)}dt = +\infty$. This means that it is expected that the Brownian motion will never hit the level a, no matter how small the value of a.

6 Maximum of Brownian motion

The reflection principle is applied to derive the distribution of the maximum process of Brownian motion $(M_t)_{t\ge 0}$ where $M_t = \max\limits_{0\le s\le t} B_s$, which is the maximum value of Brownian motion on $[0, t]$. If at time t, $B_t \ge a > 0$, then

$M_t \geq a$ and $\tau_a \leq t$. Thus $\{M_t > a\} = \{M_t > a, B_t > a\} \cup \{M_t > a, B_t \leq a\} = \{B_t > a\} \cup \{M_t > a, B_t \leq a\}$. As the union is disjoint, we have

$$\begin{aligned} P(M_t > a) &= P(B_t > a) + P(M_t > a, B_t \leq a) = P(B_t > a) + P(\tau_a \leq t, B_t \leq a) \\ &= 2P(B_t > a) = \int_a^{+\infty} \sqrt{\frac{2}{\pi t}} e^{-x^2/(2t)} dx, \quad \forall a > 0. \end{aligned}$$

Another alternative way to derive the distribution is to use the fact that $M_t > a$ if and only if the process hits a by time t, that $\tau_a < t$. This gives:

$$P(M_t > a) = P(\tau_a < t) = \int_0^t \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/(2s)} ds = \int_a^{+\infty} \sqrt{\frac{2}{\pi t}} e^{-x^2/(2t)} dx, \quad a^2/s = x^2/t.$$

Example 6.1. A laboratory instrument takes annual temperature measurements. Measurement errors are assumed to be independent and normally distributed. As precision decreases over time, errors are modeled as standard Brownian motion. For how many years can the lab be guaranteed that there is at least 90% probability that all errors are less than 4 degrees, given that the 95th percentile of the standard normal random distribution is 1.645?

Answer 6.1. The problem asks for the largest t such that $P(M_t \leq 4) \geq 0.9$. We have

$$0.9 \leq P(M_t \leq 4) = 1 - P(M_t > 4) = 1 - 2P(B_t > 4) = 2P(B_t \leq 4) - 1,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. This gives $0.95 \leq P(B_t \leq 4) = P(\sqrt{t}Z \leq 4) = P(Z \leq \frac{4}{\sqrt{t}})$, where $Z \sim N(0,1)$. The 95th percentile of the standard normal random distribution is 1.645. Thus the desired value t should satisfy $4/\sqrt{t} \geq 1.645 \Leftrightarrow t \leq 5.91$ years.

7 Zeros of Brownian motion

Brownian motion reaches level x, no matter how large x, with probability 1. Brownian motion also returns to the origin infinitely often. In fact, on any interval $(0, \epsilon)$, no matter how small ϵ, the process crosses the t-axis infinitely many times.

The times when the process crosses the t-axis are the zeros of Brownian motion.

Theorem 4. For $0 \leq r < t$, let $z_{r,t}$ be the probability that standard Brownian motion has at least one zero in (r,t). Then $z_{r,t} = \frac{2}{\pi} \arccos(\sqrt{\frac{r}{t}})$.

Proof. We have

$$z_{r,t} = P(B_s = 0), \text{for some } s \in (r,t) = \int_{-\infty}^{+\infty} P(B_s = 0 | B_r = x) \frac{1}{\sqrt{2\pi r}} e^{-x^2/(2r)} dx, \text{for some } s \in (r,t)$$

In additon, for $x < 0$,

$$P(B_s = 0|B_r = x) = P(M_t \geq 0|B_r = x) = P(M_t \geq -x|B_r = 0) = P(M_{t-r} > -x|B_0 = 0) = P(M_{t-r} > -x).$$

For $x \geq 0$, we consider the reflected Brownian motion

$$P(B_s = 0|B_r = x) = P(B_s = 0|B_r = -x) = P(M_t \geq 0|B_r = -x) = P(M_t \geq x|B_r = 0) = P(M_{t-r} > x).$$

Thus we can express $P(B_s = 0|B_r = x) = P(M_{t-r} > |x|)$.

