Decision Making

(for Financial Engineering & Risk Management program)

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February 9, 2022

Contents

Decision analysis involves the use of a rational process (with judging/thinking) for selecting the best among several alternatives. The "goodness" of a selected decision depends on the data used in describing decision making situation.

Some situations where a decision making is needed:

• Transportation problems; network flow problems, ... (the data are well defined - Decision making with certainty).

• A manufacturer introducing a new product into the marketplace: What will be the reaction of potential customers? How much should be produced? Should the product be test marketed in a small region before deciding upon full distribution? How much advertising is need to launch the product successfully?(the data only got by inspection, predict based upon the last experiences, ... and are not precise and often given by some probability distributions) - Decision making under risk

• An oil company deciding whether to drill for oil in a particular location. How likely is oil there? How much? How deep will they need to drill? Should geologists investigate the site further before drilling? Decision making under uncertainty

- An agricultural firm selecting the mix of crops and livestock for the upcoming season. What will be the weather conditions? Where are prices headed? What will costs be?

These are the kinds of decision making in the face of great uncertainty that decision analysis is designed to address. Decision analysis provides a framework and methodology for rational decision making when the outcomes are uncertain.

Three categories of decision making process:

• Decision making under certainty (in which the data are well defined),

• Decision making under risk (in which the payoffs associated with each decision alternative are described by probability distributions),

• Decision making under uncertainty (in which the data are ambiguous).

This chapter concerns all the three categories.

2. Decision making under certainty - Analytic Hierarchy process (AHP)

Decision making under certainty includes the situations that can be described as LP models considered in the previous chapters. Here the AHP is designed for situations in which ideas, feelings, and emotions are quantified based on subjective judgement to provide a numeric scale for prioritizing decision alternatives.

2.1. AHP via a prototype example

Martin Hans (who) is a bright high school student, has received full academic scholarship from three institutions: UA (University of A), UB and UC.

To select one university, Martin specifies two main criteria:

- location and
- academic reputation.

Martin's judgement: Academic reputation is FIVE times as important as location. This gives

- Weight to location: 17 % (approximately),
- Weight to Academic reputation: 83 % .

Using a systematic analysis (to be clarified later), Martin ranks the three universities as follows:

Table 1

	Percent weight estimates for		
Criterion	UA	UB	UC
Location	12.9	27.7	59.4
Reputation	54.5	27.3	18.2

Note: • The sum of each row in the previous table is 100 (%).

• The structure of the decision problem is given in the Table 1.

• This problem has a single hierarchy (level) with two criteria (location and reputation) and three decision alternatives (UA, UB, UC).

Composite weight for each university is computed as:

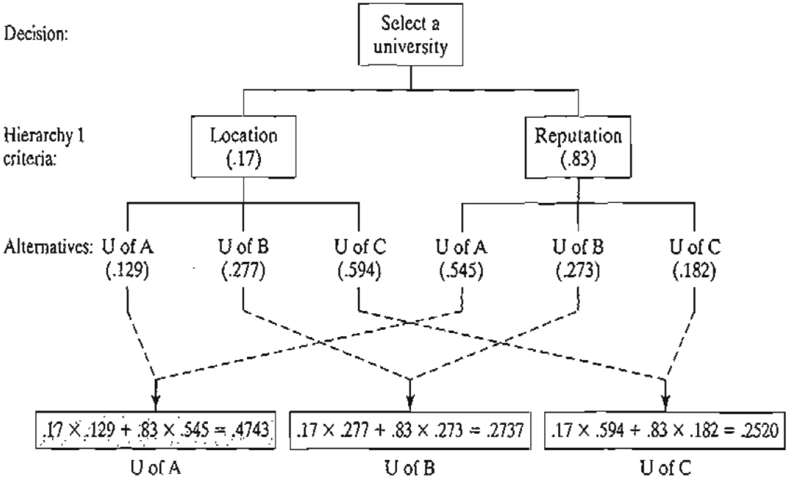
$$\begin{aligned} UA &: \quad .17 \times .129 + .83 \times .545 = .4743 \\ UB &: \quad .17 \times .277 + .83 \times .273 = .2737 \\ UC &: \quad .17 \times .594 + .83 \times .182 = .2520 \end{aligned}$$

One has

	Percent weight estimates for		
Criterion	UA	UB	UC
Location	12.9	27.7	59.4
Reputation	54.5	27.3	18.2
Composite weight	0.4743	0.2737	0.2520

Based on these calculations, UA has the highest weight, and hence, **UA is the best choice for Martin.**

Figure 1



We now turn to the **AHP with several hierarchies of criteria**.

