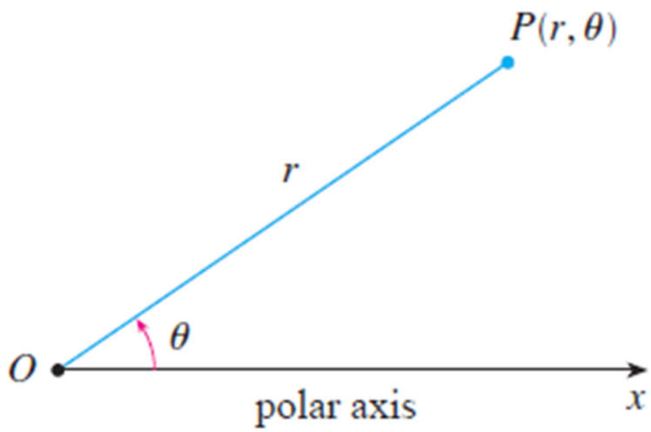
Chapter 3: MULTIPLE INTEGRALS

Lecture 11: Change of Variables in Multiple Integrals

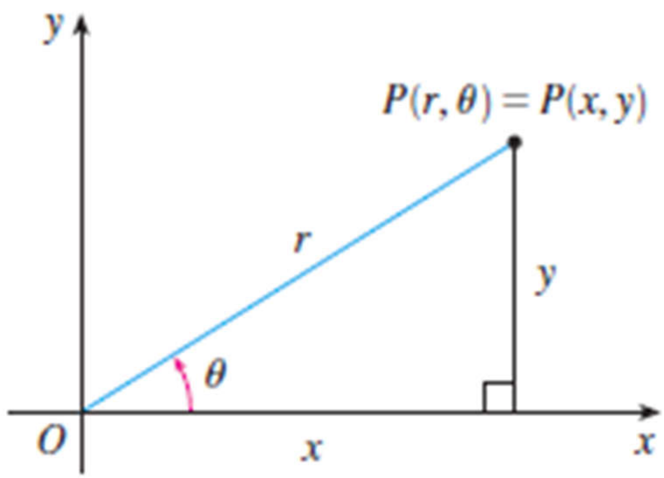
1. Double Integrals in Polar Coordinates



Polar coordinate system consists of
-a **pole** *O*
-a **polar axis**: a ray from *O* horizontal to the right

Polar coordinates of a point *P* is $P(r, \vartheta)$
r =distance from *O* to *P*
ϑ = angle between polar axis and *OP*

Polar-rectangular conversion



$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$

$$\boxed{r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}}$$

Change to Polar Coordinates in a Double Integral

By changing into polar coordinates $x = r\cos\theta, \quad y = r\sin\theta$

we can express *R* in xy-plane as

$$\boxed{R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}, \quad 0 \le \beta - \alpha \le 2\pi}$$

such a set is called a **polar rectangle**

Theorem 1: If *f* is continuous on *R*, then

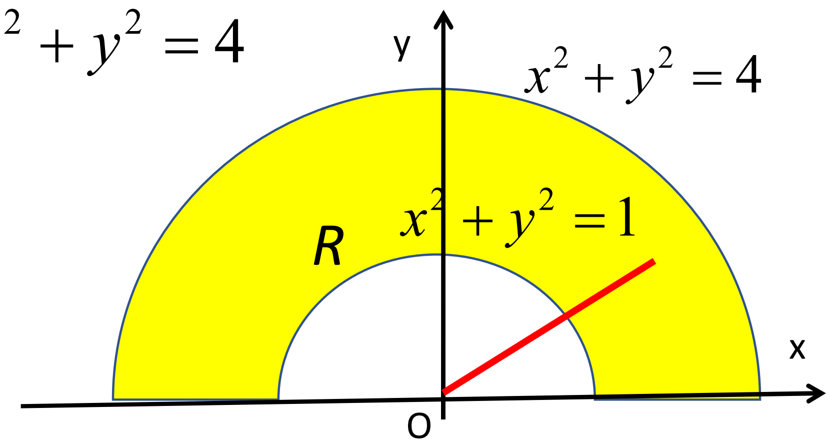
$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\; r\; dr\,d\theta$$

