ANALYSIS 2

3. Sequences and Series
(Chapter 8)

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Contents

Section 1

Sequences

Sequences

A sequence is an infinite list of real numbers:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The sequence $a_1, a_2, a_3, \ldots$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Definition 1.1

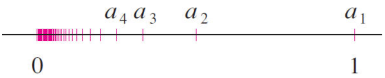
A **sequence** of numbers is a function whose domain is the set of positive integers.

For a sequence $a : \mathbb{N} \to \mathbb{R}$, it is customary to denote a_n instead of $a(n)$ for the value of the function at the number n. The number a_n is called the ***n*th term** of the sequence.

Sequences

A sequence can be pictured either by:

- Plotting its terms on a number line or
- Plotting its graph.



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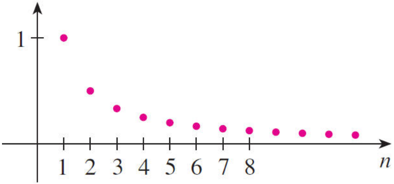


(b)

The sequence $\left\{\frac{1}{n}\right\}$.

Sequences

Descriptions of sequences

A sequence can be specified in three ways:

(i) List the first few terms followed by ... if the pattern is obvious, e.g. $1, 1/4, 1/9, \ldots$

(ii) Provide a formula for the general term a_n, e.g. $a_n = (-1)^n \sin n$.

(iii) Provide a formula for calculating the term a_n from previous terms $a_1, a_2, ..., a_{n-1}$, together with values of enough initial terms. E.g. the Fibonacci sequence $a_1 = a_2 = 1, a_n = a_{n-1} + a_{n-2}, n > 2.$

Example: The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ can be described as

- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
- $a_n = \frac{1}{n}, \quad n = 1, 2, \ldots.$
- $a_1 = 1,\ a_{n+1} = \frac{a_n}{a_n+1}.$
- $a_1 = 1,\ a_2 = \frac{1}{2},\ a_{n+1} = \frac{a_n a_{n-1}}{2a_{n-1} - a_n}.$

Section 2

Limits of Sequences

Limits of Sequences

Definition 1.3

- A sequence $\{a_n\}$ **converges** to the number L, and we write

$$\lim_{j\to\infty} a_j = L$$

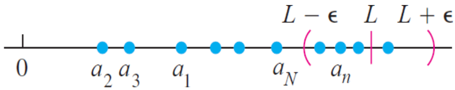
  or

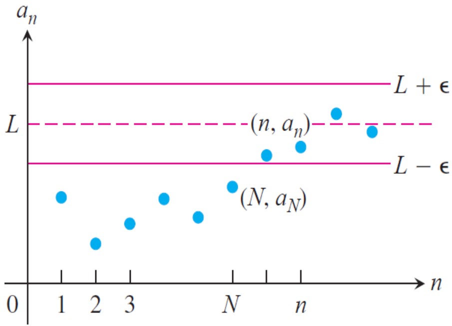
$$a_n \to L \text{ as } n \to \infty$$

  if for every $\epsilon > 0$ there is a corresponding integer N such that if $n > N$ then $|a_n - L| < \epsilon$. We call L the **limit** of the sequence.
- If $\lim_{j\to\infty} a_j$ exists, we say the sequence **converges** (or **is convergent**). Otherwise, it **diverges** (or **is divergent**).

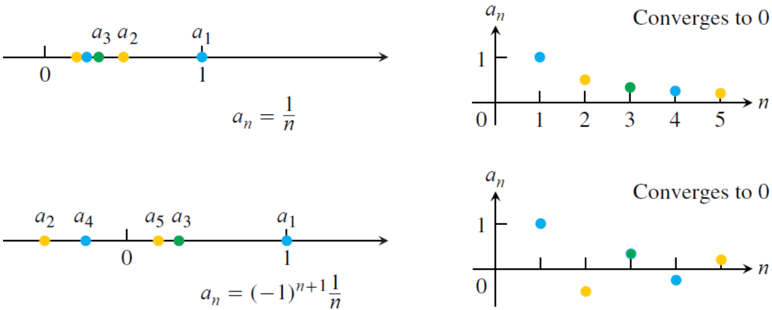
Remark: The limit of a sequence, if exists, is unique. It is determined by the behavior of the sequence at infinity, hence it does not change if we modify or drop finitely many terms of the sequence.

Limits of Sequences





Limits of Sequences



Limits of $\left\{\frac{1}{n}\right\}$ and $\left\{\frac{(-1)^n}{n}\right\}$.

