Poisson processes

A motivation - Insurance risk model

- In a portfolio insurance risk such as a portfolio of motor insurance policies, interest quantities are *number of claims* arriving in a fixed period of time and the *sizes* of those claim

A motivation - Insurance risk model

- In a portfolio insurance risk such as a portfolio of motor insurance policies, interest quantities are *number of claims* arriving in a fixed period of time and the *sizes* of those claim
- modelling number of claims by a counting process such as Poisson process
- modelling the financial losses which can be suffered by individuals and insurance companies as a result of insurable events such as storm or fire damage to property, theft of personal property and vehicle accidents. One candidate is compound Poisson process

Table of Contents

Poisson processes

A Poisson process with **intensity** (or **rate**) λ is a random (counting) process $(N_t)_{t \geq 0}$ with the following properties:

1. $N_0 = 0$.
2. For all $t > 0$, N_t has a **Poisson distribution** with parameter λt.
3. **(Stationary increments)** For all $s, t > 0, N_{s+t} - N_s$ has the same distribution as N_t. That is,
   $P(N_{s+t} - N_s = k) = P(N_t = k) = \dfrac{e^{-\lambda t}(\lambda t)^k}{k!}$, for $k = 0, 1, \dots$
4. (**Independent increments**) For $0 \leq s < t, N_t - N_s$ and $N_r - N_q$ are independent random variables.

Some properties of Poisson processes

- The distribution of the number of arrivals in an interval depends only on the length of the interval

Some properties of Poisson processes

- The distribution of the number of arrivals in an interval depends only on the length of the interval
- The number of arrivals on disjoint intervals are independent random variables

Some properties of Poisson processes

- The distribution of the number of arrivals in an interval depends only on the length of the interval
- The number of arrivals on disjoint intervals are independent random variables
- $E(N_t) = \lambda t$

Some properties of Poisson processes

- The distribution of the number of arrivals in an interval depends only on the length of the interval
- The number of arrivals on disjoint intervals are independent random variables
- $E(N_t) = \lambda t$
- λ is the average number of arrivals per unit of time.

Some properties of Poisson processes

- The distribution of the number of arrivals in an interval depends only on the length of the interval
- The number of arrivals on disjoint intervals are independent random variables
- $E(N_t) = \lambda t$
- λ is the average number of arrivals per unit of time.
- when $h \ll 0$ (very small) then

$$P(N_h = 1) = \lambda h + o(h)$$

and

$$P(N_h \geq 2) = o(h)$$

Construct by tossing a low-probability coin very fast

- Pick n large
- A coin with low Head probability $\frac{\lambda}{n}$
- Toss this coin at times which are positive integer multiples of $\frac{1}{n}$
- N_t be number of Head on $[0, t]$. Then N_t is binomial distributed and converges to $Poiss(\lambda t)$ as $n \to \infty$
- For, $N_{t+s} - N_s$ is indepdendent of the past and Poisson distributed $Poiss(\lambda t)$

Construct by tossing a low-probability coin very fast

- Pick n large
- A coin with low Head probability $\frac{\lambda}{n}$
- Toss this coin at times which are positive integer multiples of $\frac{1}{n}$
- N_t be number of Head on $[0,t]$. Then N_t is binomial distributed and converges to $Poiss(\lambda t)$ as $n \to \infty$
- For, $N_{t+s} - N_s$ is indepdendent of the past and Poisson distributed $Poiss(\lambda t)$

Simulation Poisson process?

Example

Starting at 6 a.m., customers arrive at Martha's bakery according to a Poisson process at the rate of 30 customers per hour. Find the probability that more than 2 customers arrive between 9 a.m and 11 a.m.

