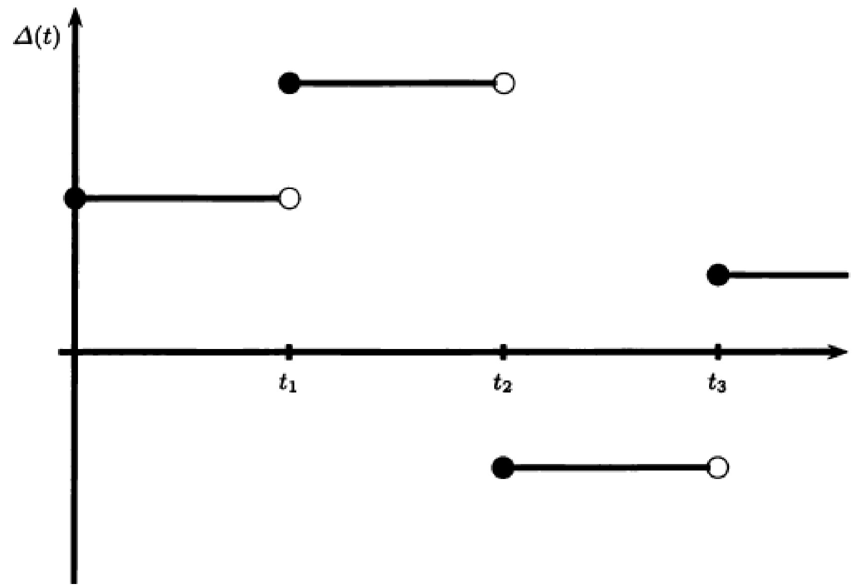

Fig. 4.2.1. A path of a simple process.

Figure 1: Simple process

1 Ito Integrals

We fixed a positive number T and seek to make sense of the integral concept $\int_0^T \Delta(t)dW_t$. The basic ingredients here are a Brownian motion W_t, $t \leq 0$, together with a filtration $\mathcal{F}_t$, for this Brownian motion. We will let the integrand $\Delta(t)$ be an adapted stochastic process. Our reason for doing this is that $\Delta(t)$ will eventually be the position we take in an asset at time t, and this typically depends on the price path of the asset up to time t. The information available at time t is sufficient to evaluate Δ_t at that time. When we are standing at time 0 and $t > 0$, Δ_t is unknown to us. It is a random variable. When we get to time t, we have sufficient information to evaluate Δ_t; its randomness has been resolved.

To define the Ito's integral $\int_0^T \Delta(t)dW_t$, Ito first defined it for simple integrands and then extended it to non-simple integrands as a limit of the integral of simple integrands.

1.1 Ito integrals for simple processes

Let $\Pi = \{t_0, t_1, ..., t_n\}$ be a partition of $[0, T]$, i.e., $0 = t_0 < t_1 < t_2 < ... < t_n = T$. Let $(\Delta_t)_{t \geq 0}$ be a simple process on the partition Π, i.e., Δ_t is constant in t on each subinterval $[t_j, t_{j+1})$.

We shall think of the interplay between the simple process Δ_t and the Brownian motion W_t in the notation $\int_0^T \Delta(t)dW_t$ the following way. Regard W_t as the price per share of an asset at time t. Think of $t_0, t_1, ..., t_n$ as the trading dates in the asset, and think of $\Delta_{t_0}, \Delta_{t_1}, ..., \Delta_{t_{n-1}}$ as the position (number of shares) taken in the asset at each trading date and held to the next trading date. The gain from trading at each time t is given by:

$$I_t = \Delta_{t_0}[W_t - W_{t_0}], 0 \le t \le t_1,$$
$$I_t = \Delta_{t_0}[W_{t_1} - W_{t_0}] + \Delta_{t_1}[W_t - W_{t_1}], t_1 \le t \le t_2,$$
$$I_t = \Delta_{t_0}[W_{t_1} - W_{t_0}] + \Delta_{t_1}[W_{t_2} - W_{t_1}] + \Delta_{t_2}[W_t - W_{t_2}], t_2 \le t \le t_3,$$
$$I_t = \Delta_{t_0}[W_{t_1} - W_{t_0}] + \Delta_{t_1}[W_{t_2} - W_{t_1}] + ... + \Delta_{t_{k-1}}[W_{t_k} - W_{t_{k-1}}] + \Delta_{t_k}[W_t - W_{t_k}], t_k \le t \le t_{k+1},$$

The process $(I_t)_{t\ge 0}$ is called the Ito integral of the simple process Δ_t and denote $I_t = \displaystyle\int_0^t \Delta_u dW_u$.

1.2 Ito integral for general integrands

In this section, we define the Ito integral $\int_0^t \Delta_u dW_u$ for integrands Δ_t that are allowed to vary continuously with time and also to jump. In particular, we no longer assume that Δ_t is a simple process. We still assume that Δ_t, $t \ge 0$, is adapted to the filtration $\mathcal{F}_t$. We also assume the square-integrability condition $E[\int_0^t \Delta_u^2 du] < \infty$.

In order to define $\int_0^t \Delta_u dW_u$, we approximate Δ_t by simple processes. Figure **??** suggests how this can be done. In that figure, the continuously varying Δ_t is shown as a solid line and the approximating simple integrand is dashed. Notice that Δ_t is allowed to jump. The approximating simple integrand is constructed by choosing a partition $0 = t_0 < t_1 < ... < t_n$, setting the approximating simple process equal to Δ_{t_j} at each t_j, and then holding the simple process constant over the subinterval $[t_j, t_{j+1})$. As the maximal step size of the partition approaches zero, the approximating integrand will become a better and better approximation of the continuously varying one.

In general, it is possible to choose a sequence Δ_t^n of simple processes such that as $n \to \infty$, these processes converge in mean square to Δ_t, i.e., $\displaystyle\lim_{n\to\infty} E[\int_0^T |\Delta_t^n - \Delta_t|^2 dt] = 0$. For each Δ_t^n, the Ito integral $\int_0^t \Delta_u^n dW_u$ has already been defined for $0 \le t \le T$. We define the Ito integral for the continuously varying integrand Δ_t by the formula $\displaystyle\int_0^t \Delta_u dW_u = \lim_{n\to\infty} \int_0^t \Delta_u^n dW_u, \ \forall 0 \le t \le T$.