Examples

1. If $\alpha > 0$ then
$$\lim_{n\to\infty} \frac{1}{n^\alpha} = 0.$$
2. Evaluate
$$\lim_{n\to\infty} n \tan^{-1}\left(\frac{1}{n}\right).$$
3. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.
4. For what values of r is the sequence $\{r^n\}$ convergent? (Answer: $-1 < r \leq 1$)

Properties of Limit

Theorem 2.3

Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences (to finite limits) and c is a constant. Then,

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_n$$

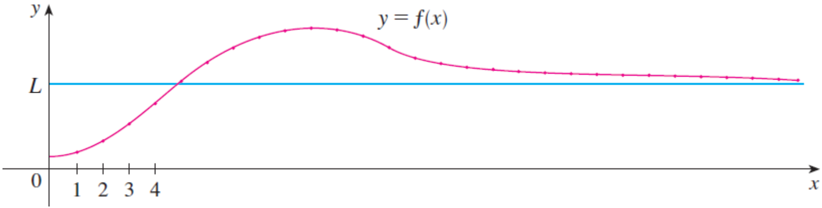
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \qquad \text{if} \quad \lim_{n\to\infty} b_n \neq 0$$

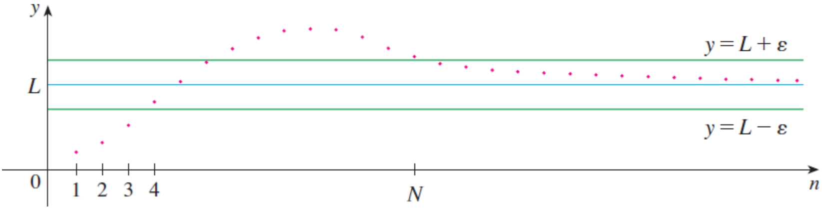
$$\lim_{n\to\infty} a_n^{\alpha} = \Big[\lim_{n\to\infty} a_n\Big]^{\alpha} \quad \text{if } \alpha > 0 \text{ and } a_n > 0.$$

Properties of Limit

Theorem 2.1

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{j\to\infty} a_j = L$.





Properties of Limit

Theorem 2.2

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

In particular, if $a_n \to L$ then $|a_n| \to |L|$.

Example: Find

$$\lim_{n\to\infty} \cos\left(\frac{3n^2 - 12n + 7}{5n^2 + n - 9}\right).$$

Example: Find

$$\lim_{n\to\infty} n^{1/n}.$$

Properties of Limit

Squeeze Theorem for Sequences

Suppose

$$a_n \leq b_n \leq c_n \quad \text{for } n \geq N$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then

$$\lim_{n\to\infty} b_n = L.$$

Corollary 2.1

(a) If $|a_n| \leq b_n$ and $b_n \to 0$, then $a_j \to 0$.

(b) $a_n \to 0$ if and only if $|a_n| \to 0$.

Bounded Sequences

Definition 2.3

• A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \le M \qquad \text{for all} \quad n.$$

In this case, the number M is an **upper bound** for the sequence.

• $\{a_n\}$ is **bounded below** if there is a number m such that

$$a_n \ge m \qquad \text{for all} \quad n.$$

In this case, the number m is a **lower bound** for the sequence.

• If it is bounded above and below, then $\{a_n\}$ is a **bounded** sequence.

Remark $\{a_n\}$ is bounded if and only if there is a constant K such that $|a_n| \le K$ for every n.

Bounded Sequences

- The sequence $a_n = 2^n$ is not bounded above, but it is bounded below (by 0).
- The sequence $a_n = 9 - n$ is not bounded below, but it is bounded above (by 9).
- The sequence $a_n = (-1)^n n$ is neither bounded above nor below.
- The sequence $a_n = \sqrt{n+1} - \sqrt{n}$ is bounded.

Theorem

- If $\{a_n\}$ converges then it is bounded.
- Conversely, an unbounded sequence is divergent.

Monotonic Sequences

Definition 2.2

A sequence $\{a_n\}$ is called:

- **increasing**, if $a_n < a_{n+1}$ for all n, that is,

$$a_1 < a_2 < a_3 < \cdots ;$$

- **decreasing**, if $a_n > a_{n+1}$ for all n, that is,

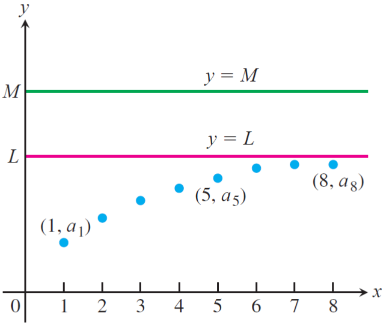
$$a_1 > a_2 > a_3 > \cdots ;$$

- **monotonic**, if it is either increasing or decreasing.

Monotonic Sequence Theorem

Theorem 2.5

- If $\{a_n\}$ is increasing and $a_n \leq M$ for all n, then $\{a_n\}$ converges and $\lim_{j\to\infty} a_j \leq M$.
- If $\{a_n\}$ is decreasing and $a_n \geq m$ for all n, then $\{a_n\}$ converges and $\lim_{j\to\infty} a_j \geq m$.