Solution

- Initial time $t = 0$ (corresponding to 6 a.m)
- Number of customers: Poisson process $(N_t)_{t\geq 0}$ at rate of 30 customers **per hour**
- Number of customers up to 9 a.m ($t = 3$): N_3
- Number of customers up to 11 a.m ($t = 5$): N_5
- Number of customers between 9 a.m and 11 a.m: $N_5 - N_3 \hookrightarrow Poiss((5-3)\lambda) = Poiss(60)$
-

$$
\begin{aligned}
P(N_5 - N_3 > 2) &= 1 - P(N_5 - N_2 \leq 2) \\
&= 1 - [P(N_5 - N_2 = 0) + P(N_5 - N_2 = 1)] \\
&= 1 - \left[e^{-60}\frac{60^0}{0!} + e^{-60}\frac{60^1}{1!}\right]
\end{aligned}
$$

Example

Joe receives text messages starting at 10 a.m. at the rate of 10 texts per hour according to a Poisson process. Find the probability that he will receive exactly 18 texts by noon (12 a.m.) and 70 texts by 5 p.m.

Example

Joe receives text messages starting at 10 a.m. at the rate of 10 texts per hour according to a Poisson process. Find the probability that he will receive exactly 18 texts by noon (12 a.m.) and 70 texts by 5 p.m.

Solution

- Initial time $t = 0$ (10 a.m)
- Text message arrival: Poisson process $(N_t)_{t \geq 0}$ with rate $\lambda = 10$

Example

Joe receives text messages starting at 10 a.m. at the rate of 10 texts per hour according to a Poisson process. Find the probability that he will receive exactly 18 texts by noon (12 a.m.) and 70 texts by 5 p.m.

Solution

- Initial time $t = 0$ (10 a.m)
- Text message arrival: Poisson process $(N_t)_{t \geq 0}$ with rate $\lambda = 10$
- Number of message up to 12a.m (corresponding to time $t = 2$) is N_2
- Number of message up to 5 p.m (corresponding to time $t = 7$) is N_7

Example

Joe receives text messages starting at 10 a.m. at the rate of 10 texts per hour according to a Poisson process. Find the probability that he will receive exactly 18 texts by noon (12 a.m.) and 70 texts by 5 p.m.

Solution

- Initial time $t = 0$ (10 a.m)
- Text message arrival: Poisson process $(N_t)_{t \geq 0}$ with rate $\lambda = 10$
- Number of message up to 12a.m (corresponding to time $t = 2$) is N_2
- Number of message up to 5 p.m (corresponding to time $t = 7$) is N_7
- Need to find $P(N_2 = 18, N_7 = 70)$

$$P(N_2 = 18, N_7 = 70) = \underbrace{P(N_7 = 70 | N_2 = 18) P(N_2 = 18)}_{\text{multiplication rule}}$$

$$= P(N_7 - N_2 + N_2 = 70 | N_2 = 18) P(N_2 = 18)$$

$$= P(\underbrace{N_7 - N_2}_{\text{independent of } N_2} + \underbrace{18}_{\text{substitute } N_2 \text{ by } 18} = 70 | N_2 = 18) P(N_2 = 18)$$

$$= P(\underbrace{N_7 - N_2}_{Pois((7-2)\lambda) = Poiss(50)} = 52) P(\underbrace{N_2}_{Pois(2\lambda) = Poiss(20)} = 18)$$

$$= \frac{e^{-50}(50)^{52}}{52!} \frac{e^{-20}(20)^{18}}{18!}$$

Another approach

$$P(N_2 = 18, N_7 = 70) = P(\underbrace{N_2 = 18, N_7 - N_2 = 52}_{\color{red}{independent}})$$

$$= P(N_2 = 18)P(\underbrace{N_7 - N_2}_{\color{red}{\text{same distribution as } N_5}} = 52)$$

$$= P(N_2 = 18)P(N_5 = 52)$$

$$= \frac{e^{-20}(20)^{18}}{18!}\frac{e^{-50}(50)^{52}}{52!} = 0.0045 = 0.45\%,$$

Example

On election day, people arrive at a voting center according to a Poisson process. On average, 100 voters arrive every hour. If 150 people arrive during the first hour, what is the probability that at most 350 people arrive before the third hour?

