Probability Review

September 11, 2020

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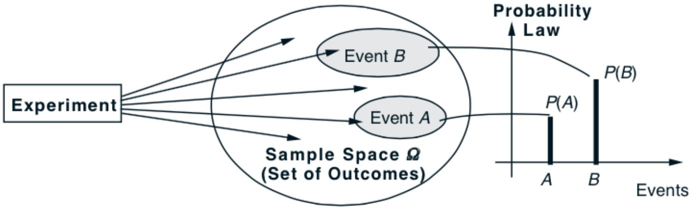
Each numerical value in the sequence is modeled by a random variable, so a stochastic process is simply a (finite or infinite) **sequence of random variables**

Why study random processes?

- Many phenomenon can be modelled by random processes: understand random processes help to understand the phenomenon.
- Wide application: finance, actuary...
- Provide foundation for later: financial mathematics 1 and 2.

Plan

Probability models



- Random experiment: produce uncertain outcomes under the same condition.
- An outcome ω: a result of experiment.
-
- A sample space Ω: all results of experiment.
- An event: a collection of some outcomes
-
- Probability measure P: estimate the likelihood of each event.

Probability measure on finite space

- Ω has finite elements
- $\mathcal{G}$: set of all subset of Ω
- A probability measure P on Ω is a function mapping $\mathcal{G}$ into $[0,1]$ with the following properties
  1. $P(\Omega) = 1$
  2. $P(A^c) = 1 - P(A)$
  3. If $A_1, A_2, \ldots$ are pairwise disjoint sets in $\mathcal{G}$ then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Example 1

- A coin has probability 1/3 for H and 2/3 for T
- All possible outcome of three coin tosses

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

- Probability of individual outcomes?

Example 1

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- Probability of individual outcomes?
- Probability of H on the first toss?

$\sigma - algebra$: record of information

Let Ω be a sample space of a random experiment. A collection of subsets of Ω is called an $\sigma - algebra$ over Ω if it satisfies the following conditions:

1. $\Omega \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3. $A_i \in \mathcal{F},\ \forall i = 1, 2 \cdots \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$

Example 2

Some important $\sigma - algebra$ of subsets of Ω in Example 2

1. $\mathcal{F}_0 = \{\emptyset, \Omega\}$: trial $\sigma - algebra$ - contains **no information**. Knowing whether the outcome w of the three tosses is in $\emptyset$ and whether it is in Ω tells you nothing about w.

Example 2

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2.

$$\mathcal{F}_1 = \{0, \Omega, \{HHH, \{HHT, HTH, HTT\}, \{THH, THT, TTH, TTT\}\}$$
$$= \{0, \Omega, A_H, H_T\}$$

where

$$A_H = \{HHH, HHT, HTH, HTT\} = \{\text{ H on first toss }\}$$

$$A_T = \{THH, THT, TTH, TTT\} = \{\text{ H on first toss }\}$$

$\mathcal{F}_1$**: information of the first coin or "information up to time 1"**. For example, you are told that the first coin is H and no more.

Example 2

3.

$$\mathcal{F}_2 = \{\emptyset, \Omega, \{HHH, HHT\}, \{HTH, HTT\}, \{THH, THT\}, \{TTH, TTT\}$$

and all sets which can be built by taking unions of these }

$$= \{\emptyset, \Omega, A_{HH}, A_{HT}, A_{TH}, A_{TT}\}$$

and all sets which can be built by taking unions of these }

where

$$A_{HH} = \{HHH, HHT\} = \{\text{HH on the first two tosses}\}$$

$$A_{HT} = \{HTH, HTT\} = \{\text{HT on the first two tosses}\}$$

$$A_{TH} = \{THH, THT\} = \{\text{TH on the first two tosses}\}$$

$$A_{TT} = \{TTH, TTT\} = \{\text{TT on the first two tosses}\}$$

$\mathcal{F}_2$: **information of the first two tosses or "information up to time 2"**

4. $\mathcal{F}_3 = \mathcal{G}$ set of all subsets of Ω: **“full information”** about the outcome of all three tosses

σ-algebra generated by a set

Defintion

The σ-algebra generated by A, denoted $\sigma(A)$, is the collection of possible events from the experiment at hand (i.e. all element in A and compliment, countable union, intersection of elements in A)

Example

Come back to the experiment of tossing a coin three times

1. $\mathcal{F}_1 = \sigma(A_T, A_H)$
2. $\mathcal{F}_2 = \sigma(A_{HH}, A_{HT}, A_{TH}, A_{TT})$
3. Which set generates $\mathcal{F}_3$?

Borel $\sigma - algebra$

Consider a random experiment of picking a random real number $\mathbb{R}$.

- $\Omega = \mathbb{R}$.
- Open set (a, b)
- Borel σ - algebra $\sigma(\mathbb{R}) = \sigma($ all open sets in $\mathbb{R})$

Filtration

A filtration is a sequence of $\sigma - algebra$ $\mathcal{F}_0, \mathcal{F}_2, \ldots, \mathcal{F}_n$ such that each $\sigma - algebra$ in the sequence contains all the sets contained by the previous $\sigma - algebra$.

Probability space

Probability space is a triple $(\Omega, \mathcal{F}, P)$ with

- Ω is a sample space
- $\mathcal{F}$ is a $\sigma - algebra$ over Ω.
- A function

$$P : \mathcal{F} \rightarrow [0, 1]$$

  satisfies
  - $P(\Omega) = 1$;
  - $P(A) = 1 - \mathcal{P}(A^c)$ for all $A \in \mathcal{F}$;
  - For all pairwise disjoint sets $A_i \in \mathcal{F}$,

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

.

P is called a probability (measure) on $\mathcal{F}$

Plan

Random variables

- Outcomes of a random experiment might take on numerical values.

