**EXERCISES AND PROBLEMS FOR CHAPTER 1:
METRIC SPACES**

A. Problems and Exercises for everyone:
All problems and exercises in parts B and C.

B. Non-assessed Problems and Exercises (corrected in class):[1]

0.1.7; 0.1.8; 0.1.10; 0.1.13; 0.1.20; 0.2.8; 0.3.2; 0.3.5; 0.3.7; 0.4.1; 0.5.2; 0.5.4–0.5.6; 0.5.8; 0.5.9; 0.5.11; 0.6.1; 0.6.7; 0.6.13; 0.6.16; 0.7.1; 0.7.2; 0.7.9; 0.7.10.

C. Assessed Assignments (to be submitted):

0.1.1; 0.1.2; 0.1.3; 0.1.4–0.1.6; 0.1.16; 0.1.18; 0.1.19; 0.2.1; 0.2.2; 0.2.4; 0.2.9; 0.3.1; 0.3.3; 0.4.2; 0.5.1; 0.5.3; 0.5.10; 0.6.6; 0.7.3; 0.7.8.

D. Bonus Problems and Exercises:

Remaining problems and exercises.

0.1 LOGIC. SETS AND CARDINAL NUMBERS

Exercise 0.1.1. Consider the following statement:

$$\forall \text{ integers } n, \text{ if } n^2 \text{ is even then } n \text{ is even.}$$

Which of the following are equivalent ways of expressing this statement?

(a) All integers have even squares and are even.

(b) Given any integer whose square is even, that integer is itself even.

(c) For all integers, there are some whose square is even.

(d) Any integer with an even square is even.

(e) If the square of an integer is even, then that integer is even.

[1] 0.5.4–0.5.6 means "from Exercise 0.5.4 to Exercise 0.5.6".

(f) All even integers have even squares.

Exercise 0.1.2. The real number r is **rational** if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called **irrational**. Show that $\sqrt{2}$ is irrational.

Exercise 0.1.3. If 41 balls are chosen from a collection of red, white, blue, garnet, and gold colored balls, then one of the following statement is true:

(a) there are at least 12 red balls,

(b) there are at least 15 white balls,

(c) there are at least 4 blue,

(d) there are at least 10 garnet,

(e) there are at least 4 gold balls chosen.

Exercise 0.1.4. For two sets A and B show that the following statements are equivalent.

(a) $A \subset B$;

(b) $A \cup B = B$;

(c) $A \cap B = A$.

Exercise 0.1.5. Let $f : X \to Y$ be a function. Show that $A \subset f^{-1}\big(f(A)\big)$ for all $A \subset X$ and $f\big(f^{-1}(B)\big) \subset B$ for all $B \subset Y$.

Exercise 0.1.6. Assume $f : X \to Y$ and $g : Y \to Z$. If $A \subset Z$, show that $(g \circ f)^{-1}(A) = f^{-1}\big(g^{-1}(A)\big)$.

Exercise 0.1.7. Let $\{A_i\}_{i\in I}$ be a family of subsets of X and $\{B_i\}_{i\in I}$ a family of subsets of Y, and let $f : X \to Y$ be a mapping. Prove the following relationships

$$f\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} f(A_i), \qquad f\Big(\bigcap_{i\in I} A_i\Big) \subset \bigcap_{i\in I} f(A_i)$$
$$f^{-1}\Big(\bigcup_{i\in I} B_i\Big) = \bigcup_{i\in I} f^{-1}(B_i), \qquad f^{-1}\Big(\bigcap_{i\in I} B_i\Big) = \bigcap_{i\in I} f^{-1}(B_i).$$

If B be a subset of Y, then $f^{-1}(B^c) = \big(f^{-1}(B)\big)^c$.

Exercise 0.1.8. Assume that A and B are nonempty subsets of $\overline{\mathbb{R}}$ and satisfy $B \subset A$. Show that $\sup B \le \sup A$ and $\inf A \le \inf B$.

