EERCISES FOR CHAPTER 4: CONSTRAINED PROBLEMS

Exercises for everyone: All exercises in parts **A** and **B**.

A. Non-assessed Exercises (corrected in class):

0.1.1; 0.1.3; 0.1.11; 0.2.5; 0.2.6; 0.2.9; 0.2.11 (a); 0.3.1; 0.3.3; 0.3.5; 0.4.3; 0.4.5.

B. Assessed Assignments (to be submitted):

0.1.2; 0.1.4; 0.1.6; 0.1.7; 0.1.10; 0.1.13; 0.1.15–0.1.18;[1] 0.2.1; 0.2.2; 0.2.3; 0.2.7; 0.2.8; 0.2.10; 0.2.11 (b); 0.2.13; 0.2.15; 0.3.4; 0.3.6; 0.4.1; 0.4.2; 0.4.4.

C. Bonus Exercises: Remaining exercises.

0.1 FIRST AND SECOND-ORDER CONDITIONS (EQUALITY CONSTRAINTS)

Exercise 0.1.1. Consider the problem

$$\begin{array}{ll} \text{minimize} & f(x_1, x_2) = 2x_1^2 - x_2^2 \\ \text{subject to} & h(x_1, x_2) = x_1^2 x_2 - x_2^3 = 0. \end{array}$$

Show that the point $\mathbf{x}^* = (0, 0)$ is a global minimizer but $\mathbf{x}^*$ is not regular.

Exercise 0.1.2. Solve problem

$$\begin{array}{ll} \text{minimize} & x^2 + 4y^2 + 16z^2 \\ \text{subject to} & xyz = 1. \end{array}$$

ANS. Global minimizers

$$(2, 1, 1/2), \quad (2, -1, -1/2), \quad (-2, 1, -1/2), \quad (-2, -1, 1/2).$$

[1] This means all exercises from 0.1.15 to 0.1.18.

Exercise 0.1.3. Find the points of $\mathbb{R}^3$ lying on the surface defined by $x^4+y^4+z^4 = 1$ with smallest and largest distance from the origin.

(*Hint:* In problems involving the minimization of a norm or of a distance it is often simpler to minimize the square of the norm or distance.)

ANS. Points with shortest distance:

$$(\pm 1, 0, 0), \quad (0, \pm 1, 0), \quad (\pm 1, 0, 0), \quad \text{distance} = 1.$$

Points with farthest distance:

$$\left(\alpha\frac{1}{\sqrt[4]{3}}, \beta\frac{1}{\sqrt[4]{3}}, \gamma\frac{1}{\sqrt[4]{3}}\right), \quad \text{distance} = \sqrt{3}$$

where $\alpha, \beta, \gamma \in \{\pm 1\}$.

Exercise 0.1.4. Find the distance from a given point $\mathbf{x}_0$ in $\mathbb{R}^n$ to a given hyperplane $\mathbf{a}^T\mathbf{x} = \alpha$.

ANS. $d = |\mathbf{a}^T\mathbf{x}_0 - \alpha|/|\mathbf{a}|$ at the point $\mathbf{x}^* = \mathbf{x}_0 - \frac{\mathbf{a}^T\mathbf{x}_0 - \alpha}{|\mathbf{a}|^2}\mathbf{a}$.

Exercise 0.1.5. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and let H be the hyperplane $H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = \alpha\}$. Find $\mathbf{x} \in H$ such that $f(\mathbf{x}) = |\mathbf{x}_1 - \mathbf{x}|^2 + |\mathbf{x}_2 - \mathbf{x}|^2$ is minimum.

(*Hint:* Apply Exercise 0.1.4 with $\mathbf{x}_0 = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$.)

Exercise 0.1.6. Let $a_1, \ldots, a_n$ be positive numbers. Determine the minimum value that the expression

$$y = \sum_{i=1}^{n} a_i x_i^2$$

under the constraint

$$\sum_{i=1}^{n} x_i = c,$$

where c is another given constant.

ANS. $x_j = c/(aa_j)$, where $a = \sum_{i=1}^{n} \frac{1}{a_i}$.

Exercise 0.1.7. Let $\mathbf{A}$ be a matrix of full row rank. Find the point in the set $\mathbf{A}\mathbf{x} = \mathbf{b}$ which minimizes $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{x}$.

