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Probability, Homework 11.

A/ Problems:

I/ Covariance:

1/ Marginal pmf of X: $p_X(x) = \sum_y p_{X,Y}(x,y) \Rightarrow p_X(1) = 0.1, p_X(2) = 0.35, p_X(3) = 0.55$

Marginal pmf of Y: $p_Y(y) = \sum_{x=1}^{3} p_{X,Y}(x,y) \Rightarrow p_Y(1) = 0.2, \ p_Y(3) = 0.5, \ p_Y(5) = 0.3$

$E(XY) = \sum_{x,y} xy\, p_{X,Y}(x,y) = 7.85, \ E(X) = \sum_x x p_X(x) = 2.45, \ E(Y) = \sum_y y p_Y(y) = 3.2$

$E(X^2) = \sum_x x^2 p_X(x) = 6.45, \ E(Y^2) = \sum_y y^2 p_Y(y) = 12.2$

$\Rightarrow Var(X) = E(X^2) - E(X)^2 = 0.4475, \ Var(Y) = E(Y^2) - E(Y)^2 = 1.96$

$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = 0.01, \ \rho_{XY} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = 0.0107$

2/ Marginal density of X:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_0^x f(x,y)\,dy + \int_{[0,x]^c} f(x,y)\,dy = \int_0^x 8xy\,dy + \int_{[0,x]^c} 0\,dy = 4x^3, \ \forall x \in [0,1]$$

Marginal density of Y:

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \int_y^1 f(x,y)\,dx + \int_{[y,1]^c} f(x,y)\,dx = \int_y^1 8xy\,dx + \int_{[y,1]^c} 0\,dx = 4y - 4y^3, \ \forall y \in [0,1]$$

$$E(XY) = \iint_{\mathbb{R}^2} xy f(x,y)\,dx\,dy = \iint_{\{0 \le y \le x \le 1\}} xy f(x,y)\,dx\,dy + \iint_{\{0 \le y \le x \le 1\}^c} 0\,dx\,dy = \int_0^1 \int_0^x 8x^2y^2\,dy\,dx = \frac{4}{9}.$$

$$E(X) = \int_{\mathbb{R}} x f_X(x)\,dx = \int_0^1 x f_X(x)\,dx + \int_{[0,1]^c} 0\,dx = \int_0^1 4x^4\,dx = \frac{4}{5}.$$

$$E(Y) = \int_{\mathbb{R}} y f_Y(y)\,dy = \int_0^1 y f_Y(y)\,dy + \int_{[0,1]^c} 0\,dy = \int_0^1 4y^2 - 4y^4\,dy = \frac{8}{15}.$$

$$E(X^2) = \int_{\mathbb{R}} x^2 f_X(x)\,dx = \int_0^1 4x^5\,dx = \frac{2}{3}, \ E(Y^2) = \int_{\mathbb{R}} y^2 f_Y(y)\,dy = \int_0^1 4y^3 - 4y^5\,dy = \frac{1}{3}$$

$\Rightarrow Var(X) = E(X^2) - E(X)^2 = \frac{2}{75}$, $Var(Y) = E(Y^2) - E(Y)^2 = \frac{11}{225}$

$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{4}{225}$, $\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = 0.4924$.

3/ Marginal density of X:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)dy = \int_x^1 f(x,y)dy + \int_{[x,1]^c} f(x,y)dy = \int_x^1 2\,dy + \int_{[x,1]^c} 0\,dy = 2-2x, \ \forall x \in [0,1]$$

Marginal density of Y.

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)dx = \int_0^y f(x,y)dx + \int_{[0,y]^c} f(x,y)dx = \int_0^y 2dx + \int_{[0,y]^c} 0dx = 2y \quad , \forall y \in [0,1].$$

$$E(XY) = \iint_{\mathbb{R}^2} xy f(x,y)\,dxdy = \iint_{\{0\le x\le y\le 1\}} xy f(x,y)dxdy + \iint_{\{0\le x\le y\le 1\}^c} 0\,dxdy = \int_0^1\int_x^1 2xy\,dy\,dx = \frac{1}{4}.$$

$$E(X) = \int_{\mathbb{R}} x f_X(x)dx = \int_0^1 x f_X(x)dx + \int_{[0,1]^c} 0dx = \int_0^1 2x - 2x^2 dx = \frac{1}{3}$$

$$E(Y) = \int_{\mathbb{R}} y f_Y(y)dy = \int_0^1 y f_Y(y)dy + \int_{[0,1]^c} 0dy = \int_0^1 2y^2 dy = \frac{2}{3}.$$

$$E(X^2) = \int_{\mathbb{R}} x^2 f_X(x)dx = \int_0^1 2x^2 - 2x^3 dx = \frac{1}{6}, \quad E(Y^2) = \int_{\mathbb{R}} y^2 f_Y(y)dy = \int_0^1 2y^3 dy = \frac{1}{2}.$$

$\Rightarrow Var(X) = E(X^2) - E(X)^2 = \frac{1}{18}$, $Var(Y) = E(Y^2) - E(Y)^2 = \frac{1}{18}$.

$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{36}$, $\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{1}{2}$.

4/ $Cov(aX, bY) = E(aX \cdot bY) - E(aX) - E(bY) = ab\,(E(XY) - E(X)E(Y)) = ab \cdot Cov(X,Y)$.

5/ $Cov(X-Y, X+Y) = Cov(X, X+Y) - Cov(Y, X+Y) = Cov(X,X) + Cov(X,Y) - Cov(Y,X) - Cov(Y,Y)$
$= Var(X) - Var(Y) = 0$, thus $X-Y$ and $X+Y$ are uncorrelated.

6/ $Cov(X,Y) = E(XY) - E(X)E(Y) = E(aX + bX^2 + cX^3) - E(X)\,E(a + bX + cX^2)$
$= a \cdot E(X) + b \cdot E(X^2) + c\,E(X^3) = b.$

II/ Sum of RVs:

1/ Let X be the random variable representing the result while rolling a fair dice.
Then $E(X) = \frac{7}{2}$ and by the additivity of expectation, $E(T_n) = n \cdot E(X) = \frac{7n}{2}$.

2/ For each $j = \overline{1,n}$, let X_j be the random variable attaining the values 1 if component j is working, and 0 otherwise. Then $X = \sum_{j=1}^{n} X_j$ and for each j,
$E(X_j) = 1 \cdot P(\{X_j = 1\}) = P_j \Rightarrow E(X) = \sum_{j=1}^{n} E(X_j) = \sum_{j=1}^{n} P_j$.

3/ $E\left[(X+Y)^2\right] = E(X^2 + 2XY + Y^2) = E(X^2) + 2E(XY) + E(Y^2) = 11$.

4/ $E\left[(X-Y)^2\right] = E(X^2 - 2XY + Y^2) = E(X^2) - 2E(X)E(Y) + E(Y^2) = p - 2pr + r$.

B/ Simulation: