Optimization 2

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November 28, 2021

Part II

MULTIOBJECTIVE FUZZY/STOCHASTIC LINEAR PROGRAMMING

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References. (For Part II)

Textbook

[1] M. Sakawa, H. Yani, I. Nishizaki, Linear and multiobjective programming with fuzzy stochastic extension. Springer, New York, 2013.

Other references

[2] D. T. Luc, Multiobjective linear programming - An Introduction. Springer, 2016.

[3] J. R. Birge, F. Louveaux, Introduction to Stochastic Programming, 2nd ed. Springer, 2011

Contents

[a] Content of this section depends on the time spent on the previous sections.

1. Multiobjective LP: Problem formulation

• Multiobjective LP: The problem to optimize *multiple conflicting linear objective functions simultaneously* under the given linear constraints is called the multiobjective linear programming problem (MLP).

• Learning Objectives. After learning this chapter, students will be able to (understand and know):

- The formulation of a multiobjective linear programming problems (MLP) and can model a practical problem as a MLP.
- Different notions of solutions to MLP
- Some methods for solving MLP
- Some ways of modification and extension.

Example 1:

Example 2: The example is inspired from the classic portfolio model for a finite set I of assets with expected return r_i, and estimated covariance v_{ij} of the returns of assets $i, j \in I$:

$$
\begin{array}{lll}
(\mathrm{P}) & \max & \sum_{i \in I} r_i x_i \\
 & \min & \sum_{i,j \in I} v_{ij} x_i x_j \\
 & \text{s.t.} & \sum_{i \in I} x_i = 1, x_i \geq 0, i \in I.
\end{array}
$$

Taking into account the almost unlimited number of existing assets in the global economy, it is natural to replace I by $\mathbb{N}$ in (P), $X := \mathbb{R}^{\mathbb{N}}$, and

- $f(x) := \sup_{J \in \mathcal{F}(\mathbb{N})} \sum_{i \in J} f_i(x)$, with $f_i(x) := -r_i x_i$,
- $g(x) := \sup_{K \in \mathcal{F}(\mathbb{N}^2)} \sum_{(i,j) \in K} g_{ij}(x)$, with $g_{ij}(x) = v_{ij} x_i x_j$,
- $h(x) := \sup_{J \in \mathcal{F}(\mathbb{N})} \sum_{i \in J} x_i = 1.$

• Introductory Example.

Production planning problem.

Example 3.1 (Production planning problem). A manufacturing company desires to maximize the total profit from producing two products P_1 and P_2 utilizing three different materials M_1, M_2, and M_3. The company knows that to produce 1 ton of product P_1 requires 2 tons of material M_1, 3 tons of material M_2, and 4 tons of material M_3, while to produce 1 ton of product P_2 requires 6 tons of material M_1, 2 tons of material M_2, and 1 ton of material M_3. The total amounts of available materials are limited to 27, 16, and 18 tons for M_1, M_2, and M_3, respectively. It also knows that product P_1 yields a profit of 3 million yen per ton, while P_2 yields 8 million yen. Given these limited materials, the company is trying to figure out how many units of products P_1 and P_2 should be produced to maximize the total profit. This production planning problem can be formulated as the following linear programming problem:

1. Multiobjective LP: Problem formulation

$$\left.\begin{array}{ll}\text{minimize} & z_1 = -3x_1 - 8x_2 \\ \text{subject to} & 2x_1 + 6x_2 \leq 27 \\ & 3x_1 + 2x_2 \leq 16 \\ & 4x_1 + x_2 \leq 18 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}\right\}$$

In the case where people concerns some more "goal" which also must be minimized (or maximized), such as, to minimize the pollution poured out to the environment in the production process, one must consider some other objective to be minimized. In such a case we have a problem of "minimizing" (in some suitable sense) 2 objective functions.

1. Multiobjective LP: Problem formulation

The following is a continuation of Example 3.1 and illustrates the situation mentioned at the end of the previous slide.

Example 3.2 (Production planning with environmental consideration). Unfortunately, however, in production process, it is pointed out that product P_1 yields 5 units of pollution per ton and product P_2 yields 4 units of pollution per ton. Thus, the manager should not only maximize the total profit but also minimize the amount of pollution.

For simplicity, assume that the amount of pollution is a linear function of two variables x_1 and x_2 such as

$$5x_1 + 4x_2,$$

where x_1 and x_2 denote the numbers of tons produced of products P_1 and P_2, respectively.

1. Multiobjective LP: Problem formulation

Considering the environment quality, the production planning problem can be reformulated as the two-objective linear programming problem:

$$\left.\begin{array}{lrcl}
\text{minimize} & z_1 = & -3x_1 - 8x_2 & \\
\text{minimize} & z_2 = & 5x_1 + 4x_2 & \\
\text{subject to} & & 2x_1 + 6x_2 & \leq 27 \\
 & & 3x_1 + 2x_2 & \leq 16 \\
 & & 4x_1 + x_2 & \leq 18 \\
 & & x_1 \geq 0, \quad x_2 & \geq 0.
\end{array}\right\}$$

Question: What does it means "minimize 2 objective functions"? Think of it?

