# Chapter 2: Partial derivatives

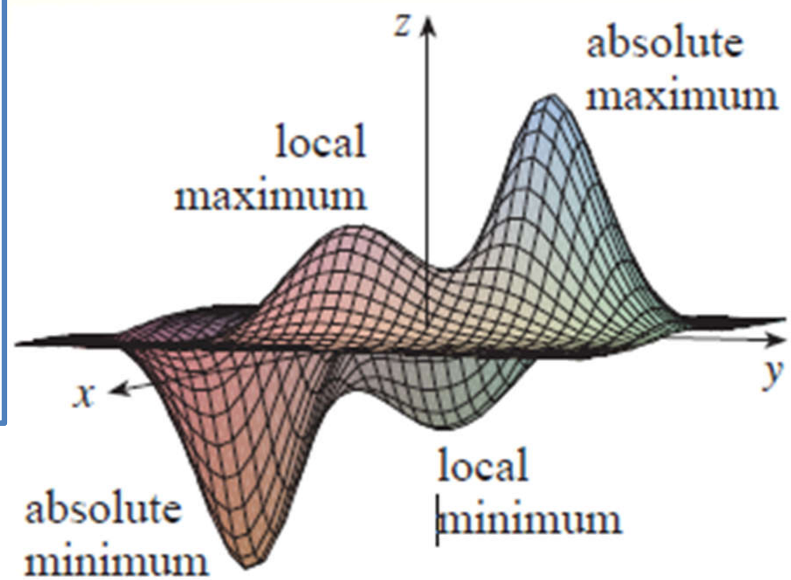
## **Lecture 6:** Maximum and Minimum Values

# Local maximum and minimum values

- **Definition 1a.** A function of two variables *f(x,y)* has a **local maximum value** at *(a,b)* if $f(x,y) \le f(a,b)$ when *(x,y)* near *(a,b)*.

$$\exists r > 0, \text{ such that } f(x,y) \le f(a,b)$$

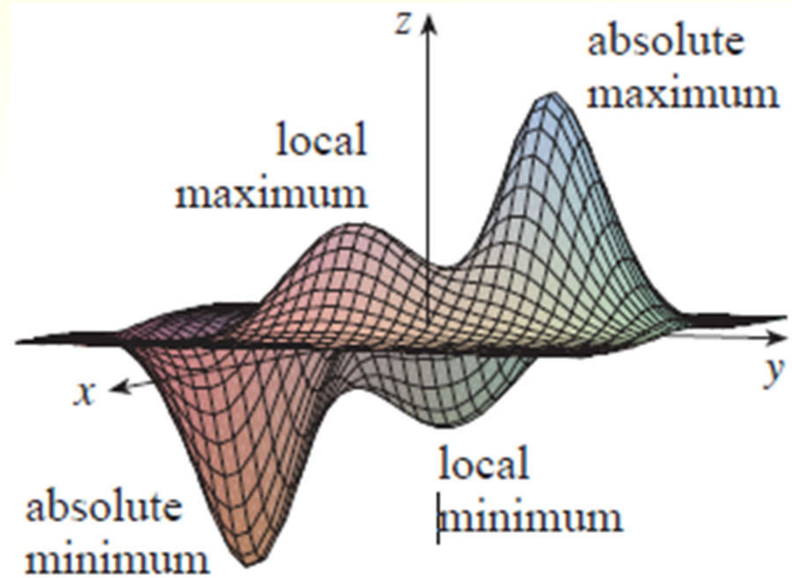
$$\text{if } d((x,y),(a,b)) = \sqrt{(x-a)^2 + (y-b)^2} \le r$$




- **Definition 1b.** A function of two variables *f(x,y)* has a **local minimum value** at *(a,b)* if $f(x,y) \ge f(a,b)$ when *(x,y)* near *(a,b)*.

