INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

FINAL EXAMINATION
June 2018
Duration: 90 minutes

SUBJECT: REAL ANALYSIS	
Deputy Head of Dept. of Mathematics:	Lecturer:
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INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 *(25 marks)* Let μ^* be an outer measure on X, and let $\emptyset \neq Y \subset X$. Define $\nu(A) = \mu^*(Y \cap A)$, $A \subset X$. Is ν an outer measure on X?

Question 2 (a) *(15 marks)* Let

$$F(x) = \begin{cases} x^3 & \text{if} \quad x < 0, \\ x^2 + 1 & \text{if} \quad x \geq 0. \end{cases}$$

If μ_F is the Lebesgue-Stieltjes measure corresponding to F, compute $\mu_F\big((-1, 0]\big)$ and $\mu_F(\{0\})$.

(b) *(10 marks)* Let

$$X = \{1, 2, 3, 4, 5, 6\}, \quad \mathcal{M} = \big\{\emptyset, X, \{1, 3, 5\}, \{2, 4, 6\}\big\},$$

and let f, g be functions on X defined by

$$f(x) = x, \qquad g(x) = \begin{cases} 1 & \text{if } x \in \{1, 3, 5\}, \\ 0 & \text{if } x \in \{2, 4, 6\}. \end{cases}$$

Determine which of f, g is measurable.

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Question 3 Let $(X, \mathcal{M}, \mu)$ be a measure space and let f be a nonnegative integrable function on X.

(a) *(10 marks)* Let

$$A = \{x \in X : f(x) > 0\} \text{ and } A_n = \left\{x \in X : f(x) \geq \frac{1}{n}\right\}, \; n = 1, 2, \ldots.$$

Show that A, A_n are measurable and $\bigcup_{n=1}^{\infty} A_n = A$.

(b) *(10 marks)* Show that $\lim_{n\to\infty} \displaystyle\int_{A_n} f d\mu = \int_X f d\mu$.

(c) *(10 marks)* Show that for each $\epsilon > 0$, there exists a measurable set E such that $\mu(E) < \infty$ and $\displaystyle\int_E f d\mu > \int_X f d\mu - \epsilon$.

Question 4

(a) *(10 marks)* Let $(X, \mathcal{M}, \mu)$ be a measure space and $A, B \in \mathcal{M}$ with $\mu(A \cap B) = 0$. Define ν_1 and ν_2 by

$$\nu_1(E) = \mu(E \cap A) \quad \text{and} \quad \nu_2(E) = \nu(E \cap B), \quad E \in \mathcal{M}.$$

Show that $\nu_1 \perp \nu_2$.

(b) *(10 marks)* Let $f : \mathbb{R} \to \mathbb{R}$ define by $f(x) = x^2 - 2x$ and let $\nu(E) = \displaystyle\int_E f(x)dx$, $E \in \mathcal{L}$. Determine f^- and evaluate $\nu^-([-5, 1])$.

*** END OF QUESTION PAPER ***

SOLUTIONS
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Question 1 Obviously, $\nu \geq 0$ on $\mathcal{P}(X)$ and $\nu(\emptyset) = 0$. If $A \subset B \subset X$, then $(Y \cap A) \subset (Y \cap B)$, so $\nu(A) = \mu^*(Y \cap A) \leq \mu^*(Y \cap B) = \nu(B)$. Finally, if $\{A_n\} \subset \mathcal{P}(X)$, then by σ-subadditivity of μ^*,

$$\nu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \mu^*\Big(Y \cap \bigcup_{n=1}^{\infty} A_n\Big) = \mu^*\Big(\bigcup_{n=1}^{\infty}(Y \cap A_n)\Big)$$
$$\leq \sum_{n=1}^{\infty} \mu^*(Y \cap A_n) = \sum_{n=1}^{\infty} \nu(A_n).$$

Thus ν is σ-subadditive and therefore ν is an outer measure on X.

Question 2 (a) We first observe that F is increasing on $\mathbb{R}$. Moreover, $x = 0$ is the only discontinuity of F and $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+}(x^2+1) = 1 = F(0)$. Hence F is right continuous at 0 and hence, F is right continuous on $\mathbb{R}$. By definition, $\mu_F\big((-1,0]\big) = F(0) - F(-1) = 1 - (-1) = 2$ and

$$\mu_F(\{0\}) = \mu_F\Big(\bigcap_{n=1}^{\infty}\Big(-\frac{1}{n}, 0\Big]\Big) = \lim_{n\to\infty} \mu_F\Big(\Big(-\frac{1}{n}, 0\Big]\Big)$$
$$= \lim_{n\to\infty}\Big[F(0) - F\Big(-\frac{1}{n}\Big)\Big] = \lim_{n\to\infty}\Big[1 - \Big(-\frac{1}{n}\Big)^3\Big] = 1.$$

(b) Since

$$\{x \in X : f(x) < 2\} = \{x \in X : x < 2\} = \{1\} \notin \mathcal{M},$$

f is not measurable. The set $A = \{1, 3, 5\}$ is measurable and $g = \chi_A$, so g is measurable.

