MIDTERM EXAMINATION
April 2018
Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Deputy head of Dept. of Mathematics:	Lecturer:
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INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 Let (X, d) be a metric space, let A and B be subsets of X.

(a) *(15 marks)* Show that

$$\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B).$$

(b) *(15 marks)* Suppose that A and B are nonempty and compact. Prove that there exist $x_0 \in A$ and $y_0 \in B$ such that

$$d(x_0, y_0) = \sup\big\{d(x, y) : x \in A,\ y \in B\big\}.$$

Question 2 Let (X, d) be a metric space.

(a) *(15 marks)* Let A and B be nonempty subsets of X. Show that the set

$$C = \big\{x \in X : d(x, A) < d(x, B)\big\}$$

is an open set.

(*Hint:* The functions $f(x) = d(x, A)$ and $g(x) = d(x, B)$ are continuous.)

(b) *(15 marks)* We call $x \in X$ a *cluster point* of a sequence $\{x_n\} \subset X$ if every ball $B(x, r)$ contains infinitely many terms of the sequence, that is $\{n \in \mathbb{N} : x_n \in B(x, r)\}$ is infinite for every fixed $r > 0$. Show that x is a cluster point of a sequence $\{x_n\}$ if there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x$.

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Question 3 The symmetric difference of two sets A and B is

$$A\Delta B = (A \setminus B) \cup (B \setminus A).$$

Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) *(10 marks)* Show that if A and B are measurable sets, then so is $A\Delta B$.

(b) *(10 marks)* Show that if A and B are measurable sets and $\mu(A\Delta B) = 0$, then $\mu(A) = \mu(B)$.

Question 4

(a) *(10 marks)* Suppose that $(X, \mathcal{M}, \mu)$ is a probability space. Show that for every $E \in \mathcal{M}$, $\mu(E^c) = 1 - \mu(E)$.

(b) *(10 marks)* Show that the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is generated by the family $\mathcal{E} = \big\{[a, b) : a, b \in \mathbb{R},\ a < b\big\}$, that is, $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E})$.

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MIDTERM EXAMINATION
November 2017
Duration: 120 minutes

SOLUTIONS

Question 1 (a) Suppose that $x \in \text{int}\,(A \cap B)$. Then there is a ball $B(x,r) \subset A \cap B$. It follows that $B(x,r) \subset A$ and $B(x,r) \subset B$, that is, $x \in \text{int}\,(A) \cap \text{int}\,(B)$. Hence

$$\text{int}\,(A \cap B) \subset \text{int}\,(A) \cap \text{int}\,(B). \tag{1}$$

Conversely, if $x \in \text{int}\,(A) \cap \text{int}\,(B)$, then there are balls $B(x,r_1)$ and $B(x,r_2)$ such that $B(x,r_1) \subset A$ and $B(x,r_2) \subset B$. Setting $r = \min\{r_1, r_2\}$ we have $B(x,r) \subset A \cap B$. Thus $x \in \text{int}\,(A \cap B)$ and hence,

$$\text{int}\,(A) \cap \text{int}\,(B) \subset \text{int}\,(A \cap B). \tag{2}$$

(1) and (2) give $\text{int}\,(A \cap B) = \text{int}\,(A) \cap \text{int}\,(B)$.

(b) Set

$$\alpha := \sup\big\{d(x,y) : x \in A,\ y \in B\big\}.$$

There exist sequences $\{x_n\} \subset A$ and $\{y_n\} \subset B$ such that $d(x_n, y_n) \to \alpha$ as $n \to \infty$. Since A is compact, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ that converges to a point $x_0 \in A$. Likewise, there is a subsequence $\{y_{n_{k_p}}\}_{p=1}^{\infty}$ of $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to a point $y_0 \in B$. Then $x_{n_{k_p}} \to x_0$ and $y_{n_{k_p}} \to y_0$ so that

$$d(x_0, y_0) = \lim_{p \to \infty} d\big(x_{n_{k_p}}, y_{n_{k_p}}\big) = \alpha.$$

Question 2 (a) Since the functions $f(x) := d(x,A)$ and $g(x) := d(x,B)$ are continuous, so is their difference $h(x) := f(x) - g(x)$. Thus

$$C = \big\{x \in X : d(x,A) < d(x,B)\big\} = \big\{x \in X : h(x) < 0\big\} = h^{-1}\big((-\infty, 0)\big),$$

which is an open set.

(b) Assume that there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ such that

$$\lim_{k \to \infty} x_{n_k} = x.$$

For a fixed positive number r, there is $k_0 \in \mathbb{N}$ such that $d(x_{n_k}, x) < r$ for all $k \geq k_0$. Thus $\{n_k : k \geq k_0\} \subset \{n : x_n \in B(x,r)\}$, that is, the set $\{n : x_n \in B(x,r)\}$ is infinite. Therefore x is a cluster point of $\{x_n\}$.

Question 3 (a) Since $A, B \in \mathcal{M}$ and $\mathcal{M}$ is a σ-algebra, we have $A \setminus B, B \setminus A \in \mathcal{M}$. Thus $A \Delta B = (A \setminus B) \cup (B \setminus A) \in \mathcal{M}$.

(b) Since $A \setminus B, B \setminus A \in \mathcal{M}$ and since $A \setminus B, B \setminus A$ are subsets of $A \Delta B$, we obtain

$$\mu(A \setminus B) \leq \mu(A \Delta B) \quad \text{and} \quad \mu(B \setminus A) \leq \mu(A \Delta B).$$

By assumption, $\mu(A \Delta B) = 0$ so that $\mu(A \setminus B) = \mu(B \setminus A) = 0$. From the disjoint unions $A = (A \cap B) \cup (A \setminus B)$ and $B = (A \cap B) \cup (B \setminus A)$, it follows that

$$\mu(A) = \mu(A \cap B) + \mu(A \setminus B) = \mu(A \cap B)$$
$$\mu(B) = \mu(A \cap B) + \mu(B \setminus A) = \mu(A \cap B).$$

Thus $\mu(A) = \mu(B)$.

(Note also that since $A \setminus B$ and $B \setminus A$ are disjoint measurable sets and $A \Delta B = (A \setminus B) \cup (B \setminus A)$ we have

$$0 = \mu(A \cup B) = \mu(A \setminus B) + \mu(B \setminus A).$$

It follows that $\mu(A \setminus B) = \mu(B \setminus A) = 0$.)

Question 4 (a) Every measurable set of a probability space has finite measure. Thus

$$\mu(E^c) = \mu(X \setminus E) = \mu(X) - \mu(E) = 1 - \mu(E).$$

(b) Since for every $[a, b) \in \mathcal{E}$,

$$[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right) \in \mathcal{B}(\mathbb{R}).$$

Hence $\mathcal{E} \subset \mathcal{B}(\mathbb{R})$ so that $\sigma(\mathcal{E}) \subset \mathcal{B}(\mathbb{R})$.

Conversely, for each open interval (a, b), $a, b \in \mathbb{R}$, $a < b$, we have

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right) \in \sigma(\mathcal{E}).$$

If V is an open set in $\mathbb{R}$, V can be represented as $V = \bigcup_n (a_n, b_n)$. Thus $V \in \sigma(\mathcal{E})$ for every open set $V \subset \mathbb{R}$. Hence $\mathcal{B}(\mathbb{R}) \subset \sigma(\mathcal{E})$. Therefore $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E})$.