Conditional Probablity

Plan

1. Conditional probability
2. Independence
3. Multiplication formula

Conditional probability provides us with a way to reason about the outcome of an experiment, based on **partial information**.

Example

In an experiment involving two successive rolls of a fair die, you are told that the sum of the two rolls is 9. How likely is it that the first roll was a 6?

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In some applications, the practitioner is interested in the probability structure under certain restrictions. For instance, in epidemiology, rather than studying the chance that a person from the general population has diabetes, it might be of more interest to know this probability for a distinct group such as Asian women in the age range of 35 to 50 or Hispanic men in the age range of 40 to 60. This type of probability is called a conditional probability.

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seek to construct a new probability law, which takes into account this knowledge and which, for any event A, gives us the conditional probability of B given A, denoted by $P(B|A)$

Example

Toss 2 dice. Suppose the first dice is 3. What is the probability of the sum of two dice to be 8?

- Sample space: 36 outcomes (x, y)
- $A =$ {first dice is 3}, $B =$ {sum is 8}
- If A happens, there are only 6 possible outcomes $(3, y)$
- New sample space: $(3, y),\ y = 1, \ldots, 6$
- Probability of B in new space: **conditional probability** $= P(B|A) = P\{(3,5)|A\} = 1/6$

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Example

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

Select an individual randomly

- M: a man is chosen, E: the one chosen is employed
- Using the reduced sample space E,

$$P(M|E) = \frac{460}{600}$$

All outcomes are equally likely,

$$P(A|B) = \frac{\text{number of elements of } AB}{\text{number of elements of } A} = \frac{P(AB)}{P(A)}$$

In general

- If A happens, then in order for B happen, the sample must be in AB
- A becomes the new sample space

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if $P(A) > 0$

Measure the likelihood of B in the new sample space A

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Meaning

The notion of conditional probability provides the capability of reevaluating the idea of probability of an event in light of additional information, that is, when it is known that another event has occurred. The probability $P(A|B)$ is an updating of $P(A)$ based on the knowledge that event B has occurred.

Properties

1. Complement rule

$$P(B^c|A) = 1 - P(B|A)$$

2. Additive rule

$$P(B \cup C|A) = P(B|A) + P(C|A) - P(BC|A)$$

Example

Joint probabilities appear in the body of the table.

	Men (M)	Women (W)	Total
Promoted (A)	.24	.03	.27
Not Promoted (A^c)	.56	.17	.73
Total	.80	.20	1.00

Marginal probabilities appear in the margins of the table.

The probability that an officer is promoted given that the officer is a man is $P(A|M) = \frac{P(AM)}{P(M)} = \frac{.24}{.8} = .3$

In other words, given that an officer is a man, that officer had a 30% chance of receiving a promotion.

Example

The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(DA) = 0.78$. Find the probability that a plane

1. arrives on time, given that it departed on time
2. departed on time, given that it has arrived on time.

Solution

1. $P(A|D) = \frac{P(DA)}{P(D)} = \frac{.78}{.83} \approx .94$
2. $P(D|A) = \frac{P(DA)}{P(A)} = \frac{.78}{.82} \approx .95$

Example

A fair coin is flipped twice. what is the conditional probability that both flips land on heads, given that
(a) the first flip lands on heads?
(b) at least one flip lands on heads?

Solution for (a)

- $A = \{HH\}$ (both head)
- $F = \{HH, HT\}$ (first is head)

$$P(A|F) = \frac{P(AF)}{P(F)} = \frac{P(\{HH\})}{P(\{HH, HT\})}$$
$$= \frac{1/4}{2/4} = \frac{1}{2}$$

Solution for (b)

- $B = \{HH, HT, TH\}$ (at least one head)

$$\begin{aligned} P(A|B) &= \frac{P(AB)}{P(B)} \\ &= \frac{P(\{HH\})}{P(\{HH, HT, TH\})} = \frac{1}{3} \end{aligned}$$

Comment

$$P(A) = \frac{1}{4}$$

while

$$P(A|B) = \frac{1}{3} \neq P(A)$$

indicates that A depends on B

$$
\begin{aligned}
&P(\text{2nd coin is Head} \mid \text{1st coin is Head}) \\
&= P(\text{2nd coin is Head}) \\
&= \frac{1}{2}
\end{aligned}
$$

Result of the 2nd coin does not the depend of the 1st coin result

Idea

- Usually $P(A|B) \neq P(A)$.
- If $P(A|B) = P(A)$, B has no effect on A or knowing B does not change the probability that A happens then
- A and B have no relation

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Plan

Independent events

A and B are independent if

$$P(A|B) = P(A)$$

or

$$P(AB) = P(A)P(B).$$

Complement

If A is independent of B then it is independent of B^c.

Example

Two successive rolls of a fair 6-sided die

A: the 1st roll results in 2

B: the 2nd roll results in 4

Are A and B independent?

Solution

- $P(A) = \frac{1}{6}$
- $P(B) = \frac{1}{6}$
- $P(AB) = \frac{1}{36} = P(A)P(B)$
- A and B are independent

Example

A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

Solution

- $A:$ fire engine is available, $P(A) = .98$
- B: ambulance is available, $P(B) = .92$
- Both will be available: AB
- Because A and B are independent

$$P(AB) = P(A)P(B) = (.98)(.92)$$

Plan

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Multiplication Formula

$$P(AB) = P(B|A)P(A).$$

Think of AB as event with 2 steps, then probability equals probability of first step multiply with the conditional probability of second step given first step

Example

The probability that the first stage of a numerically controlled machining operation for high-rpm pistons meets specifications is 0.90. Given that the first stage meets specifications, the probability that a second stage of machining meets specifications is 0.95. What is the probability that both stages meet specifications?

