OPTIMIZATION 1

CHAPTER 2

THE SIMPLEX METHOD

In this chapter we answer some key questions on feasibility and optimality:

1. How can we tell if the linear program has a feasible solution?
2. How do we find a basic feasible solution (if one exists)?
3. How can we recognize whether a basic feasible solution is optimal?
4. What should we do if a basic feasible solution is known (or believed) to be nonoptimal?

- To solve linear programs with large sizes we need a solution algorithm that is easily programmed for computer use.
- The method that will be developed in this chapter for solving linear programming problems is called the **simplex method**.
- This method and its various modifications remain among the primary means used today to solve linear optimization problems.

  Problems with thousands of variables and constraints are routinely solved by the simplex algorithm.

To apply the simplex algorithm, the system of constraints must be in canonical form and the associated basic solution must be feasible.

Definition 1.1

A system of m equations and n unknowns, with $m \leq n$, is in **canonical form** with a distinguished set of m basic variables if each basic variable has coefficient 1 in one equation and 0 in the others, and each equation has exactly one basic variable with coefficient 1.

For example, the following system is canonical form:

$$\begin{array}{l} x_1 \quad + \bar{a}_{1,m+1}x_{m+1} + \bar{a}_{1,m+2}x_{m+2} + \cdots + \bar{a}_{1n}x_n = \bar{a}_{10} \\ \quad x_2 \quad + \bar{a}_{2,m+1}x_{m+1} + \bar{a}_{2,m+2}x_{m+2} + \cdots + \bar{a}_{2n}x_n = \bar{a}_{20} \\ \qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \quad \vdots \\ \qquad x_m + \bar{a}_{m,m+1}x_{m+1} + \bar{a}_{m,m+2}x_{m+2} + \cdots + \bar{a}_{mn}x_n = \bar{a}_{m0}. \end{array} \tag{1}$$

Corresponding to this canonical representation of the system, the variables $x_1, x_2, \ldots, x_m$ are basic and the other variables are nonbasic.

The corresponding basic solution is then:

$$x_1 = \bar{a}_{10}, \; x_2 = \bar{a}_{20}, \ldots, \; x_m = \bar{a}_{m0}, \; x_{m+1} = 0, \ldots, \; x_n = 0.$$

The basic step in the simplex method is derived from the pivot operation used to solve linear equations.

In short, the pivot operation swaps a basic variable and a non-basic variable.

Example 1.1 Consider the following system

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 6 \\ 3x_2 + x_3 + 8x_4 &= 3. \end{aligned} \tag{2}$$

Convert the system in canonical form with the following basic variables

(a) x_1 and x_2;

(b) x_1 and x_3;

(c) x_2 and x_4.

Solution (a) Pivoting at the $3x_2$ term of the second equation gives the equivalent system

$$\begin{aligned} x_1 \qquad + \frac{5}{3}x_3 - \frac{5}{3}x_4 &= 5 \\ x_2 + \frac{1}{3}x_3 - \frac{8}{3}x_4 &= 1. \end{aligned}$$

This system is in canonical form with basic variables x_1 and x_2.

(b) Adding -2 times the second equation to the first equation in system (2) yields

$$\begin{aligned} x_1 - 5x_2 + \qquad -15x_4 &= 0 \\ 3x_2 + x_3 + 8x_4 &= 3. \end{aligned}$$

Also it is customary to represent the system (1) by its corresponding array of coefficients or **tableau**:

x_1	x_2	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n	rhs
1	0	$\cdots$	0	$\bar{a}_{1,m+1}$	$\bar{a}_{1,m+2}$	$\cdots$	$\bar{a}_{1n}$	$\bar{a}_{10}$
0	1	$\cdots$	0	$\bar{a}_{2,m+1}$	$\bar{a}_{2,m+2}$	$\cdots$	$\bar{a}_{2n}$	$\bar{a}_{20}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	0	$\cdots$	1	$\bar{a}_{m,m+1}$	$\bar{a}_{m,m+2}$	$\cdots$	$\bar{a}_{mn}$	$\bar{a}_{m0}$

Given a system in canonical form, suppose a basic variable is to be made nonbasic and a nonbasic variable is to be made basic.

Question:

What is the new canonical form corresponding to the new set of basic variables?

Suppose in the canonical system (1) we wish to replace the basic variable x_p, $1 \leq p \leq m$, by the nonbasic variable x_q.

This can be done if and only if $\bar{a}_{pq}$ is nonzero.

It is accomplished by dividing row p by $\bar{a}_{pq}$ to get a unit coefficient for x_q in the pth equation, and then subtracting suitable multiples of row p from each of the other rows in order to get a zero coefficient for x_q in all other equations.

Denoting the coefficients of the new system in canonical form by $\bar{a}'_{ij}$, we have explicitly

$$\begin{cases} \bar{a}'_{ij} = \bar{a}_{ij} - \dfrac{\bar{a}_{pj}}{\bar{a}_{pq}} \bar{a}_{iq}, \quad i \neq p \\ \bar{a}'_{pj} = \dfrac{\bar{a}_{pj}}{\bar{a}_{pq}}. \end{cases} \tag{3}$$

- Equations (3) are the **pivot equations** that arise frequently in linear programming.
- The element $\bar{a}_{pq}$ in the original system is said to be the **pivot element**.

Example 1.2 Consider the system in canonical form:

$$\begin{array}{ccccccccccccc} x_1 & & & + & x_4 & + & x_5 & - & x_6 & = & 5 \\ & x_2 & & + & 2x_4 & - & 3x_5 & + & x_6 & = & 3 \\ & & x_3 & - & x_4 & + & 2x_5 & - & x_6 & = & -1. \end{array}$$

Find the basic solution having basic variables x_3, x_4, x_5.

