REAL ANALYSIS

Chapter 2 MEASURES

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References

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2. H. L. Royden, P. M. Fitzpatrick, *Real Analysis*, 4th ed. Pearson Education, 2010 (**pp. 337–358; 424–429**)

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Definition 1.1

Let X be an arbitrary nonempty set. A collection $\mathcal{M}$ of subsets of X is called an **algebra** (or a **field**) if it satisfies the following conditions:

(i) $X \in \mathcal{M}$,

(ii) $A \in \mathcal{M} \Longrightarrow A^c \in \mathcal{M}$,

(iii) $A, B \in \mathcal{M} \Longrightarrow A \cup B \in \mathcal{M}$.

In other words, an algebra of sets in X is a nonempty collection of subsets of X that is closed under complements and finite unions.

Proposition 1.1

If $\mathcal{M}$ is an algebra of subsets of X, then

(a) $\emptyset \in \mathcal{M}$;

(b) $A_1, \dots, A_n \in \mathcal{M} \Longrightarrow \cup_{i=1}^{n} A_i \in \mathcal{M}$;

(c) $A_1, \dots, A_n \in \mathcal{M} \Longrightarrow \cap_{i=1}^{n} A_i \in \mathcal{M}$;

(d) $A, B \in \mathcal{M} \Longrightarrow A \setminus B \in \mathcal{M}$;

(e) $A, B \in \mathcal{M} \Longrightarrow A \Delta B \in \mathcal{M}$.

In words,

- If $\mathcal{M}$ is an algebra then $\emptyset \in \mathcal{M}$.
- An algebra is closed under finite unions.
- An algebra is closed under finite intersections.
- An algebra is closed under set difference.
- An algebra is closed under symmetric difference.

σ-Algebras

Definition 1.2

A system $\mathcal{M}$ of subsets of a nonempty set X is called a **σ-algebra** or **σ-field** (over X) if it has the following properties:

(i) $X \in \mathcal{M}$;

(ii) $A \in \mathcal{M} \Longrightarrow A^c \in \mathcal{M}$;

(iii') $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{M} \Longrightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{M}$.

In other words, a σ-algebra of sets in X is a nonempty collection of subsets of X that is closed under complements and countable unions.

Remark 1.1

(a) A σ-algebra is an algebra. Thus if $\mathcal{M}$ is a σ-algebra, then $\emptyset \in \mathcal{M}$.

(b) σ-algebras are also closed under countable intersections.

Example 1.1 In any set X, $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are always σ-algebras.

- $\{\emptyset, X\}$ is the smallest possible σ-algebras and
- $\mathcal{P}(X)$ is the largest possible σ-algebras.

That is, if $\mathcal{M}$ is a σ-algebra of subsets of X, then

$$\{\emptyset, X\} \subset \mathcal{M} \subset \mathcal{P}(X).$$

Example 1.2 If $A \subset X$, then

$$\mathcal{M} = \{\emptyset, A, A^c, X\}$$

is a σ-algebra in X.

Example 1.3 If $\mathcal{M}$ is a σ-algebra in a set X and Y is a nonempty subset of X, then

$$\mathcal{A} = \{Y \cap A : A \in \mathcal{M}\}$$

is a σ-algebra in Y.

In case $Y \in \mathcal{M}$, $\mathcal{A}$ consists simply of all the subsets of Y which are elements of $\mathcal{M}$:

$$Y \in \mathcal{M} \Longrightarrow \mathcal{A} = \{B \in \mathcal{M} : B \subset Y\}.$$

Example 1.4 For any set X the system of all its subsets which are either countable or co-countable (that is, the $A \subset X$ such that either A or A^c is countable) constitute a σ-algebra.

Example 1.5 For any set X the system of all sets $A \subset X$ which are either finite or co-finite (i.e., have finite complement in X) is an algebra, but is a σ-algebra only if X is finite.

Example 1.6 Let X, Y be nonempty sets and $f : X \to Y$ a mapping. Further let $\mathcal{A}$ and $\mathcal{B}$ be σ-algebras in X and Y, respectively. Then the systems of sets

$$\mathcal{M} = f^{-1}(\mathcal{B}) := \left\{f^{-1}(B) : B \in \mathcal{B}\right\}$$

and

$$\mathcal{N} = \left\{B \subset Y : f^{-1}(B) \in \mathcal{A}\right\}$$

are respectively σ-algebras over X and Y.
One says $f^{-1}(\mathcal{B})$ is the **inverse image of $\mathcal{B}$ under f**. Thus,

The inverse image of a σ-algebra is a σ-algebra.

