Chapter 3: MULTIPLE INTEGRALS

Lecture 8:

- **Double Integrals over Rectangles**
- **Iterated Integrals**

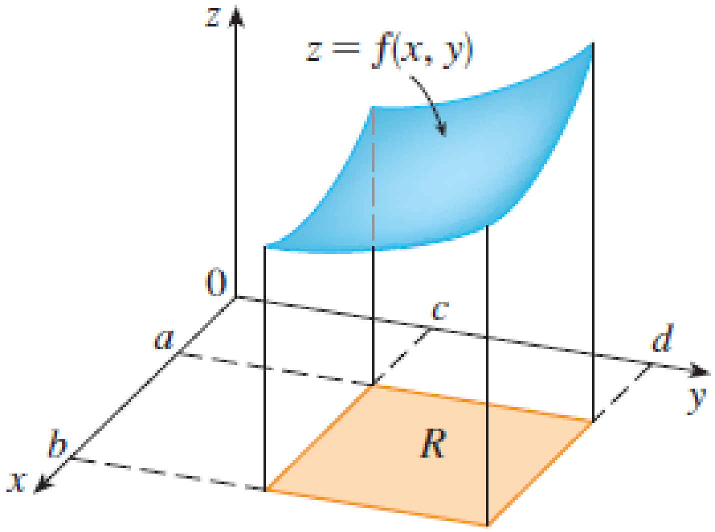
How can you evaluate the water amount in a lake?

1. *Double Integrals over Rectangles:* Volume of a Solid under a Surface

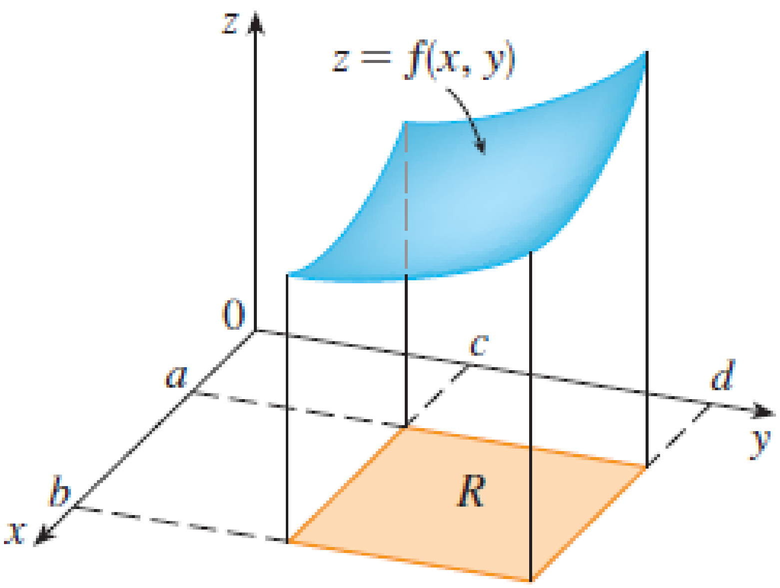
$z = f(x, y) \geq 0$ defined on

$$R = [a,b] \times [c,d] = \{(x,y) \mid a \leq x \leq b,\ \ c \leq y \leq d\}$$

$$S = \{(x,y,z) \mid \ \ 0 \leq z \leq f(x,y)\}$$



Problem: What is the volume of S?



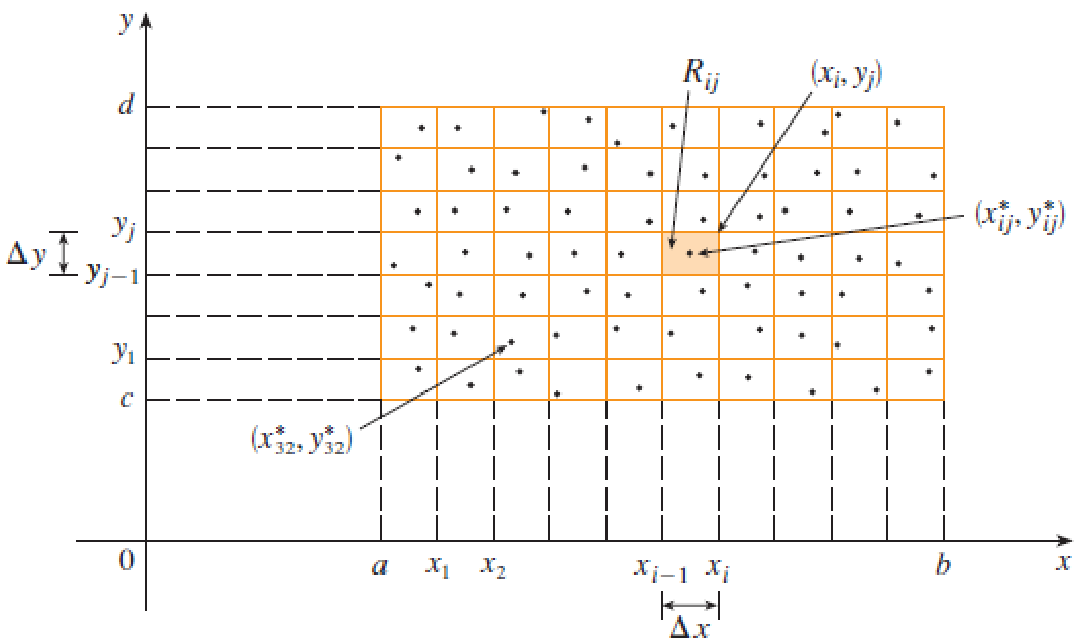
Divide rectangle R into mn subrectangles:

