Random Processes, Lecture Notes

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1 Conditional Expectation

1. **Linearity**: $\mathbb{E}(aY + bZ|X) = a \cdot \mathbb{E}(Y|X) + b \cdot \mathbb{E}(Z|X)$;
2. **Taking Out What We Know**: $\mathbb{E}(f(X) \cdot Y|X) = f(X) \cdot \mathbb{E}(Y|X)$;
3. **Iterated Conditioning**: if $\mathcal{G} \subset \mathcal{H}$ then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G})$;
4. **Independence**: if X and Y are independent then $\mathbb{E}(Y|X) = \mathbb{E}(Y)$;
5. **Tower Property:** $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X))$.

2 Random Process

2.1 Definitions

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1. A **random process** (or **stochastic process**) is a collection of random variables $\{X_t\}_{t\in I}$;
2. I is called the **index set** (or **parameter set**) of the (random) process, and each $t \in I$ is called a **time**;
3. For each $\omega \in \Omega, X_t(\omega)$ is a function relative to t, called the **sample path**;
4. The set $S = \{X_t(\omega) : t \in I, \omega \in \Omega\}$ is called the **state space** of the process;
5. A **filtration** is an increasing collection of $\sigma-$algebras $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying
$$\mathcal{F}_t \subset \mathcal{F}, \forall t \geq 0 \text{ and } \mathcal{F}_s \subset \mathcal{F}_t, \forall 0 \leq s \leq t.$$
The quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is called a **filtered probability space**;
6. Given a process $\{X_t\}_{t\in I}$, the collection
$$\sigma(\{X_t\}) = \sigma\left(\left\{X_t^{-1}(A) : t \in I, A \in \mathcal{B}(\mathbb{R})\right\}\right)$$
is a $\sigma-$algebra on Ω, called the **$\sigma-$algebra generated by** $\{X_t\}$.

2.2 Martingale

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and a process $\{X_t\}_{t\geq 0}$.

1. We say that $\{X_t\}$ is **adapted** to $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t-$measurable, $\forall t \geq 0$;
2. $\{X_t\}$ is called a **martingale** if it is adapted and
$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s, \forall 0 \leq s \leq t.$$

2.3 Discrete Case

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a process $\{X_n\}_{n\in\mathbb{N}}$.

1. A **filtration** is an increasing collection of $\sigma-$algebras $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ satisfying

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}, \forall n \in \mathbb{N}.$$

   The quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ is called a **filtered probability space**;

2. We say that $\{X_n\}$ is **adapted** to $\{\mathcal{F}_n\}$ if X_n is $\mathcal{F}_n-$measurable, $\forall n \in \mathbb{N}$;

3. $\{X_n\}$ is called a **martingale** if it is adapted and

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n, \forall n \in \mathbb{N}.$$

3 Poisson Process

3.1 Definitions

A **Poisson process** with **intensity** (or **rate**) λ is a process $\{N_t\}_{t\geq 0}$ satisfying:

1. $N_0 = 0$;
2. $N_t \sim \text{Pois}(\lambda t), \forall t \geq 0$;
3. **Stationary Increments**: $N_{s+t} - N_s \sim N_t, \forall s, t > 0$;
4. **Independent Increments**: if $s, t, u, v > 0$ and $s + t < u$, then

$$N_{s+t} - N_s \text{ and } N_{u+v} - N_u \text{ are independent.}$$

N_t is called the **arrival count up to time** t**.**

3.2 Arrival Time & Inter-Arrival Time

Consider a Poisson process $\{N_t\}_{t\geq 0}$. Suppose S_n is the time of the n^{th} arrival, $\forall n \in \mathbb{N}$.

1. $N_t = \max\{n : S_n \leq t\}, \forall t \geq 0$;
2. $S_n \sim \text{Gamma}(n, \lambda), \forall n \in \mathbb{N}$;
3. The **inter-arrival time** is defined by

$$X_n = S_n - S_{n-1}, \forall n \in \mathbb{N}.$$

   Then X_n are independent and $X_n \sim \text{Exp}(\lambda), \forall n \in \mathbb{N}$;

4. Given n random variables $X_1, ...X_n$, we define the **order statistic** $\{X_{(i)}\}_{i=1}^n$ as

$$X_{(k)} = \min\left(\{X_i\}_{i=1}^n \setminus \{X_{(j)}\}_{j=1}^{k-1}\right), \forall k = \overline{1, n}.$$

5. If $N_t = m$ and $U_1, ..., U_m$ are independent, identically distributed random variables with

$$U_i \sim U([0, t]), \forall i = \overline{1, m}$$

   then

$$[(S_1, ..., S_m)|N_t = m] \sim (U_{(1)}, ..., U_{(m)}).$$

3.3 Compound Poisson Process

Let $\{W_i\}_{i\in\mathbb{N}}$ be independent, identically distributed random variables of some distribution F and are independent of a Poisson process $\{N_t\}_{t\geq 0}$ with rate $\lambda > 0$.