The Borel σ-algebra

Theorem 1.2

The intersection $\bigcap_{i\in I}\mathcal{M}_i$ of any family $(\mathcal{M}_i)_{i\in I}$ of σ-algebras on a common set X is itself a σ-algebra on X.

For every nonempty system $\mathcal{E}$ of subsets of X, there is at least one σ-algebra of subsets of X containing $\mathcal{E}$, namely $\mathcal{P}(X)$.

Let

$$\sigma(\mathcal{E}) = \bigcap \{\mathcal{M} \subset \mathcal{P}(X) : \mathcal{M} \supset \mathcal{E}, \ \mathcal{M} \text{ is a } \sigma\text{-algebra in } X\}.$$

Then, by Theorem 1.1, $\sigma(\mathcal{E})$ is a σ-algebra in X and it is the smallest σ-algebra containing $\mathcal{E}$.

$\sigma(\mathcal{E})$ is called the **$\boldsymbol{\sigma}$-algebra generated by $\boldsymbol{\mathcal{E}}$**.

$\sigma(\mathcal{E})$ is a σ-algebra with the following properties:

(i) $\mathcal{E} \subset \sigma(\mathcal{E})$;

(ii) for every σ-algebra $\mathcal{M}$ in X with $\mathcal{E} \subset \mathcal{M}$, we have $\sigma(\mathcal{E}) \subset \mathcal{M}$.

Remark 1.2

If $\mathcal{E} \subset \sigma(\mathcal{F})$ then $\sigma(\mathcal{E}) \subset \sigma(\mathcal{F})$.

Example 1.7

(a) If $\mathcal{E}$ itself is a σ-algebra in X, then $\sigma(\mathcal{E}) = \mathcal{E}$.

(b) If $\mathcal{E}$ consists of a single set $A \subset X$, then

$$\sigma(\mathcal{E}) = \{\emptyset, A, A^c, X\}.$$

(c) The σ-algebra in Example 1.4 ($\mathcal{M}$ consists of all countable subsets of X and their complements) is generated by the system of all finite subsets of X.

Definition 1.3

If X is any metric space, the σ-algebra generated by the family of open sets in X is called the **Borel σ-algebra** on X and is denoted by $\mathcal{B}(X)$. Its members are called **Borel sets**.

Example 1.8 Let (X, d) be a metric space. The following are Borel sets:

(i) any open or closed set;

(ii) any one-point set $\{x\}$, $x \in X$;

(iii) any countable set.

(iv) If $X = \mathbb{R}$ with the usual metric

$$d(x, y) = |x - y|,$$

then all intervals (open, closed, semiclosed, finite or infinite) are Borel sets.

Let (X, d) be a metric space. $\mathcal{B}(X)$ includes open sets, closed sets, countable intersections of open sets, countable unions of closed sets, and so forth.

It is clear that

$\mathcal{B}(X)$ is also generated by the class of all closed sets in X.

Example 1.9 The Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is generated by each of the following three systems:

$$\mathcal{E} = \big\{(a, b) : a < b,\ a, b \in \mathbb{R}\big\},$$
$$\mathcal{F} = \big\{(-\infty, b] : b \in \mathbb{R}\big\}, \quad \text{and}$$
$$\mathcal{G} = \big\{(-\infty, b) : b \in \mathbb{R}\big\}.$$

Definition 2.1

By a **measurable space** we mean a couple $(X, \mathcal{M})$ consisting of a set X and a σ-algebra $\mathcal{M}$ of subsets of X. A subset E of X is called **measurable** (or measurable with respect to $\mathcal{M}$) provided E belongs to $\mathcal{M}$.

♠

Definition 2.2

Let X be a set equipped with a σ-algebra $\mathcal{M}$. A **measure** on $\mathcal{M}$ (or on $(X, \mathcal{M})$, or simply on X if $\mathcal{M}$ is understood) is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$;

(ii) if $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Property (ii) is called **countable additivity** (or **σ-additivity**).

It implies **finite additivity**:

(ii') if $\{E_i\}_{i=1}^n$ are *disjoint* sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i).$$

Definition 2.3

If μ is a measure on a measurable space $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a **measure space**.

