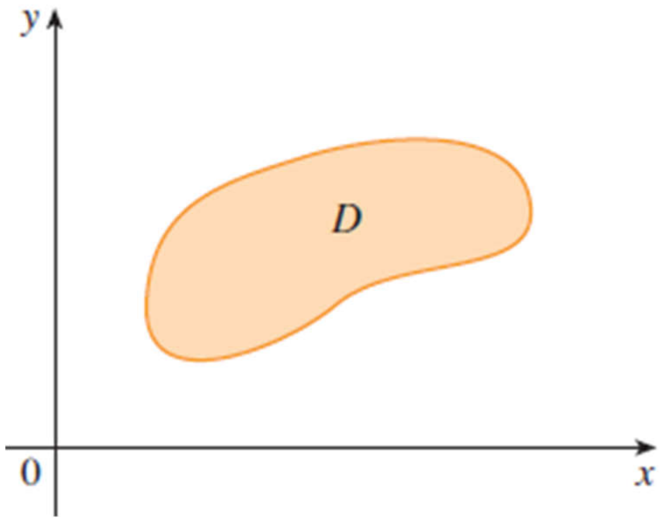
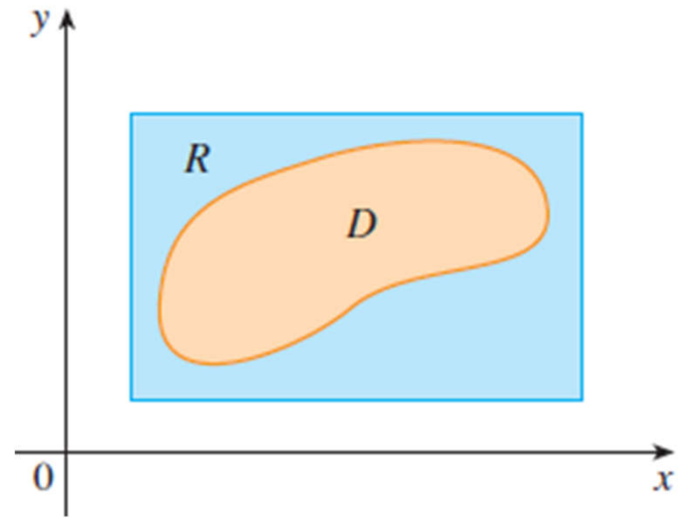
Chapter 3: MULTIPLE INTEGRALS