As a result, $z_{r,t} = \int_{-\infty}^{+\infty} P(M_{t-r} > |x|) \frac{1}{\sqrt{2\pi r}} e^{-x^2/(2r)} dx = \frac{2}{\pi} \arccos(\sqrt{\frac{r}{t}})$. (see [**?** , p.342-343] for detailed calculation) $\square$

Important note 7.1. Applying the above theorem, we have $z_{0,\epsilon} = 2/\pi \arccos(0) = 1$. That is $B_t = 0$ for some $0 < t < \epsilon$. By the strong Markov property, for Brownian motion restarted at t, there is at least one zero in (t, ϵ), with probability 1. Continuing this way, there are infinitely many zeros in $(0, \epsilon)$.

8 BROWNIAN MOTION WITH DRIFT

Standard Brownian motion is often too simple a model for real-life applications. Many variations arise in practice. Brownian motion started in x has a constant mean function. A common variant of Brownian motion has linear mean function as well as an additional variance parameter.

Definition 8.1. For real value μ and $\sigma > 0$, the process defined by $W_t = \mu t + \sigma B_t$, for $t \geq 0$, is called Brownian motion with drift parameter μ and variance parameter σ^2.

Important note 8.1. Brownian motion with drift is a Gaussian process with continuous sample paths and independent and stationary increments. For $s, t > 0$, $W_{t+s} - W_t$ is normally distributed with mean μs and variance $\sigma^2 s$.

Proof of independent increments:

$P(W_t - W_s \leq k|\mathcal{F}_s) = P(\mu(t-s) + \sigma(B_t - B_s) \leq k|\mathcal{F}_s) = P(B_t - B_s \leq \frac{k - \mu(t-s)}{\sigma}|\mathcal{F}_s) = P(B_t - B_s \leq \frac{k - \mu(t-s)}{\sigma}) = P(\mu(t-s) + \sigma(B_t - B_s) \leq k) = P(W_t - W_s \leq k)$. Thus, $W_t - W_s$ is independent of $\mathcal{F}_s$.

Proof of stationary increments:

$W_t - W_s = \mu(t-s) + \sigma(B_t - B_s) \stackrel{d}{=} \mu(t-s) + \sigma B_{t-s} \stackrel{d}{=} W_{t-s}, \ \forall\ 0 < s < t.$

Example 8.1. Find the probability that Brownian motion with drift parameter $\mu = 0.6$ and variance $\sigma^2 = 0.25$ takes values between 1 and 3 at time $t = 4$.

Answer 8.1. Brownian motion with drift parameter $\mu = 0.6$ and variance $\sigma^2 = 0.25$ is a process $(W_t)_{t \geq 0}$ defined by $W_t = 0.6t + 0.5B_t$, where $(B_t)_{t\geq 0}$ is a Brownian motion.

The desired probability is:

$$P(1 \leq W_4 \leq 3) = P(1 \leq 0.6 * 4 + 0.5 * B_4 \leq 3) = P(-2.8 \leq B_4 \leq 1.2) = \int_{-2.8}^{1.2} \frac{e^{-x^2/8}}{\sqrt{8\pi}} dx = 0.645.$$

Example 8.2. (**Home team advantage.**) A novel application of Brownian motion to sports scores is given by **?**]. The goal is to quantify the home team advantage by finding the probability in a sports match that the home team wins the game given that they lead by l points after a fraction $0 \leq t \leq 1$ of the game is completed. The model is applied to basketball where scores can be reasonable approximated by a continuous distribution.