Exercise 0.1.9. If $A \subset \overline{\mathbb{R}}$ and $c \in \mathbb{R}$, $c \ge 0$, then

$$\sup(cA) = c \cdot \sup A \quad \text{and} \quad \inf(cA) = c \cdot \inf A$$

where $cA = \{ca : a \in A\}$. Postulate a similar type of statement for $\sup(cA)$ if $c < 0$.

Exercise 0.1.10. Prove that if a is an upper bound for $A \subset \mathbb{R}$, and if a is also an element of A, then it must be that $a = \sup A$.

Exercise 0.1.11. If $\sup A < \sup B$, then show that there exists an element $b \in B$ that is an upper bound for A.

Exercise 0.1.12. Let A and B be subsets of $\mathbb{R}$. Decide if the following statements about suprema and infima are true or false. For any that are false, supply an example where the claim in question does not appear to hold.

(a) A finite, nonempty set always contains its supremum.

(b) If $a < u$ for every element a in A, then $\sup A < u$.

(c) If $a < b$ for every $a \in A$ and every $b \in B$, then it follows that $\sup A < \inf B$.

(d) Define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Show that

$$\inf(A + B) = \inf A + \inf B \quad \text{and} \quad \sup(A + B) = \sup A + \sup B,$$

(e) If $\sup A \le \sup B$, then there exists an element $b \in B$ that is an upper bound for A.

Exercise 0.1.13. Let $f : X \to \overline{\mathbb{R}}$ be finite functions. Show that

$$\sup_{x \in X} \big(f(x) + g(x)\big) \le \sup_{x \in X} f(x) + \sup_{x \in X} g(x)$$

and that

$$\inf_{x \in X} \big(f(x) + g(x)\big) \ge \inf_{x \in X} f(x) + \inf_{x \in X} g(x).$$

Exercise 0.1.14. Let X and Y be nonempty sets and let $h : X \times Y \to \overline{\mathbb{R}}$ and let $f, g : X \to \overline{\mathbb{R}}$ be defined by

$$f(x) := \sup_{y \in Y} h(x,y) \quad \text{and} \quad g(y) := \inf_{x \in X} h(x,y).$$

Prove that

$$\sup_{y \in Y} g(y) \le \inf_{x \in X} f(x),$$

that is,

$$\sup_{y \in Y} \inf_{x \in X} h(x,y) \le \inf_{x \in X} \sup_{y \in Y} h(x,y).$$

Show that the inequality may be either an equality or a strict inequality.

Exercise 0.1.15. Let X and Y be nonempty sets and let $h : X \times Y \to \overline{\mathbb{R}}$ and let $F, G : X \to \overline{\mathbb{R}}$ be defined by

$$F(x) := \sup_{y \in Y} h(x,y) \quad \text{and} \quad G(y) := \sup_{x \in X} h(x,y).$$

Establish the **Principle of the Iterated Suprema**:

$$\sup\{h(x,y) : (x,y) \in X \times Y\} = \sup_{x \in X} F(x) = \sup_{y \in Y} G(y).$$

We sometimes express this in symbols by

$$\sup_{x,y} h(x,y) = \sup_{x} \sup_{y} h(x,y) = \sup_{y} \sup_{x} h(x,y).$$

Exercise 0.1.16. Alternate the terms of the sequences $\{1+1/n\}$ and $\{-1/n\}$ to obtain the sequence $\{x_n\}$ given by

$$2, -1, \frac{3}{2}, -\frac{1}{2}, \frac{4}{3}, -\frac{1}{3}, \frac{5}{4}, -\frac{1}{4}, \ldots.$$

Determine the values of $\limsup x_n$ and $\liminf x_n$. Also find $\sup x_n$ and $\inf x_n$.

Exercise 0.1.17. This exercise establish the basic properties of the limit superior and the limit inferior of extended real sequences.

Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\overline{\mathbb{R}}$. Prove the following statements.

(a) $\liminf x_n \leq \limsup x_n$.