- Problem formulation

The problem to optimize *multiple conflicting linear objective functions simultaneously* under the given linear constraints is called the multiobjective linear programming problem (MLP). A general form of MLP is as follows:

$$\left.\begin{array}{lrl} \text{minimize} & z_1(x) = & c_1 x \\ \text{minimize} & z_2(x) = & c_2 x \\ & \cdots\cdots\cdots\cdots & \\ \text{minimize} & z_k(x) = & c_k x \\ \text{subject to} & & Ax \le b \\ & & x \ge 0, \end{array}\right\}$$

1. Multiobjective LP: Problem formulation

A general form of MLP:

$$\left.\begin{array}{lrcl}
\text{minimize} & z_1(x) & = & c_1 x \\
\text{minimize} & z_2(x) & = & c_2 x \\
 & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\
\text{minimize} & z_k(x) & = & c_k x \\
\text{subject to} & & & Ax \leq b \\
 & & & x \geq 0,
\end{array}\right\}$$

where

$$\boldsymbol{c}_i = (c_{i1}, \ldots, c_{in}), \quad i = 1, \ldots, k$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

1. Multiobjective LP: Problem formulation

The general MLP in the previous form is sometimes written in the form of vector minimization problem:

$$
\begin{array}{lll}
\text{(MLP)} & \text{minimize} & \mathbf{z}(x) = (z_1(x), z_2(x), \cdots, z_k(x))^T \\
& \text{subject to} & Ax \leq b \\
& & x \geq 0,
\end{array}
$$

where $\mathbf{z}(x) = (z_1(x), z_2(x), \cdots, z_k(x))^T = (c_1 x, c_2(x), \cdots, c_k x)^T$ is a k-dimensional vector.[1]

[1] Here $(z_1(x), z_2(x), \cdots, z_k(x))^T$ denotes the transpose of the given vector.

Now, if we introduce the $(k \times n)$-matrix $C = (c_1, c_2, \cdots, c_k)^T$.
Then we have

$$Cx = (z_1(x), z_2(x), \cdots, z_k(x))^T = (c_1x, c_2(x), \cdots, c_kx)^T = \mathbf{z}(x),$$

and the MLP can be expressed as:

$$\text{(MLP)} \quad \begin{array}{ll} \text{minimize} & \mathbf{z}(x) = Cx \\ \text{subject to} & Ax \leq b \\ & x \geq 0, \end{array}$$

1. Multiobjective LP: Problem formulation

Coming back to the Introductory Example, the MLP in this example can be expressed in the matrix form as:

$$\text{(MLP1)} \quad \text{minimize} \quad \mathbf{z}(x) = \begin{pmatrix} -3 & -8 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{subject to} \quad \begin{pmatrix} 2 & 6 \\ 3 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 27 \\ 16 \\ 18 \end{pmatrix}.$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

1. Multiobjective LP: Problem formulation

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$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Remark 1. What does it mean the inequality " $\leq$ " in (MLP) and (MLP1)?

Consider the closed, convex cone $\mathbb{R}^n_+$ in the space $\mathbb{R}^n$, and the partial order generated by this cone:

$$x \leq_{\mathbb{R}^n_+} y \quad \Leftrightarrow \quad (x_i \leq y_i, \ \forall i = 1, 2, ..., n).$$

A remaining question:

"Minimize" a vector function, how should we understand this? In other words, what does it mean "solution" of (MLP) or (MLP1)?

This question will be answered in the next section.

• Complete optimal solution

Let $X := \{x \in \mathbb{R}^n \mid Ax \leq b, \;\; x \geq 0\}$ be the feasible set of (MLP).

Definition 1 (Complete optimal solution)

A point $x^* \in X$ is said to be a complete optimal solution of the Problem (MLP) if

$$z_i(x^*) \leq z_i(x), \text{ for all } i = 1, 2, \cdots, k, \text{ and for all } x \in X.$$

Remark 2. • [Recall] Consider the closed, convex cone $\mathbb{R}^n_+$ in the space $\mathbb{R}^n$, and the partial order generated by this cone:

$$x \leq_{\mathbb{R}^n_+} y \quad \Leftrightarrow \quad (x_i \leq y_i, \; \forall i = 1, 2, ..., n)\,.$$

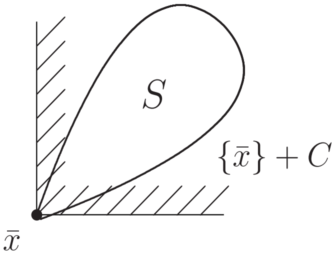
Remark 2. • [Recall] Consider the closed, convex cone $\mathbb{R}^n_+$ in the space $\mathbb{R}^n$, and the partial order generated by this cone:

$$x \leq_{\mathbb{R}^n_+} y \quad \Leftrightarrow \quad (x_i \leq y_i,\ \forall i = 1, 2, ..., n)\,.$$

• $x^* \in X$ is a complete optimal solution of (MLP) means that $C(x^*)$ is a strong minimal element of the set $C(X)$ w.r.t. the ordering cone $\mathbb{R}^n_+$ in the sense that

$$C(x^*) \leq_{\mathbb{R}^n_+} C(x),\ \forall x \in X.$$

• Strong minimal element of a set S. In the figure below, C is a cone that defines the order in the space in consideration (e.g., $\mathbb{R}^n$).