- ☐ Dividing [a, b] into m subintervals $[x_{i-1}, x_i]$ of equal width

$$\Delta x = (b-a)/m$$

- ☐ Dividing [c, d] into n subintervals $[y_{j-1}, y_j]$ of equal width

$$\Delta y = (d-c)/n$$

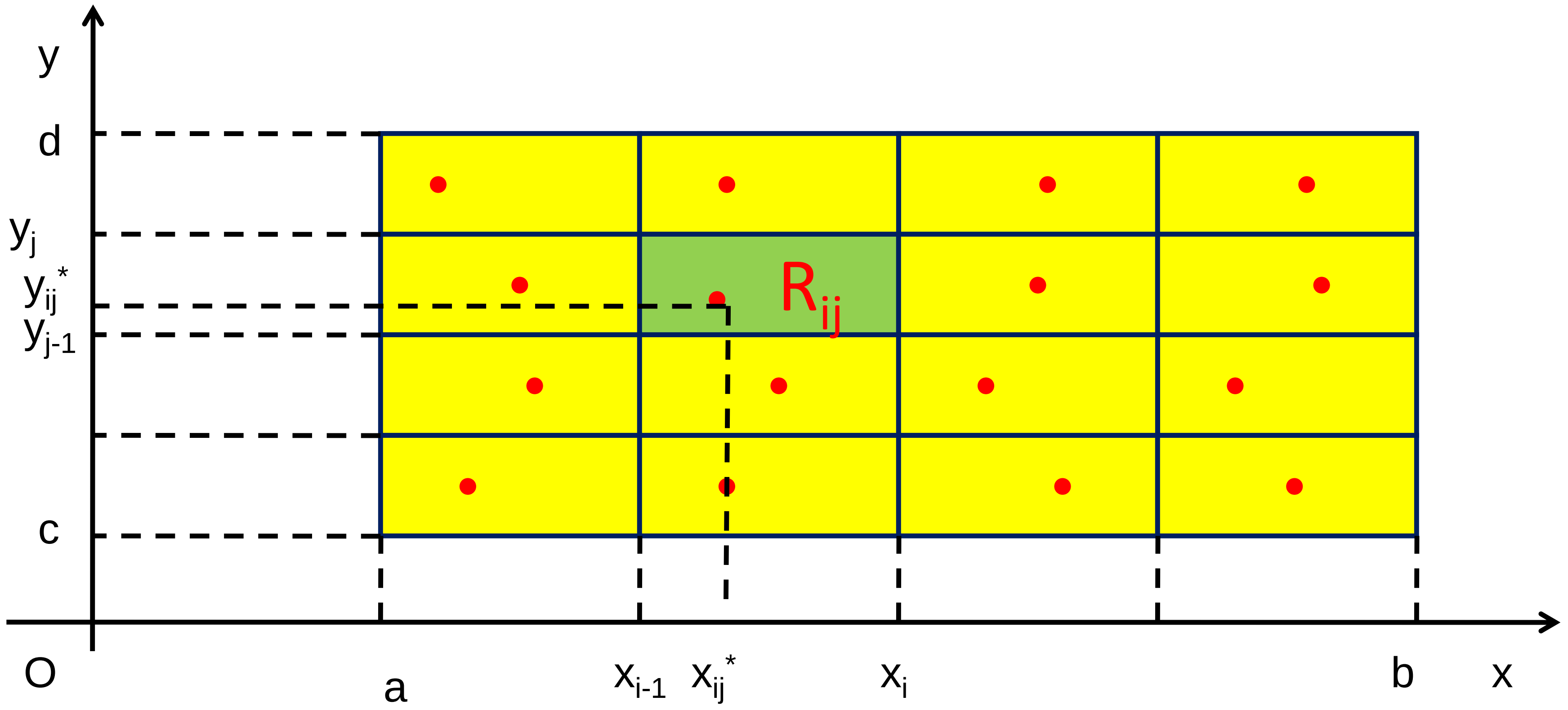


- Form *m.n* subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \le x \le x_i, \;\; y_{j-1} \le y \le y_j\}$$