Exercise 0.1.8. Let $\alpha_i \geq 1$, $i = 1, \dots n$, and $\sigma > 0$ be fixed numbers. Solve the problem

$$\begin{aligned} \text{maximize} \quad & \prod_{i=1}^{n} x_i^{\alpha_i} \\ \text{subject to} \quad & \sum_{i=1}^{n} \alpha_i x_i = \sigma \\ & x_i \geq 0. \end{aligned}$$

From the solution of the problem derive the inequality for the arithmetic and the geometric means.

ANS. $x_1^* = \cdots = x_n^* = \sigma / \sum_{i=1}^{n} \alpha_i$.

Exercise 0.1.9. Solve the problem

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} |x_i|^p \\ \text{subject to} \quad & \sum_{i=1}^{n} |x_i|^q = 1,\ x_i \in \mathbb{R}, \end{aligned}$$

where $p > 1$ and $q > 1$ are fixed numbers.

ANS. $\mathbf{x}^* = \pm \mathbf{e}_i$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis in $\mathbb{R}^n$ if $q < p$; $\mathbf{x}^* = (\pm n^{-1/q}, \dots, \pm n^{-1/q})$ if $q > p$; the case $p = q$ is trivial.

Exercise 0.1.10. Let $\mathbf{a} \in \mathbb{R}^n$. Solve the problem

$$\begin{aligned} \text{maximize} \quad & \mathbf{a}^T \mathbf{x} \\ \text{subject to} \quad & |\mathbf{x}| = 1,\ \mathbf{x} \in \mathbb{R}^n. \end{aligned}$$

Use the result to derive the Cauchy-Schwarz Inequality.

ANS. If $\mathbf{a} \neq \mathbf{0}$, then $x_i^* = a_i / |\mathbf{a}|,\ i = 1, \dots, n$.

Exercise 0.1.11. Let $\mathbf{a} \in \mathbb{R}^n$. Solve the problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = \mathbf{a}^T \mathbf{x} \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i = 0 \\ & \sum_{i=1}^{n} x_i^2 = 1. \end{aligned}$$

Exercise 0.1.12. Find the maximum and minimum values of

$$f(x_1, x_2) = \int_{x_1}^{x_2} \frac{1}{1+t^4} dt$$

over the region determined by $x_1^2 x_2^2 = 1$.

Exercise 0.1.13. Determine all maxima and minima of

$$f(x_1, x_2, x_3) = \frac{1}{x_1^2 + x_2^2 + x_3^2}$$

subject to

$$\begin{aligned} h_1(x_1, x_2, x_3) &= 1 - x_1^2 - 2x_2^2 - 3x_3^2 = 0 \\ h_2(x_1, x_2, x_3) &= x_1 + x_2 + x_3 = 0. \end{aligned}$$

Exercise 0.1.14. Determine all maxima and minima of

$$f(x, y, z) = xz + y^2$$

on the sphere $x^2 + y^2 + z^2 = 1$.

Exercise 0.1.15. Show that the problem

$$\begin{aligned} \text{minimize} \quad & f(x, y) = x^2 + y^2 \\ \text{subject to} \quad & (x-2)^3 - y^2 = 0 \end{aligned}$$

admits no Lagrange multipliers and explain why. Solve this problem graphically.

Exercise 0.1.16. Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be coercive functions with continuous first partial derivatives on $\mathbb{R}^n$. Suppose the equations $f(\mathbf{x}) = 1$ and $g(\mathbf{x}) = 1$ define nonintersecting surfaces in $\mathbb{R}^n$. Show that the vector $\mathbf{x}^* - \mathbf{y}^*$ of minimum norm satisfying $f(\mathbf{x}^*) = g(\mathbf{y}^*) = 1$ is perpendicular to both surfaces.

Exercise 0.1.17. Find the point on the ellipse

$$5x^2 - 6xy + 5y^2 = 4$$

for which the tangent line is at a maximum distance from the origin.

State and solve its general problem in $\mathbb{R}^n$.

Exercise 0.1.18. Let $\mathbf{A}$ be a positive definite 3×3-matrix. Let $\mathbf{x}^*$ be the vector of largest norm on the ellipsoid

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = 1.$$

Show that $\mathbf{x}^*$ is an eigenvector of $\mathbf{A}$ and relate $|\mathbf{x}^*|$ to the corresponding eigenvalue of $\mathbf{A}$. State and solve its general problem in $\mathbb{R}^n$.

Exercise 0.1.19. Show that if $\mathbf{A}$ is an $n \times n$-symmetric matrix, then $\mathbf{A}$ has n mutually orthogonal eigenvectors of unit length.

