Random Walk

Outline

- Textbook: Chapter 5 Shreve I and Section 3.2 Shreve II
- Content
  - Basic property of symmetric random walk: martingale, first passage time distribution and reflection principle, Quadratic variation
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  - Limiting scaled symmetric random walk to a continuous time random process: Brownian motion
  - Limiting binomial asset pricing to a geometric Brownian motion

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Symmetric Random Walk

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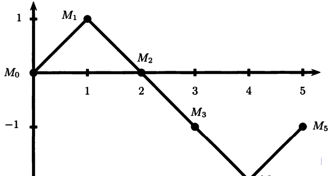
- **Symmetric random walk** $(M_k)_{k \geq 0}$: $M_0 = 0$ and

$$M_k = \sum_{n=0}^{k} X_n = X_1 + \cdots + X_k$$



Martingale Property

$\mathcal{F}_k$: σ - algebra of information corresponding to the first k coin tosses then

$$E(M_{n+1}|\mathcal{F}_n) = M_n$$

for all n.

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Proof.

$$\begin{aligned} E(M_{n+1}|\mathcal{F}_n) &= E(\underbrace{(M_{n+1} - M_n)}_{X_{n+1} \text{ independent of } \mathcal{F}_n} + \underbrace{M_n}_{\mathcal{F}_n \text{ measurable}} |\mathcal{F}_n) \\ &= E(X_{n+1}|\mathcal{F}_n) + E(M_n|\mathcal{F}_n) \\ &= E(X_{n+1}) + M_k \\ &= 0 + M_k = M_k \end{aligned}$$

$\square$

First passage time

The first time the random walk reaches level m

$$\tau_m = \inf\{n : M_n = m\}$$

If the random walk never reaches m then denote $\tau_m = \infty$

Example

- $P(\tau_1 = 2) = 0$
- $P(\tau_1 = 4) = 0$
- $P(\tau_1 = 2j) = 0$
- $P(\tau_1 = 1) = P(H) = \frac{1}{2}$
- $P(\tau_1 = 3) = P(THH)$
- $P(\tau_1 = 5) = P(THTHH) + P(TTHHH) = 2\left(\frac{1}{2}\right)^5$
- $P(\tau_1 = 2j - 1) =$
  "Number of path that first reaches 1 on the $(2j-1)$ step" $*$ $\left(\frac{1}{2}\right)^{2j-1}$

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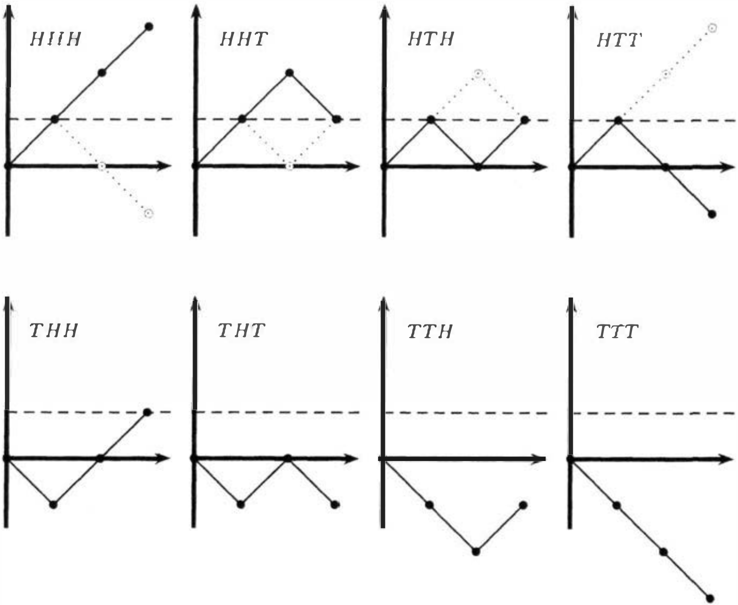
Find pmf of first passage time

Reflection Principle

- toss a coin an odd number $(2j-1)$ of times
- Some of the paths of the random walk will reach the level 1 in the first $2j-1$ steps and others will not
- Consider a path that reaches 1 at some time $\tau_1 \leq 2j-1$
- From that moment, we can create a "reflected" path which steps up each time the original path steps down and steps down each time the original path steps up.
- If the original path ends above 1 at the final time $2j-1$, the reflected path ends below 1, and vice versa. If the original path ends at 1, the reflected path does also.


HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

- Number of path that reaches 1 by time $2j-1$
  - number of paths that are at 1 at $2j-1$: $M_{2j-1}=1$
  - number of paths that exceeds 1 at $2j-1$: $M_{2j-1}>1$
  - number of paths that below 1 at $2j-1$: $M_{2j-1}<1$

  By total law probability

$$
\begin{aligned}
P(\tau_1 \leq 2j-1) = & P(\tau_1 \leq 2j-1, M_{2j-1}=1) \\
& + P(\tau_1 \leq 2j-1, M_{2j-1}>1) \\
& + P(\tau_1 \leq 2j-1, M_{2j-1}<1)
\end{aligned}
$$

- The reflected paths that exceed 1 correspond to path that are below 1

$$P(\tau_1 \leq 2j-1, M_{2j-1}>1) = P(\tau_1 \leq 2j-1, M_{2j-1}<1)$$

- Random walk surely reaches 1 before $2j-1$ if $M_{2j-1}>1$ (because it starts at 0)

$$P(\tau_1 \leq 2j-1, M_{2j-1}>1) = P(M_{2j-1}>1)$$

- If $M_{2j-1} = 1$ then $\tau_1 \leq 2j-1$. Hence

$$P(\tau_1 \leq 2j-1, M_{2j-1} = 1) = P(M_{2j-1} = 1)$$

- It implies that

$$P(\tau_1 \leq 2j-1) = P(M_{2j-1} = 1) + 2P(M_{2j-1} > 1)$$

- By symmetric

$$P(M_{2j-1} > 1) = P(M_{2j-1} < -1)$$

-

$$\begin{aligned}
P(\tau_1 \leq 2j-1) &= P(M_{2j-1} = 1) + P(M_{2j-1} > 1) \\
&\quad + P(M_{2j-1} < -1) \\
&= 1 - P(M_{2j-1} = -1)(M_{2j-1} \text{ only takes odd values} \\
&= 1 - P(M_{2j-1} = 1)(\text{by symmetric})
\end{aligned}$$

- $M_{2j-1} = 1$: there must be j Heads and $j-1$ Tails in the first $2j-1$ tosses
- Number of path with $M_{2j-1} = 1$

$$\binom{2j-1}{j}$$

- $$P(M_{2j-1} = 1) = \binom{2j-1}{j}\left(\frac{1}{2}\right)^{2j-1}$$
- $$P(\tau_1 \leq 2j-1) = 1 - \binom{2j-1}{j}\left(\frac{1}{2}\right)^{2j-1}$$

Probability mass function of τ_1

$$
\begin{aligned}
P(\tau_1 = 2j-1) &= P(\tau_1 \leq 2j-1) - P(\tau_1 \leq 2j-3) \\
&= \binom{2j-1}{j}\left(\frac{1}{2}\right)^{2j-1} - \binom{2j-3}{j}\left(\frac{1}{2}\right)^{2j-3} \\
&= \left(\frac{1}{2}\right)^{2j-1} \frac{(2j-2)!}{j!(j-1)!}
\end{aligned}
$$

Increments

$$M_n - M_m = X_{m+1} + X_{m+2} + \cdots + X_n = \sum_{j=m+1}^{n} X_j$$

the change in position if the random walk between times m and n

Properties of Increments

1. X_j are independent and identically distributed with $EX_j = 0$, $Var(X_j) = 1$. Hence

$$E(M_n - E(M_m)) = 0$$

and

$$Var(M_n - E(M_m)) = n - m$$

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2. **Independent increments:** if $0 = k_0 < k_1, \cdots < k_m$ then the RVs

$$M_{k_1} - M_{k_0}, M_{k_2} - M_{k_1}, \ldots, M_{k_m} - M_{k_{m-1}}$$

are independent.
increments over nonoverlapping time intervals are independent because they depend on different coin tosses