Example 2.1 For every σ-algebra $\mathcal{M}$ in X and every point $a \in X$ the function δ_a defined on $\mathcal{M}$ by

$$\delta_a(A) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{if } a \notin A, \end{cases}$$

is a measure. It is called the **unit point mass** or **Dirac measure** concentrated at $\boldsymbol{a}$.

Example 2.2 Let X be an uncountable set, and let $\mathcal{M}$ be the σ-algebra of countable or co-countable sets, that is,

$$\mathcal{M} = \{E \subset X : E \text{ countable or } E^c \text{ countable}\}.$$

The function μ on $\mathcal{M}$ defined by

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ 1 & \text{if } E \text{ is co-countable,} \end{cases}$$

is a measure.

Note There exist non-empty sets which have measure 0.

Example 2.3 Let X be an arbitrary set, and let $\mathcal{M}$ be a σ-algebra on X. Define $\mu : \mathcal{M} \to [0, \infty]$ by

$$\mu(E) = \begin{cases} \text{the number of elements of } E \text{ if } E \text{ is finite,} \\ \infty \text{ if } E \text{ is infinite.} \end{cases}$$

Then μ is a measure, called the **counting measure** on $\mathcal{M}$.

Example 2.4 Let $(X, \mathcal{M}, \mu)$ be a measure space and take $\emptyset \neq A \in \mathcal{M}$. Let

$$\mathcal{M}_A = \{E \subset A : E \in \mathcal{M}\} \quad \text{and} \quad \mu_A(E) = \mu(E).$$

Then $(A, \mathcal{M}_A, \mu_A)$ is also a measure space.

Note that

$$\mathcal{M}_A = \{F \cap A : F \in \mathcal{M}\}.$$

Example 2.5 Let X be an infinite set and $\mathcal{M} = \mathcal{P}(X)$. Define

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is finite,} \\ \infty & \text{if } E \text{ is infinite,} \end{cases}$$

Then μ is finitely additive but not σ-additive.

Definition 2.4

Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) If $\mu(X) < \infty$, μ is called **finite**.

(b) If $X = \cup_{n=1}^{\infty} A_n$ where $A_n \in \mathcal{M}$ and $\mu(A_n) < \infty$ for all n, μ is called **σ-finite**.

(c) If $E = \cup_{n=1}^{\infty} E_n$ where $E_n \in \mathcal{M}$ and $\mu(E_n) < \infty$ for all n, the set E is said to be **σ-finite**.

(d) If $\mu(X) = 1$, we also call μ a **probability measure** and $(X, \mathcal{M}, \mu)$ a **probability space**.

Note In probability contexts, measurable sets are called **events**.

Example 2.6 Let $(X, \mathcal{M}, \mu)$ be a measure space and let c be a positive real number. Then

- $c\mu$ is a measure;
- in particular, if $0 < \mu(X) < \infty$, $\left(X, \mathcal{M}, \frac{1}{\mu(X)}\mu\right)$ is a probability space.

Example 2.7

(a) Let $\mathcal{M} = \{X, \emptyset\}$. Define $\mu(\emptyset) = 0$ and $\mu(X)$ be any number in $[0, \infty]$. Then μ is a measure on $\mathcal{M}$.

(b) Let A be a subset of X such that $\emptyset \neq A \neq X$ and let $\mathcal{M} = \{\emptyset, A, A^c, X\}$.

Then all probability measures on $(X, \mathcal{M})$ have the form

$$\mathbf{P}(\emptyset) = 0, \quad \mathbf{P}(A) = p, \quad \mathbf{P}(A^c) = 1-p, \quad \mathbf{P}(X) = 1,$$

where $0 \leq p \leq 1$.

Example 2.8 Let $X = \{x_1, x_2, \dots\}$ be a countable (finite or countably infinite) set, $x_k \neq x_m$ for $k \neq m$, and let $p_1, p_2, \dots$ be extended nonnegative numbers. For each $A \subset X$, define

$$\mu(A) = \sum_{x_k \in A} p_k.$$

(If $A = \emptyset$, we set $\mu(A) = 0$.) Then

(a) μ is a measure on $\mathcal{P}(X)$ and

$$\mu(\{x_k\}) = p_k, \quad k = 1, 2, \dots;$$

(b) μ is σ-finite if $p_k < \infty$ for each k;

(c) μ is a probability measure if $\sum_k p_k = 1$;

(d) μ is the counting measure if $p_k = 1$ for all k.

