Nguyen Minh Quan MAMAIU19036

I/ Linear 1st order.

$$a_0(x)\frac{dy}{dx} + a_1(x)y = b(x),$$

$$\iff \frac{dy}{dx} + P(x)y = Q(x).$$

S : Rewrite into standard form.

· Integrating factor:

$$\mu(x) = e^{\int P(x)dx}.$$

Step 3: Rewrite as

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)Q(x)$$

$$\iff y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right]$$

II/ Separable:

$$\frac{dy}{dx} = g(x)h(y) \iff M(x)dx = N(y)dy$$

$$\Rightarrow \int M dx = \int N dy + C.$$

III/ Exact Equation:

$$M(x,y)dx + N(x,y)dy = 0 \quad (1)$$

Case 1. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then

(1) is exact. $\Rightarrow \exists F(x,y)$: $\begin{cases}\frac{\partial F}{\partial x} = M \\ \frac{\partial F}{\partial y} = N.\end{cases}$

(1) $\iff dF = 0 \iff F = c$: constant.

Case 2: If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then

(1) is not exact.

- If $\frac{M_y - N_x}{N} = f(x)$ then $\mu(x) = e^{\int f(x)dx}$

- If $\frac{M_y - N_x}{-M} = g(y)$ then $\mu(y) = e^{\int g(y)dy}$

$\Rightarrow \mu M dx + N\mu dy = 0$ is exact.

IV/ Wronskian determinant:

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

V/ 2nd order ~~non~~ homogeneous with constant cafficients:

$$ay'' + by' + cy = 0, \quad (1).$$

Characteristic equation of (1):

$$ar^2 + br + c = 0. \quad (2)$$

⊛ If $\Delta = b^2 - 4ac > 0$, then (2) has 2 distinct real roots

$$r_1 = \frac{-b+\sqrt{\Delta}}{2a}, \quad r_2 = \frac{-b-\sqrt{\Delta}}{2a}$$

$\Rightarrow$ General solution for (1):

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

⊛ If $\Delta = 0$, then (2) has a repeat root $r = \frac{-b}{2a}$.

$\Rightarrow$ General solution for (1):

$$y(x) = (c_1 + c_2 x)e^{rx}.$$

⊛ If $\Delta < 0$, let

$$\alpha = \frac{-b}{2a} \text{ and } \beta = \frac{\sqrt{4ac - b^2}}{2a}.$$

$\Rightarrow$ General solution for (1):

$$y(x) = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x.$$

VI/ 2nd order non-homogeneous with constant coefficients:

$$ay'' + by' + cy = g(x). \quad (1)$$

Step 1: Find a particular solution $y_p(x)$ for (1).

Case 1. $g(x) = P_n(x) = \sum_{i=0}^{n} a_i x^i$.

- $c \neq 0 \Rightarrow y_p(x) = \sum_{i=0}^{n} A_i x^i$.
- $b \neq 0, c = 0 \Rightarrow y_p(x) = x\left(\sum_{i=0}^{n} A_i x^i\right)$
- $b = c = 0 \Rightarrow y_p(x) = x^2\left(\sum_{i=0}^{n} A_i x^i\right)$.

Case 2: $g(x) = P_n(x)e^{\alpha x}$.

Consider the characteristic equation: $ar^2 + br + c = 0$. (2)

- α is not a root of (2): $y_p(x) = Q_n(x)e^{\alpha x}$.
- α is a single root of (2): $y_p(x) = xQ_n(x)e^{\alpha x}$
- α is a double root of (2): $y_p(x) = x^2 Q_n(x)e^{\alpha x}$.

Case 3: $g(x) = P_n(x)e^{\alpha x}\begin{cases}\cos\beta x \\ \sin\beta x\end{cases}$

Consider (2).

- $\alpha + i\beta$ is not a root of (2):
$y_p(x) = \left[Q_n(x)\cos\beta x + R_n(x)\sin\beta x\right]e^{\alpha x}$
- $\alpha + i\beta$ is a root of (2):
$y_p(x) = x\left[Q_n(x)\cos\beta x + R_n(x)\sin\beta x\right]e^{\alpha x}$

Case 4: $g(x) = g_1(x) + \dots + g_n(x)$, where each g_i is of case 1/2/3.

Consider the following equation:

$$ay'' + by' + cy = g_1(x).$$

$$\vdots$$

$$ay'' + by' + cy = g_n(x).$$

Use the previous methods to get particular solutions $y_1, \dots, y_n$ for the above equations.

Then by the Super-posision principle, a particular solution of (1) is

$$y_p = y_1 + \dots + y_n.$$

Step 2 Find the general solution $y'(x)$ of $ay'' + by' + cy = 0$.

Step 3 General solution for (1):

$$y(x) = y'(x) + y_p(x).$$

VII/ 2nd order homogeneous with non-constant coefficients:

$$y'' + p(x)y' + q(x)y = 0. \quad (1)$$

Step 1 Find a particular solution y_1 for (1). y_1 may be given, or have special forms (like x^α).

Step 2 Calculate $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}dx$

Step 3 General solution for (1):

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

VIII/ 2nd order non-homogeneous with non-constant coefficients:

$$y'' + p(x)y' + q(x)y = g(x) \quad (1).$$

Step 1 Find a fundamental solution set $\{y_1, y_2\}$ of $y'' + p(x)y' + q(x)y = 0$.

Step 2 Let $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

then $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(x). \end{cases}$ (2)

Step 3 Solve (2) for u_1', u_2', then calculate y_p from u_1, u_2.

Step 4 General solution for (1):

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x).$$