Homework 2, Probability

Tran Viet Hang
Instructor: Dr. Nguyen Minh Quan

February 9, 2022

---

1. Chapter 3

1.1. Problems

Problem 44. 12% of all U.S. households are in California. A total of 1.3% of all U.S. households earn more than $250,000 per year, while a total of 3.3% of all California households earn more than $250,000 per year.

(a) What proportion of all non-California households earn more than $250,000 per year?

(b) Given that a randomly chosen U.S. household earns more than $250,000 per year, what is the probability it is a California household?

Solution.

(a) The proportion of all California households earn more than $250,000 per year is $12\% \times 3.3\% = 3.96 \times 10^{-3}$. We know that 1.3 percent of all U.S. households earn more than $250,000 per year. Therefore, the proportion of all non-California households earn more than $250,000 per year is: $\frac{0.013 - 3.96 \times 10^{-3}}{0.88} = 0.01027$, implying a total of 1.027 percent of all non-California households earn more than $250,000 per year.

(b) Let C be California households, NC be non-California households, G be households' income which is more than $250,000 and L be households' income which is less than $250,000. Applying Bayes' Theorem,

$$\mathbb{P}(C|G) = \frac{\mathbb{P}(C) \cdot \mathbb{P}(G|C)}{\mathbb{P}(C) \cdot \mathbb{P}(G|C) + \mathbb{P}(NC) \cdot \mathbb{P}(G|NC)} = \frac{0.033 \cdot 0.12}{0.033 \cdot 0.12 + 0.01027 \cdot 0.88} \approx 0.3047.$$

□

Problem 52. Suppose that an insurance company classifies people into one of three classes: good risks, average risks, and bad risks. The company's records indicate that the probabilities that good-, average-, and bad-risk persons will be involved in an accident over a 1-year span are, respectively, .05, .15, and .30. If 20 percent of the population is a good risk, 50 percent an average risk, and 30 percent a bad risk, what proportion of people have accidents in a fixed year? If policyholder A had no accidents in 2012, what is the probability that he or she is a good risk? is an average risk?

Solution. Consider the following events:

A : A randomly chosen policyholder have accidents in a fixed year;
R_1 : A randomly chosen person is good risk;
R_2 : A randomly chosen person is average risk;
R_3 : A randomly chosen person is bad risk.

The proportion of people having accidents in a fixed year is:

$$\mathbb{P}(A) = 0.2 \times 0.05 + 0.5 \times 0.15 + 0.3 \times 0.3 = 0.175.$$

If policyholder A had no accidents in 2012, the probability that he or she is a good risk is:

$$\mathbb{P}(R_1|A^c) = \frac{\mathbb{P}(R_1) \cdot \mathbb{P}(A^c|R_1)}{\mathbb{P}(A^c)} = \frac{0.2 \cdot 0.95}{1 - 0.175} \approx 0.2303.$$

The probability that he or she is an average risk is:

$$\mathbb{P}(R_2|A^c) = \frac{\mathbb{P}(R_2) \cdot \mathbb{P}(A^c|R_2)}{\mathbb{P}(A^c)} = \frac{0.5 \cdot 0.85}{1 - 0.175} \approx 0.51515.$$

□

Problem 53. A worker has asked her supervisor for a letter of recommendation for a new job. She estimates that there is an 80 percent chance that she will get the job if she receives a strong recommendation, a 40 percent chance if she receives a moderately good recommendation, and a 10 percent chance if she receives a weak recommendation. She further estimates that the probabilities that the recommendation will be strong, moderate, and weak are .7, .2, and .1, respectively.

(a) How certain is she that she will receive the new job offer? Consider the following events:

(b) Given that she does receive the offer, how likely should she feel that she received a strong recommendation? a moderate recommendation? a weak recommendation?

(c) Given that she does not receive the job offer, how likely should she feel that she received a strong recommendation? a moderate recommendation? a weak recommendation?

Solution.

(a) Consider the following events:

R : She does receive the offer;
S : A letter of recommendation is good;
M : A letter of recommendation is moderately good;
W : A A letter of recommendation is weak.

The probability that she will receive the new job offer is:

$$\mathbb{P}(R) = 0.8 \times 0.7 + 0.4 \times 0.2 + 0.1 \times 0.1 = 0.65.$$

(b) The probability that she received a strong recommendation given that she does receive the job offer is:

$$\mathbb{P}(S|R) = \frac{\mathbb{P}(R|S) \cdot \mathbb{P}(S)}{\mathbb{P}(R)} = \frac{0.8 \cdot 0.7}{0.65} \approx 0.8615.$$

The probability that she received a moderately strong recommendation given that she does receive the job offer is:

$$\mathbb{P}(M|R) = \frac{\mathbb{P}(M|R) \cdot \mathbb{P}(M)}{\mathbb{P}(R)} = \frac{0.4 \cdot 0.2}{0.65} \approx 0.1231.$$

The probability that she received a weak recommendation given that she does receive the job offer is:

$$\mathbb{P}(W|R) = \frac{\mathbb{P}(W|R) \cdot \mathbb{P}(W)}{\mathbb{P}(R)} = \frac{0.1 \cdot 0.1}{0.65} \approx 0.0154.$$

(c) The probability that she received a strong recommendation given that she does not receive the job offer is:

$$\mathbb{P}(S|R^c) = \frac{\mathbb{P}(R^c|S) \cdot \mathbb{P}(S)}{\mathbb{P}(R^c)} = \frac{(1 - 0.8) \cdot 0.7}{1 - 0.65} = 0.4.$$

The probability that she received a moderately strong recommendation given that she does not receive the job offer is:

$$\mathbb{P}(M|R^c) = \frac{\mathbb{P}(R^c|M) \cdot \mathbb{P}(M)}{\mathbb{P}(R^c)} = \frac{(1 - 0.4) \cdot 0.2}{1 - 0.65} \approx 0.3429.$$

The probability that she received a weak recommendation given that she does not receive the job offer is:

$$\mathbb{P}(W|R^c) = \frac{\mathbb{P}(R^c|W) \cdot \mathbb{P}(W)}{\mathbb{P}(R^c)} = \frac{(1-0.1) \cdot 0.1}{1-0.65} \approx 0.2571.$$

□

2. Chapter 4

2.1. Problems

Problem 1. Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings. What are the possible values of X, and what are the probabilities associated with each value?

Solution. We list all possible outcomes for X as in the following table. Here W, B, O stand for white, black and orange ball respectively.

Outcome	X	Probability
(W, W)	-2	4/13
(W, B)	1	32/91
(W, O)	-1	16/91
(B, B)	4	6/91
(B, O)	2	8/91
(O, O)	0	1/91

It yields the distribution of X as

x	-2	-1	0	1	2	4
$\mathbb{P}(X = x)$	4/13	16/91	1/91	32/91	8/91	6/91

□

Problem 6. Let X represent the difference between the number of heads and the number of tails obtained when a fair coin is tossed 3 times. What are the probabilities associated with the values that X can take on?

Solution. Let Y be the number of heads occur in 3 tosses, then $Y \sim \text{Bino}(3, 0.5)$. The probability mass function of Y is given by

$$\mathbb{P}(Y = k) = \binom{3}{k} 0.5^k \cdot 0.5^{3-k}, \forall k = \overline{0, 3}.$$

We list all possible outcomes for Y as in the following table.

Y	X	Probability
0	3	0.125
1	1	0.375
2	1	0.375
3	3	0.125

It yields the distribution of X as

x	1	3
$\mathbb{P}(X = x)$	0.75	0.25

□

Problem 13. A salesman has scheduled two appointments to sell vacuum cleaners. His first appointment will lead to a sale with probability .3, and his second will lead independently to a sale with probability .6. Any sale made is equally likely to be either for the deluxe model, which costs \$1000, or the standard model, which costs \$500. Determine the probability mass function of X, the total dollar value of all sales.

Solution. We list all possible outcomes for X as in the following table. Here N, S, D stand for no sale, standard model sold and deluxe model sold respectively.

Outcome	X	Probability
NN	0	0.28
NS	500	0.21
SN	500	0.06
ND	1000	0.21
DN	1000	0.06
SS	1000	0.045
SD	1500	0.045
DS	1500	0.045
DD	2000	0.045

It yields the distribution of X as

x	0	500	1000	1500	2000
$\mathbb{P}(X = x)$	0.28	0.27	0.315	0.09	0.045

□

Problem 17. Suppose that the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ b/4 & 0 \leq b < 1 \\ 1/2 + (b-1)/4 & 1 \leq b < 2 \\ 11/12 & 2 \leq b < 3 \\ 1 & b \geq 3 \end{cases}$$

Determine (a) $\mathbb{P}(X = i), i = \overline{1,3}$ and (b) $\mathbb{P}(1/2 < X < 3/2)$.

