FINAL EXAMINATION
January 2019
Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Deputy Head of Dept. of Mathematics:	Lecturer:
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INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 *(25 marks)* Let $F(x) = \tan^{-1} x$, $x \in \mathbb{R}$. If μ_F is the Lebesgue-Stieltjes measure corresponding to F, determine $\mu_F\big((0,1]\big)$ and $\mu_F\big((-\infty, 0]\big)$.

Question 2 Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} e^x & \text{if } x \leq 0, \\ 2x+3 & \text{if } x > 0. \end{cases}$$

(a) *(15 marks)* Show that $g(x)$ is Borel measurable on $\mathbb{R}$.
(*Hint:* Express g in terms of e^x, $(2x+3)$, $\chi_{(-\infty,0]}$, and $\chi_{(0,\infty)}$.)

(b) *(10 marks)* Determine $E = g^{-1}\big([2,5]\big)$ and the Lebesgue measure $m(E)$.

Question 3 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{A_n\}$ be a sequence of measurable sets such that $A_n \subset A_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} A_n = X$. Suppose f is integrable on X and $h_n = f\chi_{A_n}$. Show that

(a) *(10 marks)* $f - h_n$ is defined a.e. on X;

(b) *(10 marks)* $\lim_{n\to\infty} \int_X |f - h_n| d\mu = 0$, and

(c) *(5 marks)* $\lim_{n\to\infty} \int_{A_n} |f - h_n| d\mu = 0$.

(*Hint:* Use the Dominated Convergence Theorem.)

Question 4 Suppose that X is a nonempty countable set and μ and ν are two measures on $\mathcal{P}(X)$.

(a) *(10 marks)* Show that $M = \{x \in X : \mu(\{x\}) = 0\}$ is the largest μ-null set, that is, $\mu(M) = 0$ and if $\mu(A) = 0$ then $A \subset M$. Show that $N = \{x \in X : \nu(\{x\}) = 0\}$ is the largest ν-null set.

(b) *(10 marks)* Show that $\nu \ll \mu$ if and only if $M \subset N$.

(c) *(5 marks)* If $\mu(\{x\}) < \infty$, $\nu(\{x\}) < \infty$ for all $x \in X$, and $\nu \ll \mu$, find $\dfrac{d\nu}{d\mu}$.

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SOLUTIONS
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Question 1 Since $F(x)$ is continuous and increasing, the Lebesgue-Stieltjes measure μ_F corresponding to F exists. By definition,

$$\mu_F\big((0,1]\big) = F(1) - F(0) = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

As $\{(-n,0]\}$ is an increasing sequence and $(-\infty,0] = \bigcup_{n=1}^{\infty}(-n,0]$, we have

$$\begin{aligned}\mu_F\big((-\infty,0]\big) &= \lim_{n\to\infty}\mu_F\big((-n,0]\big) = \lim_{n\to\infty} F(0) - F(-n)\\ &= \lim_{n\to\infty}\big[0 - \tan^{-1}(-n)\big] = \lim_{n\to\infty}\tan^{-1}(n) = \frac{\pi}{2}.\end{aligned}$$

Question 2 (a) Since $(-\infty,0]$ and $(0,\infty)$ are Borel measurable, so are the functions $\chi_{(-\infty,0]}$ and $\chi_{(0,\infty)}$. As e^x and $2x+4$ are continuous, they are Borel measurable. Thus $g(x) = e^x\chi_{(-\infty,0]} + (2x+3)\chi_{(0,\infty)}$ is Borel measurable.

(b) Since $g(x) = e^x \leq 1$ for $x \leq 0$, it follows that

$$\begin{aligned}E &= g^{-1}\big([2,5]\big) = \big\{x : g(x) \in [2,5]\big\} = \big\{x > 0 : g(x) \in [2,5]\big\}\\ &= \big\{x > 0 : 2 \leq 2x+3 \leq 5\big\} = (0,1].\end{aligned}$$

Thus $m(E) = m\big((0,1]\big) = 1$.

Question 3 (a) Since f is integrable, it is finite a.e. on X. In addition, h_n is defined everywhere, hence $f - h_n$ is defined a.e.

(b) For each $x \in X = \bigcup_{k=1}^{\infty} A_k$, there is $n_x \in \mathbb{N}$ such that $x \in A_{n_x}$. As $A_k \subset A_{k+1}$, we have $x \in A_n$ for all $n \geq n_x$ so that $h_n(x) = f(x)$ for all $n \geq n_x$. This implies $\lim_{n\to\infty} h_n(x) = f(x)$ for all $x \in X$ and hence, $\lim_{n\to\infty}|f(x) - h_n(x)| \to 0$ a.e. Moreover, $|h_n| = \big|f\chi_{A_n}\big| \leq |f|$, so

$$0 \leq |f - h_n| \leq |f - h_n| \leq |f| + |h_n| \leq 2|f| \quad \text{a.e.}$$

The function $2f$ is integrable on X and we can apply the DCT to obtain

$$\lim_{n\to\infty}\int_X |f - h_n| d\mu = \int_X 0 d\mu = 0.$$

(c) Since $0 \leq |f - h_n|\chi_{A_n} \leq |f - h_n|$ a.e.,

$$0 \leq \int_{A_n} |f - h_n| d\mu = \int_X |f - h_n|\chi_{A_n} d\mu \leq \int_X |f - h_n| d\mu \to 0.$$

Thus $\lim_{n\to\infty}\int_{A_n}|f - h_n| d\mu = 0$.

Question 4 (a) Since X is countable, so is M. By assumption, $\mu(\{x\}) = 0$ for each $x \in M$. Thus $M = \bigcup_{x\in M}\{x\}$ is a countable union of μ-null sets $\{x\}$, so M is a μ-null set. If E is a μ-null set, then $\mu(\{x\}) = 0$ for all $x \in E$, hence $x \in M$, i.e., $E \subset M$. Thus M is the largest μ-null set. In the same manner we can see that N is the largest ν-null set.

(b) Suppose that $\nu \ll \mu$. Since $\mu(M) = 0$, then $\nu(M) = 0$. By part (a), N is the largest ν-null set, hence $M \subset N$. Conversely, if $M \subset N$, then for all μ-null set E, we have $E \subset M \subset N$. As $E \subset N$ and $\nu(N) = 0$, we have $\nu(E) = 0$, i.e., $\nu \ll \mu$. Thus $\nu \ll \mu$ if and only if $M \subset N$.

(c) Since X is countable and $\mu(\{x\}) < \infty$, $\nu(\{x\}) < \infty$ for all $x \in X$, $X = \bigcup_{x \in X} \{x\}$ is the countable union of sets of finite measures. Thus μ and ν are σ-finite measures. Moreover, $\nu \ll \mu$, so by the Radon-Nikodym theorem, $\frac{d\nu}{d\mu}$ exists. Let $f = \frac{d\nu}{d\mu}$. Then

$$\nu(\{x\}) = \int_{\{x\}} f d\mu = f(x)\mu(\{x\}),$$

implying $f(x) = \frac{\nu(x)}{\mu(x)}$ for all $x \notin M$. Since X is equipped with the σ-algebra $\mathcal{P}(X)$, any function defined on X is measurable. Since $\frac{d\nu}{d\mu}$ is unique a.e. $[\mu]$ and $\mu(M) = 0$, any function f on X satisfying $f(x) = \frac{\nu(x)}{\mu(x)}$ for $x \notin M$ is the Radon-Nikodym derivative of ν with respect to μ.