Chapter 5: Numerical Methods for Differential equations
Lecture 1

1. **Introduction to differential equations**
2. **One-step methods**
3. **Euler's Method: A simple one-step method**

1. Introduction to differential equations

Basic Concepts

- A differential equation may involve derivatives of higher-order, such as y', y'', y''', etc

$$\boxed{f(x, y(x), y'(x), y''(x), \ldots, y^{(n)}(x)) = 0}$$

- The **order of a differential equation** is that of the highest-order derivative in the equation
- A **linear first-order differential equation** is an equation of the form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

$$\boxed{\begin{aligned} & mv'(t) = -pv(t) - mg \\ & v'(t) + \frac{p}{m}v(t) = -g \end{aligned}}$$

How to solve 1st-order diff. eq.?

- Set the **integrating factor** $I(x) = e^{\int P(x)dx}$
- We have $I'(x) = e^{\int P(x)dx} P(x) = I(x)P(x)$
- Multiply both sides the diff. eq. by I(x) to get

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

$$I(x)y'(x) + I(x)P(x)y = I(x)Q(x)$$

$$I(x)y' + I'(x)y = I(x)Q(x)$$

$$\frac{d}{dx}\big(I(x)y\big) = I(x)Q(x)$$

$$I(x)y = \int I(x)Q(x)dx \quad \Rightarrow \quad \boxed{y = e^{-\int P(x)dx} \int I(x)Q(x)dx}$$

Separable Differential Equations

- Differential equation of the form

$$\boxed{\frac{dy}{dx} = \frac{f(x)}{g(y)}} \quad \Rightarrow \quad \boxed{g(y)y'(x) = f(x)}$$

is called a **separable differential equation**.

- A separable differential equation can be solved by integrating both sides. This method is known as **separation of variables**

$$\int g(y)y'(x)dx = \int f(x)dx \Rightarrow \int g(y)dy = \int f(x)dx$$

$$\boxed{G(y) = F(x) + C}$$

F and G are anti-derivatives of f and g

Ex:

Solve the initial-value problem for differential equation

1) $y' - 2y = 2x + 1, \qquad x > 0, \qquad y(0) = 2$

2) $y' = \dfrac{4x^3}{y}, \qquad x > 1, \qquad y(1) = 3$

3) $y' = \sin(xy), \qquad x > 0, \qquad y(0) = 1$

YOU CANNOT SOLVE IT!
THIS IS THE REASON WHY YOU NEED THIS LECTURE!

2. One-step Methods

A class of methods

Approximations of solution of DEs

Approximate solution of the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad a \le x \le b$$

$$y(a) = y_0$$

Sequence of approximate values $y_i \approx y(x_i)$ will be generated at various values x_i, called **mesh points**

Assumption: mesh points are equally distributed in [a, b]

$$x_i = a + ih, \quad i = 0, 1, 2, \ldots, N, \qquad h = \frac{b-a}{N}: \text{ step size}$$