Strongly minimal element of a set S.

• **Further note on an order generates by a cone.**

Remark 3. The concept of Complete optimal solution for (MLP) is very strict. It does not always exist when the objective functions conflict with each other. It is often not applicable in practice.

- Pareto optimal solution

Definition 1 (Pareto optimal solution)

A point $x^* \in X$ is said to be a Pareto optimal solution of the Problem (MLP) if there does not exist another $x \in X$ such that $z_i(x) \leq z_i(x^*)$, for all $i \in I := \{1, 2, \cdots, k\}$ and $z_j(x) \neq z_j(x^*)$ for at least one $j \in I$.

How can we understand this concept?

Note: Saying that "$x^* \in X$ is a Pareto optimal solution of the Problem (MLP)" is the same as: "Cx^* is a Pareto minimal point of the set $C(X)$".

Pareto minimal points of a set in $\mathbb{R}^n$ with the order cone $\mathbb{R}^n_+$.

Let $\emptyset \neq A \subset \mathbb{R}^n$.

- $\bar{y} \in A$ is called a Pareto minimal point of A (w.r.t. $\mathbb{R}^n_+$) if

$$A \cap \left(\bar{y} - \mathbb{R}^n_+\right) = \{\bar{y}\}.$$

- $\bar{y} \in A$ is called a weak Pareto minimal point of A (w.r.t. $\mathbb{R}^n_+$) if

$$A \cap \left(\bar{y} - \mathrm{int}\ \mathbb{R}^{\mathrm{n}}_+\right) = \emptyset.$$

How can we understand these notions?

[Some illustration figures are needed]

• Minimal (Pareto minimal) element of a set S. In the figure below, C is a cone that defines the order in the space in consideration (e.g., $\mathbb{R}^n$).

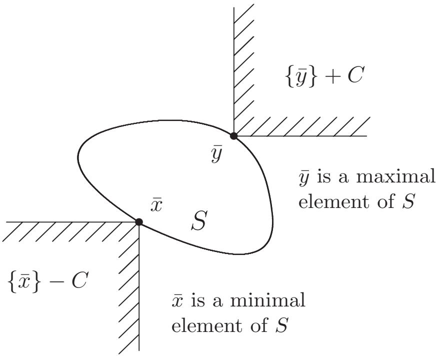


Figure 4.1: Minimal and maximal elements of a set S.

Note 1: Saying that "$x^* \in X$ is a Pareto optimal solution of the Problem (MLP)" is the same as: "Cx^* is a Pareto minimal point of the set $C(X)$". **Is it true? Explain?**

2. Solution concepts

Note 1: Saying that "$x^* \in X$ is a Pareto optimal solution of the Problem (MLP)" is the same as: "Cx^* is a Pareto minimal point of the set $C(X)$". **Is it true? Explain?**

Note 2: What are the Pareto minimal point(s), strong minimal point(s) of the set:

$$S := \{(x_1, x_2) \in [0, 2] \times [0, 2] \mid x_2 \geq 1 - \sqrt{1 - (x_1 - 1)^2} \text{ for } x_1 \in [0, 1]\}?$$

Note 1: Saying that "$x^* \in X$ is a Pareto optimal solution of the Problem (MLP)" is the same as: "Cx^* is a Pareto minimal point of the set $C(X)$". **Is it true? Explain?**

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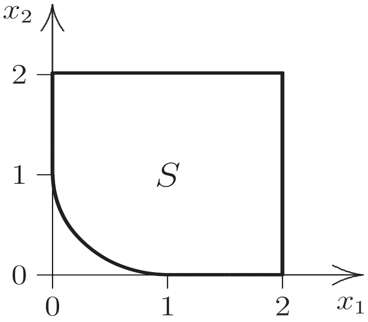

2. Solution concepts

In the Figure in "Note 2" in previous page, is the points of the form $(x, 0)$ with $x \in (1, 2]$ a Pareto minimal point of S? Similar question for the points of the form $(0, y)$ with $y \in (1, 2]$.

In the Figure in "Note 2" in previous page, is the points of the form $(x, 0)$ with $x \in (1, 2]$ a Pareto minimal point of S? Similar question for the points of the form $(0, y)$ with $y \in (1, 2]$.

• Weak Pareto minimal point of set in $\mathbb{R}^n$ with the order cone $\mathbb{R}^n_+$.

Recall: Let $\emptyset \neq A \subset \mathbb{R}^n$.

• $\bar{y} \in A$ is called a weak Pareto minimal point of A (w.r.t. $\mathbb{R}^n_+$) if $A \cap \left(\bar{y} - \text{int } \mathbb{R}^n_+\right) = \emptyset$.

In the Figure in "Note 2" in previous page, is the points of the form $(x, 0)$ with $x \in (1, 2]$ a Pareto minimal point of S? Similar question for the points of the form $(0, y)$ with $y \in (1, 2]$.

- Weak Pareto minimal point of set in $\mathbb{R}^n$ with the order cone $\mathbb{R}^n_+$.

Recall: Let $\emptyset \neq A \subset \mathbb{R}^n$.

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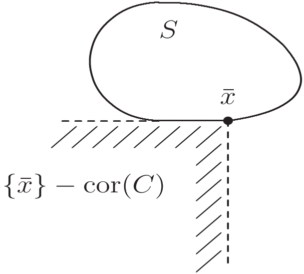


Weakly minimal element of a set S.

- Weak Pareto optimal solution

Definition 1 (Weak Pareto optimal solution)

A point $x^* \in X$ is said to be a weak Pareto optimal solution of the Problem (MLP) if there does not exist another $x \in X$ such that $z_i(x) < z_i(x^*), \quad$ for all $i \in I := \{1, 2, \cdots, k\}$.

Comments. ...

Note 3: "$x^* \in X$ is a weak Pareto optimal solution of the Problem (MLP)" means that "Cx^* is a weak Pareto minimal point of the set $C(X)$".

We have

$$X^{CO} \subset X^{P} \subset X^{WP},$$

where

- $X^{C)}$ is the set of all complete optimal solutions,
- X^{P} is the set of all Pareto solutions, and
- X^{WP} is the set of all weal Pareto solutions of the problem

(MLP).

Question: How can we determine the set $C(X)$?

Introductory Example (Revisited)

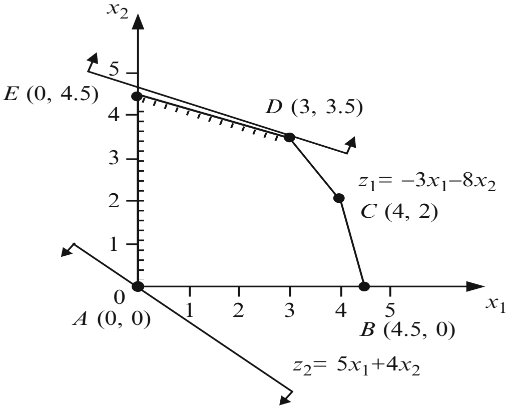
$$\text{(MLP1)} \quad \text{minimize} \quad \mathbf{z}(x) = \begin{pmatrix} -3 & -8 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{subject to} \quad \begin{pmatrix} 2 & 6 \\ 3 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 27 \\ 16 \\ 18 \end{pmatrix}.$$

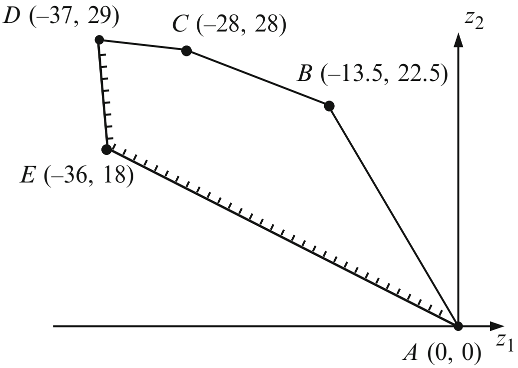
$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The feasible set X and the set $C(X)$ are as follows:

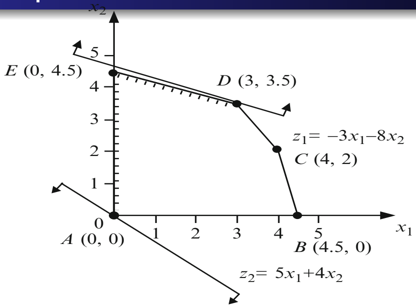
The feasible set X of (MLP1)

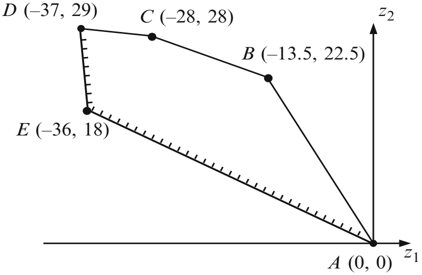


The set $c(X)$
(i.e., the image of the feasible set X through the linear map c)



2. Solution concepts





Exercise. Find the set X^{CO}, X^{P}, and X^{WP} of the problem (MLP1) when the objective functions of this problems are changed to:

(a) $z_1 = -x_1 - 4x_2$, $z_2 = 2x_1 - x_2$,

(b) $z_1 = -4x_1 - 5x_2$, $z_2 = 2x_1 - x_2$.

Several computational methods have been proposed for characterizing Pareto optimal solutions depending on the different scalarizations of the multiobjective linear programming problems. Among the many possible ways of scalarizing the multiobjective linear programming problems, the weighting method, the constraint method, and the weighted minimax method have been studied as a means of characterizing Pareto optimal solutions of the multiobjective linear programming problems.