(b) If $0 \leq \alpha \leq \infty$, then

$$\liminf(\alpha \cdot x_n) = \alpha \cdot \liminf x_n \quad \text{and}$$
$$\limsup(\alpha \cdot x_n) = \alpha \cdot \limsup x_n.$$

(c) If $-\infty \leq \beta \leq 0$, then

$$\liminf(\beta \cdot x_n) = \beta \cdot \limsup x_n \quad \text{and}$$
$$\limsup(\beta \cdot x_n) = \beta \cdot \liminf x_n.$$

(d) $\liminf x_n + \liminf y_n \leq \liminf(x_n + y_n)$.

(e) $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$.

(f) If $x_n \leq y_n$ for all $n \in \mathbf{N}$, then

$$\liminf x_n \leq \liminf y_n \quad \text{and} \quad \limsup x_n \leq \limsup y_n.$$

(g) If $\{x_n\}$ is convergent, then

$$\liminf(x_n + y_n) = \lim x_n + \liminf y_n,$$
$$\limsup(x_n + y_n) = \lim x_n + \limsup y_n.$$

Exercise 0.1.18. Prove that $5^n - 4n - 1$ is divisible by 16 for all n.

Exercise 0.1.19. Prove the following equality by mathematical induction.

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

Exercise 0.1.20. Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

0.2 METRIC SPACES AND EXAMPLES

Exercise 0.2.1. Show that $\rho(x,y) = |e^x - e^y|$ is a metric on $\mathbb{R}$.

Exercise 0.2.2. Let $X = (0, \infty)$. Show that

$$d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right|, \quad x, y \in X$$

is a distance on X.

Exercise 0.2.3. Let $c_1, c_2, \ldots, c_n$ be positive numbers. For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, define

$$\rho(x,y) = \sum_{i=1}^{n} c_1 |x_i - y_i|.$$

Show that ρ is a metric on $\mathbb{R}^n$.

Exercise 0.2.4. Let Ω be a nonempty set. We shall denote by $B(\Omega)$ the set of all real-valued functions defined on Ω that are bounded. That is, a function $f : \Omega \to \mathbb{R}$ belongs to $B(\Omega)$ if and only if there exists a number $M > 0$ (depending on f) such that $|f(\omega)| \leq M$ for all $\omega \in \Omega$. For each $f \in B(\Omega)$ define

$$\|f\| = \sup\{|f(\omega)| : \omega \in \Omega\}.$$

Show that $\|\cdot\|$ is a norm on $B(\Omega)$.

Exercise 0.2.5. Two metrics d_1 and d_2 on a set X are said to be **equivalent** if, for each $x \in X$ and $\epsilon > 0$, there are positive numbers r_1 and r_2 such that

$$B_1(x, r_1) \subset B_2(x, \epsilon) \quad \text{and} \quad B_2(x, r_2) \subset B_1(x, \epsilon).$$

Here B_j denotes the ball in (X, d_j), $j = 1, 2$.

Now let (X, d) be a metric space and

$$\rho(x,y) := \frac{d(x,y)}{1 + d(x,y)}, \quad x, y \in X.$$

Prove that d and ρ are equivalent metrics on X. (Hint: The function $t \to t/(1+t)$ is increasing.)

Exercise 0.2.6. For $X := (0,1)$, prove the following:

(a) $\rho(x,y) = |(1/x) - (1/y)|$ is a metric on X.

(b) The usual metric and ρ are equivalent.

(c) There is no metric on $\mathbb{R}$ which is equivalent to the usual metric and which induces the metric ρ on X.

Exercise 0.2.7. Given a metric space (X,d). For $x, y \in X$, define $\rho(x,y) = \min\{d(x,y), 1\}$.

(a) Show that ρ is a metric on X.

(b) Show that $\lim_{n\to\infty} x_n = x$ in (X,d) if and only if $\lim_{n\to\infty} x_n = x$ in (X,ρ).

