Real Analysis, Chapter 3
Worksheet 6: Radon-Nikodym Derivative

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In this worksheet, we assume the $\sigma-$finite space $(X, \mathcal{M}, \mu)$.

Definition 3.6.1 (Absolutely Continuity)

A signed measure ν is absolutely continuous w.r.t. μ, denoted $\nu \ll \mu$, if

$$\forall A \in \mathcal{M}, \text{ if } \mu(A) = 0 \text{ then } \nu(A) = 0.$$

Lemma 3.6.1

Assume that μ and ν are finite and ν is a measure.

(a) There are measures ν_1, ν_2 on $\mathcal{M}$ with $\nu = \nu_1 + \nu_2, \nu_1 \ll \mu, \nu_2 \perp \mu$;

(b) There is a function $g \in \mathcal{M}$ such that $\nu_1(A) = \int_A g \, d\mu, \forall A \in \mathcal{M}$.

Guidelines:

- Let $\mathcal{H} = \left\{ f \geq 0 : \int_A f \, d\mu \leq \nu(A), \forall A \in \mathcal{M} \right\}$ and $\alpha = \sup_{f \in \mathcal{H}} \int_X f \, d\mu$;
- For each $n \in \mathbb{N}$, take $f_n \in \mathcal{M}$ such that $\int_X f_n \, d\mu \in [\alpha - 1/n, \alpha]$;

- Let $g_n = \max_{1 \le k \le n} f_k$, then $g_n \in \mathcal{H}$ and $g_n \nearrow g \in \mathcal{H}$;
- Then $\int_X g \, d\mu = \alpha$. Set $\nu_1(A) = \int_A g \, d\mu, \forall A \in \mathcal{M}$ and $\nu_2 = \nu - \nu_1$;
- Let (P_n, N_n) be a Hahn decomposition of $\sigma_n = \nu_2 - \mu/n$;
- Set $P = \bigcup P_n$, then $g + \chi_{P_n} \in \mathcal{H}$, so $\mu(P_n) = \mu(P) = \nu_2(P^c) = 0$.

Theorem 3.6.2

The conclusion of Lemma 3.6.1 also holds if μ and ν are $\sigma-$finite only.

Guidelines: By assumption, $X = \bigcup E_n$ where $\mu(E_n) < \infty, \nu(E_n) < \infty$.

- Apply Lemma 3.6.1 on each set E_n to obtain $\nu_{1,n}, \nu_{2,n}$;
- Set $\nu_i = \sum \nu_{i,n}$, then $\nu = \nu_1 + \nu_2, \nu_1 \ll \mu, \nu_2 \perp \mu$.

Theorem 3.6.3

The conclusion of Theorem 3.6.2 also holds if ν is a signed measure only.

Hint. Use Theorem 3.6.2 on ν^+ and ν^-.

Theorem 3.6.4 (Lebesgue Decomposition Theorem)

If there are measures $\nu_1, \xi_1, \nu_2, \xi_2$ on $\mathcal{M}$ satisfying

$$\nu = \nu_\mathrm{i} + \xi_\mathrm{i}, \nu_\mathrm{i} \ll \mu, \xi_\mathrm{i} \perp \mu, \mathrm{i} = \overline{1, 2}$$

then $\nu_1 = \nu_2$ and $\xi_1 = \xi_2$.

The pair (ν_1, ξ_1) is called the Lebesgue decomposition of ν w.r.t. μ.

Corollary 3.6.5 (Radon-Nikodym Theorem)

If $\nu \ll \mu$, then there exists uniquely $\mu-$a.e. a function $\mathrm{f} \in \mathcal{M}$ satisfying

$$\nu(\mathrm{A}) = \int_\mathrm{A} \mathrm{f}\, \mathrm{d}\mu, \forall \mathrm{A} \in \mathcal{M}.$$

f is called the Radon-Nikodym derivative (or density) of ν w.r.t. μ, denoted

$$\mathrm{f} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \frac{\mathrm{d}}{\mathrm{d}\mu}\nu.$$