Example

On election day, people arrive at a voting center according to a Poisson process. On average, 100 voters arrive every hour. If 150 people arrive during the first hour, what is the probability that at most 350 people arrive before the third hour?

Solution

- Number of arrivals $(N_t)_{t\geq 0}$ is a Poisson process at rate of $\lambda = 100$ per hour.
- 150 people arrive during the first hour: $N_1 = 150$
- at most 350 people arrive before the third hour: $N_3 \leq 350$
- Need to find $P(N_3 \leq 350 | N_1 = 150)$

$$
\begin{aligned}
P(N_3 \leq 350 | N_1 = 150) &= P(N_3 - N_1 + N_1 \leq 350 | N_1 = 150) \\
&= P(N_3 - N_1 \leq 200 | N_1 = 150) \\
&= P(N_3 - N_1 \leq 200) = P(N_2 \leq 200) \\
&= \sum_{k=0}^{200} P(N_2 = k) \\
&= \sum_{k=0}^{200} \frac{e^{-100*2}(100*2)^k}{k!} = 0.519.
\end{aligned}
$$

Painful to compute these term directly (out of memory). One solution is using normal approximation for Poisson distribution

Example

You get email according to a Poisson process at a rate of $\lambda = 5$ messages per hour. You check your email every thirty minutes. Find

1. P(no message)
2. P(one message)

Arrival, inter-arrival time

Arrival, inter-arrival time

- Inter-arrival time $X_1, X_2, X_3 \ldots$

Arrival, inter-arrival time

- Inter-arrival time $X_1, X_2, X_3 \ldots$
- Arrival time

$$
\begin{aligned}
S_0 &= 0 \\
S_1 &= X_1 \\
S_2 &= X_1 + X_2 \\
&\ldots
\end{aligned}
$$

Arrival, inter-arrival time

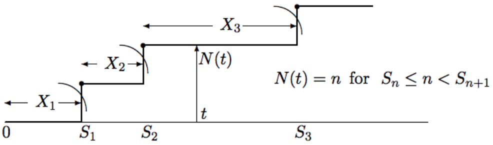
- Inter-arrival time $X_1, X_2, X_3 \ldots$
- Arrival time

$$
\begin{aligned}
S_0 &= 0 \\
S_1 &= X_1 \\
S_2 &= X_1 + X_2 \\
&\ldots
\end{aligned}
$$

- Number of arrival up to time t

$$N_t = \sum_{n=1}^{\infty} 1_{\{S_n \leq t\}} = \max\{n : S_n \leq t\}$$

Inter-arrival time are i.i.d exponential RVs



- $P(X_1 > t) = P(N_t = 0) = e^{-\lambda t}$. Hence $X_1 \hookrightarrow Exp(\lambda)$
-

$$
\begin{aligned}
& P(X_2 > t | X_1 = s) = P(\text{no event in (s,s+t)} | X_1 = s) \\
& = P(\text{no event in (s,s+t)}) \text{ (by independent increment)} \\
& = P(\text{no event in (0,t)}) \text{ (by stationary increment)} \\
& = P(N_t = 0) = e^{-\lambda t}
\end{aligned}
$$

Hence $X_2 \hookrightarrow Exp(\lambda)$ and independent of X_1

Construction by exponential interarrival times

- $X_1, X_2, \ldots, X_n$ are i.i.d $Exp(\lambda)$
- Arrival time

$$\begin{aligned}
S_0 &= 0 \\
S_1 &= X_1 \\
S_2 &= X_1 + X_2 \\
&\ldots \\
S_n &= X_1 + X_2 + \cdots + X_n
\end{aligned}$$