Answer 8.2. For $0 \leq t \leq 1$, let W_t denote the difference in scores between the home and visiting teams after $100t$ percent of the game has been completed. The process is modeled as a Brownian motion with drift, where

TABLE 8.2 Table for Basketball Data Probabilities $p(l,t)$ that the Home Team Wins the Game Given that they are in the Lead by l Points After a Fraction t of the Game is Completed

	Lead						
Time t	$l=-10$	$l=-5$	$l=-2$	$l=0$	$l=2$	$l=5$	$l=10$
0.00				0.62			
0.25	0.32	0.46	0.55	0.61	0.66	0.74	0.84
0.50	0.25	0.41	0.52	0.59	0.65	0.75	0.87
0.75	0.13	0.32	0.46	0.56	0.66	0.78	0.92
0.90	0.03	0.18	0.38	0.54	0.69	0.86	0.98
1.00	0.00	0.00	0.00		1.00	1.0	1.0

Source: Stern (1994).

the mean parameter μ is a measure of home team advantage. The probability that the home team wins the game, given that they have an l point lead at time $t<1$, is:

$$\begin{aligned} P(l,t) &= P(W_1>0|W_t=l) = P(W_1-W_t+W_t>0|W_t=l) = P(W_1-W_t>-l|W_t=l) = P(W_1-W_t>-l) \\ &= P(W_{1-t}>-l) = P(\mu(1-t)+\sigma B_{1-t}>-l) = P(B_{1-t}<\frac{l+\mu(1-t)}{\sigma}) = P(B_t<\frac{\sqrt{t}[l+\mu(1-t)]}{\sigma\sqrt{1-t}}). \end{aligned}$$

The last equality is because B_t has the same distribution as $\sqrt{t/(1-t)}B_{1-t}$. This can be explain as

$$\frac{B_t}{\sqrt{t}} \stackrel{d}{=} N(0,1), \frac{B_{1-t}}{\sqrt{1-t}} \stackrel{d}{=} N(0,1) \rightarrow \frac{B_t}{\sqrt{t}} \stackrel{d}{=} \frac{B_{1-t}}{\sqrt{1-t}}.$$

The model is applied to the results of 493 national Basketball Association games in 1992. Drift and variance parameters are fit from the available data with estimates $\hat{\mu}=4.87, \hat{\sigma}=15.82$. The results give the probability of a home team win for several values of l and t. Due to the home court advantage, the home team has a greater than 50% chance of winning even if it is behind by two points at half time ($t=0.5$). Even in the last five minutes of play ($t=0.9$), home team comebacks from five points are not that unusual, according to the model, with probability 0.18.

8.1 Simulating

Simulating Brownian motion with drift is very similar to that of standard Brownian motion. Suppose we want to model a generalized Brownian motion in a time interval $[0,T]$. We divide the interval into n equal subintervals by discrete time points $0=t_0<t_1<\cdots<t_n=T$, with the time step $h=t_i-t_{i-1}$. Following the model, we have $W_{t_i}=\mu t_i+\sigma B_{t_i}$ and $W_{t_{i-1}}=\mu t_{i-1}+\sigma W_{t_{i-1}}$. The value of W_{t_i} is determined by a recursive formula:

$$W_{t_i}=W_{t_{i-1}}+\mu*(t_i-t_{i-1})+\sigma(B_{t_i}-B_{t_{i-1}})=W_{t_{i-1}}+\mu*h+\sigma\sqrt{h}*X,$$

where $X\sim N(0,1)$. All we need to simulate a generalized Brownian motion is to simulate a standard normal random variable, which can be done by using command "rnorm(1)" in software R. The recursive (iteration) process occurs as follows:

- $W_0=0$;
- $W_1=W_0+\mu*h+\sigma\sqrt{h}*X$
- $W_2=W_1+\mu*h+\sigma\sqrt{h}*X$
- $\cdots$

- $W_n = W_{n-1} + \mu * h + \sigma\sqrt{h} * X$

For each iteration, as X is a random variable, the value of X may change.

```
BM=function(mu,sigma,T,N) {
### we need to specify the input of the Brownian process
# mu=0.05  the mean return of the stock
# sigma=0.3 the volatility of the return of the stock
# T=1  expiry time
# N=100  number of simulation points
h=T/N  # the timestep of the simulation
t=(0:T*N)/N # discrete time points  #may be not used in this code
X=rep(0, (N+1)) # generate Brownian vector, with length N+1
X[1]=0
for(i in 1:N) { X[i+1]=X[i] +mu*h+sigma*sqrt(h)*rnorm(1)}
return(X)
}
### after running the above code, to produce a figure for Brownian motion, we use
## the following command: BM1=BM(0.05,0.3,1,1000)  t=(0:1000)/1000    plot(t,BM1,type='l')
```