(*Hint:* Prove by induction: if $\mathbf{x}_1, ..., \mathbf{x}_k$, $(1 \leq k \leq n-1)$ are k mutually orthogonal unit eigenvectors, show that the problem

$$\begin{aligned} \text{maximize} \quad & f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} \\ \text{subject to} \quad & h_1(\mathbf{x}) = |\mathbf{x}|^2 - 1 = 0, \\ & h_2(\mathbf{x}) = \mathbf{x}_1^T \mathbf{x} = 0, \\ & \vdots \\ & h_{k+1}(\mathbf{x}) = \mathbf{x}_k^T \mathbf{x} = 0. \end{aligned}$$

has a solution.)

0.2 INEQUALITY CONSTRAINTS

Exercise 0.2.1. Consider the constraint

$$g(\mathbf{x}) = \left(\frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - 1 \right)^3 \geq 0.$$

(a) Determine which feasible points are regular points and which are not.

(b) Consider the constraint $\tilde{g}(\mathbf{x}) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - 1 \geq 0$. Show that the constraint $\tilde{g}(\mathbf{x}) \geq 0$ is equivalent to the constraint $g(\mathbf{x}) \geq 0$. Show also that every feasible point for $\tilde{g}(\mathbf{x}) \geq 0$ is also a regular point of this constraint. Thus regularity is not a result of the geometry of the feasible region, but a result of its algebraic representation.

Exercise 0.2.2. Show that the point $\mathbf{x}^* = \mathbf{0} \in \mathbb{R}^2$ is a global minimum point for the problem

$$\begin{aligned} \text{minimize} \quad & x_1 \\ \text{subject to} \quad & x_2 - x_1^3 \leq 0 \\ & -x_2 - x_1^3 \leq 0. \end{aligned}$$

Show that the KKT conditions have no solution. Explain!

Exercise 0.2.3. Determine the solutions to the KKT conditions for problem

$$\begin{aligned} \text{minimize} \quad & -2x_1 - x_2 \\ \text{subject to} \quad & x_1 + 2x_2 - 6 \le 0 \\ & x_1^2 - 2x_2 = 0. \end{aligned}$$

Which of these solutions represent minimum points? Illustrate the problem graphically.

Exercise 0.2.4. Solve the problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = x_1 + x_2 \\ \text{subject to} \quad & \log x_1 + 4 \log x_2 \ge 1. \end{aligned}$$

(**Note:** $\log t$ is another notation for $\ln t$.)

Exercise 0.2.5. Consider the quadratic problem

$$\begin{aligned} \text{minimize} \quad & q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b}. \end{aligned} \tag{0.2.1}$$

Show that if $\mathbf{d}^T\mathbf{Q}\mathbf{d} \ge 0$ for all directions $\mathbf{d}$ satisfying the condition that $\mathbf{a}_i^T\mathbf{d} \le 0$ for all $i \in I(\mathbf{x}^*)$, where $\mathbf{a}_i$ $(1 \le i \le n)$ are columns of $\mathbf{A}$, then $\mathbf{x}^*$ is a local minimizer of problem (0.2.1).

Exercise 0.2.6. Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{x}^T\mathbf{Q}\mathbf{x} \le 1, \end{aligned}$$

where $\mathbf{Q}$ is a positive-definite symmetric matrix.

(i) Solve the problem. What is the optimal objective value?

(ii) What is the solution when the objective function is to be maximized?

Exercise 0.2.7. Let $\mathbf{Q}$ be an $n \times n$ symmetric matrix.

(i) Find all stationary points of the problem

$$\begin{array}{ll} \text{maximize} & f(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}\mathbf{x} \\ \text{subject to} & \mathbf{x}^T\mathbf{x} = 1. \end{array}$$

Determine which of the stationary points are global maximizers.

(ii) How do your results in part (i) change if the constraint is replaced by $\mathbf{x}^T\mathbf{A}\mathbf{x} \leq 1$, where $\mathbf{A}$ is symmetric and positive definite?

Exercise 0.2.8. Use the optimality conditions to find all local solutions to the problem

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) = x_1 + x_2 \\ \text{subject to} & (x_1 - 1)^2 + x_2^2 \leq 2 \\ & (x_1 + 1)^2 + x_2^2 \geq 2. \end{array}$$

Exercise 0.2.9. We have a rope of length a to tie a box from top to bottom along the two perpendicular directions. What is the maximum volume that such a box can contain?