Example

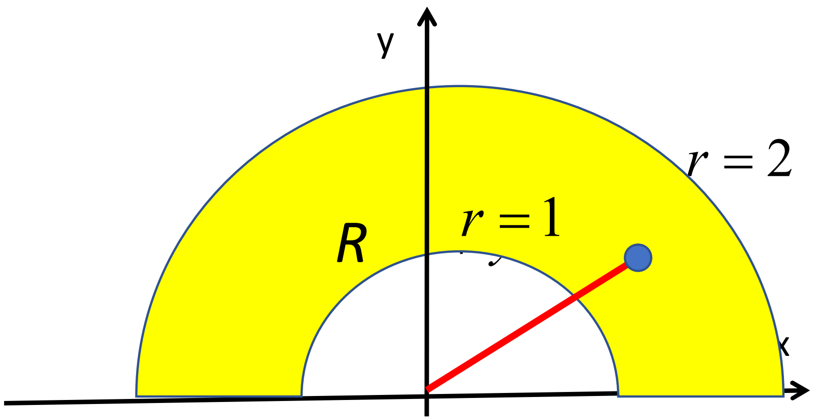
Evaluate
$$\iint_R (3x+4y^2)\,dA$$

where R is the region in the upper half-plane bounded by the circles

$$x^2+y^2=1 \text{ and } x^2+y^2=4$$



Solution



- Change into polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

- Then,

$$R = \{(r,\theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$$

$$\iint_R (3x + 4y^2)\,dA = \int_0^{\pi}\int_1^{2} (3r\cos\theta + 4(r\sin\theta)^2)\; r\; dr d\theta$$

$$= \int_0^{\pi}\int_1^{2} (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr d\theta = \int_0^{\pi} (r^3\cos\theta + r^4\sin^2\theta)\Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\,d\theta = 7\sin\theta\Big|_0^{\pi} + 15\int_0^{\pi} \frac{1-\cos 2\theta}{2} d\theta$$

$$= 0 + \frac{15}{4}(2\theta - \sin 2\theta)\Big|_0^{\pi} = \frac{15\pi}{2}$$

Change to Polar Coordinates in a Double Integral

Theorem 2: If f is continuous on a polar region D of the form

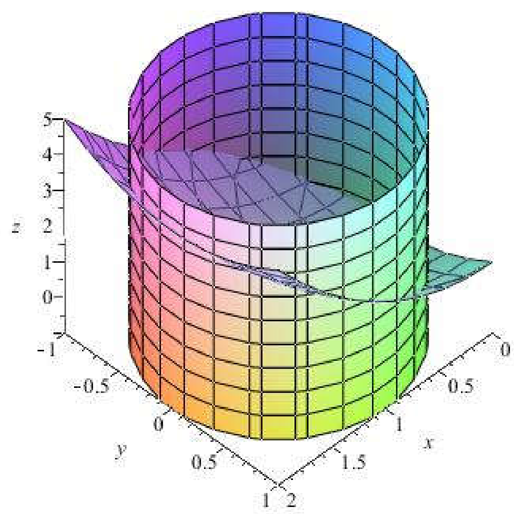
$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

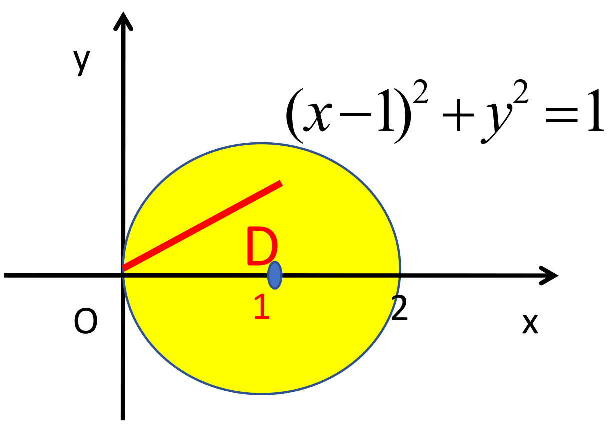
then

$$\boxed{\iint_D f(x,y)\,dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta}$$

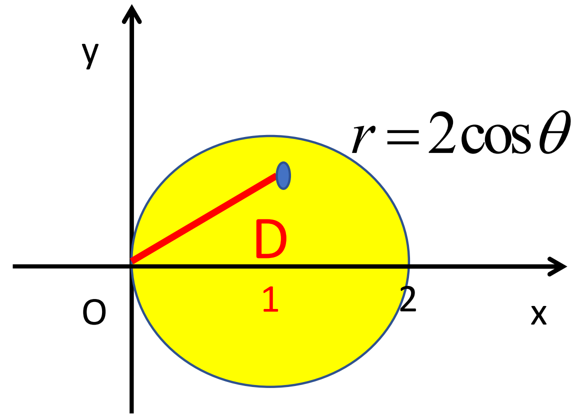
Example

- Find the volume of the solid that lies under the paraboloid above the xy-plane $z = x^2 + y^2$ **a**nd inside the cylinder $x^2 + y^2 = 2x$





