VIETNAM NATIONAL UNIVERSITY-HCMC
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Chapter 5. Linear transformations

Linear Algebra

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Introduction to linear transformation

Definition

Let V and W be vector spaces. A linear transformation is a function

$$L : V \to W$$

with the following properties:
(a) For any $u, v \in V$, we have $L(u + v) = L(u) + L(v)$.
(b) For any $u \in V$, $c \in \mathbf{R}$, we have $L(cu) = cL(u)$.

If $V = W$, the linear transformation $L : V \to W$ is also called a linear operator on V.

Introduction to linear transformation

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If $V = W$, the linear transformation $L : V \to W$ is also called a linear operator on V.

Remark: Namely, $L : V \to W$ is a linear transformation if and only if $L(au + bv) = aL(u) + bL(v)$ for any real numbers a, b and any vectors u, v in V.

Linear transformation

Example: Projection into the xy-plane

$L : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Linear transformation

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Example: Reflection with respect to the x-axis

$L : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}$$

Linear transformation

Example

Let A be an $m \times n$ matrix. We define the function $L : \mathbb{R}^n \to \mathbb{R}^m$ by $L(u) = Au$ (also called matrix transformation). Then L is a linear transformation since

$$L(u + v) = A(u + v) = Au + Av = Lu + Lv$$

$$L(cu) = A(cu) = cAu = cL(u)$$

Linear transformation

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Example: Rotation

If

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

then L is the rotation counterclockwise through an angle φ.

Linear transformation

Example

Let W be the vector space of all real-valued functions and let V be the subspace of all differentiable functions. Let $L : V \to W$ be defined by

$$L(f) = f',$$

where f' is the derivative of f. Then L is a linear transformation.

Linear transformation

Example

Let W be the vector space of all real-valued functions and let V be the subspace of all differentiable functions. Let $L : V \to W$ be defined by

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Example

Let $V = C[a, b]$ be the vector space of all real-valued functions that are integrable over the interval $[a, b]$. Let $W = \mathbb{R}$ and let $L : V \to W$ be defined by

$$L(f) = \int_a^b f(x)dx$$

Then L is a linear transformation.

Example of not a linear transformation

- $f(x) = sin\ x$
- $f(x) = x^2$
- $f(x) = x + 1$

Linear transformation

Properties

Let $L : V \to W$ be a linear transformation. Then

- $L(0_V) = 0_W$.
- $L(u - v) = L(u) - L(v)$

Linear transformation

Properties

Let $L : V \to W$ be a linear transformation. Then

- $L(0_V) = 0_W$.
- $L(u - v) = L(u) - L(v)$

Theorem

Let $L : V \to W$ be a linear transformation of an n-dimensional vector space V into a vector space W. Let $S = \{v_1, v_2, ..., v_n\}$ be a basis for V. If v is any vector in V, then $L(v)$ is completely determined by $\{L(v_1), L(v_2), ..., L(v_n)\}$.

$\to$ This theorem tell us that once we say what a linear transformation L does to a basis for V, then we have completely specified L!

Linear transformation

Theorem

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and consider the natural basis $\{e_1, e_2, ..., e_n\}$ for $\mathbb{R}^n$. Let A be the $m \times n$ matrix whose jth column is $L(e_j)$.

The matrix A has the following property: If $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any vector in $\mathbb{R}^n$, then $L(x) = Ax$.

The matrix A is called the standard matrix representing L.

Linear transformation

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 - 2x_3 \end{bmatrix}$$

Find the standard matrix representing L.

Linear transformation

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 - 2x_3 \end{bmatrix}$$

Find the standard matrix representing L.

Solution:

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Linear transformation

Solution (Cont.):

$$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$L(e_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} L(e_1) & L(e_2) & L(e_3) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \end{bmatrix}$$

Linear transformation

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation for which we know

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(a) What is $L\left(\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\right)$?

(b) What is $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$?

The kernel of Linear transformation

Definition: Kernel

Let $L : V \to W$ be a linear transformation of a vector space V into a vector space W. The kernel of L, *ker* L is the subset of V consisting of all elements v of V such that $L(v) = 0_W$.

$$\ker L = L^{-1}(0)$$

Theorem

Let $L : V \to W$ be a linear transformation of a vector space V into a vector space W. Then
(a) $\ker L$ is a subspace of V.
(b) L is one-to-one if and only if $\ker L = \{0_V\}$.

Linear transformation

Example: Kernel

Let $L : P_2 \to R$ be a linear transformation defined by

$$L\left(at^2 + bt + c\right) = \int_0^1 \left(at^2 + bt + c\right) dt$$

then

$$\ker L = \left\{at^2 + bt + (-a/3 - b/2) : a, b \in R\right\}$$

and L is not a one-to-one since

$$\dim \ker L = 2$$

Range

Definition: Range

If $L : V \to W$ is a linear transformation of a vector space V into a vector space W, then the range of L or image of V under L, denoted by range L, consists of all those vectors in W that are images under L of vectors in V.

Thus w is in range L if there exists some vector v in V such that $L(v) = w$. The linear transformation L is called onto if range $L = W$.

Theorem

If $L : V \to W$ is a linear transformation of a vector space V into a vector space W, then range L is a subspace of W.

Linear transformation

Example

If $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

then L is onto.

Example

If $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

then L is not onto.

Linear transformation

Example

If $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Find $dim\ker L$ and $dim\ range\ L$.

Linear transformation

Theorem

If $L : V \to W$ is a linear transformation of an n-dimensional vector space V into a vector space W then

$$dim \ker L + dim\ range\ L = dim\ V$$

.

Example

If $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_3 \\ u_1 + u_2 \\ u_2 - u_3 \end{bmatrix}$$

Find dim ker L and dim $range\ L$.

Linear transformation

Solution

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \ker L \Leftrightarrow L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_3 \\ u_1 + u_2 \\ u_2 - u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies a basis for $\ker L$ is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

So $\dim \ker L = 1$.

Linear transformation

Solution

Next, every vector in range L is of the form

$$\begin{bmatrix} u_1 + u_3 \\ u_1 + u_2 \\ u_2 - u_3 \end{bmatrix}$$

which can be written as

$$u_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

A basis of range L is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

So dim range $L = 2$.

Invertible

Corollary

If $L : V \to W$ is a linear transformation of a vector space V into a vector space W and $dim\ V = dim\ W$, then the following statements are true:

(a) If L is one-to-one, then it is onto.

(b) If L is onto, then it is one-to-one.

Remark: A linear transformation $L : V \to W$ is invertible if and only if L is one-to-one and onto.