We now try to vectorise the above code using the fact that $B_t - B_s$ is a normal random variable with mean 0 and variance (t-s) and

$$B_{t_i} = (B_{t_i} - Bt_{i-1}) + (B_{t_{i-1}} - Bt_{i-2}) + \cdots (B_1 - B_0).$$

In other words, each B_{t_i} is a sum of i i.i.d random variables with mean μh and variance $\sigma^2 h$, where h is the time step. This idea is the key for producing a vectorized code. We first generate n i.i.d normal random variable by using command in R, "rnorm(n,μ h,$\sigma\sqrt{h}$)", then we use "cumsum" command to cumulatively sum up the resulted vector. The code is provided as follows:

```
### an alternative code for Brownian motion
## this is R code for simulating generalized Brownian motion.
BM=function(mu,sigma,T,N) {
### we need to specify the input of the Brownian process
# mu=0.05  the mean return of the stock
# sigma=0.3 the volatility of the return of the stock
# T=1  expiry time
# N=100  number of simulation points
#  h=T/N  # the timestep of the simulation
X <- c(0,cumsum(rnorm(N,mu*T/n,sigma*sqrt(T/n))))
return(invisible(X))}
```

It is not easy as it looks to run many sample paths of Brownian motions. The following code will allow us to do it.

```
### this is R code for generating n sample paths of Brownian motion
BMSamplepaths <- function(mu,sigma,T,N,nt)
{h=T/N
t=seq(0,T,by=h) # a way to produce a sequence of time points
X=matrix(rep(0,length(t)*nt), nrow=nt)
```

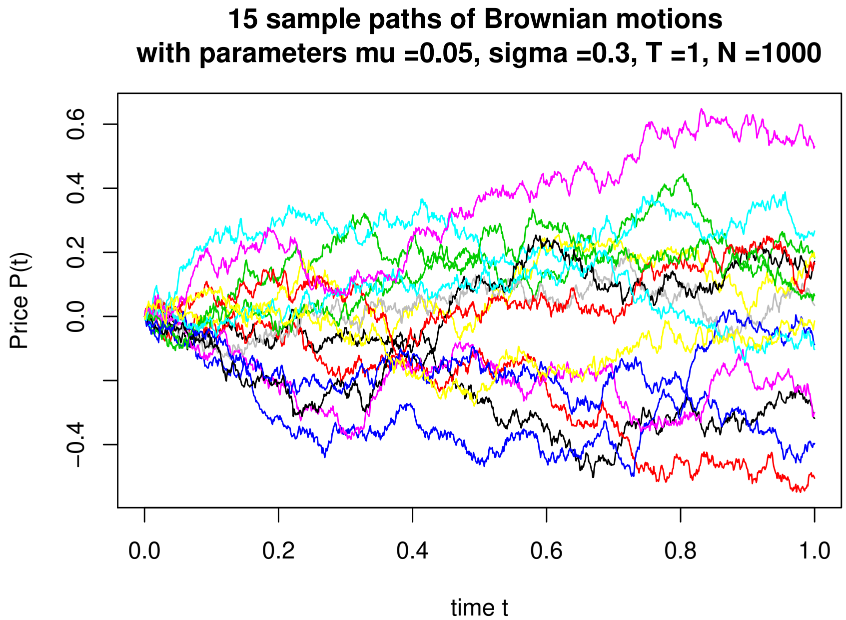

Figure 5: Sample paths of generalized Brownian motion

```
# #return(X)
for (i in 1:nt) {X[i,]= BM(mu=mu,sigma=sigma,T=T,N=N)}
# # ##Plot
ymax=max(X); ymin=min(X) #bounds for simulated prices
plot(t,X[1,],t='l',main='15 sample paths of Brownian motions
with parameters mu =0.05, sigma =0.3, T =1, N =1000',ylim=c(ymin, ymax), col=1,
ylab="Price P(t)",xlab="time t")
for(i in 2:nt){lines(t,X[i,], t='l',ylim=c(ymin, ymax),col=i)}
}
```

Running the above code, we will obtain Figure 5.