2.2. AHP with several hierarchies of criteria

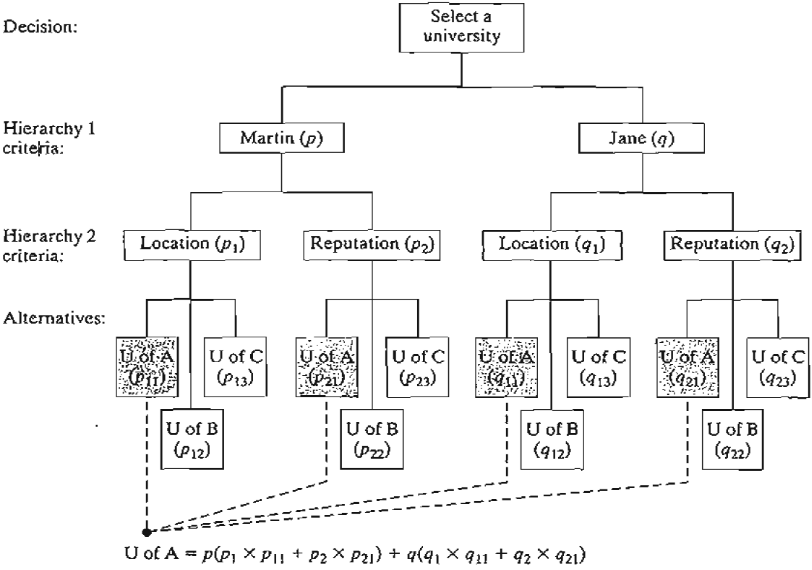
Consider again the Martin's decision making problem. Suppose further that Martin's twin sister, Jane, also was accepted with full fellowship to the three mentioned universities. However, their parents stipulate that they both must attend the same university.

The structure of the new decision problem is summarized in the Figure 2 (next page).

• p and q (presumable to be equal) represent the relative weights given to Martin' and Jane's opinions about the selection process. Note: $p + q = 1$.

• (p_1, p_2), (q_1, q_2) reflect Martin's and Jane's individual opinions about the criteria location and reputation of each university. Also, $p_1 + p_2 = 1$, $q_1 + q_2 = 1$.

Figure 2


Decision:
Select a university
Hierarchy 1 criteria:
Martin (p)
Jane (q)
Hierarchy 2 criteria:
Location (p1)
Reputation (p2)
Location (q1)
Reputation (q2)
Alternatives:
U of A (p11)
U of B (p12)
U of C (p13)
U of A (p21)
U of B (p22)
U of C (p23)
U of A (q11)
U of B (q12)
U of C (q13)
U of A (q21)
U of B (q22)
U of C (q23)
U of A = p(p1 × p11 + p2 × p21) + q(q1 × q11 + q2 × q21)

- The numbers $p_{11}, p_{12}, p_{13}, p_{21}, p_{22}, p_{23}$, and q_{11}, q_{12}, q_{13}, q_{21}, q_{22}, q_{23} are opinions of Martin's and Jane's on the rank of the three universities (as in the case without Jane), respectively. Note that:

$$\begin{aligned} p_{11} + p_{12} + p_{13} &= 1, \quad p_{21} + p_{22} + p_{23} = 1, \\ q_{11} + q_{12} + q_{13} &= 1, \quad q_{21} + q_{22} + q_{23} = 1. \end{aligned}$$

- Composite weight (W) is computed as ($W(A)$ stands for the composite weight of U of A)

$$\begin{aligned} W(A) &= p(p_1 \times p_{11} + p_2 \times p_{21}) + q(q_1 \times q_{11} + q_2 \times q_{21}), \\ W(B) &= p(p_1 \times p_{12} + p_2 \times p_{22}) + q(q_1 \times q_{12} + q_2 \times q_{22}), \\ W(C) &= p(p_1 \times p_{13} + p_2 \times p_{23}) + q(q_1 \times q_{13} + q_2 \times q_{23}). \end{aligned}$$

The largest composite weight represents the best choice for both Martin and Jane.

Under the condition of risk, the payoffs associated with each decision alternative are described by probability distributions.