Random variables

- Outcomes of a random experiment might take on numerical values.
- Outcomes of a random experiment may be not numbers.
- A random variable assigns a number to each outcome.
- A random variable on a probability space $(\Omega, \mathcal{F}, P)$ is a function

$$\begin{aligned} X: \quad & \Omega \to R \\ & \omega \to X(\omega) \end{aligned}$$

Random variables

- Given a random variable

$$X : \quad \Omega \to R$$
$$\omega \to X(\omega)$$

  The set $\mathbf{S} = \{X(\omega) | \omega \in \Omega\}$ is called the state of the random variable X

- If $\mathbf{S}$ is a countable set then X is called a discrete random variable.
- If $\mathbf{S}$ is an uncountable set then X is called a continuous random variable.

Example 3 - Binomial Asset Pricing Model

- Initial stock price S_0
- Next period
  - **Upward**: uS_0
  - **Downward**: dS_0

  where $0 < d < 1 < u$, $(d = \frac{1}{u}\ )$
- Toss a coin
  - Head: move up
  - Tail: move down
- Outcome of 2 tosses?

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u: **up factor,** d: **down factor**

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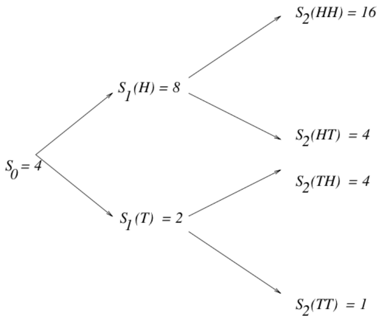

Binomial tree of stock prices with $S_0 = 4$, $u = 1/d = 2$.

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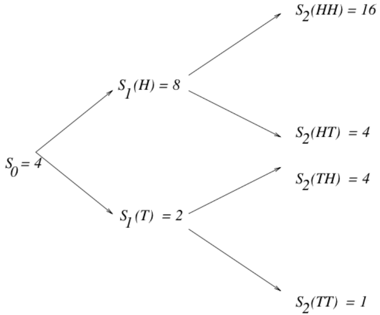


Binomial tree of stock prices with $S_0 = 4$, $u = 1/d = 2$.

The likelihood of stock price S_4 at perdiod 4?

Example 4

Consider Binomial Asset Pricing in Example 1 (with $S_0 = 4$, $u = 2$, $d = 1/2$) then S_1, S_2, S_3 are random variables.
Consider S_2

$$\begin{aligned}
S_2(HHH) &= S_2(HHT) = 16 \\
S_2(HTH) &= S_2(HTT) = S_2(THH) = S_2(THT) = 4 \\
S_2(TTH) &= S_2(TTT) = 1
\end{aligned}$$

Preimage under S_2 of the interval $[4, 27]$

$$\{w : 4 \leq S_2(w) \leq 27\} = A^c_{TT}$$

The complete list of subsets of Ω we can get as preimages of Borel sets in R is:

$$\emptyset, \Omega, A_{HH}, A_{HT} \cup A_{TH}, A_{TT}$$

and sets which can be built by taking unions of these.

Example 4 - cont

- This collection of sets is a $\sigma - algebra$, called the $\sigma - algebra$ generated by the random variable S_2 , and is denoted by $\sigma(S_2)$.
- Information content of $\sigma(S_2)$ is exactly the information learned from S_2
- $S_2(w) = 4$ tells you that w in $A_{HT} \cup A_{TH}$. So you know that in the first two tosses, there was a head and a tail, and you know nothing more.
- the information in the first two tosses is greater than the information in S_2
- if you see the first two tosses, you can distinguish A_{HT} from A_{TH}, but you cannot make this distinction from knowing the value of S_2 alone.

σ-algebra generated by a random variable $\sigma(X)$

- X: a random variable on $(\Omega, \mathcal{F})$
- $\{X \leq x\} = \{w : X(w) \leq x\}$: core information learned from X
- σ - algebra generated by X is

$$\sigma(X) = \sigma(\{X \leq x\}, x \in \mathbb{R}) =$$

- Let $\mathcal{G}$ be a $\sigma - algebra$ of $\mathcal{F}$. If $\sigma(X) \subset \mathcal{G}$ then X is called $\mathcal{G}$ - measurable

Cumulative distribution function (cdf)

Probability that X does not exceed x

$$F(x) = P(X \leq x)$$

Probability mass function (pmf) of a discrete random variable

- $Range(X)$: countable set $x_1, x_2, \ldots$
- pmf of X is the set of value $p_1, p_2, \ldots$ given by

$$p_i = P(X = x_i)$$

  which satisfies

  1. $0 \leq p_i \leq 1$ for all i
  2. $\sum_i p_i = 1$

- $P(a \leq x \leq b) = \sum_{i : a \leq x_i \leq b} p_i$
- cdf $F(x) = \sum_{i : x_i \leq x} p_i$

Example - Bernoulli distribution

Toss a fair coin. Let X be the number of H.
Probability mass function (pmf) of X

x	0	1
$P(X=x)$	$P(T)=1/2$	$P(H)=1/2$

Bernoulli distribution

A random variable X is Bernoulli distribution if it is an indicator random variable of a trial which has success probability p and failure probability $1-p$.

$$P(X=0)=1-p$$
$$P(X=1)=p$$

Denote $X \hookrightarrow Ber(p)$

Example - Binomial distribution

Toss a fair coin twice. Pmf of the number of H

x	$P(X = x)$
0	$P(TT) = 1/4$
1	$P(TH) + P(TT) = 1/2$
2	$P(HH) = 1/4$

Example - Binomial distribution

Toss a fair coin 3 times. Pmf of the number of H

x	$P(X = x)$
0	$P(TTT) = 1/8$
1	$P(TTH) + P(THT) + P(HTT) = 3/8$
2	$P(HHT) + P(HTH) + P(THH) = 3/8$
3	$P(HHH) = 1/8$

Binomial distribution

- n independent trials, each is Ber(p).
- X is the number of success
- X is called Binomial RV with parameter (n, p)
- Denote $X \sim Bino(n, p)$.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Example - Payoff of European Call Option

- **European call option**: confers the **right to buy** the stock at **maturity or expiration time** $T = 2$ for **strike price** $K = 14$ dollars. It is worth
  - $S_2 - K$ if $S_2 - K > 0$
  - 0 otherwise
- Value (payoff) of the option at maturity

$$C_2 = \max(S_2 - K, 0) = (S_2 - K, 0)^+$$

- Stock price: binomial model with $S_0 = 4$, $d = 1/2$, $u = 2$, $p = p(H)1/2$,$q = p(T) = 1 - p = 1/2$
- Find probability distribution for payoff C_2 of the European call option.

