Decision Making

(for Financial Engineering & Risk Management program)

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Contents

We generally represent games using tables.

- Row player
- Column player
- Each cell represent of the payoffs for two player. The first number is the payoff for the row player, the second number is the payoff of column player for that outcome. Consider an example:

Example: Prisoners' Dilemma's problem. Two prisoners (players 1 and 2) have committed a crime together and are interrogated separately. Each prisoner has two possible choices:

- he may "cooperate" (C) which means "not betray his partner",

or

- he may "defect" (D), which means "betray his partner".

The punishment for the crime is 10 years of prison. Betrayal yields a reduction of 1 year for the defector (traitor). If a prisoner is not betrayed, he is convicted to 1 year for a minor offense

Model This situation can be summarized as follows:

Strategy	*C*	*D*
C	-1, -1	-10, 0
D	0, -10	-9, -9

This table is read in the same way as before, but now there are two payoffs at each position: by convention the first number is the payoff for player 1 (the row player) and the second number is the payoff for player 2 (the column player).

Observe that the game is no longer zero-sum, and we have to write down both numbers at each matrix position. ,

Solution • For both players C is a strictly dominated choice:
• D is better than C, whatever the other player does. So it is natural to argue that the outcome of this game will be the pair of choices (D, D) leading to the payoffs $(-9, -9)$.
• Thus, due to the existence of strictly dominated choices, the Prisoners' Dilemma game is easy to analyze.

Remark. • The payoff $(-9, -9)$ is inferior. The players could obtain the higher payoffs of -1 for each by cooperating, i.e., both playing C.

• If the game is played repeatedly, other (higher) payoffs are possible.

Cooperative games

In a cooperative game the focus is on payoffs and coalitions, rather than on strategies.

In contrast to the equilibrium analysis of noncooperative theory, the implicit assumption is that players can make binding agreements.

Example Three Cooperating Cities Story Cities 1, 2 and 3 want to be connected with a nearby power source. The possible transmission links and their costs are shown in the following figure. Each city can hire any of the transmission links. If the cities cooperate in hiring the links they save on the hiring costs (the links have unlimited capacity). The situation is represented in Figure 1.6 below.

Model: The players in this situation are the three cities. Denote the player set by $N = \{1, 2, 3\}$. These players can form coalitions: any subset S of N is called a coalition. Table 1.1 presents the costs as well as the savings of each coalition.
The costs $c(S)$ are obtained by calculating the cheapest routes connecting the cities in the coalition S with the power source. •
The cost saving $v(S)$ are determined by

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S), \quad \text{for each non-empty } S \subset N.$$

The cost savings $v(S)$ for coalition S are equal to the difference in costs corresponding to the situation where all members of S work alone and the situation where all members of S work together.
The pair (N, v) is called a cooperative game.

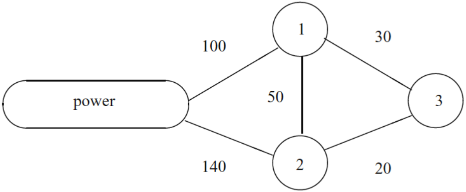

Fig. 1.6 Situation leading to the three cities game

Table 1.1 The three cities game

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S)$	100	140	130	150	130	150	150
$v(S)$	0	0	0	90	100	120	220

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The cost savings $v(S)$ for coalition S are equal to the difference in costs corresponding to the situation where all members of S work alone and the situation where all members of S work together.

How to solve the game? Basic questions in a cooperative game (N, v) are:

- Q1: which coalitions will actually be formed, and
- Q2: how should the worth (savings) of such a coalition be distributed among its members?

To form a coalition, the consent of every member is needed, but it is likely that the willingness of a player to participate in a coalition depends on what that player obtains in that coalition.

Therefore, the second question seems to be the more basic one. We will focus on this question.

Specifically, it is usually assumed that the grand coalition N of all players is formed.

The question is then reduced to the problem of distributing the amount $v(N)$ among all the players.

Let x_i be the saving that the player i obtains. W find $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. It is clear that one should have

$$x_1 + x_2 + x_3 = 220 = v(N).$$

The easy way is to take: $x_1 = x_2 = x_3 = 220/3$. However, this does not really reflect the asymmetry of the situation: some players save more than others. (Students check, please) 1

There are quite different solutions proposed in the literature. Among them: some are interested the most: the *core*, the *Shapley value*, and the *nucleolus* (solution methods). We consider here the two first methods. The last one is a bit complicated and will be ignored.

☐ **The core:** The core consists of payoff distributions that cannot be improved upon by any smaller coalition.
For the three cities example, this means that the core consists of those vectors (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = 220$, $x_1, x_2, x_3 \geq 0$, $x_1 + x_2 \geq 90$, $x_1 + x_3 \geq 100$, $x_2 + x_3 \geq 120$. (a large set, can not give a solution).

☐ **The Shapley value method:** Consists of only one value (a vector (x_1, x_2, x_3)). Each player receives his average contribution to the worth (savings) of coalitions.

More precisely, imagine the players entering the "bargaining room" one at a time, say firstly, player 1, then player 2, and player 3.

- Player 1 enters, a coalition of himself. Worth: 0.

- Player 2 enters: form a coalition $\{1,2\}$. The contribution of Player 2 is:

$$v(\{1,2\}) - v(\{1\}) = 90 - 0 = 90.$$

- Player 3 enters: form a coalition $\{1,2,3\}$. The contribution of Player 3 is:

$$v(\{1,2,3\}) - v(\{1,2\}) = 220 - 90 = 130.$$

This results in a payoff vector: $(0, 90, 130)$.

What happen if interchange the order of players entering the room? We get another vector!

How many ways of interchanging? answer: equal to the number of permutations of elements in N. In our example: $3! = 6$.

• Realizing the Five other permutation, and take the average. We get the vector of distribution: $(65, 75, 80)$.

Student continue this task as an exercise.

Student make a lecture note by themselves as an Assignment 1. Deadline for submission will be announce at class.

Material: is supplied on BlackBoard.

References

[1] Hans Peters, Game theory.

[2] Ragan Nicole Brackin, *N*-Person Cooperative Game Theory Solutions, Coalitions, and Applications. Master thesis, University of Tennessee, 2002.