3.1. Decision making without experimentation

Example 1. Jack wants to invest $ 10,000 in the stock market by buying shares of one of the two companies: A and B. Jack have got the information:

• Shares in company A, though risky, could yield a 50% return on investment during the next year. If the stock market conditions are not favorable (i.e., "bear" market) the stock may loose 20% of its value.

• Company B provides safe investments with 15% return in a "bull" market and only 5% in a "bear" market.

- Shares in company A, though risky, could yield a 50% return on investment during the next year. If the stock market conditions are not favorable (i.e., "bear" market) the stock may loose 20% of its value.

- Company B provides safe investments with 15% return in a "bull" market and only 5% in a "bear" market.

- All the publications Jack have consulted are predicting a 60% chance for a "bull" market and 40% chance for a "bear" market.

Question: Where should Jack invest his money (company A or B)?

The previous decision making problem can be summarized as:

Table A

Decision alternative	1-year return on $10,000 investment "Bull" market ($)	"Bear" market ($)
Company A stock	5000	-2000
Company B stock	1500	500
Probability of occurrence	0.6	0.4

In this example, the states "bear" or "bull" market are referred to as states of nature. They are random variables whose probabilities of occurrence are 0.4 and 0.6, respectively.

3.1.1. Maximim payoff criterion

Maximim payoff criterion: For each possible action (alternative), find the minimum payoff over all possible states of nature. Next, find the maximum of these minimum payoffs. Choose the action whose minimum payoff gives this maximum.

3.1.1. Maximim payoff criterion

Decision alternative	1-year return on $10,000 investment: "Bull" market ($)	"Bear" market ($)	Min
Company A	5000	-2000	- 2000
Company B	1500	500	500
Probability of occurrence	0.6	0.4	

Applying this criterion to the Example 1, as Table A shows, the minimum payoff of the alternative Company A stock is -2000, the minimum payoff of the alternative Company B stock is 500. The maximum of these two numbers is 500. So by the Maximim payoff criterion, the choice is to invest in stock B.

Note: The Maximim payoff criterion gives a choice regardless the probabilities of the states of nature.

3.1.2. Maximum likely-hood criterion

Maximum likelihood criterion: Choose the most likely state of nature (the one with the largest probability). For this state of nature, choose the action with the maximum pay-off.

Applying this criterion to the Example 1, as Table A shows, the state of "Bull" market has the largest prior probability (0.6). Choose this state. In the column of "Bull" market, the alternative "company A stock" has the largest payoff ($5000). So the choice is to invest in stock A.

3.1.3. Expected value criterion

Expected value criterion is the most commonly chosen criterion for decision making under risk. By this criterion, decision alternatives are compared based on maximization of expected (average) payoffs.

• Calculate the expected (average) value of the payoff for each of the possible alternatives. Choose the alternative with the maximum expected payoff.

Applying the Expected value criterion to Example 1, the expected values (payoffs for 1-year return) for the two alternatives are:

$$\begin{aligned} &\text{For stock A}: && 5000 \times 0.6 + (-2000) \times 0.4 = 2200(\$) \\ &\text{For stock B}: && 1500 \times 0.60 + 500 \times 0.40 = 1100(\$) \end{aligned}$$

Based on these results, Jack's decision is to invest in stock A.

General situation

In general situation, a decision problem may include n states of nature and m alternatives.

• Let $p_j > 0$ be the probability of occurrence for the state of nature j. Note that $p_1 + p_2 + \cdots + p_n = 1$.

• Let a_{ij} be the payoff of alternative i given state of nature j $(i = 1, 2, \cdots, m,\ j = 1, 2, 3, \cdots, n)$.

Then the expected payoff for the alternative i (denoted by EV_i) is:

$$EV_i = a_{i1}p_1 + a_{i2}p_2 + \cdots + a_{in}p_n, \quad i = 1, 2, \cdots m.$$

The best alternative V_{i_0} is the one with

$$EV_{i_0} = \max\{EV_i \mid i = 1, 2, \cdots, m\}.$$

Figure 3

State of nature / Alternative	SN1	SN2	SN3	...	SNn	Expecte payoff
A1	a_{11}	a_{12}	a_{13}	... a_{1j} .	a_{1n}	EV_1
A2	a_{21}	a_{22}	a_{23}	.. a_{2j}	a_{2n}	EV_2
A3	a_{31}	a_{32}	a_{33}	....	a_{3n}	EV_3
⋮	⋮					⋮
Am	a_{m1}	a_{m2}	a_{m3}	... a_{mj}	a_{mn}	EV_m
probability of occurence	P_1	P_2	P_3	.	P_n	