Lecture 9:
Double Integrals over General Regions
Applications

1. Double Integrals over General Region

Let D be a bounded region, enclosed in a rectangle R





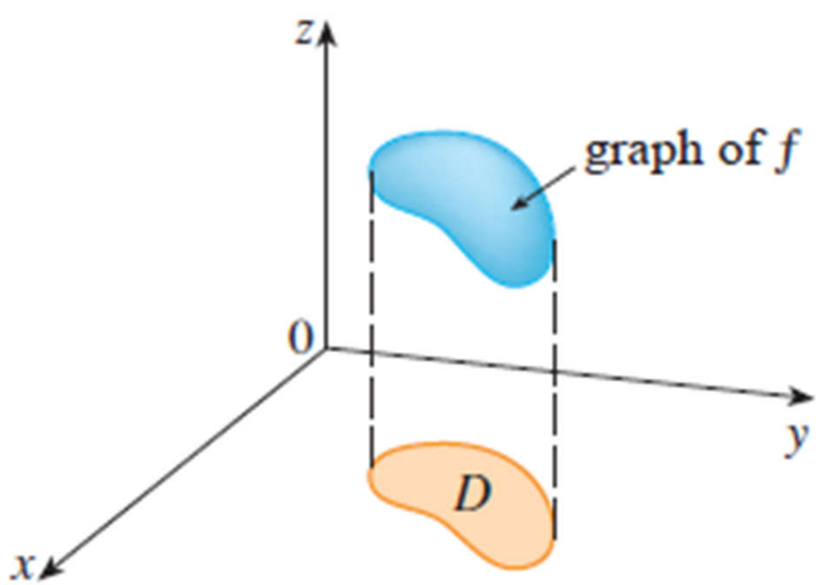
We define a new function F on R by

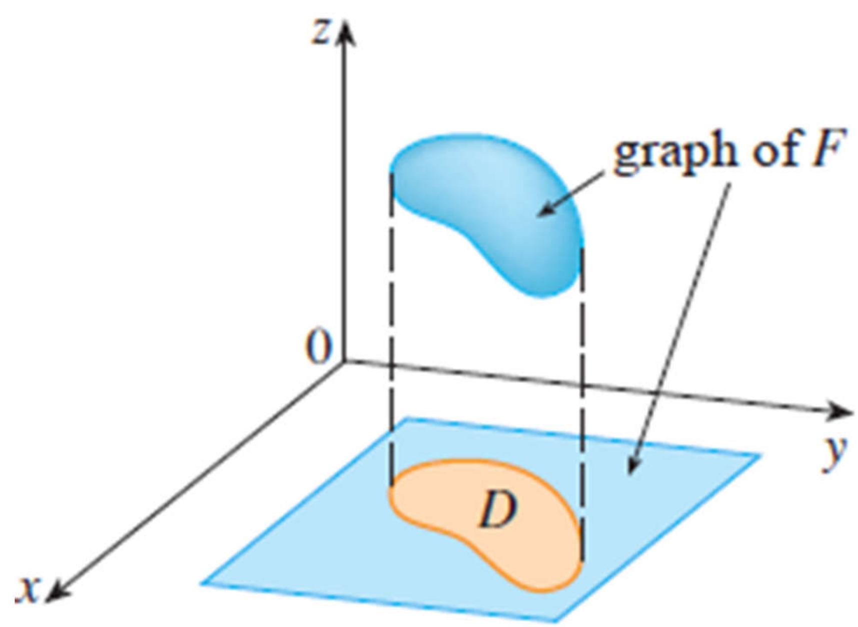
$$F(x,y)=\begin{cases} f(x,y), & \text{if } (x,y)\in D \\ 0, & \text{if } (x,y)\notin D \end{cases}$$

Double Integrals over General Region

If the double integral of F exists over R, then we define the **double integral of f over D by**

$$\iint_D f(x, y)dA = \iint_R F(x, y)dA$$





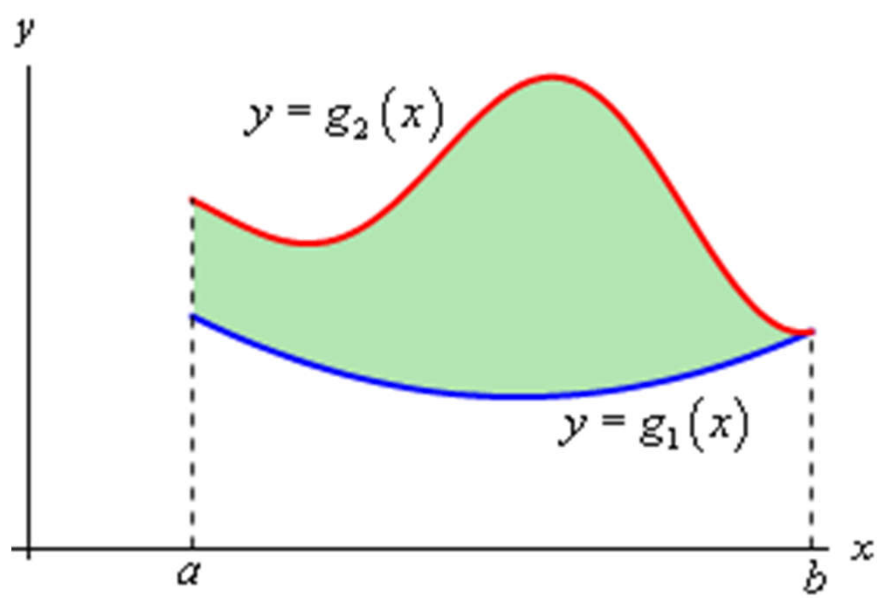
Case 1: Region of Type I

- D is region of **type I if it lies between the graphs of two continuous** functions of x, that is

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

- where g_1 and g_2 are
- continuous on [a, b]

Case 1

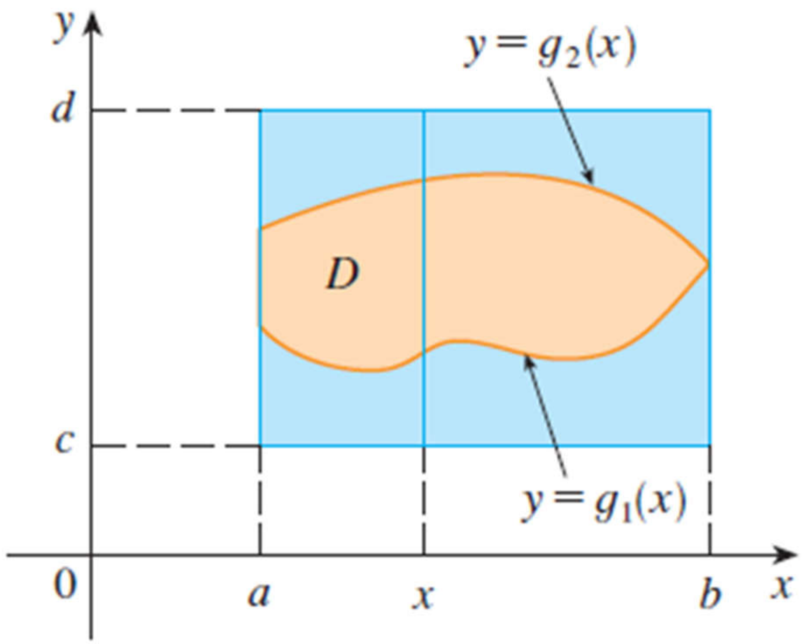


Region of Type I

- In order to evaluate double integral of f over D we choose a rectangle R=[a, b]x[c, d] that contains D
- Then