Quadratic Variation

Quadratic Variation up to time k is

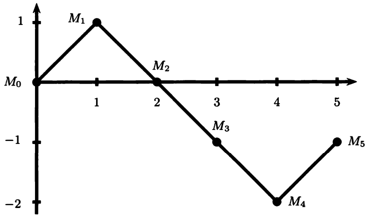
$$\langle M, M \rangle_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2$$

quadratic variance is computed path - by - path by taking all the one - step increment $M_j M_{j-1}$ along that path, squaring these increment and summing them

Example

Along this path

- Step 1: $M_1 - M_0 = 1$
- Step 2: $M_2 - M_1 = -1$
- Step 3: $M_3 - M_2 = -1$
- Step 4: $M_4 - M_3 = -1$
- Step 5: $M_5 - M_4 = 1$



Find $\langle M, M\rangle_k$ for $k = 1, 2, 3, 4, 5$ along this path.

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- Step 2: $M_2 - M_1 = -1$
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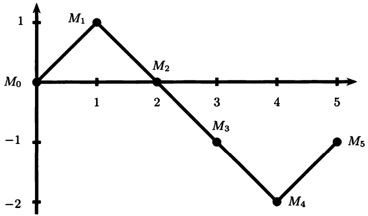
- Step 5: $M_5 - M_4 = 1$



Find $\langle M, M \rangle_k$ for $k = 1, 2, 3, 4, 5$ along this path.
Generate another path of random walk up to time period 5. And can calculate the about quadratic variations along the new path.

Property Quadratic Variation of Symmetric Random Walk

$$\langle M, M \rangle_k = k$$

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$$\langle M, M \rangle_k = k$$

Proof.

Hint

$$(M_j - M_{j-1})^2 = X_j^2 = 1$$

for all j $\square$

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Scaled Symmetric Random Walk

Speed up time and scale down the size of a symmetric random walk

$$W_t^{(n)} = \frac{1}{\sqrt{n}} M_{nt}$$

provided for nt is an integer.

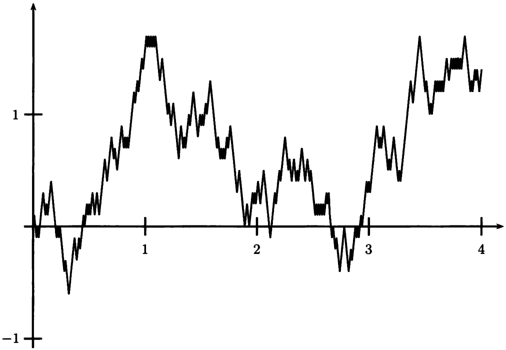

Fig. 3.2.2. A sample path of $W^{(100)}$.