Example 2.9 (a) Let $X = \{x_1, x_2, \dots, x_N\}$ be a finite set of exactly N points, and let $\mathcal{M}$ be the power set of X, that is, $\mathcal{M} = \mathcal{P}(X)$. If we choose

$$p_k = \mu(\{x_k\}) = \frac{1}{N},$$

then the function μ defined in Example 2.8 reduces to

$$\mu(A) = \frac{\text{number of elements of } A}{N},$$

which is a probability measure on $(X, \mathcal{M})$.

- μ is called the **uniform probability**.

(b) If $X = \{0, 1, \dots, N\}$, $p \in (0, 1)$ is a fixed number, and

$$p_k = \binom{N}{k} p^k (1-p)^{N-k}, \qquad k = 0, 1, \dots, N,$$

we call μ (defined in Example 2.8) the **binomial probability** with parameters N, p.

(c) If $X = \{0, 1, 2, \ldots\} = \{0\} \cup \mathbb{N}$, $\lambda \in (0, \infty)$, and

$$p_k = e^{-\lambda} \frac{\lambda^k}{k!}, \qquad k = 0, 1, \ldots$$

we get the **Poisson probability** with parameter λ.

(d) If $X = \mathbb{N}$, p is a fixed number in $(0, 1)$, and

$$p_k = (1 - p)^{k-1} p, \qquad k = 1, 2 \ldots,$$

then μ is the **geometric probability**.

Theorem 2.1

Let $(X, \mathcal{M}, \mu)$ *be a measure space.*

(a) (Monotonicity) *If* $E, F \in \mathcal{M}$ *and* $E \subset F$, *then*

$$\mu(E) \leq \mu(F).$$

(b) (Excision) *If* $E \subset F$ *and* $\mu(E) < \infty$, *then*

$$\mu(F \setminus E) = \mu(F) - \mu(E).$$

In particular, if $\mu(E) = 0$, *then*

$$\mu(F \setminus E) = \mu(F).$$

(c) (σ-Subadditivity) *If* $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$, *then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Remark 2.1

1. If μ is finite, then $\mu(E) < \infty$ for all $E \in \mathcal{M}$.
2. σ-subadditivity implies **finite subadditivity**:
   If $\{E_n\}_{n=1}^k \subset \mathcal{M}$, then
   $$\mu\left(\bigcup_{n=1}^{k} E_n\right) \leq \sum_{n=1}^{k} \mu(E_n).$$

Theorem 2.2 (Continuity of Measure)

Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) (Continuity from below) *If* $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$ *and* $E_1 \subset E_2 \subset \cdots$, *then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

(b) (Continuity from above) *If* $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$, $E_1 \supset E_2 \supset \cdots$, *and* $\mu(E_1) < \infty$, *then*

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Let $\{A_n\}$ be a sequence of subsets of a set X.

- If
$$A_1 \subset A_2 \subset A_3 \subset \cdots \quad \text{and} \quad \cup_{n=1}^{\infty} A_n = A,$$
we say that the A_n form an **increasing** sequence of sets with limit A, or that the A_n **increase** to A and we write $A_n \nearrow A$.

- If
$$A_1 \supset A_2 \supset A_3 \supset \cdots \quad \text{and} \quad \cap_{n=1}^{\infty} A_n = A,$$
we say that the A_n form a **decreasing** sequence of sets with limit A, or that the A_n **decrease** to A and we write $A_n \searrow A$.

Example 2.10 (**The Borel-Cantelli Lemma**)
Let $\{E_k\}$ be a sequence of measurable sets in X, such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

Then

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 0.$$

Note Let $\{E_k\}$ be a sequence of sets in X. We define **limit superior** and **limit inferior** of $\{E_k\}$ to be the sets

$$\limsup_n E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

and

$$\liminf_n E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k,$$

respectively. It is easy to verify that

$$\limsup E_n = \Big\{x : x \in E_n \text{ for infinitely many } n\Big\},$$
$$\liminf E_n = \Big\{x : x \in E_n \text{ for all but finitely many } n\Big\}.$$

The Borel-Cantelli Lemma can be restated as follows:

> If $\sum_{k=1}^{\infty} \mu(E_k) < \infty$, then almost all $x \in X$ lie in at most finitely many of the sets E_k.