- Stop when $S_n \leq t < S_{n+1}$

Arrival time or waiting time S_n

The density function of S_n is given by

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

i.e. S_n is Gamma distributed with parameter (n, λ) (also called Erlang)

Proof

- the nth event will occur prior to or at time t if and only if the number of events occurring by time t is at least n

$$S_n \leq t \Leftrightarrow N_t \geq n$$

- cdf of S_n

$$F_{S_n}(t) = P(S_n \leq t) = P(N_t \geq n) = \sum_{k=n}^{\infty} P(N_t = k) = \sum_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)}{k!}$$

- pdf of S_n

$$f_{S_n}(t) = \frac{dF_{S_n}(t)}{dt} = \sum_{k=n}^{\infty} \left(-\lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \right)$$

$$= -\lambda e^{-\lambda t} \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t} \sum_{k=n}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} =$$

We have

$$\sum_{k=n}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} = \sum_{k=n-1}^{\infty} \frac{(\lambda t)^{k}}{k!} = \frac{(\lambda t)^{n-1}}{(n-1)!} + \sum_{k=n}^{\infty} \frac{(\lambda t)^{k}}{k!}$$

So

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Example

Consider a Poisson process with rate $\lambda = 1$. Compute

1. E(time of the 10'th event),
2. P(the 10th event occurs 2 or more time units after the 9th event),
3. P(the 10th event occurs later than time 20)

Solution

1.

$$S_{10} = X_1 + \cdots + X_{10}$$

where $X_i \hookrightarrow Exp(\lambda)$
We have

$$E(X_i) = \int_0^\infty x\lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

So

$$E(S_{10}) = E(X_1) + \cdots + E(X_{10}) = \frac{10}{\lambda} = 10$$

2.

$$P(S_{10} - S_9 \geq 2) = P(X_{10} > 2) = \int_2^\infty \lambda e^{-\lambda x dx} = e^{-2\lambda} = e^{-2}$$

3.

$$P(S_{10} > 20) = \int_{20}^\infty f_{S_{10}}(t)dt = \int_{20}^\infty \lambda e^{-\lambda t} \frac{(\lambda t)^{10-1}}{(10-1)!} dt$$

Example

LetN_t be a Poisson process with intensity $\lambda = 2$, and let $X_1, X_2, \ldots$ be the corresponding inter-arrival times.

1. Find the probability that the first arrival occurs after $t = 0.5$, i.e., $P(X_1 > 0.5)$.
2. Given that we have had no arrivals before $t = 1$, find $P(X_1 > 3)$.
3. Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrival occurs after $t = 4$.
4. I start watching the process at time $t = 10$. Let T be the time of the first arrival that I see. In other words, T is the first arrival after $t = 10$. Find $E(T)$ and $Var(T)$.
5. I start watching the process at time $t = 10$. Let T be the time of the first arrival that I see. Find the conditional expectation and the conditional variance of T given that I am informed that the last arrival occurred at time $t = 9$.

Order statistic

Let $X_1, X_2, ..., X_n$ be rv then $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are the order statistics corresponding to $X_1, X_2, ..., X_n$ if $X_{(k)}$ is the k-smallest value among $X_1, X_2, ..., X_n$.

Property

If $U_1, U_2, ..., U_n$ are i.i.d uniformly distributed $U([0, t])$ then the joint pdf of $U_{(1)}, U_{(2)}, ..., U_{(n)}$ is

$$f(x_1, ..., x_n) = \frac{t^n}{n!}$$

for $0 \leq x_1 \leq x_2 ... \leq x_n \leq t$

Proof

- joint pdf of $U_1, U_2, ..., U_n$ is $\frac{1}{t^n}$
- $U_{(1)}, U_{(2)}, ..., U_{(n)}$ is equal to $(x_1, x_2 ..., x_n)$ when $U_1, U_2, ..., U_n$ is equal to any of $n!$ permutation of $(x_1, x_2 ..., x_n)$.
- the joint pdf of $U_{(1)}, U_{(2)}, ..., U_{(n)}$ is

$$f(x_1, ..., x_n) = \frac{t^n}{n!}$$

Conditional distribution of arrival times

Given that $N_t = n$, the n arrival times $S_1, ..., S_n$ have the same distribution as order statitics corresponding to n independent random variables uniformly distributed on $[0, t]$.

