EXERCISES AND PROBLEMS FOR CHAPTER 2: MEASURES

A. Problems and Exercises for everyone:
All problems and exercises in parts B and C.

B. Non-assessed Problems and Exercises (corrected in class):

0.1.1; 0.1.3; 0.1.5; 0.1.7 (a), (b); 0.2.1; 0.2.2; 0.2.3; 0.2.4; 0.2.9; 0.2.11; 0.2.13; 0.3.1; 0.3.3; 0.4.2; 0.5.4, 0.5.6.

C. Assessed Assignments (to be submitted):

0.1.2; 0.1.4; 0.1.8 (a), (b); 0.2.1; 0.2.6; 0.2.8; 0.2.12; **??**; 0.3.2; 0.3.8; 0.5.3; 0.5.5.

D. Bonus Problems and Exercises: Remaining exercises and problems.

0.1 ALGEBRAS AND σ-ALGEBRAS

Exercise 0.1.1. Show that a nonempty family $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra provided that for all $A, B \in \mathcal{A}$ we have $A^c \in \mathcal{A}$ and $A \cap B \in \mathcal{A}$.

Exercise 0.1.2. Prove that for any class $\mathcal{E}$ of sets in X and any mapping $f : X \to X$, one has $\sigma\big(f^{-1}(\mathcal{E})\big) = f^{-1}\big(\sigma(\mathcal{E})\big)$, where $f^{-1}(\mathcal{E}) = \{f^{-1}(E) : E \in \mathcal{E}\}$.

Exercise 0.1.3. Prove that every countable set in $\mathbb{R}$ is a Borel set.

Exercise 0.1.4. If Y is a nonempty Borel subset of $\mathbb{R}$, show that the Borel algebra of the subspace Y is $\{A \in \mathcal{B}(\mathbb{R}) : A \subset Y\}$.

Exercise 0.1.5. An F_σ-set is any countable union of closed sets, and a G_δ-set is any countable intersection of open sets. Prove that both types of sets are Borel sets.

Exercise 0.1.6. Let $\{E_n\}$ be a sequence in an algebra $\mathcal{A}$, then there is a sequence $\{F_n\}$ of disjoint sets of $\mathcal{A}$ such that $F_n \subset E_n$ for each n, $\bigcup_{n=1}^{k} B_n = \bigcup_{n=1}^{k} A_n$ for each n, and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.

Exercise 0.1.7. Prove that $\mathcal{B}(\mathbb{R})$ is generated by each of the following:

(a) the open intervals $\mathcal{E}_1 = \{(a, b) : a < b\}$, $a, b \in \mathbb{R}$;

(b) the closed intervals $\mathcal{E}_2 = \{[a, b] : a < b\}$, $a, b \in \mathbb{R}$;

(c) the half-open intervals $\mathcal{E}_3 = \{(a, b] : a < b\}$ or $\mathcal{E}_4 = \{[a, b) : a < b\}$ $(a, b \in \mathbb{R})$;

(d) the open rays $\mathcal{E}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, b) : b \in \mathbb{R}\}$;

(e) the open rays $\mathcal{E}_7 = \{[a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, b] : b \in \mathbb{R}\}$.

Exercise 0.1.8. Let D be an arbitrary dense set in $\mathbb{R}$ (say $D = \mathbf{Q}$). Prove that $\mathcal{B}(\mathbb{R})$ is generated by any of the following classes of sets:

(a) the open intervals $\mathcal{F}_1 = \{(a, b) : a < b\}$, $a, b \in D$;

(b) the closed intervals $\mathcal{F}_2 = \{[a, b] : a < b\}$, $a, b \in D$;

(c) the half-open intervals $\mathcal{F}_3 = \{(a, b] : a < b\}$ or $\mathcal{F}_4 = \{[a, b) : a < b\}$, $a, b \in D$;

(d) the open rays $\mathcal{F}_5 = \{(a, \infty) : a \in D\}$ or $\mathcal{F}_6 = \{(\infty, b) : b \in D\}$;

(e) the open rays $\mathcal{F}_7 = \{[a, \infty) : a \in D\}$ or $\mathcal{F}_8 = \{(\infty, b] : b \in D\}$.