Solution

- A: first stage meets specification, $P(A) = .9$
- B: second stage meets specification, $P(B|A) = .95$
- Both stages meet specification AB

$$P(AB) = P(A)P(B|A) = (.9)(.95)$$

Example

Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution

- A: the first fuse is defective
- B: the second fuse is defective
- Both are defective: AB
- The probability of first removing a defective fuse is $P(A) = 1/4$

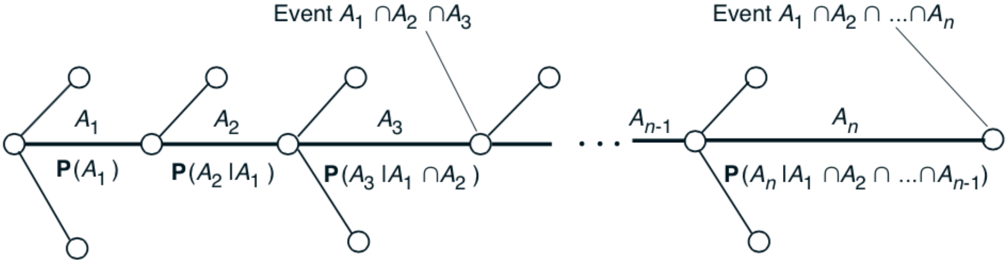
- After removing a defective fuse at the first removing, it remains 4 defectives among 19 fuses, then the probability of removing a second defective fuse is $P(B|A) = 4/19$
- $P(AB) = P(A)P(B|A) = \frac{1}{4} \times \frac{4}{19} = \frac{1}{19}$

Practice

Multistep

Sequence of events $A_1, A_2, \ldots, A_k$

$$P(A_1 A_2 \ldots A_k) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 A_2) \ldots \times P(A_k|A_1 \ldots A_{k-1})$$



Example

An urn contains 8 red and 4 blue balls. Draws 3 balls without replacement. What is probability that all 3 are red?

- E_i= ball number i is red, $i = 1, 2, 3$
- $P(E_1) = 8/12$
- $P(E_2|E_1) = 7/11$
- $P(E_3|E_1E_2) = 6/10$

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$$
\begin{aligned}
P(E_1E_2E_3) &= P(E_1)P(E_2|E_1)P(E_3|E_1E_2) \\
&= \frac{8}{12}\frac{7}{11}\frac{6}{10} = \frac{14}{55}
\end{aligned}
$$

Practice

A class consisting of 4 graduate and 12 undergraduate students is randomly divided into 4 groups of 4. What is the probability that each group includes a graduate student?

Multiplication law for independent events

If $A_1, \ldots, A_k$ are independent then

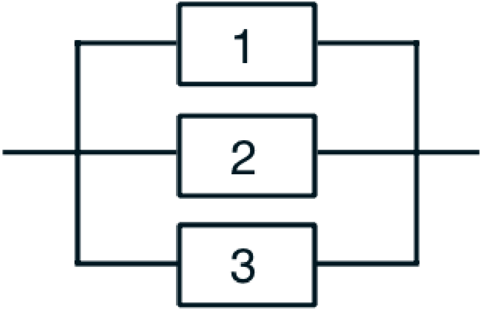
$$P(A_1 \ldots A_k) = P(A) \ldots P(A_k)$$

Example

- A series system is up if all of its component is up
- p_i: prob that component i is up
- Prob that series system is up: $p_1 p_2 \ldots p_n$

Example

A parallel system is up if any one of its component is up



- p_i: probability that component i is up
- Probability that the parallel system is down:

$$(1 - p_1)(1 - p_2)...(1 - p_n)$$

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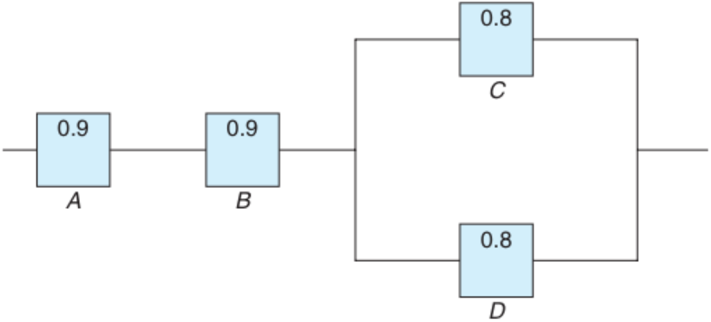
Example

An electronic system consists of 4 independent components. Find the probability that entire system works.



Solution

- Probability that the subsystem CD in parallel is up

$$p_{CD} = 1-(1-p_C)(1-p_D) = 1-(1-.8)(1-.8) = .96$$

- Probability that the whole system is up

$$p_A p_B p_{CD} = (.9)(.9)(.96) =$$

Summary

- Conditional prob $P(A|B) = \frac{P(AB)}{P(B)}$
- A and B are independent if

$$P(AB) = P(A)P(B) \text{ or } P(A)|B) = P(A)$$

- Multiplication rule $P(AB) = P(A)P(B|A)$
- Multiplication rule for **independent** events

$$P(AB) = P(A)P(B)$$