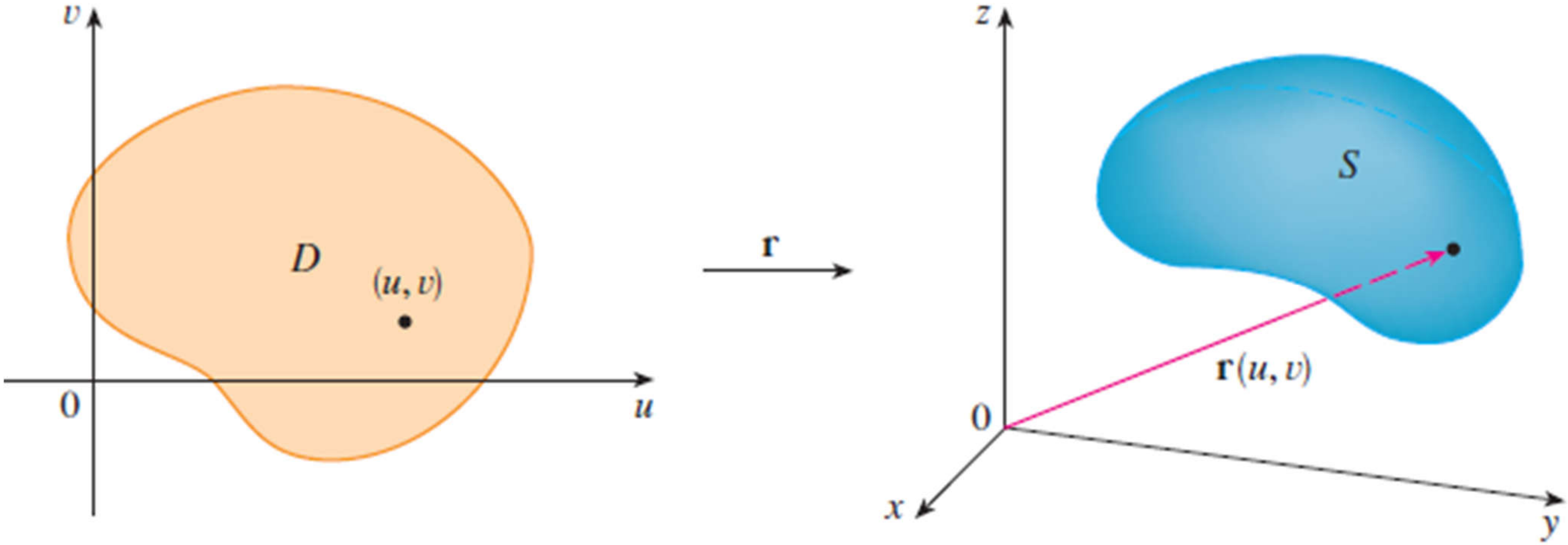
Chapter 4: Vector Calculus

Lecture 14

Surface Integrals and Applications

1. Parametric Surfaces



Vector-valued function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k},\ (u,v) \in D \subset \mathbb{R}^2$$

Set of all points $(x,y,z) \in \mathbb{R}^3$ such that

$$\boxed{x = x(u,v),\ y = y(u,v),\ z = z(u,v),\ (u,v) \in D \subset \mathbb{R}^2}$$

is called a **parametric surface** *S*

Parametric equations for S

Example

Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + v\,\mathbf{j} + 2\sin u\,\mathbf{k}$$

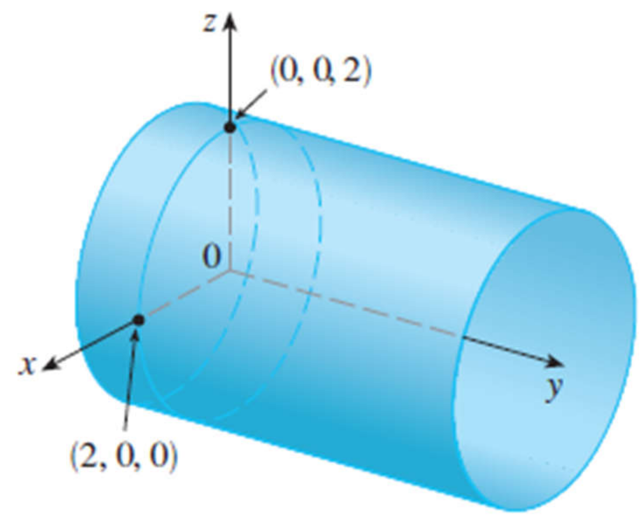
Solution: The parametric equations for this surface are

$$x = 2\cos u,\ y = v,\ z = 2\sin u$$

So, for any point (x, y, z) on this surface, we have

$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4$$

Vertical cross-sections parallel to the xz-plane
(*y constant)* are all circles with radius 2.
No restriction is placed on y=v
The surface is a circular cylinder with radius 2
whose axis is the *y-axis*