each with area ΔA= Δx Δy

- Choose a sample point (x_{ij}^*, y_{ij}^*) in each R_{ij}



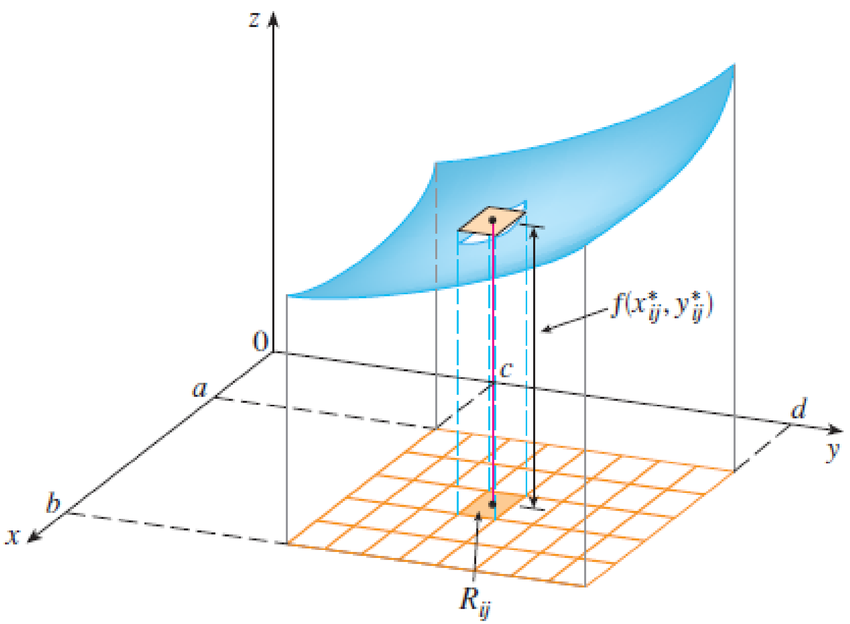
Approximate the part of S that lies above each R_{ij} by a thin rectangular box with base R_{ij} and height

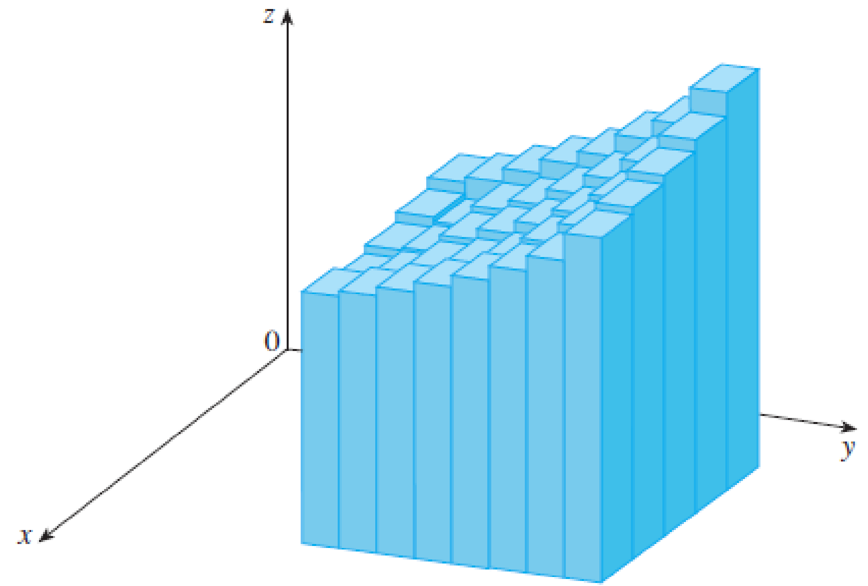
$$f(x_{ij}^*, y_{ij}^*)$$

The volume of this box is

$$f(x_{ij}^*, y_{ij}^*)\Delta A$$

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$





Volumes and Double Integrals

- Our intuition suggests that the approximation becomes better as *m* and *n* become larger and so we would expect that

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \Delta A$$

- We use this expression to define the **volume** of the solid that lies under the graph of *f* and above the rectangle *R*

Volumes and Double Integrals

Definition: The double integral of *f* over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

if this limit exists

- **Remark:**
  - It can be proved that the limit exists if *f* is a continuous function.
  - It also exists for some discontinuous functions as long as they are reasonably “well-behaved”

Volume of a solid under a surface

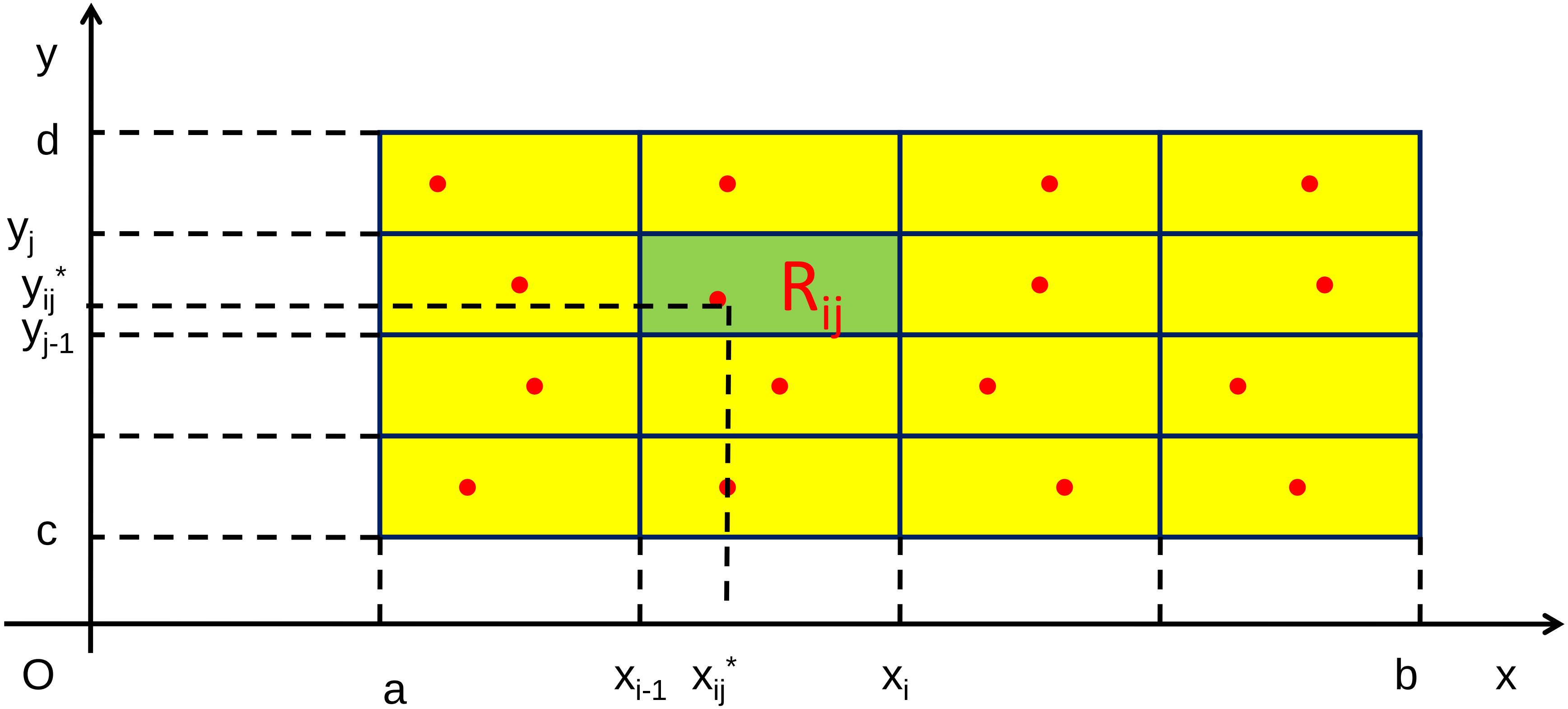
- If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z=f(x,y)$ is

$$V = \iint_R f(x, y)\,dA$$

Approximation by double Riemann sum

Double Riemann sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$

is used as an approximation of double integral



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The choice of Sample Points: Upper right-hand corners