Solution



$$V = \iint_D f(x,y)\,dA = \iint_D (x^2 + y^2)\,dA$$

- Change in polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$$

- Boundary circle $x^2 + y^2 = 2x$ becomes

- Thus,

$$r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$$

$$D = \{(r,\theta) \mid -\pi/2 < \theta < \pi/2, 0 \le r \le 2\cos\theta\}$$

$$V = \iint_D (x^2 + y^2)\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \; r \; dr d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left. \frac{r^4}{4} \right|_{r=0}^{r=2\cos\theta} d\theta = 4 \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta = 8 \int_0^{\pi/2} \cos^4 \theta d\theta$$

$$= 2 \int_0^{\pi/2} (1 + \cos 2x)^2 \, dx = 2 \int_0^{\pi/2} (1 + 2\cos 2x + \cos^2 2x) dx$$

$$= 2(x + \sin 2x)\Big|_0^{\pi/2} + \int_0^{\pi/2} (1 + \cos 4x) dx$$

$$= \pi + (x + \sin 4x / 4)\Big|_0^{\pi/2} = 3\pi / 2$$

Change of Variables in Integral of one variable

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

Transformation in 2D

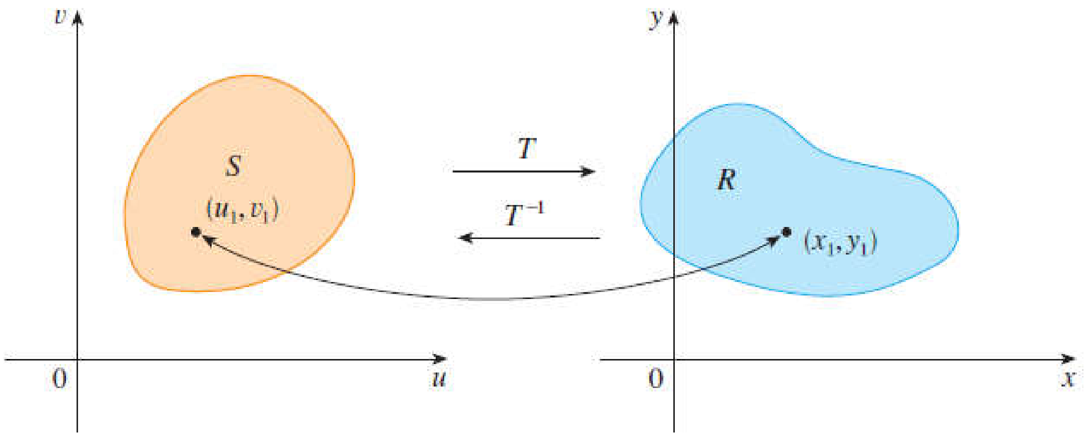


FIGURE 1

$$T(u,v) = (x,y) \text{ or } x = g(u,v),\ y = h(u,v)$$

T is C^1-transformation: g, h have continuous partial derivatives

T is one-to-one, its inverse: T^{-1}

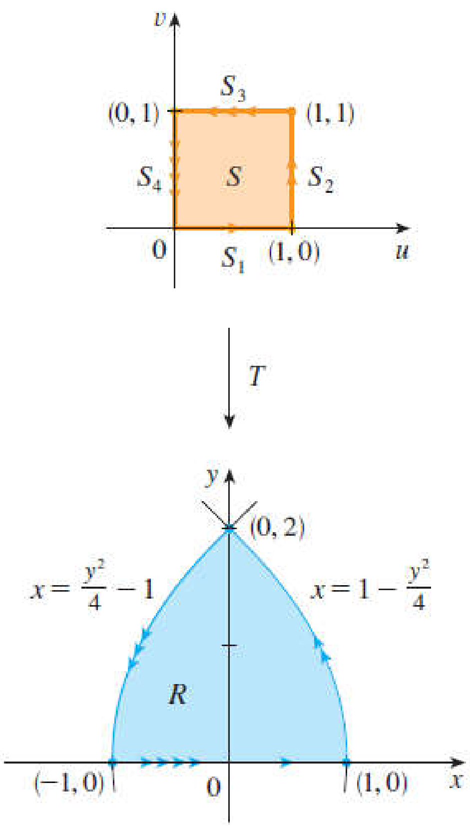


FIGURE 2

V EXAMPLE 1 A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$.