Think about Scaled Symmetric Random Walk

- To see a single sample path of a scaled symmetric random walk

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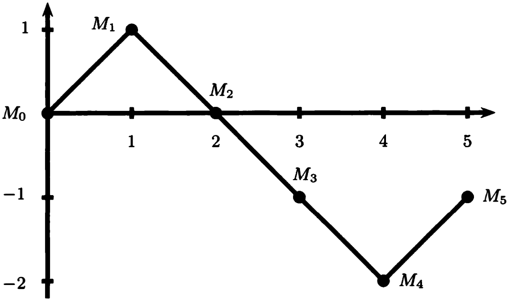
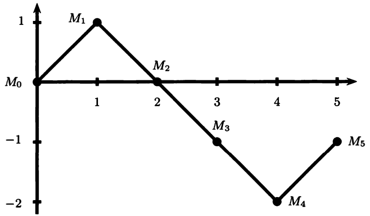
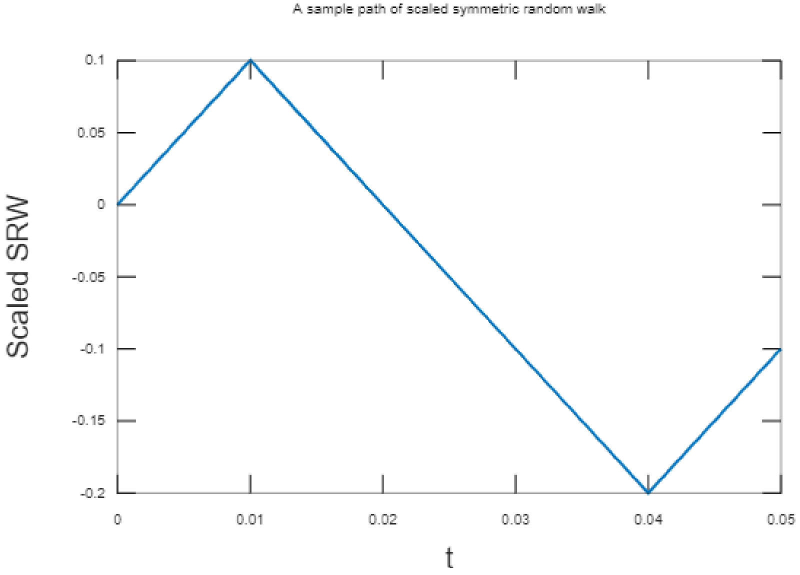


?? Graph of the sample path of the corresponding scaled random walk $W^{(100)}$



Select t such that $100t$ is an integer, e.g, 0, $\frac{1}{100}$, $\frac{2}{100}$, $\frac{3}{100}$

- $t = 0$: $W_0^{(100)} = \frac{1}{\sqrt{100}} M_0 = 0$
- $t = 1$: $W_{\frac{1}{100}}^{(100)} = \frac{1}{\sqrt{100}} M_1 = \frac{1}{10}$
- $t = 2$: $W_{\frac{2}{100}}^{(100)} = \frac{1}{\sqrt{100}} M_2 = 0$
- $t = 3$: $W_{\frac{3}{100}}^{(100)} = \frac{1}{\sqrt{100}} M_3 = -\frac{1}{10}$
- $t = 4$: $W_{\frac{4}{100}}^{(100)} = \frac{1}{\sqrt{100}} M_4 = -\frac{2}{10}$
- $t = 5$: $W_{\frac{5}{100}}^{(100)} = \frac{1}{\sqrt{100}} M_5 = -\frac{1}{10}$


A sample path of scaled symmetric random walk
Scaled SRW
t
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
0
0.01
0.02
0.03
0.04
0.05

Properties of Scaled Symmetric Random Walk

1. **Independent increment**: for all $0 = t_0 < t_1 < \cdots < t_m$ then

$$W_{t_1}^{(n)} - W_{t_0}^{(n)}, W_{t_2}^{(n)} - W_{t_1}^{(n)}, \ldots, W_{t_m}^{(n)} - W_{t_{m-1}}^{(n)}$$

   are independent

2. For $t > s$

$$E(W_t^{(n)} - W_s^{(n)}) = 0$$

$$Var(W_t^{(n)} - W_s^{(n)}) = t - s$$

   because $(W_t^{(n)} - W_s^{(n)}$ is the sum of $n(t-s)$ independent RVs with means 0 and variance $\frac{1}{n}$

3. **Martingale**

$$E(M_t^{(n)} | \mathcal{F}_s) = M_s^{(n)}, \forall t > s$$

   where $\mathcal{F}_s$ is σ - algebra of information available at time s (knownlegde of the first ns coin tosses)

Quadratic Variance of Scaled Symmetric Random Walk

Definition

$$\left\langle W^{(n)}, W^{(n)} \right\rangle (t) = \sum_{j=1}^{nt} \left(W^{(n)}_{\frac{j}{n}} - W^{(n)}_{\frac{j-1}{n}} \right)^2$$

