FINANCIAL RISK MANAGEMENT 2

Ta Quoc Bao

Department of Mathematics,
International University-VNUHCM

Chapter 6. Analytical VaR for Bonds and Options

In this chapter we study some important methods to approximate Value-at-Risk, the so-called Delta, Gamma and Theta approximations and then apply to evaluate VaR for non-linear position such as Bonds and Options. There are two main approach used to calculate VaR in the non-linear case:

- Using Monte Carlo simulation to obtain a numerical estimate of VaR. This method is very accurate, however, it can be computationally expensive for large portfolios.
- The second approach(delta-gamma methods) consists of analytical approximations of the true distribution of changes in the portfolio value. It can provide an approximate but fast parametric solution to the problem.

1. The Delta approximation

Assume that we have K risk factors denote by $f = f_1, f_2, ..., f_K$. Consider a portfolio whose value V is a non-linear function of the risk factors f. Then the change in value ΔV, can be approximate by a first-order Taylor expansion

$$\Delta V \approx \sum_{i=1}^{K} \frac{\partial V}{\partial f_i} \Delta f_i = \sum_{i=1}^{K} \delta_i \Delta f_i$$

where $\delta_i := \frac{\partial V}{\partial f_i}$ the first-order partial derivative of the value function V with respect to the risk factor f_i can be viewed as a sensitivity measure of the value of the financial instrument to changes or shocks in underlying risk factors .

Define

$$\Delta x_i := \frac{\Delta f_i}{f_i}$$

as the proportional changes or returns in the underlying risk factor in one day. Then we obtain

$$\Delta V \approx \sum_{i=1}^{K} \delta_i \times f_i \times \Delta x_i$$

The volatility of ΔV is

$$\sigma(\Delta V) = \sqrt{\delta' \Sigma \delta}$$

where vector $\delta := (\delta_1 f_1, \delta_2 f_2 ..., \delta_K f_K)$ and Σ is the covariance matrix of Δx_i and Δx_j.

Remark. For a portfolio consisting of N assets, then we have

$$\Delta V \approx \sum_{j=1}^{N} \omega_j \sum_{i=1}^{K} \frac{\partial V_j}{\partial f_i} \times f_i \times \Delta x_i$$

Example. Consider a forward contract on a foreign currency. Let S_t be the current spot price, measured in dollars, of one unit of the foreign currency and F_t be the forward price, measured in dollars, of one unit of the foreign currency. A foreign currency has the property that the holder of the currency can earn interest at the risk-free rate prevailing in the foreign country. Denote r_f as the value of a foreign risk-free interest rate for a maturity τ with continuous compounding and r as the domestic risk-free interest rate for the same maturity. We have

$$F_t = S_t e^{-r_f \tau} - F_T e^{-r\tau}$$

where F_T is the contracted forward price at maturity. Then we have the change price

$$\Delta F_t = \frac{\partial F_t}{\partial S}\Delta S_t + \frac{\partial F_t}{\partial r_f}\Delta r_f + \frac{\partial F_t}{\partial r}\Delta r$$

Hence,

$$\Delta F_t = e^{-r\tau}\Delta S - \tau S_t e^{-r_f \tau}\Delta r_f + \tau F_T e^{-r\tau}\Delta r$$

Special case: Delta-Normal VaR

For simple, we assume that the value of portfolio is a function of one the risk factor which is the price P_t of the asset at time t. The the loss can be approximated by

$$L = -\Delta V = -\frac{\partial V}{\partial P}\Delta P = -\delta_V \times \Delta P$$

We now assume that the stock returns are normal distribution

$$\frac{\Delta P}{P_t} \sim N(\mu, \sigma^2) \Rightarrow \Delta P \sim N(P_t\mu, P_t^2\sigma^2)$$

Then we get

$$VaR_\alpha(L) = -\delta_V VaR_\alpha(\Delta P)$$

Be careful with the sign of δ_V. Note that we are working with loss then α is big

Hence we obtain

$$VaR_\alpha(L) = \begin{cases} -\delta_V P_t \mu + \delta_V P_t \sigma \Phi^{-1}(\alpha) & \text{if} \quad \delta_V \geq 0 \\ -\delta_V P_t \mu - \delta_V P_t \sigma \Phi^{-1}(\alpha) & \text{if} \quad \delta_V < 0 \end{cases}$$