We will study these methods in this section, one after another.

3.1. Weighting method. We consider the general multiobjective linear programming problem of the form:

$$
\text{(MLP)} \quad \begin{array}{ll} \text{minimize} & \mathbf{z}(x) = (z_1(x), z_2(x), \cdots, z_k(x))^T \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}
$$

and denote by X its feasible set, i.e., $X = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$.

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- The weighting method is for obtaining a Pareto optimal solution of (MLP).

3.1. Weighting method. We consider the general multiobjective linear programming problem of the form:

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and denote by X its feasible set, i.e.,
$X = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$.

• The weighting method is for obtaining a Pareto optimal solution of (MLP).

• The weighting method is to solve the weighting problem formulated from (MLP) by taking the weighted sum of all the objective functions in the original (MLP).

- The weighting problem associated to (MLP) is defined by

$$\text{(WP)} \quad \underset{x \in X}{\text{minimize}} \;\; wz(x) = \sum_{i=1}^{k} w_i z_i(x),$$

where $w = (w_1, \cdots, w_k)$ is a vector of weighting coefficients assigned to the objective functions and assumed to be:

$$w = (w_1, \cdots, w_k) \geq 0, \quad w \neq 0.$$

The relationships between the optimal solution x^* of (WP) and the Pareto optimal concept of the (MLP) can be given in the next theorems.

3.1. Weighting method

In the following, $w > 0$ means that $w_i > 0$ for all $i = 1, ..., k$.

Theorem 3.1

If $x^* \in X$ is an optimal solution of the weighting problem for some $w > 0$, then x^* is a Pareto optimal solution of the (MLP).

3.1. Weighting method

In the following, $w > 0$ means that $w_i > 0$ for all $i = 1, ..., k$.

Theorem 3.1

If $x^* \in X$ is an optimal solution of the weighting problem for some $w > 0$, then x^* is a Pareto optimal solution of the (MLP).

Proof. • $x^* \in X$ is an optimal solution of (WP) with $w > 0$. If x^* is NOT a Pareto solution of (MLP), then there exists $\bar{x} \in X$ and an index j such that

$$z_j(\bar{x}) < z_j(x^*) \text{ and } z_i(\bar{x}) \leq z_i(x^*), \ \forall i \neq j. \tag{1}$$

• As $w = (w_1, \cdots, w_k) > 0$, it follows from (1) that

$$\sum_{i=1}^{k} w_i z_i(\bar{x}) < \sum_{i=1}^{k} w_i z_i(x^*),$$

which contradicts the fact that x^* is a solution of (WP).

• Thus, x^* must be a Pareto solution of (MLP). □

Remark. It is possible that $x^* \in X$ is a solution of (WP) for some $w \geq 0$, $w \neq 0$ and x^* is a Pareto solution of (MLP). For details, see the example at the end of this subsection.

[2] [1] M. Sakawa, H. Yani, I. Nishizaki, Linear and multiobjective programming with fuzzy stochastic extension. Springer, New York, 2013.

Remark. It is possible that $x^* \in X$ is a solution of (WP) for some $w \geq 0$, $w \neq 0$ and x^* is a Pareto solution of (MLP). For details, see the example at the end of this subsection.

Theorem 3.2

If $x^* \in X$ is a Pareto optimal solution of (MLP), then x^* is an optimal solution to the weighting problem (WP) for some $w = (w_1, \cdots, w_k) \geq 0$, $w \neq 0$.

[2] [1] M. Sakawa, H. Yani, I. Nishizaki, Linear and multiobjective programming with fuzzy stochastic extension. Springer, New York, 2013.

Remark. It is possible that $x^* \in X$ is a solution of (WP) for some $w \geq 0$, $w \neq 0$ and x^* is a Pareto solution of (MLP). For details, see the example at the end of this subsection.

Theorem 3.2

If $x^* \in X$ is a Pareto optimal solution of (MLP), then x^* is an optimal solution to the weighting problem (WP) for some $w = (w_1, \cdots, w_k) \geq 0$, $w \neq 0$.

The proof of this Theorem 3.2 is a bit complicated, using the strong duality between linear programs and their dual problems. For details, see the textbook[2] pages 78-80.

[2] [1] M. Sakawa, H. Yani, I. Nishizaki, Linear and multiobjective programming with fuzzy stochastic extension. Springer, New York, 2013.

Remark. Let $w = (w_1, \cdots, w_k) > 0$.

- The equation

$$wz = w_1 z_1(x) + w_2 z_2(x) + \cdots + w_k z_k(x) = c \quad \text{(constant)}$$

in $z = (z_1, z_2, ..., z_k)$ space represents a hyperplane. Concretely, when $k = 2$ it is a line, when $k = 3$ it is a plane, etc.

- Solving the (WP) with such a weighting coefficient $w > 0$ yields the minimum c such that this hyperplane has at least one common point with the feasible region. The corresponding Pareto solution x^* is then obtained.