Monotonic Sequence Theorem

Example

- The sequence $a_n = n$ is increasing.
- The sequence $a_n = 2^{-n}$ is decreasing.
- The sequence $a_n = n - \frac{(-1)^n}{n+2}$ is increasing.
- The sequence $a_n = \sin n$ is not monotonic.

Monotonic Sequence Theorem

Example: Let $a_1 = 1$ and $a_n = \sqrt{6 + a_{n-1}}$ for $n > 1$. Prove that $\{a_n\}$ converges and find its limit.
Solution:

- We prove $\{a_n\}$ is increasing and $a_n < 3$ by induction.
  We have $3 > \sqrt{7} > 1$, i.e. $3 > a_2 > a_1$. Suppose $3 > a_{n+1} > a_n$, then
  $$3 = \sqrt{6+3} > \sqrt{6 + a_{n+1}} > \sqrt{6 + a_n},$$
  i.e. $3 > a_{n+2} > a_{n+1}$.
- Since $\{a_n\}$ is increasing and bounded, it converges.
  Suppose $\lim_{n\to\infty} a_n = L$, then
  $$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6 + a_n} = \sqrt{6 + L}.$$
  Solving $L = \sqrt{6 + L}$ gives $L = 3$.

Infinite Limits

Definition 2.1

- The sequence $\{a_n\}$ **diverges to infinity** if for every positive number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write
$$\lim_{j\to\infty} a_j = \infty \quad \text{or} \quad a_n \to \infty.$$
- Similarly if for every negative number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write
$$\lim_{j\to\infty} a_j = -\infty \quad \text{or} \quad a_n \to -\infty.$$

Section 3

Series

Series

Definition

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- Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \text{ or } \sum_{j=1}^{\infty} a_j$$

  is called a **series**.
  The numbers $a_1, a_2, \ldots$ are called the **terms** of the series. The number a_n is the ***n*th term** or the **general term** of the series.
- The ***n*th partial sum** of the series is

$$s_n = \sum_{j=1}^{n} a_j = a_1 + a_2 + a_3 + \cdots + a_n.$$

Series

Definition

If the sequence $\{s_n\}$ converges to a finite limit s we say the series $\sum_{j=1}^{\infty} a_j$ is **convergent** and write:

$$\sum_{j=1}^{\infty} a_j = s \quad \text{or} \quad a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s.$$

The number s is called the **sum** of the series.
If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Thus, the sum of a series is the limit of the sequence of partial sums:

$$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j.$$

Series

Note

- Any letter may be used for the index. Thus,

$$\sum_{j=1}^{\infty} a_j = \sum_{l=1}^{\infty} a_l = \sum_{\alpha=1}^{\infty} a_\alpha.$$

- Infinite series may begin with any index. For example,

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}, \quad \sum_{n=9}^{\infty} \frac{1}{(n-8)^2}.$$

- The convergent property of a series depends only on the behavior of the series at infinity. It does not change if we can add/delete/modify a finite number of terms of the series.

Examples

1. Compute

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)}$$

2. Show that the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{j}} + \cdots$$

is divergent.

3. Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{j=0}^{\infty} ar^j, \quad a \neq 0,$$

in which a and r are fixed numbers. Each term is obtained by multiplying the preceding one with the **common ratio** $\boldsymbol{r}$.

Theorem

(a) *If* $|r| < 1$, *then* $\sum_{j=0}^{\infty} ar^j$ *is convergent, with sum* $\frac{a}{1-r}$.

(b) *If* $|r| \geq 1$, *and* $a \neq 0$, *then* $\sum_{j=0}^{\infty} ar^j$ *is divergent.*

In words: The sum of a convergent geometric series is

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

Properties of Series

Theorem 3.2

If $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ are convergent series, then so are the series $\sum_{j=1}^{\infty} ca_j$ (where c is a constant), $\sum_{j=1}^{\infty}(a_j + b_j)$, and $\sum_{j=1}^{\infty}(a_j - b_j)$, and

(i) $\sum_{j=1}^{\infty} ca_j = c \sum_{j=1}^{\infty} a_j$;

(ii) $\sum_{j=1}^{\infty}(a_j + b_j) = \sum_{j=1}^{\infty} a_j + \sum_{j=1}^{\infty} b_j$;

(iii) $\sum_{j=1}^{\infty}(a_j - b_j) = \sum_{j=1}^{\infty} a_j - \sum_{j=1}^{\infty} b_j$.

Section 4

Tests of Convergence

Test of Divergence

Theorem

- If $\lim_{j\to\infty} a_j$ does not exist or if $\lim_{j\to\infty} a_j \neq 0$, then the series $\sum_{j=1}^{\infty} a_j$ diverges.