The mathematical problem is

$$\begin{array}{ll} \text{maximize} & V(x_1, x_2, x_3) = x_1x_2x_3 \\ \text{subject to} & 2x_1 + 2x_2 + 4x_3 \leq a, \\ & x_1, x_2, x_3 \geq 0, \end{array}$$

where x_1, x_2, x_3 are the sides of the boxed and x_3 is the height.

ANS. $x_1 = x_2 = a/6,\ x_3 = a/12.$

Exercise 0.2.10. Let $\mathbf{a} \in \mathbb{R}^n$. Solve the problem

$$\begin{array}{ll} \text{minimize} & |\mathbf{x}|^4 + |\mathbf{x} - \mathbf{a}|^2 \\ \text{subject to} & |\mathbf{x}|^2 \leq 1. \end{array}$$

Exercise 0.2.11. Let $\alpha_1, \ldots, \alpha_k$ be positive numbers and $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ be given points.

(a) Find a point $\mathbf{x}$ lying on the unit sphere in $\mathbb{R}^n$ for which $\sum_{i=1}^k \alpha_i |\mathbf{x} - \mathbf{x}_i|^2$ is smallest.

(b) Find a point $\mathbf{x}$ belonging to the unit ball in $\mathbb{R}^n$ for which $\sum_{i=1}^k \alpha_i |\mathbf{x} - \mathbf{x}_i|^2$ is smallest.

ANS.

$$\text{(a)} \quad \begin{cases} \mathbf{x}^* = \bar{\mathbf{x}}/|\bar{\mathbf{x}}| & \text{if } \bar{\mathbf{x}} \neq \mathbf{0} \\ |\mathbf{x}^*| = 1 & \text{is arbitrary if } \bar{\mathbf{x}} = \mathbf{0}, \end{cases} \qquad \text{(b)} \quad \mathbf{x}^* = \begin{cases} \bar{\mathbf{x}} & \text{if } |\bar{\mathbf{x}}| \leq 1 \\ \bar{\mathbf{x}}/|\bar{\mathbf{x}}| & \text{if } |\bar{\mathbf{x}}| > 1, \end{cases}$$

where $\bar{\mathbf{x}} = \left(\sum_{i=1}^k \alpha_i \mathbf{x}_i\right) / \sum_{i=1}^k \alpha_i$.

Exercise 0.2.12. Consider the problem

$$\text{minimize} \quad \max_{1 \leq i \leq m} f_i(\mathbf{x})$$

This problem is called a **minimax** problem.

(i) Formulate the minimax problem as a constrained optimization problem.

(ii) Use the optimality conditions for the constrained optimization problem to derive the optimality conditions for the minimax problem.

Exercise 0.2.13. Apply the Karush-Kuhn-Tucker Theorem to locate all solutions of the following problem

$$\begin{aligned} \text{minimize} \quad & f(x, y) = e^{-(x+y)} \\ \text{subject to} \quad & e^x + e^y \leq 20, \\ & x \geq 0. \end{aligned}$$

Exercise 0.2.14. Apply the Karush-Kuhn-Tucker Theorem to locate all solutions of the following problem

$$\begin{aligned} \text{minimize} \quad & f(x, y) = x^2 + y^2 - 4x - 4y \\ \text{subject to} \quad & x^2 - y \leq 0, \\ & x + y \leq 2. \end{aligned}$$

Exercise 0.2.15. Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$ be a fixed vector. Suppose the convex program

$$\begin{aligned} \text{minimize} \quad & |\mathbf{x}|^2 \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

has solution $\mathbf{x}^*$ and there is $\bar{\mathbf{x}}$ with $\mathbf{A}\bar{\mathbf{x}} < \mathbf{b}$. Use the Karush-Kuhn-Tucker conditions to show that there is a vector $\mathbf{y}$ in $\mathbb{R}^m$ such that $\mathbf{x}^* = \mathbf{A}^T\mathbf{y}$.

0.3 PENALTY AND BARRIER METHODS

Exercise 0.3.1. Prove the following statement: *If* $g(\mathbf{x})$ *has continuous first partial derivatives on* $\mathbb{R}^n$*, the same is true of* $h(\mathbf{x}) = [g^+(\mathbf{x})]^2$*. Moreover,*

$$\frac{\partial h}{\partial x_i}(\mathbf{x}) = 2g^+(\mathbf{x})\frac{\partial g}{\partial x_i}(\mathbf{x}), \quad i = 1, 2, \ldots, n, \tag{0.3.1}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Exercise 0.3.2. Solve the following problems by the penalty method:

(a) $$\begin{array}{ll} \text{minimize} & (x_1 - 1)^2 + x_2^2, \\ \text{subject to} & x_2 \geq x_1 + 1. \end{array}$$

(b) $$\begin{array}{ll} \text{minimize} & x_1 + x_2, \\ \text{subject to} & x_2 = x_1^2. \end{array}$$

Exercise 0.3.3. Consider the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2, \\ \text{subject to} & x_1 + x_2 \geq 1. \end{array}$$

Suppose that the logarithmic barrier method is used to solve this problem.