Definition 2.5

If $(X, \mathcal{M}, \mu)$ is a measure space, a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called a **null set**.

Remark 2.2

- Every measurable subset of a null set is also a null set.
- By σ-subadditivity, *any countable union of null sets is a null set*.

Definition 2.6

A measure whose domain includes all subsets of null sets is called **complete**.

$$\mu \text{ complete} \iff \Big(\big[\mu(A) = 0 \wedge (B \subset A)\big] \Longrightarrow B \in \mathcal{M}\Big).$$

Example 2.11

(a) Every measure μ on $\mathcal{P}(X)$ is complete.

(b) The counting measure (defined on any σ-algebra) is complete.

(c) If X has more than one point and $\mathcal{M} = \{\emptyset, X\}$, then the measure $\mu(\emptyset) = \mu(X) = 0$ is not complete.

- Let $\mu : \mathcal{E} \to \overline{\mathbb{R}}$ and $\nu : \mathcal{F} \to \overline{\mathbb{R}}$. If

$$\mathcal{E} \subset \mathcal{F} \quad \text{and} \quad \mu(E) = \nu(E) \quad \text{for all} \quad E \in \mathcal{E},$$

  ν is said to be an **extension** of μ to $\mathcal{F}$, or equivalently, μ is the **restriction** of ν to $\mathcal{E}$.

- Let $\mathcal{A}$ be an algebra on X and $\mu : \mathcal{A} \to [0, \infty]$ a σ-additive function on $\mathcal{A}$ with $\mu(\emptyset) = 0$.

Question:

Under what conditions does there exist a σ-algebra $\mathcal{M}$ on X and a measure $\bar{\mu}$ on $\mathcal{M}$ such that μ is the restriction of $\bar{\mu}$ to $\mathcal{A}$?

Definition 3.1

An **outer measure** on a nonempty set X is a set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies three properties:

1) $\mu^*(\emptyset) = 0$,
2) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$, that is, μ^* is monotone,
3) $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ holds for every sequence $\{A_n\}$ of subsets of X; that is, μ^* is **σ-subadditive**.

Remark 3.1 • Conditions 2) and 3) are equivalent to the following:

If $A, A_n \subset X$ and $A \subset \bigcup_{n=1}^{\infty} A_n$, then $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

• Every outer measure is finitely subadditive.

• Any measure on $\mathcal{P}(X)$ is also an outer measure on X.

Example 3.1 For $A \subset X$ set

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \neq \emptyset. \end{cases}$$

Then μ^* is an outer measure on X.

- Note that μ^* is *not* additive.

Theorem 3.1

Suppose $\mathcal{E}$ is a family of subsets of X with $\emptyset \in \mathcal{E}$ and $\mu : \mathcal{E} \to [0, \infty]$ satisfies $\mu(\emptyset) = 0$. For any $A \subset X$, define

$$\mu^*(A) = \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathcal{E} \text{ and } A \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$

If there is no sequence $\{A_n\}$ of $\mathcal{E}$ such that $A \subset \cup_{n=1}^{\infty} A_n$, then we let $\mu^*(A) = \infty$. Then μ^* is an outer measure, called the ***outer measure induced by $(\mathcal{E}, \mu)$***.

Definition 3.2

Let μ^* be an outer measure on X. A subset A of X is called **measurable** (more precisely, **μ^*-measurable**) if,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all } E \subset X.$$

Remark 3.2 • $\emptyset$ and X are measurable.

• If $\mu^*(N) = 0$, we say the set N is **μ^*-null** (or of **μ^*-measure zero**). Every μ^*-null set is measurable.

• A is measurable if and only if

$$\mu^*(E) \geq \mu^*(A \cap E) + \mu^*(A^c \cap E) \quad \text{for all } E \subset X.$$

Example 3.2 Let X be a set having more than one point, $\mathcal{M} = \{\emptyset, X\}$, $\mu(\emptyset) = 0$, $\mu(X) = 1$. Then μ is a measure on the σ-algebra $\mathcal{M}$.

(a) The outer measure μ^* induced by $(\mathcal{M}, \mu)$ is

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \neq \emptyset. \end{cases}$$

(b) The family of μ^*-measurable sets is also the σ-algebra $\mathcal{M}$.