We can approximate an Ito integral as:
$\displaystyle\int_0^t \Delta_u dW_u \approx \sum_{i=0}^n \Big[\Delta_{t_0}(W_{t_1} - W_{t_0}) + \Delta_{t_1}(W_{t_2} - W_{t_2}) + \cdots + \Delta_{t_{n-1}}(W_{t_n} - W_{t_{n-1}}) + \Delta_{t_n}(W_t - W_{t_n})\Big]$, where $\{0 = t_0 < t_1 < \cdots < t_n < t\}$ is a partition of the interval $[0, t]$.

Depending on n and the corresponding partitions, we have different approximation of the Ito integral. In general, the larger n, the better approximation.

Definition 1.1. Let $\{(W_t)_{t\ge 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and let $\mathcal{F}_t$ be the associated filtration. We consider a function $f(W_t, t)$, which is $\mathcal{F}_t$-measurable (i.e., $f(W_t, t)$ is a stochastic process adapted to the filtration of a Brownian motion W_t) and is square-integrable $E[\displaystyle\int_0^t f(W_s, s)^2 ds] < \infty$, for all $t > 0$. The Ito integral of $f(W_t, t)$, with respect to the Brownian motion W_t is defined as:

$$I_t = \int_0^t f(W_s, s) dW_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})$$

where $t_i = it/n, i = 0...n$.

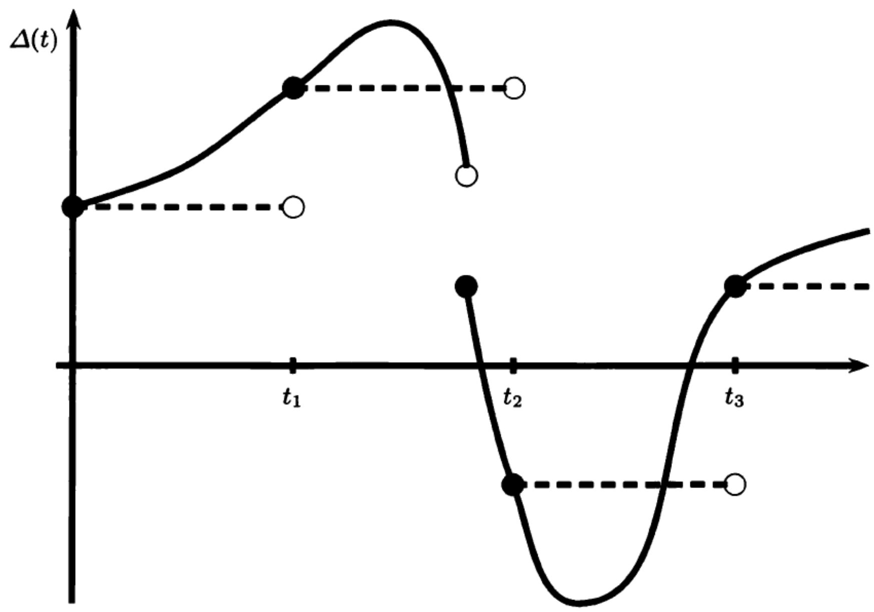

Fig. 4.3.1. Approximating a continuously varying integrand.

Figure 2:

Definition 1.2. Let $\{(W_t)_{t\geq 0}\}$ be a standard Wiener process on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and let $\mathcal{F}_t$ be the associated filtration. We consider a function $f(W_t, t)$, which is $\mathcal{F}_t$-measurable (i.e., $f(W_t, t)$ is a stochastic process adapted to the filtration of a Brownian motion W_t) and is square-integrable $E[\int_0^t f(W_s, s)^2 ds] < \infty$, for all $t > 0$. The Stratonovich integral of $f(W_t, t)$, with respect to the Brownian motion W_t is defined as:

$$I_t = \int_0^t f(W_s, s) dW_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\frac{W_{t_i} + W_{t_{i+1}}}{2}, t_i)(W_{t_{i+1}} - W_{t_i})$$

where $t_i = it/n, i = 0...n$.

Theorem 1. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $\{W_t, t \geq 0\}$ be a standard Wiener process. Show that it has finite quadratic variation such that $< W, W >_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = t$, where $t_i = it/n, i = 0...n$. Finally, deduce that $dW_t^2 = dt$.

Proof. Since the quadratic variation is a sum of random variables, we need to show that its expected value is t and its variance converges to zero as $n \to \infty$.

Recall here that if $X \sim N(\mu, \sigma)$, then $E[X^2] = \mu^2 + \sigma^2$, and $E[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$.

Let $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i} \sim N(0, t/n)$, then $< W, W >_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta W_{t_i}^2]$.

As $E[\Delta W_{t_i}^2] = t/n$, $E[\lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta W_{t_i}^2] = \lim_{n\to\infty} \sum_{i=0}^{n-1} E[\Delta W_{t_i}^2] = t$.

In additon, as $E[\Delta W_{t_i}^4] = 3t^2/n^2$, $Var[\lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta W_{t_i}^2] = \lim_{n\to\infty} \sum_{i=0}^{n-1} Var[\Delta W_{t_i}^2] = \lim_{n\to\infty} \sum_{i=0}^{n-1} (E[W_{t_i}^4] - E[W_{t_i}^2]^2) = 3t^2/n^2 - t^2/n^2 = \lim_{n\to\infty} 2t^2/n^2 = 0$. We now conclude that $< W, W >_t = t$.

Using Ito integral definition, we have $\int_0^t dW_s dW_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = t$. Also, $\int_0^t ds = t$, thus $\int_0^t dW_s dW_s = \int_0^t ds$. Differentiating two sides with respect to t, we obtain $dW_t dW_t = dt$ or $dW_t^2 = dt$. ☐

Theorem 2. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $\{W_t, t \geq 0\}$ be a standard Wiener process. Show that the following cross-variation between W_t and t, and the quadratic variation of t are:
$\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) = 0$ and $\lim_{n\to\infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 = 0$,
where $t_i = it/n$, $i = 0, ..., n$. Finally, deduce that $dW_t.dt = 0$ and $dt^2 = 0$.

Proof. Since $W_{t_{i+1}} - W_{t_i} \sim N(0, t/n)$ then taking expectation and variance we have:
$E[\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i)] = \lim_{n\to\infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i) E[W_{t_{i+1}} - W_{t_i}] = 0$
$Var[\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i)] = \lim_{n\to\infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 Var[W_{t_{i+1}} - W_{t_i}] = \lim_{n\to\infty} \sum_{i=0}^{n-1} (t/n)^3 = 0.$

We thus conclude that $\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i)] = 0$.

