THE INTERNATIONAL UNIVERSITY - VIETNAM NATIONAL UNIVERSITY - HCMC

FINAL EXAMINATION

Date: August 2016 • Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Chair of Department of Mathematics	Lecturer
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INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1. *[20 marks]*

Suppose that $X = A \cup B$ where A, B are nonempty and measurable. Show that a function $f : X \to \overline{\mathbb{R}}$ is measurable if and only if f is measurable on A and on B.

Question 2. *[15 marks]*

Show that if f is integrable on E, g is measurable on E, and there exists a finite constant M such that $|g| \leq M$ in E, then fg is integrable on E and $\int_E |fg| d\mu \leq M \int_E |f| d\mu$.

Question 3. *[20 marks]*

Let f be Lebesgue integrable on $[-1, 1]$. Show that the functions $f_n(x) = x^n f(x)$, $n = 1, 2, ...$, are Lebesgue integrable on $[-1, 1]$ and that

$$\lim_{n\to\infty} \int_{-1}^{1} x^n f(x) dx = 0.$$

(*Hint:* Use the Dominated Convergence Theorem.)

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Question 4. *[20 marks]*

If f has a continuous derivative on $[a, b]$, show that $V_a^b(f) \leq \int_a^b |f'(t)|dt$.

(*Hint:* If $a \leq x_i < x_{i+1} \leq b$, then $|f(x_{i+1}) - f(x_i)| = \left| \int_{x_i}^{x_{i+1}} f'(t)dt \right|$. Derive that for any partition P of $[a, b]$, $V(f, P) \leq \int_a^b |f'(t)|dt$.)

Question 5. *[25 marks]*

(a) *[10 marks]* Let S be a negative set of a signed measure μ. Show that if A and B are measurable subsets of S with $A \subset B$, then $\mu(A) \geq \mu(B)$.

(b) *[15 marks]* Suppose that μ and ν are finite signed measures satisfying $\nu \ll \lambda$, $\mu \ll \lambda$. Show that $\nu + \mu \ll \lambda$ and the Radon-Nikodym derivative of $\nu + \mu$ with respect to λ is

$$\frac{d}{d\lambda}(\nu + \mu) = \frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda}.$$

$* * *$ END OF QUESTION PAPER $* * *$

SOLUTIONS

Question 1.

Suppose that f is measurable on X. For each $\alpha \in \mathbb{R}$,

$$\begin{aligned}\{x \in A : f(x) < \alpha\} &= A \cap \{x \in X : f(x) < \alpha\} \quad \text{and} \\ \{x \in B : f(x) < \alpha\} &= B \cap \{x \in X : f(x) < \alpha\}\end{aligned}$$

are measurable sets. Hence f is measurable on A and on B. *[10 marks]*

Conversely suppose that f is measurable on A and on B. Since $X = A \cup B$,

$$\{x \in X : f(x) < \alpha\} = \{x \in A : f(x) < \alpha\} \cup \{x \in B : f(x) < \alpha\}.$$

As $\{x \in A : f(x) < \alpha\}$ and $\{x \in B : f(x) < \alpha\}$ are measurable, so are $\{x \in X : f(x) < \alpha\}$. *[10 marks]*

Question 2.

Since f and g are measurable, so is fg *[5 marks]*. Condition $|fg| \leq M|f|$ a.e. in E implies that

$$\int_E |fg| d\mu \leq \int_E M|f| d\mu = M \int_E |f| d\mu < \infty. \qquad [6 \text{ marks}]$$

Hence $|fg|$ is integrable on E and so is fg. *[5 marks]*

Question 3.

Since x^n is continuous on $[-1.1]$, it is Lebesgue measurable. Furthermore, f is Lebesgue measurable, so $x^n f(x)$ is Lebesgue measurable on $[-1, 1]$ *[3 marks]*. As f is Lebesgue integrable on $[-1, 1]$, it is finite a.e. on $[-1, 1]$, that is, the set $N = \{x \in [-1, 1] : |f(x)| = \infty\}$ has Lebesgue measure zero *[4 marks]*. Thus $m(N \cup \{-1, 1\}) = 0$. For each $x \in [-1, 1] \setminus (N \cup \{-1, 1\})$, $x^n f(x) \to 0$ and hence, $x^n f(x) \to 0$ a.e. on $[-1, 1]$. *[5 marks]* Moreover, $|x^n f(x)| \leq |f(x)|$ on $[-1, 1]$ and f is integrable *[2 marks]*. Therefore by the Dominated Convergence Theorem, $\int_{-1}^1 x^n f(x) dx \to \int_{-1}^1 0 dx = 0$ *[6 marks]*.

Question 4.

Let $P = \{a = x_0 < x_1 < \cdots < x_k = b\}$ be any partition of $[a, b]$. We have

$$\begin{aligned}V(f, P) = \sum_{k=1}^n |f(x_k) - f(x_{k-1})| &= \sum_{k=1}^n \int_{x_{i-1}}^{x_i} f'(t) dt \qquad [10 \text{ marks}] \\ \leq \sum_{k=1}^n \int_{x_{i-1}}^{x_i} |f'(t)| dt &= \int_a^b |f'(t)| dt. \qquad [8 \text{ marks}]\end{aligned}$$

It follows that $V_a^b(f) = \sup_P V(f, P) \leq \int_a^b |f'(t)| dt$ *[2 marks]*.

Question 5.
(a) *[10 marks]* We have $B = A \cup (B \setminus A)$ for $A \subset B$. Since $B \setminus A$ is a measurable subset of B and since B is a negative set for μ, $\mu(B \setminus A) \leq 0$. Thus

$$\mu(B) = \mu\big(A \cup (B \setminus A)\big) = \mu(A) + \mu(B \setminus A) \leq \mu(A).$$

(b) Since $\nu \ll \lambda$ and $\mu \ll \lambda$, by the Radon-Nikodym theorem, there are λ-integrable functions f, g such that

$$\mu(A) = \int_A f d\lambda, \quad \nu(A) = \int_A g d\lambda \quad \text{for every measurable set } A. \text{ [}\textit{7 marks}\text{]}$$

This implies that for every measurable set A,

$$(\mu + \nu)(A) = \mu(A) + \nu(A) = \int_A f d\lambda + \int_A g d\lambda = \int_A (f + g) d\lambda. \quad [\textit{4 marks}]$$

Thus, $\mu + \nu \ll \lambda$ *[2 marks]* and

$$\frac{d}{d\lambda}(\nu + \mu) = \frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda}. \qquad [\textit{2 marks}]$$