Solution We set up the coefficient array below:

x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	0	0	1	1	- 1	5
0	1	0	2	-3	1	3
0	0	1	-1	2	-1	-1

The circle indicated is our first pivot element and corresponds to the replacement of x_1 by x_4 as a basic variable. After pivoting we obtain the array

x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	0	0	1	1	- 1	5
-2	1	0	0	-5	3	-7
1	0	1	0	3	-2	4

The variable that goes from nonbasic to basic is called the **entering variable**, the variable that goes from basic to nonbasic is called the **leaving variable**.

Our next pivot element is -5 and we will replace x_2 by x_5. We then obtain

x_1	x_2	x_3	x_4	x_5	x_6	rhs
3/5	1/5	0	1	0	- 2/5	18/5
2/5	-1/5	0	0	1	-3/5	7/5
-1/5	3/5	1	0	0	-1/5	-1/5

From this last canonical form we obtain the new basic solution

$$(0, 0, -1/5, 18/5, 7/5, 0).$$

Note The tableaus (and the use of explicit matrix inverses) are merely notational devices that assist our explanations of the simplex method.

Computer implementations of the simplex method use other techniques more suitable for large sparse problems.

Determination of Vector to Leave Basis

Nondegeneracy assumption

Every basic feasible solution of

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

is a nondegenerate basic feasible solution.

In this section the simplex method for solving linear programming problems will be introduced.

The idea of the simplex method is to select the column so that the resulting new basic feasible solution will yield a lower value to the objective function than the previous one.

2.2 DETERMINING A MINIMUM FEASIBLE SOLUTION

The simplex method is a systematic and effective way to examine basic feasible solutions to solve a linear program.

By an elementary calculation it is possible to determine

- ◇ which vector should enter the basis so that the objective value is reduced, and
- ◇ which vector should leave in order to maintain feasibility.

An introductory example

Example 2.1 Consider the following problem

$$\begin{aligned} \text{maximize} \quad & z = 3x + 2y \\ \text{subject to} \quad & 2x + 3y \le 12 \\ & 2x + y \le 8 \\ & x \ge 0, \ y \ge 0. \end{aligned}$$

Solution We first convert the problem to canonical form:

$$\begin{array}{ll} \text{minimize} & w = -3x - 2y \\ \text{subject to} & \begin{aligned} 2x + 3y + u &= 12 \\ 2x + y + v &= 8 \\ x, y, u, v &\geq 0. \end{aligned} \end{array} \tag{4}$$

Observation

- Basic variables are u and v.
- If either x or y is increased from zero, then w will decrease. Thus the current basis

$$(x, y, u, v) = (0, 0, 12, 8)$$

is not optimal.

- The system of equalities is in canonical form.
- The objective function is expressed in terms of the nonbasic variables alone.
- The coefficient of x is greater in absolute value than the coefficient of y, so w decreases more rapidly when x is increased. (w will decrease by 3 for every unit of increase in x.)

The idea: keep y at 0 and increase x.
Then Problem (4) reduces to

$$\begin{aligned} \text{minimize} \quad & w = -3x \\ \text{subject to} \quad & u = 12 - 2x \\ & v = \ \ 8 - 2x \\ & x, u, v \geq \ 0. \end{aligned}$$

Observation In this problem,

- There is only one nonbasic variable, namely x;
- The objective function and the basic variables are expressed in terms of the nonbasic variable.

Question How much can x be increased?

Since $12 - 2x \geq 0$ and $8 - 2x \geq 0$, it follows that

$$x \leq \min\left\{\frac{12}{2}, \frac{8}{2}\right\} = 4.$$

Consequently x can only be increased to the value $x = 4$.

For $x = 4$, $y = 0$, we obtain the basic feasible solution

$$(x, y, u, v) = (4, 0, 4, 0).$$

Thus x is the *entering variable* and v is the *leaving variable*. (New basic variables now are u and v.)

Next we rewrite Problem (4) in canonical form with new basic variables x and u:

$$\begin{array}{rrcl}
\text{minimize} & w & = & -12 - \frac{1}{2}y + \frac{3}{2}v \\
\text{subject to} & 2y + u - v & = & 4 \\
 & x + \frac{1}{2}y \quad + \frac{1}{2}v & = & 4 \\
 & x, y, u, v & \geq & 0.
\end{array} \tag{5}$$

Express new basic variables in terms of the nonbasic variables:

$$\begin{aligned} \text{minimize} \quad & w = -12 - \frac{1}{2}y + \frac{3}{2}v \\ \text{subject to} \quad & u = 4 - 2y + v \\ & x = 4 - \frac{1}{2}y - \frac{1}{2}v \\ & x, y, u, v \geq 0. \end{aligned}$$

Observe that if y is increased from zero, then w will decrease.

Keeping $v = 0$ we get

$$\begin{aligned} \text{minimize} \quad & w = -12 - \frac{1}{2}y \\ \text{subject to} \quad & u = 4 - 2y \\ & x = 4 - \frac{1}{2}y \\ & x, y, u \geq 0. \end{aligned}$$

Since $4 - 2y \geq 0$ and $4 - \frac{1}{2}y \geq 0$, we have

$$y \leq \min\left\{\frac{4}{2}, \frac{4}{1/2}\right\} = 2.$$

For $y = 2$ and $v = 0$, the corresponding basic feasible solution is

$$(x, y, u, v) = (3, 2, 0, 0).$$

Problem (5) now is equivalent to the following

$$\begin{aligned}
\text{minimize} \quad & w = -13 + \frac{1}{4}u + \frac{5}{4}v \\
\text{subject to} \quad & y + \frac{1}{2}u - \frac{1}{2}v = 2 \\
& x - \frac{1}{4}u + \frac{3}{4}v = 3 \\
& x, y, u, v \geq 0,
\end{aligned}$$

which is in canonical form with basic variables x and y.

Now clearly $w = -13 + \frac{1}{4}u + \frac{5}{4}v \geq -13$. Therefore the optimal value is $w = -13$ and the optimal solution is

$$(x, y, u, v) = (3, 2, 0, 0).$$

The original problem has optimal value 13 corresponding to the optimal solution

$$(x, y) = (3, 2).$$

In geometric terms, the method moved along edges of the feasible region.