In addition, $\lim_{n\to\infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 = \lim_{n\to\infty} \sum_{i=0}^{n-1} (t/n)^2 = 0$.

Finally, because $\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i)] = \int_0^t dW_s ds = 0$ and $\lim_{n\to\infty} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 = \int_0^t ds ds = 0$, by differentiating both sides with respect to t, we can deduce that $dW_t.dt = dt.dt = 0$. ☐

Theorem 3. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $\{W_t, t \geq 0\}$ be a standard Wiener process. Show that it has unbounded first variation such that $\lim_{n\to\infty} \sum_{i=0}^{n-1} |W_{t_{i+1}} - W_{t_i}| = \infty$, where $t_i = it/n$, $i = 0, ..., n$.

Proof. Since $\sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 \leq \max_{0\leq k\leq n-1} |W_{t_{k+1}} - W_{t_k}| \sum_{i=0}^{n-1} |W_{t_{i+1}} - W_{t_i}|$,

then $\sum_{i=0}^{n-1} |W_{t_{i+1}} - W_{t_i}| \geq \dfrac{\sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2}{\max_{0\leq k\leq n-1} |W_{t_{k+1}} - W_{t_k}|}$.

As W_t is continuous, $\lim_{n\to\infty} \max_{0\leq k\leq n-1} |W_{t_{k+1}} - W_{t_k}| = 0$. In addition, $\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = t < \infty$. We thus conclude that $\lim_{n\to\infty} \sum_{i=0}^{n-1} |W_{t_{i+1}} - W_{t_i}| = \infty$. ☐

Theorem 4. The Ito integral has the following properties:
i) If $I_t = \int_0^t f(W_s, s) dW_s$ and $J_t = \int_0^t g(W_s, s) dW_s$ then

$$I_t \pm J_t = \int_0^t [f(W_s, s) \pm g(W_s, s)] dW_s$$

and for a constant c, $cI_t = \int_0^t c f(W_s, s) dW_s$.
ii) I_t is a martingale.
iii) $E[I_t^2] = E(\int_0^t f(W_s, s)^2 ds)$.

Proof. We have $I_t = \int_0^t f(W_s, s)dW_s = \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})$ and

$J_t = \int_0^t g(W_s, s)dW_s = \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} g(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})$, where $t_i = it/n, i = 0...n$.

i) $I_t \pm J_t = \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) \pm \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} g(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) = \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} [f(W_{t_i}, t_i) \pm g(W_{t_i}, t_i)](W_{t_{i+1}} - W_{t_i}) = \int_0^t [f(W_s, s) \pm g(W_s, s)]dW_s$.

$cI_t = c \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) = \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} cf(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) = \int_0^t cf(W_s, s)dW_s$

ii) To prove I_t is a martingale, we prove $E[I_t|\mathcal{F}_u] = I_u$ for all $u \le t$. We have:

$$\begin{aligned}
I_t &= \int_0^t f(W_s, s)dW_s = \int_0^u f(W_s, s)dW_s + \int_u^t f(W_s, s)dW_s \\
&= \lim_{n\to\infty} \sum_{i=0}^{m-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) + \lim_{n\to\infty} \sum_{i=m}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) = I_u + \lim_{n\to\infty} \sum_{i=m}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})
\end{aligned}$$

where $t_i = it/n, i = 0...n$, and $t_m = u$.
Recall the tower properties of filtrations: $E[X|\mathcal{F}] = E[[X|\mathcal{G}]|\mathcal{F}]$ for all $\mathcal{F} \subseteq \mathcal{G}$. It is also clear that $\mathcal{F}_u \subseteq \mathcal{F}_{t_j}$ for all $j \ge m$. Thus,

$$\begin{aligned}
E[I_t|\mathcal{F}_u] &= E[I_u + \lim_{n\to\infty} \sum_{i=m}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})|\mathcal{F}_u] = I_u + E[\lim_{n\to\infty} \sum_{i=m}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})|\mathcal{F}_u] \\
&= I_u + \lim_{n\to\infty} \sum_{i=m}^{n-1} E[f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})|\mathcal{F}_u] = I_u + \lim_{n\to\infty} \sum_{i=m}^{n-1} E[E[f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})|\mathcal{F}_{t_i}]|\mathcal{F}_u] \\
&= I_u + \lim_{n\to\infty} \sum_{i=m}^{n-1} E[f(W_{t_i}, t_i)(E[W_{t_{i+1}}|\mathcal{F}_{t_i}] - W_{t_i})|\mathcal{F}_u] = I_u + \lim_{n\to\infty} \sum_{i=m}^{n-1} E[f(W_{t_i}, t_i)(W_{t_i} - W_{t_i})|\mathcal{F}_u] = I_u
\end{aligned}$$

iii) Recall that W_t and $W_t^2 - t$ are martingale processes. Let $t_i = it/n, i = 0...n$, wee have:

$$E[I_t^2] - E[\int_0^t f(W_s, s)^2 ds] = E[(\lim_{n\to\infty} \sum_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}))^2] - E[\lim_{n\to\infty} \sum_{i=0}^{n-1} f(W_{t_i}, t_i)^2 (t_{i+1} - t_i)]$$

We now focus on simplifying $E[(\lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}))^2]$. Note that

$$\begin{aligned}
(\sum_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}))^2 &= [\sum_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})][\sum_{j=0}^{n-1} f(W_{t_j}, t_j)(W_{t_{j+1}} - W_{t_j})] \\
&= \sum_{i=0}^{n-1} f^2(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})^2 + 2 \sum_{0 \le i < j \le n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) f(W_{t_j}, t_i)(W_{t_{j+1}} - W_{t_j})
\end{aligned}$$

and for $i < j$,

$$\begin{aligned}
&E[f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) f(W_{t_j}, t_i)(W_{t_{j+1}} - W_{t_j})] = E[E[f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) f(W_{t_j}, t_i)(W_{t_{j+1}} - W_{t_j})|\mathcal{F}_{t_j}]] \\
&= E[f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) f(W_{t_j}, t_i) E[(W_{t_{j+1}} - W_{t_j})|\mathcal{F}_{t_j}]] = E[f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) f(W_{t_j}, t_i) 0] = 0.
\end{aligned}$$

