MIDTERM EXAMINATION
RANDOM PROCESS

Semester 1, 2020-21 • November 2020 • Total duration: 90 minutes

Student's name:	Student ID:

Chair of Mathematics Department	Lecturer	Score and Examiner
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INSTRUCTIONS: Each student is allowed calculators and one double-sided sheet of reference material (size A4 or similar) marked with their name and ID. All other documents and electronic devices are forbidden.

1. *(20 points)* Consider a binomial asset pricing model with initial stock price $S_0 = 4$, up factor $u = 2$, down factor $d = \frac{1}{2}$, interest rate $r = 0.05$ and the risk neutral probability measure

$$\tilde{p} = \frac{1+r-d}{u-d}, \qquad \tilde{q} = \frac{u-1-r}{u-d}.$$

   (a) Prove that the discounted stock price process $\left(\frac{S_n}{(1+r)^n}\right)_{n\geq 0}$ is $(\mathscr{F}_n)_{n\geq 0}$ - martingale where $\mathscr{F}_n$ is σ - algebra generated by tossing results up to time n.

   (b) Compute $E(S_{100})$

2. *(30 points)* The number of failures N_t, which occur in a computer network over the time interval $[0,t)$, can be described by a Poisson process $\{N_t\}_{t\geq 0}$. On average, there is a failure after every 4 hours, i.e. the intensity of the process is equal to $\lambda = 0.25$. Find the probability that

   (a) at most 1 failure in $[0,8)$ (time unit: hour).

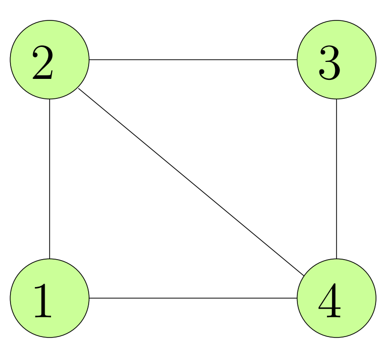
   (b) at least 2 failures in $[8,16)$ (time unit: hour).

   (c) at most 1 failure in $[0,8)$ and at least 2 failures in $[8,16)$ (time unit: hour)?

   (d) the third failure occurs after 8 hours?

   (e) 3 failures in $[0,8]$ and 5 failures in $[4,10]$.

3. *(30 points)* Assume that a graph with undirected edges is given by



At every step, a random walker moves to a randomly chosen neighbor. Let X_n be the position of the walker after n steps. Then $(X_n)_{n\geq 0}$ is a Markov chain with the transition matrix

$$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

The walker chooses a starting vertex at random (i.e. $P(X_0 = i) = \frac{1}{4}$, $i = 1,2,3,4$).

(a) Compute $P(X_2 = 3, X_1 = 2|X_0 = 1)$.

(b) Evaluate $P(X_3 = 4|X_0 = 1)$.

(c) What is the probability mass function of X_2?

4. *(20 points)* **(Random walk with reflecting barriers)** A person walks along a straight line and, at each time period, takes a step to the right with probability 0.4, and a step to the left with probability 0.6. The person starts in one of the positions $1,2$ but if he reaches position 0 (or position 4), his step is instantly reflected back to position 1 (or position 3, respectively). Equivalently, we may assume that when the person is in positions 1 or 3, he will stay in that position with corresponding probability 0.6 and 0.4, respectively.
Denote X_n be his position after n steps.

   (a) Determine the transition matrix of the Markov chain $(X_n)_{n\geq 0}$.

   (b) Suppose that $P(X_0 = 1) = P(X_0 = 2) = \frac{1}{2}$. Find $E(X_3)$.

5. **(Optional) Discrete - time stochastic integral** *(10 points)* Suppose that $(M_n)_{0\leq n\leq N}$ is a martingale and $(\Delta_n)_{0\leq n\leq N-1}$ is an adapted process associated with the filtration $(\mathscr{F}_n)_{0\leq n\leq N}$ (i.e Δ_n is $\mathscr{F}_n$ - measurable for all n). Define the discrete - time stochastic integral $I_0, I_1, \ldots, I_N$ by setting $I_0 = 0$ and

$$I_n = \sum_{k=0}^{n-1} \Delta_k (M_{k+1} - M_k).$$

Prove that $(I_n)_{0\leq n\leq N}$ is a - $(\mathscr{F}_n)$ - martingale.

—— THE END ——