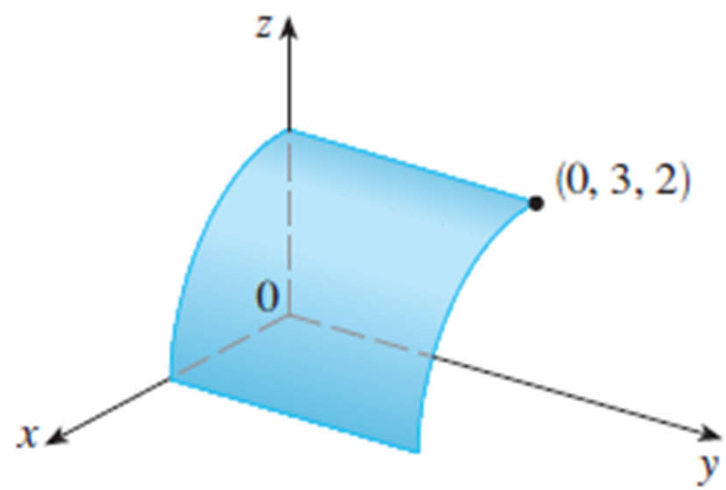
Note

- In last Example we placed no restrictions on the parameters *u* and *v* and so we obtained the entire cylinder. If, for instance, we restrict *u* and *v* by writing the parameter domain as

$$0 \le u \le \pi / 2, \quad 0 \le v \le 3$$

➡ $x \ge 0, z \ge 0, 0 \le y \le 3$

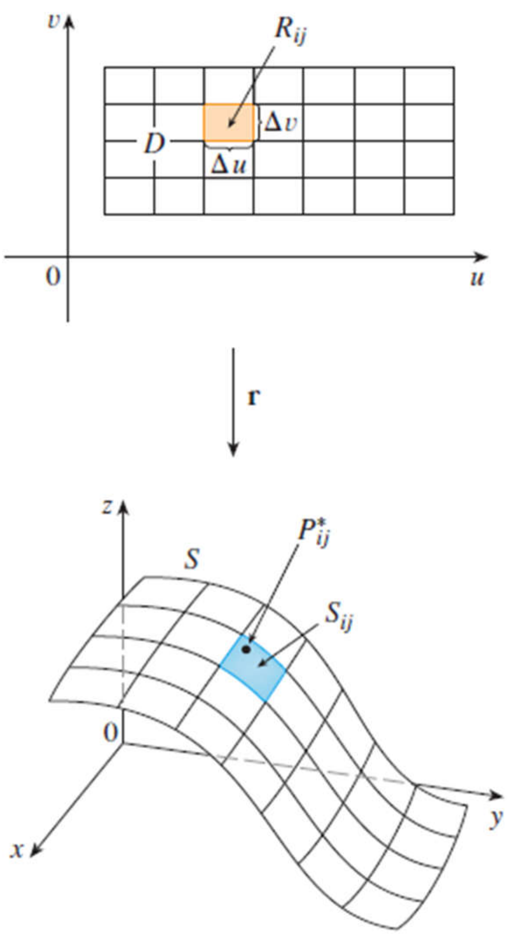
➡ We get a quarter-cylinder with length 3



2. Surface Integrals

f is a function defined on a surface S given by

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k},\ (u,v) \in D \subset \mathbb{R}^2$$



