Real Analysis, Chapter 1
Worksheet 1: Metrics & Norms

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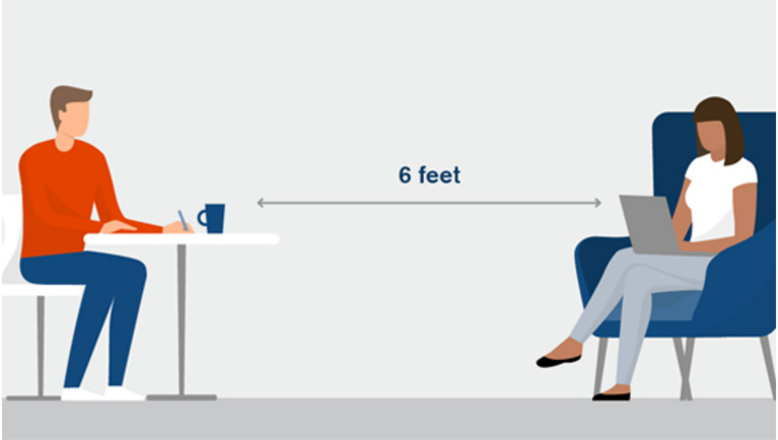


Figure 1: Social Distancing

- The very first "distance" concept was due to Euclid (300 B.C.):

  Distance between A and B = length of segment AB.

- Euclidean "distance" concept started taking its central role in algebraic geometry after the invention of Cartesian coordinates (René Descartes, 1637).

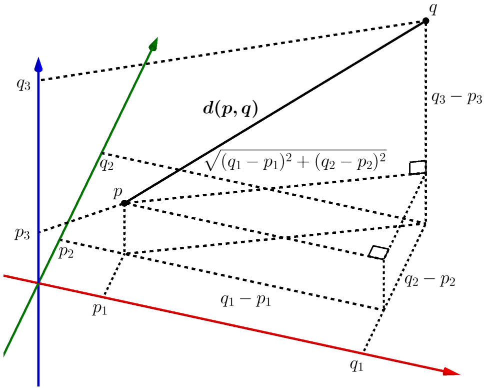
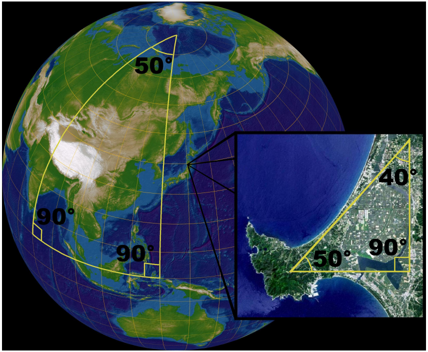


Figure 2: Euclidean Distance in $\mathbb{R}^3$

However, with the development of mathematics, several inadequacies of Euclidean distance have become apparent.

- The distance formula, in high-dimensional space, is complicated;
- The invention of non-Euclidean geometry (Lobachevsky, 1830);
- Euclidean distance is not applicable in many real-life situations.



Thus, it is necessary to mathematically generalize the Euclidean distance.

The generalized concept of "distance" should preserves several important properties of the Euclidean distance, namely:

- Positivity: $AB \geq 0$ and $AB = 0$ only if $A \equiv B$;
- Symmetry: $AB = BA$;
- Triangle Inequality: $AB + AC \geq BC$.

The following definition arises naturally from the above prerequisites.

Definition 1.1.1 (Metric Space)

A metric (or a distance) d on a set $X \neq \emptyset$ is a function $d : X \times X \to \mathbb{R}$ satisfying the three properties:

(a) Positivity: $\forall x, y \in X : d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$;

(b) Symmetry: $d(x, y) = d(y, x), \forall x, y \in X$;

(c) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X$.

The pair (X, d) is called a metric space.

The same process can be used to obtain the concept of normed space, concerning the fact that $\mathbb{R}^n$ is a vector space.

Definition 1.1.2 (Normed Space)

A norm $||\cdot||$ on a vector space X is a function $||\cdot|| : X \times X \to [0, \infty)$ satisfying the three properties:

(a) Positivity: $\forall x \in X : ||x|| \geq 0$ and $||x|| = 0 \Leftrightarrow x = 0$;

(b) Scalar Multiplication: $||\alpha x|| = |\alpha| \cdot ||x||, \forall x \in X, \alpha \in \mathbb{R}$;

(c) Triangle Inequality: $||x + y|| \leq ||x|| + ||y||, \forall x, y \in X$.

The pair $(X, ||\cdot||)$ is called a normed space.

Theorem 1.1.1

Given a normed space $(X, ||\cdot||)$, the function $\rho : X \times X \to \mathbb{R}$ defined by

$$\rho(x, y) = ||x - y||, \forall x, y \in X$$

is a metric on X.

In words, **every normed space is also a metric space**.

Problem 1

Given a metric space (X, d) and a function $d' : X \times X \to \mathbb{R}$ defined by

$$d'(x, y) = \sqrt{d(x, y)}, \forall x, y \in X.$$

Show that d' is a metric on X.

