Conditional expectation

1 Conditional distribution

1. Suppose that $p(x, y, z)$, the joint probability mass function of (X, Y, Z) is

$$\begin{array}{cc} p(1,1,1) = \frac{1}{8} & p(2,1,1) = \frac{1}{4} \\ p(1,1,2) = \frac{1}{8} & p(2,1,2) = \frac{3}{16} \\ p(1,2,1) = \frac{1}{16} & p(2,2,1) = 0 \\ p(1,2,2) = 0 & p(2,2,2) = \frac{1}{16} \end{array}$$

   $E(X|Y = 2, Z = 1)$

2. An unbiased die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a six and a five. Find
   (a) $E[X]$
   (b) $E[X|Y = 1]$
   (c) $E[X|Y = 5]$.

3. If X and Y are independent Poisson random variables with respective means λ_1 and λ_2, calculate the conditional expected value of X given that $X + Y = n$.

4. Suppose the joint density of X and Y is given by

$$f(x, y) = \begin{cases} 6xy(2 - x - x) & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Compute the conditional expectation $E(X|Y = y)$, where $0 < y < 1$.

5. Suppose the joint density of X and Y is given by

$$f(x, y) = \begin{cases} 4y(x - y)e^{-x-y} & \text{if } 0 < x < \infty, 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

   Compute $E(X|Y = y)$ and deduce $E(X|Y)$.

6. Let (X_1, X_2) be bivariate normal distributed with $\mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 4 & 12 \\ 12 & 9 \end{pmatrix}$.
   (a) Find the conditional distribution of $X_2|X_1 = 3$.
   (b) Find $E(X_2|X_1 = 3)$
   (c) Find formula of $E(X_2|X_1 = x_1)$ and then deduce $E(X_2|X_1)$.

7. You visit a random number N of stores and in the ith store, you spend a random amount of money X_i. Let

$$T = X_l + X_2 + \cdots + X_N$$

   be the total amount of money that you spend. We assume that N is a positive integer random variable with a given PMF, and that the X_l are random variables with the same mean $E[X]$ and variance $var(X)$. Furthermore, we assume that N and all the X_i are independent. Show that

$$E[T] = E[X]E[N]$$

   and

$$var(T) = var(X)E[N] + (E[X])^2 var(N)$$

   .