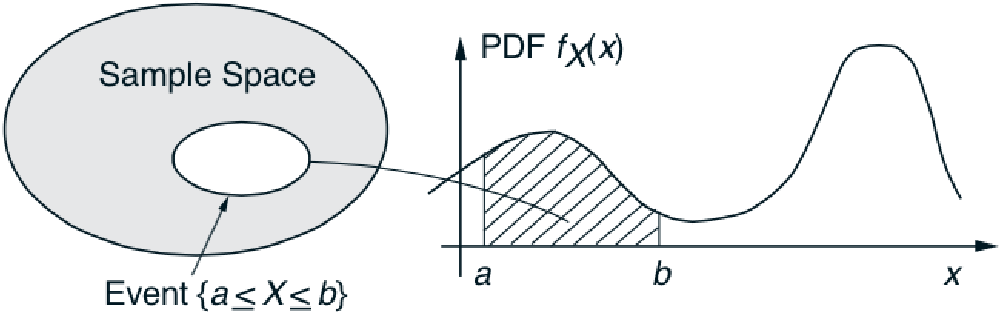
Continuous random variables

March 28, 2021

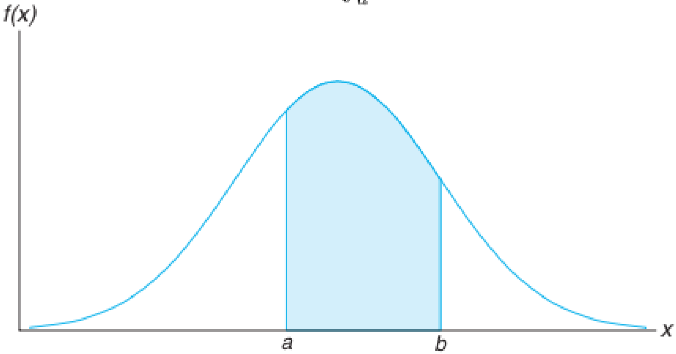
Suppose $Range(X)$ is uncountable.
X is *continuous* if there is a non negative function $f(x)$ so that

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

$f(x)$ is called the *probability density function* (pdf) of X.


Sample Space
Event $\{a \le X \le b\}$
PDF $f_X(x)$
a
b
x

$$P(a < X < b) = \int_a^b f(x)\, dx.$$

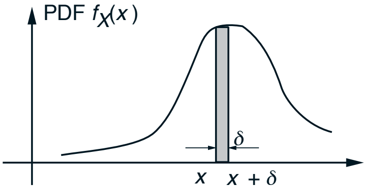


Note that probability of any individual value is 0

- $P(X = a) = 0$
- $$\begin{aligned} P(a \le X \le b) &= P(a < X \le b) \\ &= P(a \le X < b) \\ &= P(a < X < b) \end{aligned}$$

$$P\left(x \leq X \leq x+\delta\right) = \int_x^{x+\delta} f(u)du \approx \delta f(x)$$

$f(x)$ is not a probability of X at x but
a measure of how likely it is that the random variable will be near x - "pmf per unit length"



Density has the same role as the probability mass function for discrete random variables: it tells which values x are relatively more probable for X than others.

- $f(x) \geq 0$ for all x
- $\int\limits_{-\infty}^{\infty} f(x)dx (= P(-\infty < X < \infty)) = 1$

Suppose that the error in the reaction temperature, in $\circ C$, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3} & \text{if } -1 < x < .2 \\ 0 & \text{elsewhere} \end{cases}$$

1. Verify that $f(x)$ is a density function.
2. Find $P(0 < X \leq 1)$.

1. Obviously $f(x) \geq 0$. Need to verify the 2nd condition

$$\int_{-\infty}^{\infty} f(x)dx = 0 \text{ or } \int_{-\infty}^{-1} 0dx + \int_{-1}^{2} \frac{x^2}{3}dx + \int_{2}^{\infty} 0dx = 1$$

2. $P(0 < X \leq 1) = \int_0^1 f(x)dx = \int_0^1 \frac{x^2}{3}dx = \left.\frac{x^3}{9}\right|_0^1 = \frac{1}{9}$

A gambler spins a wheel of fortune, continuously calibrated between 0 and 1, and observes the resulting number. Assuming that all subintervals of $[0, 1]$ of the same length are equally likely. The observed number is a random variable X with pdf

$$f(x) = \begin{cases} c, & 0 \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