Thus,

$$E[(\lim_{n\to\infty}\sum_{i=0}^{n-1} f(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i}))^2] = \lim_{n\to\infty} E[(\sum_{i=0}^{n-1} f(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i}))^2]$$

$$= \lim_{n\to\infty} E[\sum_{i=0}^{n-1} f^2(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i})^2] + 2\lim_{n\to\infty}\sum_{0\le i<j\le n-1} E[f(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i})f(W_{t_j},t_i)(W_{t_{j+1}}-W_{t_j})]$$

$$= \lim_{n\to\infty} E[\sum_{i=0}^{n-1} f^2(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i})^2]$$

As a result,

$$E[I_t^2] - E[\int_0^t f(W_s,s)^2 ds] = \lim_{n\to\infty} E[\sum_{i=0}^{n-1} f^2(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i})^2] - \lim_{n\to\infty} E[\sum_{i=0}^{n-1} f(W_{t_i},t_i)^2(t_{i+1}-t_i)]$$

$$= \lim_{n\to\infty} E[\sum_{i=0}^{n-1} f^2(W_{t_i},t_i)[(W_{t_{i+1}}-W_{t_i})^2 - (t_{i+1}-t_i)]$$

$$= \lim_{n\to\infty} E[\sum_{i=0}^{n-1} f^2(W_{t_i},t_i)[(W_{t_{i+1}}^2 - t_{i+1}) + (W_{t_i}^2 + t_i) - 2W_{t_{i+1}}W_{t_i}]$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} E[f^2(W_{t_i},t_i)[(W_{t_{i+1}}^2 - t_{i+1}) + (W_{t_i}^2 + t_i) - 2W_{t_{i+1}}W_{t_i}]]$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} E[E[f^2(W_{t_i},t_i)[(W_{t_{i+1}}^2 - t_{i+1}) + (W_{t_i}^2 + t_i) - 2W_{t_{i+1}}W_{t_i}|\mathcal{F}_{t_i}]]]$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} E[f^2(W_{t_i},t_i)[E[(W_{t_{i+1}}^2 - t_{i+1})|\mathcal{F}_{t_i}] + (W_{t_i}^2 + t_i) - 2E[W_{t_{i+1}}|\mathcal{F}_{t_i}]W_{t_i}]]$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} E[f^2(W_{t_i},t_i)[W_{t_i}^2 - t_i + (W_{t_i}^2 + t_i) - 2W_{t_i}W_{t_i}]] = 0$$

We now can conclude that $E[(\int_0^t f(W_s,s)dW_s)^2] = E[\int_0^t f(W_s,s)^2 ds]$

□

Example 1.1. Let W_t be a standard Brownian motion. Compute the following Ito integral with respect to W_t: $\int_0^t W_s dW_s$ and then deduce that the Ito integral is a martingale.

Proof. Let $\Pi : 0 = t_0 < t_1 < ... < t_{n-1} < t_n = t$ be a partition of the interval $[0,t]$, where $t_i = it/n$. We have

$$\int_0^t W_s dW_s = \lim_{n\to\infty}\sum_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i}) = \lim_{n\to\infty}\sum_{i=0}^{n-1}[W_{t_i}W_{t_{i+1}} - W_{t_i}^2]$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1}[-\frac{1}{2}W_{t_{i+1}}^2 + W_{t_i}W_{t_{i+1}} - \frac{1}{2}W_{t_i}^2 - \frac{1}{2}W_{t_i}^2 + \frac{1}{2}W_{t_{i+1}}^2] = \lim_{n\to\infty}\sum_{i=0}^{n-1}[-\frac{1}{2}[W_{t_{i+1}} - W_{t_i}]^2 + \frac{1}{2}[W_{t_{i+1}}^2 - W_{t_i}^2]]$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{2}[W_{t_{i+1}}^2 - W_{t_i}^2] - \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{2}[W_{t_{i+1}} - W_{t_i}]^2 = \frac{1}{2}(W_t^2 - t)$$

As it can be prove that $W_t^2 - t$ is a martingale, the Ito is a martingale. □

1.3 Exercise

1. Consider the Stratonovich integral $S(t) = \int_0^t W_s dW_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2}(W_{t_{i+1}} + W_{t_i})(W_{t_{i+1}} - W_{t_i})$, where $t_i = it/n, i = 0...n$. Show that $S(t) = \frac{1}{2}W_t^2$ and show also that the Stratonovich integral is not a martingale.

2. Using the equality: $\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = t$ to compute the following limit $\lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_{i+1}}(W_{t_{i+1}} - W_{t_i})$. Then deduce that if the integral $\int_0^t W_s dW_s$ is defined as the above limit then it is not a martingale.

3. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $\{W_t, t \geq 0\}$ be a standard Brownian motion. The stochastic Ito integral with respect to a standard Brownian motion can be defined as

$$\int_0^t s dW_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} t_i (W_{t_{i+1}} - W_{t_i})$$

where $t_i = it/n, i = 0...n$. Using the above Ito integral's definition to prove that

$$\int_0^t s dW_s = tW_t - \int_0^t W_s ds.$$

Proof. We have

$$\int_0^t s dW_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} t_i (W_{t_{i+1}} - W_{t_i}) = \lim_{n\to\infty} \sum_{i=0}^{n-1} [t_i (W_{t_{i+1}} - W_{t_i}) + t_{i+1} W_{t_{i+1}} - t_{i+1} W_{t_{i+1}}]$$
$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} [t_{i+1} W_{t_{i+1}} - t_i W_{t_i} + t_i W_{t_{i+1}} - t_{i+1} W_{t_{i+1}}]$$
$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} [t_{i+1} W_{t_{i+1}} - t_i W_{t_i}] + \lim_{n\to\infty} \sum_{i=0}^{n-1} [t_i W_{t_{i+1}} - t_{i+1} W_{t_{i+1}}] = tW_t - \lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_{i+1}} (t_{i+1} - t_i)$$

We now need to prove $\int_0^t W_s ds = \lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_{i+1}}(t_{i+1} - t_i)$. As $\int_0^t W_s ds = \lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_i}(t_{i+1} - t_i)$, we have

$$\int_0^t W_s ds - \lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_{i+1}}(t_{i+1} - t_i) = \lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_i}(t_{i+1} - t_i) - \lim_{n\to\infty} \sum_{i=0}^{n-1} W_{t_{i+1}}(t_{i+1} - t_i)$$
$$= - \lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) = 0$$

We now can conclude that $\int_0^t s dW_s = tW_t - \int_0^t W_s ds$.