Exercise 0.2.8. There are other norms of interest on the space $\mathbb{R}^n$ besides the Euclidean norm. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \quad \text{and} \quad \|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $\mathbb{R}^n$. Sketch the unit balls

$$B_1 = \{x \in \mathbb{R}^2 : \|x\|_1 \le 1\} \quad \text{and} \quad B_\infty = \{x \in \mathbb{R}^2 : \|x\|_\infty \le 1\}.$$

Exercise 0.2.9. Show that $\|(x,y,z)\| = |x| + 2\sqrt{y^2+z^2}$ is a norm on $\mathbb{R}^3$. Sketch the unit ball $B = \{(x,y,z) \in \mathbb{R}^3 : \|(x,y,z)\| \le 1\}$.

Exercise 0.2.10. Let (X_i, d_i), $1 \le i \le m$, be metric spaces and $X := X_1 \times \cdots \times X_m$. For $x = (x_1, \ldots, x_m) \in X$ and $y = (y_1, \ldots, y_m) \in X$, define

$$d(x,y) = \sqrt{d_1(x_1,y_1)^2 + \cdots + d_m(x_m,y_m)^2}\,.$$

Show that d is a metric on X called the **product metric**. The metric space (X,d) is called the **product of the metric spaces** (X_i, d_i).

0.3 OPEN SETS, CLOSED SETS, INTERIORS, AND CLOSURES

Exercise 0.3.1. Show that if $a, b, c, d \in \mathbb{R}$, $a < b$ and $c < d$, then the set $\{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$ is closed in $\mathbb{R}^2$.

Exercise 0.3.2. Determine whether the following sets are open in $\mathbb{R}^2$, closed in $\mathbb{R}^2$, or neither.

(a) $\{(x, y) : |x| + |y| < 1\}$.

(b) $\{(x, y) : 0 \leq y \leq e^x\}$.

(c) $\{(x, y) : \max\{x, y\} = 2\}$.

(d) $\{(x, y) : 0 < y \leq e^x\}$.

Exercise 0.3.3. Show that any finite set is closed.

Exercise 0.3.4. For subsets A and B of a metric space (X, d), show that:

(a) $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$.

(b) $\operatorname{int}(A \cup B) \supset \operatorname{int}(A) \cup \operatorname{int}(B)$.

(c) If $A \subset B$ then $\overline{A} \subset \overline{B}$.

(d) $\overline{(\overline{A})} = \overline{A}$.

(e) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(f) If B is open, then $\overline{A} \cap B \subset \overline{A \cap B}$.

Exercise 0.3.5. Show that in a normed space $(X, \|\cdot\|)$, the closure of any open ball $B(a, r) = \{x \in X : \|x - a\| < r\}$ is the closed ball $\overline{B}(a, r)$. Give an example of a metric space for which the corresponding statement is false.

Exercise 0.3.6. For a subset E of a metric space X, a point $x \in X$ is called an **exterior point** of E provided there is an open ball centered at x that is contained in E^c: the collection of exterior points of E is called the **exterior** of E and denoted by $\operatorname{ext} E$. Show that $\operatorname{ext} E$ is always open. Show that E is closed if and only if $E^c = \operatorname{ext} E$.

Exercise 0.3.7. Let E be a subset of a metric space X. Show that

(a) ∂E is always closed;

(b) E is open if and only if $E \cap \partial E = \emptyset$;

(c) E is closed if and only if $\partial E \subset E$.

Exercise 0.3.8. A point $a \in X$ is a **limit point** or **cluster point** of the set $F \subset X$ if every open ball $B(x, r)$ contains a pointy $x \in E$ with $x \neq a$ (a need not be in F). All points of F that are not limit points of F are called **isolated points** of F. Show that

(a) $\overline{F}$ is the union of F and all its limit points in X.

(b) F is closed in X if and only if it contains all its limit points.

(c) Let F' be the set of limit points of F. Show that $\overline{F} = F \cup F'$.

Exercise 0.3.9. Let E be a subset of a metric space X and E' the set of limit points of E (see Exercise 0.3.8).

(a) Show that E' is closed.

(b) Show that x is a limit point of E if and only if x is a limit point of $\overline{E}$.

(c) If x is a limit point of E', show that x is a limit point of E.

(d) Let $E = (0, 1) \cup \left\{2 + \frac{1}{n} : n \in \mathbf{N}\right\}$. Find E' and exhibit a limit point of E that is not a limit point of E'.

