THE INTERNATIONAL UNIVERSITY(IU) - VIETNAM NATIONAL UNIVERSITY HCMC

FINAL EXAMINATION
January, 2015
Duration: 120 minutes

SUBJECT:	**ANALYSIS II**
Vice Chair of Department of Mathematics Signature: Assoc. Prof. Dr. Pham Huu Anh Ngoc	Lecturers: Signature: Dr. Nguyen Ngoc Hai

INSTRUCTIONS:

- *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1. (a) *(10 marks)* Find the volume of the solid generated when the region $\{(x, y) : \ 0 \le x \le 2, \ 0 \le y \le x^2\}$ is rotated about the line $y = -1$.

(b) *(10 marks)* Find the length of the curve

$$y = \int_{-2}^{x} \sqrt{3t^4 - 1}\, dt, \quad -2 \le x \le -1.$$

Question 2. *(20 marks)* Find the following limits:

(a) $\displaystyle \lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}}$. (b) $\displaystyle \lim_{n\to\infty} \left(\sqrt{n^2 + 3n} - n\right)$.

Question 3. *(20 marks)* Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\displaystyle \sum_{k=1}^{\infty} \frac{(-2)^k}{3^{k+1}}$ (b) $\displaystyle \sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2 + 4}}$.

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Question 4. (a) *(15 marks)* Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n}}.$$

(b) *(5 marks)* Evaluate

$$\sum_{n=0}^{\infty} \frac{n+1}{3^n}.$$

Question 5. (a) *(10 marks)* Find a vector orthogonal to the two vectors $\mathbf{u} = -\mathbf{i} + 6\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$.

(b) *(10 marks)* Find an equation of the line through $(1, -3, 4)$ that is parallel to the line

$$\mathbf{r}(t) = \langle 3 + 4t, 5 - t, 7 \rangle.$$

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SOLUTIONS OF FINAL EXAMINATION
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Question 1. (a) *(10 marks)* Find the volume of the solid generated when the region $\{(x,y) : 0 \le x \le 2,\ 0 \le y \le x^2\}$ is rotated about the line $y = -1$.

(b) *(10 marks)* Find the length of the curve $y = \int_{-2}^{x} \sqrt{3t^4 - 1}\, dt, \quad -2 \le x \le -1$.

Solution (a) At each x in the interval $0 \le x \le 2$, the cross section of the solid is a washer with outer radius $x^2 + 1$ and inner radius 1. Hence the volume of the solid is

$$V = \pi \int_0^2 \big[(x^2+1)^2 - 1\big] dx \qquad [5\ marks]$$

$$= \pi \int_0^2 (x^4 + 2x^2) dx = \pi \left(\frac{1}{5}x^5 + \frac{2}{3}x^3 \Big|_0^2 \right) = \frac{176}{15}\pi. \qquad [5\ marks]$$

(b) By the FTC, $y'(x) = \sqrt{3x^4 - 1},\ -2 \le x \le -1$. Thus the length of the curve is

$$L = \int_{-2}^{-1} \sqrt{1 + y'(x)^2}\, dx = \int_{-2}^{-1} \sqrt{1 + 3x^4 - 1}\, dx \qquad [5\ marks]$$

$$= \int_{-2}^{-1} \sqrt{3} x^2 dx = \frac{\sqrt{3}}{3} x^3 \Big|_{-2}^{-1} = \frac{7\sqrt{3}}{3}. \qquad [5\ marks]$$

Question 2. *(20 marks)* Find the following limits:

(a) $\displaystyle \lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}}$. (b) $\displaystyle \lim_{n\to\infty} \big(\sqrt{n^2 + 3n} - n\big)$.

Solution Consider the function $f(x) = \ln(x+1)/\sqrt{x}$ on the interval $[1, \infty)$. Applying L'Hôspital's Rule we get

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{2\sqrt{x}}} \qquad [4\ marks]$$

$$= \lim_{x\to\infty} \frac{2\sqrt{x}}{x+1} = \lim_{x\to\infty} \frac{2/\sqrt{x}}{1 + 1/x} = 0. \qquad [4\ marks]$$

Thus, *[2 marks]*

$$\lim_{n\to\infty}\frac{\ln(n+1)}{\sqrt{n}} = \lim_{n\to\infty} f(n) = 0.$$

(b)

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n^2+3n}-n\right) &= \lim_{n\to\infty}\frac{(n^2+3n)-n^2}{\sqrt{n^2+3n}+n} && [5\ marks]\\ &= \lim_{n\to\infty}\frac{3}{\sqrt{1+3/n}+1} = \frac{3}{2}. && [5\ marks]\end{aligned}$$

Question 3. *(20 marks)* Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{(-2)^k}{3^{k+1}}$ (b) $\displaystyle\sum_{k=1}^{\infty}\frac{k}{\sqrt{k^2+4}}$.