Examples: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2n-3}{5n+1}, \quad \text{and} \quad \sum_{n=1}^{\infty} \cos n$$

are convergent or divergent.

Nonnegative Series

Definition

A nonnegative series is a series whose terms are all nonnegative.

Theorem 4.1

A nonnegative series converges if and only if its partial sums are bounded from above.

Example 4.1 Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 5n}$$

is convergent or not.

Integral Test

Theorem

Suppose f is a continuous, positive, non-increasing function on $[N, \infty)$ and $a_j = f(j)$. Then

$$\sum_{j=1}^{\infty} a_j \text{ converges} \iff \int_N^{\infty} f(x)dx \text{ converges.}$$

Example: Determine if the following series converges

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

p-Series and Harmonic Series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is a ***p*-series**, where p is a positive constant.

For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

is called the **harmonic series**.

Theorem 4.3

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

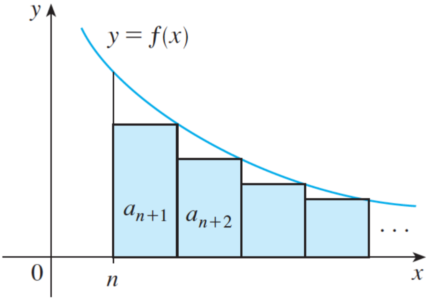
That is

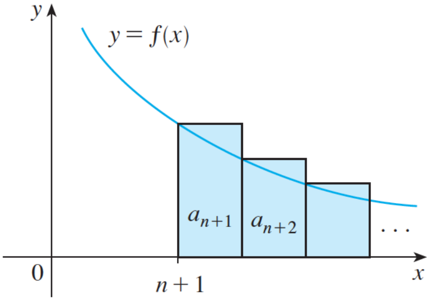
$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \iff p > 1.$$

Remainder estimate

- Suppose $\sum_{j=1}^{\infty} a_j$ converges to s.
- Then $R_n = s - s_n$ is the **remainder** when we use s_n to approximate s
- If $a_n = f(n)$ and f is positive and decreasing function then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$$





Comparison Test

The Comparison Test

Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be series with nonnegative terms such that $a_j \leq b_j$ for all $j \geq N$.

(a) If $\sum_{j=1}^{\infty} b_j$ is convergent, then $\sum_{j=1}^{\infty} a_j$ is also convergent.

(b) If $\sum_{j=1}^{\infty} a_j$ is divergent, then $\sum_{j=1}^{\infty} b_j$ is also divergent.

In words,

(i) Convergence of bigger series implies convergence of smaller series,

(ii) Divergence of smaller series implies divergence of bigger series.

Example 5.1 Determine if the following series converges

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} \text{ and } \sum_{n=1}^{\infty} \frac{1 + \sin n}{n^2}.$$

Limit Comparison Test

Theorem

Suppose that $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ are nonnegative series and

$$\lim_{j\to\infty} \frac{a_j}{b_j} = K$$

where K is a finite number and $K > 0$. Then either both series converge or both diverge.

Example Decide whether the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{2n^2 - 5\sqrt{n} + 1}{7n^4 - 3n^3 + 9n - 1}.$$

Alternating Series

Definition

$\sum_{j=1}^{\infty} a_j$ is an **alternating series** if its terms are alternately positive and negative, i.e. $a_j a_{j+1} < 0$ for all $j \geq 1$.

The general term of an alternating series is of the form

$$a_j = (-1)^{j+1} c_j \qquad \text{or} \qquad a_j = (-1)^j c_j$$

where $c_j = |a_j|$ are positive numbers.
Examples: The following are alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$-1 + \frac{1}{2} - \frac{1}{2^2} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n}$$

Alternating Series Test

Theorem

If

(i) $0 < c_{j+1} \leq c_j$ for all j and

(ii) $\lim_{j\to\infty} c_j = 0$,

then the series

$$\sum_{j=1}^{\infty}(-1)^{j+1}c_j = c_1 - c_2 + c_3 - c_4 + \cdots$$

converges. Furthermore, if the series converges to s then $|R_n| = |s - s_n| < c_{n+1}$.

Example

1. The **alternating harmonic series**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

   satisfies

   i/ $0 < c_{n+1} < c_n$ because $\frac{1}{n+1} < \frac{1}{n}$;

   ii/ $c_n = \frac{1}{n} \to 0$.

   Thus, it converges by the Alternating Series Test.

2. Similarly, the **alternating** $p-$**series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$$

   converges for any $p > 0$.

Alternating Series Test - Proof

We have

- $s_{2k+2} > s_{2k}$ and
- $s_{2k+1} < s_{2k-1}$.

Thus,

$$s_2 < s_4 < \cdots < s_{2k} < s_{2k-1} < \cdots < s_1.$$

The sequence of even partial sums is increasing and is bouded above by the sequence of odd partial sums. Thus $\lim s_{2k}$ and $\lim s_{2k-1}$ exists. Since

$$\lim s_{2k} - \lim s_{2k-1} = \lim(s_{2k} - s_{2k-1}) = \lim c_{2k} = 0,$$

these two limits equal. Thus $\sum_{n=1}^{\infty}(-1)^{n+1}c_n$ converges.
From the proof, we can see s lies between s_n amd s_{n+1}, so

$$|s - s_n| < |s_{n+1} - s_n| = c_{n+1}$$

Alternating Series Test

Example: How big should n be so that the error in using s_n to approximate the following series is less than 10^{-3}?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+2^n}$$

Answer: The error is less than $|a_{n+1}|$, so we need

$$1/(1+2^{n+1}) < 10^{-3},$$

or

$$n \geq 9.$$

Absolute Convergence

Definition

The series $\sum_{j=1}^{\infty} a_j$ **converges absolutely** (is **absolutely convergent**) if $\sum_{j=1}^{\infty} |a_j|$ converges.

Theorem

If a series is absolutely convergent, then it is convergent.

Proof: Suppose $\sum_{j=1}^{\infty} |a_j|$ converges. Let $b_j = a_j + |a_j|$ the $2|a_j| \geq b_j \geq 0$. By the Comparison Test, $\sum_{j=1}^{\infty} b_j$ converges. Thus,

$$\sum_{j=1}^{\infty} a_j = \sum_{j=1}^{\infty} b_j - \sum_{j=1}^{\infty} |a_j|$$

also converges.

Examples

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{n^2} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2 \cdot 2^2} + 0 - \frac{\sqrt{3}}{2 \cdot 4^2} - \frac{\sqrt{3}}{2 \cdot 5^2} + 0 + \cdots$$

is convergent or divergent.

Conditional Convergence

Definition 7.2

A series $\sum_{j=1}^{\infty} a_j$ **converges conditionally** (is **conditionally convergent**) if it converges but the series $\sum_{j=1}^{\infty} |a_j|$ diverges.

Example 7.2 The alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad p > 0$$

converges

- absolutely if $p > 1$,
- conditionally if $0 < p \leq 1$.

Ratio Test

Theorem

Suppose that

$$\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = \rho.$$

Then the series $\sum_{j=1}^{\infty} a_j$

(a) absolutely converges if $\rho < 1$,

(b) diverges if $\rho > 1$,

(c) may converge or diverge if $\rho = 1$.

Example: For the geometric series

$$\sum_{n=0}^{\infty} r^n,$$

it is clear that $\rho = |r|$. Thus, it converges absolutely if $|r| < 1$ and diverges if $r > 1$.

When $r = \pm 1$, the series diverges by the Divergence Test.

Examples

Determine the convergence of

1. $\sum_{n=0}^{\infty} \frac{1}{n!}$.
2. $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$.

Solution:

1. We have
$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$
Thus the series converges by the Ratio Test.
2. $$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1\end{aligned}$$
so the series absolutely converges.

Root Test

Theorem

Suppose that

$$\lim_{j\to\infty} |a_j|^{\frac{1}{j}} = \rho.$$

Then the series $\sum_{j=1}^{\infty} a_j$

(a) absolutely converges if $\rho < 1$,

(b) diverges if $\rho > 1$,

(c) may converge or diverge if $\rho = 1$.

Notes:

- If $\rho = 1$ in the Ratio Test, don't try the Root Test because ρ will again be 1.
- For the p-series $\sum \frac{1}{n^p}$, it is clear that $\rho = 1$. But the series converges iff $p > 1$.

Examples

- For the series $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n^n}$,

$$\rho = \lim_{n\to\infty} \frac{2^{(n+1)/n}}{n} = 0 < 1$$

so the series converges absolutely.

- For the series $\sum_n (1 + \frac{1}{n})^{n^2}$

$$\rho = \lim_{n\to\infty} (1 + \frac{1}{n})^n = e > 1$$

so the series diverges.

- Determine whether $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ converges.

Guidelines for Testing Convergence

1. Check if $a_j \to 0$. If not, the series diverges. Otherwise go to Step 2.
2. Is the Alternating Series Test applicable? If not, go to Step 3.
3. Take absolute values of the terms and check if they can be compared to a geometric or a p-series.
4. Try the Integral Test, the Ratio Test, or the Root Test.