Exercise 0.3.10. Prove that if d and ρ are equivalent metrics on a nonempty set X, then a subset of X is open in the metric space (X, d) if and only if it is open in the metric space (X, ρ). (See Problem 0.2.5 for the definition of equivalent metrics.)

0.4 CONVERGENCE

Exercise 0.4.1. Suppose that a sequence $\{x_n\}$ in a normed vector space $(X, \|\cdot\|)$ converges to x. Prove that the sequence of nonnegative numbers $\{\|x_n\|\}$ converges to $\|x\|$.

Exercise 0.4.2. Let X be the product of the metric spaces (X_i, d_i), $1 \leq i \leq m$ (see Exercise 0.2.10). Then the sequence $\{x_n\} = \{(x_{n,1}, \ldots, x_{n,m})\}$ converges in X to the point $x_0 = (x_{0,1}, \ldots, x_{0,m})$ if and only if, for each $i = 1, \ldots, m$, the sequence $\{x_{n,i}\}$ converges in X_i to $x_{0,i} \in X_i$.

0.5 CONTINUOUS MAPPINGS BETWEEN METRIC SPACES

Exercise 0.5.1. Let $f : X \to Y$ be continuous and E a nonempty subset of X. Show that the restriction $f_{|E} : E \to Y$ of f to E is always continuous.

Exercise 0.5.2. ($\mathbb{R}^n$**-Valued Functions**) Let X be a metric space, and let $\phi_1, \ldots, \phi_n$ be real-valued functions on X. Define $\Phi : X \to \mathbb{R}^n$ by $\Phi(x) = \big(\phi_1(x), \ldots, \phi_n(x)\big)$. Then for $a \in X$, Φ is continuous at a if and only if each ϕ_i is continuous at a.

Exercise 0.5.3. Let (X, d) be a discrete space. Show that every mapping from (X, d) to a metric space (Y, ρ) is continuous.

Exercise 0.5.4. For a metric space (X, d), show that the metric $d : X \times X \to \mathbb{R}$ is continuous, where $X \times X$ has the product metric.

Exercise 0.5.5. Let $f : X \times Y \to Z$ be continuous. Show that, for each y in Y, the mapping $x \mapsto f(x, y)$ from X into Z is continuous. Similarly, $y \mapsto f(x, y)$ is continuous for each x in X.

Exercise 0.5.6. Suppose $(X, \|\cdot\|)$ is a norm space. Show that the function $f(x) = \|x\|^2$ is continuous.

Exercise 0.5.7. Show that a continuous mapping between metric spaces remains continuous if an equivalent metric is imposed on the domain and an equivalent metric is imposed on the range.

Exercise 0.5.8. Let f be a mapping from the metric space (X, d) to the metric space (Y, ρ). The **graph** of the mapping f is the set of ordered pairs

$$\{(x, y) : \; x \in X \text{ and } f(x) = y\} = \{\big(x, f(x)\big) : x \in X\}.$$

Show that if f is continuous, then the graph of f is closed in the product space $X \times Y$.

Exercise 0.5.9. A mapping from a metric space (X, d) to a metric space (Y, ρ) is said to be **uniformly continuous**, provided for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\rho\big(f(x), f(x')\big) < \epsilon \quad \text{whenever} \quad x, x' \in X, \; d(x, x') < \delta.$$

(a) Show that Lipschitz continuous functions are uniformly continuous.

(b) Show that the function $h : (0, 1] \to \mathbb{R}$, $h(x) = \frac{1}{x}$ is continuous, but not uniformly continuous.

Exercise 0.5.10. Let a be a point in the metric space (X, d). Define the function $f : X \to \mathbb{R}$ by $f(x) = d(x, a)$. Show that f is uniformly continuous.

Exercise 0.5.11. For a nonempty subset A of the metric space (X, d) and a number $\epsilon > 0$.

(a) Show that $\{x \in X : d(x, A) = 0\} = \overline{A}$.