Solution.

(a) By the continuity of a probability measure,

$$\begin{aligned}\mathbb{P}(X=1) &= \lim_{n\to\infty} \mathbb{P}\left(1 - \frac{1}{2n} < X \leq 1 + \frac{1}{2n}\right) = \lim_{n\to\infty}\left(F\left(1+\frac{1}{2n}\right) - F\left(1 - \frac{1}{2n}\right)\right) \\ &= \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{8n} - \frac{1}{4} + \frac{1}{8n}\right) = \lim_{n\to\infty}\left(\frac{1}{4} + \frac{1}{4n}\right) = \frac{1}{4}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{P}(X=2) &= \lim_{n\to\infty} \mathbb{P}\left(2 - \frac{1}{2n} < X \leq 2 + \frac{1}{2n}\right) = \lim_{n\to\infty}\left(F\left(2+\frac{1}{2n}\right) - F\left(2 - \frac{1}{2n}\right)\right) \\ &= \lim_{n\to\infty}\left(\frac{11}{12} - \frac{1}{2} - \frac{1}{8n}\right) = \frac{11}{12} - \frac{1}{2} = \frac{5}{12}\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(X=3) &= \lim_{n\to\infty} \mathbb{P}\left(3 - \frac{1}{2n} < X \leq 3 + \frac{1}{2n}\right) = \lim_{n\to\infty}\left(F\left(3+\frac{1}{2n}\right) - F\left(3 - \frac{1}{2n}\right)\right) \\ &= \lim_{n\to\infty}\left(1 - \frac{11}{12}\right) = 1 - \frac{11}{12} = \frac{1}{12}.\end{aligned}$$

(b) By the continuity of a probability measure,

$$\begin{aligned}\mathbb{P}\left(\frac{1}{2} < X < \frac{3}{2}\right) &= \lim_{n\to\infty} \mathbb{P}\left(\frac{1}{2} < X \leq \frac{3}{2} + \frac{1}{4n}\right) = \lim_{n\to\infty}\left(F\left(\frac{3}{2}+\frac{1}{4n}\right) - F\left(\frac{1}{2}\right)\right) \\ &= \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{8} + \frac{1}{16n} - \frac{1}{8}\right) = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{16n}\right) = \frac{1}{2}.\end{aligned}$$

$\square$

Problem 18. Four independent flips of a fair coin are made. Let X denote the number of heads obtained. Plot the probability mass function of the random variable $Z = X - 2$.

Solution. Note that $X \sim \text{Bino}(4, 0.5)$, the probability mass function of Z is given by

$$\mathbb{P}(Z = k) = \mathbb{P}(X = k + 2) = \binom{4}{k+2} 0.5^{k+2} \cdot 0.5^{2-k}, \forall k = \overline{-2, 2}.$$

$\square$

Problem 19. If the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ 1/2 & 0 \le b < 1 \\ 3/5 & 1 \le b < 2 \\ 4/5 & 2 \le b < 3 \\ 9/10 & 3 \le b < 3.5 \\ 1 & b \ge 3.5 \end{cases}$$

calculate the probability mass function of X.

Solution. The probability mass function of X is given by

x	0	1	2	3	3.5
$\mathbb{P}(X = x)$	1/2	1/10	1/5	1/10	1/10

$\square$

Problem 21. Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students who were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

(a) Which of $\mathbb{E}[X]$ or $\mathbb{E}[Y]$ do you think is larger? Why?

(b) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

Solution.

(a) In the first sampling method, we choose one of 148 students randomly. The probability of choosing a student from the fullest bus is higher than the probability of picking a student from the other buses. Thus, the fuller buses are weighted higher than the other buses, so we would expect $\mathbb{E}[X]$ to be larger. In the second method, one of the four bus drivers is randomly selected. Each bus driver is equally likely to be chosen! Each bus is weighted the same in this case, therefore, $\mathbb{E}[Y] < \mathbb{E}[X]$. Equality holds when each bus contains the same number of students.