SOLUTION The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side, S_1, is given by $v = 0$ $(0 \leqslant u \leqslant 1)$. (See Figure 2.) From the given equations we have $x = u^2$, $y = 0$, and so $0 \leqslant x \leqslant 1$. Thus S_1 is mapped into the line segment from $(0, 0)$ to $(1, 0)$ in the xy-plane. The second side, S_2, is $u = 1$ $(0 \leqslant v \leqslant 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \qquad y = 2v$$

Eliminating v, we obtain

$$\boxed{4} \qquad x = 1 - \frac{y^2}{4} \qquad 0 \leqslant x \leqslant 1$$

which is part of a parabola. Similarly, S_3 is given by $v = 1$ $(0 \leqslant u \leqslant 1)$, whose image is the parabolic arc

$$\boxed{5} \qquad x = \frac{y^2}{4} - 1 \qquad -1 \leqslant x \leqslant 0$$

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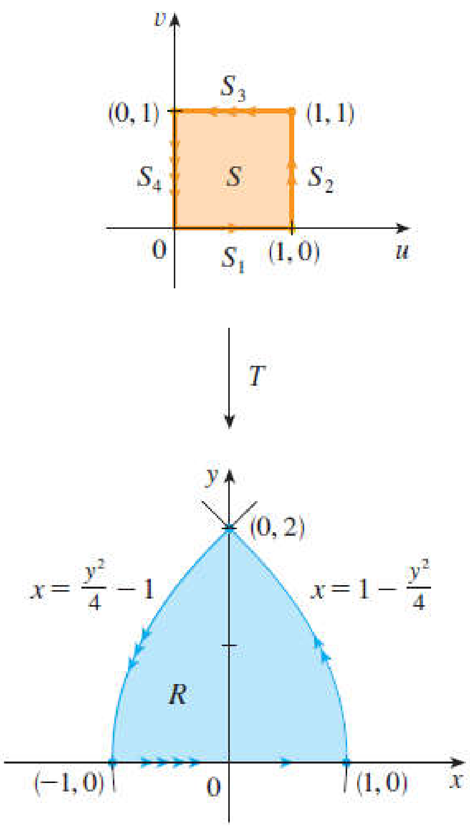


FIGURE 2

Finally, S_4 is given by $u = 0$ $(0 \leqslant v \leqslant 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \leqslant x \leqslant 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region R (shown in Figure 2) bounded by the x-axis and the parabolas given by Equations 4 and 5.

Change of Variables in Double Integrals

Now let's see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv. (See Figure 3.)