# Absolute maximum and minimum values

- **Definition 2a**. A function of two variables $f(x,y)$ has an **absolute maximum value** at $(a,b)$ if $f(x,y) \leq f(a,b)$ *for all* $(x,y)$ in the domain of $f$




**Definition 2b**. A function of two variables $f(x,y)$ has an **absolute minimum value** at $(a,b)$ if $f(x,y) \geq f(a,b)$ *for all* $(x,y)$ in the domain of $f$

# Extreme Values

We use the word "**extremum value**" to indicate that *f* has either max or min value

**Theorem.** ***If f has a local extreme value at (a,b) and $f_x(a,b)$ and $f_y(a,b)$ exist, then $f_x(a,b)=0$ and $f_y(a,b)=0$.***

Proof. Let g(x)=f(x,b). If f has a local extreme value at (a,b), then g has a local extreme value at x=a, so by Fermat's Theorem g'(a)=0. But g'(a)= $f_x(a,b)$ . So $f_x(a,b)=0$

Similarly, by applying Fermat's Theorem to the function h(y)=f(a,y), we obtain $f_y(a,b)=0$

# Critical points

- A point $(a,b)$ is called a **critical point (*stationary point*) *of f if*** $f_x(a,b)=0$ *and* $f_y(a,b)=0$ , or if one of these partial derivatives does not exist (***singular point***).
- If $f$ has a local maximum or minimum at $(a,b)$, then $(a,b)$ is a critical point of $f$.
- However, as in single-variable calculus, not all critical points are maxima or minima.
- At a critical point, a function could have a local maximum or a local minimum or neither.

# Second Derivatives Test

**Theorem**: Suppose that all second partial derivatives of $f$ are Continuous on a disk centered at $(a,b)$

Let $f_x(a,b) = f_y(a,b) = 0 \quad (i.e.\ (a,b)\ is\ a\ critical\ point)$

$$D = f_{xx}(a,b) f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2 = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

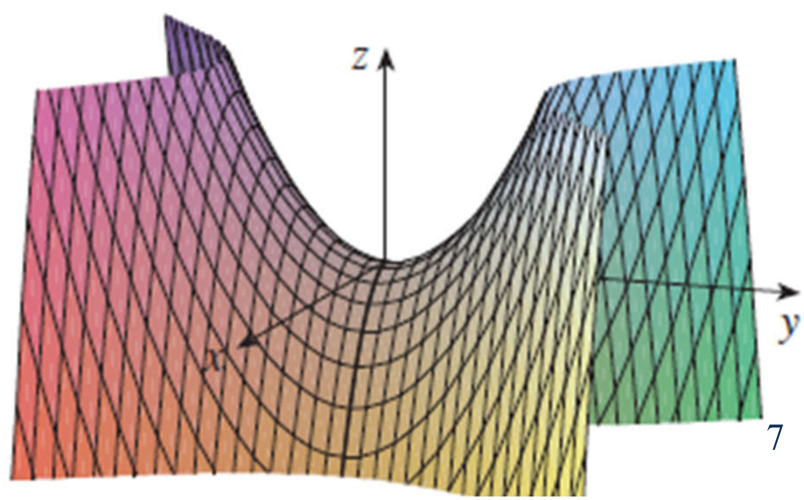
(a) If D>0 and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum value

(b) If D>0 and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum value

(c) If D<0, then $f(a,b)$ is not an extreme value

# Notes

- In case (c), where D<0, the point $(a,b)$ is called a **saddle point of** $f$ **and the graph of** $f$ crosses its tangent plane at (a,b).
- If D=0, the test gives no information: $f$ could have a local maximum or local minimum at $(a,b)$, or $(a,b)$ could be a saddle point of $f$.