- The sample point (x_{ij}^*, y_{ij}^*) can be the upper right-hand corner of R_{ij}, namely (x_i, y_j). Then the expression for the double integral looks simpler:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta A$$

$$x_i = a + i\frac{b-a}{m}, \qquad y_j = c + j\frac{d-c}{n},$$

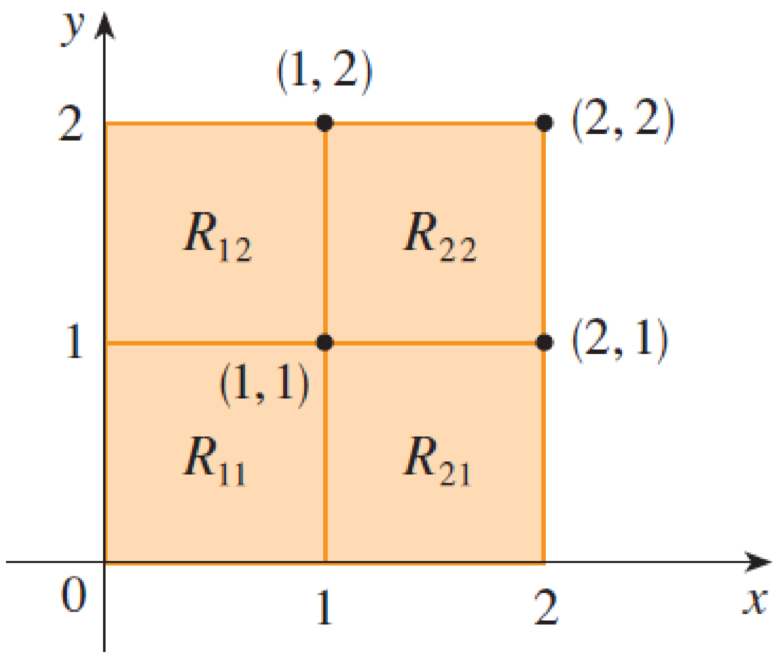
$$\Delta A = \Delta x \Delta y = \frac{b-a}{m}\frac{d-c}{n}$$

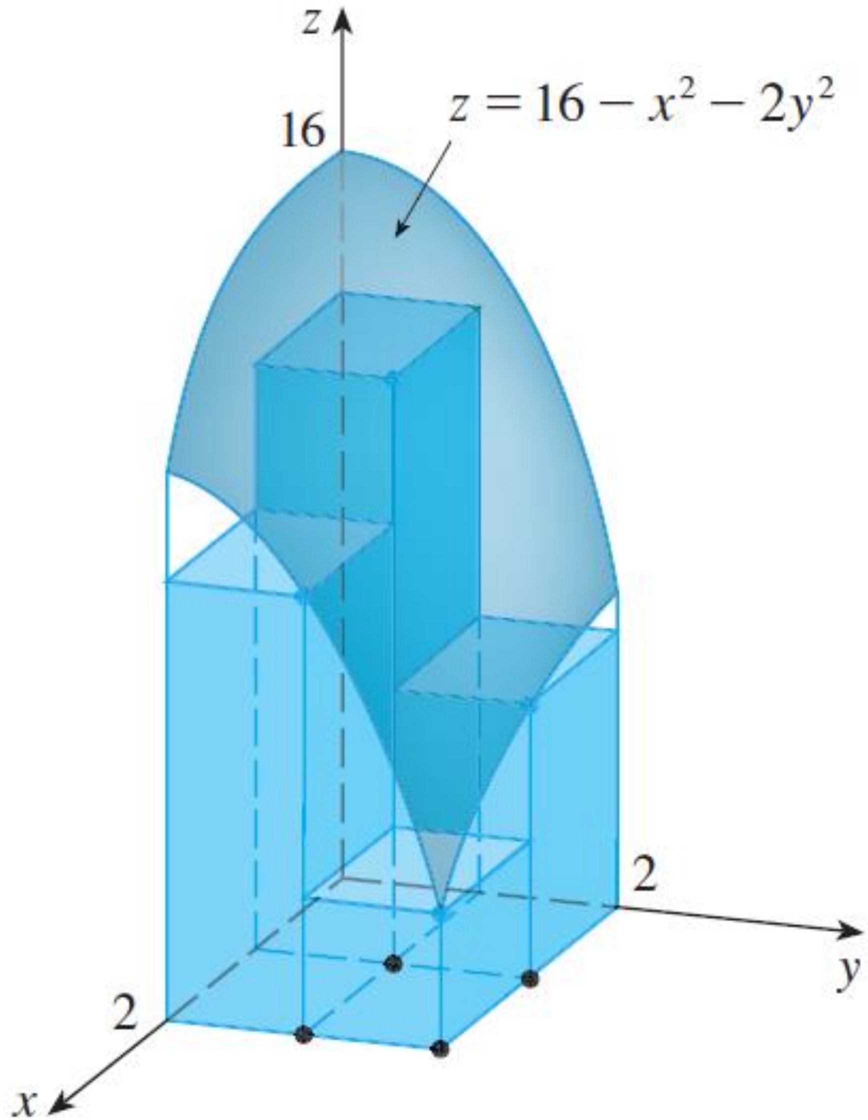
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta A$$

Example: Estimate the volume of the solid that lies above the square R=[0, 2]x[0, 2] and below the elliptic paraboloid

$$z = 16 - x^2 - 2y^2$$

Divide R into four equal squares and choose the sample point to be the upper right corner of each subsquare.