Note that the equality $\int_0^t s dW_s = tW_t - \int_0^t W_s ds$ can be proved very easily if we apply the technique: integrations by part. □

4. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $\{W_t, t \geq 0\}$ be a standard Brownian motion. The stochastic

Ito integral with respect to a standard Brownian motion can be defined as

$$\int_0^t f(W_s,s)dW_s = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i})$$

where $t_i = it/n, i = 0...n$. Using the properties of standard Brownian motion to prove that

$$E[\int_0^t f(W_s,s)dW_s] = 0.$$

5. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space and let $\{W_t, t\geq 0\}$ be a standard Brownian motion. The stochastic Ito integral with respect to a standard Brownian motion can be defined as

$$\int_0^t (t-s)dW_s = \lim_{n\to\infty}\sum_{i=0}^{n-1}(t-t_i)(W_{t_{i+1}}-W_{t_i})$$

where $t_i = it/n, i = 0...n$. Using the properties of standard Brownian motion to prove that

$$E[\int_0^t (t-s)dW_s] = 0.$$

6. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space and let $\{W_t, t\geq 0\}$ be a standard Brownian motion.
i) Use integration by parts to show that:

$$\int_0^t W_s ds = \int_0^t (t-s)dW_s.$$

ii) Use the Isometry property of Ito integral to prove that $E[(\int_0^t W_s ds)^2] = t^3/3$.

Proof. i) Using the integration by parts, we have

$$\int_0^t W_s ds = sW_s|_0^t - \int_0^t sdW_s = tW_t - \int_0^t sdW_s = t\int_0^t dW_s - \int_0^t sdW_s = \int_0^t (t-s)dW_s$$

It can be proved that $E[\int_0^t f(W_s,s)dW_s] = 0$ for any well-defined function f. Thus

$$E[\int_0^t (t-s)dW_s] = 0.$$

ii) $E[(\int_0^t W_s ds)^2] = E[(\int_0^t (t-s)dW_s)^2] = E[(\int_0^t (t-s)^2 ds)] = t^3/3$ □

2 Ito-Deoblin formula

The modern theory of stochastic calculus developed from the work of Ito in 1944. Not only did Ito define the integral with respect to Brownian motion, but he also developed the change-of-variable formula, commonly called Ito's rule or Ito's formula. An amazing twist to the story of stochastic calculus has recently emerged. In February 1940, the French national Academy of Sciences received a document from W.Doeblin, a French soldier on the German front. Doeblin died shortly thereafter, and the document remain sealed until May 2000. When it was opened, the document was found to contain a construction of the stochastic integral slightly different from Ito and a clear statement of the change-of-variable formula. Because of this remarkable development, the change of variable formula is called Ito-Doeblin formula.

We want a rule to differentiate expressions of the form $f(W_t)$, where $f(x)$ is a differentiable function and W_t is a Brownian motion. If W_t were also differentiable, then the chain rule from ordinary calculus would give: $f_t'(W_t) = f'(W_t)(W_t)'t$, which could be written in differential notation as: $df(W_t) = f'(W_t)dW_t$. However, W_t is nowhere differentiable and has a nonzero quadratic variation, the correct formula has an extra term, namely, $df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt$. This is the Ito-Doeblin formula in differential form. Integrating this from 0 to t, we obtain the Ito-Doeblin formula in integral form:

$$f(W_t) - f(W_0) = \int_0^t f'(W_u)dW_u + \frac{1}{2}\int_0^t f''(W_u)du.$$

The mathematically meaningful form of the Ito-Doeblin formula is the integral form. This is because we have pricise definitions for both terms appearing on the right-hand side. The first $\int_0^t f'(W_u)dW_u$ is an Ito integral. The second $\int_0^t f''(W_u)du$ is an ordinary integral with respect to the time variable.

For pencil and paper computation, the more convenient form of Ito-Doeblin formula is the differential form. There is an intuitive meaning but no precise definition for the terms $df(W_t), dW_t, dt$ appearing in this formula. The intuitive meaning is that $df(W_t)$ is the change in $f(W_t)$ when t changes a “little bit” dt, dW_t is the change in the Brownian motion when t changes a “little bit” in dt, and the whole formula is exact only when the “little bit” is “infinitesimally small”. Because there is no precise definition for “little bit” and “infinitesimally small”, we rely on the integral form to give the mathematical meaning of the Ito-Doeblin formula.

2.1 Heuristic derivation of the Ito-Doeblin formula

2.1.1 Ito process

Let $(W_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ be an associated filtration. An Ito process is a stochastic process of the integral form:

$$X_t = X_0 + \int_0^t \Delta_u dW_u + \int_0^t \Phi_u du$$

or the differential form:

$$dX_t = \Delta_t dW_t + \Phi_t dt.$$

Here Δ_t and $\Phi_t dt$ are adapted process of the filtration $(\mathcal{F}_t)_{t\geq 0}$.

2.1.2 Ito-Doeblin formula for one-variable functions of Ito process

We now find the differential form of $df(X_t)$, where f is a differential function and X_t is an Ito stochastic process.

For one-variable functions, their second-order Taylor series expansions have the form:

$$df(x) = f'(x)dx + \frac{1}{2}f''(x)dx^2.$$

Recall that standard Brownian motion accumulates quadratic variation at rate one per unit time, i.e., $dW_t^2 = dt$. This is the key for our derivation.

Substituting $x = X_t$ in the above formula, we obtain:

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)dX_t^2 = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(\Delta_t dW_t + \Phi_t dt)^2 = f'(X_t)dX_t + \frac{1}{2}f''(X_t)\Delta_t^2 dt.$$

Thus we obtain $df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)\Delta_t^2 dt$. By integrating both sides, we obtain the integral form:

$$\int_0^t df(X_s) = \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)\Delta_s^2 ds$$

or

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)\Delta_s^2 ds$$

Especially, substituting $x = W_t$ in the above formula, we obtain:

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dW_t^2 = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.$$

By integrating both sides, we obtain the integral form:

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)ds$$

Example 1. Compute dW_t^2.

Answer. Let $f(x) = x^2$, then $f'(x) = 2x,\ f''(x) = 2$. Thus $f'(W_t) = 2W_t,\ f''(W_t) = 2$. Thus applying the Ito-Doeblin formula, we have: $dW_t^2 = 2W_t dW_t + dt$.

Example 2. Compute $d(\sin W_t)$.

Example 3. Compute $d(tW_t^2)$.

Example 4. Compute $d(W_t^3)$.