(b) Using the probability mass function of X,

$$\mathbb{E}[X] = 40 \cdot \frac{40}{148} + 33 \cdot \frac{33}{148} + 25 \cdot \frac{25}{148} + 50 \cdot \frac{50}{148} \approx 39.2837.$$

As Y takes on any of the given values with equal chance,

$$\mathbb{E}[Y] = 40 \cdot \frac{1}{4} + 33 \cdot \frac{1}{4} + 25 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = 37.$$

$\square$

Problem 22. Suppose that two teams play a series of games that ends when one of them has won i games. Suppose that each game played is, independently, won by team A with probability p. Find the expected number of games that are played when (a) $i = 2$ and (b) $i = 3$. Also, show in both cases that this number is maximized when $p = 1/2$.

Solution. Denote X the expected number of games that are played.

(a) When $i = 2$, we list all possible outcomes for team A as in the following table.

Outcome	X	Probability
WW	2	p^2
LL	2	$(1-p)^2$
WLW	3	$p^2(1-p)$
WLL	3	$p(1-p)^2$
LWW	3	$p^2(1-p)$
LWL	3	$p(1-p)^2$

It yields the distribution of X as

x	2	3
$\mathbb{P}(X = x)$	$p^2 + (1-p)^2$	$2p(1-p)$

implying

$$\mathbb{E}(X) = 2p^2 + 2(1-p)^2 + 6p(1-p) = 2 + 2p(1-p).$$

Note that

$$2 \cdot \mathbb{E}(X) - 4 = 4p(1-p) \leq (p + 1 - p)^2 = 1,$$

the maximum value of $\mathbb{E}(X)$ is $5/2$, obtained at $p = 1 - p = 1/2$.

(b) Similarly when $i = 3$, the distribution of X is

x	3	4	5
$\mathbb{P}(X = x)$	$p^3 + (1-p)^3$	$3p(1-p)[p^2 + (1-p)^2]$	$6p^2(1-p)^2$

implying $\mathbb{E}(X) = 3 + 3p(1-p) + 6p^2(1-p)^2$. Note that

$$\frac{8}{3} \cdot \mathbb{E}(X) - 8 = 8p(1-p) + 16p^2(1-p)^2 \leq 2(p + 1 - p)^2 + (p + 1 - p)^4 = 3,$$

the maximum value of $\mathbb{E}(X)$ is $33/8$, obtained at $p = 1 - p = 1/2$.

□

Problem 23. You have \$1000, and a certain commodity presently sells for \$2 per ounce. Suppose that after one week the commodity will sell for either \$1 or \$4 an ounce, with these two possibilities being equally likely.

(a) If your objective is to maximize the expected amount of money that you possess at the end of the week, what strategy should you employ?

(b) If your objective is to maximize the expected amount of the commodity that you possess at the end of the week, what strategy should you employ?

Solution.

(a) The random variable X of the amount of money at the end of the week satisfies

$$\mathbb{P}(X = 1000 - 2k + k) = \mathbb{P}(X = 1000 - 2k + 4k) = \frac{1}{2}.$$

So the expectation of it is

$$\mathbb{E}[X] = \frac{1}{2}(1000 - 2k + k + 1000 - 2k + 4k) = 1000 + \frac{k}{2}.$$

To maximize it, we use all the money to buy 500 ounce of commodity at the beginning of the week and sell all of it at the end.

(b) To maximize the commodity, we should buy $0 \le k \le 500$ ounce of the commodity at the beginning of the week and buy it with the remaining money at the end. Then the random variable Y of the amount of commodity at the end of the week satisfies

$$\mathbb{P}(Y = k + (1000 - 2k)/1) = \mathbb{P}(Y = k + (1000 - 2k)/4) = \frac{1}{2}.$$

So the expectation of it is

$$\mathbb{E}[Y] = \frac{1}{2}(k + (1000 - 2k)/1 + k + (1000 - 2k)/4) = 625 - \frac{k}{4}.$$

To maximize it, we should let $k = 0$ and use all the money to buy commodity at the end of the week.

□

Problem 27. An insurance company writes a policy to the effect that an amount of money a must be paid if some event E occurs within a year. If the company estimates that E will occur within a year with probability p, what should it charge the customer in order that its expected profit will be 10 percent of a?