The constant c is determined by

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

So

$$\int_{0}^{1} cdx = 1$$

then $c = 1$

RV X takes values in an interval $[a, b]$ such that all subintervals of the same length are equally likely. X is an uniform RV with pdf

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & otherwise \end{cases}$$

Denote $X \sim Uni([a, b])$

Suppose the p.d.f of X is

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

1. What is the value of C?
2. Find $P(X > 1)$?

Solution

1. By property of p.d.f: $\int_0^2 C(4x - 2x^2)dx = 1$, which implies $C = 3/8$.
2. Then

$$P(X > 1) = \int_1^2 \frac{3}{8}(4x - 2x^2)dx = \frac{1}{2}$$

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The total number of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a continuous random variable X that has the density function

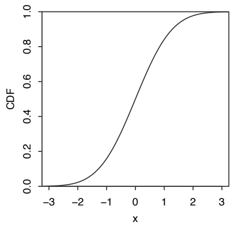
$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 0, & elsewhere \end{cases}$$

1. less than 120 hours
2. between 50 and 100 hours.

Cumulative distribution function (cdf)

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

Properties of cdf of a continuous RV



- $F'(x) = f(x)$ for all x
- $P(a \leq X \leq b) = F(b) - F(a)$
- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
- continuous

Suppose that the error in the reaction temperature, in $^\circ C$, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3} & -1 < x < 2 \\ 0 & elsewhere \end{cases}.$$

Find cdf F of X

- For $x < -1$

$$F(x) = \int_{-\infty}^{x} f(u)du = \int_{-\infty}^{x} 0du = 0$$

- For $-1 \leq x \leq 2$

$$F(x) = \int_{-\infty}^{x} f(u)du = \int_{-1}^{x} \frac{u^3}{3}du = \frac{x^3+1}{9}$$

- For $x > 2$

$$F(x) = \int_{-\infty}^{x} f(u)du = \int_{-1}^{2} \frac{u^3}{3} du = 1$$

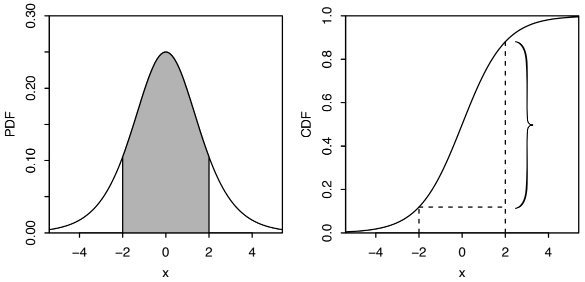
- Hence

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{x^3+1}{3}, & -1 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

The logistic distribution has cdf

$$F(x) = \frac{e^x}{1+e^x}$$

1. Compute $P(-2 < X < 2)$
2. Find pdf of X



Practice

A certain river floods every year. Suppose that the low-water mark is set at 1 and a high-water mark X has distribution function

$$F(x) = \begin{cases} 0, & x, 1 \\ 1 - \frac{1}{x^2}, & x \geq 1 \end{cases}$$

1. Verify that $F(x)$ is a cdf.
2. Find the pdf of X.
3. Calculate the probability that the high-water mark is between 3 and 4.

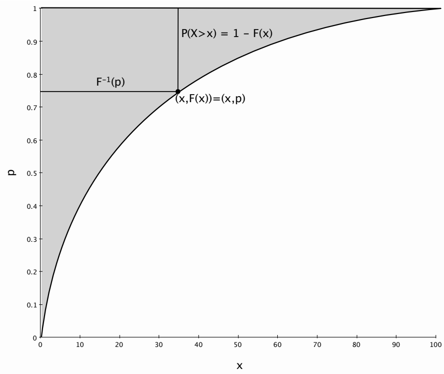
Suppose a certain type of small data processing firm is so specialized that some have difficulty making a profit in their first year of operation. The probability density function that characterizes the proportion X that make a profit is given by

$$f(x) = \begin{cases} ky^4(1-y)^3, & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$$

1. What is the value of k that renders the above a valid density function?
2. Find cdf $F(x)$
3. Find the probability that at most 50% of the firms make a profit in the first year.
4. Find the probability that at least 80% of the firms make a profit in the first year.

Quantile function of continuous random variable

$$F^{-1}(p) = x \Leftrightarrow P(X \leq x) = p$$



Find quantile function F^{-1} of logistic distribution.