2.1.3 Ito-Doeblin formula for two-variable functions

We now find the differential form of $df(X_t, Y_t)$, where f is a twice differential function and X_t, Y_t are Ito processes as follows: $dX_t = a_t dW_t + b_t dt$, $dY_t = c_t dW_t + d_t dt$.
For two-variable functions, their second-order Taylor series expansions have the form:

$$df(x,t) = f'_x(x,t)dx + f'_t(x,t)dt + \frac{1}{2}f''_{x^2}(x,t)dx^2 + f''_{xt}(x,t)dxdt + \frac{1}{2}f''_{t^2}(x,t)dt^2.$$

Substituting $x = X_t, t = Y_t$ in the above formula, we obtain:

$$\begin{aligned} df(X_t,Y_t) &= f'_x(X_t,Y_t)dX_t + f'_t(X_t,Y_t)dY_t + \frac{1}{2}f''_{x^2}(X_t,Y_t)dX_t^2 + f''_{xt}(X_t,Y_t)dX_t dY_t + \frac{1}{2}f''_{t^2}(X_t,Y_t)dY_t^2 \\ &= f'_x(X_t,Y_t)dX_t + f'_t(X_t,Y_t)dY_t + \frac{1}{2}f''_{x^2}(X_t,Y_t)a_t^2 dt + f''_{xt}(X_t,Y_t)a_t c_t dt + \frac{1}{2}f''_{t^2}(X_t,Y_t)c_t^2 dt \end{aligned}$$

We now consider some special cases. Let $f(x,t) = xt$ then:

$$\begin{aligned} dX_tY_t &= Y_t dX_t + X_t dY_t + dX_t dY_t \\ &= Y_t dX_t + X_t dY_t + a_t c_t dt \end{aligned}$$

Substituting $x = W_t, y = t$ in the general formula and ignore all the terms associated with dt^m, with $m > 1$, we obtain:

$$df(W_t,t) = f'_x(W_t,t)dW_t + f'_t(W_t,t)dt + \frac{1}{2}f''_{x^2}(W_t,t)dW_t^2 = f'_x(W_t,t)dW_t + [f'_t(W_t,t) + \frac{1}{2}f''_{x^2}(W_t,t)]dt.$$

By integrating both sides, we obtain the integral form:

$$f(W_t,t) - f(W_0,0) = \int_0^t f'_x(W_s,s)dW_s + \int_0^t f'_t(W_s,s)ds + \int_0^t \frac{1}{2}f''_{x^2}(W_s,s)ds$$

Example 1. Compute $d(\mu t + \sigma W_t)$.

Answer. Let $f(x,t) = \mu t + \sigma x$, then $f'_x(x,t) = \sigma,\ f'_t(x,t) = \mu\ f''_{x^2}(x) = 0$. Thus $f'_x(W_t,t) = \sigma,\ f'_t(W_t,t) = \mu,\ f''(W_t,t) = 0$. Thus applying the Ito-Doeblin formula, we have: $d(\mu t + \sigma W_t) = \sigma dW_t + \mu dt$.

2.2 Exercise

1. Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and let $\mathcal{F}_t$ be the associated filtration. Using Ito formula to prove that

$$\int_0^t W_s^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s ds$$

2. Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and let $\mathcal{F}_t$ be the associated filtration. Using Ito formula to prove that

$$\int_0^t (W_s^2 + s^2)dW_s = W_t(W_t^2 + t^2) - \int_0^t (2sW_s + 3W_s)ds - \int_0^t 2W_s^2 dW_s$$

3. Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and let $\mathcal{F}_t$ be the associated filtration. Using Ito formula to prove that

$$\int_0^t (W_s^2 + s^2)ds = t(W_t^2 + t^2) - (\frac{2}{3}s^3 + \frac{s^2}{2}) - \int_0^t 2sW_s dW_s$$

4. Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and let $\mathcal{F}_t$ be the associated filtration. Using Ito formula to prove that

$$\int_0^t \frac{W_s^2}{s^2}ds = \frac{W_t^3}{t^2} - \int_0^t [\frac{3W_s}{s^2} - \frac{2W_s^3}{s^3}]ds - \int_0^t \frac{2W_s^2}{s^2}$$

5. Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and let $\mathcal{F}_t$ be the associated filtration. Using Ito formula to prove that

$$\int_0^t W_s^2 s^2 ds = W_t^2 t^3 - \int_0^t [2W_s^2 s^2 + s^3]ds - 2\int_0^t s^3 W_s dW_s$$

6. * Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and let $\mathcal{F}_t$ be the associated filtration. We consider a function $f(x,t)$ such that $f(W_t,t)$ is $\mathcal{F}_t$-measurable and is square-integrable, i.e., $E[\int_0^t f(W_s,s)^2 ds] < \infty$, for all $t > 0$. Prove that

$$\int_0^t f(W_s,s)dW_s = W_t f(W_t,t) - \int_0^t [W_s f'_t(W_s,s) + f'_x(W_s,s) + \frac{1}{2} W_s f''_{x^2}(W_s,s)]ds - \int_0^t W_s f'_x(W_s,s)dW_s$$

and

$$\int_0^t f(W_s,s)ds = tf(W_t,t) - \int_0^t s[f'_t(W_s,s) + \frac{1}{2} f''_{x^2}(W_s,s)]ds - \int_0^t s f'_x(W_s,s)dW_s$$

3 Stochastic differential equations

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms has a random component. Within the context of mathematical finance, SDEs are frequently used to model diverse phenomena such as stock prices, interest rates or volatilities. Typically, SDEs have continuous paths with both random and non-random components and to drive the random component of the model they usually incorporate a Brownian motion. To enrich the model further, other types of random fluctuations are also employed in conjunction with the Brownian motion, such as the Poisson process when modelling discontinuous jumps. In this lecture, we will concentrate solely on SDEs having only a Brownian motion.

Definition 3.1. A diffusion-type stochastic differential equation (SDE) can be described as

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t,$$

where W_t is a standard Brownian motion. We call $\mu(X_t,t)$ the drift and $\sigma(X_t,t)$ be the volatitility of the process. The integral form of the SDE is

$$X_t = X_0 + \int_0^t \mu(X_s,s)ds + \int_0^t \sigma(X_s,s)dW_s,$$

with $X_0 = x_0$ be the initial state of the solution process.

Important note 3.1. Our objective to solve a SDE:

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t$$

is to find all stochastic processes satisfying the SDE. Such processes are called Ito processes.