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Example

$$\begin{aligned}
\left\langle W^{(100)}, W^{(100)} \right\rangle (1.37) &= \sum_{j=1}^{137} \left(W^{(100)}_{\frac{j}{100}} - W^{(100)}_{\frac{j-1}{100}} \right)^2 \\
&= \sum_{j=1}^{137} \left(\frac{1}{10} X_j \right)^2 = \sum_{j=1}^{137} \frac{1}{100} = 1.37
\end{aligned}$$

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\end{aligned}$$

This is computed path - by - path: go from time 0 to time t along the path of the scaled random walk, evaluate the increment over each time step and square these increments before summing them

Quadratic Variance of Scaled Symmetric Random Walk

$$\langle W^{(n)}, W^{(n)} \rangle (t) = t$$

Quadratic Variance of Scaled Symmetric Random Walk

$$\langle W^{(n)}, W^{(n)} \rangle (t) = t$$

Proof.

$$\begin{aligned}
\langle W^n, W^n \rangle (t) &= \sum_{j=1}^{nt} \left(W^{(n)}_{\frac{j}{n}} - W^{(n)}_{\frac{j-1}{n}} \right)^2 \\
&= \sum_{j=x}^{nt} \left(\frac{1}{\sqrt{n}} X_j \right)^2 \\
&= \sum_{j=1}^{nt} \frac{1}{n} = t
\end{aligned}$$

$\square$

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Another way to see scaled symmetric random walk at a given time

- Fix time t
- Consider all possible paths evaluated at t
- In other words, fix t and think about scaled random walk corresponding to different sequence of coin tosses, w.

Example

- Set $t = .25$ and consider set of all possible values of $W^{(100)}_{.25} = \frac{1}{10} M_{25}$

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$$-2.5 \ -2.3, \ldots, , -.1, \ -.1, 2.3, , 2.5$$

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Must get 13 heads and 12 tails,

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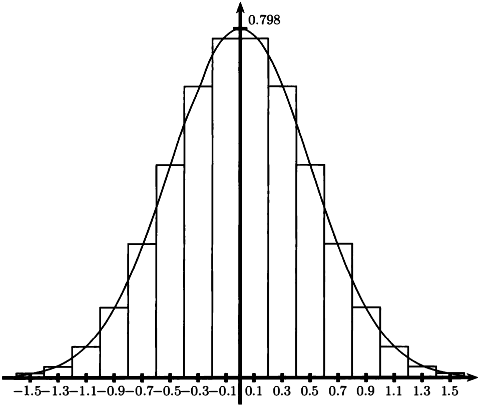
- $P(W_{.25}^{(100)} = .1) = ?$
  Must get 13 heads and 12 tails,

$$P(W_{.25}^{(100)} = .1) = \binom{13}{25} \left(\frac{1}{2}\right)^{13} \left(\frac{1}{2}\right)^{12} = .1555$$

Distribution of $W_{.25}^{(100)}$



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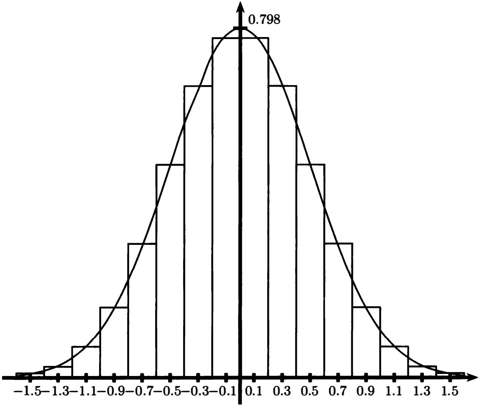
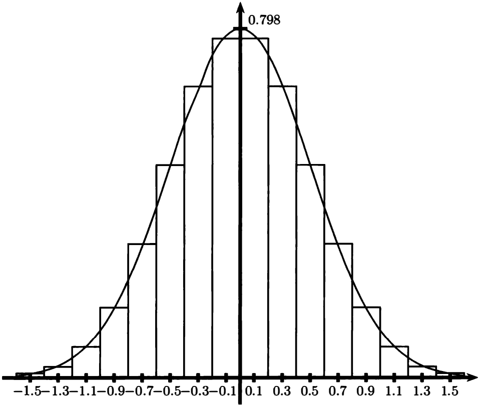


Normal??