Solution

pmf of S_2

x	1	4	16
$P(S_2 = x)$	$1/4$	$1/2$	$1/4$

pmf of C_2

x	0	2
$P(C_2 = x)$	$3/4$	$1/4$

Example - European Put Option

- **European put option**: confers the **right to sell** the stock at **maturity or expiration time** $T = 2$ for **strike price** $K = 3$ dollars. It is worth
  - $K - S_2$ if $K - S_2 > 0$
  - 0 otherwise
- Value (payoff) of the option at maturity

$$P_2 = \max(K - S_2, 0) = (K - S_2, 0)^+$$

- Stock price: binomial model with $S_0 = 4$, $d = 1/2$, $u = 2$, $p(H) = p(T) = 1/2$
- Find probability distribution for payoff P_2 of the European put option.

Solution

pmf of S_2

x	1	4	16
$P(S_2 = x)$	$1/4$	$1/2$	$1/4$

pmf of C_2

x	0	2
$P(P_2 = x)$	$3/4$	$1/4$

Two different random variables can have the same distribution

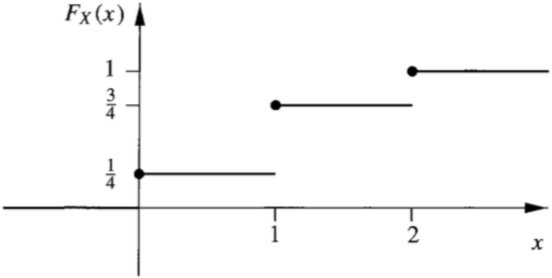
Example - A random variable can have more than one distribution

- Binomial Asset Pricing in Example 1
- If $p = q = \frac{1}{2}$ then $P(S_2 = 16) = \frac{1}{4}$
- If $p = 2/3$ and $q = \frac{1}{3}$ then $P(S_2 = 16) = \frac{4}{9}$

Example

Cdf for number of H when tossing a fair coint twice

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/4 & \text{if } 0 \leq x < 0 \\ 3/4 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$



Practice

Find c.d.f of stock price S_2, European call option C_2 and European put option P_2 in the previous example

Probability density function (pdf) of a continuous random variable

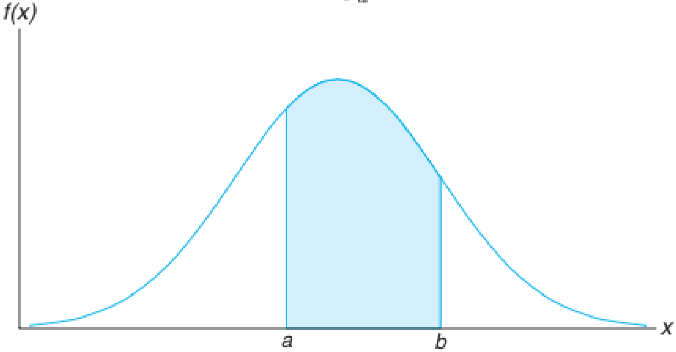
- $Range(X)$: uncountable
- The pdf of X is a function f that satisfies
  1. $f(x) \geq 0$ for all x
  2. $P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$
- cdf

$$F(a) = P(X \leq a = \int_{-\infty}^{a} f(x)dx)$$

- $P(X = a) = P(a \leq X \leq a) = \int_{a}^{a} f(x)dx = 0$
- $P(a \leq X \leq b) = \int_{a}^{b} f(x)dx$
- $P(a \leq X \leq b) = F(b) - F(a)$

Probability as an Area

$$P(a < X < b) = \int_a^b f(x)\,dx.$$



Note that probability of any individual value is 0

Interpretation of p.d.f

$$P\left(a-\frac{\epsilon}{2} \leq X \leq a+\frac{\epsilon}{2}\right) = \int_{a-\frac{\epsilon}{2}}^{a+\frac{\epsilon}{2}} f(x)dx$$
$$\approx \epsilon f(a)$$

$f(a)$ is a measure of how likely it is that the random variable will be near a.

Continuous uniform distribution

- X is uniformly distributed on $[a, b]$ if its pdf is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Continuous uniform distribution

- X is uniformly distributed on $[a, b]$ if its pdf is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

  Any value in $[a, b]$ is equally likely to be value of X.
  Denote $X \sim Uni[a, b]$.

- cdf

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

Exponential distribution with parameter λ

- pdf

$$f(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$$

- cdf

$$F(x) = \begin{cases} 0, & x < 0 \\ e^{-\lambda x}, & x \geq 0 \end{cases}$$

Normal distribution

Definition

Continuous RV X is said to be normally distributed with parameter μ and σ^2 if its pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

Denote $X \sim \mathcal{N}(\mu, \sigma^2)$.

Properties

1. $E(X) = \mu$
2. $Var(X) = \sigma^2$

Properties of cdf

- Increasing
- $$\lim_{x \to -\infty} F(x) = 0$$

  and

  $$\lim_{x \to \infty} F(x) = 1$$
- has left-limit
- Right- continuous

Simulate random number

Simulate uniform random number with Scilab

- Simulate 1 random number from uniform distribution: **rand**
- Simulate a $m \times n$ matrix of random number from unifrom distribution **rand(m,n)**

Simulate random number from typical distribution

search Google for binomial, poisson, exponential, normal ....