The simplex method begins with the problem in canonical form.

Basic idea of the simplex method

With the nondegeneracy assumption, we move from one basic feasible solution to another that gives a smaller value for the objective function by replacing exactly one basic variable at each step.

2.2 DETERMINING A MINIMUM FEASIBLE SOLUTION

Consider the problem in canonical form

$$
\begin{array}{ll}
\text{minimize} & z = c_{m+1}x_{m+1} + \cdots + c_n x_n + z_0 \\
\text{subject to} & x_1 + \quad \cdots \quad + a_{1,m+1}x_{m+1} + \cdots + a_{1n}x_n = b_1 \\
& \qquad x_2 + \quad \cdots \quad + a_{2,m+1}x_{m+1} + \cdots + a_{2n}x_n = b_2 \\
& \qquad\qquad \vdots \\
& \qquad\qquad x_m + a_{m,m+1}x_{m+1} + \cdots + a_{mn}x_n = b_m \\
& \qquad\qquad x_1, x_2, \ldots, x_n \geq 0,
\end{array}
\tag{6}
$$

a_{ij}, b_i, c_j, and z_0 are constants, and $b_i > 0$ for all $i = 1, \ldots, m$.

Note In Problem (6), z is expressed in terms of nonbasic variables.

Theorem 2.1 (Optimality Criterion)

For the linear programming problem of (6), if $c_j \geq 0$, $j = m+1, \ldots, n$, then the minimal value of the objective function is z_0 and is attained at the basic feasible solution $(b_1, b_2, \ldots, b_m, 0, \ldots, 0)$.

Theorem 2.2 (The minimum ratio test)

Assume that in problem (6) $\mathbf{b} > \mathbf{0}$, $c_q < 0$ $(q > m)$, *and there is at least one* $a_{iq} > 0$, $i = 1, ..., m$. *If*

$$\frac{b_p}{a_{pq}} = \min_i \left\{ \frac{b_i}{a_{iq}} : a_{iq} > 0 \right\},$$

then the problem can be put into canonical form with basic variables $x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_m, x_q$. *The value of the objective function at the associated basic feasible solution is*

$$z_0 + \frac{c_q b_p}{a_{pq}} < z_0.$$

- x_q is called the **entering variable** and x_p the **leaving variable**.
- The column of x_q **enters** the basis, and the column of x_p **leaves** the basis.

Remark

For the linear programming problem of (6), if there is an index $q > m$ such that

$$c_q < 0 \text{ and } a_{iq} \leq 0 \text{ for all } i = 1, 2, \dots, m,$$

then the objective function is not bounded below.

The Simplex Tableau

Example 3.1

$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 + 3x_3 \\ \text{subject to} & \end{array}$$

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & + & x_3 & \leq & 2 \\ x_1 & + & 2x_2 & + & 3x_3 & \leq & 5 \\ 2x_1 & + & 2x_2 & + & x_3 & \leq & 6 \\ x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq & 0. \end{array}$$

Solution To transform the problem into canonical form, we change the maximization to minimization and introduce three nonnegative slack variables x_4, x_5, x_6.

$$
\begin{array}{lrcl}
\text{minimize} & z = -3x_1 - x_2 - 3x_3 & & \\
\text{subject to} & 2x_1 + x_2 + x_3 + x_4 & = 2 & \\
& x_1 + 2x_2 + 3x_3 + x_5 & = 5 & (7) \\
& 2x_1 + 2x_2 + x_3 + x_6 & = 6 & \\
& x_1, x_2, x_3, x_4, x_5, x_6 & \geq 0. &
\end{array}
$$

This problem is equivalent to

$$
\begin{array}{lrcrcrcrcrcrcrcr}
\text{minimize} & z \\
\text{subject to} & 2x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & & & = & 2 \\
 & x_1 & + & 2x_2 & + & 3x_3 & & & + & x_5 & & & & & = & 5 \\
 & 2x_1 & + & 2x_2 & + & x_3 & & & & & + & x_6 & & & = & 6 \\
 & -3x_1 & - & x_2 & - & 3x_3 & & & & & & & - & z & = & 0 \\
\end{array}
$$

$$z \text{ free}, \; x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

The row $-3x_1 - x_2 - 3x_3 - z = 0$ is called the **objective row**.

We then have the initial tableau

x_1	x_2	x_3	x_4	x_5	x_6	z	rhs
2	1	1	1	0	0	0	2
1	2	3	0	1	0	0	5
2	2	1	0	0	1	0	6
-3	-1	-3	0	0	0	-1	0

Since the objective row of this tableau has negative entries, the basic feasible solution is not optimal.

We select the second column, pivot on 1 and result in

x_1	x_2	x_3	x_4	x_5	x_6	z	rhs
2	1	1	1	0	0	0	2
-3	0	1	-2	1	0	0	1
-2	0	-1	-2	0	1	0	2
-1	0	-2	1	0	0	-1	2

Again we pivot on 1

x_1	x_2	x_3	x_4	x_5	x_6	z	rhs
5	1	0	3	-1	0	0	1
-3	0	1	-2	1	0	0	1
-5	0	0	-4	1	1	0	3
-7	0	0	-3	2	0	-1	4

We select 5.

x_1	x_2	x_3	x_4	x_5	x_6	z	rhs
1	1/5	0	3/5	-1/5	0	0	1/5
0	3/5	1	-1/5	2/5	0	0	8/5
0	1	0	-1	0	1	0	4
0	7/5	0	6/5	3/5	0	-1	27/5

Since the last row has no negative elements, we conclude that the solution corresponding to this tableau is optimal.