Theorem 3.2 (Carathéodory's Theorem)

If μ^* is an outer measure on X, then the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

The measure $\bar{\mu}$ that is the restriction of μ^* to the σ-algebra $\mathcal{M}$ of μ^*-measurable sets is called the **measure induced by the outer measure μ^***.

$$\bar{\mu} = \mu^*_{|\mathcal{M}}$$

Definition 3.3

Let $\mathcal{M} \subset \mathcal{P}(X)$ be an algebra. A function $\mu : \mathcal{M} \to [0, \infty]$ will be called a **premeasure** if

(i) $\mu(\emptyset) = 0$;

(ii) μ is σ-additive, that is, if $\{A_n\}_{n=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{M}$ such that $\cup_{n=1}^{\infty} A_n \in \mathcal{M}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

A premeasure is finitely additive.

The notions of finite and σ-finite premeasures are defined just as for measures.

If μ is a premeasure on an algebra $\mathcal{M} \subset \mathcal{P}(X)$, it induces an outer measure on X in accordance with Theorem 3.1, namely,

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathcal{M},\ E \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$

A premeasure induces an outer measure and an outer measure induces a measure:

premeasure $\mu \Longrightarrow$ outer measure $\mu^* \Longrightarrow$ measure $\bar{\mu}$

Question:

What is the relation between μ and $\bar{\mu}$?

Theorem 3.3

Let μ be a premeasure on an algebra $\mathcal{M}$, let μ^* be the outer measure generated by μ, and let $\mathcal{M}_{\mu^*}$ be the σ-algebra of μ^*-measurable sets.

(i) *Every set in $\mathcal{M}$ is μ^*-measurable and the restriction of μ^* to $\mathcal{M}_{\mu^*}$ is an extension of μ:*

$$\mathcal{M} \subset \mathcal{M}_{\mu^*} \quad \text{and} \quad \mu^*_{|\mathcal{M}} = \mu.$$

(ii) *If μ is σ-finite and $\mathcal{A}$ is any σ-algebra with $\mathcal{M} \subset \mathcal{A} \subset \mathcal{M}_{\mu^*}$, then the restriction of μ^* to $\mathcal{A}$ is the only measure on $\mathcal{A}$ that is an extension of μ.*

If $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$, f is called

- **increasing** if $f(x) \leq f(y)$ whenever $x \leq y$;
- **strictly increasing** if $f(x) < f(y)$ whenever $x < y$;
- **decreasing** if $f(x) \geq f(y)$ whenever $x \leq y$;
- **strictly decreasing** if $f(x) > f(y)$ whenever $x < y$.

A function that is either increasing or decreasing is called **monotone**.

- Let f be a function on $(a, b) \subset \mathbb{R}$. For $c \in [a, b)$, the limit

$$f(c+) = \lim_{x \searrow c} f(x)$$

(provided it exists) is called the **right-hand limit** of f at c.

- Similarly, for $c \in (a, b]$ the limit

$$f(c-) = \lim_{x \nearrow c} f(x)$$

(provided it exists) is called the **left-hand limit** of f at c.

A function f is said to be **right continuous** at c if

$$f(c+) = f(c)$$

and **left continuous** at c if

$$f(c-) = f(c).$$

If $f(c-)$ and $f(c+)$ are both finite and f is discontinuous at c, then we say that f has a **simple discontinuity** at c.

Theorem 4.1

Let $f : (a, b) \to \mathbb{R}$ be increasing and let $c \in (a, b)$. Then

(a) $f(c-)$ *and* $f(c+)$ *both exist,*

(b) $f(c-) = \sup\{f(x) : a < x < c\}$,

(c) $f(c+) = \inf\{f(x) : c < x < b\}$,

(d) $-\infty < f(c-) \le f(c) \le f(c+) < \infty$,

(e) $a < c < d < b$ *implies* $f(c+) \le f(d-)$.

A similar theorem obtains for decreasing functions.

Note Let $f : \mathbb{R} \to \mathbb{R}$ be monotone.

(i) If $x_n < a \in \mathbb{R}$, $x_n \to a$, then

$$\lim_{n\to\infty} f(x_n) = f(a-).$$

(ii) If $x_n \to +\infty$, then $\lim_{n\to\infty} f(x_n) = f(+\infty)$.