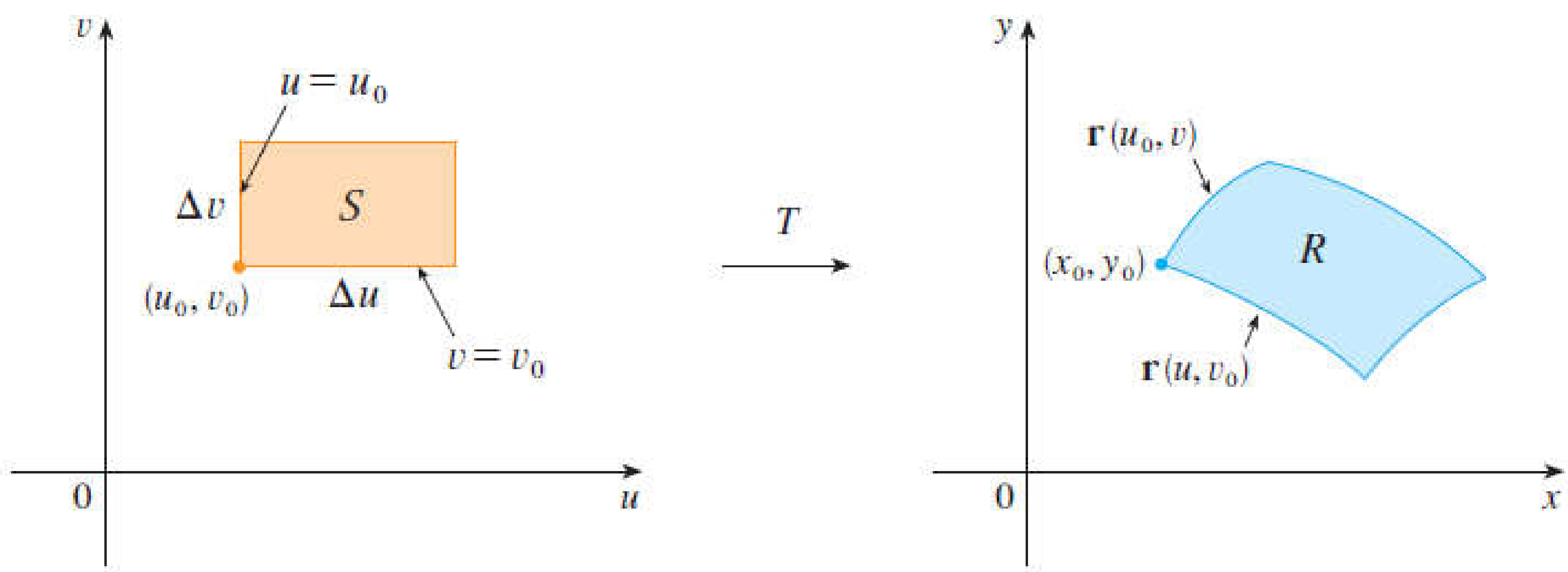


FIGURE 3

The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\mathbf{r}(u, v) = g(u, v)\,\mathbf{i} + h(u, v)\,\mathbf{j}$$

is the position vector of the image of the point (u, v). The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$. The tangent vector at (x_0, y_0) to this image curve is

$$\mathbf{r}_u = g_u(u_0, v_0)\,\mathbf{i} + h_u(u_0, v_0)\,\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}$$

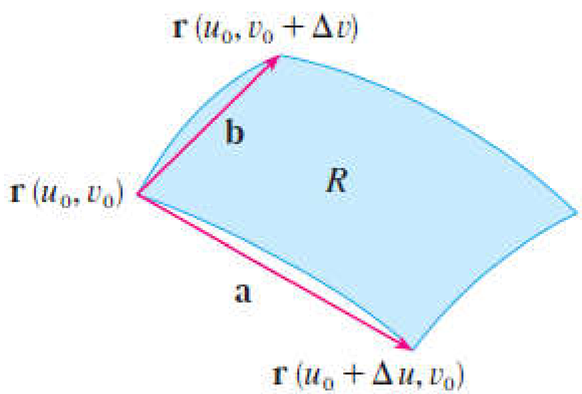


FIGURE 4

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\mathbf{r}_v = g_v(u_0, v_0)\,\mathbf{i} + h_v(u_0, v_0)\,\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

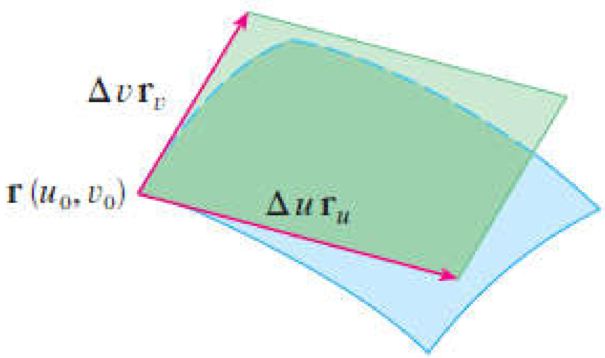


FIGURE 5

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and so
$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u\, \mathbf{r}_u$$

Similarly
$$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v\, \mathbf{r}_v$$

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u\, \mathbf{r}_u$ and $\Delta v\, \mathbf{r}_v$. (See Figure 5.) Therefore we can approximate the area of R by the area of this parallelogram, which, from Section 12.4, is

$$\boxed{6} \qquad |(\Delta u\, \mathbf{r}_u) \times (\Delta v\, \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

7 Definition The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation we can use Equation 6 to give an approximation to the area ΔA of R:

8
$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \, \Delta v$$

where the Jacobian is evaluated at (u_0, v_0).

Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}. (See Figure 6.)