9 Brownian bridge

A useful and interesting manipulation of the Wiener process is the so-called Brownian bridge, which is a Brownian motion starting at x at time t_0 and passing through some point y at time $T > t_0$. In particular, it is defined as:

$$W_{t_0,x}^{T,y}(t) = x + W_{t-t_0} - \frac{t-t_0}{T-t_0}(W_{T-t_0} - y + x).$$

In other words, this is the process $\{W_t, 0 \leq t \leq T | W_{t_0} = x, W_t = y\}$. This process is easily simulated using the simulated trajectory of the Brownian motion.

```
Brownian_bridge=function(x,y,N,T){
#x the starting point
#y the end point at T
# number of time steps
#T the end point of time
```

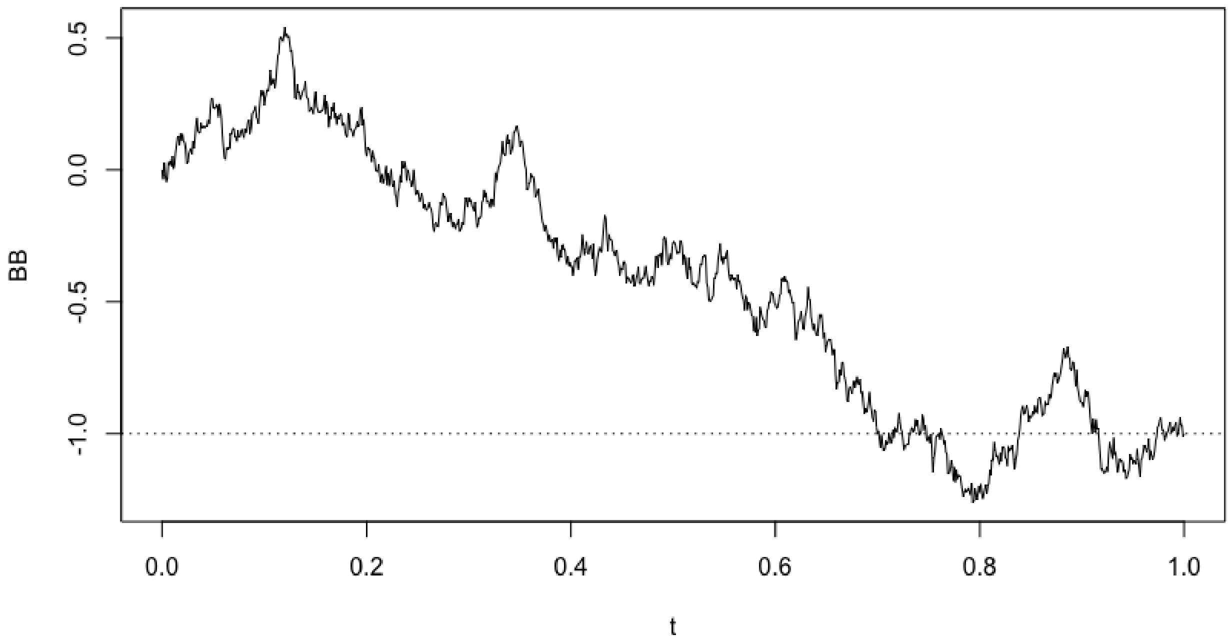

Figure 6: Brownian bride between $(0, 0)$ and $(1, -1)$.

```
#  set.seed (123)  fix the sample path in case needed
#N <- 100 # number of end points of the grid including T
#T <- 1 # length of the interval [0 ,T] in time units
Delta <- T/N # time increment
W <- numeric (N +1) # initialization of the vector W
t <- seq (0,T, length =N +1)
W <- c(0, cumsum ( sqrt ( Delta ) * rnorm (N )))
BB <- x + W - t/T * (W[N +1] - y +x)
plot (t,BB , type ="l")
abline (h=-1, lty =3)}
```

Using the code with parameters $(x = 0, y = -1, N = 1000, T = 1)$ in R, we obtain Figure 6.