Divide domain D into subrectangles

$$R_{ij}, i = 1,2,\ldots,m; j = 1,2,\ldots,n$$

Then surface S is divided into corresponding patches

$$S_{ij}, i = 1,2,\ldots,m; j = 1,2,\ldots,n$$

$$P^*_{ij} \in S_{ij}, \forall i,j$$

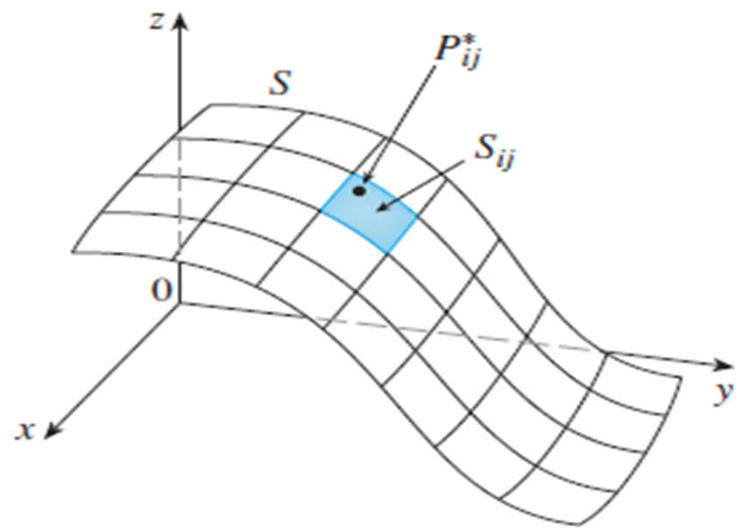
$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P^*_{ij})\Delta S_{ij}$$

Definition of Surface Integrals of a function

The surface integral of a function f over the surface S is defined by

$$\boxed{\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^{*})\Delta S_{ij}}$$

if this limit exists



Application:

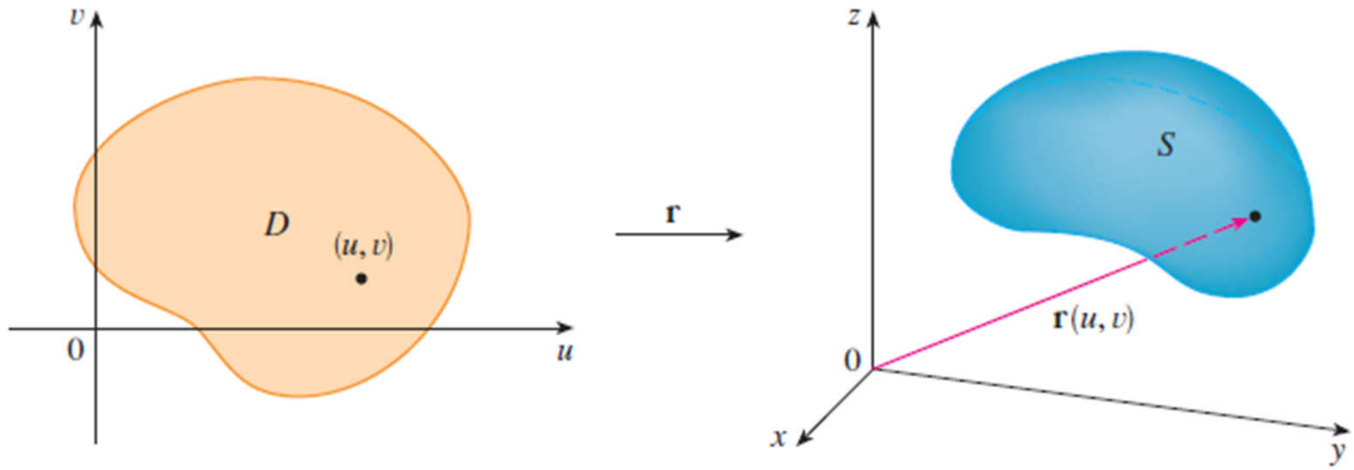
S has density $\rho(x,y,z)$

Mass of S: $\quad m = \iint_S \rho(x,y,z)\,dS$

Center of mass of S at point $(\bar{x},\bar{y},\bar{z})$:

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x,y,z)\,dS,\quad \bar{y} = \frac{1}{m}\iint_S y\rho(x,y,z)\,dS,\quad \bar{z} = \frac{1}{m}\iint_S z\rho(x,y,z)\,dS$$

Evaluating Surface Integrals



$$S: \quad \mathbf{r}(u,v) = < x(u,v), y(u,v), z(u,v) >, (u,v) \in D \subset \mathbb{R}^2$$

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$$

where $\mathbf{r}_u = < x_u, y_u, z_u >, \qquad \mathbf{r}_v = < x_v, y_v, z_v >$