Example 3.1. Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that the process $(X_t)_{t\geq 0}$ is governed by the SDE: $dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t$, where μ and σ are functions of X_t and t. Show that X_t is a martingale if $\mu(X_t,t) = 0$.

Proof. By taking the integrals both sides of the SDE, we have:

$$\int_s^t dX_u = \int_s^t \mu(X_u,t)du + \int_s^t \sigma(X_u,u)dW_u \Leftrightarrow X_t - X_s = \int_s^t \mu(X_u,t)du + \int_s^t \sigma(X_u,u)dW_u$$

If $\mu(X_t,t) = 0$ then $X_t - X_s = \int_s^t \sigma(X_u,u)dW_u$. Thus $E[X_t|\mathcal{F}_s] = E[X_s|\mathcal{F}_s] + E[\int_s^t \sigma(X_u,u)dW_u|\mathcal{F}_s]$. Let

$\Pi : s = t_0 < t_1 < ... < t_{n-1} = t_n = t$ be a partition of the interval $[s, t]$, where $t_i = s + i(t-s)/n$. We have

$$E[\int_s^t \sigma(X_u, u)dW_u|\mathcal{F}_s] = E[\lim_{n\to\infty}\sum_{i=0}^{n-1}\sigma(X_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})|\mathcal{F}_s] = E[E[\lim_{n\to\infty}\sum_{i=0}^{n-1}\sigma(X_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})|\mathcal{F}_{t_i}]|\mathcal{F}_s]]$$
$$= E[\lim_{n\to\infty}\sum_{i=0}^{n-1}\sigma(X_{t_i}, t_i)(W_{t_i} - W_{t_i})|\mathcal{F}_s] = 0.$$

As a result $E[X_t|\mathcal{F}_s] = E[X_s|\mathcal{F}_s] = X_s$. We conclude that X_t is a martingale if $\mu(X_t, t) = 0$. $\square$

Example 3.2. (Bachelier Model: Arithmetic Brownian motion). Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that the process $(X_t)_{t\geq 0}$ is governed by the SDE:

$$dX_t = \mu dt + \sigma dW_t$$

where μ and σ are constant. Show that the random variable $(X_T|X_t = x)$ follows a normal distribution with mean $x + \mu(T-t)$ and variance $\sigma^2(T-t)$.

Proof. By integrating both sides of the SDE, we obtain:

$$\int_t^T dX_s = \int_t^T \mu ds + \int_t^T \sigma dW_s \Leftrightarrow X_T - X_t = \mu(T-t) + \sigma(W_T - W_t).$$

As $W_T - W_t \sim N(0, T-t)$ (a normal random variable with mean 0 and variance $T-t$), $(X_T|X_t = x) \sim N(x + \mu(T-t), \sigma^2(T-t))$. $\square$

Example 3.3. (Lognormal Distribution)

i) Let $Z \sim N(0,1)$, show that the moment generating function of a standard normal distribution is $E(e^{\theta Z}) = e^{\frac{1}{2}\theta^2}$ for a constant θ.

ii) Show that if $X \sim N(\mu, \sigma^2)$ then $Y = e^X$ follows a lognormal distribution, i.e., $Y \sim logN(\mu, \sigma^2)$, with probability density function $f_Y(y) = \dfrac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{\log y - \mu}{\sigma})^2}$ with mean $E[Y] = e^{\mu + \frac{1}{2}\sigma^2}$ and variance $Var(Y) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$.

Proof. i) As $Z \sim N(0,1)$, the probability density function of Z is

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}.$$

Recall that if $U = g(Z)$ then $E[U] = E[g(Z)] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} g(z)e^{-\frac{z^2}{2}}dz$.
Thus $E[e^{\theta Z}] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{\theta z}e^{-\frac{z^2}{2}}dz = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-\frac{(z-\theta)^2}{2} + \frac{\theta^2}{2}}dz = e^{\frac{\theta^2}{2}}\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-\frac{(z-\theta)^2}{2}}d(z-\theta) = e^{\frac{\theta^2}{2}}$.

ii) For $y > 0$, by definition we have the cumulative distribution $F(y)$ of Y is

$$F(y) = P(Y < y) = P(e^X < y) = P(X < \log y) = \int_{-\infty}^{\log y} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}dx.$$

The probability density function f_Y of Y is

$$f_Y(y) = F'(y) == \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\log y - \mu}{\sigma})^2}$$

As $X \sim N(\mu, \sigma^2)$, we can express $X = \mu + \sigma Z$, where $Z \sim N(0,1)$. Thus

$$E[Y] = E[e^X] = E[e^{\mu+\sigma Z}] = e^{\mu} E[e^{\sigma Z}] = e^{\mu} e^{\frac{\sigma^2}{2}} = e^{\mu + \frac{\sigma^2}{2}}.$$

Also,

$$E[Y^2] = E[e^{2X}] = E[e^{2\mu+2\sigma Z}] = e^{2\mu} E[e^{2\sigma Z}] = e^{2\mu} e^{\frac{4\sigma^2}{2}} = e^{2\mu+2\sigma^2}.$$

As a result,

$$Var[Y] = E[Y^2] - (E[Y])^2 = e^{2\mu+2\sigma^2} - e^{2\mu+2\frac{\sigma^2}{2}} = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$$

□

Example 3.4. (Black-Scholes model: Geometric Brownian motion). Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that the process $(X_t)_{t\geq 0}$ is governed by the geometric Brownian motion:

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

where μ and σ are constant.

i) By applying Ito formula to $Y_t = \log X_t$, show that $X_T = X_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$.

ii) Show that the random variable $(X_T|X_t = x)$ follows a lognormal distribution with mean $xe^{\mu(T-t)}$ and variance $x^2(e^{\sigma^2(T-t)} - 1)e^{2\mu(T-t)}$.