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

- by Fubini's Theorem



- It holds that

$$F(x,y)=0,\ \text{if } y<g_1(x),\ \text{or } y>g_2(x),\ \text{because } (x,y)\notin D$$

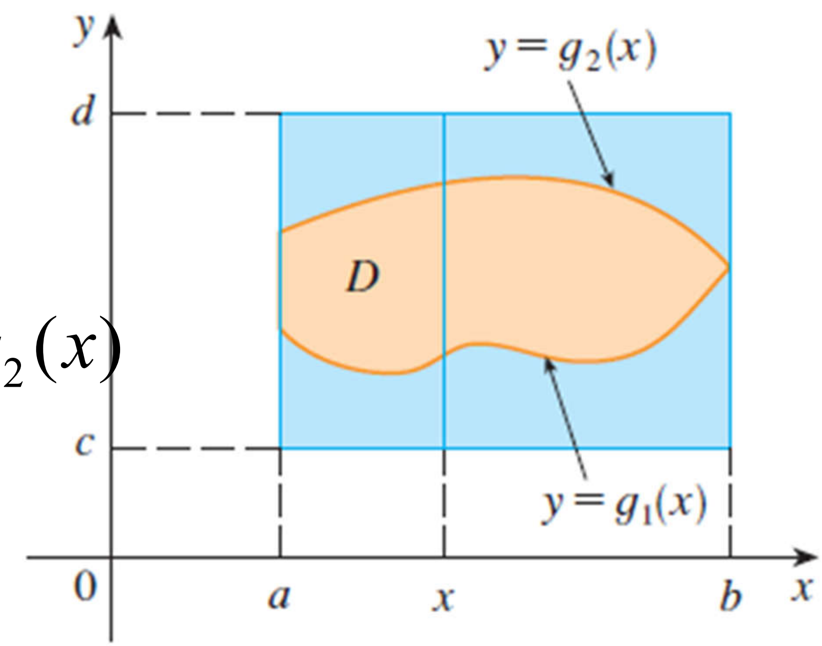
- Therefore

$$\int_c^d F(x,y)dy=\int_c^{g_1(x)} F(x,y)dy+\int_{g_1(x)}^{g_2(x)} F(x,y)dy+\int_{g_2(x)}^{d} F(x,y)dy$$

$$=\int_{g_1(x)}^{g_2(x)} F(x,y)dy=\int_{g_1(x)}^{g_2(x)} f(x,y)dy$$

- Because

$$F(x,y)=f(x,y),\ \text{for } g_1(x)\le y\le g_2(x)$$



Region of Type I

- **<u>Theorem</u>**: If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

- then

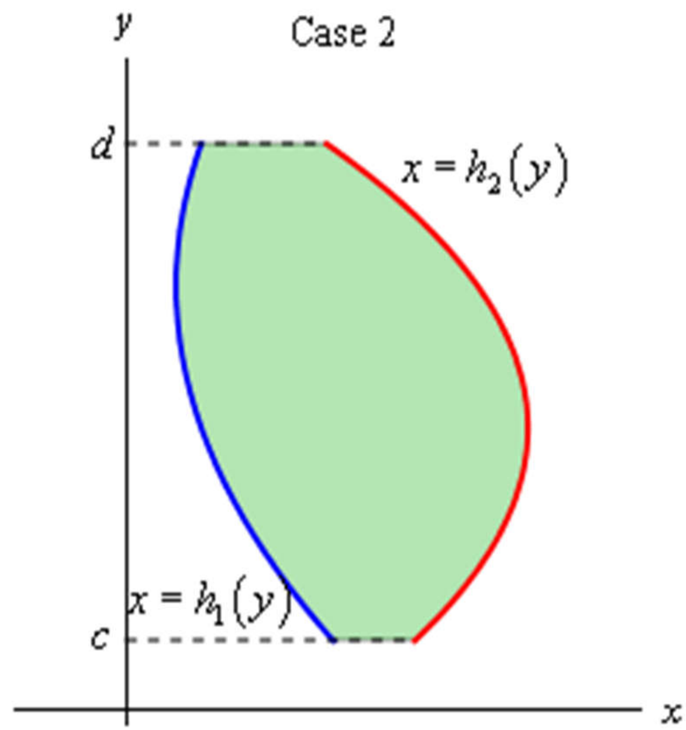
$$\boxed{\iint_D f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx}$$