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA}$$

Special Surfaces: Graph

If S is the graph of a function $z=g(x,y)$ with domain D, then

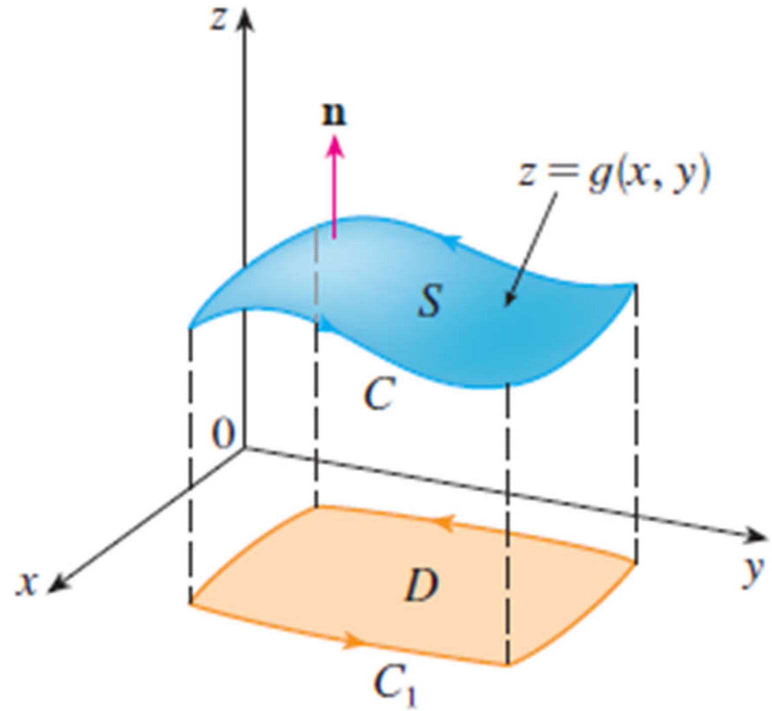
$$\mathbf{r}(x,y) = < x, y, g(x,y) >$$

$$\mathbf{r}_x(x,y) = < 1, 0, g_x(x,y) >$$

$$\mathbf{r}_y(x,y) = < 0, 1, g_y(x,y) >$$

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & g_x(x,y) \\ 0 & 1 & g_y(x,y) \end{vmatrix} = < -g_x(x,y), -g_y(x,y), 1 >$$

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{g_x(x,y)^2 + g_y(x,y)^2 + 1}$$



$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{(g_x(x,y))^2 + (g_y(x,y))^2 + 1}\,dA}$$