Solution

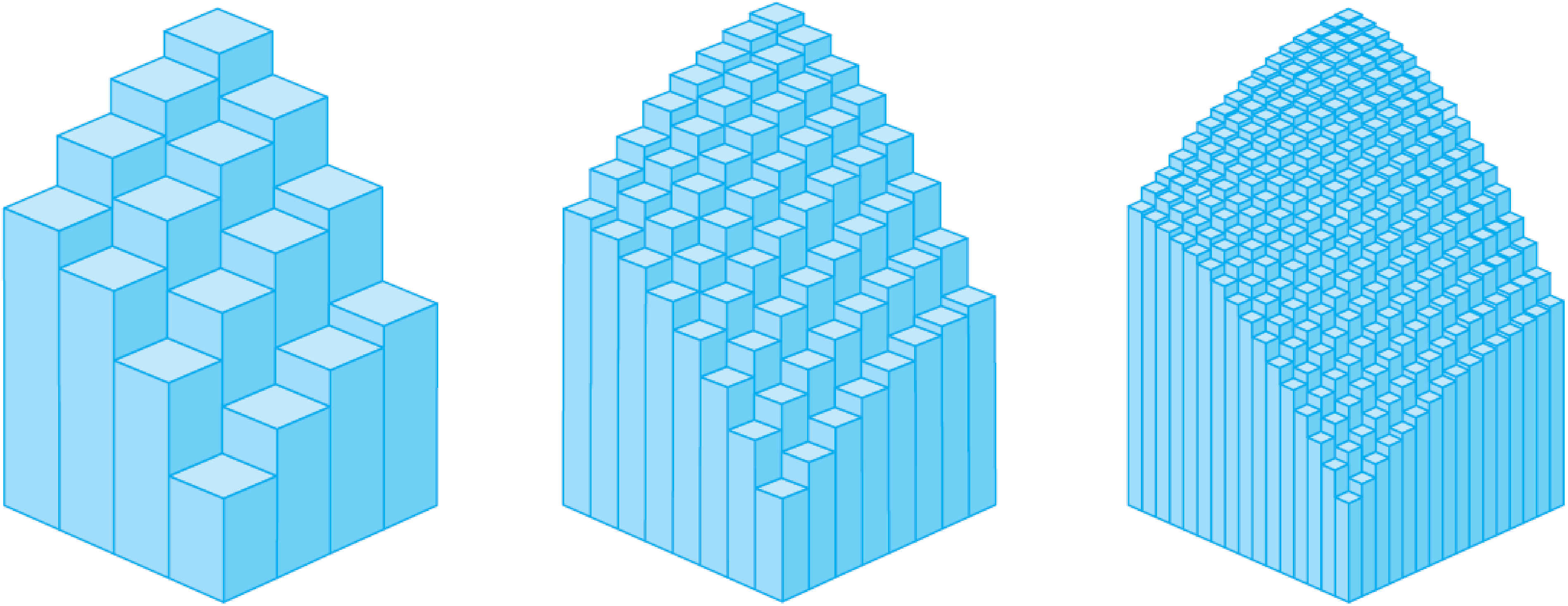
- Approximate the double integral by the Riemann sum with m=n=2:

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\Delta A$$

$$= f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$

The Riemann sum approximations become more accurate as *m* and *n* increase



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Approximation: Midpoint Rule

- We take the sample point (x_{ij}^*, y_{ij}^*) to be the center $(\overline{x_i}, \overline{y_j})$ of R_{ij}, where

$$\overline{x_i} = \frac{x_{i-1} + x_i}{2}, \qquad \overline{y_j} = \frac{y_{j-1} + y_j}{2}$$