One-step Methods

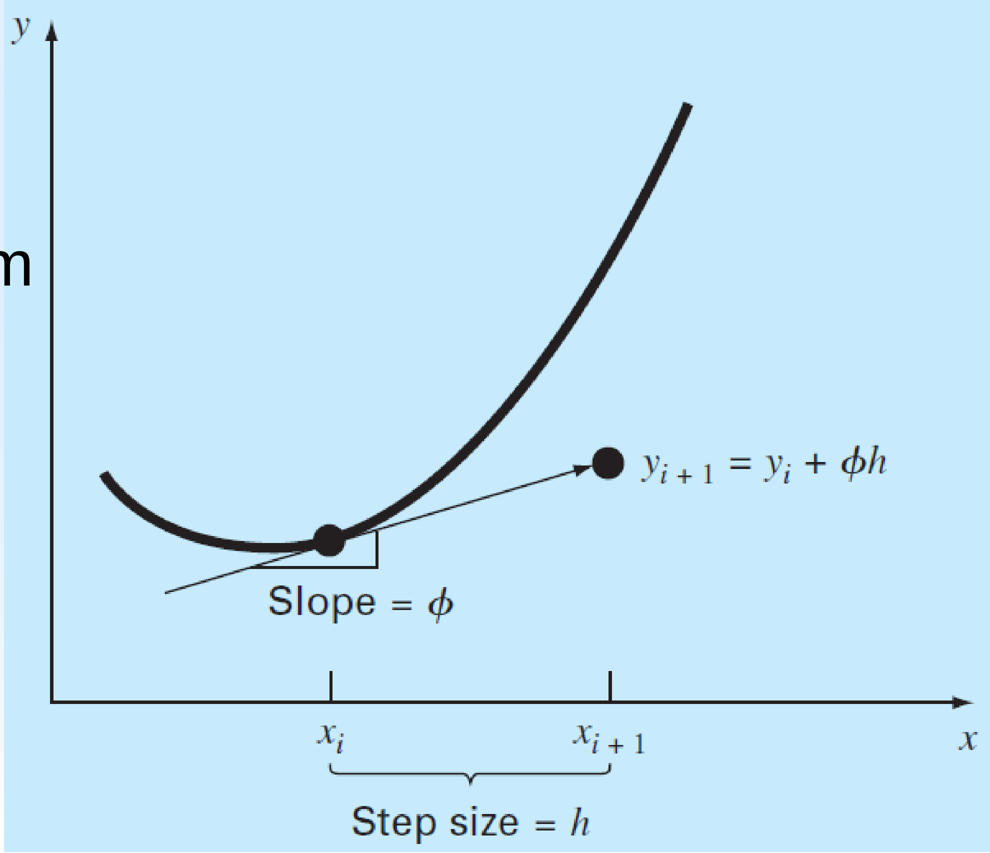
For problem

$$\frac{dy}{dx} = f(x, y), \qquad a \le x \le b$$

$$y(a) = y_0$$

A one-step method is of the form

$$y_{i+1} = y_i + h\varphi$$



3. Euler's method

It is a one-step method

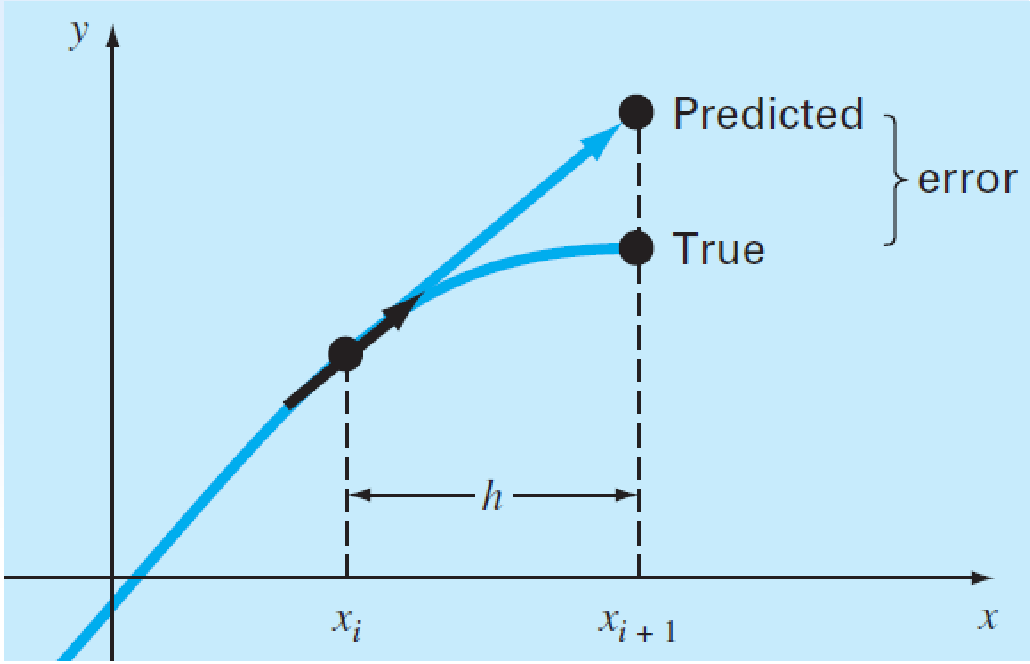
$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0$$

$$x_i = x_0 + ih, \text{ step size } h > 0, \; y_i \approx y(x_i), \quad i = 0, 1, 2, \ldots$$