Distribution of $W_{.25}^{(100)}$



Normal?? Central limit theorem

Limiting distribution fo $W_t^{(n)}$ as $n \to \infty$

- $W_t^{(n)} = \sum_{j=1}^{nt} \frac{1}{\sqrt{n}} M_{jn}$
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- By central limit theorem

$$W_t^{(n)} \stackrel{distribution}{\longrightarrow} N(0, t)$$

as $n \to \infty$

Binomial Asset Pricing vs Random Walk

- Model for stock price on the time interval $[0, t]$

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- Model for stock price on the time interval $[0, t]$
- Fix n and construct a binomial model that takes n steps per unit time
- up factor $u_n = 1 + \frac{\sigma}{\sqrt{n}}$, down factor $d_n = 1 - \frac{\sigma}{\sqrt{n}}$, $\sigma > 0$
- $p = q = \frac{1}{2}$
- Let H_{nt} and T_{nt} be number of Heads and Tails in the first nt coin tosses then

$$H_{nt} + T_{nT} = nt$$

and state of the random walk at nt is

$$M_{nt} = H_{nt} - T_{nt}$$

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- $\begin{cases} H_{nt} = \frac{1}{2}(nt + M_{nt}) \\ T_{nt} = \frac{1}{2}(nt - M_{nt}) \end{cases}$

Binomial Asset Pricing vs Random Walk

Stock price at time t

$$S_n(t) = S(0)u_n^{H_{nt}}d_n^{T_{nt}} = S(0)\left(1+\frac{\sigma}{n}\right)^{H_{nt}}\left(1-\frac{\sigma}{n}\right)^{T_{nt}}$$
$$= S(0)\left(1+\frac{\sigma}{n}\right)^{\frac{1}{2}(nt+M_{nt})}\left(1-\frac{\sigma}{n}\right)^{\frac{1}{2}(nt-M_{nt})}$$

Take logarithm both sides

$$\log S_n(t) = \log S(0) + \frac{1}{2}(nt+M_{nt})\log\left(1+\frac{\sigma}{\sqrt{n}}\right) + \frac{1}{2}(nt-M_{nt})\log\left(1-\frac{\sigma}{\sqrt{n}}\right)$$

Limit of Binomial Model

- n is large enough then $\frac{\sigma}{n} \approx 0$. Therefore by Taylor's expansion

$$\log\left(1+\frac{\sigma}{\sqrt{n}}\right) \approx \frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n} \text{ and } \log\left(1-\frac{\sigma}{\sqrt{n}}\right) \approx -\frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n}$$

-

$$\log S_n(t) \approx \log S(0) - \frac{1}{2}\sigma^2 t + \sigma \frac{1}{\sqrt{n}} M_{nt} = \log S(0) - \frac{1}{2}\sigma^2 t + \sigma W_t^{(n)}$$

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- $$\log S_n(t) \approx \log S(0) - \frac{1}{2}\sigma^2 t + \sigma \frac{1}{\sqrt{n}} M_{nt} = \log S(0) - \frac{1}{2}\sigma^2 t + \sigma W_t^{(n)}$$

- As $n \to \infty$, $W_t^{(n)} \xrightarrow{D} \mathcal{N}(0, t)$ implies that the distribution of $\log S_n(t)$ converges to a normal distribution or $S_n(t)$ converges to the distribution of

$$S(t) = S(0) e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)}$$

where $W(t) \hookrightarrow \mathcal{N}(0, t)$.
We say that $S(t)$ has log-normal distribution.