0.2 MEASURES

Exercise 0.2.1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that if μ is σ-finite, then for every set $E \in \mathcal{M}$, there exists a sequence $\{E_n\} \subset \mathcal{M}$ such that $E = \bigcup_n E_n$ and $\mu(E_n) < \infty$ for each n, i.e., every $E \in \mathcal{M}$ is σ-finite.

Exercise 0.2.2. Show that a countable union of null sets is again a null set.

Exercise 0.2.3. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}$. Prove that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right).$$

Exercise 0.2.4. Let $\mathcal{M}$ be a σ-algebra of subsets of a set X and the set function $\mu : \mathcal{M} \to [0, \infty)$ be finitely additive.

(a) Prove that μ is a measure if and only if whenever $\{A_n\} \subset \mathcal{M}$, $A_1 \subset A_2 \subset \cdots$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(b) Suppose that μ is finite. Prove that μ is a measure if and only if whenever $\{A_n\} \subset \mathcal{M}$, $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then

$$\lim_{n\to\infty} \mu(A_n) = 0.$$

Exercise 0.2.5. Let $\mathcal{A}$ be the algebra of sets $A \subset \mathbf{N}$ such that either A or $\mathbf{N} \setminus A$ is finite. For finite A, let $\mu(A) = 0$, and for A with a finite complement let $\mu(A) = 1$. Then μ is an additive, but not countably additive set function.

Exercise 0.2.6. Let X be a countably infinite set, and let $\mathcal{A}$ be the algebra consisting of all finite subsets of X and their complements. If A is finite, set $\mu(A) = 0$, and if A^c is finite, set $\mu(A) = 1$.

(a) Show that μ is finitely additive but not countably additive on $\mathcal{A}$.

(b) Show that X is the limit of a sequence of sets $A_n \in \mathcal{A}$, $A_1 \subset A_2 \subset \cdots$ such that $\mu(A_n) = 0$ for all n but $\mu(X) = 1$.

Exercise 0.2.7. Let μ be counting measure on X, where X is an infinite set. Show that there is a sequence of sets $A_1 \supset A_2 \supset \cdots$ with $\bigcap_{n=1}^{\infty} A_n = \emptyset$ and $\lim_{n\to\infty} \mu(A_n) \neq 0$.

Exercise 0.2.8. Let $\mu_1, \ldots, \mu_n$ be measures on $(X, \mathcal{M})$ and $c_1, \ldots, c_n$ positive numbers. Show that $\mu := c_1\mu_1 + \cdots + c_n\mu_n$ is a measure on $(X, \mathcal{M})$.

Exercise 0.2.9. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that for $A, B \in \mathcal{M}$,

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B). \tag{0.2.1}$$

Applications: Show that if μ is a probability measure, then for any measurable sets A, B we have

(i) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$, and

(ii) $\min\{\mu(A), \mu(B)\} \geq \mu(A \cap B) \geq \mu(A) + \mu(B) - 1$.

Exercise 0.2.10. Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Show that μ_E is a measure on $\mathcal{M}$.

Exercise 0.2.11. Let $(X, \mathcal{M}, P)$ be a probability space and $B \in \mathcal{M}$ with $P(B) > 0$. The number

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

is called the **conditional probability of $\boldsymbol{A}$ given $\boldsymbol{B}$**.

Show that the function $A \mapsto P(A|B)$ is a probability measure on the σ-algebra $\mathcal{M}$.

Exercise 0.2.12. Given a probability space $(X, \mathcal{M}, P)$ we say that the elements of $\mathcal{M}$ are **events**. The events A, B are **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

Show that if A and B are independent events, then A^c and B are also independent.

Exercise 0.2.13. The **symmetric difference** of two sets A and B is $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Show that if A and B are measurable and $\mu(A \Delta B) = 0$, then $\mu(A) = \mu(B)$.

(b) Show that if μ is complete, $A \in \mathcal{A}$ and $\mu(A \Delta B) = 0$, then $B \in \mathcal{A}$.

Exercise 0.2.14. Let $(X, \mathcal{M})$ be a measurable space. Verify the following:

(a) If μ and μ are measures defined on $\mathcal{M}$, then the set function λ defined on $\mathcal{M}$ by $\lambda(E) = \mu(E) + \nu(E)$ also is a measure. We denote λ by $\mu + \nu$.

(b) If μ and ν are measures on $\mathcal{M}$ and $\mu \geq \nu$, then there is a measure ξ on $\mathcal{M}$ for which $\mu = \nu + \xi$.

(c) If ν is σ-finite, the measure ξ in (b) is unique.

(d) Show that in general the measure ξ in (b) need not be unique but that there is always a smallest such ξ.

0.3 OUTER MEASURES

Exercise 0.3.1. Let $X = \{a, b\}$ and define $\mu^*(\emptyset) = 0$, $\mu^*(\{a\}) = 1$, $\mu^*(\{b\}) = 2$, and $\mu^*(X) = 2$. Show that μ^* is an outer measure but is not additive.

Exercise 0.3.2. Let X be any set. Define $\nu : \mathcal{P}(X) \to [0, \infty]$ by defining $\nu(\emptyset) = 0$ and for $E \subset X$, $E \neq \emptyset$, defining $\nu(E) = \infty$. Show that ν is an outer measure.

Exercise 0.3.3. Prove that for any outer measure μ^* and any set A such that $\mu^*(A) = 0$, A is μ^*-measurable.

Exercise 0.3.4. Let $X = \mathbf{N}$ and $\mathcal{E}$ be the family of all singletons and the whole set $\mathbf{N}$. Let $\mu(\emptyset) = 0$, $\mu(\{n\}) = \frac{1}{2^n}$, and $\mu(\mathbf{N}) = 2$. Determine $\mu^*(\mathbf{N})$ and all μ^*-measurable sets.

Exercise 0.3.5. Prove that if μ^* is an outer measure on X and if $B \subset X$, $\mu^*(B) = 0$, then $\mu^*(A \cup B) = \mu^*(A \setminus B) = \mu^*(A)$.

Exercise 0.3.6. Let μ^* be an outer measure on X, and let $Y \subset X$. Define $\nu^*(A) = \mu^*(A)$ when $A \subset Y$. Is ν^* an outer measure on Y?

Exercise 0.3.7. Let μ^* be an outer measure on X, and let $Y \subset X$. Define $\nu^*(A) = \mu^*(Y \cap A)$. Is ν^* an outer measure on X?

Exercise 0.3.8. Show that a subset E of X is μ^*-measurable if and only if for each $\epsilon > 0$ there exists a measurable set F such that $F \subset E$ and $\mu(E \setminus F) < \epsilon$.

0.4 THE LEBESGUE MEASURE ON $\mathbb{R}^n$

Exercise 0.4.1. Let $I_1, I_2, \ldots, I_n$ be a finite set of intervals covering the rationals in $[0,1]$. Show that $\sum_{k=1}^{n} m(I_k) \geq 1$.

Exercise 0.4.2. Let S be a subset of $\mathbb{R}^n$ such that for each $\epsilon > 0$ there is a closed set F contained in S for which $m^*(S \setminus F) < \epsilon$. Prove that S is Lebesgue measurable.

Exercise 0.4.3. Prove that a subset E of $\mathbb{R}^n$ is Lebesgue measurable if for each $\epsilon > 0$, there exists an open set U such that $E \subset U$ and $m^*(U \setminus E) < \epsilon$.

