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Probability, Homework 10.

1/

a) Let $A = [30, 50] \times [30, 50]$. Then:

$$1 = \iint_{\mathbb{R}} f(x)\,dx\,dy = \iint_{A} f(x)\,dx\,dy + \iint_{A^c} f(x)\,dx\,dy = \iint_{A} k(x^2+y^2)\,dx\,dy + \iint_{A^c} 0\,dx\,dy$$

$$\Leftrightarrow 1 = \int_{30}^{50}\int_{30}^{50} k(x^2+y^2)\,dx\,dy = \frac{3920000}{3}k \Rightarrow k = \frac{3}{3920000}.$$

b) Let $B = [30, 40] \times [40, 50]$. Then:

$$P(\{30 \le X \le 40\} \cap \{40 \le Y < 50\}) = \iint_{B} f(x)\,dx\,dy = \int_{30}^{40}\int_{40}^{50} \frac{3(x^2+y^2)}{3920000}\,dx\,dy = \frac{1}{4}.$$

c) Let $C = [30, 40] \times [30, 40]$. Then:

$$P(\{30 \le X \le 40\} \cap \{30 \le Y \le 40\}) = \iint_{C} f(x)\,dx\,dy = \int_{30}^{40}\int_{30}^{40} \frac{3(x^2+y^2)}{3920000}\,dx\,dy = \frac{37}{196}.$$

2/

a) Marginal density of X:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_0^1 f(x,y)\,dy + \int_{[0,1]^c} f(x,y)\,dy = \int_0^1 \frac{2}{3}(x+2y)\,dy + \int_{[0,1]^c} 0\,dy = \frac{2x+2}{3}, \forall x \in [0,1]$$

$(f_X(x) = 0 \text{ if } x \notin [0,1])$

b) Marginal density of Y:

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \int_0^1 f(x,y)\,dx + \int_{[0,1]^c} f(x,y)\,dx = \int_0^1 \frac{2}{3}(x+2y)\,dx + \int_{[0,1]^c} 0\,dx = \frac{4y+1}{3}, \forall y \in [0,1].$$

$(f_Y(y) = 0 \text{ if } y \notin [0,1])$

c) $$P(\{X < 0.5\}) = \int_{-\infty}^{0.5} f_X(x)\,dx = \int_{-\infty}^{0} f_X(x)\,dx + \int_0^{0.5} f_X(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{0.5} \frac{2x+2}{3}\,dx = \frac{5}{12}.$$

3/

a) Marginal densiry of X:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_0^1 f(x,y)dy + \int_{[0,1]^c} f(x,y)dy = \int_0^1 \frac{3}{2}(x^2+y^2)dy + \int_{[0,1]^c} 0\,dy = \frac{3x^2+1}{2}, \forall x \in [0,1].$$

$(f_X(x) = 0 \text{ if } x \notin [0,1])$

Marginal density of Y:

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)dx = \int_0^1 f(x,y)dx + \int_{[0,1]^c} f(x,y)dx = \int_0^1 \frac{3}{2}(x^2+y^2)dx + \int_{[0,1]^c} 0\,dx = \frac{3y^2+1}{2}, \forall y \in [0,1].$$

$(f_Y(y) = 0 \text{ if } y \notin [0,1])$.

Since $f(x,y) \neq f_X(x) \cdot f_Y(y)$, X and Y are not independent.

b) $$E(X) = \int_{\mathbb{R}} x f_X(x)\,dx = \int_0^1 x f_X(x)dx + \int_{[0,1]^c} x f_X(x)dx = \int_0^1 \frac{3x^3+x}{2}dx + \int_{[0,1]^c} 0\,dx = \frac{5}{8}.$$

$$E(X^2) = \int_{\mathbb{R}} x^2 f_X(x)\,dx = \int_0^1 x^2 f_X(x)dx + \int_{[0,1]^c} x^2 f_X(x)dx = \int_0^1 \frac{3x^4+x^2}{2}dx + \int_{[0,1]^c} 0\,dx = \frac{7}{15}.$$

$\Rightarrow Var(X) = E(X^2) - E(X)^2 = \frac{73}{960}$. By symmetry of X and Y, $Var(Y) = \frac{73}{960}$.

4/ Marginal densiy of X:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_0^{1-x} f(x,y)dy + \int_{[0,1-x]^c} f(x,y)dy = \int_0^{1-x} 6x\,dy + \int_{[0,1-x]^c} 0\,dy = 6x - 6x^2, \forall x \in [0, 1-y].$$

$(f_X(x) = 0 \text{ if } x \notin [0, 1-y])$

Marginal densiry of Y:

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \int_0^{1-y} f(x,y)dx + \int_{[0,1-y]^c} f(x,y)dx = \int_0^{1-y} 6x\,dx + \int_{[0,1-y]^c} 0\,dx = 3(1-y)^2, \forall y \in [0, 1-x].$$

$(f_Y(y) = 0 \text{ if } y \notin [0, 1-x])$

Since $f(x,y) \neq f_X(x) \cdot f_Y(y)$, X and Y are not independent.