This tableau corresponds to the problem

$$
\begin{array}{ll}
\text{minimize} & z = \frac{7}{5}x_2 + \frac{6}{5}x_4 + \frac{3}{5}x_5 - \frac{27}{5} \\
\text{subject to} & x_1 + \frac{1}{5}x_2 + \frac{3}{5}x_4 - \frac{1}{5}x_5 = \frac{1}{5} \\
& \frac{3}{5}x_2 + x_3 - \frac{1}{5}x_4 + \frac{2}{5}x_5 = \frac{8}{5} \\
& \frac{7}{5}x_2 - x_4 + x_6 = 4 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{array}
$$

Thus

$$x_1 = \frac{1}{5}, \;\; x_2 = 0, \;\; x_3 = \frac{8}{5}, \;\; x_4 = 0, \;\; x_5 = 0, \;\; x_6 = 4$$

is the optimal solution of Problem (7) with a corresponding optimal value $-27/5$.

To summarize:

1. The simplex method begins with the problem in canonical form.
2. We move from one basic feasible solution to another by replacing exactly one basic variable at each step.

Note The z column always appears in the form

$$\begin{array}{|r|}
\hline
z \\
\hline
0 \\
0 \\
\vdots \\
0 \\
\hline
-1 \\
\hline
\end{array}$$

in any simplex tableau.

Thus from now on we will *not* include the z column in tableaux.

Consider the problem in canonical form

$$
\begin{array}{ll}
\text{minimize} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & x_1 + \quad \cdots \quad + \bar{a}_{1,m+1}x_{m+1} + \cdots + \bar{a}_{1n}x_n = \bar{a}_{10} \\
& \qquad x_2 + \ \cdots \ + \bar{a}_{2,m+1}x_{m+1} + \cdots + \bar{a}_{2n}x_n = \bar{a}_{20} \\
& \qquad\qquad\qquad\qquad \vdots \\
& \qquad\qquad x_m + \bar{a}_{m,m+1}x_{m+1} + \cdots + \bar{a}_{mn}x_n = \bar{a}_{m0} \\
& \qquad\qquad\qquad x_1, x_2, \ldots, x_n \geq 0,
\end{array}
\tag{8}
$$

where $\bar{a}_{i0} > 0$ for all $i = 1, 2, \ldots, m$.

The basic feasible solution is

$$(\mathbf{x_B}, \mathbf{0}) = (\bar{a}_{10}, \bar{a}_{20}, \ldots, \bar{a}_{m0}, 0, 0, \ldots, 0).$$

The objective function corresponding to the basic solution $\mathbf{x} = (\mathbf{x_B}, \mathbf{0})$ is

$$z_0 := \mathbf{c}^T\mathbf{x} = c_1\bar{a}_{10} + c_2\bar{a}_{20} + \cdots + c_m\bar{a}_0 = \mathbf{c}_\mathbf{B}^T\mathbf{x_B},$$

where $\mathbf{c}_\mathbf{B}^T = [c_1, c_2, \ldots, c_m]$.

We can express the objective function z in terms of the nonbasic variables $x_{m+1}, \ldots, x_n$:

$$\mathbf{c}^T\mathbf{x} = z_0 + \sum_{j=m+1}^{n} (c_j - z_j)x_j,$$

where

$$z_j = \mathbf{c}_\mathbf{B}^T \bar{\mathbf{a}}_j.$$

The coefficients

$$r_j = c_j - z_j$$

are called **relative cost coefficients** or **reduced cost coefficients**.

Thus Problem (8) is equivalent to

$$
\begin{array}{ll}
\text{minimize} & z = r_{m+1}x_{m+1} + r_{m+2}x_{m+2} + \cdots + r_n x_n + z_0 \\
\text{subject to} & x_1 + \quad \cdots \quad + \bar{a}_{1,m+1}x_{m+1} + \cdots + \bar{a}_{1n}x_n = \bar{a}_{10} \\
& x_2 + \ \cdots \ + \bar{a}_{2,m+1}x_{m+1} + \cdots + \bar{a}_{2n}x_n = \bar{a}_{20} \\
& \vdots \\
& x_m + \bar{a}_{m,m+1}x_{m+1} + \cdots + \bar{a}_{mn}x_n = \bar{a}_{m0} \\
& x_1, x_2, \ldots, x_n \geq 0.
\end{array}
\tag{9}
$$

The simplex tableau takes the initial form as follows

x_1	x_2	$\cdots$	x_m	x_{m+1}	$\cdots$	x_j	$\cdots$	x_n	rhs
1	0	$\cdots$	0	$\bar{a}_{1,m+1}$	$\cdots$	$\bar{a}_{1j}$	$\cdots$	$\bar{a}_{1n}$	$\bar{a}_{10}$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
0	0	$\cdots$	0	$\bar{a}_{i,m+1}$	$\cdots$	$\bar{a}_{ij}$	$\cdots$	$\bar{a}_{in}$	$\bar{a}_{i0}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
0	0	$\cdots$	1	$\bar{a}_{m,m+1}$	$\cdots$	$\bar{a}_{mj}$	$\cdots$	$\bar{a}_{mn}$	$\bar{a}_{m0}$
0	$\cdots$	0	0	r_{m+1}	$\cdots$	r_j	$\cdots$	r_n	$-z_0$

By Theorem 2.1, if in Problem (9) all relative cost coefficients r_j are nonnegative, then the basic feasible solution $(\mathbf{x_B}, \mathbf{0})$ is optimal.

The simplex algorithm

Step 0. Form a tableau corresponding to a basic feasible solution. The relative cost coefficients r_j can be found by row reduction.

Step 1. If each $r_j \geq 0$, stop; the current basic feasible solution is optimal.

Step 2. Select q such that $r_q < 0$ to determine which nonbasic variable is to become basic.

Step 3. Calculate the ratios

$$\frac{\bar{a}_{i0}}{\bar{a}_{iq}} \quad \text{for} \quad \bar{a}_{iq} > 0, \; i = 1, 2, \ldots, m.$$

If no $\bar{a}_{iq} > 0$, stop; the problem is unbounded. Otherwise, select p as the index i corresponding to the minimum ratio:

$$\frac{\bar{a}_{p0}}{\bar{a}_{pq}} = \min_i \left\{ \frac{\bar{a}_{i0}}{\bar{a}_{iq}} : \bar{a}_{iq} > 0 \right\}.$$

Step 4. Pivot on the pqth element, updating all rows including the last. Return to Step 1.