# General Case

Suppose that $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a critical point of $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ and is interior to the domain of $f$. Also, suppose that all the second partial derivatives of $f$ are continuous throughout a neighbourhood of $\mathbf{a}$, so that the **Hessian matrix**

$$\mathcal{H}(\mathbf{x}) = \begin{pmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \cdots & f_{1n}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \cdots & f_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\mathbf{x}) & f_{n2}(\mathbf{x}) & \cdots & f_{nn}(\mathbf{x}) \end{pmatrix}$$

(where $f_i(x) = f_{x_i}(x)$

$f_{ij}(x) = f_{x_i x_j}(x)$)

is also continuous in that neighbourhood. Note that the continuity of the partials guarantees that $\mathcal{H}$ is a symmetric matrix.

(a) If $\mathcal{H}(\mathbf{a})$ is positive definite, then $f$ has a local minimum at $\mathbf{a}$. $(x^T H(\mathbf{a})x > 0, \forall x \neq 0)$

(b) If $\mathcal{H}(\mathbf{a})$ is negative definite, then $f$ has a local maximum at $\mathbf{a}$. $(x^T H(\mathbf{a})x < 0, \forall x \neq 0)$

(c) If $\mathcal{H}(\mathbf{a})$ is indefinite, then $f$ has a saddle point at $\mathbf{a}$.

(d) If $\mathcal{H}(\mathbf{a})$ is neither positive nor negative definite nor indefinite, this test gives no information.

***PROOF*** Let $g(t) = f(\mathbf{a} + t\mathbf{h})$ for $0 \le t \le 1$, where $\mathbf{h}$ is an $n$-vector. Then

$$g'(t) = \sum_{i=1}^{n} f_i(\mathbf{a} + t\mathbf{h})\, h_i$$

$$g''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(\mathbf{a} + t\mathbf{h})\, h_i\, h_j = \mathbf{h}^T \mathcal{H}(\mathbf{a} + t\mathbf{h})\mathbf{h}.$$

(In the latter expression, $\mathbf{h}$ is being treated as a column vector.) We apply Taylor's Formula with Lagrange remainder to $g$ to write

$$g(1) = g(0) + g'(0) + \frac{1}{2} g''(\theta)$$

for some $\theta$ between 0 and 1. Thus,

$$f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \sum_{i=1}^{n} f_i(\mathbf{a})\, h_i + \frac{1}{2}\mathbf{h}^T \mathcal{H}(\mathbf{a} + \theta\mathbf{h})\mathbf{h}.$$

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = \frac{1}{2}\mathbf{h}^T \mathcal{H}(\mathbf{a}+\theta\mathbf{h})\mathbf{h}.$$

If $\mathcal{H}(\mathbf{a})$ is positive definite, then, by the continuity of $\mathcal{H}$, so is $\mathcal{H}(\mathbf{a}+\theta\mathbf{h})$ for $|\mathbf{h}|$ sufficiently small. Therefore, $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) > 0$ for nonzero $\mathbf{h}$, proving (a).

Parts (b) and (c) are proved similarly. The functions $f(x, y) = x^4 + y^4$, $g(x, y) = -x^4 - y^4$, and $h(x, y) = x^4 - y^4$ all fall under part (d) and show that in this case a function can have a minimum, a maximum, or a saddle point.

$$x = \begin{pmatrix} h \\ k \end{pmatrix}$$

$$x^T H(\mathbf{a})x = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2)$$

$$= f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}D$$

# Example 1

- Find the local maximum and minimum values and saddle points of

$$f(x,y)=x^3y + 12x^2 -8y$$

-Critial points satisfy:

$$f_x(x,y)=3x^2y+24x=0, f_y(x,y)=x^3-8=0 \Leftrightarrow x=2$$

$$\Rightarrow f_x(x,y)=12y+48=0 \Rightarrow y=-4$$

Unique critical point: $(2,-4)$

$$f_{xx}(x,y)=6xy+24, f_{xy}(x,y)=3x^2, f_{yy}(x,y)=0$$

$$\Rightarrow D=-12^2=-144<0.$$

Thus, (2,-4) is a saddle point of $f$

$\boxed{f \text{ has no extreme value!}}$

**Example 2**: Find the local maximum and minimum values and saddle points of the function

$$f(x,y)=x^2 + y^2+ x^2y + 4$$

Solution

Critial points:

$$\left.\begin{array}{l} f_x(x,y) = 2x + 2xy = 0 \\ f_y(x,y) = 2y + x^2 = 0 \end{array}\right\} \Leftrightarrow x = 0, y = 0, \text{ or } x = \pm\sqrt{2}, y = -1$$

$$P(0,0), Q(\sqrt{2},-1), R(-\sqrt{2},-1)$$

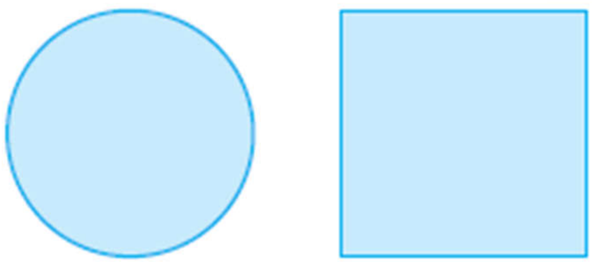
$$f_{xx}(x,y) = 2 + 2y, f_{xy}(x,y) = 2x, f_{yy}(x,y) = 2$$

$$D = 2(2+2y) - 4x^2$$

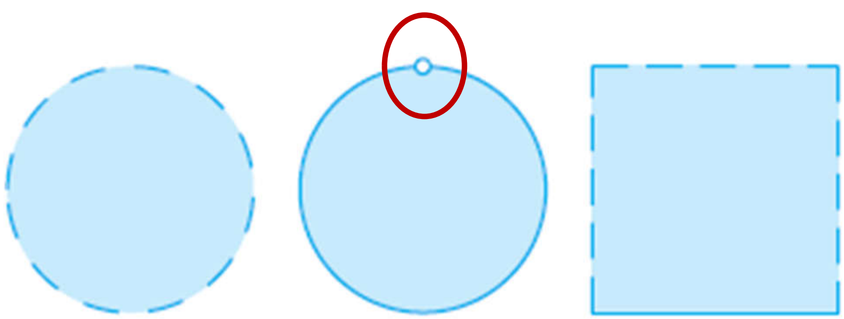
$At\ P: D = 4 > 0, f_{xx}(0,0) = 2 > 0. So, f(0,0) = 4\ is\ a\ local\ \min$

$At\ Q\ and\ R: D = -8 < 0.$ So, $Q$ and $R$ are saddle points

# Closed Sets and Bounded Sets



(a) Closed sets



(b) Sets that are not closed

A **boundary point P** of D is a point such that every disk with center P contains points in D and also points not in D.

A **closed set** in $R^2$ is one that contains all its boundary points.

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk. In other words, it is finite in extent.

Ex: If $f$ is continuous, then the following sets are closed

$$A=\{(x,y)\mid f(x,y)\le c\},\quad \partial A=\{(x,y)\mid f(x,y)=c\}$$

$$B=\{(x,y)\mid f(x,y)\ge c\},\quad \partial B=\{(x,y)\mid f(x,y)=c\}$$

$$C=\{(x,y)\mid d\le f(x,y)\le c\},$$

$$\partial C=\{(x,y)\mid f(x,y)=c\}\cup\{(x,y)\mid f(x,y)=d\}$$

**Theorem**: If $f$ is continuous on a closed, bounded set $D$ in $\mathbb{R}^2$, then $f$ attains an absolute maximum value $f(a,b)$ and an absolute minimum value $f(c,d)$ at some points $(a,b)$ and $(c,d)$ on $D$.