Guidelines:

- Positivity: $\sqrt{a} \geq 0$ and $\sqrt{a} = 0 \Leftrightarrow a = 0$;
- Symmetry: $\sqrt{d(x, y)} = \sqrt{d(y, x)}$, by the symmetry of d;
- Triangle Inequality: since $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$, by the triangle inequality of d, $\sqrt{d(x, z)} \leq \sqrt{d(x, y) + d(y, z)} \leq \sqrt{d(x, y)} + \sqrt{d(y, z)}$.

Problem 2 (Linearity of Metrics)

Let d_1, d_2 be two metrics on X and $c > 0$.

(a) Show that the function $d_3 : X \times X \to \mathbb{R}$ defined by

$$d_3(x, y) = c \cdot d_1(x, y), \forall x, y \in X$$

is a metric on X;

(b) Show that the function $d_4 : X \times X \to \mathbb{R}$ defined by

$$d_4(x, y) = d_1(x, y) + d_2(x, y), \forall x, y \in X$$

is a metric on X.

Guidelines for 2a:

- Positivity: if $ca \geq 0$ then $a \geq 0$, and if $ca = 0$ then $a = 0$;
- Symmetry: if $a = b$ then $ca = cb$;
- Triangle Inequality: if $m \leq n + p$ then $cm \leq cn + cp$.

Guidelines for 2b:

- Positivity: if $a, b \geq 0$ and $a + b = 0$ then $a = b = 0$;
- Symmetry: if $a = b$ and $c = d$ then $a + c = b + d$;
- Triangle Inequality: $m \leq n + p, a \leq b + c$ then $m + a \leq n + p + b + c$.

Use the guidelines with appropriate a, b, c, m, n, p for each part.

Problem 3 (Linearity of Norms)

Let $||\cdot||_1, ||\cdot||_2$ be two norms on a vector space X and $c > 0$.

(a) Show that the function $||\cdot||_3 : X \to \mathbb{R}$ defined by

$$||x||_3 = c \cdot ||x||_1, \forall x \in X$$

is a norm on X.

(b) Show that the function $||\cdot||_4 : X \to \mathbb{R}$ defined by

$$||x||_4 = ||x||_1 + ||x||_2, \forall x \in X$$

is a norm on X.

The proof for Problem 3 is similar to Problem 2.

Problem 4 (Mole Metric)

Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & x \in [0, 1) \\ 1, & x \geq 1 \end{cases}, \forall x \in [0, \infty).$$

Show that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x, y) = f(|x - y|), \forall x, y \in \mathbb{R}$$

is a metric on $\mathbb{R}$.

Hint. See **Exercise 0.2.7a, HW1**. Note that $d(x, y) = \min \{|x - y|, 1\}$.

Problem 5

Show that the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d((x_a, y_a), (x_b, y_b)) = \begin{cases} |x_a - x_b|, & y_a = y_b \\ |x_a - x_b| + 1, & y_a \neq y_b \end{cases}, \forall (x_a, y_a), (x_b, y_b) \in \mathbb{R}^2$$

is a metric on $\mathbb{R}^2$.

Guidelines: Let $A(x_A, y_A), B(x_B, y_B), C(x_C, y_C) \in \mathbb{R}^2$ be arbitrary.

- Positivity: if $y_A \neq y_B$ then $d(A, B) = |x_A - x_B| + 1 > 0$;
- Symmetry: trivial since $|x_A - x_B| = |x_B - x_A|$;
- Triangle Inequality: consider two cases as follows:
  If $y_A = y_C$ then $d(A, C) \leq |x_A - x_B| + |x_B - x_C| \leq d(A, B) + d(B, C)$.
  Else, either $y_A \neq y_B$ or $y_C \neq y_B$ and the result follows.

Problem 6

Show that the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d(x, y) = \begin{cases} 0, & x = y \\ ||x|| + ||y||, & x \neq y \end{cases}, \forall x, y \in \mathbb{R}^2$$

is a metric on the Euclidean space $(\mathbb{R}^2, ||\cdot||)$.

The proof for Problem 6 is similar to Problem 5.

Problem 7

If $B = \{b_i\}_{i=1}^n$ is a basis of $\mathbb{R}^n$, then the function $\varphi_B : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi_B(x) = \sum_{i=1}^{n} |\lambda_i|, \forall x \in \mathbb{R}^n$$

where $\sum_{i=1}^{n} \lambda_i b_i$ is the coordinate of x with respect to B, is a norm on $\mathbb{R}^n$.

The proof for Problem 7 is left as an exercise to the reader.

If B is the standard basis, then φ_B is called the **Manhattan norm** and its induced metric is called the **Manhattan distance**.

Problem 8 (British Railway Metric)

If (X, d) is a metric space and $z \in X$, then the function $d_1 : X \times X \to \mathbb{R}$ defined by

$$d_1(x, y) = \begin{cases} 0, & x = y \\ d(x, z) + d(z, y), & x \neq y \end{cases}, \forall x, y \in X$$

is a metric on X.

The proof for Problem 8 is left as an exercise to the reader.