Proof. i) Applying Ito formula to $Y_t = \log X_t$, we have:

$$dY_t = d(logX_t) = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2} dX_t^2 = \mu dt + \sigma dW_t - \frac{1}{2}(\mu dt + \sigma dW_t)^2 = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

Integrating both sides of $d(logX_t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$, we obtain:

$$\int_t^T d(\log X_s) = \int_t^T (\mu - \frac{1}{2}\sigma^2) ds + \int_t^T \sigma dW_s \Leftrightarrow \log X_T - \log X_t = (\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)$$
$$\Leftrightarrow X_T = X_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}$$

ii) We can express $X_T = X_t e^{(\mu-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)} = e^{\log X_t + (\mu-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}$. Thus $(X_T|X_t = x) = e^U$, where $U = \log x + (\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)$, $U \sim N(\log x + (\mu - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))$. As U is a normal random variable with mean $\log x + (\mu - \frac{1}{2}\sigma^2)(T-t)$ and variance $\sigma^2(T-t))$, $(X_T|X_t = x)$ is a lognormal random variable with mean and variance as follows:

$$E[X_T|X_t = x] = e^{E[U]+\frac{1}{2}Var[U]} = e^{\log x + (\mu-\frac{1}{2}\sigma^2)(T-t)+\frac{1}{2}\sigma^2(T-t)} = e^{\log x + \mu(T-t)} = xe^{\mu(T-t)}$$

$$Var[X_T|X_t = x] = (e^{Var[U]} - 1)e^{2E[U]+Var[U]} = (e^{Var[U]} - 1)(E[X_T|X_t = x])^2 = x^2(e^{\sigma^2(T-t)} - 1)e^{2\mu(T-t)}$$

□

Example 3.5. (Ornstein-Ulenbeck Process). Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that the process $(X_t)_{t\geq 0}$ is governed by the Ornstein-Uhlenbeck process:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

where κ, θ and σ are constant.

i) By applying Ito formula to $Y_t = e^{\kappa t}X_t$, show that $X_T = X_t e^{(-\kappa)(T-t)} + \theta[1 - e^{-\kappa(T-t)}] + \int_t^T \sigma e^{-\kappa(T-s)} dW_s$.

ii) Show that the random variable $(X_T|X_t = x)$ follows a normal distribution with mean

$$xe^{-\kappa(T-t)} + \theta[1 - e^{-\kappa(T-t)}]$$

and variance

$$\frac{\sigma^2}{2\kappa}[1 - e^{-2\kappa(T-t)}].$$

Proof. i) Applying Ito formula to $Y_t = e^{\kappa t}X_t$, we have:

$$dY_t = d(e^{\kappa t}X_t) = \kappa e^{\kappa t}X_t dt + e^{\kappa t}dX_t = \kappa e^{\kappa t}X_t dt + e^{\kappa t}\kappa(\theta - X_t)dt + e^{\kappa t}\sigma dW_t = e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma dW_t$$

Integrating both sides of $d(e^{\kappa t}X_t) = e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma dW_t$, we obtain:

$$\int_t^T d(e^{\kappa t}X_t) = \int_t^T e^{\kappa s}\kappa\theta ds + \int_t^T e^{\kappa t}\sigma dW_s \Leftrightarrow e^{\kappa T}X_T - e^{\kappa t}X_t = \theta(e^{\kappa T} - e^{\kappa t}) + \int_t^T \sigma e^{\kappa t} dW_s$$
$$\Leftrightarrow X_T = X_t e^{(-\kappa)(T-t)} + \theta[1 - e^{-\kappa(T-t)}] + \int_t^T \sigma e^{-\kappa(T-s)} dW_s$$

ii) We have $(X_T|X_t = x) = xe^{(-\kappa)(T-t)} + \theta[1 - e^{-\kappa(T-t)}] + \int_t^T \sigma e^{-\kappa(T-s)} dW_s$. The Ito integral $\int_t^T \sigma e^{-\kappa(T-s)} dW_s$ can be expressed as $\lim_{n\to\infty} \sum_{i=0}^{n-1} \sigma e^{-\kappa(T-t_i)}(W_{i+1} - W_i)$, where $t_i = t + i(T-t)/n$. As $\{(W_{i+1} - W_i)\}_{i=0,\ldots,n-1}$ are independent normal random variables, the Ito integral can be considered as a limit of a sum of independent normal random variables. Thus, Ito integral is also a normal random variable. As a result, $(X_T|X_t = x)$ follows a normal distribution. Its mean is computed as:

$$E[(X_T|X_t = x)] = xe^{(-\kappa)(T-t)} + \theta[1 - e^{-\kappa(T-t)}] + E[\int_t^T \sigma e^{-\kappa(T-s)} dW_s] = xe^{(-\kappa)(T-t)} + \theta[1 - e^{-\kappa(T-t)}]$$

because $E[\int_t^T \sigma e^{-\kappa(T-s)} dW_s] = E[\int_0^T \sigma e^{-\kappa(T-s)} dW_s] - E[\int_0^t \sigma e^{-\kappa(T-s)} dW_s] = 0$ due to the fact that Ito integrals are martingale. The variance of $(X_T|X_t = x)$ is:

$$Var[(X_T|X_t = x)] = Var[\int_t^T \sigma e^{-\kappa(T-s)} dW_s] = E[(\int_t^T \sigma e^{-\kappa(T-s)} dW_s)^2] = E[\int_t^T \sigma^2 e^{-2\kappa(T-s)} ds] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(T-t)})$$

□

3.1 Exercise

1. Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that the process $(X_t)_{t\geq 0}$ is governed by the SDE:

$$dX_t = 5dt + 2dW_t.$$

   Show that the random variable $(X_7|X_3 = 2)$ follows a normal distribution with mean 22 and variance 16.

2. Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that the process $(X_t)_{t\geq 0}$ is governed by the geometric Brownian motion:

$$dX_t = 3X_t dt + 2X_t dW_t.$$

   Find the distribution of the random variable $(X_7|X_2 = 2)$.

3. Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that the process $(X_t)_{t\geq 0}$ is governed by the Ornstein-Uhlenbeck process:

$$dX_t = 7(4 - X_t)dt + 3dW_t.$$

   Find the distribution of the random variable $(X_7|X_3 = 4)$.

4. (Generalized Geometric Brownian motion). Let $\{(W_t)_{t\geq 0}\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that the process $(X_t)_{t\geq 0}$ is governed by the geometric Brownian motion:

$$dX_t = \mu_t X_t dt + \sigma_t X_t dW_t$$

   where μ and σ are time-dependent functions.

   i) By applying Ito formula to $Y_t = \log X_t$, show that $X_T = X_t e^{\displaystyle\int_t^T (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_t^T \sigma_s dW_s}$.

   ii) Show that the random variable $(X_T|X_t = x)$ follows a lognormal distribution with mean $xe^{\int_t^T \mu_s ds}$ and variance $x^2(e^{\int_t^T \sigma_s^2 ds} - 1)e^{2\int_t^T \mu_s ds}$.