Practice

- Generate 10000 uniform random number
- Plot histogram of the generated sample to make a comparison with pdf of uniform distribution
- Similar for binomial, poisson, exponential, normal random number

How to generate value of European Call/Put Option on binomial asset pricing model

Inverse Transform Method - Non-uniform Random Numbers

Let X be a RV with cdf F.
For all $y \in [0, 1]$, define

$$F^{-1}(y) = \inf\{x : F(x) \geq y\}$$

If $U \sim Uni([0, 1])$ then $F^{-1}(U)$ has the same distribution as X

Practice

Generate 10000 random number from $Exp(2)$ by inverse transform method. **Hint** cdf of $Exp(2)$ is

$$F(x) = e^{-2x},\ x \geq 0$$

Inverse function of $F(x)$ is

$$F^{-1}(x) = -\frac{1}{2}\ln x$$

Simulate Value for Payoff of European Call Option

- European Call Option: $T = 2$, $K = 14$
- Binomial stock price model: $S_0 = 4$, $u = 2$, $d = 1/2$, $p(H) = p(T) = 1/2$
- pmf of C_2: $P(C_2 = 0) = 3/4$, $P(C_2 = 2) = 1/4$
- cdf of C_2

$$F(x) = \begin{cases} 0, & x < 0 \\ 3/4, & 0 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

-

$$F^{-1}(y) = \begin{cases} 0, & y \le 3/4 \\ 2, & y > 3/4 \end{cases}$$

Simulate Value for Payoff of European Call Option - cont

- Generate U
- If $U \leq 3/4$ then $C_2 = 0$ else $C_2 = 2$.

Generate random number from a discrete distribution

- X takes on n distinct values $x_1 < x_2 < ... < x_n$
- X has pmf $P(X = x_i) = p_i$
- Simulate a random number of X
  - Generate U
  - If $F(x_{i-1} < U \leq F_{x_i})$ (it means that $\sum_{k=1}^{i-1} p_k < U \leq \sum_{k=1}^{i} p_k$) then set $X = x_i$

Practice

Generate 10000 value for payoff of European Put Option with $T = 2$, $K = 3$, binomial stock price model $S_0 = 4$, $u = 1/d = 2$, $p = q = 1/2$
Plot histogram of generated value and compare the result with pmf of P_3

Practice

- Consider a binomial asset pricing model $S_0 = 4$, $u = 1/d = 2$, $p = 1/3$, $q = 2/3$
- Simulate a path of stock price up to $T = 4$. It means that a sequence of value $(S_0, S_1, S_2, S_3, S_4)$

Expectation and variance

Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The unconditional expectation of X is

$$E(X) = \begin{cases} \sum_{k=1}^{\infty} x_k P(X = x_k), \text{if } X \text{ is discrete with values } \{x_1, x_2, ...\} \\ \int_{-\infty}^{\infty} x f(x) dx, \text{if } X \text{ is continuous with density function } f \end{cases}$$

Let g is a real-valued function. $E(g(X)) = \begin{cases} \sum_{k=1}^{\infty} g(x_k) P(X = x_k), \\ \int_{-\infty}^{\infty} g(x) f(x) dx \end{cases}$

The variance of X is

$$Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2.$$

Some properties of expectation and variance

- $E[aX + b] = aE[X] + b$
- $Var[aX + b] = a^2 Var[X]$

Example

Find expectation and variance for stock price S_2, European call option C_2 and European put option P_2 in the previous example

Plan

Joint distribution of two discrete random variables

- For constants $a < b$ and $c < d$,

$$P(a \leq X \leq b, c \leq Y \leq d) = \sum_{x=a}^{b} \sum_{y=c}^{d} \mathbf{P}(X = x, Y = y)$$

Joint distribution of two discrete random variables

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$$P(a \leq X \leq b, c \leq Y \leq d) = \sum_{x=a}^{b} \sum_{y=c}^{d} \mathbf{P}(X = x, Y = y)$$

- if X and Y are discrete then the probability mass function P_X, P_Y are determined as:

  $P_X(X = x) = \sum_{y=-\infty}^{\infty} \mathbf{P}(X = x, Y = y)$ and

  $P_Y(Y = y) = \sum_{x=-\infty}^{\infty} \mathbf{P}(X = x, Y = y)$.

Joint distribution of two continuous random variables

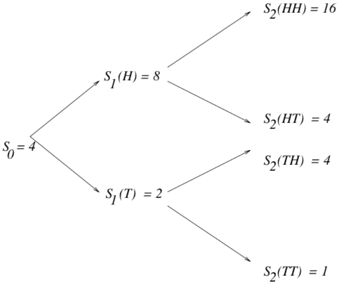
Given two continuous random variables X and Y. The joint probability density function, denoted by $f_{X,Y}(x,y)$ of X and Y satisfies the following properties:

- $f_{X,Y}(x,y) > 0, \forall x, y$
- $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dx dy = 1$
- $P(a \leq x \leq b, c \leq y \leq d) = \int_{a}^{b} \int_{c}^{d} f_{X,Y}(x,y) dy dx$

Example

Consider binomial asset pricing model

- $S_0 = 4$, $u = 2$, $d = 1/2$
- $p(H) = 1/3$, $p(T) = 2/3$



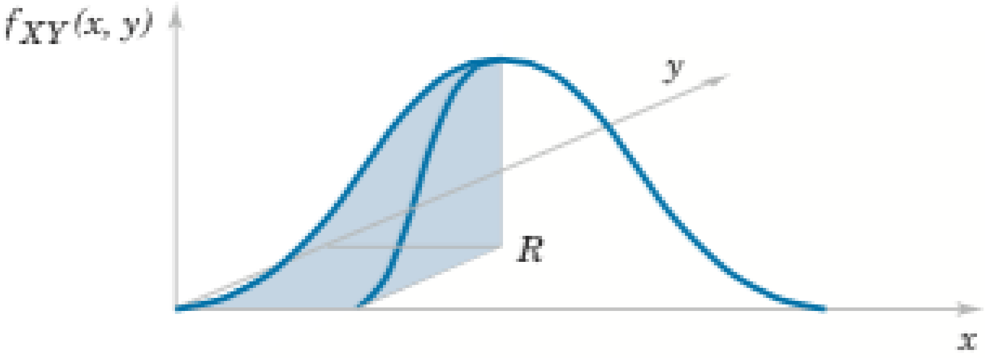
Binomial tree of stock prices with $S_0 = 4$, $u = 1/d = 2$.