Case 2: Region of Type II

- D is region of type II if

$$D = \{(x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

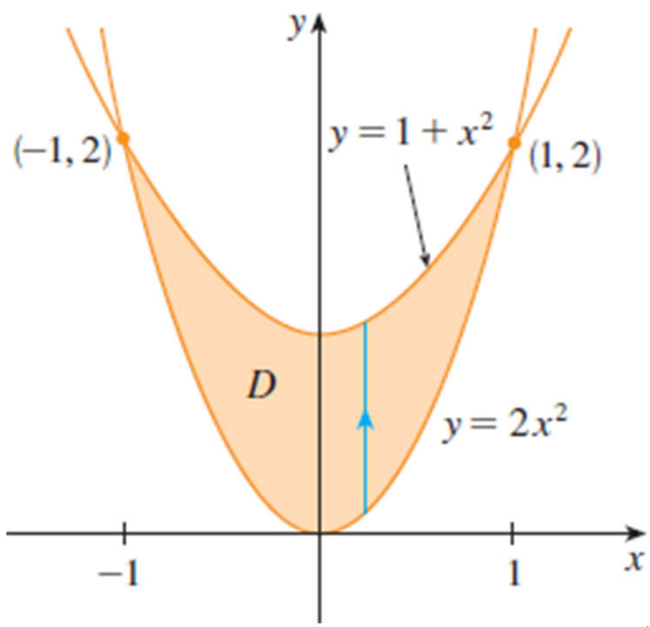
where h_1 and h_2 are continuous on $[c, d]$



- **Theorem**: If D is a region of type II, then

$$\iint_D f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy$$

Example: Evaluate $\iint_D (2x+y)\,dA$, where D is the region bounded by the parabolas $y=2x^2$ and $y=x^2+1$



Solution

Intersection points satisfy:

$$2x^2 = 1 + x^2 \Leftrightarrow x = \pm 1 \Rightarrow A(-1,2), B(1,2)$$

Therefore, the domain D can be expressed as region of type I:

$$D = \{(x,y) \mid -1 \le x \le 1,\ \ 2x^2 \le y \le 1+x^2\}$$

$$\iint_D (2x+y)dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (2x+y)dydx = \int_{-1}^{1} (2xy + y^2/2)\Big|_{y=2x^2}^{y=1+x^2} dx$$

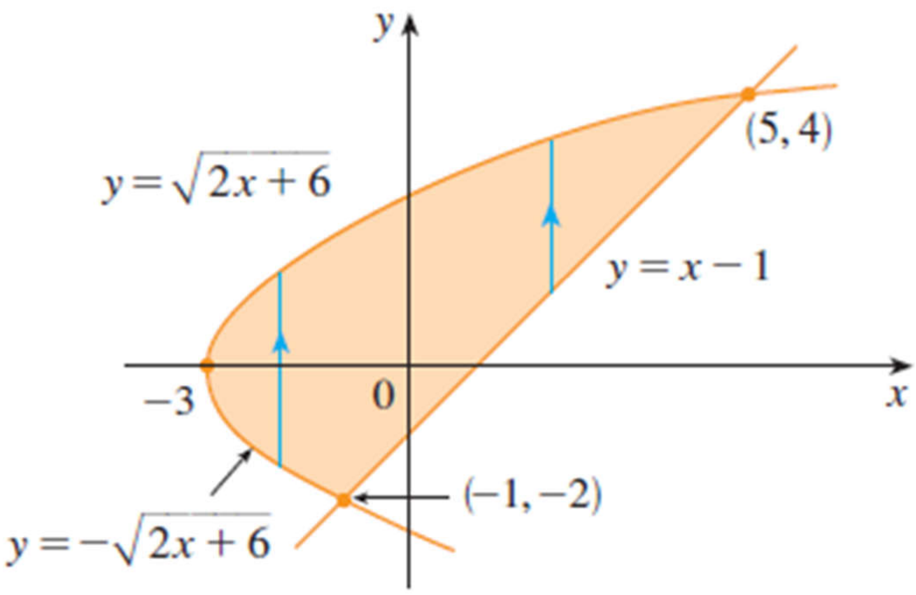
$$= \int_{-1}^{1} [2x(1+x^2) + ((1+x^2))^2/2 - 4x^3 - 2x^4]dx = \int_{0}^{1} (1+2x^2-3x^4)dx$$

$$= (x + 2x^3/3 - 3x^5/5)\Big|_0^1 = 1 + 2/3 - 3/5 = 16/15$$

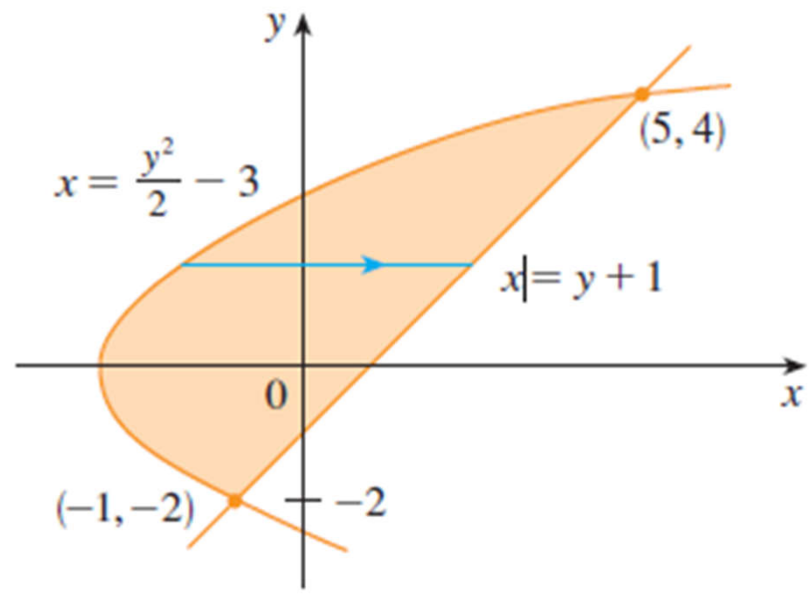
Example 2: Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line y=x-1 and the parabola $y^2 = 2x + 6$