(iii) If $x_n > a \in \mathbb{R}$, $x_n \to a$, then

$$\lim_{n\to\infty} f(x_n) = f(a+).$$

(iv) If $x_n \to -\infty$, then $\lim_{n\to\infty} f(x_n) = f(-\infty)$.

Here

$$f(+\infty) := \lim_{x\to\infty} f(x) \quad \text{and}$$

$$f(-\infty) := \lim_{x\to-\infty} f(x).$$

Theorem 4.2

The set of all discontinuities of a monotone function is countable and each discontinuity is simple.

If $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, then by Theorem 4.1, f has right- and left-hand limits at each point:

$$f(a+) = \lim_{x \to a^+} f(x) = \inf_{x > a} f(x),$$
$$f(a-) = \lim_{x \to a^-} f(x) = \sup_{x < a} f(x).$$

Moreover,

$$f(\infty) := \lim_{x \to +\infty} f(x) = \sup_{x \in \mathbb{R}} f(x)$$

and

$$f(-\infty) := \lim_{x \to -\infty} f(x) = \inf_{x \in \mathbb{R}} f(x)$$

(these values possibly equal $\pm\infty$).

Let

$$I_{a,b} = (a, b] \quad \text{if } -\infty \le a \le b < \infty$$
$$I_{a,b} = (a, \infty) \quad \text{if } -\infty \le a < b = \infty$$

(if $a = b \in \mathbb{R}$, $(a, b] = \emptyset$).

Theorem 4.3

Let $\mathcal{M}$ be the collection of finite disjoint unions of intervals in $\mathbb{R}$ of the form $I_{a,b}$. Then

(i) *$\mathcal{M}$ is an algebra;*

(ii) *The σ-algebra generated by $\mathcal{M}$ is $\mathcal{B}(\mathbb{R})$.*

Definition 4.1

Measures on $\mathbb{R}$ whose domain is the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ are called **Borel measures**.

Let μ be a finite Borel measure on $\mathbb{R}$.

Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \mu((-\infty, x]).$$

Then F is increasing, right continuous, and

$$\mu((a, b]) = F(b) - F(a), \quad \mu((a, \infty)) = F(\infty) - F(a).$$

We will show that:

> There is a $1-1$ correspondence between Borel measures on $\mathbb{R}$ (that are finite on every bounded Borel sets) and the class of increasing and right continuous functions.

Theorem 4.4

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. If I_{a_i,b_i} $(i = 1, 2, \dots, k)$ are disjoint, let

$$\mu_F\left(\bigcup_{i=1}^{k} I_{a_i,b_i}\right) = \sum_{i=1}^{k}\left[F(b_i) - F(a_i)\right].$$

Then μ_F is a premeasure on $\mathcal{M}$, the algebra constructed as in Theorem 4.3.

Let μ^* be the outer measure generated by μ_F.

Definition 4.2

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. The space $(\mathbb{R}, \mathcal{M}_{\mu^*}, \bar{\mu}_F)$, where $\bar{\mu}_F = \mu^*{}_{|\mathcal{M}_{\mu^*}}$, is called a **Lebesgue-Stieltjes measure space** and $\bar{\mu}_F$ is called the **Lebesgue-Stieltjes measure** generated by F.

Remark 4.1

$\mathcal{B}(\mathbb{R}) \subset \mathcal{M}_{\mu^*}$, that is, every Borel set in $\mathbb{R}$ is μ^*-measurable.

Definition 4.3

When $F(x) = x$ for all $x \in \mathbb{R}$, the measure $\bar{\mu}_F$ is called the **Lebesgue measure** and the σ-algebra $\mathcal{M}_{\mu^*}$ is called the class of **Lebesgue measurable sets**.

- The Lebesgue measure is denoted by $\boldsymbol{m}$ and the Lebesgue σ-algebra $\mathcal{M}_{\mu^*}$ is denoted by $\boldsymbol{\mathcal{L}}$.
- $\mathcal{L} \supset \mathcal{B}(\mathbb{R})$ and m is complete.
- If $a, b \in \mathbb{R}$, then

$$m\big([a,b]\big) = m\big((a,b]\big) = m\big([a,b)\big) = m\big((a,b)\big) = b - a, \ldots.$$

Theorem 4.5

If $F : \mathbb{R} \to \mathbb{R}$ is any increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for all a, b. If G is another such function, we have $\mu_F = \mu_G$ iff $F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((0,x]) & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -\mu((x,0]) & \text{if} \quad x < 0, \end{cases}$$

then F is increasing and right continuous, and $\mu = \mu_F$.