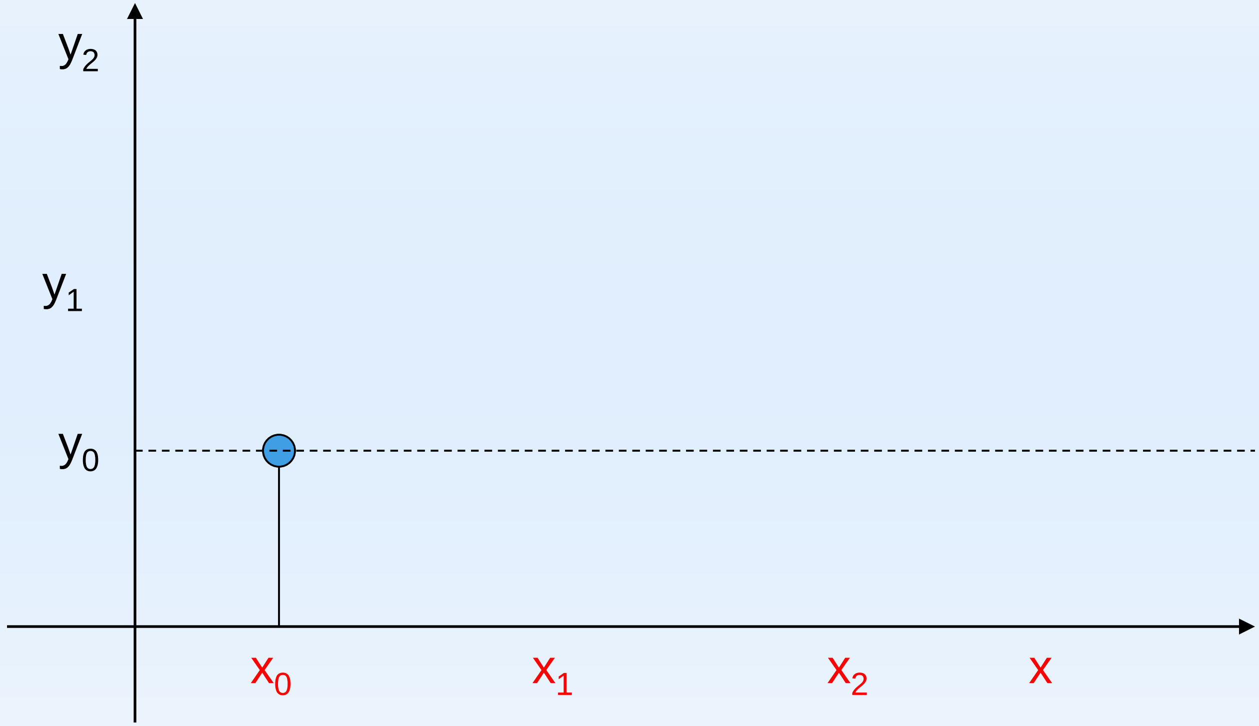
$$\frac{y(x_i + h) - y(x_i)}{h} \approx y'(x_i) = f(x_i, y(x_i))$$