Example 3.2 Solve the following problem

$$
\begin{aligned}
\text{minimizing} \quad & -4x_1 + x_2 + x_3 + 7x_4 + 3x_5 \\
\text{subject to} \quad & -6x_1 \qquad\quad + x_3 - 2x_4 + 2x_5 = 6 \\
& 3x_1 + x_2 - x_3 + 8x_4 + \quad x_5 = 9 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}
$$

The information of the problem is recorded in tableau form in Table 1.

x_1	x_2	x_3	x_4	x_5	rhs
-6	0	1	-2	2	6
3	1	-1	8	1	9
-4	1	1	7	3	0

Table 3.1

We now first write the problem in canonical form and express the objective function in terms of nonbasic variables (basic variables are x_2 and x_3).

x_1	x_2	x_3	x_4	x_5	rhs
-6	0	1	-2	2	6
-3	1	0	6	3	15
5	0	0	3	-2	-21

Table 3.2

This table corresponds to the problem in canonical form

$$\begin{aligned}
\text{minimize} \quad & z = 5x_1 + 3x_4 - 2x_5 + 21 \\
\text{subject to} \quad & -6x_1 + x_3 - 2x_4 + 2x_5 = 6 \\
& -3x_1 + x_2 + 6x_4 + 3x_5 = 15 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}$$

Since x_2 and x_3 are basic variables, our initial basic feasible solution is

$$(x_1, x_2, x_3, x_4, x_5) = (0, 15, 6, 0, 0)$$

and the corresponding value of the objective function $z = 21$.

The x_5 column is the pivot column. Since

$$\min\left\{\frac{6}{2}, \frac{15}{3}\right\} = 3,$$

we should pivot at the 2 in the first row, replacing the basic variable x_3 with the variable x_5.

Dividing the first row by 2 and then adding multiples of this row to the remaining rows:

x_1	x_2	x_3	x_4	x_5	rhs
-3	0	$1/2$	-1	1	3
-3	1	0	6	3	15
5	0	0	3	-2	-21
-3	0	$1/2$	-1	1	3
6	1	$-3/2$	9	0	6
-1	0	1	1	0	-15

Table 3.3

The latter tableau represents the following problem

$$\begin{aligned} \text{minimize} \quad & z = -x_1 + x_3 + x_4 + 15 \\ \text{subject to} \quad & -3x_1 + \frac{1}{2}x_3 - x_4 + x_5 = 3 \\ & 6x_1 + x_2 - \frac{3}{2}x_3 + 9x_4 = 6. \end{aligned}$$

The associated basic feasible solution is

$$(0, 6, 0, 0, 3),$$

and the value of the objective function at this point is $z = -(-15) = 15$.

Pivoting at the 6 in the x_1 column of the second row gives the tableau of Table 3.4.

x_1	x_2	x_3	x_4	x_5	rhs
0	$1/2$	$-1/4$	$7/2$	1	6
1	$1/6$	$-1/4$	$3/2$	0	1
0	$1/6$	$3/4$	$5/2$	0	-14

Table 3.4

The minimum value of the objective function has been attained. This value, $z = -(-14) = 14$, is attained at the basic feasible solution $(1, 0, 0, 0, 6)$.

Example 3.3

$$
\begin{aligned}
\text{maximize} \quad & 2x_2 + x_3 \\
\text{subject to} \quad & x_1 + x_2 - 2x_3 \le 7 \\
& -3x_1 + x_2 + 2x_3 \le 3 \\
& x_1, x_2, x_3 \ge 0.
\end{aligned}
$$

Solution The standard form of the problem is

$$
\begin{array}{rrcl}
\text{minimize} & -2x_2 - x_3 & & \\
\text{subject to} & x_1 + x_2 - 2x_3 + x_4 & & = 7 \\
 & 3x_1 + x_2 + 2x_3 & + x_5 & = 3 \\
 & & x_1, x_2, x_3, x_4, x_5 & \geq 0.
\end{array}
$$

This problem is in canonical form with basic variables x_4 and x_5, and the steps of the simplex algorithm are displayed in the following table.

x_1	x_2	x_3	x_4	x_5	rhs
1	1	−2	1	0	7
−3	1	2	0	1	3
0	−2	−1	0	0	0
4	0	−4	1	−1	4
−3	1	2	0	1	3
−6	0	3	0	2	6
1	0	−1	1/4	−1/4	1
0	1	−1	3/4	1/4	6
0	0	−3	3/2	1/2	12

The three negative entries in the third column of the previous tableau indicate that the objective function is unbounded below.

Example 3.4 **(Degeneracy)**
Solve the linear programming problem

$$\begin{aligned} \text{maximize} \quad & z = 5x_1 + 3x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 2 \\ & 2x_1 + x_2 \leq 4 \\ & -3x_1 + 2x_2 \leq 6 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Solution We rewrite the problem in canonical form

$$
\begin{array}{lrcl}
\text{minimize} & w = -5x_1 - 3x_2 & & \\
\text{subject to} & x_1 - x_2 + x_3 & = & 2 \\
 & 2x_1 + x_2 + x_4 & = & 4 \\
 & -3x_1 + 2x_2 + x_5 & = & 6 \\
 & x_1, x_2, x_3, x_4, x_5 & \geq & 0.
\end{array}
$$

The simplex method leads to the following tableaux.

x_1	x_2	x_3	x_4	x_5	rhs
1	−1	1	0	0	2
2	1	0	1	0	4
−3	2	0	0	1	6
−5	−3	0	0	0	0
1	−1	1	0	0	2
0	3	−2	1	0	0
0	−1	3	0	1	12
0	−8	5	0	0	10

x_1	x_2	x_3	x_4	x_5	rhs
1	0	1/3	1/3	0	2
0	1	−2/3	1/3	0	0
0	0	$\boxed{7/3}$	1/3	1	12
0	0	−1/3	8/3	0	10
1	0	0	2/7	−1/7	2/7
0	1	0	3/7	2/7	24/7
0	0	1	1/7	3/7	36/7
0	0	0	19/7	1/7	82/7

The optimal solution of the original problem is

$$x_1 = \frac{2}{7}, \quad x_2 = \frac{24}{7}$$

with the optimal value being $z = \frac{82}{7}$.