- If we have only one stock in our portfolio then $\delta_V = 1$.
- If we have K stock in our portfolio then $\delta_V = K$

Example. *The yearly volatility of the S& P 500 is assumed to be 20%. Calculate the VaR with the confidence level at 95% for 5-day horizon. We know that the current price is $P_t = 2800$. The 5-day return is assumed to be normally distributed with zero mean*

We have $\sigma = 0.2\sqrt{5/250}$, $\alpha = 0.95$ and $\delta_V = 1$. So we get the delta-approximation of VaR is

$$VaR_{0.95}(L) = P_t \times \sigma \times \Phi^{-1}(0.95) = 2800 \times 0.0283 \times (1.645) = 130.3$$

2. Delta-Gamma approximation

When the changes of the factors are two much then the first-order Taylor expansion may not be too accurate. We can improve the accuracy by using Delta-Gamma approximation. We have the Delta-Gamma approximation with K risk factors

$$\Delta V \approx \sum_{i=1}^{K} \delta_i \Delta f_i + \frac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} \frac{\partial^2 V}{\partial f_i \partial f_j} \Delta f_i \Delta f_j$$

Putting $\delta = (\delta_1 f_1, \delta_2 f_2, ..., \delta_K f_k)$ and $\Gamma_{ij} = f_i f_i \frac{\partial^2 V}{\partial f_i \partial f_j}$, $\Delta x = (\Delta x_1, \Delta x_2, ..., \Delta x_K)$ then we have

$$\Delta V = \delta(\Delta x)' + \frac{1}{2} \Delta x \Gamma (\Delta x)'$$

Special case: Delta-Gamma-Normal VaR

In case we have only one risk factor, it is the price of asset and we assume that the return is normally distributed. Then the loss

$$L = -\Delta V = -\Big(\frac{\partial V}{\partial P}\Delta P + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}(\Delta P)^2\Big)$$
$$= -\Big(\delta_V \Delta P + \frac{1}{2}\Gamma_V (\Delta P)^2\Big)$$

Note that since ΔP is normally distributed then $(\Delta P)^2$ is a squared normally distributed, i.e., chi-squared distribution. Therefore, we cannot use the analytical formula we used previously. We evaluate VaR in the following two steps

(i) First, calculate the VaR for the stock price changes

$$VaR_\alpha(\Delta P) = P_t \sigma \Phi^{-1}(1-\alpha)$$

(ii) Then use this value to calculate the portfolio VaR

$$VaR_\alpha(L) = -\left(\delta_V VaR_\alpha(\Delta P) + \frac{1}{2}\Gamma_V (VaR_\alpha(\Delta P))^2\right)$$

3. Calculating VaR for Bonds

Consider a default-free coupon bond maturing in T years with cash flows $c_1, c_2, .., c_T$. Assuming a yield to maturity with annual interest rate r, and the par value at time T is B_T,the bond price is the present value of the future cash flows

$$P(r,T) = \sum_{t=1}^{T} \frac{c_t}{(1+r)^t} + \frac{B_T}{(1+r)^T}$$

The bond price change when:

- interest rate r change, this is main risk
- time changes (although not for all coupon bonds), this is secondary risk
- Hence, for short-time periods, the risk of holding bonds is interest rate risk

Let $P(r)$ denote the price of a default free bond as a function of r. For a small interest rate change $dr \approx 0$ then

$$P(r+dr) \approx P(r) + P'(r)dr$$

or

$$P'(r) \approx \frac{d}{dr}P(r)$$

Hence

$$\Delta P(r) \approx P(r+dr) - P(r) = P'(r)dr$$

Example. Pure discount bond with face value \$ 1 maturing it time T.