Find the joint pmf of the stock price (S_1, S_2)

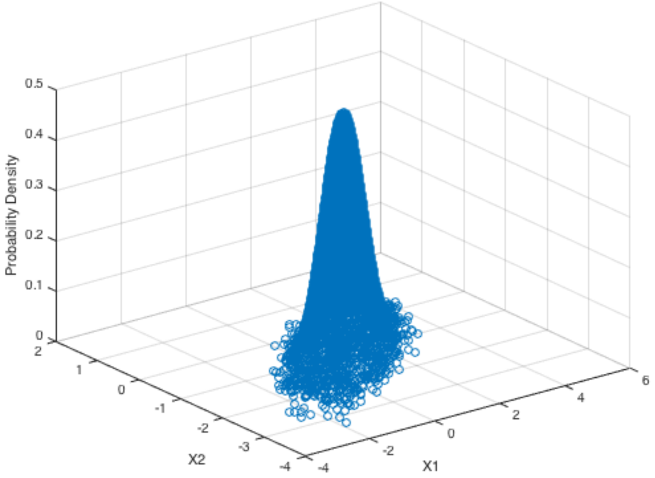
Joint distribution of two continuous random variables

- Cumulative distribution function CDF is defined as $F(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y) dy dx$ and
$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$$
- The marginal probability density f_X of X is computed as: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$
- The marginal probability density f_Y of Y is computed as: $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$
- $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy$

$$P((X,Y) \in R = \iint_R f_{X,Y}(x,y)dxdy$$



Example - multivariate normal random vector



Plan

Conditional probability

For events A and B, the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

defined for $P(B) > 0$.

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Measure the likelihood of A in the new sample space B

Example

Roll a fair dice twice. What is the probability that **the first roll is a** 2 *given that the sum of the roll is* 7?

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- A*: outcome of the 1st dice*

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- $A = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)\}$
- $AB = \{(2, 5)\}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/36}{6/36} = \frac{1}{6}$$

Practice

Roll a fair dice twice. What is the probability that **the first roll is a** 2 *given that the second roll is even*?

Does the result of the first roll effect on the second?

Independence

Two events A and B are independent if

$$P(A|B) = P(A),\ P(B) > 0$$

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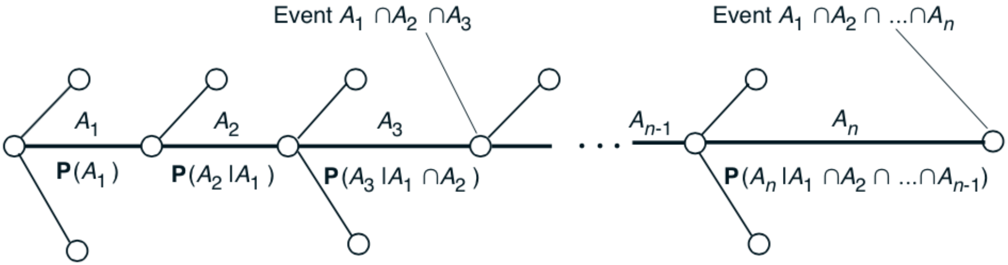
$$P(AB) = P(A)P(B)$$

the second condition is used to check the dependency between A and B even for $P(B) = 0$

Multiplication rule

1. $P(AB) = P(B)P(A|B)$
2. General case

$$P(A_1A_2...A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)...P(A_k|A_1...A_{k-1})$$



Example

Consider a binomial asset pricing model with $S_0 = 4$, $u = 2$, $d = 1/2$, $p = 1/3$, $q = 2/3$. Find $P(S_1 = 2, S_2 = 4, S_3 = 2)$

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Solution

$$
\begin{aligned}
&P(S_1 = 2, S_2 = 4, S_3 = 2)\\
&= P(S_1 = 2)P(S_2 = 4|S_1 = 2)P(S_3 = 2|S_1 = 2, S_2 = 4)\\
&= qpq = pq^2
\end{aligned}
$$

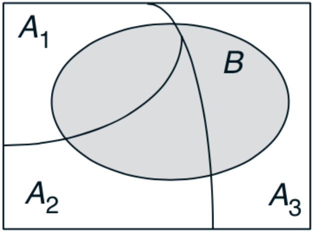
Law of total probability

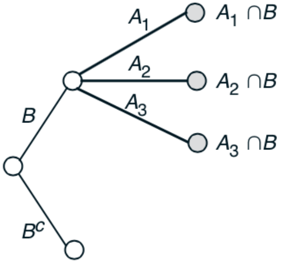
Let $A_1, \ldots, A_k$ be a partition the sample space

- $A_i \cap A_j = \emptyset$ for all $i \neq j$ (*mutually exclusive (disjoint)*)
- $\cup_i A_i = \Omega$

Then, for any event B, we have

$$P(B) = \sum_{i=1}^{k} P(B \cap A_i) = \sum_{i=1}^{k} P(B|A_i)P(A_i).$$





Example

Consider a binomial asset pricing model with $S_0 = 4$, $u = 1/d = 2$, $p(H) = 2/3$, $p(T) = 1/3$. Find $P(S_2) = 4$

Example

An insurance company thinks that there are some people who are accident prone and others who are not. An accident-prone person will have an accident at some time within 1 year with probability .4, whereas this probability is .2 for other person. Assume that 30 percent of the population is accident prone, what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

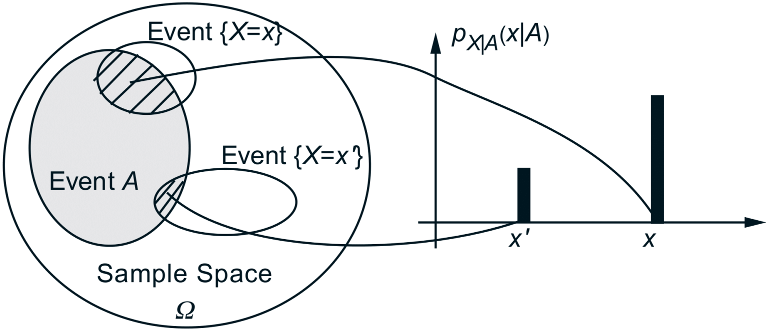
Conditioning a RV on an event

The conditional pmf of a RV X given an event A is

$$p_{X|A}(x) = P(X = x|A) = \frac{P((X = x) \cap A)}{P(A)}$$

if $P(A) > 0$

Visualization



Example

Let X be the roll of a fair die and let A be the event that the roll is an even number. Then

$$p_{X|A}(1) = \frac{P(X = 1 \text{ and roll is even})}{P(\text{roll is even})} = 0$$