$$\frac{y_{i+1} - y_i}{h} = f(x_i, y_i)$$

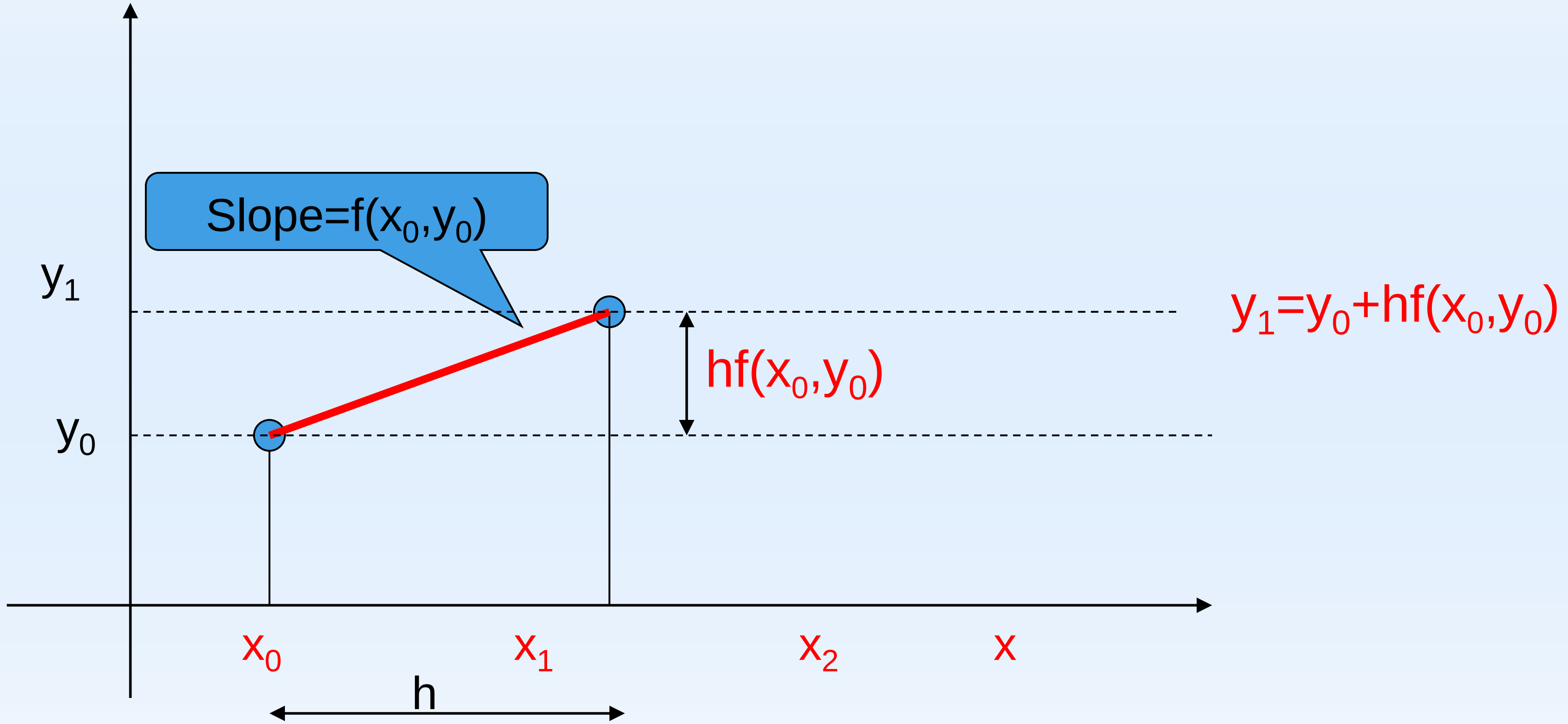
$$\boxed{y_{i+1} = y_i + h \; f(x_i, y_i)}$$