$$\sum_{i=1}^{n} p_i = 1$$

$$EV_1 = a_{11}p_1 + a_{12}p_2 + a_{13}p_3 + \dots + a_{1j}p_j + \dots + a_{1n}p_n$$

$$EV_i = a_{i1}p_1 + a_{i2}p_2 + a_{i3}p_3 + \dots + a_{ij}p_j + \dots + a_{in}p_n$$

$i = 1, m$

The best alternative V_{i_0} is the one with

$$EV_{i_0} = \max\{EV_i \mid i = 1, 2, \dots m\}$$

3.2. Decision making with experimentation

In the previous Subsection 3.1, the expected value criterion (also, other criterions) relies solely on the probabilities available from historical data (called: prior probabilities).

It is natural to think of doing some more experimentation to **improve** the preliminary estimates of these prior probabilities.

The probabilities improved after experimentation are called posterior probabilities. We will **learn the way how to find posterior probabilities via an example**.

Example 2 [Example 1 revisited] Suppose that Jack does not rely solely on the probabilities of “bull” (0.6) or “bear” (0.4) markets available on the financial publications. He wants to conduct a personal investigation by consulting an expert in the stock market.

• The expert offers the general opinion of “for” or “against” the investment.

• The expert’s opinion is further quantified in the following manner:

☐ If it is a “bull” market, there is a 90% chance the vote will be “for”.

☐ If it is a “bear” market, the chance of a “for” vote is only 50%.

Questions:

1. How to make use of this additional information?

2. If the expert's recommendation is "for", would Jack invest in stock A or B?

3. If the expert's recommendation is "against", would Jack invest in stock A or B?

3.2. Decision making with experimentation

To process the extra information, we introduce the followings:

- Findings after experimentation:

$$\nu_1 = \text{“for” vote}, \quad \nu_2 = \text{“against” vote}.$$

- States of nature:

$$m_1 = \text{“bull” market}, \quad m_2 = \text{“bear” market}.$$

Then the expert statements can be rewritten as conditional probabilities:

☐ If it is a “bull” market, there is a 90% chance the vote will be “for”.

$$P(\nu_1 \mid m_1) = 0.9, \quad P(\nu_2 \mid m_1) = 1 - 0.9 = 0.1,$$

☐ If it is a “bear” market, the chance of a “for” vote is only 50%.

$$P(\nu_1 \mid m_2) = 0.5, \quad P(\nu_2 \mid m_2) = 1 - 0.5 = 0.5.$$

3.2. Decision making with experimentation

To process the extra information, we introduce the followings:

- Findings after experimentation:

$$\nu_1 = \text{"for" vote}, \quad \nu_2 = \text{"against" vote}.$$

- States of nature:

$$m_1 = \text{"bull" market}, \quad m_2 = \text{"bear" market}.$$

☐ If - "bull" market, there is a 90% chance the vote will be "for".

☐ If - "bear" market, the chance of a "for" vote is only 50%.

These can be summarized as (here $P(\nu_i \mid m_j)$):

	ν_1	ν_2
m_1	0.9	0.1
m_2	0.5	0.5

Table 1.

3.2. Decision making with experimentation

Problem: How to calculate the posterior (conditional) probabilities $P\{m_i \mid \nu_j\}$? (for what?). Recall

	1-year return on \$10,000 investment		
Decision alternative	"Bull" market (\$)	"Bear" market (\$)	"for" /"against"
Company A	5000	-2000	
Company B	1500	500	
Prob. occur.	?(α)	?(β)	"for"
Prob. occur.	?	?	"against"

- $\alpha = P\{m_1 \mid \nu_1\} = P\{\text{"bull"} \mid \text{"for"}\}$ (known: $P(\nu_i \mid m_j)$)

- If the opinion of the expert is "for" then What are the expectation of A and of B?

- If "for", then $E(A) = 5000\alpha - 2000\beta$
  $E(B) = 1500\alpha + 500\beta$

Problem: How to calculate the posterior (conditional) probabilities $P\{m_i \mid \nu_j\}$?