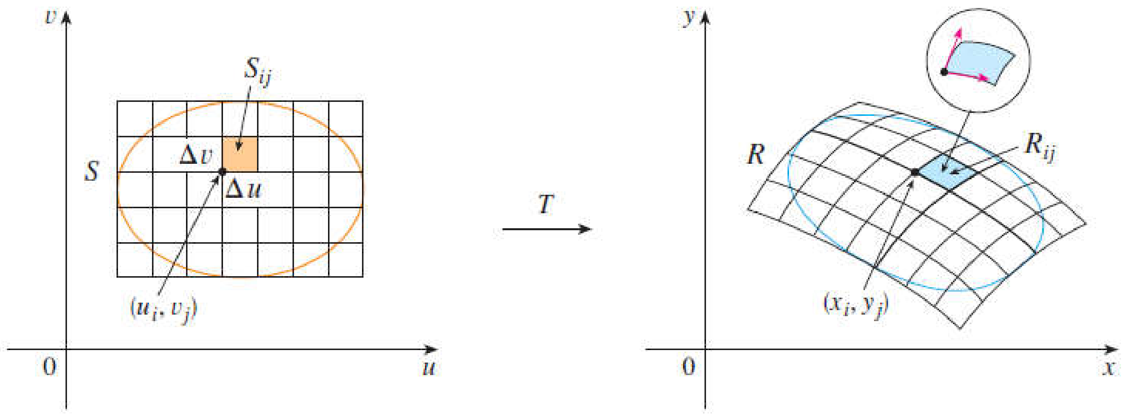


FIGURE 6

Applying the approximation $\boxed{8}$ to each R_{ij}, we approximate the double integral of f over R as follows:

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\big(g(u_i, v_j), h(u_i, v_j)\big) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v$$

where the Jacobian is evaluated at (u_i, v_j). Notice that this double sum is a Riemann sum for the integral

$$\iint_S f\big(g(u, v), h(u, v)\big) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

9 Change of Variables in a Double Integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Theorem 9 says that we change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

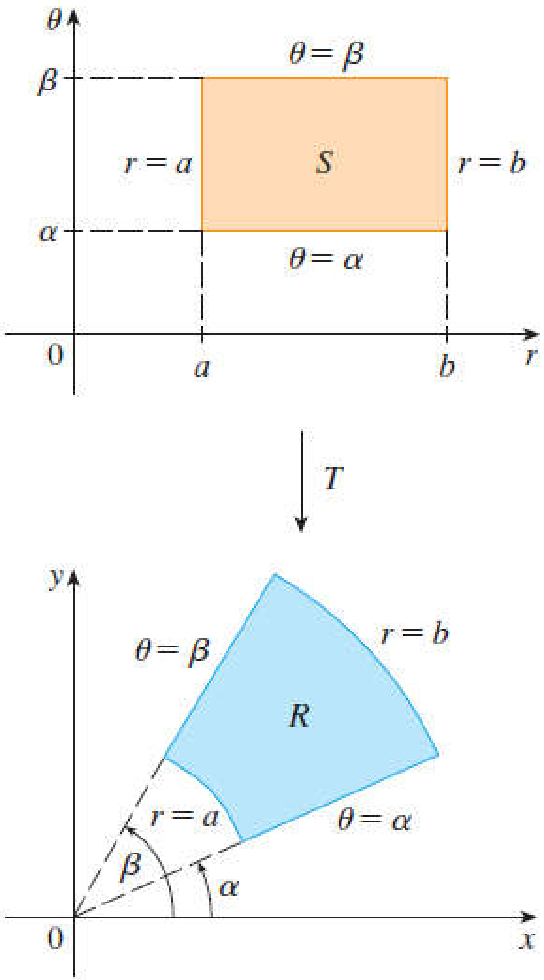


FIGURE 7

As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates is just a special case. Here the transformation T from the $r\theta$-plane to the xy-plane is given by

$$x = g(r, \theta) = r\cos\theta \qquad y = h(r, \theta) = r\sin\theta$$

and the geometry of the transformation is shown in Figure 7. T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane. The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

Thus Theorem 9 gives

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr\, d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.