Solution

$$D = \left\{(x, y) \mid -2 \leqslant y \leqslant 4,\ \tfrac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1\right\}$$



(a) D as a type I region



(b) D as a type II region

$$\iint_D xy\, dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy\, dx\, dy = \int_{-2}^{4} \left[\frac{x^2}{2} y \right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \tfrac{1}{2} \int_{-2}^{4} y\left[(y+1)^2 - \left(\tfrac{1}{2}y^2 - 3\right)^2\right] dy$$

$$= \tfrac{1}{2} \int_{-2}^{4} \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y \right) dy$$

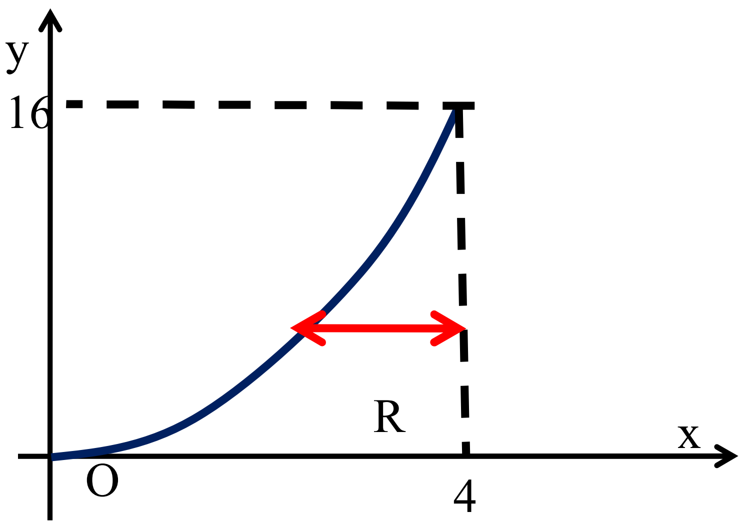
$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2 \right]_{-2}^{4} = 36$$

If D is expressed as region of type I:

$$\iint_D xy\, dA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\, dy\, dx + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy\, dy\, dx$$

Interchanging Limits of Integration

- Sometimes it is easier to integrate first with respect to x, and then y, while with other integrals the reverse process is easier.
- So, we need to interchange limits of integration
- **<u>Example</u>**: Evaluate



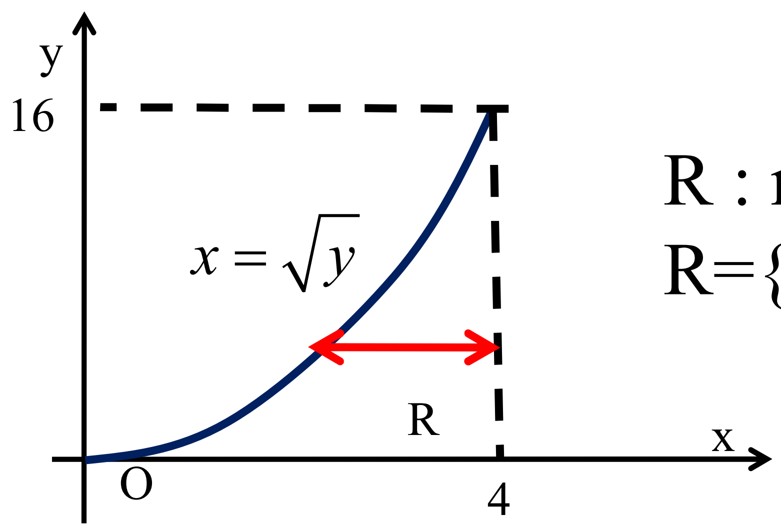
$$\int_0^{16}\int_{\sqrt{y}}^{4}\sqrt{x^3+4}\,dx\,dy$$

Interchanging Limits of Integration

- **<u>Can you evaluate</u>** $\displaystyle\int_{\sqrt{y}}^{4} \sqrt{x^3+4}\,dx$, or $\displaystyle\int \sqrt{x^3+4}\,dx$?