Examples

1. $\sum_n \frac{(-1)^n n^2}{4^n}$.
2. $\sum_n \left(\frac{2n-1}{4n+3}\right)^n$
3. $\sum_n \frac{(-1)^n \sqrt{n^3+1}}{n^2+9}$
4. $\sum_{n=1}^{\infty} \frac{3n-1}{2n^2+1}$
5. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^s}$
6. $\sum \frac{3^n}{5^n-4^n}$
7. $\sum \frac{n+1}{2n^3-2n+3}$
8. $\sum_n \frac{n^4}{e^n}$
9. $\sum_n \frac{\ln n}{n+10}$

Section 5

Power Series

Power Series

Definition

1. A **power series centered at** 0 is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

where x is a variable and the constants c_n's are called the **coefficients** of the series.

2. More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + \cdots + c_n (x-a)^n + \cdots$$

is called a **power series in** $(x - a)$ or a **power series centered at** a.

Power Series

For each fixed x, a power series becomes a series of constants that we can test for convergence or divergence.
A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

whose domain is the set of all x for which the series converges. The domain contains a since the power series converges for $x = a$.

Power Series

Example 8.1 The power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Thus,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}, \ -1 < x < 1.$$

Power Series

Example 8.2 For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots;$

(b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots;$

(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots;$

(d) $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots.$

Answers: (a) $-1 < x \le 1$ (b) $-1 \le x \le 1$
(c) $-\infty < x < \infty$ (d) $\{0\}$

Power Series

Theorem 8.1

For a given power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

one of the following three possibilities holds:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series absolutely converges if $|x-a| < R$ and diverges if $|x-a| > R$.

In Case (i), set $R = 0$, and in Case (ii), set $R = \infty$. We call R the **radius of convergence** of the power series.

Power Series

Definition 1

The set of x for which the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges is called **interval of convergence** of the power series.
It is an interval of length R centered at a. The interval can be open, closed, or half-open.

Theorem

Let

$$L = \lim_{n\to\infty} |c_{n+1}|/|c_n|$$

or

$$L = \lim_{n\to\infty} |c_n|^{1/n},$$

then

$$R = \frac{1}{L}.$$

Power Series

How to Find the Interval of Convergence

1. Find the radius of convergence R, using the Root Test or the Ratio Test.
2. If $R = \infty$, the interval of convergence is $(-\infty, \infty)$. If $R < \infty$, test for convergence at each endpoint $a \pm R$, using the Comparison Test, the Integral Test, or the Alternating Series Test.

Example: Determine the center, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}.$$

Answer: Interval of convergence is $[-4, -1]$.

Section 6

Power Series Representation

Power Series Representation

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all x in the interval $(a-R, a+R)$, we say that the power series is a **representation** of $f(x)$ on that interval and f is **expanded into the power series**.

Theorem 9.1

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$

Power Series Representation

Theorem 9.1 (cont'd)

and

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \\ &= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \\ \int f(x)dx &= C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \cdots \\ &= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}. \end{aligned}$$

These series have the same radius of convergence R.

Power Series Representation

Note

1. It follows from the first part of Theorem 9.1 that

   Every power series is infinitely termwise differentiable inside its interval of convergence.

2. Although Theorem 9.1 says that the radius of convergence remains the same when a power series is differentiated or integrated, the interval of convergence may NOT remain the same.

Example

Differentiating each side of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

we get

$$\begin{aligned}\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots \\ &= \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n, \quad -1 < x < 1.\end{aligned}$$

According to the theorem, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, namely, $R = 1$.

Example

Find a power series representation for $\ln(1-x)$ and its radius of convergence.

Solution: We have

$$-\ln(1-x) = \int \frac{1}{1-x}dx = \int \left(1+x+x^2+\cdots\right)dx$$

$$= C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = C + \sum_{n=1}^{\infty} \frac{x^n}{n}. \qquad |x| < 1$$

Setting $x = 0$ gives $C = 0$, hence

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad |x| < 1.$$

The radius of convergence is the same as that of the original series: $R = 1$.

Example

Identify the function

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \ -1 \leq x \leq 1.$$

Solution We differentiate the original series term by term and get

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \\ &= \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}, \qquad -1 < x < 1, \\ f(x) &= \int f'(x)dx = \int \frac{1}{1+x^2} dx = \tan^{-1} x + C. \end{aligned}$$

The series for f is zero when $x = 0$, so $C = 0$. Hence
$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \tan^{-1} x, \quad -1 \leq x \leq 1.$

Section 7

Taylor and Maclaurin Series

Taylor and Maclaurin Series

In this section we introduce methods to:

1. Decide if a function have power series representations.
2. Find such representations.

Theorem

If f has a power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Taylor and Maclaurin Series

Definition

Let f be an indefinitely differentiable function. The series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

is called the **Taylor series of the function f at a** (or **about a** or **centered at a**).

Note: By the previous theorem, *if* f has a power series expansion at a, then the series must be the Taylor series of f at a.
However, even if the Taylor series of f exists, it may not be the power series of f. It can

- diverges for all $x \neq a$, or
- converges to a function different from f.