Solution. Let $c > 0$ be the charge fee and X be the profit. The distribution of X is given by

x	$c - a$	c
$\mathbb{P}(X = x)$	p	$1 - p$

implying

$$\frac{a}{10} = \mathbb{E}(X) = p \cdot (c - a) + (1 - p) \cdot c = c - p \cdot a.$$

Hence the desired charge fee is $c = a \cdot (p + 0.1)$. □

Problem 35. A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win -\$1 (i.e. you lose \$1). Calculate

(a) the expected value of the amount you win;

(b) the variance of the amount you win.

Solution.

(a) The probability of choosing two marbles having the same color is 4/9, the probability of choosing two marbles having different colors is 5/9.

$$\mathbb{E}[X] = \frac{4}{9} \cdot 1.1 + \frac{5}{9} \cdot (-1) \approx -0.0667.$$

(b) We have

$$\mathbb{E}[X^2] = \frac{4}{9}(1.1)^2 + \frac{5}{9}(-1)^2 \approx 1.0933.$$

Hence,

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1.0933 - (-0.0667)^2 \approx 1.0889.$$

□

Problem 38. Find $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$ where X, Y are given in **Problem 21**.

Solution.

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 40^2 \cdot \frac{40}{148} + 33^2 \cdot \frac{33}{148} + 25^2 \cdot \frac{25}{148} + 50^2 \cdot \frac{50}{148} - 39.2837^2 \approx 82.2098.$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = 40^2 \cdot \frac{1}{4} + 33^2 \cdot \frac{1}{4} + 25^2 \cdot \frac{1}{4} + 50^2 \cdot \frac{1}{4} - 37^2 = 84.5.$$

□

Problem 39. If $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 5$, find (a) $\mathbb{E}((X+2)^2)$ and (b) $\mathrm{Var}(3X+4)$.

Solution.

(a) Note that

$$\mathbb{E}[X^2] = \mathrm{Var}(X) + \mathbb{E}[X]^2 = 5 + 1^2 = 6$$

and therefore by the linearity of expectation,

$$\mathbb{E}[(X+2)^2] = \mathbb{E}[X^2 + 4X + 4] = \mathbb{E}[X^2] + 4 \cdot \mathbb{E}[X] + 4 = 6 + 4 \cdot 1 + 4 = 14.$$

(b) By the linearity of expectation,

$$\mathbb{E}[3X+4] = 3 \cdot \mathbb{E}[X] + 4 = 3 \cdot 1 + 4 = 7$$

and

$$\mathbb{E}[(3X+4)^2] = \mathbb{E}[9X^2 + 24X + 16] = 9 \cdot \mathbb{E}[X^2] + 24 \cdot \mathbb{E}[X] + 16 = 9 \cdot 6 + 24 \cdot 1 + 16 = 94,$$

implying

$$\mathrm{Var}(3X+4) = \mathbb{E}[(3X+4)^2] - \mathbb{E}[3X+4]^2 = 94 - 7^2 = 45.$$

□

Problem 42. A man claims to have extrasensory perception. As a test, a fair coin is flipped 10 times and the man is asked to predict the outcome in advance. He gets 7 out of 10 correct. What is the probability that he would have done at least this well if he did not have ESP?

Solution. Since the probability that he will predict correctly the outcome is 0.5, $X \sim \mathrm{Bino}(10, 0.5)$. The probability mass function of X is given by

$$\mathbb{P}(X = k) = \binom{10}{k} 0.5^k \cdot 0.5^{10-k}, \forall k = \overline{0, 10}.$$

Hence, the probability of correctly guessing 7 or more flips is

$$\mathbb{P}(X \geq 7) = \mathbb{P}(X = 7) + \mathbb{P}(X = 8) + \mathbb{P}(X = 9) + \mathbb{P}(X = 10) \approx 0.171875.$$

□

Problem 43. A and B will take the same 10-question examination. Each question will be answered correctly by A with probability 0.7, independently of her results on other questions. Each question will be answered correctly by B with probability 0.4, independently both of her results on the other questions and on the performance of A.

(a) Find the expected number of questions that are answered correctly by both A and B.

(b) Find the variance of the number of questions that are answered correctly by either A or B.

Solution.

(a) Because they're independent, the probability of A and B both getting a particular question correct is:

$$\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B) = (0.7)(0.4) = 0.28.$$

Let X denote the number of questions both students answer correctly. X is given by a binomial distribution of 10 trials, each with a probability of 0.28, giving us: $X \sim \mathrm{Bino}(10, 0.28)$, hence

$$\mathbb{E}[X] = 10 \times 0.28 = 2.8.$$

(b) The variance of the number of questions that are answered correctly by either A or B is:

$$\mathrm{Var}(X) = np(1-p) = 10(0.28)(1-0.28) = 2.016.$$

$\square$

Problem 44. A communications channel transmits the digits 0 and 1. However, due to static, the digit transmitted is incorrectly received with probability .2. Suppose that we want to transmit an important message consisting of one binary digit. To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1. If the receiver of the message uses "majority" decoding, what is the probability that the message will be wrong when decoded? What independence assumptions are you making?

Solution. Let random variable X denote the number of binary digits that are in error, if individual errors are independent, then $X \sim \text{Bino}(5, 0.2)$. The probability mass function of X is given by

$$\mathbb{P}(X = k) = \binom{5}{k} 0.2^k \cdot 0.8^{5-k}, \forall k = \overline{0, 5}.$$

Hence,

$$\mathbb{P}(X = 3) = 0.0512, \mathbb{P}(X = 4) = 0.0064, \mathbb{P}(X = 5) = 0.00032$$

, so

$$\mathbb{P}(3 \leq X \leq 5) = P(X = 3) + P(X = 4) + P(X = 5) = 0.05792$$

$\square$

Problem 49. It is known that diskettes produced by a certain company will be defective with probability .01, independently of one another. The company sells the diskettes in packages of size 10 and offers a money-back guarantee that at most 1 of the 10 diskettes in the package will be defective. The guarantee is that the customer can return the entire package of diskettes if he or she finds more than 1 defective diskette in it. If someone buys 3 packages, what is the probability that he or she will return exactly 1 of them?

Solution. Let X be the number of defective diskettes in one package, then $X \sim \text{Bino}(10, 0.01)$. The probability mass function of X is given by

$$\mathbb{P}(X = k) = \binom{10}{k} 0.01^k \cdot 0.99^{10-k}, \forall k = \overline{0, 10}.$$

Hence, the probability that more than 1 diskette is defective is given by:

$$\mathbb{P}(X > 1) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) \approx 0.004.$$

Now let Y be the number of packages returned, then since a package being returned is independent of any other, $Y \sim \text{Bino}(3, 0.004)$. The probability mass function of X is given by

$$\mathbb{P}(X = k) = \binom{3}{k} 0.004^k \cdot 0.996^{3-k}, \forall k = \overline{0, 3}.$$

Hence, the probability that he or she will return exactly 1 of them is $\mathbb{P}(Y = 1) \approx 0.012$. $\square$

Problem 50. When coin 1 is flipped, it lands on heads with probability 0.4; when coin 2 is flipped, it lands on heads with probability 0.7. One of these coins is randomly chosen and flipped 10 times.

(a) What is the probability that the coin lands on heads on exactly 7 of the 10 flips?

(b) Given that the first of these 10 flips lands heads, what is the conditional probability that exactly 7 of the 10 flips land on heads?

Solution. Consider an experiment of tossing the chosen coin one time, with the following associated events:

H : The coin lands on head;
F : Coin 1 is chosen.

Then by the Total Probability formula,

$$\mathbb{P}(H) = \mathbb{P}(H|F) \cdot \mathbb{P}(F) + \mathbb{P}(H|F^c) \cdot \mathbb{P}(F^c) = 0.4 \cdot 0.5 + 0.7 \cdot 0.5 = 0.55.$$

(a) Let X_1 be the number of head occurs in 10 flips, then $X_1 \sim \text{Bino}(10, 0.55)$. The probability mass function of X_1 is given by

$$\mathbb{P}(X_1 = k) = \binom{10}{k} 0.55^k \cdot 0.45^{10-k}, \forall k = \overline{0, 10}.$$

Hence $\mathbb{P}(X_1 = 7) = 0.166$.