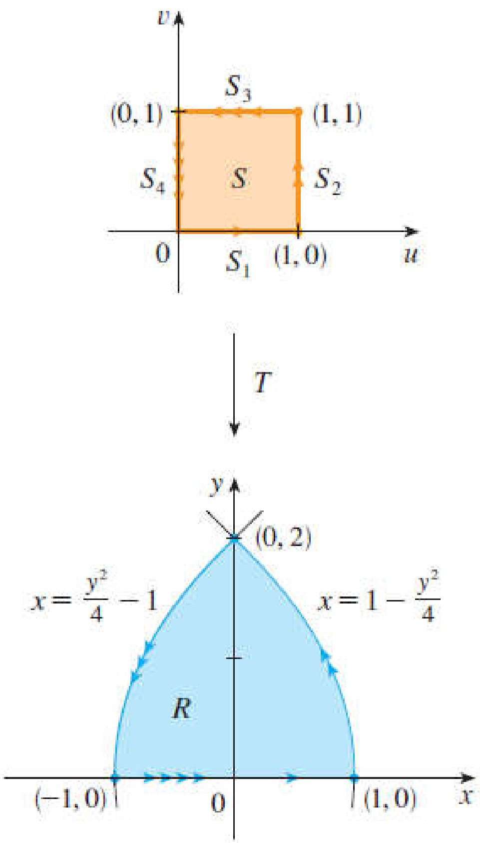


FIGURE 2

Solution

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

$$\iint\limits_R y\, dA = \iint\limits_S 2uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA = \int_0^1 \int_0^1 (2uv)4(u^2 + v^2)\, du\, dv$$

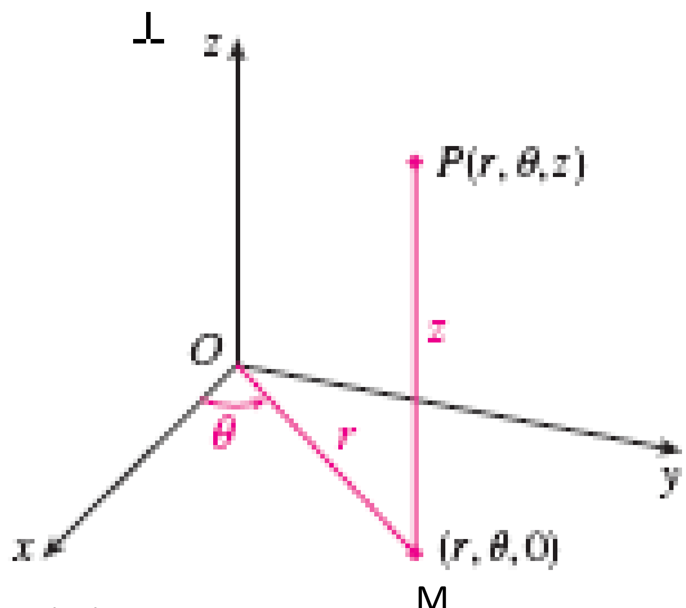
$$= 8 \int_0^1 \int_0^1 (u^3 v + uv^3)\, du\, dv = 8 \int_0^1 \left[\tfrac{1}{4}u^4 v + \tfrac{1}{2}u^2 v^3 \right]_{u=0}^{u=1} dv$$

$$= \int_0^1 (2v + 4v^3)\, dv = \left[v^2 + v^4 \right]_0^1 = 2$$

3. Triple Integrals in Cylindrical and Spherical Coordinates

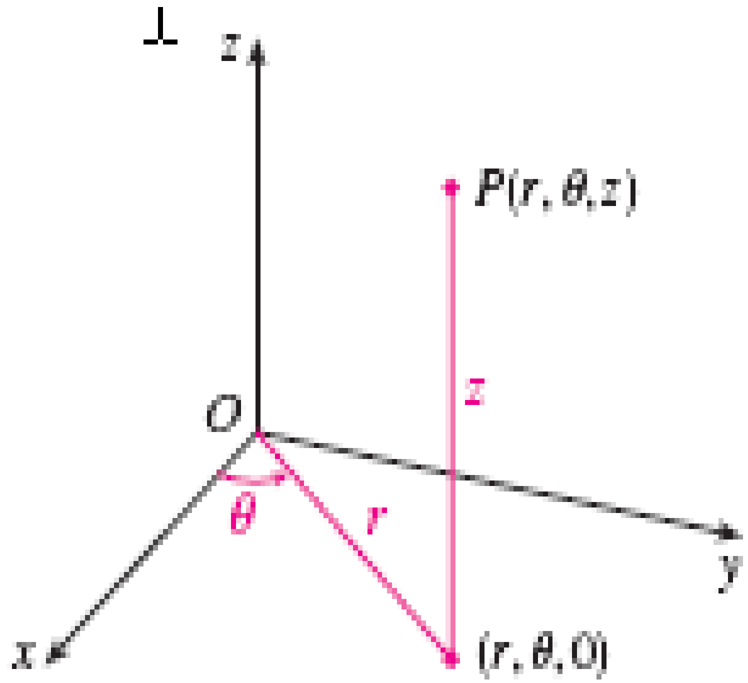
Cylindrical Coordinates

- A point *P(x, y, z)* is represented by the ordered triple (r, θ, z) , where:
- r and θ are polar coordinates of the projection of P onto the xy-plane