- **No!**

$$\int_0^{16}\int_{\sqrt{y}}^{4} \sqrt{x^3+4}\,dx\,dy = \iint_R \sqrt{x^3+4}\,dA = I$$



R : region of type II

R={(x,y)| 0≤y ≤ 16, √y ≤x ≤4 }

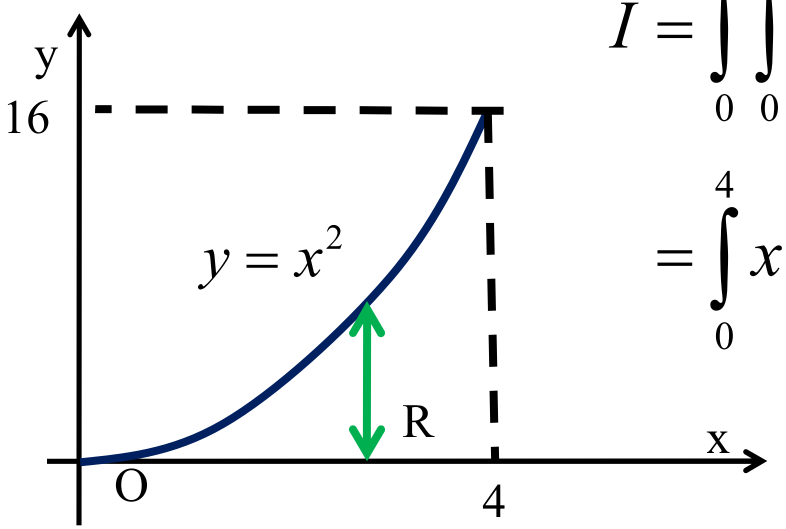
Solution

$$I = \int_0^{16} \int_{\sqrt{y}}^{4} \sqrt{x^3+4}\,dx\,dy$$

- We re-write R: region of type I:

  R={(x,y) | 0 ≤x ≤4, 0≤y ≤ x^2}

- Thus, the double integral can be written as



$$I = \int_0^4 \int_0^{x^2} \sqrt{x^3+4}\,dy\,dx = \int_0^4 y\sqrt{x^3+4}\Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^4 x^2\sqrt{x^3+4}\,dx = \frac{2}{9}(x^3+4)^{3/2}\Big|_0^4 = 122.83$$

Properties of double integrals

- We assume that all of the integrals exist. It holds that:

1) $\iint_D (f(x,y)+g(x,y))dA = \iint_D f(x,y)dA + \iint_D g(x,y)dA$

2) $\iint_D cf(x,y)dA = c\iint_D f(x,y)dA,$ where c is a constant

3) If $f(x,y) \geq g(x,y), \forall (x,y) \in D$, then

$$\iint_D f(x,y)dA \geq \iint_D g(x,y)dA$$

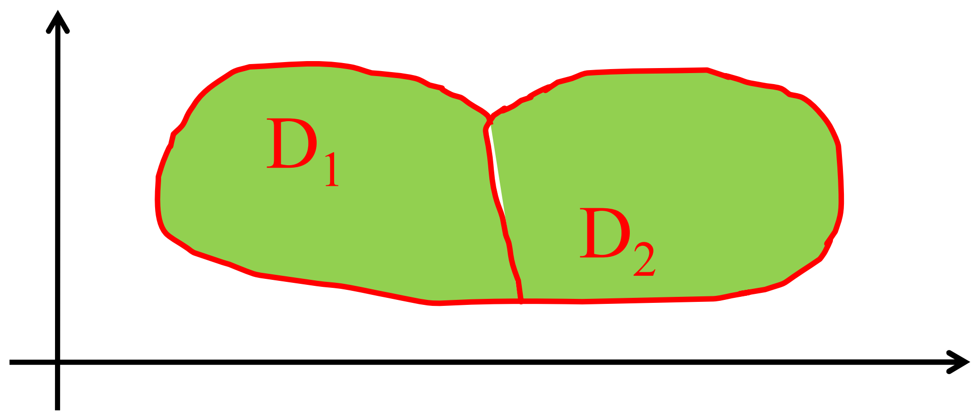
Properties of Double Integrals

If $D = D_1 \cup D_2$, where D_1 and D_2 do NOT overlap except perhaps on their boundaries. Then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