Algorithm (for finding $P\{m_i \mid \nu_j\}$)

Step 1. Compute the joint probabilities as

$$P\{m_i, \nu_j\} = P\{\nu_j \mid m_i\}P\{m_i\}, \text{ for all } i, j. \qquad (1)$$

Note that $P\{m_i, \nu_j\}$ is the probability for the two events m_i and ν_j occurring at the same time.

Taking the prior probabilities $P\{m_1\} = 0.6$, $P\{m_2\} = 0.4$ and the conditional probabilities given in Table 1 into account, the joint probabilities can be computed by (1) and are given in the Table 2 (multiply rows 1 and 2 in Table 1 by 0.6 and 0.4, resp.).

$P\{m_i, \nu_j\}$:

	ν_1	ν_2
m_1	$0.54 = P(m_1, \nu_1)$	$0.06 = P(m_1, \nu_2)$
m_2	0.20	0.20

Table 2.

Step 2. Compute the absolute probabilities $P\{\nu_j\}$ as

$$P\{\nu_j\} = \sum_i P\{m_i, \nu_j\}, \text{ for all } j. \qquad (2)$$

- Ex. $P(\nu_1) = P(m_1, \nu_1) + P(m_2, \nu_1)$

(computed from Table 2 by summing respective columns).

$P\{\nu_1\}$	$P\{\nu_2\}$
0.74	0.26

Table 3.

Recall:

$P\{m_i, \nu_j\}$:

	ν_1	ν_2
m_1	$0.54 = P(m_1, \nu_1)$	$0.06 = P(m_1, \nu_2)$
m_2	$0.20 = P(m_2, \nu_1)$	0.20

Table 2.

Step 3. Determine the desired posterior probabilities $P\{m_i \mid \nu_j\}$ as

$$P\{m_i \mid \nu_j\} = \frac{P\{m_i, \nu_j\}}{P\{\nu_j\}}. \quad (3)$$

These posterior probabilities for Example 2 are shown in the Table 4 (by dividing each column of Table 2 by the element of the corresponding column of Table 3).

$P\{m_i \mid \nu_j\}$:

	ν_1	ν_2
m_1	0.730	0.231
m_2	0.270	0.769

Table 4.

Recall

$sP\{m_i, \nu_j\}$:

	ν_1	ν_2
m_1	$0.54 = P(m_1, \nu_1)$	$0.06 = P(m_1, \nu_2)$
m_2	$0.20 = P(m_2, \nu_1)$	0.20

Table 2.

$P\{\nu_1\}$	$P\{\nu_2\}$
0.74	0.26

Table 3.

$P\{m_i \mid \nu_j\}$:

	ν_1 "for"	ν_2 "against"
m_1	0.730 (bull, α)	0.231
m_2	0.270 (bear, β)	0.769

Table 4.

Note: These are different from the prior probabilities $P\{m_1\} = 0.6$, $P\{m_2\} = 0.4$.

$P\{m_i \mid \nu_j\}$:

	ν_1 "for"	ν_2 "against"
m_1	0.730 (bull, α)	0.231
m_2	0.270 (bear, β)	0.769

Table 4.

Coming back to the problem and supply with posterior Probability:

Decision alternative	1-year return on \$10,000 investment "Bull" market (\$)	"Bear" market (\$)	"for" /"against"
Company A	5000	-2000	
Company B	1500	500	
Prob. occur.	$\alpha = 0.730$	$\beta = 0.270$	"for"
Prob. occur.	0.231	0.769	"against"

- $\alpha = P\{m_1 \mid \nu_1\} = P\{\text{"bull"} \mid \text{"for"}\}$
- If "for", $E(A) = 5000 \times 0.730 - 2000 \times 0.270 = 3110(\$)$
  $E(B) = 1500 \times 0.730 + 500 \times 0.270 = 1230\$$
- If "against", $E(A) = 5000 \times 0.231 - 2000 \times 0.769 = -383(\$)$
  $E(B) = 1500 \times 0.231 + 500 \times 0.769 = 731(\$)$

Some Questions:

Q1: One question remains unanswered: Is it worth conducting an experimentation? (e.g., for the problem in Example 2).
For an answer of this question, see [1], Chapter 16, page 694, Subsection "The value of experimentation".

Q2: What would be changed in our previous calculation if Jack is asked to pay $ 200 for the opinion of the expert?

To be continued.