$$P(r,T) = (1+r)^{-T}$$

Hence

$$P'(r) = -T(1+r)^{-T-1} = -T(1+r)^{-1}P(r,T)$$

Consequently,

$$\Delta P = -T(1+r)^{-T-1}dr$$

Define Modified Duration as

$$D^* := \frac{-P'(r)}{P(r)}$$

Then the change value

$$\Delta P \approx -D^* P(r, T) dr$$

Assume that $dr = r_t - r_{t-1}$ the daily change in the risk-free annual interest rate is normally distributed,i.e.,

$$dr = r_t - r_{t-1} \sim N(0, \sigma_r^2)$$

then the bond returns $R_{bond} := \frac{\Delta P}{P(r,T)}$ have the distribution

$$R_{bond} \sim N(0, (D^* \sigma_r)^2)$$

So the daily VaR for bond at confidence level α is

$$VaR^{\alpha}(bond) = (D^{*}\sigma_r)\Phi^{-1}(\alpha)P(r,T)$$

Example. A company has a position in bonds worth \$6 million. The modified duration of the portfolio is 5.2 years. Assume that only parallel shifts in the yield curve can take place and that the standard deviation of the daily yield change (when yield is measured in percent) is 0.09. Use the duration model to estimate the 20-day 90% VaR for the portfolio.

Solution. The change in the value of the portfolio for a small change ΔP is approximately $-DP(r,T)\Delta r$. The standard deviation of the daily change in the value of the bond portfolio equals $DP(r,T)\sigma_r$
We have: $D = 5.2, P(r,T) = 6 \times 10^6, \sigma_r = 0.0009$. So the 20-day 90% VaR for the bond is

$$VaR^{0.1}(bond) = -5.2 \times 6 \times 10^6 \times 0.0009 \times \Phi^{-1}(0.1) \times \sqrt{20} = \$160990$$

4. Calculating VaR for options

4.1 Introduction to options

Option types:

- Call option: A call option gives the holder of the option the right to buy an asset by a certain date for a certain price.
- Put option: A put option gives the holder the right to sell an asset by a certain date for a certain price.
- The date specified in the contract is known as the expiration date or the maturity date.
- The price specified in the contract is known as the exercise price or the strike price.

Exercise styles: There are two types of Options which are American and European option.

- European option: Gives owner the right to exercise the option only on the expiration date.
- American option: Gives owner the right to exercise the option on or before the expiration date.
  Note that: Most of the options that are traded on exchanges are American options. However, European options are generally easier to analyze than American options, and some of the properties of an American option are frequently deduced from those of its European counterpart.

The following notations are usually used:

- Underlying asset and its price S
- Exercise price or strike price K
- Expiration date or maturity date T (today is 0)

Option Payoff. The payoff of an option on the expiration date is determined by the price of the underlying asset. At time t the price is S_t and the strike price K. So we have the payoff functions for call and put options. *More precisely, Difference between the market price and the strike price depending on derivative type*.
The payoff (value) of call and put options at time T are

$$\Psi(S_T) = \max(S_T - K, 0) := \begin{cases} S_T - K, & \text{if} \quad S_T > K \\ 0, & \text{if} \quad S_T < K \end{cases}$$

and

$$\psi(S_T) = \max(K - S_T, 0) := \begin{cases} K - S_T, & \text{if} \quad S_T < K \\ 0, & \text{if} \quad S_T > K \end{cases}$$

Option trading

- We now consider the call option. It is easily to see that
  - if $S_T > K$ at expiry, then the buyer of the call option should exercise the option by paying a lower amount K to obtain an asset worth S_T.
  - However, if $S_T \leq K$ then the buyer of the call option should not exercise the option because it would not make any financial sense to pay a higher amount K to obtain an asset which is of a lower value S_T . Here, the option expires worthless.

So in general, the profit earned by the buyer of the call option is

$$\gamma(S_T) = \max(S_T - K, 0) - C(S_t, t, K, T) \qquad (1)$$

where $C(S_t, t, K, T)$is the premium paid at time $t < T$.