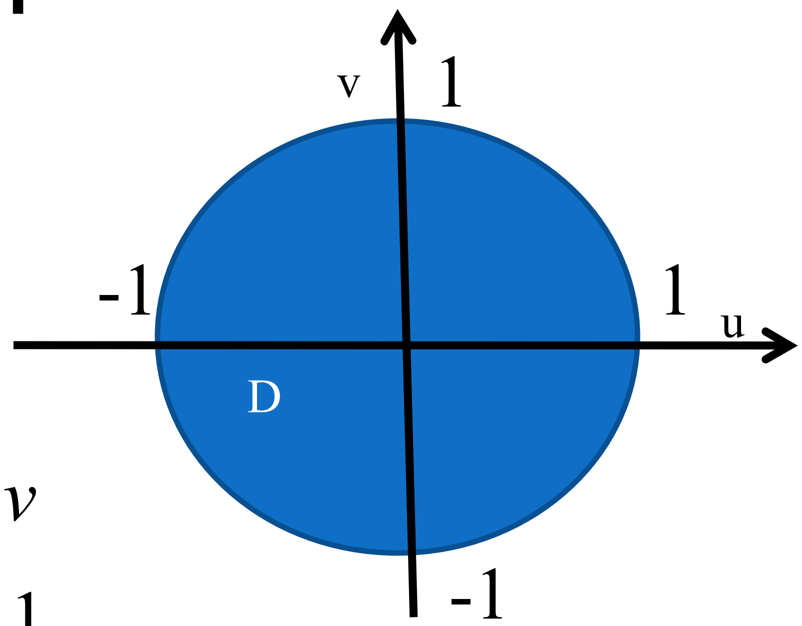
Example

Evaluate

$$\iint_S yz\,dS,$$

$$S: x = uv, \quad y = u + v, \quad z = u - v$$

where (u, v) satisfies $u^2 + v^2 \leq 1$



Solution

$$S:\mathbf{r}=<uv,u+v,u-v> \text{ defined on } D:u^2+v^2\le 1,$$

$$\mathbf{r}_u=<v,1,1>,\quad \mathbf{r}_v=<u,1,-1>$$

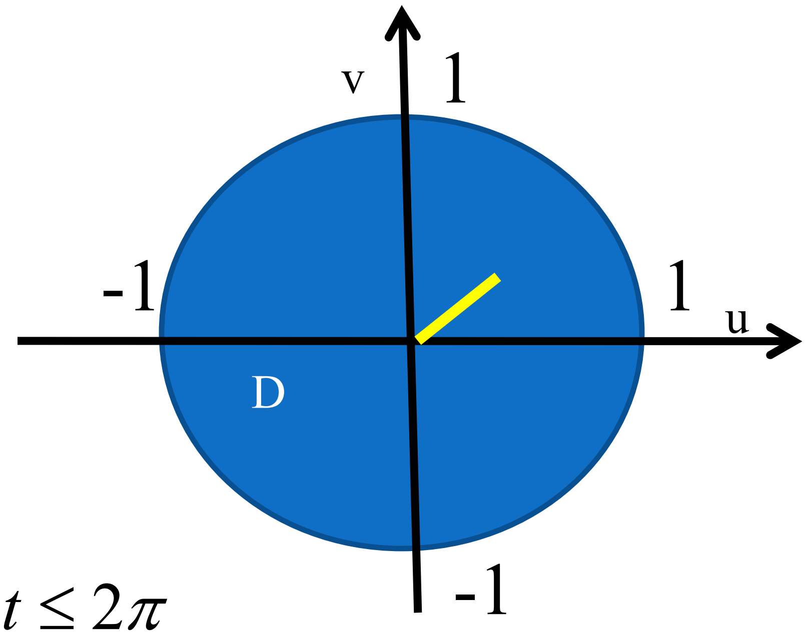
$$\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ v & 1 & 1\\ u & 1 & -1\end{vmatrix}=<-2,u+v,v-u>$$

$$|\mathbf{r}_u\times\mathbf{r}_v|=\sqrt{4+(u+v)^2+(v-u)^2}=\sqrt{2u^2+2v^2+4}$$

$$I=\iint_S yz\,dS=\iint_D yz\,|r_u\times r_v|\,dA=\iint_D (u+v)(u-v)\sqrt{2u^2+2v^2+4}\,dA$$

$$=\iint_D (u^2-v^2)\sqrt{2u^2+2v^2+4}\,dA$$

$$I = \iint_D (u^2 - v^2)\sqrt{2u^2 + 2v^2 + 4}\,dA$$



Change into polar coordinates:

$$D : u = r\cos t, v = r\sin t, 0 \le r \le 1, 0 \le t \le 2\pi$$

$$I = \int_0^1 \int_0^{2\pi} r^2(\cos^2 t - \sin^2 t)\sqrt{2r^2 + 4}\; r\; dt dr$$

$$= \int_0^1 \int_0^{2\pi} (\cos(2t))r^3\sqrt{2r^2 + 4}\;\; dt dr = \int_0^1 (\sin(2t)/2)r^3\sqrt{2r^2 + 4}\;\;)\Big|_{t=0}^{t=2\pi}\; dr$$

