Counting techniques

November 13, 2020

Why need counting

- Suppose (Ω, P) has equally likely outcomes.
- To calculate $P(E)$, we need to count the number of elements in E and Ω.
- Need to learn counting technique.

Why need counting

- Suppose (Ω, P) has equally likely outcomes.
- To calculate $P(E)$, we need to count the number of elements in E and Ω.
- Need to learn counting technique.

Why need counting

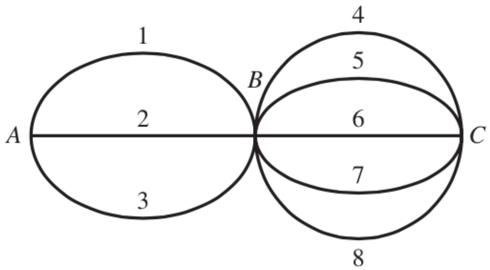
- Suppose (Ω, P) has equally likely outcomes.
- To calculate $P(E)$, we need to count the number of elements in E and Ω.
- Need to learn counting technique.

Methods

- Multiplication
  - counting with order
- Combination
  - counting without order

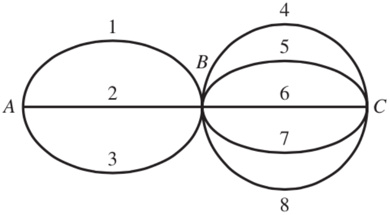
Multiplication rule

Suppose that there are three different routes from city A to city B and five different routes from city B to city C.



Count the number of different routes from A to C that pass through B

Solution



All possible routes is

$$\begin{Bmatrix} (1,4) & (1,5) & (1,6) & (1,7) & (1,8) \\ (2,4) & (2,5) & (2,6) & (2,7) & (2,8) \\ (3,4) & (3,5) & (3,6) & (3,7) & (3,8) \end{Bmatrix}$$

The number of routes is

$3 \times 5 = 15$

Example - Experiment in Two Parts

Consider an experiment that has the following two characteristics:

1. The experiment is performed in two parts.
2. The first part of the experiment has m possible outcomes $x_1, \ldots, x_m$, and, regardless of which one of these outcomes x_i occurs, the second part of the experiment has n possible outcomes $y_1, \ldots, y_n$.

Each outcome in the sample space S of such an experiment will therefore be a pair having the form (x_i, y_j)

$$S = \left\{ \begin{array}{cccc} (x_1, y_1) & (x_1, y_2) & \dots & (x_1, y_n) \\ (x_2, y_1) & (x_2, y_2) & \dots & (x_2, y_n) \\ \dots & & & \\ (x_m, y_1) & (x_m, y_2) & \dots & (x_m, y_n) \end{array} \right\}$$

Total number of outcomes in S is $|S| = mn$

Multiplication rule

- Suppose there is a job that has 2 steps
- There are m ways to do step 1
- There are n ways to do step 2
- There are $m \times n$ ways to do the job.

Multiplication rule

- Suppose there is a job that has 2 steps
- There are m ways to do step 1
- There are n ways to do step 2
- There are $m \times n$ ways to do the job.

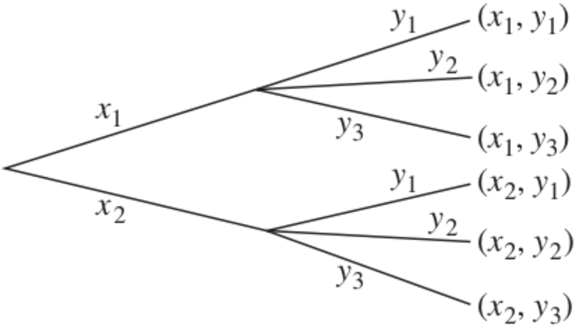
Multiplication rule

- Suppose there is a job that has 2 steps
- There are m ways to do step 1
- There are n ways to do step 2
- There are $m \times n$ ways to do the job.