Example

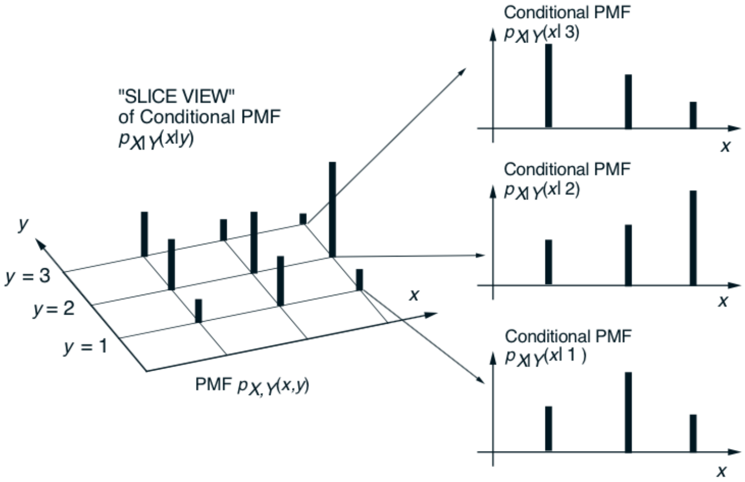
Let X be the roll of a fair die and let B be the event that the roll is an even number. Then

$$p_{X|B}(2) = \frac{P(X = 2 \text{ and roll is even})}{P(\text{roll is even})} = \frac{1}{3}$$

Conditional of a discrete RV on another

- 2 RVs X and Y
- given $Y = y$ with $P(Y = y) > 0$
- conditional pmf of X

$$
\begin{aligned}
p_{X|Y}(x|y) &= P(X = x|Y = y) \\
&= \frac{P(X = x, Y = y)}{P(Y = y)}
\end{aligned}
$$

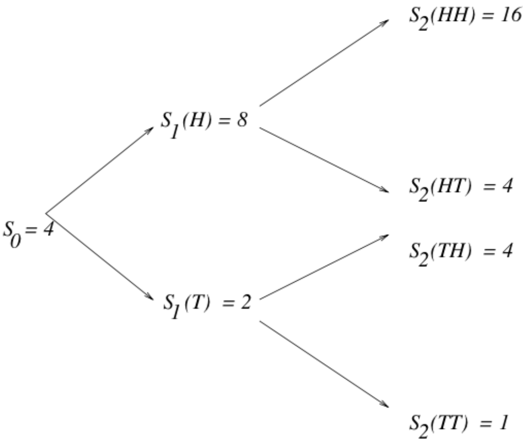


For each y, we view the joint pmf along the slice $Y = y$ and renormalize such that

$$\sum_x p_{X|Y}(x|y) = 1$$

Example

Consider a binomial asset pricing model with $S_0 = 4$, $d = 2$, $u = \frac{1}{2}$, $p = 2/3$ and $q = 1/3$. Find conditional pmf of S_2 given $S_1 = 2$.



Solution

x	1	4	8
$P_{S_2\|S_1=2}(x\|2)$	$\frac{1}{3}$	$\frac{2}{3}$	0

Practice

Consider a binomial asset pricing model with $S_0 = 4$, $d = 2$, $u = \frac{1}{2}$, $p = 2/3$ and $q = 1/3$. Find conditional pmf of S_1 given $S_2 = 4$.

Practice

Consider binomial asset pricing model

- $S_0 = 4$, $u = 2$, $d = 1/2$
- $p = 1/3$, $q = 2/3$

Find

1. The conditional probability of $S_2 = 16$ given that $S_1 = 8$
2. The conditional probability of $S_3 = 8$ given that $S_1 = 8$

Conditional distributions

- If X and Y are jointly distributed discrete random variables, then the conditional probability mass function of Y given $X = x$ is
$P(Y = y|X = x) = \dfrac{P(X = x, Y = y)}{P(X = x)}$ defined when $P(X = x) > 0$.

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- For continuous random variable X and Y, the conditional density function of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)},$$

provide the likelihood that X takes values near x given that Y takes values near by y

Example

Suppose that $W_1 \hookrightarrow N(0,1)$ and $W_2(0,1)$ are log-return of the first and ther second year of stock A. Suppose W_1 and W_2 are independent. The cumulative log-return of this stock is

$$B_1 = W_1$$

and

$$B_2 = W_1 + W_2.$$

Given that $B_1 = 2$, find the conditional pdf of B_2.

Example

Alvin goes to a bus stop where the time T between two successive buses has an exponential PDF with parameter λ. Suppose that Alvin arrives t secs after the preceding bus arrival and let us express this fact with the event $A = \{T > t\}$. Let X be the time that Alvin has to wait for the next bus to arrive. What is the conditional CDF $F_{X|A}(x|A) = P(X \leq x|A)$?

Practice

Let X be exponentially distributed with mean 1. Once we observe the experimental value x of X, we generate a normal random variable Y with zero mean and variance $x + 1$. What is the joint pdf of X and Y ?