Taylor and Maclaurin Series

Definition

The Taylor series of f at $a = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

is called the **Maclaurin series** of f.

Example 10.1 Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Taylor and Maclaurin Series

We now try to find conditions that guarantee that f equals its Taylor series, i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

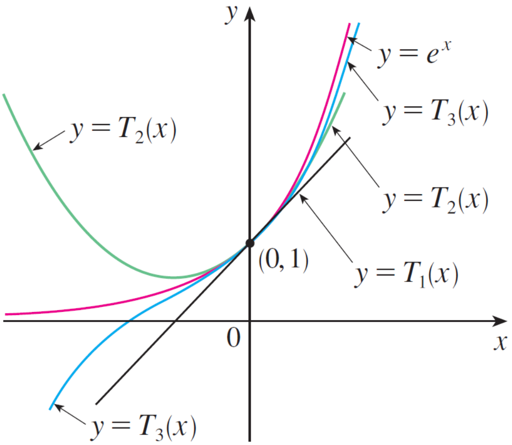
Definition 10.1

The **Taylor polynomial of order *n*** of f at a is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \\ + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Thus, T_n is the n-th partial sum of the Taylor series of f at a.

Taylor and Maclaurin Series



$f(x) = e^x$ and and its Taylor polynomials $T_1(x)$, $T_2(x)$, and $T_3(x)$.

Taylor and Maclaurin Series

Example 10.2 Find the Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$\begin{array}{rclrcl} f(x) & = & \cos x, & f'(x) & = & -\sin x \\ f''(x) & = & -\cos x, & f''(x) & = & \sin x \\ & \vdots & & & \vdots & \\ f^{(2n)}(x) & = & (-1)^n \cos x, & f^{(2n+1)}(x) & = & (-1)^{n+1} \sin x. \end{array}$$

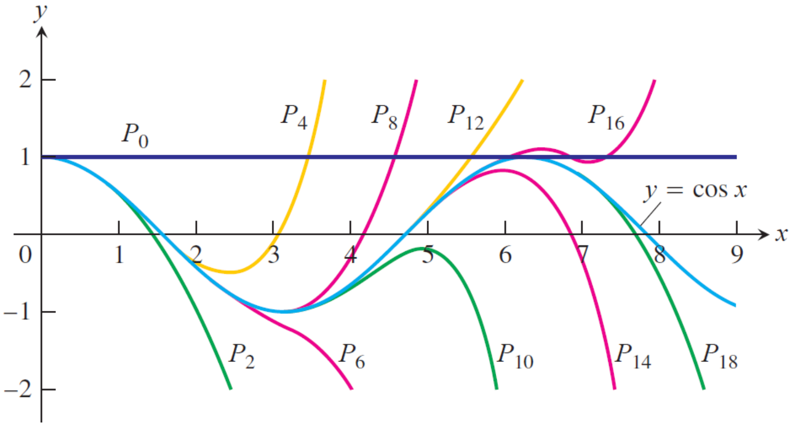
When $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

Taylor and Maclaurin Series

Hence the Taylor polynomials of orders $2n$ and $2n+1$ are identical:

$$T_{2n}(x) = T_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$



Taylor and Maclaurin Series

Clearly, f is the sum of its Taylor series at a if

$$f(x) = \lim_{n\to\infty} T_n(x), \qquad |x-a| < R. \tag{1}$$

If we let

$$R_n(x) = f(x) - T_n(x)$$

then $R_n(x)$ is called the **remainder of the Taylor series** and (1) is equivalent to

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \text{for} \quad |x-a| < R.$$

Remainder Formulas

Theorem

The following holds for the Taylor remainders of f

- (The integral form of the remainder term)

$$R_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

- (Lagrange's form of the remainder term)

$$R_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-a)^{n+1}$$

for some s between x and a.

Taylor and Maclaurin Series

Corollary

Let $f(x)$ be an infinitely differentiable function on the open interval $I = (a-R, a+R)$ with $R > 0$. Assume there exists a constant $K \geq 0$ such that for all n,

$$\left|f^{(n)}(x)\right| \leq K \quad \text{for all} \quad x \in I.$$

Then $f(x)$ is represented by its Taylor series in I:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{for all} \quad x \in I.$$

Examples

- For $f(x) = e^x$, we have
$$|f^{(n)}(x)| = |e^x| \leq e^{a+R}$$
for any $n \in \mathbb{N}$ and $x \in (a-R, a+R)$. Thus, for any a, the Taylor series of e^x at a converges to e^x, i.e.
$$e^x = \sum_{k=0}^{\infty} \frac{e^a}{n!}(x-a)^n, \qquad -\infty < x < \infty$$
In particular, when $a = 0$ we have
$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\,, \ -\infty < x < \infty}.$$
Letting $x = 1$ gives
$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots.$$

Example

- By similar arguments, we can derive the following formulas

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad (\forall x)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (\forall x)$$

Example

Find Maclaurin series for

$$f(x) = e^{-x^2/3}.$$

Using the power series representation of e^x, we have

$$\begin{aligned} e^{-x^2/3} &= \sum_{n=0}^{\infty} \frac{(-x^2/3)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n n!} x^{2n} \end{aligned}$$

for all real x.

