LINEAR ALGEBRA – FINAL EXAMINATION

Semester 2, 2021-22 • Duration: 90 minutes

Student's name: Student ID:	Proctor's signature

Chair of Dept. of Mathematics	Lecturer	Score and Examiner
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INSTRUCTIONS:

- Use of calculator is allowed.
- This is an open book exam.
- All other documents and electronic devices are forbidden.
- You must explain your answers in detail; no marks will be given for the answer alone.
- There are a total of 5 (five) questions in this midterm examination.
- If you have issue with your internet connection or encounter problem submitting on Blackboard, call me at **0363761054** or email me at kkbhan@hcmiu.edu.vn
- GOODLUCK!

Question 1. (20 points) Let V be the Euclidean space space R^3 with the standard inner product. Let

$$u = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \; v = \begin{bmatrix} a \\ -1 \\ -b \end{bmatrix}.$$

For what value of a and b is u, v an orthonormal set?

Question 2. Let $T : R^2 \to R^2$ be a transformation defined a reflection about the x-axis.
a) (5 points) Find $T(x,y)$.
b) (10 points) Show that T is a linear transformation.
b) (10 points) Find the matrix A of T relative to the standard basis B and nonstandard basis $B' = \{(1,1), (-1,1)\}$.

Question 3. (20 points) Find the equation of the least-square lines $y = a + bx$ that best fits the data set:

x	-1	0	1	2	3
y	-2	1	1	1	4

Question 4. Let A be the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

a) (10 points) Find the eigenvalues and eigenvectors of A.
b) (10 points) Is the matrix A diagonalizable? If it is, find an invertible matrix P such that the matrix $D = P^{-1}AP$ is a diagonal matrix.
c) (10 points) Find D^5.

Question 5. (5 points) Prove that if λ is an eigenvalue of the matrix A, then λ^2 is an eigenvalue of the matrix A^2.

—END—

1/ $\{u, v\}$ is an orthonormal set

$\Leftrightarrow \langle u, v \rangle = 0 \Leftrightarrow u \cdot v = 0 \Leftrightarrow \dfrac{a}{\sqrt{2}} - \dfrac{b}{\sqrt{2}} = 0$

$\Leftrightarrow a = b$., and

$\|u\| = \|v\| = 1 \Leftrightarrow \sqrt{\langle v, v \rangle} = 1$

$\Leftrightarrow \sqrt{a^2 + 1^2 + b^2} = 1 \Leftrightarrow a^2 + 1 + a^2 = 0 \Leftrightarrow a = 0$.

Thus $\{u, v\}$ is an orthonormal set if $a = b = 0$

2/

a) $T(x,y) = (x,-y), \ \forall (x,y) \in \mathbb{R}^2$.

b) ⊛ Take any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Then

$$T((x_1,y_1)+(x_2,y_2)) = T(x_1+x_2, y_1+y_2)$$
$$= (x_1+x_2, -(y_1+y_2)) = (x_1, -y_1) + (x_2, -y_2) = T(x_1,y_1) + T(x_2,y_2).$$

⊛ Take any $(x,y) \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Then

$$T(c(x,y)) = T(cx, cy) = (cx, -cy) = c(x,-y)$$
$$= c\,T(x,y).$$

Thus T is a linear transformation, as desired.

c) $B = \{(1,0), (0,1)\}$, $B' = \{(1,1), (-1,1)\}$

Let $v_1 = (1,0)$, $v_2 = (0,1)$ then

$T(v_1) = (1,0)$, $T(v_2) = (0,-1)$. Assume

$[T(v_1)]_{B'} = (a,b)$ and $[T(v_2)]_{B'} = (c,d)$, then

$\cancel{T(v_1) = a(1,1) + b(-1,1) \Rightarrow}$

$$\begin{cases} 1 = a - b \\ 0 = a + b \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2} \\ b = -\frac{1}{2} \end{cases} \text{ and } \begin{cases} 0 = c - d \\ -1 = c + d \end{cases} \Rightarrow \begin{cases} c = -\frac{1}{2} \\ d = -\frac{1}{2} \end{cases}$$

$$\Rightarrow [T]_{BB'} = \left([T(v_1)]_{B'}, [T(v_2)]_{B'} \right) = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix}.$$

3/ From the given dataset,

$$\begin{cases} a - b = -2 \\ a + 0b = 1 \\ a + b = 1 \\ a + 2b = 1 \\ a + 3b = 4 \end{cases} \Leftrightarrow Ax = b_1, \text{ where } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \text{ and } b_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 4 \end{pmatrix}.$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 15 \end{pmatrix}$$

$$A^T b_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 17 \end{pmatrix}.$$

$\Rightarrow$ Corresponding normal system : $A^T A \hat{x} = A^T b_1$, where $\hat{x} = \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} 5 & 5 \\ 5 & 15 \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} 5 \\ 17 \end{pmatrix} \Rightarrow \text{Least square solution: } \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} -1/5 \\ 6/5 \end{pmatrix}$$

$\Rightarrow$ Equation of the least-square line : $y = \hat{a} + \hat{b}x = -\frac{1}{5} + \frac{6}{5}x$.

4/ $A = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

a) Characteristic equation:

$$\det(\lambda I_3 - A) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Leftrightarrow \lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3.$$

For $\lambda_1 = 1$: $\begin{pmatrix} -3 & 0 & 2 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow a = c = 0$

$\Rightarrow$ corresponding eigenvector: $(0, 1, 0)$.

For $\lambda_2 = 2$: $\begin{pmatrix} -2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{cases} a = c \\ b = 0 \end{cases}$

$\Rightarrow$ cor. eigenvector $(1, 0, 1)$

For $\lambda_3 = 3$: $\begin{pmatrix} -1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{cases} a = 2c \\ b = 0 \end{cases}$

$\Rightarrow$ cor. eigenvector: $(2, 0, 1)$.

b) Let $P = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, since $\det P = 1$,

the eigenvectors are linearly independent.

Hence P is invertible and

$$D = P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$ i.e. A is diagonalizable.

c) $$D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$D^3 = D^2 D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{pmatrix}$$

$$D^5 = D^2 D^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{pmatrix}$$

5/ Since λ is an eigenvalue of A, for any non zero vector x,

$Ax = \lambda x \Rightarrow A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$.

This implies λ^2 is an eigenvalue of A^2, as desired.