Question 3 (a) Since f is measurable, the sets A and A_n are measurable. Furthermore, it is clear that $A_n \subset A_{n+1} \subset A$. Hence $\bigcup_{n=1}^{\infty} A_n \subset A$. Conversely, if $x \in A$ then $f(x) > 0$ so that there is $k \in \mathbb{N}$ such that $f(x) > \frac{1}{k}$. Thus $x \in A_k$ and therefore, $\bigcup_{n=1}^{\infty} A_n \supset A$. It follows that $\bigcup_{n=1}^{\infty} A_n = A$.

(b) From part (a) we conclude that $\chi_{A_n} \nearrow \chi_A$, hence $0 \leq \chi_{A_n} f \nearrow \chi_A f$ (since $f \geq 0$). But $\chi_A f = f = 0$ on A^c. Thus $\chi_A f = f$ on X. Applying the Monotone Convergence Theorem we get

$$\lim_{n\to\infty} \int_{A_n} f d\mu = \lim_{n\to\infty} \int_X \chi_{A_n} f d\mu = \int_X \chi_A f d\mu = \int_X f d\mu.$$

(We can also use the Dominated Convergence Theorem since $\chi_{A_n} f \to f$, $|\chi_{A_n} f| \leq f$, and f is integrable on X.)

Alternative solution. Since $f \geq 0$, the set function $\nu(E) = \int_E f d\mu$ is a measure. By part (a), $A \nearrow A$. Thus $\nu(A_n) \to \nu(A)$ and therefore,

$$\lim_{n\to\infty} \int_{A_n} f d\mu = \lim_{n\to\infty} \nu(A_n) = \nu(A) = \int_A f d\mu = \int_A f d\mu + \int_{A^c} f d\mu = \int_X f d\mu$$

because $f = 0$ on A^c.

(c) It follows from part (b) that for every $\epsilon > 0$, there exists an N such that $\int_{A_N} f d\mu > \int_X f d\mu - \epsilon$. Moreover, $\int_X f d\mu \geq \int_{A_N} f d\mu \geq \int_{A_N} \frac{1}{N} d\mu = \frac{1}{N}\mu(A_N)$, giving $\mu(A_N) \leq N \int_X f d\mu < \infty$ (Chebychev's Inequality). The set $E = A_N$ satisfies all requirements.

Question 4 (a) Clearly, ν_1 and ν_2 are measures. Furthermore, $\nu_1(B) = \mu(B \cap A) = 0$ and $\nu_2(B^c) = \mu(B^c \cap B) = \mu(\emptyset) = 0$. Thus $\nu_1 \perp \nu_2$.

(b) Since $f(x) \leq 0$ if and only if $0 \leq x \leq 2$,

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ -x^2 + 2x & \text{if} \quad 0 \leq x \leq 2, \\ 0 & \text{if} \quad x > 2. \end{cases}$$

Thus,

$$\begin{aligned} \nu^-([-5,1]) &= \int_{-5}^{1} f^-(x)dx = \int_{-5}^{0} f^-(x)dx + \int_0^1 f^-(x)dx = \int_0^1 f^-(x)dx \\ &= \int_0^1 (-x^2 + 2x)dx = -\frac{1}{3}x^3 + x^2\Big|_0^1 = -\frac{1}{3} + 1 = \frac{2}{3}. \end{aligned}$$

Alternative solution. Let $A = \{x \in X : f(x) > 0\}$ and $B = \{x \in X : f(x) \leq 0\} = [0,2]$. Then $\{A, B\}$ is a Hahn decomposition for ν. Thus

$$\begin{aligned} \nu^-\big([-5,1]\big) &= -\nu\big([-5,1] \cap B\big) = -\nu\big([-5,1] \cap [0,2]\big) = -\nu\big([0,1]\big) \\ &= -\int_0^1 (x^2 - 2x)dx = -\frac{1}{3}x^3 + x^2\Big|_0^1 = -\frac{1}{3} + 1 = \frac{2}{3}. \end{aligned}$$