(b) Let A be the event that the first toss lands on head and X_2 be the number of head occurs in the remaining 9 flips, then $X_2 \sim \text{Bino}(9, 0.55)$. The probability mass function of X_2 is given by

$$\mathbb{P}(X_2 = k) = \binom{9}{k} 0.55^k \cdot 0.45^{9-k}, \forall k = \overline{0, 9}.$$

Hence $\mathbb{P}(X_2 = 6) = 0.212$ and since the tosses are independent, $\{X_2 = 6\}$ and A are independent and thus

$$\mathbb{P}(X_1 = 7|A) = \mathbb{P}(X_2 = 6|A) = \mathbb{P}(X_2 = 6) = 0.212.$$

□

Problem 54. The expected number of typographical errors on a page of a certain magazine is .2. What is the probability that the next page you read contains (a) 0 and (b) 2 or more typographical errors? Explain your reasoning!

Solution. Let X be the number of typographical errors on the next page, then $X \sim \text{Pois}(0.2)$. The probability mass function of X is given by

$$\mathbb{P}(X = k) = \frac{0.2^k \cdot e^{-0.2}}{k!}, \forall k \in \mathbb{N}.$$

Hence $\mathbb{P}(X = 0) = 0.819, \mathbb{P}(X = 1) = 0.164$ and

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) = 0.017.$$

□

Problem 55. The monthly worldwide average number of airplane crashes of commercial airlines is 3.5. What is the probability that there will be

(a) at least 2 such accidents in the next month?

(b) at most 1 accident in the next month?

Solution. Let X be the number of airline accidents next month, then $X \sim \text{Pois}(3.5)$. The probability mass function of X is given by

$$\mathbb{P}(X = k) = \frac{3.5^k \cdot e^{-3.5}}{k!}, \forall k \in \mathbb{N}.$$

Hence $\mathbb{P}(X = 0) = 0.030, \mathbb{P}(X = 1) = 0.106$ and

(a) $\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) = 0.864$;

(b) $\mathbb{P}(X \leq 1) = 1 - \mathbb{P}(X > 1) = 1 - \mathbb{P}(X \geq 2) = 0.136$.

$\square$

Problem 57. Suppose that the average number of cars abandoned weekly on a certain highway is 2.2. Approximate the probability that there will be

(a) no abandoned cars in the next week;

(b) at least 2 abandoned cars in the next week.

Solution. Let X be the number of cars abandoned next week, then $X \sim \text{Pois}(2.2)$. The probability mass function of X is given by

$$\mathbb{P}(X = k) = \frac{2.2^k \cdot e^{-2.2}}{k!}, \forall k \in \mathbb{N}.$$

Hence $\mathbb{P}(X = 0) = 0.111, \mathbb{P}(X = 1) = 0.244$ and

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) = 0.645.$$

$\square$

Problem 60. Suppose that the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$.

(a) Find the probability that 3 or more accidents occur today.

(b) Repeat part (a) under the assumption that at least 1 accident occurs today.

Solution.

(a) The probability mass function of X is given by

$$\mathbb{P}(X = k) = \frac{3^k \cdot e^{-3}}{k!}, \forall k \in \mathbb{N}.$$

Hence $\mathbb{P}(X = 0) = 0.050, \mathbb{P}(X = 1) = 0.149, \mathbb{P}(X = 2) = 0.224$ and

$$\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) = 0.577.$$

(b) Given that $X \geq 1$, the conditional probability is

$$\mathbb{P}(X \geq 3 | X \geq 1) = \frac{\mathbb{P}(X \geq 3)}{\mathbb{P}(X \geq 1)} = \frac{\mathbb{P}(X \geq 3)}{1 - \mathbb{P}(X = 0)} = 0.607.$$

$\square$

Problem 82. Suppose that a batch of 100 items contains 6 that are defective and 94 that are not defective. If X is the number of defective items in a randomly drawn sample of 10 items from the batch, find (a) $\mathbb{P}(X = 0)$ and (b) $\mathbb{P}(X > 2)$.

Solution. Since the probability that a randomly drawn item being defective is 0.06, $X \sim \text{Bino}(10, 0.06)$. The probability mass function of X is given by

$$\mathbb{P}(X = k) = \binom{10}{k} 0.06^k \cdot 0.94^{10-k}, \forall k = \overline{0, 10}.$$

Hence $\mathbb{P}(X = 0) = 0.359, \mathbb{P}(X = 1) = 0.344, \mathbb{P}(X = 2) = 0.099$ and

$$\mathbb{P}(X > 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) = 0.019.$$

$\square$