(b) Show that $\{x \in X : d(x, A) < \epsilon\}$ and $\{x \in X : d(x, A) > \epsilon\}$ are open; $\{x \in X : d(x, A) \leq \epsilon\}$ and $\{x \in X : d(x, A) \geq \epsilon\}$ are closed.

Exercise 0.5.12. Prove that every sequence $f : \mathbf{N} \to \mathbb{R}$ is uniformly continuous.

Exercise 0.5.13. Show that a subset E of a metric space X is open if and only if there is a continuous real-valued function f on X for which $E = \{x \in X : f(x) > 0\}$. (*Hint.* Use the function $x \mapsto d(x, A)$ for a suitable set A.)

Exercise 0.5.14. Show that a subset E of a metric space X is closed if and only if there is a continuous real-valued function f on X for which $E = f^{-1}(0)$. (*Hint.* Use the function $x \mapsto d(x, A)$ for a suitable set A.)

0.6 COMPLETE METRIC SPACES

Exercise 0.6.1. For a nonempty subset E of a metric space (X, d), we define the **diameter** of E, $\operatorname{diam} E$, by

$$\operatorname{diam} E = \sup\{d(x, y) : x, y \in E\}.$$

We say E is **bounded** provided that it is contained in a ball. A sequence $\{x_n\}$ in X is said to be **bounded** if the set $\{x_n : n \in \mathbf{N}\}$ is bounded.

(i) Show that E is bounded if and only if it has finite diameter.

(ii) Show that a Cauchy sequence is bounded. Hence a convergent sequence is bounded.

Exercise 0.6.2. Let d and ρ be equivalent metrics on a nonempty set X. Show that (X, d) is complete if and only if (X, ρ) is complete.

Exercise 0.6.3. Prove that the product of two complete metric spaces is complete.

Exercise 0.6.4. Let $\mathcal{D}$ be the subspace of $C([0, 1])$ consisting of the continuous functions $f : [0, 1] \to \mathbb{R}$ that are differentiable on $(0, 1)$. Is $\mathcal{D}$ complete?

Exercise 0.6.5. Let d be a metric on a set X. Define a metric ρ on X by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in X.$$

(a) Verify that convergence of sequences with respect to the d metric and the ρ metric is the same.

(b) Conclude that sets that are closed with respect to the d metric are closed with respect to the ρ metric and that sets that are open with respect to the d metric are open with respect to the ρ metric.

(c) Are the metrics d and ρ equivalent?

Exercise 0.6.6. Show that every subspace of a separable metric space is separable.

Exercise 0.6.7. Show that for a subset D of a metric space X, D is dense in the subspace $\overline{D}$.

Exercise 0.6.8. Show that the product of two separable metric spaces is again separable.

Exercise 0.6.9. Let (X, d) be a metric space. Show that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of X, then $\{d(x_n, y_n)\}$ converges in $\mathbb{R}$.

The Baire Category Theorem

Exercise 0.6.10. In a complete metric space X, is the union of a countable collection of nowhere dense sets also nowhere dense?

Exercise 0.6.11. In a complete metric space, is the union of a countable collection of sets of the first category also of the first category?

Exercise 0.6.12. Let E be a subset of a metric space X. Show that if E is closed, then the interior of ∂E is empty.

Exercise 0.6.13. In a metric space X, show that a subset E is nowhere dense if and only if for each open subset U of X, $E \cap U$ is not dense in U.

Exercise 0.6.14. A point x in a metric space X is called **isolated** provided the singleton set $\{x\}$ is open in X.

(a) Prove that a complete metric space without isolated points has an uncountable number of points.

(b) Show that if X is a complete metric space without isolated points and $\{F_n\}_n$ is a countable collection of closed sets with $\operatorname{int} F_n = \emptyset$ for all n, then $X \setminus \bigcup_{n=1}^{\infty} F_n$ is dense and uncountable.

The Banach contraction Theorem

Exercise 0.6.15. Let $[a, b]$ be a closed, bounded interval in $\mathbb{R}$ and suppose that $f : [a, b] \to [a, b]$ is a continuous function. Show that f has a fixed point.
(*Hint:* Consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - x$.)