Exercise 0.3.4. Consider the one-dimensional problem

$$\begin{array}{ll} \text{minimize} & \dfrac{-1}{x^2 + 1} \\ \text{subject to} & x \geq 1. \end{array}$$

Show that the logarithmic barrier function is unbounded below in the feasible region. Show also that the logarithmic barrier function has a local minimizer that approaches the solution $x^* = 1$ as $\mu \to 0$.

Exercise 0.3.5. Consider the problem

$$\begin{array}{ll} \text{minimize} & f(x_1, x_2) = \frac{1}{3}(x_1 + 1)^3 + x_2, \\ \text{subject to} & g_1(x_1, x_2) = -x_1 + 1 \leq 0 \\ & g_2(x_1, x_2) = -x_2 \leq 0. \end{array}$$

(i) Solve the problem using the penalty function

$$P(\mathbf{x}) = \left(g_1^+(\mathbf{x})\right)^2 + \left(g_2^+(\mathbf{x})\right)^2.$$

(ii) Solve the problem using the barrier function

$$B(\mathbf{x}) = -\frac{1}{g_1(\mathbf{x})} - \frac{1}{g_2(\mathbf{x})}.$$

ANS. (a) $x_1^*(c) = -1 - c + c\sqrt{1+4/c}$, $x_2^*(c) = -1/2c$, $\mathbf{x}^* = (1,0)$, $f_{\min} = 8/3$. (b) $x_1^*(\mu) = \sqrt{1+\sqrt{\mu}}$, $x_2^*(\mu) = \sqrt{\mu}$.

Exercise 0.3.6. Use the Penalty Method with the term $(g^+(x,y))^2$ to solve the following problem

$$\begin{aligned} \text{minimize} \quad & f(x,y) = x^2 + y^2 \\ \text{subject to} \quad & g(x,y) = 1 - x - y \leq 0. \end{aligned}$$

0.4 DUALITY AND DUAL METHODS

Exercise 0.4.1. Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x_1,x_2) = \frac{1}{2}ax_1^2 + \frac{1}{2}x_2^2 + x_1 \\ \text{subject to} \quad & x_1 \geq 0. \end{aligned}$$

Determine the solution to this problem for the cases $a = 1$ and $a = -1$. For each of the two cases, formulate the dual and determine whether its local solution gives the Lagrange multipliers at the optimal primal solution.

Exercise 0.4.2. Given the quadratic problem

$$\begin{aligned} \text{minimize} \quad & (x_1 - 2)^2 + x_2^2 \\ \text{subject to} \quad & 2x_1 + x_2 - 2 \leq 0. \end{aligned}$$

(a) Determine the optimal solution of the primal problem using the dual problem.

(b) Solve the primal problem geometrically.

Exercise 0.4.3. Given the problem

$$\begin{aligned} \text{minimize} \quad & -3x_1 - x_2 \\ \text{subject to} \quad & x_1^2 + 2x_2^2 - 2 \leq 0. \end{aligned}$$

(a) Solve the dual problem,

(b) Use part (a) to determine the solution to the primal problem.

(c) Solve the primal problem geometrically.

Exercise 0.4.4. Consider the problem

$$\begin{aligned} \underset{\mathbf{x}\in\Omega}{\text{minimize}} \quad & f(\mathbf{x}) = \sum_{i=1}^{n} x_i \log(x_i/c_i) \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned}$$

where the constants $c_i,\ i = 1, \ldots, n$, are positive, $\mathbf{A}$ is a matrix of full row rank, and $\Omega = \{\mathbf{x} : \mathbf{x} > \mathbf{0}\}$. What is the dual to this problem? Determine expressions for the first and second derivatives of the dual function.

Exercise 0.4.5. Suppose that $\mathbf{Q}$ is an $n \times n$-positive definite matrix and that $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \neq \mathbf{0}$, $c \in \mathbb{R}$. Consider the quadratic program

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ \text{subject to} \quad & \mathbf{a}^T\mathbf{x} \leq c \quad (c < 0). \end{aligned}$$

Solve the dual problem, then determine the solution to the primal problem.