4.2. Put-Call Parity

We now create a portfolio as follows: we buy a call $C(S_t, t, K, T)$ and sell a put $P(S_t, t, K, T)$ written on the same asset S_t at time t with the same expiration date $T > t$ and strike price K. We have

- At expiry time T, the payoff is

$$C(S_T, T, K, T) - P(S_T, T, K, T) \qquad (2)$$

  from (1) and (2) then the payoff now is

$$C(S_T, T, K, T) - P(S_T, T, K, T) = S_T - K \qquad (3)$$

- Assuming a constant risk-free interest rate r and by discounting the payoff back to time t, we have

$$C(S_t, t, K, T) - P(S_t, t, K, T) = S_T - Ke^{-r(T-t)} \qquad (4)$$

Example Consider a long call option with strike price $K = 100$. The current stock price is $S_t = 105$ and the call premium is \$10. What is the intrinsic value of the call option at time t? Find the payoff and profit if the spot price at the option expiration date T is $S_T = \$120$.

Solution. We have $S_t = 105, K = 100, S_T = 120$. The call price is $C(S_t, t, K, T) = 10$, and, hence, the intrinsic value(giá trị thực) of the call option at time t is

?

$$\Psi(S_t) = \max\{S_t - K, 0\} = \max\{105 - 100, 0\} = 5$$

and the payoff of call option at time T is

$$\Psi(S_T) = \max\{S_T - K, 0\} = \max\{120 - 100, 0\} = 20$$

So the profit is

$$\gamma(S_T) = \Psi(S_T) - C(S_t, t, K, T) = 20 - 10 = \$10$$

Example. Consider a short call option with strike price $K = \$100$. The current stock price is $S_t = \$105$ and the call premium is \$10. What is the intrinsic value of the call option at time t? Find the payoff and profit if the spot price at the option expiration date T is $S_T = \$120$.

Example. Consider a short call option with strike price $K = \$100$. The current stock price is $S_t = \$105$ and the call premium is \$10. What is the intrinsic value of the call option at time t? Find the payoff and profit if the spot price at the option expiration date T is $S_T = \$120$.

Solution. We have the call premium $C(S_t, t, K, T) = 10$, so the intrinsic value of the short call option at time t is

$$\Psi(S_t) = -\max\{S_t - K, 0\} = \min\{K - S_t, 0\} = \min\{100 - 105, 0\} = -5$$

Hence, at expiry time T , the payoff is

$$\Psi(S_T) = -\max\{S_t - K, 0\} = \min\{K - S_t, 0\} = \min\{100 - 120, 0\} = -10$$

Consequently, the profit is

$$\gamma(S_T) = \Psi(S_T) + C(S_t, t, K, T) = -20 + 10 = -10 \quad why?$$

4.3 Modeling stock prices

The Black and Scholes model assumes that the dynamics of the price $S(t)$ of the underlying asset is

$$dS(t) = \mu S(t) + \sigma S(t) dB_t, \quad t \in [0, \infty)$$

where μ is a the trend and σ is the volatility assumed to be constants. In example 3 we shown that

$$S(t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B(t)}$$

The $S(t)$ is called Geometric Brownian motion.
Now suppose that S_T is the price at some future time $T > t$. Note that S_T is unknown. But it can be predicted, and the best prediction will be given by the conditional expectation:

$$\mathbb{E}_t(S_T) = \mathbb{E}(S_T \mid I_t)$$

4.4 Pricing European options: Delta hedging strategy method

Consider a portfolio as: a long position in the option V and a Δ number short positions in the asset S_t. Then at time t the hedging portfolio Π_t can be expressed by

$$\Pi_t = V - \Delta \times S_t$$

We have the following assumptions

- The risk-free interest rate is known and is constant over time.
- The asset price follows a geometric Brownian motion.
- The asset pays no dividends during the life of the option.
- The option is European-style, which can only be exercised at expiry date.
- No transaction costs are associated with buying or selling the asset/option.
- Trading of the asset can take place continuously.
- Short selling is permitted.

When the price of asset changes,we have

$$d\Pi_t = dV - \Delta dS_t$$

Since S_t is a Geometric Brownian motion, it is a solution of SDE

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

Note that $V = V(S_t, t)$ is a function of S_t and t. So we have Ito formula

$$dV = \Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \mu S_t \frac{\partial V}{\partial S_t}\Big)dt + \sigma S_t \frac{\partial V}{\partial S_t} dB_t$$

Consequently,

$$d\Pi_t = \underbrace{\Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \mu S_t\big(\frac{\partial V}{\partial S_t} - \Delta\big)\Big)}_{\text{long term}}dt + \underbrace{\sigma S_t\big(\frac{\partial V}{\partial S_t} - \Delta\big)}_{\text{short term}}dB_t$$