- Then

$$\boxed{\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\,\Delta A}$$

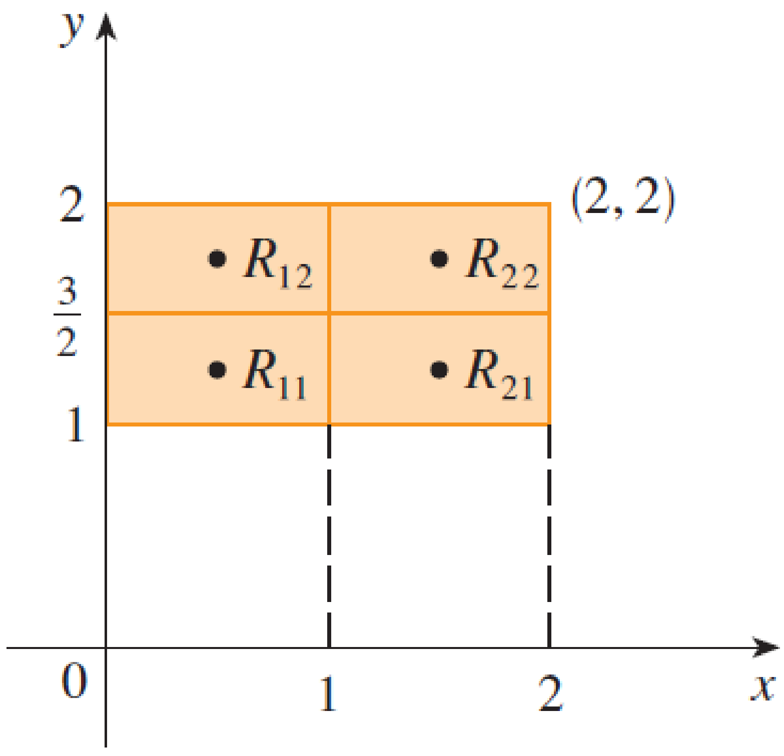
Midpoint Rule: Example

- Use midpoint rule with m=n=2 to estimate the value of the double integral

$$\iint_R (x-3y^2)\,dA$$

where

$$R=\{(x,y)\,|\,0\le x\le 2, 1\le y\le 2\}$$



$$\overline{x_1} = \frac{1}{2}, \overline{x_2} = \frac{3}{2}, \ \overline{y_1} = \frac{5}{4}, \overline{x_2} = \frac{7}{4}, \text{ and } \Delta A = \frac{1}{2}$$

$$\iint_R (x - 3y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x_i}, \overline{y_j})\Delta A$$

$$= f(\overline{x_1}, \overline{y_1})\Delta A + f(\overline{x_1}, \overline{y_2})\Delta A + f(\overline{x_2}, \overline{y_1})\Delta A + f(\overline{x_2}, \overline{y_2})\Delta A$$

$$= f(\frac{1}{2}, \frac{5}{4})\Delta A + f(\frac{1}{2}, \frac{7}{4})\Delta A + f(\frac{3}{2}, \frac{5}{4})\Delta A + f(\frac{3}{2}, \frac{7}{4})\Delta A$$

$$= (-\frac{67}{16})\frac{1}{2} + (-\frac{139}{16})\frac{1}{2} + (-\frac{51}{16})\frac{1}{2} + (-\frac{123}{16})\frac{1}{2}$$

$$= -\frac{95}{8} = -11.875$$

Average Value

- We define the **average value** of a function *f* of two variables defined on a rectangle *R* to be

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

- where *A(R)* is the area of *R*.

Properties of double integrals

- We assume that all of the integrals exist. It holds that:

1) $$\iint_R (f(x,y)+g(x,y))dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$$

2) $$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA, \quad \text{where } c \text{ is a constant}$$

3) If $f(x,y) \geq g(x,y), \forall (x,y) \in R$, then

$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$$

2. Iterated Integrals: Motivations

- **How to evaluate double integrals?**
- Let a function f(x,y) be defined on a rectangle R=[a, b]x[c, d]
- Integrate with respect to x and y separately:

$$\int_a^b f(x,y)dx = g(y), \qquad \int_c^d f(x,y)dy = h(x)$$

- These are functions of one variable. Therefore, they can also be integrated!

$$\int_c^d g(y)dy = \int_c^d \left[\int_a^b f(x,y)dx \right] dy \qquad \int_a^b h(x)dx = \int_a^b \left[\int_c^d f(x,y)dy \right] dx$$