There are options for dealing with degeneracy. For example, techniques use Bland's rule or introduce small perturbations into the right-hand sides of the constraints.

- The simplex method begins with the problem in canonical form.
- But an initial basic feasible solution is not always apparent for linear programs.

Question:

How to find an initial basic feasible solution so that the simplex method can be initiated?

By elementary operations the constraints of a linear programming problem can always be expressed in the standard form

$$\begin{aligned} a_{11}x_1 \; + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 \; + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \;\; &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \\ x_1, x_2, \ldots, x_n &\geq 0, \end{aligned} \tag{10}$$

where $\mathbf{b} \geq \mathbf{0}$.

The idea:

To enable us to obtain an initial basic feasible solution, we introduce another variable into each equation in (10), called **artificial variables**

Consider the (artificial) minimization problem

$$
\begin{array}{ll}
\text{minimize} & y_1 + y_2 + \cdots + y_m \\
\text{subject to} & \\
& \begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 & & = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & + y_2 & = b_2 \\
\vdots & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & + y_m & = b_m \\
x_1,\ x_2, \ldots, x_n \geq 0, & y_1,\ y_2, \ldots, y_m & \geq 0,
\end{array}
\end{array}
\tag{11}
$$

where $\mathbf{b} \geq \mathbf{0}$.

Then the vector $\mathbf{x} \in \mathbb{R}^n$ is a feasible solution to the problem

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{12}$$

if and only if the vector $(\mathbf{x}, \mathbf{0}) \in \mathbb{R}^{n+m}$ is a feasible solution to the problem (11).

Moreover, $(\mathbf{x}, \mathbf{0})$ is a basic feasible solution to the problem (11) if and only if $\mathbf{x}$ is a basic feasible solution to the problem (12).

Example 4.1 Find a basic feasible solution to

$$\begin{aligned} 2x_1 + \ x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + \ x_3 &= 3 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 &\geq 0. \end{aligned}$$

We introduce artificial variables $x_4 \geq 0$, $x_5 \geq 0$ and an objective function $x_4 + x_5$.

The artificial problem is

$$\begin{aligned} \text{minimize} \quad & w = x_4 + x_5 \\ \text{subject to} \quad & 2x_1 + \ x_2 + 2x_3 + x_4 \qquad = 4 \\ & 3x_1 + 3x_2 + \ x_3 \qquad + x_5 = 3 \\ & \qquad x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

The initial tableau is

x_1	x_2	x_3	x_4	x_5	rhs
2	1	2	1	0	4
3	3	1	0	1	3
0	0	0	1	1	0

Since x_4 and x_5 are basic variables, the corresponding canonical problem has the following tableau:

x_1	x_2	x_3	x_4	x_5	rhs
2	1	2	1	0	4
3	3	1	0	1	3
-5	-4	-3	0	0	-7

$$(r_3 - r_1 - r_2 \longrightarrow r_3)$$

The simplex procedure now can be applied to obtain

x_1	x_2	x_3	x_4	x_5	rhs
0	−1	4/3	1	−2/3	2
1	1	1/3	0	1/3	1
0	1	−4/3	0	5/3	−2
0	−3/4	1	3/4	−1/2	3/2
1	5/4	0	−1/4	1/2	1/2
0	0	0	1	1	0

A basic feasible solution of the original constraints is

$$(1/2, 0, 3/2).$$

Example 4.2

$$
\begin{aligned}
\text{minimize} \quad & x_1 + x_2 + x_3 \\
\text{subject to} \quad & -x_1 + 2x_2 + x_3 \leq 1 \\
& -x_1 + 2x_3 \geq 4 \\
& x_1 - x_2 + 2x_3 = 4 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

Solution The problem in standard form is

$$
\begin{aligned}
\text{minimize} \quad & x_1 + x_2 + x_3 \\
\text{subject to} \quad & \\
& -x_1 + 2x_2 + x_3 + x_4 = 1 \\
& -x_1 + 2x_3 - x_5 = 4 \\
& x_1 - x_2 + 2x_3 = 4 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}
$$

Note that the x_4 variable can serve as a basic variable.

Thus it is sufficient to add only two artificial variables, say x_6 and x_7, to the problem.

The artificial problem is then

$$
\begin{aligned}
&\text{minimize} \quad w = x_6 + x_7 \\
&\text{subject to} \\
&\qquad \begin{array}{rcl}
-x_1 + 2x_2 + x_3 + x_4 & & = 1 \\
-x_1 \qquad + 2x_3 \qquad - x_5 + x_6 & & = 4 \\
x_1 - x_2 + 2x_3 \qquad\qquad + x_7 & & = 4.
\end{array}
\end{aligned}
$$

The tableaux are given in the following table.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
-1	2	1	1	0	0	0	1
-1	0	2	0	-1	1	0	4
1	-1	2	0	0	0	1	4
0	0	0	0	0	1	1	0
-1	2	1	1	0	0	0	1
-1	0	2	0	-1	1	0	4
1	-1	2	0	0	0	1	4
0	1	-4	0	1	0	0	-8

x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
0	1	-4	0	1	0	0	-8
-1	2	1	1	0	0	0	1
1	-4	0	-2	-1	0	0	2
3	-5	0	-2	0	0	0	2
-4	9	0	4	1	0	0	-4
0	$1/3$	1	$1/3$	0	0	0	$5/3$
0	$-7/3$	0	$-4/3$	-1	0	0	$4/3$
1	$-5/3$	0	$-2/3$	0	0	0	$2/3$
0	$7/3$	0	$4/3$	1	0	0	$-4/3$

The minimal value for the function $w = x_6 + x_7$ is $\frac{4}{3} > 0$.