Now choosing the portfolio weight$\Delta = \frac{\partial V}{\partial S_t}$, from the assumption that the risk-free portfolio must have an expected return equal to the risk-free rate r, we have

$$d\Pi_t = r\Pi_t dt = r\Big(V(S_t,t) - \frac{\partial V}{\partial S_t}S_t\Big)dt$$

and

$$d\Pi_t = \Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2}\Big)dt$$

Comparing we obtain the Black–Scholes equation or an option price at time t, $V(S_t,t)$ is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + r\frac{\partial V}{\partial S_t}S_t - rV(S_t,t) = 0$$

For a constant strike $K > 0$, the boundary condition is given by the value of the claim at maturity T

$$V(S_T, T) = \begin{cases} \max\{S_T - K, 0\} & \text{for a call option} \\ \max\{K - S_T, 0\} & \text{for a put option} \end{cases}$$

and

$$V(0, t) = \begin{cases} 0 & \text{for a call option} \\ Ke^{-r(T-t)} & \text{for a put option} \end{cases}$$

So we obtain the Black-Scholes option pricing formula

$$\begin{cases} C(S_t, t, K, T) = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-) & \text{for a call option} \\ P(S_t, t, K, T) = Ke^{-r(T-t)}\Phi(-d_-) - S_t\Phi(-d_+) & \text{for a put option} \end{cases}$$

with

$$d_\pm = \frac{\log(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and Φ is a distribution function of standard normal distribution.

4.5 Delta Approximation

- First-order sensitivity of an option with respect to the underlying price is called delta, defined as (the proof is left as an exercise):

$$\Delta_C = \frac{\partial C(S)}{\partial S} = \Phi(d_+)$$

and for put option

$$\Delta_P = \frac{\partial P(S)}{\partial S} = \Phi(d_+) - 1$$

- A small change in P changes the option price by approximately Δ, but the approximation gets gradually worse as the deviation of P becomes larger

4.6 Gamma Approximation

- Second-order sensitivity of an option (call and put) with respect to the underlying price is called gamma, defined as:

$$\Gamma_C = \Gamma_P = \frac{\partial^2}{\partial S^2} = \frac{\Phi'(d_+)}{S_t \sigma \sqrt{T-t}}$$

- Gamma is highest when an option is a little out of the money and dropping as the underlying price moves away from the strike price

Example. Consider an option that expires in six months (T = 0.5) with strike price $K = 90$, price $S = 100$ and 20% volatility. Let r = 5% be the risk-free rate of return. Then

- The call delta is 0.8395 and the put delta is s -0.1605 (check!!)
- The gamma is 0.01724 (check!!)

4.7. Delta-Normal VaR

We use delta to approximate changes in the option price as a function of changes in the price of the underlying. Denote g the call or put option. The daily change in stock price is

$$dS = S_t - S_{t-1}$$

the price change dS implies that the option price change approximated by

$$dg = g_t - g_{t-1} \approx \Delta \times dS = \Delta(S_t - S_{t-1})$$

From the Black Scholes assumption, the return

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}$$

is i.i.d normally distributed

$$R_t \sim N(0, \sigma_d^2)$$

We now calculate VaR for option. We have

$$\begin{aligned}\alpha &= \mathbb{P}(g_t - g_{t-1} \leq -VaR^{\alpha}(op)) \\ &= \mathbb{P}(\Delta(S_t - S_{t-1}) \leq -VaR^{\alpha}(op)) \\ &= \mathbb{P}(\Delta S_{t-1} R_t \leq -VaR^{\alpha}(op)) \\ &= \mathbb{P}(\frac{R_t}{\sigma_d} \leq -\frac{1}{\Delta}\frac{VaR^{\alpha}(op)}{S_{t-1}\sigma_d})\end{aligned}$$

Since R_t has normal distribution, hence

$$VaR^{\alpha}(op) \approx - \mid \Delta \mid \times \sigma_d \times \Phi^{-1}(\alpha) \times S_{t-1}$$

Note that we need the absolute value of Δ because, it may be positive or negative, but VaR is always positive.