Joint continuous random variables

March 29, 2021

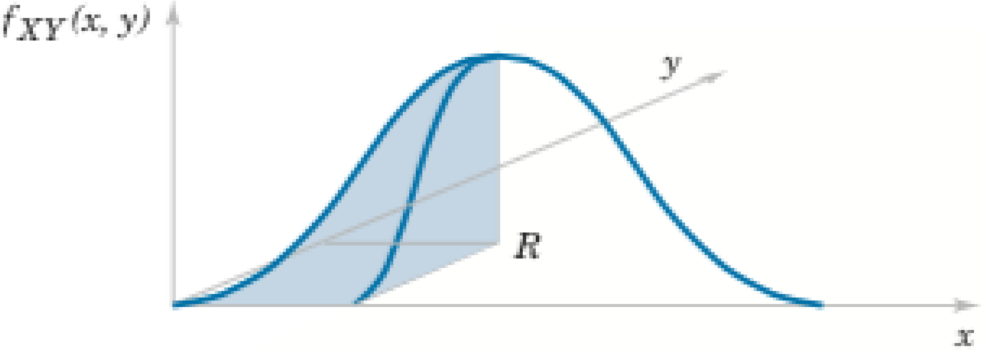
The joint p.d.f $f_{X,Y}(x,y)$ for the continuous RVs X and Y satisfies

$$P((X,Y)\in R)=\iint_R f_{X,Y}(x,y)dxdy$$

for any R in the two dimensional - plane.
In particular

$$P(a\leq X\leq b, c\leq Y\leq d)=\int_a^b\int_c^d f_{X,Y}(x,y)dxdy$$

1. $f_{X,Y}(x, y) \geq 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dxdy = 1$


$f_{XY}(x, y)$
y
R
x

When da and db are small

$$P(a < X < a + da, b < Y < b + db)$$
$$= \int_a^{a+da} \int_b^{b+db} f_{X,Y}(x, y)dxdy \approx f(a, b)dadb$$

$f_{X,Y}(a, b)$ is a measure of how likely it is that the random vector (X, Y) will be near (a, b) or "probability per unit area" in the vicinity of (a, b).

X: the time until a computer server connects to your machine (in milliseconds), Y: the time until the server authorizes you as a valid user (in milliseconds). Joint pdf of (X, Y)

$$f_{X,Y} = 6 \times 10^{-6} e^{-.001x-.002y}, \ x < y$$

Find $P(X < 1000, Y < 2000)$

$$
\begin{aligned}
&P(X < 1000, Y < 2000) \\
&= \int_{-\infty}^{1000} \int_{-\infty}^{2000} f_{X,Y}(x, y) dxdy \\
&= P(X < 1000, Y < 2000) \\
&= 6 \times 10^{-6} \int_{0}^{1000} \int_{x}^{2000} e^{-.001x-.002y} dxdy \\
&= .915
\end{aligned}
$$

Let

$$f(x,y) = \begin{cases} cx^2y, & x^2 < y < 1 \\ 0, & otherwise \end{cases}$$

be joint pdf of (X, Y)
Find

1. c
2. $P(X \geq Y)$
3. $P(X = Y)$

1.

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{-1}^{1}\int_{x^2}^{1} cx^2y = \frac{4}{21}c \Rightarrow c = \frac{21}{4}$$

2.

$$P(X \geq Y) = \iint_R f(x,y)dxdy$$

where $R = \{(x,y) : x \geq y, x^2 < y < 1\} = \{(x,y) : -1 < x < 1, x^2 < y \leq x\}$

$$P(X \geq Y) = \int_{-1}^{1}\int_{x^2}^{x} \frac{21}{4}x^2ydydx = \frac{3}{20}$$

3.

$$P(X = Y) = \iint_{y=x,\, x^2<y<1} f(x,y)dxdy = 0$$

because double integral over a line is 0.

Example

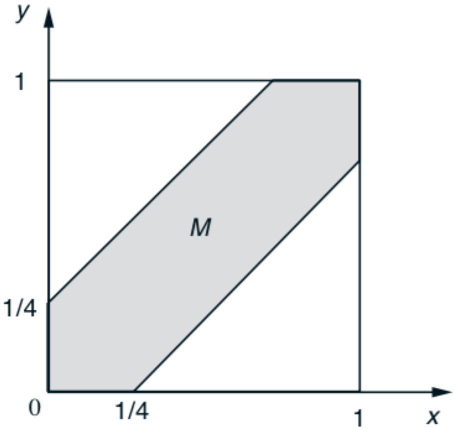
Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour. Let X and Y denote the delays of Romeo and Juliet with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 \leq x, y \leq 1 \\ 0, & otherwise \end{cases}$$

The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they will meet?