Therefore the original problem has no feasible solution.

Using artificial variables, we attack a general linear programming problem by use of the **two-phase method**.

- In **phase I** artificial variables are introduced and a basic feasible solution is found (or it is determined that no feasible solutions exist).
- In **phase II**, using the basic feasible solution resulting from phase I, the original objective function is minimized.

During phase II the artificial variables and the objective function of phase I are omitted.

Note In phase I artificial variables need be introduced only in those equations that do *not* contain slack variables.

Example 4.3 **(A free variable problem)**

$$\begin{aligned}
\text{minimize} \quad & -2x_1 + 4x_2 + 7x_3 + x_4 + 5x_5 \\
\text{subject to} \quad & \\
& -x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 7 \\
& -x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 6 \\
& -x_1 + \ \ x_2 + x_3 + 2x_4 + x_5 = 4 \\
& x_1 \text{ free}, \ x_2, \ x_3, \ x_4, \ x_5 \geq 0.
\end{aligned}$$

ANS.

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 0, \quad x_5 = 2.$$

Dual Linear Programs

Associated with every linear program is a corresponding dual linear program.

In this section we define the dual program that is associated with a given linear program. There are some interesting interpretations of the associated problem that we will discuss.

The variables of the dual problem can be interpreted as prices associated with the constraints of the original (primal) problem.

A study of duality sharpens our understanding of the simplex procedure and motivates certain alternative solution methods.

The simultaneous consideration of a problem from both the primal and dual viewpoints often provides significant computational advantage as well as economic insight.

We define duality through the following pair of programs:

$$\begin{array}{llcll} & \textbf{Primal} & & & \textbf{Dual} \\ \text{minimize} & \mathbf{c}^T\mathbf{x} & & \text{maximize} & \mathbf{y}^T\mathbf{b} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b} & & \text{subject to} & \mathbf{y}^T\mathbf{A} \leq \mathbf{c}^T \\ & \mathbf{x} \geq \mathbf{0} & & & \mathbf{y} \geq \mathbf{0}. \end{array} \tag{13}$$

If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{x}$ is an n-dimensional column vector, $\mathbf{b}$ is an m-dimensional column vector, $\mathbf{c}^T$ is an n-dimensional row vector, and $\mathbf{y}^T$ is an m-dimensional row vector.

The vector $\mathbf{x}$ is the variable of the primal program, and $\mathbf{y}$ is the variable of the dual program.

Example 5.1 Given the primal problem

$$\begin{aligned} \text{minimize} \quad & z = 6x_1 + 2x_2 - x_3 + 2x_4 \\ \text{subject to} \quad & 4x_1 + 3x_2 - 2x_3 + 2x_4 \geq 10 \\ & 8x_1 + x_2 + 2x_3 + 4x_4 \geq 18 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Determine its dual problem.

Solution Its dual is

$$\begin{aligned} \text{maximize} \quad & w = 10y_1 + 18y_2 \\ \text{subject to} \quad & 4y_1 + 8y_2 \leq 6 \\ & 3y_1 + \ \ y_2 \leq 2 \\ & -2y_1 + 2y_2 \leq -1 \\ & 2y_1 + 4y_2 \leq 2 \\ & y_1, y_2 \geq 0. \end{aligned}$$

Example 5.2 The linear programming problem

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0}, \end{aligned}$$

has as its dual problem,

$$\begin{aligned} \text{minimize} \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge \mathbf{0}. \end{aligned}$$

The pair of programs (13) is called the **symmetric form** of duality and can be used to define the dual of any linear program.

It is important to note that the role of primal and dual can be reversed.

Theorem 5.1

The dual of the dual linear program is the primal linear program.

In general, the dual of any linear program can be found by converting the program to the form of the primal shown above.

Example 5.3 The linear programming problem in standard form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

has for its dual the linear programming problem

$$\begin{aligned} \text{maximize} \quad & \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^T\mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \text{ free.} \end{aligned}$$

Similar transformations can be worked out for any linear program to first get the primal in the form (13), calculate the dual, and then simplify the dual to account for special structure.

We summarize the relationships between the primal and dual problems in the following table.

Primal problem	Dual problem
Minimization	Maximization
Right-hand sides of constraints	Coefficients of objective function
Coefficients of ith variable, one in each constraint	Coefficients of ith constraint
ith variable is ≥ 0	ith constraint is an inequality $\leq$
ith variable is unrestricted	ith constraint is an equality
jth constraint is an equality	jth variable is unrestricted
jth constraint is an inequality $\geq$	jth variable is ≥ 0
Number of constraints	Number of variables

Example 5.4 If the primal problem is

$$\begin{array}{ll} \text{minimize} & 2x_1 - 3x_2 + x_4 \\ \text{subject to} & x_1 + 2x_2 + x_3 \leq 7 \\ & x_1 + 4x_2 - x_4 = 5 \\ & x_2 + x_3 + 5x_4 \geq 3 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0, \end{array}$$

then the dual problem is

$$
\begin{array}{ll}
\text{maximize} & 7y_1 + 5y_2 + 3y_3 \\
\text{subject to} & y_1 + y_2 \leq 2 \\
& 2y_1 + 4y_2 + y_3 \leq -3 \\
& y_1 + y_3 \leq 0 \\
& -y_2 + 5y_3 \leq 1 \\
& y_1 \leq 0,\ y_3 \geq 0,\ y_2 \text{ unrestricted.}
\end{array}
$$