$$= \int_0^1 0 dr = 0$$

Oriented Surfaces: An example of one-sided strip

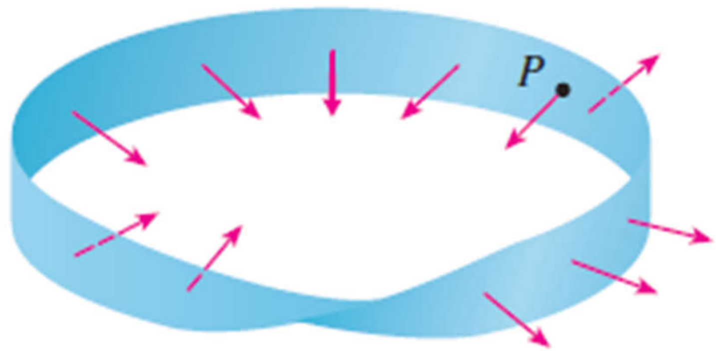
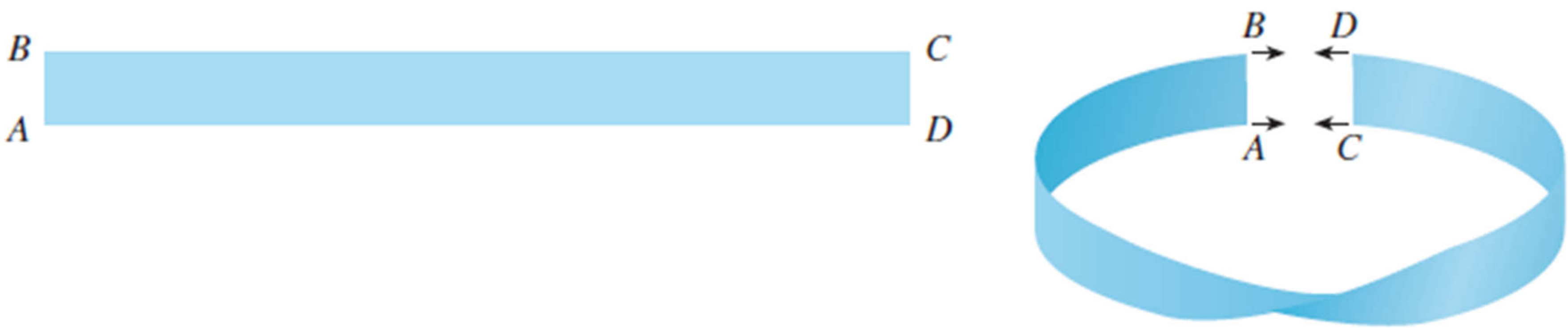


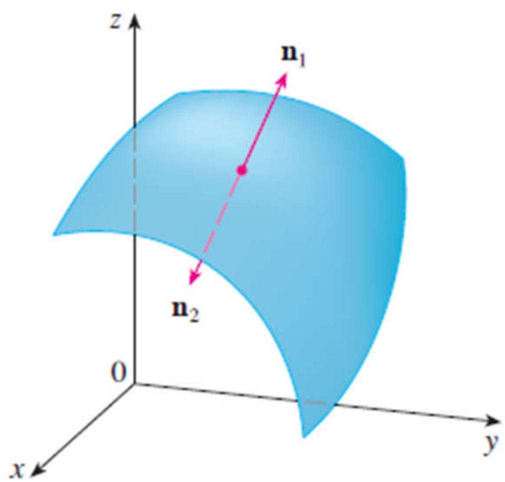
FIGURE 4
A Möbius strip

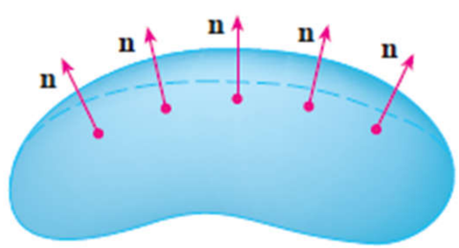


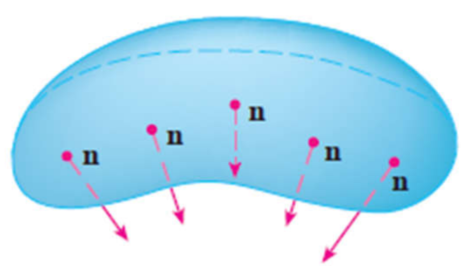
Oriented Surfaces

- Consider only orientable (two-sided) surfaces. We start with a surface that has a tangent plane at every point (x,y,z) on *S* (except at any boundary point). There are two unit normal vectors at (x,y,z)

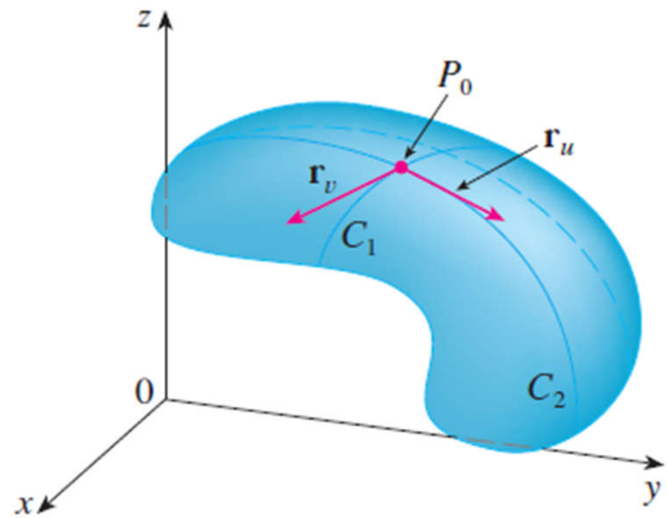
- If it is possible to choose a unit normal vector **n** at every point so that **n** varies continuously over *S*, then *S* is called an **oriented surface and the given choice of n** provides *S* with an **orientation.**