- It is worth observing that the weighting problem (for a choice of w) gives us, in general, ONE Pareto solution of the (MLP).

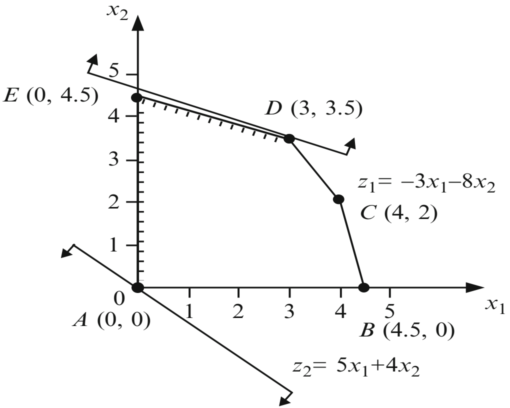
Example. Introductory example (revisted)

$$\text{(MLP1)} \quad \text{minimize} \quad \mathbf{z}(x) = \begin{pmatrix} -3 & -8 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{subject to} \quad \begin{pmatrix} 2 & 6 \\ 3 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 27 \\ 16 \\ 18 \end{pmatrix}.$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here $k = 2$, $z_1 = -3x_1 - 8x_2$, $z_2 = 5x_1 + 4x_2$. The feasible set is given in the figure in the next page.



x_2
x_1
5
4
3
2
1
0
1
2
3
4
5
E (0, 4.5)
D (3, 3.5)
$z_1 = -3x_1 - 8x_2$
C (4, 2)
A (0, 0)
B (4.5, 0)
$z_2 = 5x_1 + 4x_2$

3.1. Weighting method

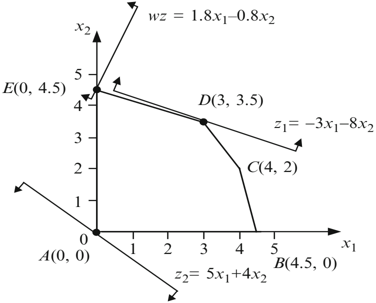
The corresponding weighting problem is as follows:

$$\begin{array}{lrl} \text{minimize} & \boldsymbol{wz}(\boldsymbol{x}) = & w_1(-3x_1 - 8x_2) + w_2(5x_1 + 4x_2) \\ \text{subject to} & & 2x_1 + 6x_2 \leq 27 \\ & & 3x_1 + 2x_2 \leq 16 \\ & & 4x_1 + x_2 \leq 18 \\ & & x_1 \geq 0,\ x_2 \geq 0. \end{array}$$

If we choose, e.g., $w_1 = 0.4$, $w_2 = 0.6$, then

$$wz(x) = 1.8x_1 - 0.8x_2.$$

As depicted in the next figure, solving (WP) yields a Pareto solution of (MLP1): $E(0, 4.5)$.



Think of this. What happens if our choice is:

- $w_1 = 1,\ w_2 = 0,$
- $w_1 = 0,\ w_2 = 1,$
- $w_1 = 1,\ w_2 = 1.$

3.2. Constraint method.

The constraint method for characterizing Pareto optimal solutions is to solve the constraint problem formulated by taking one objective function of a multiobjective linear programming problem as the objective function of the constraint problem and letting all the other objective functions be inequality constraints (Haimes and Hall 1974; Haimes et al. 1971). The constraint problem is defined by

$$\left.\begin{array}{ll} \text{minimize} & z_j(\boldsymbol{x}) \\ \text{subject to} & z_i(\boldsymbol{x}) \leq \varepsilon_i,\ i = 1, 2, \ldots, k;\ i \neq j \\ & \boldsymbol{x} \in X. \end{array}\right\}$$

Example. Introductory example (revisited). Again,

$$(\text{MLP1}) \quad \text{minimize} \quad \mathbf{z}(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} = \begin{pmatrix} -3x_1 - 8x_2 \\ 5x_1 + 4x_2 \end{pmatrix}$$

$$\text{subject to} \quad \begin{pmatrix} 2 & 6 \\ 3 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 27 \\ 16 \\ 18 \end{pmatrix}.$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The corresponding constraint problem for $j = 1$ is:

$$\begin{array}{llrcrcl}
\text{minimize} & z_1 = & -3x_1 & - & 8x_2 & & \\
\text{subject to} & & 2x_1 & + & 6x_2 & \leq & 27 \\
 & & 3x_1 & + & 2x_2 & \leq & 16 \\
 & & 4x_1 & + & x_2 & \leq & 18 \\
 & z_2 = & 5x_1 & + & 4x_2 & \leq & \varepsilon_2 \\
 & & & & \multicolumn{3}{l}{x_1 \geq 0,\ x_2 \geq 0.}
\end{array}$$

Theorem 3.3

If $x^* \in X$ is a unique optimal solution of the constraint problem for some ε_i, $i = 1, ..., k$, $i \neq j$, then x^* is a Pareto optimal solution to the (MLP).