Exercise 0.4.4. Let $\{A_k\}$ be an increasing sequence of subsets of $\mathbb{R}^n$, that is, $A_1 \subset A_2 \subset \cdots$, and let $A = \bigcup_{k=1}^{\infty} A_k$. Show that $\lim_{k\to\infty} m^*(A_k) = m^*(A)$.

(*Hint.* Let B_k be a Lebesgue measurable set with $A_k \subset B_k$ and $m(B_k) = m^*(A_k)$, $k = 1, 2, \ldots$ Set $C_m = \bigcup_{k=m}^{\infty} B_k$ and $C = \bigcap_{m=1}^{\infty} C_m$. Show that $C \supset A$, $m^*(A_k) = m(B_k) = m(C_k)$, and $\lim_{k\to\infty} m^*(A_k) = m(C)$.)

0.5 BOREL MEASURES ON $\mathbb{R}$

Exercise 0.5.1. Show that if $f : [a,b] \to [c,d]$ is both monotone and onto, then f is continuous.

Exercise 0.5.2. Show that any monotone function $f : \mathbb{R} \to \mathbb{R}$ has points of continuity in every (nonempty) open interval.

Exercise 0.5.3. Show that a strictly increasing function that is defined on an interval is Lebesgue measurable and then use this to show that a monotone function that is defined on an interval is Lebesgue measurable. (Every monotone function is measurable.)

A **distribution function** on $\mathbb{R}$ is a function $F : \mathbb{R} \to \mathbb{R}$ that is increasing and right continuous.

Exercise 0.5.4. If F is a distribution function, the measure $\mu_F(I)$ of any interval I may be expressed in terms of F: for $-\infty < a < b < \infty$,

$$\mu_F\big((a,b]\big) = F(b) - F(a), \qquad \mu_F\big([a,b]\big) = F(b) - F(a-)$$
$$\mu_F\big((a,b)\big) = F(b-) - F(a), \qquad \mu_F\big([a,b)\big) = F(b-) - F(a-).$$

Thus if F is continuous at a and b, all four expressions are equal. Show that F is continuous if and only if $\mu_F(\{y\}) = 0$ for all y.

Exercise 0.5.5. Let F be the distribution function on $\mathbb{R}$ given by

$$F(x) = \begin{cases} 0 & \text{if } x < -1; \\ 1+x & \text{if } -1 \le x < 0; \\ 2+x^2 & \text{if } 0 \le x < 2; \\ 9 & \text{if } x \ge 2. \end{cases}$$

If μ is the Lebesgue-Stieltjes measure corresponding to F, compute the measure of each of the following sets:

(a) $\{2\}$,
(b) $[-\frac{1}{2}, 3)$
(c) $(-1, 0] \cup (1, 2)$,
(d) $[0, \frac{1}{2}) \cup (1, 2]$,
(e) $\{x : |x| + 2x^2 > 1\}$.

(*Hint:* Apply Exercise 0.5.4.)

Exercise 0.5.6. A **probability distribution** is by definition a probability measure P on $\mathbb{R}$ defined on the σ-algebra of Borel sets $\mathcal{B}(\mathbb{R})$. The function $F : \mathbb{R} \to [0, 1]$ defined as

$$F(x) = P\big((-\infty, x]\big), \quad x \in \mathbb{R},$$

is called the **(cumulative) distribution function**. Prove the following properties of F.

(a) $F(x) \le F(y)$ for every $x \le y$ (that is, F is non-decreasing);

(b) $\lim_{x \to a} F(x) = F(a)$ for each $a \in \mathbb{R}$ (that is, F is right-continuous);

(c) $\lim_{x \to -\infty} F(x) = 0$.

(d) $\lim_{x \to +\infty} F(x) = 1$.

Exercise 0.5.7. Show that if $F = \chi_{[c,\infty)}$, then $m_F = \delta_c$, the Dirac measure concentrated at c.

Exercise 0.5.8. Determine the probability measure on $\mathcal{B}(\mathbb{R})$ which has $f(x) = \max\{0, \min\{x, 1\}\}$ as its distribution function.