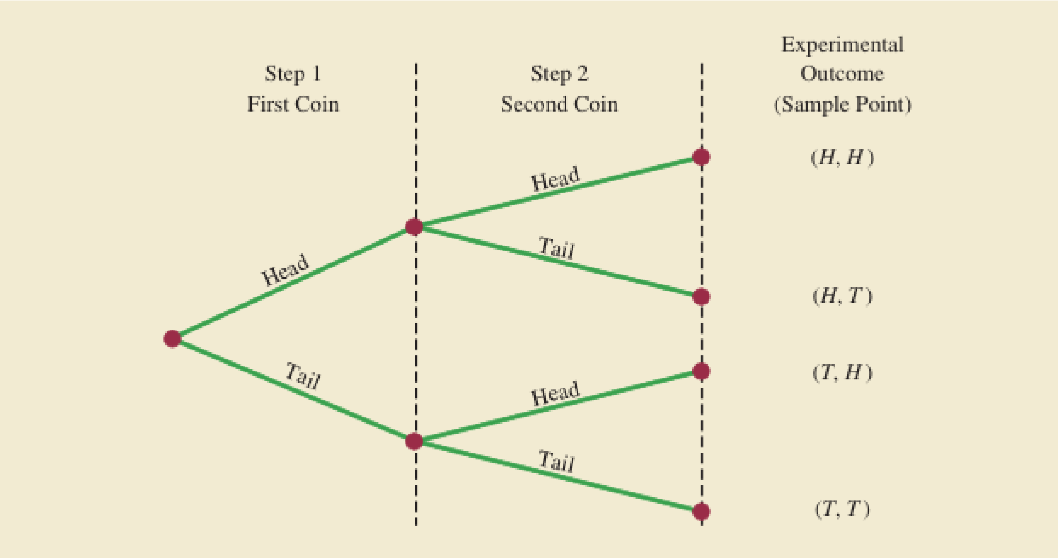
Multiplication rule

- Suppose there is a job that has 2 steps
- There are m ways to do step 1
- There are n ways to do step 2
- There are $m \times n$ ways to do the job.

Tree diagram in which end-nodes represent outcomes



Toss 2 coins

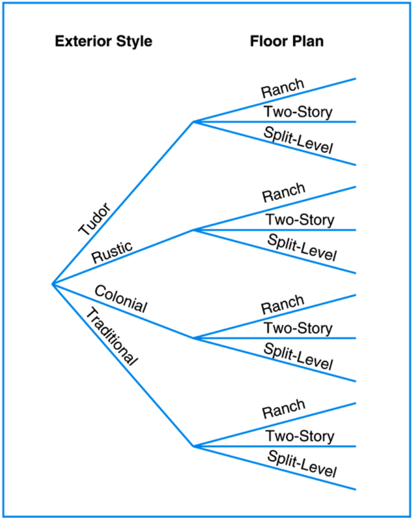


Step 1
First Coin
Step 2
Second Coin
Experimental
Outcome
(Sample Point)
Head
Tail
Head
Tail
Head
Tail
(H, H)
(H, T)
(T, H)
(T, T)

Example

A developer of a new subdivision offers prospective home buyers a choice of Tudor, rustic, colonial, and traditional exterior styling in ranch, two-story, and split-level floor plans. In how many different ways can a buyer order one of these homes?

Solution



There are $4 \times 3 = 12$ ways

General multiplication rule

Suppose that an experiment has k parts $(k \geq 2)$, that the ith part of the experiment can have n_i possible outcomes $(i = 1, ..., k)$, and that all of the outcomes in each part can occur regardless of which specific outcomes have occurred in the other parts. Then the sample space S of the experiment will contain all vectors of the form $(u_1, ..., u_k)$, where u_i is one of the n_i possible outcomes of part $i (i = 1, ..., k)$. The total number of these vectors in S will be equal to the product $n_1 n_2 ... n_k$.

Example

- Toss six fair coins
- Each outcome in S will consist of a sequence of six heads and tails, such as $HTTHHH$
- There are 2 outcomes for each coin
- Total number of outcomes in S is $2^6 = 64$
- Since there are six outcomes in S with 1 head and 5 tails, the probability of obtaining exactly one head is $6/64 = 3/32$.

Example

- Toss six fair coins
- Each outcome in S will consist of a sequence of six heads and tails, such as $HTTHHH$
- There are 2 outcomes for each coin
- Total number of outcomes in S is $2^6 = 64$
- Since there are six outcomes in S with 1 head and 5 tails, the probability of obtaining exactly one head is $6/64 = 3/32$.