Theorem 3.3

If $x^* \in X$ is a unique optimal solution of the constraint problem for some ε_i, $i = 1, ..., k$, $i \neq j$, then x^* is a Pareto optimal solution to the (MLP).

Proof. (Easy. Read the textbook, page 82).

Note. What happens if the assumption on the "unique" of solution in the Theorem 3.3 is removed?

Theorem 3.3

If $x^* \in X$ is a unique optimal solution of the constraint problem for some ε_i, $i = 1, ..., k$, $i \neq j$, then x^* is a Pareto optimal solution to the (MLP).

Proof. (Easy. Read the textbook, page 82).

Note. What happens if the assumption on the "unique" of solution in the Theorem 3.3 is removed?

Theorem 3.4

If $x^* \in X$ is a Pareto optimal solution to (MLP), then x^* is an optimal solution to the constraint problem for some ε_i, $i = 1, ..., k$, $i \neq j$.

Proof. Assume that $x^* \in X$ is a Pareto optimal solution to (MLP) but it is NOT an optimal solution for any constraint problems. So, it is NOT an optimal solution of the constraint problem

$$\left.\begin{array}{ll} \text{minimize} & z_j(\boldsymbol{x}) \\ \text{subject to} & z_i(\boldsymbol{x}) \le \varepsilon_i, \; i = 1, 2, \ldots, k; \; i \ne j \\ & \boldsymbol{x} \in X. \end{array}\right\}$$

for some j and where $\varepsilon_i := z_i(x^*)$, for all $i = 1, 2, ..., k$ and $i \neq j$. Therefore, there exists $\bar{x} \in X$ such that

$$z_j(\bar{x}) < z_j(x^*), \quad z_i(\bar{x}) \le z_i(x^*), \; \forall i = 1, 2, ..., k, \; i \neq j,$$

which contradicts the fact that x^* is a Pareto optimal solution of (MLP). $\square$

Example. Introductory example (revisited). Again,

$$\text{(MLP1)} \quad \text{minimize} \quad \mathbf{z}(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} = \begin{pmatrix} -3x_1 - 8x_2 \\ 5x_1 + 4x_2 \end{pmatrix}$$

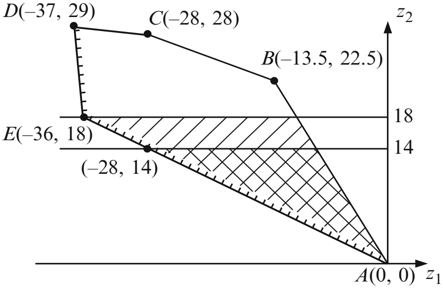
$$\text{subject to} \quad \begin{pmatrix} 2 & 6 \\ 3 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 27 \\ 16 \\ 18 \end{pmatrix}.$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The corresponding constraint problem for $j = 1$ is:

$$\begin{array}{llrcl}
\text{minimize} & z_1 = & -3x_1 - 8x_2 & & \\
\text{subject to} & & 2x_1 + 6x_2 & \leq & 27 \\
& & 3x_1 + 2x_2 & \leq & 16 \\
& & 4x_1 + x_2 & \leq & 18 \\
& z_2 = & 5x_1 + 4x_2 & \leq & \varepsilon_2 \\
& & x_1 \geq 0, \; x_2 \geq 0. & &
\end{array}$$

Example. Introductory example (revisited).



Comment. Consider the case $\varepsilon_2 = 18$; $\varepsilon_2 = 14$.

- (In the (z_1, z_2)-space) The figure shows the sets:

$$C(X) \cap \{(z_1, z_2) \mid z_2 \leq 18\} \quad \text{and} \quad C(X) \cap \{(z_1, z_2) \mid z_2 \leq 14\}.$$

- Consider the constraint problem as just a linear program.

4. Linear goal programming

The term "goal programming" first appeared in the 1961 text by Charnes and Cooper to deal with multiobjective linear programming problems that assumed the decision maker (DM) could specify goals or aspiration levels for the objective functions. Subsequent works on goal programming have been numerous, including texts on goal programming by Ijiri (1965), Lee (1972), and Ignizio (1976, 1982) and survey papers by Charnes and Cooper (1977) and Ingnizio (1983).

The key idea behind goal programming is to minimize the deviations from goals or aspiration levels set by the DM. Goal programming therefore, in most cases, seems to yield a satisficing solution in the same spirit as March and Simon (1958) rather than an optimal solution.

[Quoted from the textbook].

Recall: We consider the general multiobjective linear programming problem of the form:

$$\text{(MLP)} \quad \begin{array}{ll} \text{minimize} & \mathbf{z}(x) = (z_1(x), z_2(x), \cdots, z_k(x))^T \\ \text{subject to} & Ax \leq b, \\ & x \geq 0, \end{array}$$

where $z_1(x) = c_1 x$, ..., $z_k(x) = c_k x$.
The feasible set: $X = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$.
The (MLP)can be rewritten as:

$$\text{(MLP)} \quad \min_{x \in X}\text{imize} \ \mathbf{z}(x) = (z_1(x), z_2(x), \cdots, z_k(x))^T .$$

For linear goal programming, however, a set of k goals is specified by the DM for the k objective functions $z_i(\boldsymbol{x})$, $i = 1, \ldots, k$ and the multiobjective linear programming problem is converted into the problem of coming "as close as possible" to the set of specified goals which may not be simultaneously attainable.