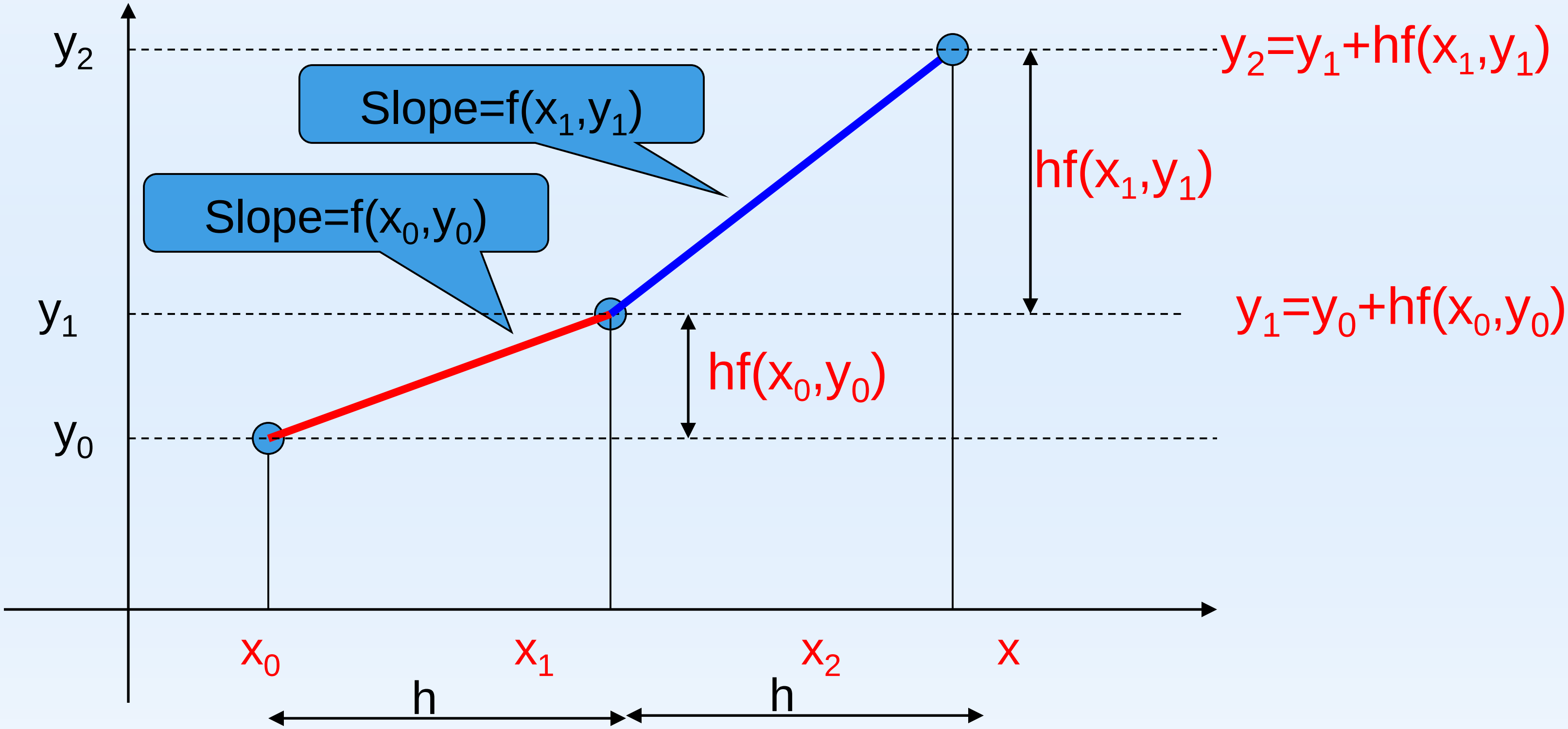
Interpretation of Euler Method



Interpretation of Euler Method



Interpretation of Euler Method



Example

Find approximate values and the errors of the solution of the initial-value problem

$$y' = 3xy, \qquad 0 < x < 1/2, \qquad y(0) = 1$$

with step size h=0.1

Solution: Exact solution

$$y' = 3xy \Rightarrow y'/y = 3x \Rightarrow \int y'/y\,dx = \int 3x\,dx$$

$$\ln(y) = 3x^2/2 + C$$

Using initial condition:

$$\ln(y(0)) = \ln(1) = 0 = C$$

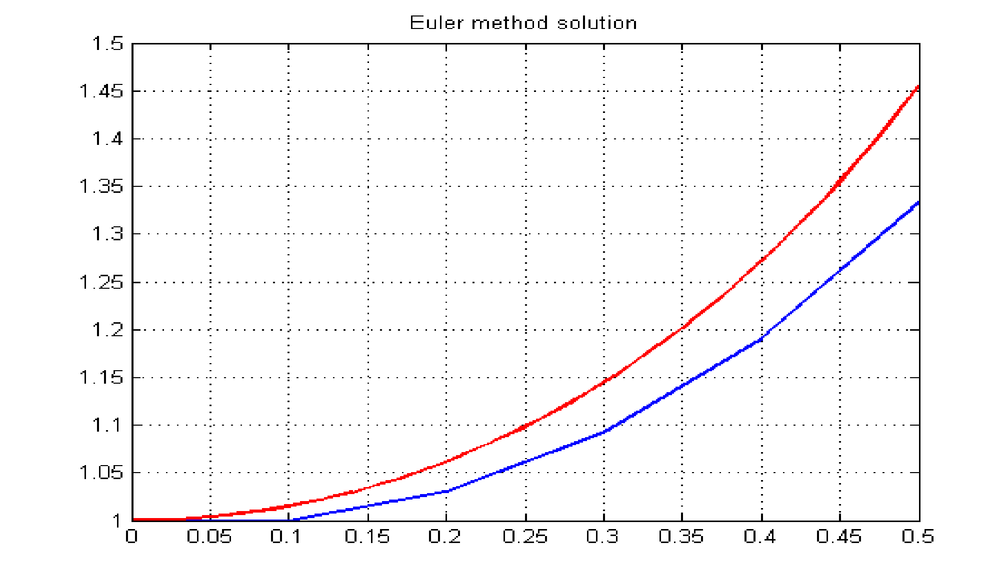
$$\Rightarrow \ln(y) = 3x^2/2 \Rightarrow y = e^{3x^2/2}$$

$$f(x,y) = 3xy, h = 0.1$$

$$x_i = 0 + ih = 0.1i, \qquad i = 0,1,2,3,4,5$$

$$y_0 = y(0) = 1$$

$$y_{i+1} = y_i + h\ f(x_i, y_i) = y_i + 3hx_i y_i$$



x	y-exact	y-Approx	Error
0	1	1	0
0.1	1.01511	1	0.0151131
0.2	1.06184	1.03	0.0318365
0.3	1.14454	1.0918	0.0527368
0.4	1.27125	1.19006	0.0811872
0.5	1.45499	1.33287	0.122122

Local truncation error

The local truncation error at a specific step measures the amount by which the exact solution to the DE fails to satisfy the method.

Method $y_{i+1} = y_i + h\,\Phi(x_i, y_i; h)$ has local truncation error:

$$\tau_{i+1}(h) = \frac{y(x_{i+1}) - (y_i + h\Phi(x_i, y_i; h))}{h} = \frac{y(x_{i+1}) - y_i}{h} - \Phi(x_i, y_i; h)$$

$$i = 0, 1, \ldots, N-1$$

Method of order p: $\tau_{i+1}(h) = O(h^p)$

Global error and Convergence

- The global error is the maximum error of the method over the entire range of approximation, assuming only that the method gives exact result at the initial value
- A one-step method is **convergent** if

$$\lim_{h\to 0} \max_{1\le i\le N} | y_i - y(x_i) | = 0$$

where $y(x_i)$: exact value

y_i: approximate value $y_i \approx y(x_i)$

Higher-order Taylor series method

Taylor expansion

$$y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(x_i) + O(h^3)$$

$$= y(x_i) + hf(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) + O(h^3)$$

where $f'(x_i, y_i) = f_x(x_i, y_i) + f_y(x_i, y_i) y'(x_i) = f_x(x_i, y_i) + f_y(x_i, y_i) f(x_i, y_i)$

The 2nd order Taylor series method is given by

$$\boxed{y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2}[f_x(x_i, y_i) + f_y(x_i, y_i) f(x_i, y_i)]}$$