Proof

Let $0 < t_1 < t_2 < ... < t_n < t_{n+1} = t$ and h_i be small enough such $t_i + h_i < t_{i+1}$

$$P(t_i \leq S_i \leq t_i + h_i, i = 1, ..., n | N_t = n)$$
$$= \frac{P(t_i \leq S_i \leq t_i + h_i, i = 1, ..., n, N_t = n)}{P(N_t = n)}$$
$$= \frac{P(\text{exact 1 event in } [t_i, t_i + h_i]), i = 1, ..., n, \text{ no event elsewhere in } [0,}{e^{-\lambda t}\frac{(\lambda t)^n}{n!}}$$
$$= \frac{\lambda h_1 e^{-\lambda h_1} ... \lambda h_2 e^{-\lambda h_2} ... \lambda h_n e^{-\lambda h_n} e^{-\lambda(t - h_1 - ... - h_n)}}{e^{-\lambda t}\frac{(\lambda t)^n}{n!}}$$
$$= \frac{n!}{t^n} h_1 h_2 ... h_n$$

Hence

$$f(t_1, ..., t_n) = \lim_{h_1, ..., h_n \to 0} \frac{P(t_i \leq S_i \leq t_i + h_i, i = 1, ..., n | N_t = n)}{h_1 ... h_n} = \frac{n!}{t^n}$$

Construction by conditional distribution of arrival times

- Determine the number of arrivals N_t which is Poisson distributed $Poiss(\lambda t)$
- N_t i.i.d $Uni([0, t])$: $U_1, U_2, \dots, U_{N_t}$
- Sort U_i to obtain arrival time $S_1, S_2, \dots$

Example

Let N_t be a Poisson process with rate $\lambda = 2$ with arrival time $S_1, S_2, \ldots$. Find

$$E(S_1 + S_2 + \cdots + S_{10} | N_4 = 10)$$

Example

Let N_t be a Poisson process with rate $\lambda = 2$ with arrival time $S_1, S_2, \dots$. Find

$$E(S_1 + S_2 + \cdots + S_{10} | N_4 = 10)$$

Solution

- Given $N_4 = 10$, $(S_1, \dots, S)10)$ has the same joint distribution as $(U_{(1)}, U_{(2)}, \dots U_{(10)})$ where U_i are i.i.d $Uni(0, 4)$
- $E(S_1 + S_2 + \cdots + S_{10} | N_4 = 10) = E(U_{(1)} + \cdots + U_{(10)})$
- $U_{(1)} + \dots U_{(10)} = U_1 + \dots U_{10}$
- $E(S_1 + S_2 + \cdots + S_{10} | N_4 = 10) = E(U_1 + \cdots + U_{10}) = E(U_1) + \cdots + E(U_{10})$
- $U_i \hookrightarrow Uni([0, 4]) \Rightarrow E(U_i) = \frac{0+4}{2} = 2$
- $E(S_1 + S_2 + \cdots + S_{10} | N_4 = 10) = 10 \times 2 = 20$

Example

Suppose that travelers arrive at a train depot accordance with a Poisson process with rate λ. If the train departs at time t, compute the expected sum of the waiting times of travelers arriving in $(0, t)$

$$E\left(\sum_{i=1}^{N_t}(t - S_i)\right)$$

Solution

Using property $E\left(\sum_{i=1}^{N_t}(t-S_i)\right) = E\left[E\left(\sum_{i=1}^{N_t}(t-S_i)|N_t\right)\right]$