Example

- Toss six fair coins
- Each outcome in S will consist of a sequence of six heads and tails, such as $HTTHHH$
- There are 2 outcomes for each coin
- Total number of outcomes in S is $2^6 = 64$
- Since there are six outcomes in S with 1 head and 5 tails, the probability of obtaining exactly one head is $6/64 = 3/32$.

Example

- Toss six fair coins
- Each outcome in S will consist of a sequence of six heads and tails, such as $HTTHHH$
- There are 2 outcomes for each coin
- Total number of outcomes in S is $2^6 = 64$
- Since there are six outcomes in S with 1 head and 5 tails, the probability of obtaining exactly one head is $6/64 = 3/32$.

Example

- Toss six fair coins
- Each outcome in S will consist of a sequence of six heads and tails, such as $HTTHHH$
- There are 2 outcomes for each coin
- Total number of outcomes in S is $2^6 = 64$
- Since there are six outcomes in S with 1 head and 5 tails, the probability of obtaining exactly one head is 6/64 = 3/32.

Example

There are 6 balls of different colors. How many ways to arrange them on a straight line?

Solution

- Think of a line with 6 positions
- 6 ways to choose ball for 1st position
- 5 ways for 2nd position, 4 for 3rd ...
- Total $6 \cdot 5 \cdot 4 \ldots 2 \cdot 1 = 6! = 720$ ways

Solution

- Think of a line with 6 positions
- 6 ways to choose ball for 1st position
- 5 ways for 2nd position, 4 for 3rd ...
- Total $6 \cdot 5 \cdot 4 \ldots 2 \cdot 1 = 6! = 720$ ways

Solution

- Think of a line with 6 positions
- 6 ways to choose ball for 1st position
- 5 ways for 2nd position, 4 for 3rd ...
- Total $6 \cdot 5 \cdot 4 \ldots 2 \cdot 1 = 6! = 720$ ways

Solution

- Think of a line with 6 positions
- 6 ways to choose ball for 1st position
- 5 ways for 2nd position, 4 for 3rd ...
- Total $6 \cdot 5 \cdot 4 \ldots 2 \cdot 1 = 6! = 720$ ways

Combination

How many ways to choose a group of 2 out of 5 people if the order of the group is not important.

Solution

- $5 \times 4 = 20$ ways to choose 2 people
- Any group AB was counted $2!$ times (AB, BA)
- If the order is not important then there are

$$\frac{5 \times 4}{2!} = \frac{5!}{3!2!}$$

Solution

- $5 \times 4 = 20$ ways to choose 2 people
- Any group AB was counted $2!$ times (AB, BA)
- If the order is not important then there are

$$\frac{5 \times 4}{2!} = \frac{5!}{3!2!}$$

Solution

- $5 \times 4 = 20$ ways to choose 2 people
- Any group AB was counted $2!$ times (AB, BA)
- If the order is not important then there are

$$\frac{5 \times 4}{2!} = \frac{5!}{3!2!}$$

Combination

The number of ways to choose an unordered group of k objects out of n distinct objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Read "n choose k"
or " the number of combinations of n objects taken k at a time"

Combination

The number of ways to choose an unordered group of k objects out of n distinct objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Read "n choose k"
or " the number of combinations of n objects taken k at a time"

Example

A committee of size 5 is to be selected randomly from a group of 6 men and 9 women. What is the probability that the committee consists of 3 men and 2 women?

Solution

- Size of sample space: $\binom{15}{5}$
- Size of event: $\binom{6}{3}$ ways to choose the men, $\binom{9}{2}$ ways to choose the women
- The probability is

$$\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}$$

Solution

- Size of sample space: $\binom{15}{5}$
- Size of event: $\binom{6}{3}$ ways to choose the men, $\binom{9}{2}$ ways to choose the women
- The probability is

$$\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}$$

Solution

- Size of sample space: $\binom{15}{5}$
- Size of event: $\binom{6}{3}$ ways to choose the men, $\binom{9}{2}$ ways to choose the women
- The probability is

$$\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}$$