# Finding the absolute maximum and absolute minimum values

For a continuous function $f$ on a closed, bounded set D:

1. Find the values of $f$ at the critical points of $f$ inside D

2. Find the extreme values of $f$ on the boundary of D

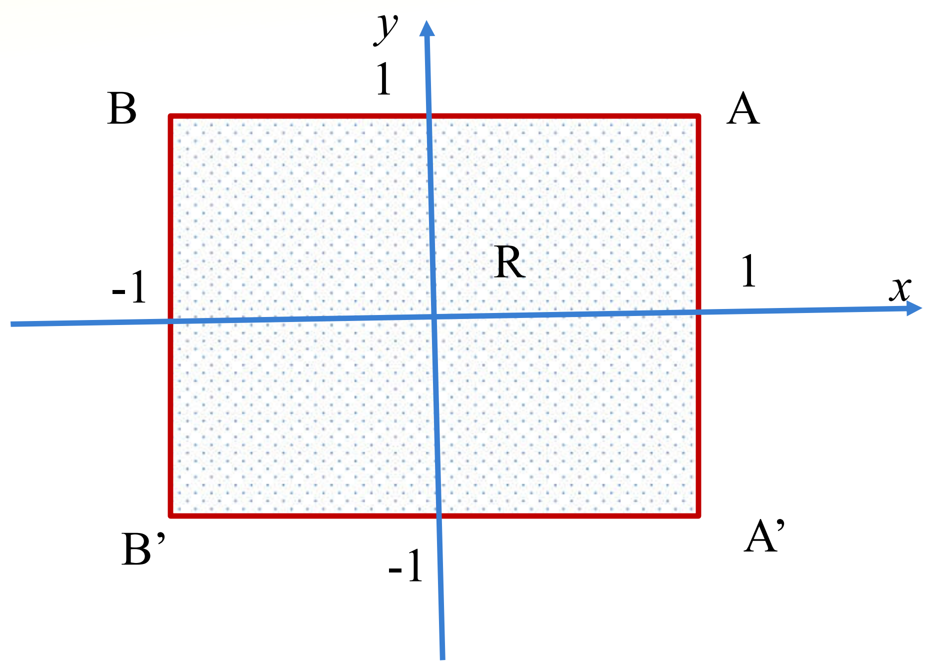
3. Compare the values from Steps 1 and 2: The largest is the absolute maximum; the smallest is the absolute minimum value.

# Example

- Find the absolute maximum and minimum values of

$$f(x,y)=x^2 + y^2 + x^2y + 4$$

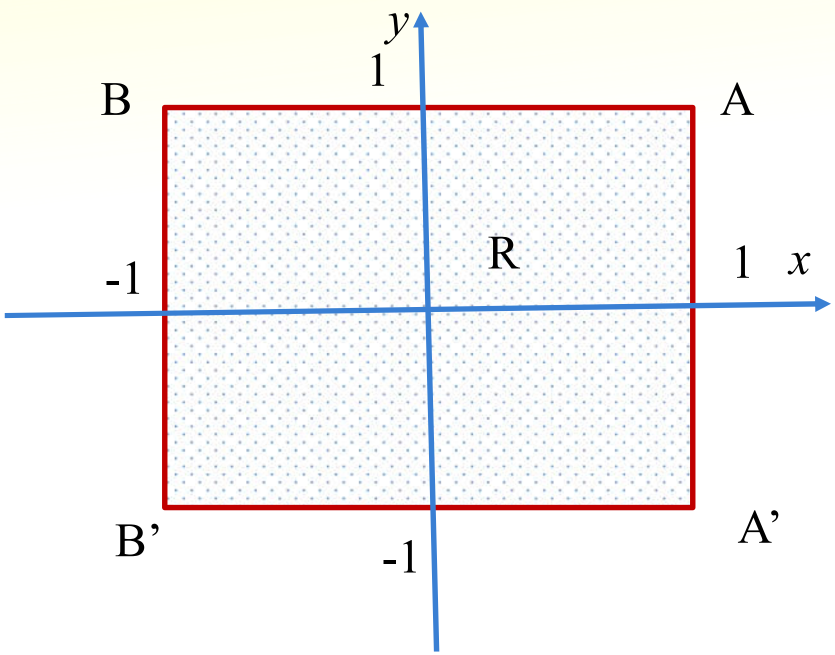
on the set $R=\{(x,y) \mid |x| \leq 1, |y| \leq 1\}$.