Solution (a) The series is geometric with the common ratio $-2/3$, so it converges *[4 marks]*. Since

$$\left|\frac{-2}{3}\right| = \frac{2}{3} < 1,$$

we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-2)^k}{3^{k+1}} &= \sum_{k=1}^{\infty}\frac{1}{3}\cdot\frac{(-2)^k}{3^k} = \frac{1}{3}\sum_{k=1}^{\infty}\frac{(-2)^k}{3^k} = \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{-2}{3}\right)^k && [4\ marks]\\ &= \frac{1}{3}\cdot\frac{-2/3}{1-(-2/3)} = \frac{1}{3}\cdot\left(-\frac{2}{5}\right) = -\frac{2}{15}. && [2\ marks]\end{aligned}$$

Thus the series converges and its sum is $-\frac{2}{15}$.

OR

$$\sum_{k=1}^{\infty}\frac{(-2)^k}{3^{k+1}} = \frac{-2/3^2}{1-(-2/3)} = \frac{-2/3^2}{\frac{5}{3}} = -\frac{2}{15}.$$

(b) *[10 marks]* Since

$$\lim_{k\to\infty}\frac{k}{\sqrt{k^2+4}} = \frac{1}{\sqrt{1+\frac{4}{k^2}}} = 1 \neq 0, \qquad [7\ marks]$$

the series $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$ diverges by the Divergence Test *[3 marks].*

Question 4. (a) *(15 marks)* Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n}}.$$

(b) *(5 marks)* Evaluate

$$\sum_{n=0}^{\infty} \frac{n+1}{3^n}.$$

Solution (a) Let $a_n = \frac{(x-2)^n}{\sqrt{n}}$. Using the Ratio Test we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{\sqrt{n+1}}}{\frac{(x-2)^n}{\sqrt{n}}} \right| \qquad [4\ marks]$$

$$= \lim_{n\to\infty} |x-2| \frac{\sqrt{n}}{\sqrt{n+1}} = |x-2|. \qquad [4\ marks]$$

The series converges if $|x-2| < 1$ and diverges if $|x-2| > 1$. Hence radius of convergence of the series is $r = 1$ *[3 marks].*

If $x - 2 = -1$, i.e., if $x = 1$, then the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which is an alternating harmonic series and hence, converges *[2 marks].*

If $x - 2 = 1$, i.e., if $x = 3$ then the series becomes $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a p-series with $p = 1/2$ and so it diverges. Therefore, the interval of convergence of the series is $[1, 3)$ *[2 marks].*

(b) We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots, \qquad -1 < x < 1.$$

Differentiating both sides of this equation gives *[3 marks]*

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \sum_{n=0}^{\infty} (n+1)x^n, \quad -1 < x < 1.$$

Putting $x = \frac{1}{3}$ gives *[2 marks]*

$$\sum_{n=0}^{\infty} \frac{n+1}{3^n} = \sum_{n=0}^{\infty} (n+1) \left(\frac{1}{3}\right)^n = \frac{1}{(1-1/3)^2} = \frac{9}{4}.$$

Question 5. (a) *(10 marks)* Find a vector orthogonal to the two vectors $\mathbf{u} = -\mathbf{i} + 6\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$.

(b) *(10 marks)* Find an equation of the line through $(1, -3, 4)$ that is parallel to the line

$$\mathbf{r}(t) = \langle 3 + 4t, 5 - t, 7 \rangle.$$

Solution (a) A vector orthogonal to the two vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 6 \\ 2 & -5 & -3 \end{vmatrix} \qquad [5\ marks]$$

$$= (0 + 30)\mathbf{i} - (3 - 12)\mathbf{j} + (5 - 0)\mathbf{k} = 30\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}. \qquad [5\ marks]$$

(b) Let L be the line through $(1, -3, 4)$ and parallel to the line $\mathbf{r}(t) = \langle 3 + 4t, 5 - t, 7 \rangle$. Then L is parallel to the vector $\mathbf{v} = \langle 4, -1, 0 \rangle$ *[5 marks]*. Hence L has a vector equation

$$\mathbf{r}_1(t) = \langle 1, -3, 4 \rangle + t\mathbf{v} = \langle 1, -3, 4 \rangle + \langle 4t, -t, 0 \rangle = \langle 1 + 4t, -3 - t, 4 \rangle.$$

Parametric equations of the line are

$$x = 1 + 4t, \quad y = -3 - t, \quad z = 4. \qquad [5\ marks]$$