Example

Find Taylor series for $\ln x$ in powers of $x - 2$.
Solution: We have

$$\begin{aligned} \ln x &= \ln(2 + (x - 2)) = \ln(2(1 + \frac{x-2}{2})) \\ &= \ln 2 + \ln(1 + (\frac{x-2}{2})) \end{aligned}$$

Use the Maclaurin series for $\ln(1 + x)$, we get

$$\begin{aligned} \ln x &= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\frac{x-2}{2})^n \\ &= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n n} (x - 2)^n \end{aligned}$$

This holds for $-1 < (x - 2)/2 \leq 1$, or $0 < x \leq 4$.

Example

Find $\cos(43^o)$ with error less than 10^{-5}.
Use the Maclaurin series for $\cos(x)$

$$\cos(43^o) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\frac{43\pi}{180})^{2n}$$

If we stop at the n-th term then error is bounded by

$$|R_{2n}| \leq \frac{1}{(2n+2)!} (\frac{43\pi}{180})^{2n+2} < 10^{-5}$$

if $n = 3$. Thus, we obtain

$$\cos(43^o) \approx \sum_{n=0}^{3} \frac{(-1)^n}{(2n)!} (\frac{43\pi}{180})^{2n}$$
$$\approx 0.7313512$$

Example

Let

$$f(x) = \int_0^x e^{-t^2} dt$$

Find the Maclaurin series of f and evaluate $f(1)$ correct up to 3 decimal places.

Solution: The power series representation of f is

$$\begin{aligned} f(x) &= \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1} \end{aligned}$$

In particular,

$$f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$$

By the Alternating Series Test, the error when using the first n terms is less than

$$\frac{1}{(2n+3)(n+1)!}.$$

This is less than 0.0005 for $n = 5$.
Thus, we obtain

$$f(1) \approx \sum_{n=0}^{5} \frac{(-1)^n}{(2n+1)n!} \approx 0.746729...$$

Binomial Series

The **Binomial Theorem** states that for any natural number k,

$$(a+b)^k = a^k + \binom{k}{1} a^{k-1}b + \binom{k}{2} a^{k-2}b^2 + \cdots + \binom{k}{k-1} ab^{k-1} + b^k.$$

Here $\binom{k}{n} = \dfrac{k(k-1)\cdots(k-n+1)}{n!} = \dfrac{k!}{n!(k-n)!}$, are the *binomial coefficients*. Letting $a=1$ and $b=x$, we obtain

$$(1+x)^k = 1 + \binom{k}{1} x + \binom{k}{2} x^2 + \cdots + \binom{k}{k-1} x^{k-1} + x^k.$$

Binomial Series

Consider the function $f(x) = (1+x)^k$ without assuming that $k \in \mathbb{N}$. To find the Maclaurin series for f, we first compute

$$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n}$$
$$f^{(n)}(0) = k(k-1)\cdots(k-n+1).$$

- If k is a nonnegative integer, then the series stops after $k+1$ terms because the coefficients from $n = k+1$ on are zero.
- If $k \notin \mathbb{N}$, then

$$\frac{f^{(n)}(0)}{n!} = \frac{k(k-1)\cdots(k-n+1)}{n!} = \binom{k}{n}.$$

is non-zero for all n.

Binomial Series

For $k \in \mathbb{R}$ and $n \in \mathbb{N}$ we define the **binomial coefficients** by

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}, \quad \binom{k}{0} = 1.$$

Then, the Maclaurin series of $f(x) = (1+x)^k$ is

$$\boxed{\begin{aligned}\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \\ + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n + \cdots\end{aligned}}$$

This series is called the **binomial series**.

Binomial Series

The Ratio Test shows that this series has radius of convergence $R = 1$. Furthermore, the binomial series converges to $(1+x)^k$ for $|x| < 1$.

Theorem

If $k \in \mathbb{R}$ and $|x| < 1$, then

$$\boxed{\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n\end{aligned}}$$

Binomial Series

Example 10.6 Find the Maclaurin series for

$$\frac{1}{\sqrt{1+x}}.$$

Answer: For $-1 < x \leq 1$,

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^n.$$

Important Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

HW 3

8.1: 16, 20, 25, 48, 54
8.2: 17, 21, 29, 62
8.3: 19, 29
8.4: 8, 33
8.5: 17, 24, 25
8.6: 8, 11, 37
8.7: 11, 16, 65