MIDTERM EXAMINATION

Semester 2, 2020-21 • Duration: 90 minutes

SUBJECT: ANALYSIS II	
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INSTRUCTIONS: Each student is allowed a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.

1. (10 points) a/ Let f be a continuous function and $F(x) = \int_{2-x}^{x^2} f(t)dt$. If $F'(1) = 3$, find $f(1)$.

b/ Suppose that f is an even function so that $\int_{-2}^{3} f(x)dx = 4$ and $\int_{0}^{2} f(x)dx = 3$. Find $\int_{2}^{3} f(x)dx$.

2. (10 points) Find the length of the parametric curve (t^2, t^3) between $t = 0$ and $t = \sqrt{2}$.

3. Compute the following integrals

(a) (10 points)

$$\int_0^\infty x^3 e^{-x^2} dx$$

(b) (10 points)

$$\int_0^2 \sqrt{4 - x^2} dx$$

(c) (20 points)

$$\int \frac{4x + 21}{(x - 2)(x^2 + 6x + 13)} dx$$

4. (20 points) Determine whether the following integral converges or diverges

$$\int_0^\infty \frac{\sin x}{x^{3/2}} dx.$$

5. (20 points) Let $f(x) = x^2(x - 1)^2$ on $[-1, 2]$. Find all partitions P of $[-1, 2]$ such that $L(f, P) = 0$.

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4/ Determine if $\mathrm{I} = \int_0^{\infty} \frac{\sin x}{x\sqrt{x}} dx$ **converges** or **diverges.**

Solution. We have $\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2$, where $\mathrm{I}_1 = \int_0^{\pi} \frac{\sin x}{x\sqrt{x}} dx$ and $\mathrm{I}_2 = \int_{\pi}^{\infty} \frac{\sin x}{x\sqrt{x}} dx.$

Since $0 = \int_0^{\pi} 0 dx \leq \mathrm{I}_1 \leq \int_0^{\pi} \frac{x}{x\sqrt{x}} dx = \int_0^{\pi} \frac{1}{\sqrt{x}} dx = 2\sqrt{\pi}, \mathrm{I}_1$ **converges.**

Since $\frac{-2}{\sqrt{\pi}} = \int_{\pi}^{\infty} \frac{-1}{x\sqrt{x}} dx \leq \mathrm{I}_2 \leq \int_{\pi}^{\infty} \frac{1}{x\sqrt{x}} dx = \frac{2}{\sqrt{\pi}}, \mathrm{I}_2$ **converges.**

Hence $\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2$ **converges.** ■

5/ Consider the function $f : [-1, 2] \to \mathbb{R}$ defined by: $f(x) = x^2(x-1)^2, \forall x \in [-1, 2]$.
Find **all** partitions P of $[-1, 2]$ satisfying $L(f, P) = 0$.

Solution. We need the following two lemmas.

Lemma 1. Let P_1 be a partition of $[-1, 2]$ satisfying $L(f, P_1) = 0$. Then $P_1 \cap (-1, 0) = P_1 \cap (1, 2) = \emptyset$.

Proof. Assume on the contrary that $\exists x \in P_1 \cap (-1, 0)$.
Consider the partition $P_2 = \{-1, x, 2\}$, then $P_2 \subset P_1$ and hence $0 = L(f, P_1) \geq L(f, P_2)$.
Since $f > 0$ on $(-1, 0), L(f, P_2) = 3f(a) > 3f(0) = 0$, a contradiction.
Thus $P_1 \cap (-1, 0) = \emptyset$. Similar arguments can be used to obtain $P_1 \cap (1, 2) = \emptyset$. ■

Lemma 2. Let P_1 be a partition of $[-1, 2]$ satisfying $L(f, P_1) = 0$. Then P_1 contains **at most one** element of $(0, 1)$.

Proof. Assume on the contrary that $\exists a, b \in P_1 \cap (0, 1) : a < b$.
Consider the partition $P_2 = \{-1, a, b, 2\}$, then $P_2 \subset P_1$ and hence $0 = L(f, P_1) \geq L(f, P_2)$.
Since $f > 0$ on $(0, 1), L(f, P_2) = (b - a) \cdot \min\{f(a), f(b)\} > (b - a) \cdot f(0) = 0$, a contradiction.
Thus P_1 contains **at most one** element of $(0, 1)$. ■

Back to the problem. Denote $\mathcal{A} = \{\{-1, 2\}, \{-1, 0, 2\}, \{-1, 1, 2\}, \{-1, 0, 1, 2\}\}$.
From the above lemmas, the family $\mathcal{B}$ of all partitions P satisfying $L(f, P) = 0$ are: $\mathcal{B} = \mathcal{A} \cup \{A \cup \{x\} : A \in \mathcal{A}, x \in (0, 1)\}$.