INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

MIDTERM EXAMINATION
November 2016
Duration: 90 minutes

SUBJECT: REAL ANALYSIS	
Deputy head of Dept. of Mathematics:	Lecturer:
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INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 *(30 marks)*

(a) Let E be a nonempty subset of $\mathbb{R}^n$. Show that E is bounded if and only if there exists a positive number K such that

$$|x_i| \leq K \quad \text{for all } i = 1, 2, ..., n \text{ and } x = (x_1, x_2, \ldots, x_n) \in E.$$

(b) Apply part (a) to show that the set

$$F = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\right\}$$

is bounded and that F is compact.

Question 2 *(25 marks)* Let (X, d) be a metric space and let $f, g : X \to \mathbb{R}$ be continuous functions. Show that the sets $\{x \in X : f(x) > g(x)\}$ and $\{x \in X : f(x) \neq g(x)\}$ are open in X.

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Question 3 *(20 marks)* Let X, Y be nonempty sets and let $f : X \to Y$ be a mapping. Let $\mathcal{M}$ be a σ-algebra in X. Show that

$$\mathcal{N} = \{F \subset Y : f^{-1}(F) \in \mathcal{M}\}$$

is a σ-algebra in Y.

Question 4 *(25 marks)* Let $(X, \mathcal{M}, \mu)$ be a probability space and let $A \in \mathcal{M}$, $\mu(A) > 0$. Show that the function $P : \mathcal{M} \to \mathbb{R}$ defined by

$$P(E) = \frac{\mu(E \cap A)}{\mu(A)}, \qquad E \in \mathcal{M}$$

is a measure and $(X, \mathcal{M}, P)$ is a probability space.

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SOLUTIONS

Question 1 (a) Suppose E is bounded. There is a ball $B(x^0, r)$ containing E. Let $K = r + \max\{|x_i^0| : 1 \leq n\}$. For each $x \in E$, we have

$$|x_i| \leq |x_i - x_i^0| + |x_i^0| \leq r + |x_i^0| \leq K, \quad 1 \leq i \leq n.$$

Conversely, suppose $|x_i| \leq K$ for all $i = 1, 2, ..., n$ and $x = (x_1, x_2, \ldots, x_n) \in E$. We then have, for every $x \in E$,

$$d(x, \mathbf{0}) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \leq \sqrt{nK^2} = \sqrt{n}\, K.$$

Hence $E \subset B(\mathbf{0}, \sqrt{n}\, K)$, that is, E is bounded.

(b) Since $0 \leq x_1, x_2$ and $x_1 + x_2 \leq 1$, we must have $0 \leq x_1, x_2 \leq 1$. By part (a), F is bounded. We now show that F is closed. Suppose $\{x^k = (x_1^k, x_2^k)\}_{k=1}^\infty \subset F$ and $x^k \to x^* = (x_1^*, x_2^*)$. Since $0 \leq x_1^k, x_2^k$ and $x_1^k + x_2^k \leq 1$, letting $k \to \infty$ gives $0 \leq x_1^*, x_2^*$ and $x_1^* + x_2^* \leq 1$. Thus $x^* \in F$ and F is closed. F is both closed and bounded in $\mathbb{R}^2$, it is compact.

Question 2 Let $h = f - g$. Then h is continuous. Thus,

$$A = \{x \in X : f(x) > g(x)\} = \{x \in X : h(x) > 0\} = h^{-1}(0, \infty)$$

is an open set in X. Similarly, $B = \{x \in X : f(x) < g(x)\}$ is also open. Therefore $\{x \in X : f(x) \neq g(x)\} = A \cup B$ is open in X.

Question 3 Since $f^{-1}(Y) = X \in \mathcal{M}$, $Y \in \mathcal{N}$.

If $F \in \mathcal{N}$, then $f^{-1}(F) \in \mathcal{M}$, hence $f^{-1}(F^c) = X \setminus f^{-1}(F) \in \mathcal{M}$ since $\mathcal{M}$ is a σ-algebra. Thus $F^c \in \mathcal{M}$.

If $\{F_n\} \subset \mathcal{N}$, then $\{f^{-1}(F_n)\} \subset \mathcal{M}$, so

$$f^{-1}\left(\bigcup_{n=1}^\infty F_n\right) = \bigcup_{n=1}^\infty f^{-1}(F_n) \in \mathcal{M}.$$

Thus $\bigcup_{n=1}^\infty F_n \in \mathcal{N}$. Therefore, $\mathcal{N}$ is a σ-algebra.

Question 4 The partial derivatives of f are

$$\begin{gathered} f_x = -6x + 6y, \qquad f_y = 6x + 6y - 6y^2 \\ f_{xx} = -6, \qquad f_{xy} = 6, \qquad f_{yy} = 6 - 12y. \end{gathered}$$

The equation $\nabla f(x, y) = \mathbf{0}$ has two solutions $(0, 0)$ and $(2, 2)$. These are critical points of f.

At $(0,0)$, the Hessian is

$$\boldsymbol{F}(0,0) = \begin{bmatrix} -6 & 6 \\ 6 & 6 \end{bmatrix}.$$

The eigenvalues of $\boldsymbol{F}(0,0)$ are $\lambda_1 = -\sqrt{72} < 0$ and $\lambda_2 = \sqrt{72} > 0$. Hence $(0,0)$ is a saddle point of f.

At $(2,2)$,

$$\boldsymbol{F}(2,2) = \begin{bmatrix} -6 & 6 \\ 6 & -18 \end{bmatrix}.$$

The eigenvalues are $\lambda = -12 \pm \sqrt{72} < 0$. Thus $\boldsymbol{F}(2,2)$ are negative definite. Hence $(2,2)$ is a local maximum point and the corresponding local maximum is $f(2,2) = 8$.

Question 5

(a) Matrix $\mathbf{A}$ is the Hessian of f,

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

(b) For $\mathbf{x} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$, $f(\mathbf{x}) = 4 > 0$ and for $\mathbf{y} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$, $f(\mathbf{y}) = -4 < 0$. Thus f and $\mathbf{A}$ are indefinite.