10 Exercise

1. Show that $f(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$ satisfies the partial differential heat equation: $\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$.
2. For standard Brownian motion, find:
   a) $P(B_2 \le 1)$
   b) $E(B_4 | B_1 = x)$
   c) $Corr(B_{t+s}, B_s)$
   d) $Var(B_4 | B_1)$
   e) $P(B_3 \le 5 | B_1 = 2)$
3. For standard Brownian motion started at $x = -3$, find:

a) $P(X_1 + X_2 > -1)$
b) The conditional density of X_2 given $X_1 = 0$.
c) $Cov(X_3, X_4)$
d) $E(X_4|X_1)$.

4. In a race between Lisa and Cooper, let X_t denote the amount of time (in seconds) by which Lisa is ahead when $100t$ percent of the race has been completed. Assume that $(X_t)_{0\leq t\leq 1}$ can be modeled by a Brownian motion with drift parameter 0 and variance parameter σ^2. If Lisa is leading by $\sigma/2$ seconds after three-fourths of the race is complete, what is the probability that she is the winner?

5. * Consider standard Brownian motion $(B_t)_{t\geq 0}$. Let $0 < s < t$.
(a) Find the joint density of (B_s, B_t).
(b) Show that the conditional distribution of B_s given $B_t = y$ is normal, with mean and variance $E(B_s|B_t = y) = \dfrac{sy}{t}$ and $Var(B_s|B_t = y) = \dfrac{s(t-s)}{t}$.

6. * Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Compute $E(X_1|X_3)$.

7. Let $(B_t)_{t\geq 0}$ be a Brownian motion started in x. Let $X_t = B_t - t(B_1 - y)$, for $0 \leq t \leq 1$. The process $(X_t)_{t\geq 0}$ is a Brownian bridge with start in x and end in y. Find the mean and covariance functions.

8. A standard Brownian motion crosses the t-axis at times $t = 2$ and $t = 5$. Find the probability that the process exceeds level $x = 1$ at time $t = 4$.

9. Let $(X_t)_{t\geq 0}$ denote a Brownian motion with drift μ and variance σ^2. For $0 < s < t$, find $E(X_sX_t)$.

10. A Brownian motion with drift parameter $\mu = -1$ and variance $\sigma^2 = 4$ starts at $x = 1.5$. Find the probability that the process is positive at $t = 3$.

11. Use the reflection principle to show $P(M_t \geq a, B_t \leq a - b) = P(B_t \geq a + b),\ \forall\ a, b > 0$.

12. a) Prove that $P(\tau_{1.5} \leq 2) = \displaystyle\int_0^2 \frac{1.5}{\sqrt{2\pi t^3}} e^{-1.5^2/(2t)} dt$.

    b) A particle moves according to Brownian motion started at $x = 1$. After $t = 2$ hours, the particle is at level 2. Find the probability that the particle reaches level 3.5 sometime in the next two hours.

13. a) Prove that $P(\tau_{-3} \leq 4) = \displaystyle\int_0^4 \frac{3}{\sqrt{2\pi t^3}} e^{-3^2/(2t)} dt$.

    b) A particle moves according to Brownian motion started at $x = 2$. After $t = 2$ hours, the particle is at level -2. Find the probability that the particle reaches level -5 sometime in the next four hours.

14. Show that the following process is a martingale with respect to Brownian motion $(B_t)_{t\geq 0}$
a) $(B_t^3 - 3tB_t)_{t\geq 0}$
b) $(B_t^4 - 6tB_t^2 + 3t^2)_{t\geq 0}$
c) $(e^{-t\frac{c^2}{2}+cB_t})_{t\geq 0}$.

15. Show that the process $(X_t)_{t\geq 0}$, $X_t = N_t - \lambda t$, is a martingale with respect to Poisson process $(N_t)_{t\geq 0}$