MIDTERM EXAMINATION
April 2017
Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Deputy Head of Dept. of Mathematics:	Lecturer:
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INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 *(25 marks)* Let $f : [a, b] \to \mathbb{R}$ be a continuous function ($a < b$). Show that the following set is closed in the Euclidean space $\mathbb{R}^2$

$$E = \{(x, y) \in \mathbb{R}^2 : a \le x \le b,\ y \ge f(x)\}.$$

Question 2 *(25 marks)* Let A be a nonempty closed bounded set in the Euclidean space $\mathbb{R}^n$. Show that there exists $\mathbf{x}_0 \in A$ such that $|\mathbf{x}_0| = \max_{\mathbf{x}\in A} |\mathbf{x}|$, where

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \quad \text{if} \quad \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Question 3 *(10 marks)* Determine which of the following sets is open in the Euclidean space $\mathbb{R}^2$. Explain your choice.

(a) $A = \{(x, y) \in \mathbb{R}^2 : y \le x^2\}$.

(b) $B = \{(x, y) \in \mathbb{R}^2 : 0 < y \le x^2 + 1\}$.

(c) $C = \{(x, y) \in \mathbb{R}^2 : -1 < x + y < 1\}$.

Question 4 *(20 marks)* Let X, Y be nonempty sets and let $f : X \to Y$ be a mapping. Let $\mathcal{A}$ be a σ-algebra in X. Show that the collection

$$\mathcal{B} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$$

is a σ-algebra over Y.

Question 5 *(20 marks)* Let $(X, \mathcal{A})$ be a measurable space. Let μ, ν be measures on $\mathcal{A}$ and c_1, c_2 be positive real numbers. Show that the function $\lambda = c_1\mu + c_2\nu$ is also a measure on $\mathcal{A}$.

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SOLUTIONS

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Question 1 Suppose $\{(x_n, y_n)\} \subset E$, $(x_n, y_n) \to (x_0, y_0) \in \mathbb{R}^2$. Then we have $x_n \to x_0$, $y_n \to y_0$, and $y_n \geq f(x_n)$ for all n. As $a \leq x_n \leq b$, we must have $a \leq x_0 \leq b$. Furthermore, since f is continuous at x_0, $f(x_n) \to f(x_0)$. It follows that $y_0 = \lim y_n \geq \lim f(x_n) = f(x_0)$, that is, $(x_0, y_0) \in E$. Consequently, E is closed.

Question 2 Since A is a closed and bounded set in $\mathbb{R}^n$, A is compact. The norm $|\cdot|$ is continuous on $\mathbb{R}^n$, hence it attains its maximum value on A, that is, there is $\mathbf{x}_0 \in A$ with $|\mathbf{x}_0| = \max_{\mathbf{x}\in A} |\mathbf{x}|$.

Question 3 A is not open since $(0,0) \in A$ but for all $r > 0$, $(0, r/2) \in B((0,0), r)$ and $(0, r/2) \notin A$.

B is not open because $(0,1) \in B$ but for all $r > 0$, $(0, 1 + r/2) \in B((0,1), r)$ and $(0, 1 + r/2) \notin B$.

C is open because the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x+y$ is continuous and C is the inverse image of an open set: $C = \{(x,y) \in \mathbb{R}^2 : x + y \in (-1,1)\} = f^{-1}(-1,1)$.

There are other several ways to prove openness of C.

Question 4 We have $f^{-1}(Y) = X \in \mathcal{A}$, so $Y \in \mathcal{B}$. If $B \in \mathcal{B}$, then $f^{-1}(B) \in \mathcal{A}$ and hence $f^{-1}(B^c) = X \setminus f^{-1}(B) \in \mathcal{A}$, that is, $B^c \in \mathcal{B}$. Finally, if $\{B_n\} \subset \mathcal{B}$, we have $f^{-1}(B_n) \in \mathcal{A}$ for all n. Thus $f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(B_n) \in \mathcal{A}$, that is, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$. Therefore $\mathcal{B}$ is a σ-algebra.

Question 5 Since μ and ν are measures, $\lambda(A) = c_1\mu(A) + c_2\nu(A) \geq 0$ for every $A \in \mathcal{A}$ and in particular, $\lambda(\emptyset) = c_1\mu(\emptyset) + c_2\nu(\emptyset) = 0$. Suppose that $\{A_n\} \subset \mathcal{A}$ is a disjoint sequence. σ-additivity of μ and ν gives,

$$\begin{aligned}
\lambda\left(\bigcup_{n=1}^{\infty} B_n\right) &= c_1\mu\left(\bigcup_{n=1}^{\infty} B_n\right) + c_2\nu\left(\bigcup_{n=1}^{\infty} B_n\right) = c_1\sum_{n=1}^{\infty}\mu(B_n) + c_2\sum_{n=1}^{\infty}\nu(B_n) \\
&= \sum_{n=1}^{\infty} c_1\mu(B_n) + \sum_{n=1}^{\infty} c_2\nu(B_n) = \sum_{n=1}^{\infty}\left[c_1\mu(B_n) + c_2\nu(B_n)\right] \\
&= \sum_{n=1}^{\infty}\lambda(B_n).
\end{aligned}$$

Thus λ is σ-additive and hence λ is a measure on $\mathcal{A}$.