$$\iint_D 1\,dA = A(D)$$

$= \text{area of } D$



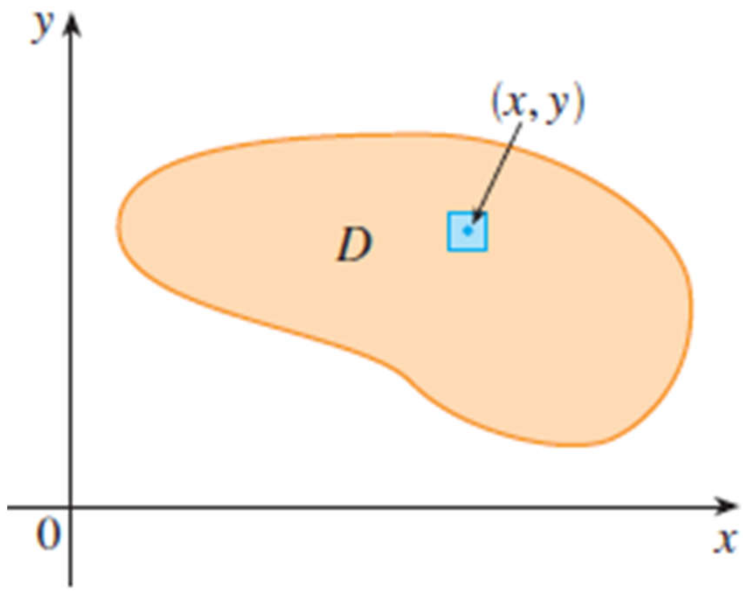
Properties of Double Integrals

If $m \le f(x, y) \le M, \forall (x, y) \in D$, then

$$m \times A(D) \le \iint_D f(x, y)\,dA \le M \times A(D)$$

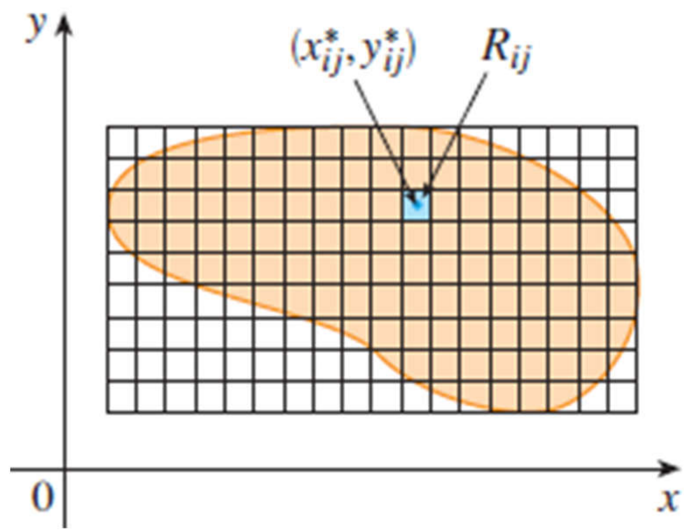
2. APPLICATIONS OF DOUBLE INTEGRALS

Mass



A lamina occupies a region D of the xy-plane and its density (in units of mass per unit area) at a point (x, y) in D is given by ρ(x, y), where ρ is continuous on D:

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$



$$m = \lim_{k,n\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

$$= \iint_D \rho(x, y)\,dA$$

Moments and Center of Mass

- The moment of a particle about an axis as the product of its mass and its directed distance from the axis.
- The mass of the part of the lamina occupying R_{ij} is approximately $\rho\ (x_{ij}{}^*, y_{ij}{}^*)\Delta A$, where ΔA = area of R_{ij}
- So, we can approximate the moment of R_{ij} with respect to the x-axis by

$$[\rho(x_{ij}^*, y_{ij}^*)\Delta A]y_{ij}^*$$

Moments and Center of Mass

- If we now add these quantities and take the limit as the number of subrectangles becomes large, we obtain the **moment of the entire lamina about the *x-axis:***

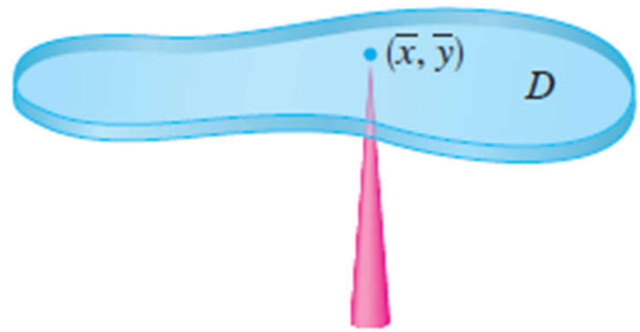
$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D y\rho(x,y)\,dA$$

- Similarly, the **moment about the *y-axis is***

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D x\rho(x,y)\,dA$$

Coordinates of Center of Mass

$(\bar{x}, \bar{y})$: center of mass



so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$

Center of mass has cooordinates $(\bar{x}, \bar{y})$:

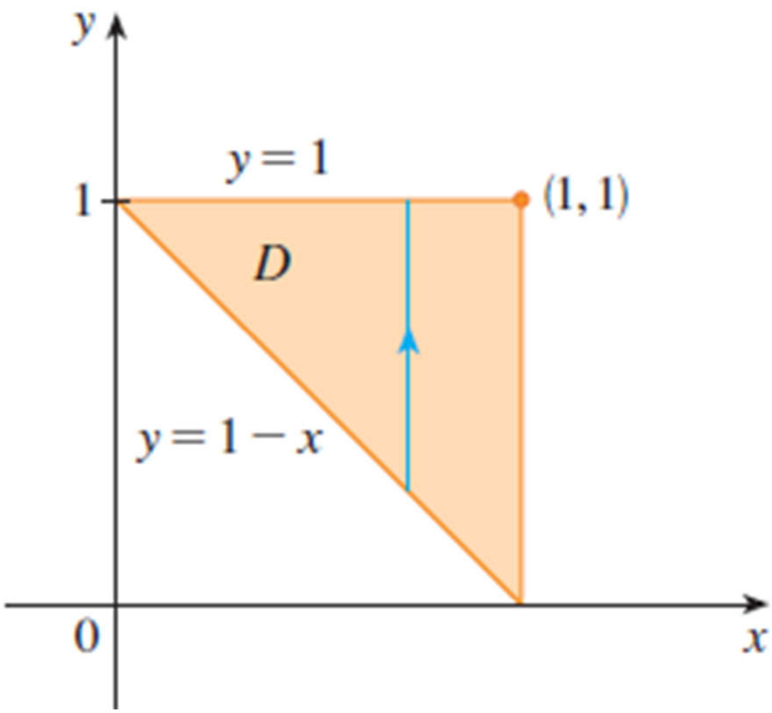
$$\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)\,dA \qquad \bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\,dA$$