The general formulation of goal programming is of the form:

$$\text{(GP)} \quad \underset{x \in X}{\text{minimize}} \ d(z(x), \hat{z}),$$

where

- $\hat{z} = (\hat{z}_1, \hat{z}_2, ..., \hat{z}_k)$ is the goal vector specified by the decision maker (DM) and
- $d(z(x), \hat{z})$ represents the distance between $z(x)$ and $\hat{z}$ in some sense (i.e., in some norm in $\mathbb{R}^k$).

The simplest form of (GP) when the l_1 norm in $\mathbb{R}^k$ is used is:

$$\text{(GP1)} \quad \underset{x \in X}{\text{minimize}} \;\; d_1(z(x), \hat{z}) = \sum_{i=1}^{k} |c_i(x) - \hat{z}_i|.$$

More generally, when the weighted vector $w = (w_1, ..., w_k) \geq 0$ is used (or weighted l_1 norm), the (GP) may has the form:

$$\text{(GP1w)} \quad \underset{x \in X}{\text{minimize}} \;\; d_1^w(z(x), \hat{z}) = \sum_{i=1}^{k} w_i |c_i(x) - \hat{z}_i|.$$

For each $i = 1, 2, ..., k$, we introduce the auxiliary variables d_i^+ and d_i^- as follows:

$$d_i^+ = \frac{1}{2}\{|z_i(\boldsymbol{x}) - \hat{z}_i| + (z_i(\boldsymbol{x}) - \hat{z}_i)\}$$

and

$$d_i^- = \frac{1}{2}\{|z_i(\boldsymbol{x}) - \hat{z}_i| - (z_i(\boldsymbol{x}) - \hat{z}_i)\}$$

Then the (GP1w) is equivalent to the following linear programming problem (Problem (3.30) in textbook):

$$\left.\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{k} w_i(d_i^+ + d_i^-) \\ \text{subject to} & z_i(\boldsymbol{x}) - d_i^+ + d_i^- = \hat{z}_i \quad i = 1, \ldots, k \\ & A\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0} \\ & d_i^+ \cdot d_i^- = 0,\ i = 1, \ldots, k \\ & d_i^+ \geq 0,\ d_i^- \geq 0,\ i = 1, \ldots, k. \end{array}\right\}$$

It is easy to get the equivalent form of d_i^+ and d_i^-:

$$d_i^+ = \begin{cases} z_i(\boldsymbol{x}) - \hat{z}_i & \text{if } z_i(\boldsymbol{x}) \geq \hat{z}_i \\ 0 & \text{if } z_i(\boldsymbol{x}) < \hat{z}_i \end{cases}$$

and

$$d_i^- = \begin{cases} \hat{z}_i - z_i(\boldsymbol{x}) & \text{if } \hat{z}_i \geq z_i(\boldsymbol{x}) \\ 0 & \text{if } \hat{z}_i < z_i(\boldsymbol{x}). \end{cases}$$

Comments. (Meaning of d_i^+, d_i^-)

• For each $i = 1, ..., k$, d_i^+ and d_i^- represent, respectively, the "overachievement" and "underachievement" of the i^{th} goal. They are called: deviation variables.

• Overachievement and underachievement never occur simultaneously: $d_i^+ > 0$ then $d_i^- = 0$ and vice versa, i.e., $d_i^+ \cdot d_i^- = 0$ for each $i = 1, ..., k$. Hence, Simplex method can be used for Problem (3.30) (in the last slide).

- ... deeper and special cases.

Depending on the decision situations, the DM may be sometimes concerned only with either the overachievement or underachievement of a specified goal. Such a situation can be incorporated into the goal programming formulation by assigning the over- and underachievement weights w_i^+ and w_i^- to d_i^+ and d_i^-, respectively. For example, if each $z_i(\boldsymbol{x})$ is a cost-type objective function with its goal $\hat{z}_i$, the overachievement is not desirable. For this case, we set $w_i^+ = 1$ and $w_i^- = 0$, and the problem (3.30) is modified as follows:

$$\left.\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{k} w_i^+ d_i^+ \\ \text{subject to} & z_i(\boldsymbol{x}) - d_i^+ + d_i^- = \hat{z}_i \quad i = 1, \ldots, k \\ & A\boldsymbol{x} \leq \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0} \\ & d_i^+ \cdot d_i^- = 0, \ i = 1, \ldots, k \\ & d_i^+ \geq 0, \ d_i^- \geq 0, \ i = 1, \ldots, k. \end{array}\right\}$$

Exercises. Ex. 3.1, 3.2, 3.3 and 3.4 (i) and (ii) [ignore 3.4 (iii)].