1. The process $\{R_t\}_{t\geq 0}$ defined by

$$R_t = \sum_{i=1}^{N_t} W_i, \forall t \geq 0$$

   is called a compound Poisson process;

2. $\mathbb{E}(R_t) = \lambda t \cdot \mathbb{E}(W_i)$ and $\mathrm{Var}(R_t) = \lambda t \cdot \mathbb{E}(W_i^2), \forall t \geq 0$.

4 Markov Chain

1. A process $\{X_n\}_{n\in\mathbb{N}}$ is called a **Markov chain** if it satisfy the **Markov property**:

$$X_{n+1}|(X_0, ..., X_n) = X_{n+1}|X_n, \forall n \in \mathbb{N}.$$

2. Associated with a Markov chain $\{X_n\}_{n\in\mathbb{N}}$ is the **transition probability**

$$p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i), \forall n \in \mathbb{N}$$

   and the **transition matrix**

From \ To	1	2	...
1	p_{11}	p_{12}	...
2	p_{21}	p_{22}	...
...	...	...	...

   or simpler,

$$P = \begin{pmatrix} p_{11} & p_{12} & ... \\ p_{21} & p_{22} & ... \\ ... & ... & ... \end{pmatrix}.$$

3. For each $n \in \mathbb{N}$, define the transition probability after n steps as

$$r_{ij}(n) = \mathbb{P}(X_{n+k} = j | X_k = i), \forall k \in \mathbb{N}.$$

   Then

$$P^n = \begin{pmatrix} p_{11} & p_{12} & ... \\ p_{21} & p_{22} & ... \\ ... & ... & ... \end{pmatrix}^n = \begin{pmatrix} r_{11}(n) & r_{12}(n) & ... \\ r_{21}(n) & r_{22}(n) & ... \\ ... & ... & ... \end{pmatrix}.$$

4. For each $n \in \mathbb{N}$, define the **unconditional distribution** of X_n as

$$\pi^{(n)} = (\pi_1^{(n)}, \pi_2^{(n)}, ...) \quad \text{where} \quad \pi_i^{(n)} = \mathbb{P}(X_n = i)$$

   then $\pi^{(n)} = \pi^{(0)} \cdot P^n$. In the long term, $\pi^{(n)}$ approaches the **stationary distribution** $\pi = (\pi_1, \pi_2, ...)$ given by

$$\pi \cdot P = \pi \quad \text{and} \quad \sum_i \pi_i = 1.$$

   Furthermore, $r_{ij}(n)$ approaches π_j as $n \to \infty$.

5 Random Walk

1. Consider a process $\{X_n\}_{n\in\mathbb{N}}$ of independent, identically distributed random variables with distribution

x	-1	1
$\mathbb{P}(X_n = x)$	1/2	1/2

then the process $\{M_n\}_{n\in\mathbb{N}}$ defined by

$$M_0 = 0 \quad \text{and} \quad M_n = \sum_{i=1}^{n} X_i, \forall n \geq 1$$

is called a **symmetric random walk**.

2. Properties of $\{M_n\}$:

   - It is a martingale and $\mathbb{E}(M_n) = 0, \mathrm{Var}(M_n) = n$;
   - The first passage time $\tau_1 = \inf\{n : M_n = 1\}$ has distribution

$$\mathbb{P}(\tau_1 = 2j-1) = \frac{1}{2^{2j-1}} \cdot \frac{(2j-2)!}{j! \cdot (j-1)!};$$

   - **Stationary Increments**: $M_{s+t} - \mathbb{E}(M_s) \sim M_t, \forall t, s \in \mathbb{N}$;
   - **Independent Increments**: if $s, t, u, v \in \mathbb{N}$ and $s+t < u$, then

$$M_{s+t} - M_s \text{ and } M_{u+v} - M_u \text{ are independent.}$$

   - **Quadratic Variation**:

$$\langle M, M\rangle_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2 = k, \forall k \in \mathbb{N}.$$

3. The process $\left\{W_t^{(n)}\right\}$ defined by

$$W_t^{(n)} = \frac{M_{nt}}{\sqrt{n}} \text{ whenever } nt \text{ is an integer}$$

is called a **scaled symmetric random walk**. It preserves all properties of $\{M_n\}$: martigale property, stationary & independent increments, quadratic variation.