Or equivalently,

$$
\begin{array}{ll}
\text{maximize} & -7y_1 + 5y_2 + 3y_3 \\
\text{subject to} & -y_1 + y_2 \leq 2 \\
& -2y_1 + 4y_2 + y_3 \leq -3 \\
& -y_1 + y_3 \leq 0 \\
& -y_2 + 5y_3 \leq 1 \\
& y_1 \geq 0,\ y_3 \geq 0,\ y_2 \text{ unrestricted.}
\end{array}
$$

Example 5.5 If the primal problem is

$$\begin{array}{ll} \text{maximize} & 3x_1 + 2x_2 + x_3 \\ \text{subject to} & x_1 + 2x_2 - x_3 \le 4 \\ & 2x_1 - x_2 + x_3 = 8 \\ & x_1 - x_2 \le 6 \\ & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \text{ unrestricted}, \end{array}$$

then the dual problem is

$$\begin{array}{ll} \text{minimize} & 4y_1 + 8y_2 + 6y_3 \\ \text{subject to} & y_1 + 2y_2 + y_3 \ge 3 \\ & 2y_1 - y_2 - y_3 \ge 2 \\ & -y_1 + y_2 = 1 \\ & y_1 \ge 0,\ y_3 \ge 0,\ y_2 \text{ unrestricted}. \end{array}$$

In this section, the deeper connection between a program and its dual, as expressed by the Duality Theorem, is derived.

Throughout this section we consider the primal program in standard form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{14}$$

and its corresponding dual

$$\begin{aligned} \text{maximize} \quad & \mathbf{y}^T\mathbf{b} \\ \text{subject to} \quad & \mathbf{y}^T\mathbf{A} \leq \mathbf{c}^T. \end{aligned} \tag{15}$$

In this section it is *not* assumed that **A** is necessarily of full rank.

Lemma 6.1 (Weak Duality Lemma)

If $\mathbf{x}$ *and* $\mathbf{y}$ *are feasible for* (14) *and* (15)*, respectively, then* $\mathbf{c}^T\mathbf{x} \geq \mathbf{y}^T\mathbf{b}$.

Primal values $\geq$ Dual values

Corollary 6.2

If $\mathbf{x}_0$ *and* $\mathbf{y}_0$ *are feasible for* (14) *and* (15)*, respectively, and if*

$$\mathbf{c}^T\mathbf{x}_0 = \mathbf{y}_0^T\mathbf{b},$$

then $\mathbf{x}_0$ *and* $\mathbf{y}_0$ *are optimal for their respective problems.*

Theorem 6.3 (Duality Theorem of Linear Programming)

Consider a pair of primal and dual linear programs.

(a) *If one of the problems has an optimal solution then so does the other, and the optimal objective values are equal.*

(b) *If either problem has an unbounded objective, the other problem has no feasible solution.*

Summary

Consider a pair of primal and dual linear programs. Exactly one of the following three alternatives holds:

(i) *Both primal and dual problems are feasible and consequently both have optimal solutions with equal extrema.*

(ii) *Exactly one of the problems is infeasible and consequently the other problem has an unbounded objective function in the direction of optimization on its feasible region.*

(iii) *Both primal and dual problems are infeasible.*

Example 6.1 Consider the problem

$$\begin{aligned} \text{maximize} \quad & z = 2x_1 + x_2 \\ \text{subject to} \quad & 3x_1 - 2x_2 \le 6 \\ & x_1 - 2x_2 \le 1 \\ & x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

Its dual problem is

$$\begin{aligned} \text{minimize} \quad & z = 6y_1 + y_2 \\ \text{subject to} \quad & 3y_1 + y_2 \ge 2 \\ & -2y_1 - 2y_2 \ge 1 \\ & y_1 \ge 0, \; y_2 \ge 0. \end{aligned}$$

The primal problem is unbounded above and the dual has no feasible solutions.

Example 6.2 Consider the problem

$$\begin{aligned} \text{maximize} \quad & 2x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 1 \\ & -x_1 - x_2 \geq 1. \end{aligned}$$

The dual problem is

$$\begin{aligned} \text{minimize} \quad & y_1 + y_2 \\ \text{subject to} \quad & y_1 - y_2 = 2 \\ & y_1 - y_2 = -1, \end{aligned}$$

which is infeasible.

Theorem 7.1 (Complementary slackness asymmetric form)

Let $\mathbf{x}$ *and* $\mathbf{y}$ *be feasible solutions for the primal and dual programs, respectively, in the pair* (14)–(15)*. A necessary and sufficient condition that they both be optimal solutions is that for all* i

(i) $x_i > 0 \Longrightarrow \mathbf{y}^T\mathbf{a}_i = c_i$;

(ii) $\mathbf{y}^T\mathbf{a}_i < c_i \Longrightarrow x_i = 0$.

Note that conditions (i) and (ii) can be combined into one equality:

$$\mathbf{x}^T(\mathbf{c} - \mathbf{A}^T\mathbf{y}) = 0.$$

Theorem 7.2 (Complementary slackness symmetric form)

Let $\mathbf{x}$ *and* $\mathbf{y}$ *be feasible solutions for the primal and dual programs, respectively, in the pair* (13). *A necessary and sufficient condition that they both be optimal solutions is that for all* i *and* j

(i) $x_i > 0 \Longrightarrow \mathbf{y}^T\mathbf{a}_i = c_i$

(ii) $\mathbf{y}^T\mathbf{a}_i < c_i \Longrightarrow x_i = 0$

(iii) $y_i > 0 \Longrightarrow \mathbf{x}^T\mathbf{a}^j = b_j$

(iv) $\mathbf{x}^T\mathbf{a}^j > b_j \Longrightarrow y_j = 0,$

(where $\mathbf{a}^j$ *is the* j*th row of* $\mathbf{A}$*).*

Note that conditions (i)–(iv) can be combined into two equalities:

$$\boxed{\mathbf{x}^T(\mathbf{c} - \mathbf{A}^T\mathbf{y}) = 0 \quad \text{and} \quad \mathbf{y}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = 0}$$