Solution

- x: arrival time of Romeo
- y: arrival time of Juliet
- They will meet if $|x - y| < 0.25$



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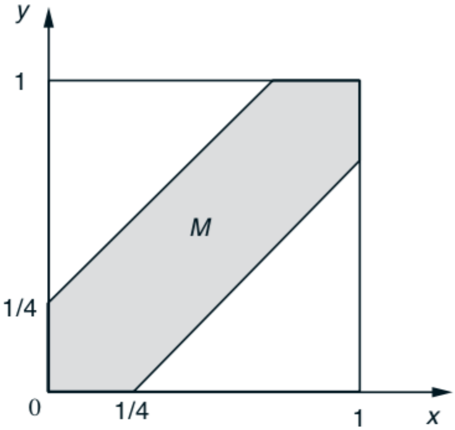

Probability that they will meet

$$P((X,Y) \in M) = \iint_M dxdy = Area(M) = \frac{7}{16}$$

X and Y are continuous: joint pdf $f_{X,Y}(x,y)$ so that

$$F_{X,Y}(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f_{X,Y}(x,y)\,dxdy$$

and

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

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$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

X and Y be described by a uniform PDF on the unit square. The joint CDF is given by $F_{X,Y}(x, y) = P(X \leq x, Y \leq y) = xy$ for $0 \leq x, y \leq 1$ then joint pdf of (X, Y) is

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) = 1$$

for $0 \leq x, y \leq 1$

Marginal c.d.f from joint p.d.f

$$F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy dx$$

$$F_Y(x) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy$$

Marginal p.d.f from joint p.d.f

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$$f_Y(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

X: the time until a computer server connects to your machine (in milliseconds), Y: the time until the server authorizes you as a valid user (in milliseconds). Joint pdf of (X, Y)

$$f_{X,Y} = 2 \times 10^{-6} e^{-.001x-.002y}, \; x, y \geq 0$$

Find pdf of X and Y

Solution

pdf of X

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= 2\times 10^{-6}\int_0^{\infty} e^{-.001x-.002y}dy \\ &= .001e^{.001x} \end{aligned}$$

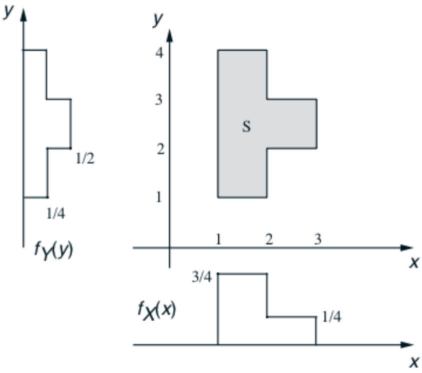
for $x > 0$

pdf of Y

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= 2 \times 10^{-6} \int_0^{\infty} e^{-.001x-.002y} dy \\ &= .002e^{.002y} \end{aligned}$$

for $y > 0$

The joint PDF of the random variables X and Y is a constant c on the set S and is zero outside. Find the value of c and the marginal PDFs of X and Y.



- Find c

$$1 = \int_S f_{X,Y}(x,y)dxdy = \int_1^2 \int_1^4 cdydx + \int_2^3 \int_2^3 cdydx = 4c$$

So $c = 1/4$

- Find pdf of X

$$f_X(x) = \int_1^4 f_{X,Y}(x,y)dy = \begin{cases} \int_1^4 \frac{1}{4}dy & \text{for } x \in [1,2] \\ \int_2^3 \frac{1}{4}dy & \text{for } x \in (2,3] \end{cases}$$

or

$$f_X(x) = \begin{cases} \frac{3}{4} & \text{for } x \in [1,2] \\ \frac{1}{4} & \text{for } x \in (2,3] \end{cases}$$

- Find c

$$1 = \int_S f_{X,Y}(x,y)dxdy = \int_1^2 \int_1^4 cdydx + \int_2^3 \int_2^3 cdydx = 4c$$

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- Find pdf of X

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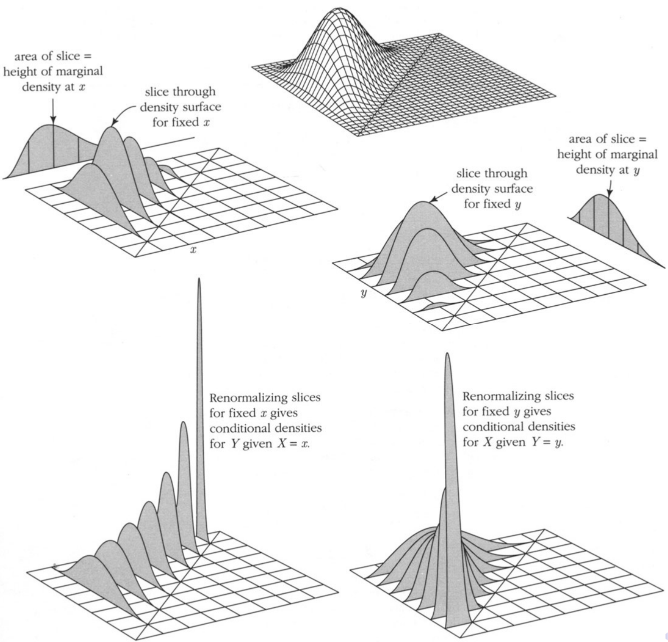
Conditional pdf of X given $Y = y$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

for $f_Y(y) > 0$


area of slice =
height of marginal
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for fixed x
x
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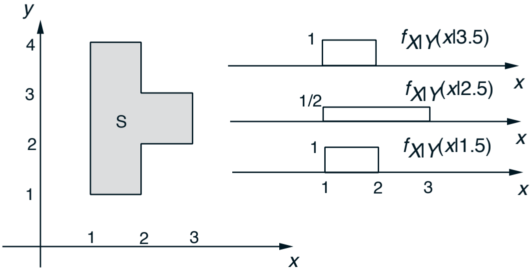

