MIDTERM EXAMINATION
July 2017
Duration: 100 minutes

SUBJECT: REAL ANALYSIS	
Head of Dept. of Mathematics:	Lecturer:
Assoc. Prof. Pham Huu Anh Ngoc	Assoc. Prof. Nguyen Ngoc Hai

INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 *(25 marks)* Given a metric space (X, d). For $x, y \in X$, define $\rho(x, y) = \min\{1, d(x, y)\}$.

(a) Show that ρ is a metric on X.

(b) Show that $\lim_{n\to\infty} x_n = x$ in (X, d) if and only if $\lim_{n\to\infty} x_n = x$ in (X, ρ).

Question 2 (a) *(15 marks)* Let A and B be subsets of a metric space (X, d), show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) *(10 marks)* Suppose that $f, g : X \to \mathbb{R}$ are continuous functions on a metric space (X, d). Show that for each real number α, the set $E = \{x \in X : \min\{f(x), g(x)\} > \alpha\}$ is open in X.

Question 3 *(25 marks)* Let A be a nonempty closed subset of $\mathbb{R}^n$. Show that for each real number $r > 0$, the set $A \cap \overline{B}(\mathbf{0}, r)$ is compact. Show that there is an $\mathbf{x}_0 \in A$ such that $\|\mathbf{x}_0\| \le \|\mathbf{x}\|$ for all $\mathbf{x} \in A$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

Question 4 *(25 marks)* A $\boldsymbol{G_\delta}$**-set** is any countable intersection of open sets. Show that for any $a, b \in \mathbb{R}$, $a < b$, the interval $(a, b]$ is a G_δ-set. Prove that the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is generated by the family

$$\mathcal{E} = \big\{(a, b] : a, b \in \mathbb{R},\ a < b\big\}.$$

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SOLUTIONS

Question 1 (a) For all $x, y \in X$, $d(x,y) \geq 0$, and hence,

$$\rho(x,y) = \min\{1, d(x,y)\} \geq 0.$$

If $\rho(x,y) = 0$, then $\rho(x,y) = d(x,y)$, so $d(x,y) = 0$, that is $x = y$. Thus it remains to prove the triangle inequality.

If $\rho(x,y) = 1$ or $\rho(y,z) = 1$, then $\rho(x,y) = \min\{1, d(x,y)\} \leq 1 \leq \rho(x,y) + \rho(y,z)$. Suppose now that $\rho(x,y) < 1$ and $\rho(y,z) < 1$. In this case, $\rho(x,y) = d(x,y)$ and $\rho(y,z) = d(y,z)$. Thus using the triangle inequality for d we get

$$\rho(x,z) \leq d(x,z) \leq d(x,y) + d(y,z) = \rho(x,y) + \rho(y,z).$$

Hence the triangle inequality always holds for ρ and so ρ is a metric on X.

(b) If $\lim_{n\to\infty} x_n = x$ in (X,d), then $d(x_n, x) \to 0$. It follows that $d(x_n,x) < 1$ and hence, $\rho(x_n,x) = d(x_n,x)$ for n large. Thus, $\lim_{n\to\infty} \rho(x_n,x) = 0$, that is $\lim_{n\to\infty} x_n = x$ in (X,ρ). Conversely, suppose $\lim_{n\to\infty} x_n = x$ in (X,ρ). As $\lim_{n\to\infty} \rho(x_n,x) = 0$, $\rho(x_n,x) < 1$ for all n large enough. Therefore, $\rho(x_n,x) = d(x_n,x)$ for all n large enough. It follows that $d(x_n,x) \to 0$, that is, $\lim_{n\to\infty} x_n = x$ in (X,d).

Question 2 (a) We note first that $\overline{A} \cup \overline{B}$ is closed. Since $\overline{A} \cup \overline{B}$ contains $A \cup B$, implies that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. On the other hand $\overline{A \cup B}$ is also closed. Since $A \subset \overline{A \cup B}$ and $B \subset \overline{A \cup B}$, we get the inclusions $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Together, these imply $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) We have

$$\begin{aligned} E &= \{x \in X : \min\{f(x), g(x)\} > \alpha\} \\ &= \{x \in X : f(x) > \alpha\} \cap \{x \in X : g(x) > \alpha\}. \end{aligned}$$

Since f and g are continuous, the sets $\{x \in X : f(x) > \alpha\}$ and $\{x \in X : g(x) > \alpha\}$ are open in X. Hence E is open.

Question 3 Let $K = A \cap \overline{B}(\mathbf{0}, r)$. As $K \subset \overline{B}(\mathbf{0}, r)$, K is a bounded set. Furthermore, since A and $\overline{B}(\mathbf{0}, r)$ are both closed, so is K. Thus K is closed and bounded in $\mathbb{R}^n$, hence it is compact.

If A is singleton, $A = \{\mathbf{x}_0\}$, then $\|\mathbf{x}_0\| = \min\{\|\bar{\mathbf{x}}\| : \bar{\mathbf{x}} \in A\}$. Assume that A has more than one point so that $A \setminus \{\mathbf{0}\} \neq \emptyset$. Fixed an $\bar{\mathbf{x}} \in (A \setminus \{\mathbf{0}\})$ and let $r = \|\bar{\mathbf{x}}\| > 0$. By the proof above, $K = A \cap \overline{B}(\mathbf{0}, r)$ is compact. The norm is a continuous function, hence there is an $\mathbf{x}_0 \in K$ with $\|\mathbf{x}_0\| = \min\{\|\mathbf{x}\| : \mathbf{x} \in K\}$. If $\mathbf{x} \in A \setminus K$, then $\mathbf{x} \notin \overline{B}(\mathbf{0}, r)$. Therefore $\|\mathbf{x}\| > r \geq \|\bar{\mathbf{x}}\| \geq \|\mathbf{x}_0\|$ and consequently, $\|\mathbf{x}_0\| \leq \|\mathbf{x}\|$ for all $\mathbf{x} \in A$.

Question 4 If $a, b \in \mathbb{R}$, $a < b$, then $(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right)$. As each interval $\left(a, b + \frac{1}{n}\right)$ is open, $(a, b]$ is a G_δ-set. Furthermore, this also shows that $(a, b]$ is a Borel set. Hence $\sigma(\mathcal{E}) \subset \mathcal{B}(\mathbb{R})$. Conversely, for each $a, b \in \mathbb{R}$, $a < b$, we have $(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right]$, implying $(a, b) \in \sigma(\mathcal{E})$. Since each open set in $\mathbb{R}$ is a countable union of open intervals, we find that every open set belongs to $\sigma(\mathcal{E})$. Thus $\mathcal{B}(\mathbb{R}) \subset \sigma(\mathcal{E})$. Therefore $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E})$.