Normal vector for parametric surfaces



S is given by

$$\mathbf{r}(u,v) =< x(u,v), y(u,v), z(u,v) >, (u,v) \in D \subset \mathbb{R}^2$$

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

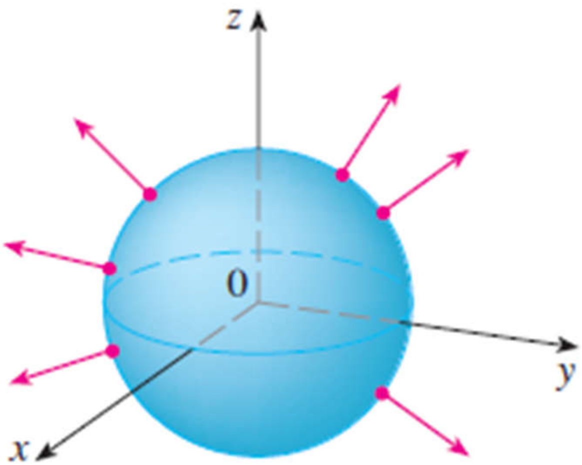
Opposite orientation: **-n**

$$S: \quad z = g(x,y), \qquad \mathbf{r}(x,y) =< x, y, g(x,y) >, \quad (x,y) \in D$$

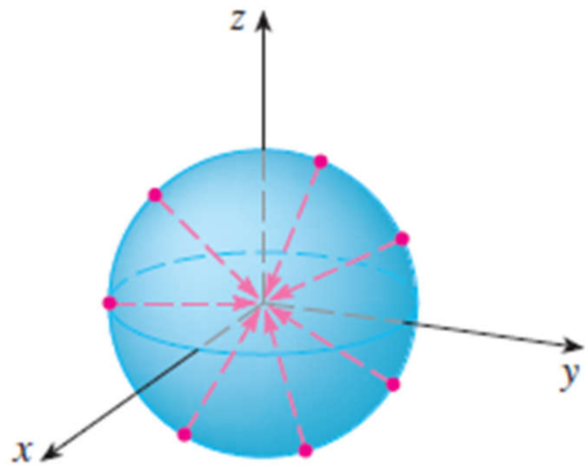
$$\mathbf{r}_x \times \mathbf{r}_y =< -g_x(x,y), -g_y(x,y), 1 >$$

upward orientation: $\mathbf{n} = \dfrac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}$, downward orientation: $\mathbf{n} = -\dfrac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}$

Positive Orientation



Positive orientation

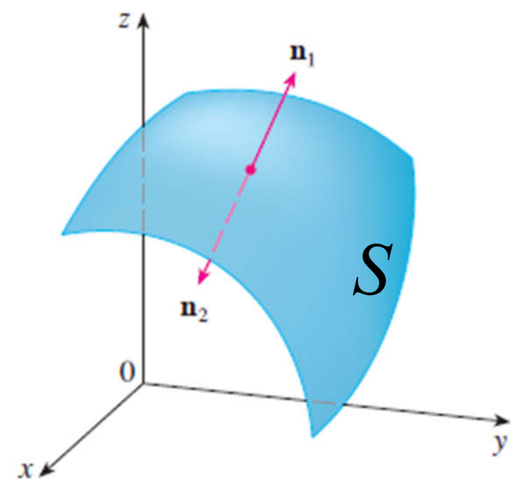


Negative orientation

For a **closed surface**, that is, a surface that is the boundary of a solid region E, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E, and inward-pointing normals give the negative orientation

Surface Integral of Vector Field

S: oriented surface
with unit normal vector **n**



F : continuous vector field on S

Surface integral of F over *S* is defined by

$$\boxed{\iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_S (\mathbf{F} \bullet \mathbf{n})\, dS}$$

This integral is also called the **flux of F across S**

Evaluation of surface integrals of vector fields

S is given by

$$\mathbf{r}(u,v) = < x(u,v), y(u,v), z(u,v) >, (u,v) \in D \subset \mathbb{R}^2$$