Note • The domain of the (complete) Lebesgue-Stieltjes measure $\bar{\mu}_F$ includes $\mathcal{B}(\mathbb{R})$ and the Borel measure μ_F equals the restriction of $\bar{\mu}_F$ to $\mathcal{B}(\mathbb{R})$.

• We also denote the complete measure $\bar{\mu}_F$ by μ_F.

Note If $\mu(\mathbb{R}) = 1$, then $\mu = \mu_F$ where

$$F(x) = \mu((-\infty, x]).$$

F is called the **cumulative distribution function of μ**; this differs from the F specified in Theorem 4.3 by $\mu((-\infty, 0])$.

Example 4.1 Show that if $F = \chi_{[c,\infty)}$, then $\mu_F = \delta_c$, the Dirac measure concentrated at c on $\mathcal{B}(\mathbb{R})$.

Let μ be a complete Lebesgue-Stieltjes measure on $\mathbb{R}$ associated to the increasing, right continuous function F.

Let $\mathcal{M}_\mu$ denote the domain of μ.

Theorem 4.6

If $E \in \mathcal{M}_\mu$, then

$$\begin{aligned}\mu(E) &= \inf\{\mu(U) : U \supset E \text{ and } U \text{ is open}\}\\ &= \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}.\end{aligned}$$

Generating the Borel σ-algebra with intervals

A subset J of $\mathbb{R}^n$ is called an **interval** in $\mathbb{R}^n$ if there are intervals $J_k \subset \mathbb{R}$ with $1 \leq k \leq n$ such that

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Theorem 5.1

Let $\mathcal{M}$ be the collection of finite disjoint unions of intervals in $\mathbb{R}^n$. Then

(i) *$\mathcal{M}$ is an algebra;*

(ii) *The σ-algebra generated by $\mathcal{M}$ is $\mathcal{B}(\mathbb{R}^n)$.*

The Lebesgue outer measure

Let $J = J_1 \times J_2 \times \cdots \times J_n$ be an interval in $\mathbb{R}^n$. The ***n*-dimensional volume** of the interval J in $\mathbb{R}^n$ is defined as

$$\text{vol}(J) := \prod_{k=1}^{n} l(J_k),$$

where $l(J_k)$ is the length of the interval J_k.

Denote by $\mathbb{J}(\boldsymbol{n})$ the collection of all intervals in $\mathbb{R}^n$.

Theorem 5.2

If $I_j \in \mathbb{J}(n)$ $(j = 1, 2, \ldots, k)$ are disjoint, let

$$\mu\left(\bigcup_{j=1}^{k} I_j\right) = \sum_{j=1}^{k} \mathrm{vol}(I_j)$$

and let $\mu(\emptyset) = 0$. Then μ is a premeasure on $\mathcal{M}$, the algebra constructed as in Theorem 5.1.

Definition 5.1

The collection of μ^*-measurable sets is denoted by $\mathcal{L}^n$ and called the **Lebesgue measurable** sets. The restriction of μ^* to $\mathcal{L}^n$ is called **Lebesgue measure** on $\mathcal{L}^n$ or ***n*-dimensional Lebesgue measure** and denoted by $\boldsymbol{m_n}$.

- When there is no danger of confusion, we shall usually omit the subscript n and write $\boldsymbol{m}$ for m_n.
- Every interval $J \in \mathbb{J}(n)$ is Lebesgue measurable and we have

$$m_n(J) = \text{vol}(J).$$

Corollary 5.3

The σ-algebra $\mathcal{L}^n$ of Lebesgue measurable subsets of $\mathbb{R}^n$ contains the Borel subsets of $\mathbb{R}^n$. Moreover, the measure space $(\mathbb{R}^n, \mathcal{L}^n, m_n)$ is both σ-finite and complete.

Theorem 5.4 (The Regularity of Lebesgue Measure)

Let E be a Lebesgue measurable subset of $\mathbb{R}^n$. Then

$$m_n(E) = \inf\left\{m_n(U) : E \subset U,\ U \text{ open}\right\}$$

and

$$m_n(E) = \sup\left\{m_n(K) : K \subset E,\ K \text{ compact}\right\}.$$