Independence of random variables

- X and Y are independence if and only if $F(x, y) = F_X(x)F_Y(y)$ for all x, y
- If X and Y are discrete RV then X and Y are independent if and only if $P(X = x, Y = y) = P(X = x)P(Y = y)$ for all x, y
- If X and Y are continuous RV then X and Y are independent if and only if $f(x, y) = f_X(x)f_Y(y)$ for all x, y

Covariance and correlation coefficient

- Covariance of X and Y is $cov(X,Y) = E[XY] - E[X]E[Y]$.
- Correlation coefficient of X and Y is given by:

$$cor(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$$

where σ_X, σ_Y are standard deviations of X and Y

$$-1 \leq corr(X,Y) \leq 1$$

Correlation coefficient is used to measure how strong linear relationship between X and Y is

Properties

- $Cov(X,Y) = Cov(Y,X)$
- $Cov(X,X) = Var(X)$
- $Cov(aX,Y) = aCov(X,Y)$
- $Cov(X+Y,Z) = Cov(X,Z) + Cov(Y,Z)$
- If X and Y are independent then $Cov(X,Y) = 0$

Cov(X,Y+Z)=cov(X,Z)+cov(Y,Z)

Variance of Sum

$X_1, \ldots, X_n$: RVs

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$$

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$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$$

Sum of normal distributions

If $X \hookrightarrow \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \hookrightarrow \mathcal{N}(\mu_Y, \sigma_Y^2)$ and $Cov(X, Y) = \sigma_{XY}$ then

$$X + Y \hookrightarrow \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY})$$

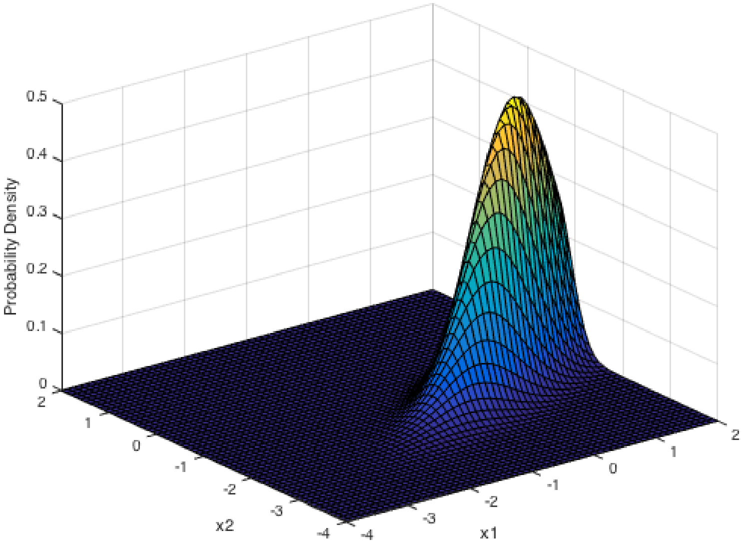
Bivariate normal distribution

Let $X_1 \hookrightarrow \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \hookrightarrow \mathcal{N}(\mu_2, \sigma_2^2)$ be 2 normal random variables with covariance σ_{12} then the random vector $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ is a bivariate normal distribution with joint pdf

$$f(x_1, x_2) = \frac{1}{2\pi \det(\Sigma)^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

where

- $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$
- $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$ is the variance - covariance matrix of X



Probability Density
0.5
0.4
0.3
0.2
0.1
0
x2
2
1
0
-1
-2
-3
-4
x1
-4
-3
-2
-1
0
1
2

Example

Let $X \hookrightarrow \mathcal{N}(0,4)$ and $Y \hookrightarrow \mathcal{N}(0,1)$ be daily return of stock A and B respectively. Suppose that $Cov(X,Y) = 2$.

1. Consider a portfolio consisting of 70% stock A and 30% stock B. Then the return of the portfolio is given by
$$W = 0.7X + 0.3Y$$
Determine the distribution of W.

2. Suppose that we aim to allocate stock A and B with weight a and $1-a$. The return of portfolio is
$$U = aX + (1-a)Y$$
Determine a to minimize risk of the portfolio.

Example

Suppose that $W_1 \hookrightarrow N(0,1)$ and $W_2(0,1)$ are log-return of the first and ther second year of stock A. Suppose W_1 and W_2 are independent. The cumulative log-return of this stock is

$$B_1 = W_1$$

and

$$B_2 = W_1 + W_2.$$

Given that $B_1 < 0.5$, find the probability that $B_2 > 0.5$.

Conditional expectation

- $E(Y|X=x) = \begin{cases} \sum_{y} yP(Y=y|X=x), \text{if X is discrete} \\ \int_{-\infty}^{\infty} yf_{Y|X}(y|x)dy, \text{if X is continuous} \end{cases}$

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- $E(Y|X=x)$ is a function of x, i.e., the result depends on the value of x

Example

Consider binomial asset pricing model

- $S_0 = 4$, $u = 2$, $d = 1/2$
- $p(H) = 1/3$, $p(T) = 2/3$

Find

1. The conditional expectation of S_2 given that $S_1 = 8$.
2. The conditional expectation of S_3 given that $S_1 = 8$

Example

Suppose that $W_1 \hookrightarrow N(0,1)$ and $W_2(0,1)$ are log-return of the first and ther second year of stock A. Suppose W_1 and W_2 are independent. The cumulative log-return of this stock is

$$B_1 = W_1$$

and

$$B_2 = W_1 + W_2.$$

Given that $B_1 = 1$, find conditional expectation of B_2.

Important note

1. If X and Y are independent then $E(Y|X=x) = E(Y)$.
2. If g is a function then $E(g(X)|X=x) = g(x)$.
3. If g is a function then

$$E(g(Y)|X=x) = \begin{cases} \sum_y g(y)P(Y=y|X=x), \text{if X is discrete} \\ \int_{-\infty}^{\infty} g(y) f_{Y|X}(y|x)dy, \text{if X is continuous} \end{cases}$$

Conditional expectation given a random variable

Let X and Y are random variable. The conditional expectation of Y given X, denoted by $E(Y|X)$ has three important properties:

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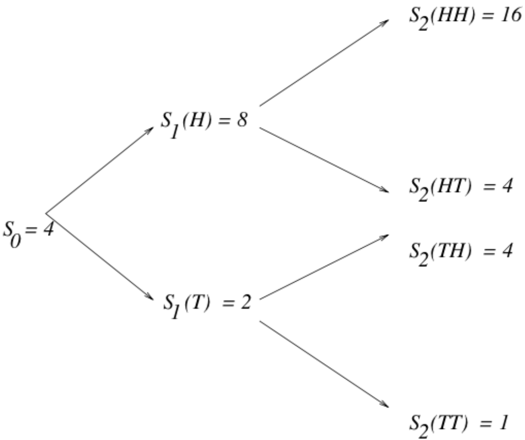
Let X and Y are random variable. The conditional expectation of Y given X, denoted by $E(Y|X)$ has three important properties:

1. $E(Y|X)$ is a random variable.
2. If $E(Y|X = x) = g(x), \ \forall x$, then $E(Y|X) = g(X)$.