Choose $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$

$$\iint_S \mathbf{F} \bullet dS = \iint_S \left(\mathbf{F} \bullet \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right) dS = \iint_D \left(\mathbf{F} \bullet \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right) |\mathbf{r}_u \times \mathbf{r}_v| \; dA$$

$$\Rightarrow \quad \boxed{\iint_S \mathbf{F} \bullet dS = \iint_D \mathbf{F} \bullet (\mathbf{r}_u \times \mathbf{r}_v) dA}$$

Surface S is a Graph

$$S: \quad z = g(x,y), \qquad \mathbf{r}(x,y) =< x, y, g(x,y) >, \quad (x,y) \in D$$

$$\mathbf{r}_x \times \mathbf{r}_y =< -g_x(x,y), -g_y(x,y), 1 >$$

upward orientation: $\mathbf{n} = \dfrac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}$, downward orientation: $\mathbf{n} = -\dfrac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}$

$$\mathbf{F} =< P, Q, R >$$

Suppose S has upward orientation, then it holds on S that

$$\mathbf{F} \bullet (\mathbf{r}_x \times \mathbf{r}_y) = -P(x,y,g(x,y))g_x(x,y) - Q(x,y,g(x,y))g_y(x,y) + R(x,y,g(x,y)))$$

$$\boxed{\iint_S \mathbf{F} \bullet dS = \iint_D \mathbf{F} \bullet (\mathbf{r}_x \times \mathbf{r}_y) dA = \iint_D \left(-Pg_x - Qg_y + R\right) dA}$$

V EXAMPLE | Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

SOLUTION S consists of a parabolic top surface S_1 and a circular bottom surface S_2.

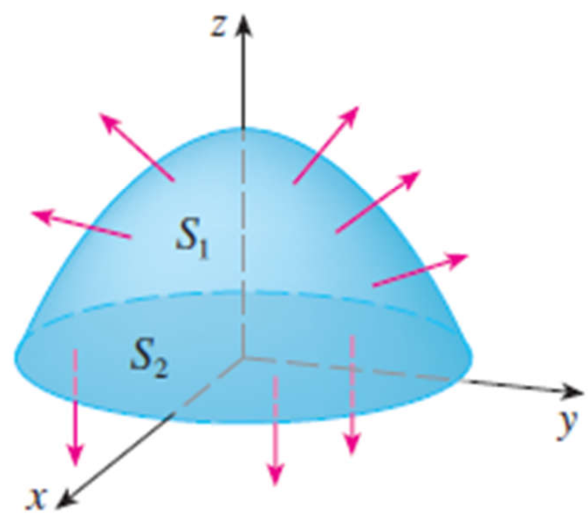
Since S is a closed surface, we use the convention of positive (outward) orientation. This means that S_1 is oriented upward and

D being the projection of S_1 onto the xy-plane, namely, the disk $x^2 + y^2 \leqslant 1$. Since

$$P(x, y, z) = y \qquad Q(x, y, z) = x \qquad R(x, y, z) = z = 1 - x^2 - y^2$$

on S_1 and

$$\frac{\partial g}{\partial x} = -2x \qquad \frac{\partial g}{\partial y} = -2y$$



$$
\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \\
&= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]\, dA \\
&= \iint_D (1 + 4xy - x^2 - y^2)\, dA \\
&= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (r - r^3 + 4r^3 \cos\theta \sin\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\tfrac{1}{4} + \cos\theta \sin\theta\right) d\theta = \tfrac{1}{4}(2\pi) + 0 = \frac{\pi}{2}
\end{aligned}
$$

The disk S_2 is oriented downward, so its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and we have

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_{D} (-z)\, dA = \iint_{D} 0\, dA = 0$$

since $z = 0$ on S_2. Finally, we compute, by definition, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ as the sum of the surface integrals of $\mathbf{F}$ over the pieces S_1 and S_2:

$$\iint_{S} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

Applications

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations. For instance, if **E** is an electric field (see Example 5 in Section 16.1), then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of **E** through the surface S. One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface S is

$$\boxed{11} \qquad Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}$.) Therefore, if the vector field **F** in Example 4 represents an electric field, we can conclude that the charge enclosed by S is $Q = \frac{4}{3}\pi\varepsilon_0$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K \nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance. The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$