Question 1.

(a) *(25 marks)* Let

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 1 \\ 2 & \text{if } x \ge 1. \end{cases}$$

Evaluate $\mu_F((-1,2])$, $\mu_F((-\infty, \frac{1}{2}])$, and $\mu_F(\{1\})$, where μ_F is the Lebesgue-Stieltjes measure associated to F.

(b) *(10 marks)* Let A and B be measurable sets. Show that $\chi_A = \chi_B$ a.e. if and only if $\mu(A\Delta B) = 0$.

Question 2. Let f_n, g, $n = 1, 2, \ldots$, be measurable functions on X. Suppose that $f_n \geq g$ for all n and $\int_X g d\mu > -\infty$.

(a) *(15 marks)* Show that $\int_X f_n d\mu$ exists for all n.
(*Hint:* Note that $f_n^- \leq g^-$.)

(b) *(20 marks)* Show that if $f_n \nearrow f$, then $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.
(*Hint:* Observe that $0 \leq g^+ \leq f_n + g^- \nearrow f + g^-$.)

Question 3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$ be a <u>non-negative integer-valued</u> measurable function. Show that

(a) *(15 marks)* $f = \sum_{n=1}^{\infty} f_n$ where $f_n = \chi_{\{f \geq n\}}$;

(b) *(15 marks)* $\int_X f d\mu = \sum_{n=1}^{\infty} \mu(\{f \geq n\})$.