where $m = \iint_D \rho(x, y)\,dA$ is the mass of the object

Electric Charge

- Physicists also consider other types of density that can be treated in the same manner.
- For example, if an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x,y)$ at a point (x,y) in D, then the **total charge Q** is given by

$$\boxed{Q(x,y) = \iint_D \sigma(x,y)\,dA}$$



Example 1: Charge is distributed over the triangular region in Figure so that the charge density

$$\sigma(x, y) = xy \qquad \left(\frac{C}{m^2}\right)$$

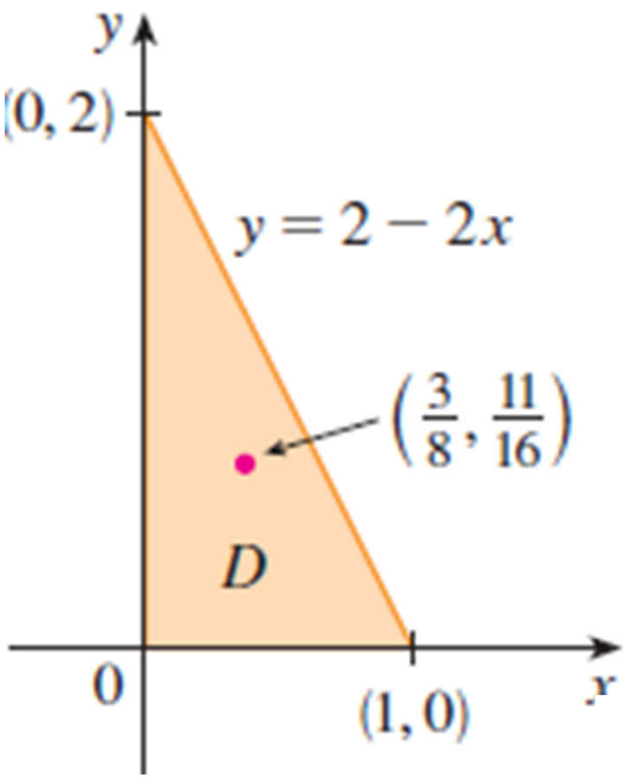
Find the total charge

Solution

$$Q = \iint_D \sigma(x, y)\, dA = \int_0^1 \int_{1-x}^1 xy\, dy\, dx$$

$$= \int_0^1 \left[x\frac{y^2}{2} \right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2}[1^2 - (1 - x)^2]\, dx$$

$$= \tfrac{1}{2} \int_0^1 (2x^2 - x^3)\, dx = \frac{1}{2} \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{5}{24}$$



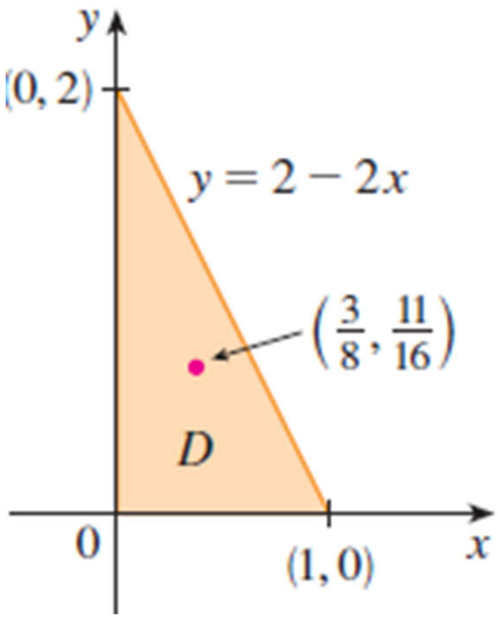
Example 2: Find the mass and center of mass of a triangular lamina with vertices (0,0), (1,0) and (0, 2) if $\rho(x, y) = 1 + 3x + y$

Solution

$$m = \iint_D \rho(x, y)\, dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= 4 \int_0^1 (1 - x^2)\, dx = 4 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3}$$



$$\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)\, dA = \tfrac{3}{8}\int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx$$

$$= \frac{3}{8}\int_0^1 \left[xy + 3x^2y + x\frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= \frac{3}{2}\int_0^1 (x - x^3)\, dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\, dA = \tfrac{3}{8}\int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx$$

$$= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3} \right]_{y=0}^{y=2-2x} dx = \tfrac{1}{4}\int_0^1 (7 - 9x - 3x^2 + 5x^3)\, dx$$

$$= \frac{1}{4}\left[7x - 9\,\frac{x^2}{2} - x^3 + 5\,\frac{x^4}{4} \right]_0^1 = \frac{11}{16}$$