As a function of x, the conditional PDF $f_{X|Y}(x|y)$ has the same shape as the joint PDF $f_{X,Y}(x,y)$ because the normalizing factor $f_Y(y)$ does not depend on x

Suppose that a point (X, Y) is chosen uniformly at random from the triangle $\{(x, y) : x \geq 0, y \geq 0, x + y \leq 2\}$. Find $P(Y > 1|X = x)$.

John throws a dart at a circular target of radius r. We assume that he always hits the target, and that all points of impact (x, y) are equally likely, so that the joint PDF of the random variables X and Y is uniform. Since the area of the circle is πr^2, we have

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\text{area of the circle}} & \text{if } (x, y) \text{ in the circle} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{\pi r^2} & \text{if } x^2 + y^2 \leq r \\ 0 & \text{otherwise} \end{cases}$$

Find $f_{X|Y}(x|y)$

For $dx, dy \approx 0$

$$P(x \leq X \leq x + dx, y \leq Y \leq y + dy | y \leq Y \leq y + dy) = \frac{P(x \leq X \leq x + dx, y \leq Y \leq y + dy)}{P(y \leq Y \leq y + dy)}$$
$$\approx \frac{f_{X,Y}(x,y)dxdy}{f_Y(y)dy} = f_{X|Y}(x|y)dx$$

In words, $f_{X|Y}(x|y)dx$ provides us with the probability that X belongs in a small interval $[x, x + dx]$, given that Y belongs in a small interval $[y, y + dy]$.
Let $dy \to 0$

$$P(x \leq X \leq x + dx | Y = y) \approx f_{X|Y}(x|y)dx$$

and then

$$P(X \in A | Y = y) = \int_A f_{X|Y}(x|y)dx$$

X and Y are independent if for any x, y

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently

$$P(X \in A, Y \in B) = P(X \in A)P(\in B)$$

for all $A, B \in \mathcal{B}(\mathbb{R})$

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Two continuous RVs X and Y are independent if for all (x, y)

- $F_{X,Y}(x, y) = F_X(x)F_Y(y)$

  or equivalent condition

- $f_{X,Y}(x, y) = f_X(x)f_Y(y)$

  In other words

$$f_{X|Y}(x|y) = f_X(x)$$

  and

$$f_{Y|X}(y|x) = f_Y(y)$$

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  and
  $$f_{Y|X}(y|x) = f_Y(y)$$

Let (X, Y) be a random point in a square of length 1 with the bottom left corner at the origin. The joint pdf of (X, Y) is given by

$$f(x, y) = \begin{cases} 1, & 0 \leq x, y \leq 1 \\ 0, & otherwise \end{cases}$$

Are X and Y independent?

- The marginal pdfs are

$$f_X(x) = \int_0^1 f(x,y)dy = x, \ \ 0 \le x \le 1$$

and

$$f_Y(y) = \int_0^1 f(x,y)dx = y, \ \ 0 \le y \le 1$$

- $f(x,y) = f_X(x)f_Y(y)$ for all x, y so X and Y are independent

Let (X, Y) be a random point in the triangle $\{(x, y) : 0 \leq x \leq y \leq 1\}$, i.e the joint pdf of (X, Y) is given by

$$f(x, y) = \begin{cases} 2, & 0 \leq x \leq y \leq 1 \\ 0, & otherwise \end{cases}$$

Are X and Y independent?

Mr. and Mrs. Smith agree to meet at a specified location between 5 and 6 p.m." Assume that they both arrive there at a random time between 5 and 6 and that their arrivals are independent.

1. Find the density for the time one of them will have to wait for the other.
2. Mrs. Smith later tells you she had to wait; given this information, compute the probability that Mr. Smith arrived before 5:30.

Among the following pdf's which are independent? (Each of the range choosen such that $\int_R f_{X,Y}(x,y)dxdy = 1$)

1. $f_{X,Y}(x,y) = 4x^2y^3$
2. $f_{X,Y}(x,y) = \frac{1}{2}(x^3y + xy^3)$
3. $f_{X,Y}(x,y) = 6e^{-2x-3y}$

X and Y are independent

- $g(X)$ and $h(Y)$ are independent for any function g and h
- $$E(XY) = E(X)E(Y)$$
- $$E(g(X)h(Y)) = E(g(X))E(h(Y))$$