Step 1: Find values of $f$ at critical points

$$\left.\begin{aligned} f_x(x,y) &= 2x + 2xy = 0 \\ f_y(x,y) &= 2y + x^2 = 0 \end{aligned}\right\} \Leftrightarrow (x=0, y=0), \text{ or } (x = \pm\sqrt{2}, y = -1) \notin R$$

$\Rightarrow$ Only one critical point $(0,0) \in R$, and $f(0,0) = 4$

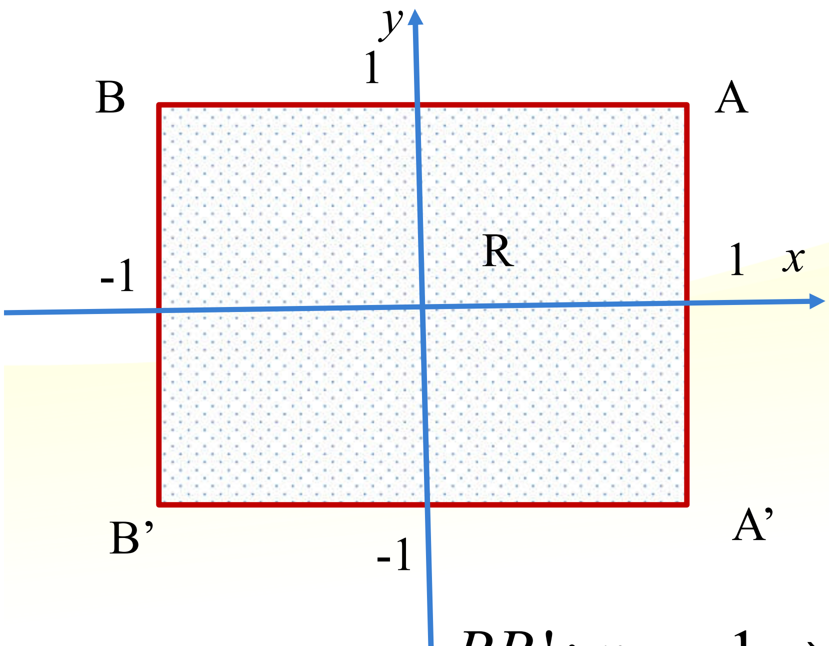



Step 2: Find max and min of $f$ on boundary of $R$

$$f(x,y) = x^2 + y^2 + x^2 y + 4$$

$$AA': x = 1 \Rightarrow f(1,y) = y^2 + y + 5, \quad -1 \le y \le 1$$

$$\min f_{AA'} = f(1, -1/2) = 19/4, \quad \max f_{AA'} = f(1,1) = 7$$




$$f(x,y) = x^2 + y^2 + x^2 y + 4$$

$$AB: y = 1 \Rightarrow f(x,1) = 2x^2 + 5, \quad -1 \le x \le 1$$

$$\min f_{AB} = f(0,1) = 5$$

$$\max f_{AB} = f(\pm 1,1) = 7$$

$$BB': x = -1 \Rightarrow f(-1,y) = y^2 + y + 5, \quad -1 \le y \le 1$$

$$\min f_{BB'} = f(-1,-1/2) = 19/4, \quad \max f_{BB'} = f(-1,1) = 7$$

$$A'B': y = -1 \Rightarrow f(x,-1) = 5, \quad -1 \le x \le 1$$

$$\min f_{A'B'} = \max f_{A'B'} = 5$$

$$Step3:$$

$$\min f = f(0,0) = 4$$

$$\max f = f(\pm 1,1) = 7$$