Example

Find approximate values and the errors of the solution of the initial-value problem

$$y' = y + 2x + 1, \quad 0 \le x \le 1/2, \quad y(0) = 2$$

with step size h=0.1

a) using Euler method
b) using Taylor series method

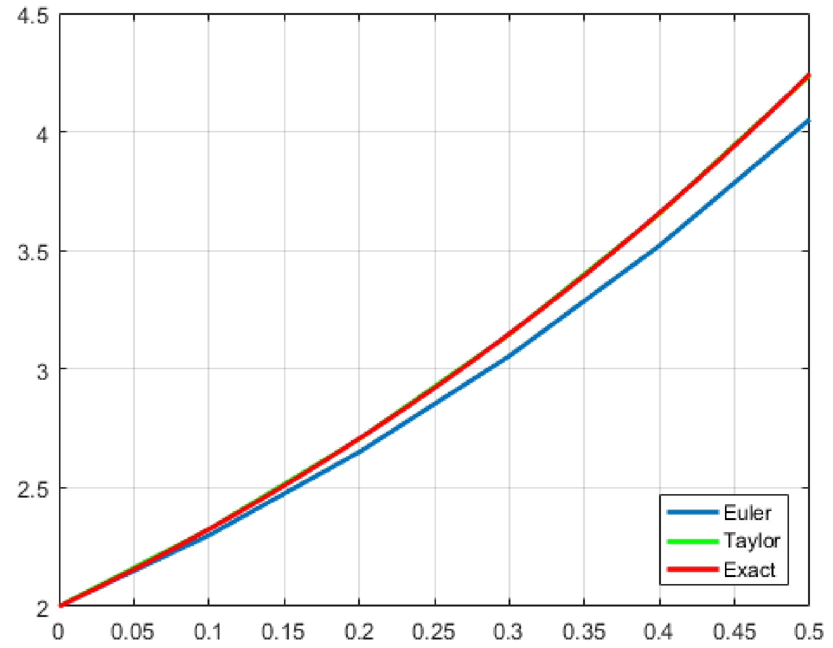
$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2}[f_x(x_i, y_i) + f_y(x_i, y_i) f(x_i, y_i)]$$

$$y' = y + 2x + 1, \quad 0 \le x \le \frac{1}{2}$$

$$y(0) = 2, \quad \text{h=0.1}$$

Exact solution:

$$y = -2x - 3 + 5e^{x}$$



x	y-exact	y-Euler	Error	y-Taylor	Error
0	2	2	0	2	0
0.1	2.326	2.3	0.02585	2.325	0.0008546
0.2	2.707	2.65	0.05701	2.705	0.001889
0.3	3.149	3.055	0.09429	3.146	0.003131
0.4	3.659	3.52	0.1386	3.655	0.004613
0.5	4.244	4.053	0.1911	4.237	0.006373

Exercise

Find approximate values and the errors of the solution of the initial-value problem

$$y' = -2xy, \qquad 0 \le x \le 1,$$

with step size h=0.2, and initial condition $y(0) = 3$

a) using Euler method
b) using Taylor series method

Homework Chapter 5

$(m-2)(n-2)$ is the two last digits of your student ID number

Problem 1: Find approximate values and the errors of the solution of the initial-value problem

$$y' = (m+n)y - mx + n, \qquad 0 \le x \le 1,$$

with step size h=1/4, and initial condition $y(0) = 2$ by

a) Euler Method
b) Midpoint Method
c) Heun method
d) 4^{th}-order Runge-Kutta method

Homework Chapter 5

Problem 2: Find approximate solution of problem by

a) Euler's method b) Midpoint method, Heun's method

$$u' = 3mu + 2v - (t^2 - n)e^{2t}$$

$$v' = -u + nv + (t^2 + mt - 4)e^{2t}, \qquad 0 \le t \le 1$$

$$u(0) = 1, \quad v(0) = 1, \qquad h = 1/4$$

Homework Chapter 5

Problem 3: Find approximate solution of the following Problem by

a) Euler's method

b) Heun's method

$$y'' - my' + ny = e^{2t} \sin t, \quad 0 \le t \le 1$$

$$y(0) = m+1, \quad y'(0) = n$$

using $h = 1/4$

Homework Chapter 5

Additional Problems:

Textbook, Page 752

Problems 25.1, 25.3, 25.4, 25.7

Deadline: 3 weeks