$E(Y|X)$ can be regared as an estimate of the value of Y based on the knownledge of X

Example

Consider a binomial asset pricing model with $S_0 = 4$, $u = 1/d = 2$, $p(H) = 2/3$ and $p(T) = 1/3$, find $E(S_2|S_1)$



Properties of conditional expectation

1. Linearity

$$E(aY + bZ|X) = aE(Y) + bE(Z)$$

2. Taking out of what we know

$$E(f(X)Y|X) = f(X)E(Y|X)$$

3. Iterated conditioning σ - algebra $\mathcal{G} \subset \mathcal{H}$

$$E(E(Z|\mathcal{H})|\mathcal{G}) = E(Z|\mathcal{G})$$

4. Independence

$$E(Y|X) = E(Y)$$

   if X and Y are independent.

5. Tower property: $E(Y) = E(E(Y|X))$

Example - Linearity

Considering a binomial asset pricing model with $S_0 = 4$, $p = 1/3$ and $q = 2/3$. Compare

$$E(S_2 + S_3|S_1)$$

and

$$E(S_2|S_1) + E(S_3|S_1)$$

Solution

- S_1 takes two values 8 and 2
- Given $S_1 = 8$

$$E(S_2|S_1 = 8) = \frac{2}{3}(16) + \frac{1}{3}(4) = 12$$

$$E(S_3|S_1 = 8) = \left(\frac{2}{3}\right)^2 (32) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(8)$$
$$+ \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)(8) + \left(\frac{1}{3}\right)^2 (2) = 18$$

So

$$E(S_2|S_1 = 8) + E(S_3|S_1 = 8) = 12 + 18 = 30$$

- Given $S_1 = 8$, (S_2, S_3) takes pair values $(16, 32)$, $(16, 8)$, $(4, 8)$, $(4, 2)$. So

$$E(S_2 + S_3 | S_1 = 8) = \left(\frac{2}{3}\right)^2 (16 + 32) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(16 + 8) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(4 + 8) + \left(\frac{1}{3}\right)^2 (4 + 2) = 30$$

- Hence $E(S_2 + S_3 | S_1 = 8) = E(S_2 | S_1 = 8) + E(S_3 | S_1 = 8) = 30$
- Similary $E(S_2 + S_3 | S_1 = 2) = E(S_2 | S_1 = 2) + E(S_3 | S_1 = 2) = 7.5$
- Regardless the outcome of S_1, we have

$$E(S_2 + S_3 | S_1) = E(S_2 | S_1) + E(S_3 | S_1)$$

Example - Take out what is known

Compare

$$E(S_1 S_2 | S_1)$$

and

$$S_1 E(S_2 | S_1)$$

Example - Iterated conditioning

Compare

$$E(S_3|S_1)$$

and

$$E(E(S_3|(S_1, S_2))|S_1)$$

Example-Independence

Compare

$$E\left(\frac{S_2}{S_1}|S_1\right)$$

and

$$E\left(\frac{S_2}{S_1}\right)$$

Practice

Suppose that $W_1 \hookrightarrow N(0,1)$ and $W_2(0,1)$ are log-return of the first and ther second year of stock A. Suppose W_1 and W_2 are independent. The cumulative log-return of this stock is

$$B_1 = W_1$$

and

$$B_2 = W_1 + W_2.$$

Find $E(B_2|B_1)$.

Example: Tower property

Angel will harvest a random number N tomatoes in her garden, where N has a Poisson distribution with parameter λ. The pmf of N is given by

$$P(N = n) = e^{-\lambda}\frac{\lambda^n}{n!}\ n \geq 0$$

Each tomato is checked for defects. The chance that a tomato has defects is p. Defects are independent from tomato to tomato. Find the expected number of tomatoes with defects.

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- The number of tomatoes with defects X depends on N.
- $N \sim P(\lambda) \Rightarrow E[N] = \lambda$.
- $(X|N = n) \sim B(p, n) \Rightarrow E[X|N = n] = np \Rightarrow E[X|N] = Np$.

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Each tomato is checked for defects. The chance that a tomato has defects is p. Defects are independent from tomato to tomato. Find the expected number of tomatoes with defects.

- The number of tomatoes with defects X depends on N.
- $N \sim P(\lambda) \Rightarrow E[N] = \lambda$.
- $(X|N = n) \sim B(p, n) \Rightarrow E[X|N = n] = np \Rightarrow E[X|N] = Np$.
- $E[X] = E[E[X|N]] = E[Np] = pE[N] = p\lambda$.

Plan

Law of Large Numbers

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of identically and independent distributed random variables with mean (expectation) μ and variance σ^2. Then the average

$$\bar{X}_n = \frac{X_1 + \ldots + X_n}{n}$$

converges almost surely to μ as n tends to infinite.

Central Limit Theorem

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of identically and independent distributed random variables with mean (expectation) μ and variance σ^2. Then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$$

is approximated by a standard normal distribution $\mathcal{N}(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

for large n

Estimate Expectation of a RV by Simulation Monte Carlo

- Simulate N random number from the given distribution $x_1, \ldots, x_N$
- Estimate sample mean $\bar{x} = \frac{x_1 + \ldots + x_N}{N}$
- Confidence Interval at level α

$$\bar{x} - z_{\frac{\alpha}{2}} \frac{s}{\sqrt{N}} < \mu < \bar{x} - z_{\frac{\alpha}{2}} \frac{s}{\sqrt{N}}$$

where $s = \sqrt{\frac{\sum_{n=1}^{n} (x_n - \bar{x})^2}{n-1}}$ is the sample standard deviation

Example

- Come back to the example of simulation value for payoff of European Call option for binomial stock pricing model.
- Simulate N values for payoff and then estimate the expected payoff of European Call Option.
- Construct CI at 95% for N increasing from 1000 to 10000.
- Compare the simulated result with exactly expectation computed from the pmf of C_2