Exercise 0.6.16. Does a mapping of a complete metric space into itself that is Lipschitz with Lipschitz constant 1 necessarily have a fixed point?

0.7 COMPACT METRIC SPACES

Exercise 0.7.1. Which of the following subsets of $\mathbb{R}$ are compact?

$$(a)\ (-\infty, 3], \qquad (b)\ (-5, 2], \qquad (c)\ [-5, 2].$$

Exercise 0.7.2. Show that the union of a finite number of compact sets is again compact.

Exercise 0.7.3. Let $f(x) = \ln x$, is f uniformly continuous on $(0, 1]$?

Exercise 0.7.4. Let $f : E \to Y$, where E is a bounded subset of $\mathbb{R}^n$. If f is uniformly continuous on E, show that the image $f(E)$ is bounded.

Exercise 0.7.5. Let K is a compact subset of the metric space X and $f : X \to Y$ is continuous on K and one-to-one. Show that f has a continuous inverse on $f(K)$. In other words, if $x_1, x_2, \ldots \in K$, $x \in K$, and $f(x_n) \to f(x)$, then $x_n \to x$.

Exercise 0.7.6. Let d and ρ be equivalent metrics on a nonempty set X. Show that the metric space (X, d) is compact if and only if the metric space (X, ρ) is compact.

Exercise 0.7.7. Show that the Cartesian product of two compact metric spaces also is compact.

Exercise 0.7.8. For a compact metric space (X, d), show that there are points $u, v \in X$ for which $d(u, v) = \operatorname{diam} X$.

Exercise 0.7.9. Let K be a nonempty compact subset of the metric space (X, d) and x_0 belong to X.

(a) Show that there are points $y_0, y_1 \in K$ for which

$$d(y_0, x_0) \leq d(x, x_0) \leq d(y_1, x_0) \quad \text{for all} \quad x \in K.$$

Thus $d(x_0, y_0) = d(x_0, K)$.

(b) Show that if $x_0 \notin K$, then $d(x_0, K) > 0$.

Exercise 0.7.10. Let A and B be nonempty subsets of a metric space (X, d). Define

$$d(A, B) := \inf\{d(x, y) : x \in A,\ y \in B\}.$$

If A is compact and B is closed, show that $A \cap B = \emptyset$ if and only if $d(A, B) > 0$. Provide closed (but *not* compact) sets A and B in $\mathbb{R}^2$ such that $d(A, B) = 0$ and $A \cap B = \emptyset$.

Exercise 0.7.11. A **homeomorphism** from a metric space X into a metric space Y is a bijection that is continuous and whose inverse mapping is also continuous. Two spaces X and Y are said to be **homeomorphic** if there exists a homeomorphism from one to the other.

Show that if $f : X \to Y$ is continuous and bijection and X is compact, then f is a homeomorphism.

Exercise 0.7.12. Suppose f is a continuous real-valued function on Euclidean space $(\mathbb{R}^n, \|\cdot\|)$ with the property that there is a number c such that $|f(x)| \geq c\|x\|$ for all $x \in \mathbb{R}$. Show that if K is a compact set of real numbers, then its inverse image under f, $f^{-1}(K)$, also is compact. (Mappings with this property are called **proper**.)

Exercise 0.7.13. Let $(X_1, d_1), \ldots, (X_n, d_m)$ be metric spaces, and let $X = X_1 \times \cdots \times X_m$. If $x = (x_1, \ldots, x_m))and y = (y_1, \ldots, y_m)$, define

$$d(x, y) = \sqrt{d_1(x_1, y_1)^2 + \cdots + d_m(x_m, y_m)^2} \quad \textit{and} \quad \rho(x, y) = \sum_{i=1}^{m} d_i(x_i, y_i).$$

(a) Show that d and ρ are distances on X.

(b) Show that d is equivalent to ρ.

(c) Show that (X, d) is complete if and only if each (X, d_i) is complete.

(d) Show that (X, d) is compact if and only if each (X, d_i) is compact.