4. By the Central Limit Theorem, the process $\left\{W_t^{(n)}\right\}$ converges in distribution to a process $\{B_t\}$ called the **Brownian motion** as $n \to \infty$.

6 Brownian Motion

1. A process $\{B_t\}_{t\geq 0}$ is called a **Brownian motion** if it has the following properties:

   - For each $\omega \in \Omega, B_t(\omega)$ is a continuous function of t;
   - $B_0 = 0$ and $B_t \sim \mathcal{N}(0, t), \forall t > 0$;
   - **Stationary Increments**: $B_{s+t} - B_s \sim B_t, \forall t, s > 0$;
   - **Independent Increments**: if $s, t, u, v > 0$ and $s+t < u$, then

$$B_{s+t} - B_s \text{ and } B_{u+v} - B_u \text{ are independent.}$$

2. Properties of $\{B_t\}$:

   - $\operatorname{cov}(B_{t+s}, B_s) = s, \forall s, t \geq 0$;
   - It is a martingale;
   - The first passage time $\tau_m = \inf\{t : B_t = m\}$ has distribution

   $$\mathbb{P}(\tau_m \leq t) = 2 \cdot \mathbb{P}(B_t \geq m).$$

     Also, $\mathbb{P}(\tau_m < \infty) = 1$ and $\mathbb{E}(\tau_m) = \infty$;
   - The maximum to date $\{M_t\}_{t\geq 0}$ where $M_t = \max\{B_s : 0 \leq s \leq t\}$ has distribution

   $$\mathbb{P}(M_t \geq x) = 2 \cdot \mathbb{P}(B_t \geq x);$$

   - **Quadratic Variation**: $\langle B \rangle (T) = T$ almost surely, i.e.

   $$(dB_t)^2 = dt, \quad dB_t dt = dtdt = 0.$$

7 Ito Integral

1. An **Ito integral** has the form

$$I_t = \int_0^t \delta_s dB_s$$

   where the integrator $\{B_s\}$ is the Brownian motion and the integrand $\{\delta_s\}$ is a square-integrable, adapted process.

2. Properties of I_t :

   - Series representation:

   $$\int_0^T f(s)dB_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(\frac{iT}{n}\right) \left[B_{(i+1)T/n} - B_{iT/n}\right];$$

   - It is a continuous function of t;
   - **Linearity**:

   $$\int_0^t (\alpha \cdot \delta_s \pm \beta \cdot \gamma_s)dB_s = \alpha \int_0^t \delta_s dB_s \pm \beta \int_0^t \gamma_s dB_s;$$

   - **Quadratic Variation**:

   $$\langle I, I \rangle (t) = \int_0^t \delta_s^2 ds;$$

   - **Isometry**: $\mathbb{E}(I_t^2) = \mathbb{E}(\langle I, I \rangle (t))$.
   - Integration by Parts: if f is continuous and deterministic then

   $$\int_0^t f(s)dB_s = f(t) \cdot B_t - \int_0^t B_s df(s);$$

   - If δ_s is deterministic then $I_t \sim \mathcal{N}(0, \langle I, I \rangle (t))$.

8 Ito-Doeblin Formula

Let $f(t,x)$ be twice continuously differentiable and $\{B_t\}$ be the Brownian motion.

1. The simple case:

$$df(t,B_t) = f_t(t,B_t)dt + f_x(t,B_t)dB_t + \frac{f_{xx}(t,B_t)dt}{2}.$$

2. Consider an Ito process $\{X_t\}_{t\geq 0}$ satisfying

$$dX_t = \mu_t dt + \sigma_t dB_t$$

   where the drift term μ_t and the diffusion term σ_t are adapted processes, then

$$df(t,X_t) = f_t(t,X_t)dt + f_x(t,X_t)dX_t + \frac{f_{xx}(t,X_t)\sigma_t^2 dt}{2}.$$

9 Stochastic Differential Equation (SDE)

We consider a general form of an SDE

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dB_t \tag{1}$$

with $X_0 = x_0$.

1. If $\mu = \mu(t), \sigma = \sigma(t)$ then integrate both sides of (1).
2. If $\mu = \mu_1(t) + \mu_2(t)\cdot X_t$ and $\sigma = \sigma(t)$ then convert (1) into

$$dX_t - \mu_2(t)\cdot X_t dt = \sigma dB_t + \mu_1(t)dt$$

   then let $Y_t = e^{a(t)}X_t$ where $a'(t) = -\mu_2(t)$ and use Ito-Doeblin formula on dY_t.
3. If σ is not a function of t only, then ... it depends. You may need to do some 'trick' to solve for an appropriate $Y_t = f(t,X_t)$.