Example

- Let f(x,y)=3xy^2 + 6x^{2}y + 2y defined on

$$R = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 1\}$$

- First, integrate f(x,y) with respect to x, to get

$$\int_0^2 (3xy^2 + 6x^2y + 2y)dx = \left(\frac{3}{2}x^2y^2 + 2x^3y + 2xy\right)\Bigg|_{x=0}^{x=2}$$

$$= 6y^2 + 20y = g(y)$$

- Second, integrate g(y) w.r.t. y to get

$$\int_0^1 (6y^2 + 20y)dy = (2y^3 + 10y)\Big|_0^1 = 12$$

Example

- On the other hand, if we take integral of f(x,y) with respect to y first, we obtain

$$\int_0^1 (3xy^2 + 6x^2y + 2y)dy = (xy^3 + 3x^2y^2 + y^2)\Big|_{y=0}^{y=1}$$

$$= 3x^2 + x + 1 = h(x)$$

- Then, we integrate h(x) w.r.t. x to get

$$\int_0^2 (3x^2 + x + 1)dx = (x^3 + \frac{x^2}{2} + x)\Big|_0^2 = 12$$

Iterated Integrals

- In general, it holds that

$$\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

- Usually the brackets are omitted, and so

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

- Each of these integrals is called an **iterated integral**

Fubini's Theorem

- If f is continuous on the rectangle

then $$R = \{(x, y) \mid a \le x \le b, c \le y \le d\}$$

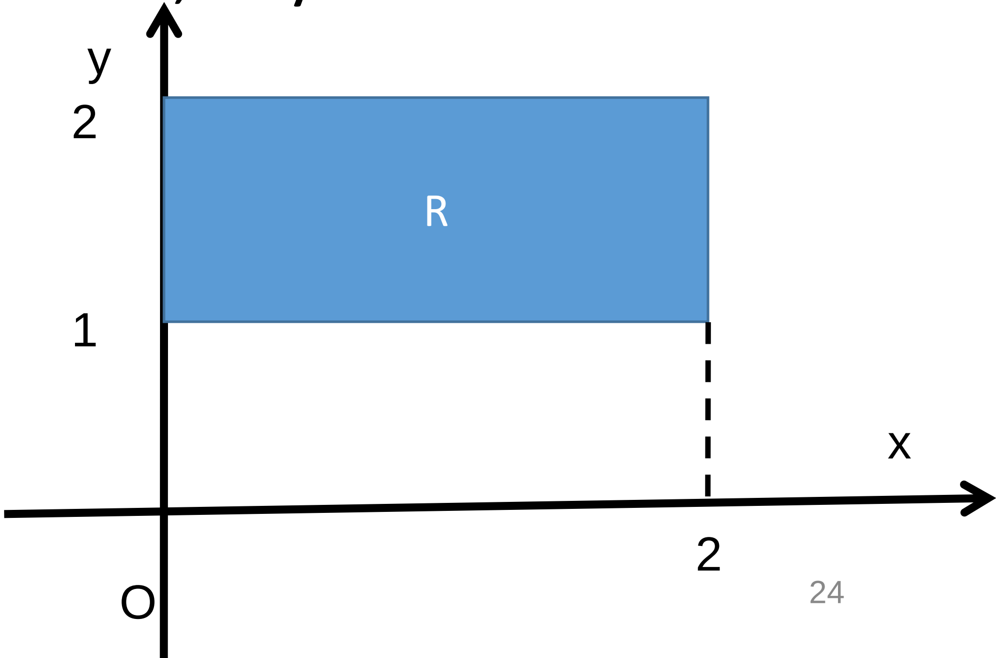
$$\boxed{\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx}$$

- More generally, this is true if f is bounded on *R*, is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 1

- Find $$\iint_R (x - 3y^2)\,dA$$

where R is the rectangle 0≤x ≤2, 1≤y ≤2



Solution

R is the rectangle 0≤x ≤2, 1≤y ≤2

- We have

$$\iint_R (x-3y^2)\,dA = \int_0^2 \boxed{\int_1^2 (x-3y^2)\,dy}\,dx$$

$$= \int_0^2 (xy-y^3)\Big|_{y=1}^{y=2} dx = \int_0^2 (2x-8-x+1)\,dx$$

$$= \int_0^2 (x-7)\,dx = \left(\frac{x^2}{2}-7x\right)\Big|_0^2 = -12$$

Homework Chapter 3

- Section 15.1 Double integrals over rectangles : 1, 3, 4, 5, 9
- Section 15.2 Iterated Integrals: 10, 12, 17, 22, 25
- Section 15.3 Double integrals over general regions: 8, 9, 16, 19, 22, 24, 28, 30
- Section 15.4 Double integrals in polar coordinates: 7, 9, 13, 20, 22
- Section 15.5 Applications of double integrals: 5, 7, 10, 12
- Section 15.7 Triple Integrals: 8, 12, 14, 37, 40