Find $E\left(\sum_{i=1}^{N_t}(t-S_i)|N_t\right)$

- $$E\left(\sum_{i=1}^{N_t}(t-S_i)|N_t=n\right) = E\left(\sum_{i=1}^{n}(t-S_i|N_t=n)\right)$$
$$= E\left(\sum_{i=1}^{n}(t-U_{(i)})\right) \text{ where } U_i \hookrightarrow U([0,t])$$
$$= E\left(\sum_{i=1}^{n}(t-U_i)\right) = nE(t-U_1) = n(t-t/2) = nt/2$$

- $E\left(\sum_{i=1}^{N_t}(t-S_i)|N_t\right) = \dfrac{tN_t}{2}$

Hence

$$E\left(\sum_{i=1}^{N_t}(t - S_i)\right) = E\left(\frac{tN_t}{2}\right) = \frac{t}{2}E(N_t) = \frac{t}{2}(\lambda t) = \frac{\lambda t^2}{2}$$

Table of Contents

Compound Poisson Processes

Let $W_1, W_2, \ldots$ be a sequence of i.i.d rv with cdf F and independent of a Poisson process $(N_t)_{t \geq 0}$ with rate λ then the process $(R_t)_{t \geq 0}$ with

$$R_t = \sum_{i=1}^{N_t} W_i$$

is called by a compound Poisson process.

Example

Suppose that health claims are filed with a health insurer at the Poisson rate per day, and that the independent severities W of each claim are exponential random variables . Then the aggregate R of claims is a compound Poisson process.

Properties of compound Poisson processes

1. $E(R_t) = \lambda t E(W)$ **(Tower property)**
2. $Var(R_t) = \lambda t E(W^2)$

Proof

Using property

$$E(X) = E(E(X|Y))$$

for $Y = N_t$

1. Compute $E(R_t|N_t)$

$$E(R_t|N_t = n) = E\left(\sum_{i=1}^{N_t} W_i\right) = E\left(\sum_{i=1}^{N_t} W_i | N_t = n\right)$$

$$= E\left(\underbrace{\sum_{i=1}^{n}}_{\text{substitute } N_t \text{ by } n} \overbrace{W_i}^{\text{independent of } N_t} |N_t = n\right)$$

$$
\begin{aligned}
E(R_t|N_t = n) &= E\left(\sum_{i=1}^{n} W_i\right) \\
&= \sum_{i=1}^{n} \underbrace{E(W_i)}_{=E(W)} = nE(W)
\end{aligned}
$$

So

$$E(R_t|N_t) = N_t E(W)$$

Hence

$$E(R_t) = E(E(R_t|N_t)) = E(N_t \underbrace{E(W}_{constant})) = E(N_t)E(W) = \lambda t E(W)$$

2.

$$Var(R_t) = E(R_t^2) - (E(R_t))^2 = E(R_t^2) - \lambda^2 t^2 (E(W))^2$$

Compute $E(R_t^2|N_t)$

$$\begin{aligned} E(R_t^2|N_t = n) &= E\left[\left(\sum_{i=1}^{N_t} W_i\right)^2 |N_t = n\right] \\ &= E\left[\left(\sum_{i=1}^{n} W_i\right)^2 |N_t = n\right] \\ &= E\left(\sum_{i=1}^{n} W_i\right)^2 \end{aligned}$$

We have

$$\left(\sum_{i=1}^{n} W_i\right)^2 = \sum_{i=1}^{n} W_i^2 + \sum_{i \neq j, i,j=1}^{n} W_i W_j$$

So

$$\begin{aligned} E\left(\sum_{i=1}^{n} W_i\right)^2 &= \sum_{i=1}^{n} E(W_i^2) + \sum_{i \neq j, i,j=1}^{n} E(W_i W_j) \\ &= \sum_{i=1}^{n} E(W_i^2) + \sum_{i \neq j, i,j=1}^{n} E(W_i) E(W_j) \\ &= nE(W^2) + n(n-1)(E(W))^2 \end{aligned}$$

Hence $E(R_t^2 | N_t = n) = nE(W^2) + n(n-1)(E(W))^2$ then $E(R_t^2 | N_t) = N_t E(W^2) + N_t (N_t - 1)(E(W))^2$

$$E(R_t^2) = E(E(R_t|N_t)) = E(N_t)E(W^2) + E(N_t(N_t-1))(E(W))^2$$

Because $N_t \hookrightarrow Poiss(\lambda t)$,

$$E(N_t) = \lambda t, \qquad\qquad Var(N_t) = E(N_t^2) - (E(N_t))^2 = \lambda t$$

we have $E(N_t^2) = Var(N_t) + (E(N_t))^2 = \lambda t + (\lambda t)^2$ and then

$$\begin{aligned}
&EN_t(N_t-1) = E(N_t^2) - E(N_t) = \lambda t + (\lambda t)^2 - \lambda t = (\lambda t)^2 \\
&\Rightarrow E(R_t^2) = \lambda t E(W^2) + (\lambda t)^2 (E(W))^2 \\
&\Rightarrow Var(R_t) = E(R_t^2) - \lambda^2 t^2 (E(W))^2 = \lambda t E(W^2)
\end{aligned}$$

Example

Consider the compound Poisson process modeling aggregate health claims; frequency N is a Poisson process with rate $\lambda = 20$ per day and severity W is an Exponential random variable with mean $\theta = 500$. Suppose that you are interested in the aggregate claims S_{10} during the first 10 days.

1. Find $E(R_{10})$
2. Find $Var(R_{10})$

Solution

1. $E(R_{10}) = E(N_{10})E(W) = (20 \times 10)(500) = 100,000$
   because $N_{10} \hookrightarrow Pois(\lambda t) = Pois(20 \times 10)$
2. $Var(R_{10}) = E(N_{10})E(W^2) = 200 \times (500^2) = 100,000,000$

An application of compound Poisson process in insurance: Cramer-Lundberg model

In insurance, compound Poisson process is used to model total claim amount on $[0, t]$. If premium arrives with rate c then the insurer's surplus level with inital surplus x is

$$U_t = x + ct - \sum_{i=1}^{N_t} W_i$$

A central object is to find the ruin probability that the insurer's surplus falls below 0 (firm bankrupts)

Table of Contents

Simulation practice

1. Simulate a path of Poisson process with rate $\lambda = 2$ on interval time $[0, 10]$ by simulating inter-arrival time
2. Simulate a path of Poisson process with rate $\lambda = 2$ on interval time $[0, 10]$ by simulating number of event N_t first and then arrival times (using conditional distribution of arrival times)
3. Simulate a path of insurance surplus on $[0, 10]$ with
   - $(N_t)_t$ is a poisson process with rate $\lambda = 2$
   - Claim size $W_k \hookrightarrow Exp(1)$
4. Estimate ruin probability of the previous problem on finite horizon time $[0, 10]$ with $c = 1$, $x = 10$, $(N_t)_t$ is a poisson process with rate $\lambda = 2$, claim size $W_k \hookrightarrow Exp(1)$
5. Which value of c should be to guarantee that the ruin probability over horizon time $[0, 10]$ is less or equal to 10^{-3}. Use set up as the previous as (except value of c)

Practice

Consider the compound Poisson process modeling aggregate health claims; frequency N is a Poisson process with rate $\lambda = 20$ per day and severity W is an Exponential random variable with mean $\theta = 500$. Simulate 10000 scenarios for the aggregate claims S_{10} during the first 10 days.

1. Estimate $E(R_{10})$ and $Var(R_{10})$ from simulated sample.
2. Plot histogram for simulated sample of R_{10}. What can you say about the distribution of R_{10}.
3. Propose an approximation or estimation for $P(R_{10} > 120,000)$.