Relationships between cylindrical and rectangular coordinates

$$\text{a) } x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = z$$
$$\text{b) } r^2 = x^2 + y^2,$$
$$\tan\theta = \frac{y}{x},$$
$$z = z$$



Example

- Find the cylindrical coordinates of the point with rectangular coordinates (3, -3, -7)

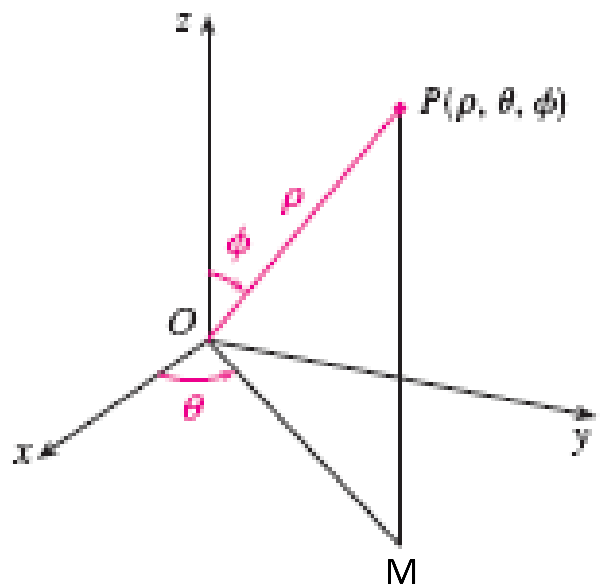
$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \Rightarrow \theta = \frac{7\pi}{4} + 2n\pi$$

- Thus, one set of cylindrical coordinates is $(3\sqrt{2}, 7\pi/4, -7)$. Another is $(3\sqrt{2}, -\pi/4, -7)$
- There are infinitely many choices

Spherical coordinates

- The spherical coordinates (ρ, θ, ϕ) of a point P in space:
- $\rho = |OP|$
- θ is the same angle as in cylindrical coordinates
- ϕ is the angle between the positive z-axis and OP



$$\rho \geq 0, \quad 0 \leq \phi \leq \pi$$

Relationship between rectangular and spherical coordinates

- Triangles OPQ and OPP' give

$$z = \rho\cos\phi, \quad r = \rho\sin\phi$$

- But

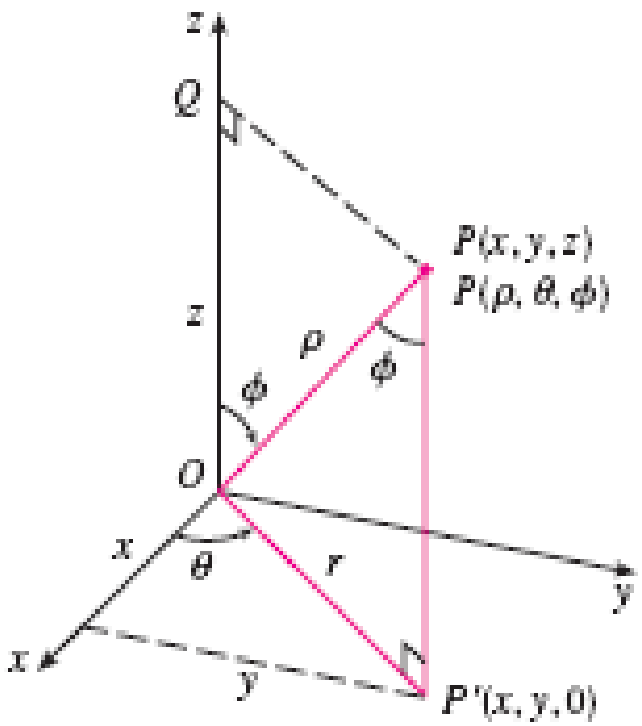
$$x = r\cos\theta, y = r\sin\theta$$

- So

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$

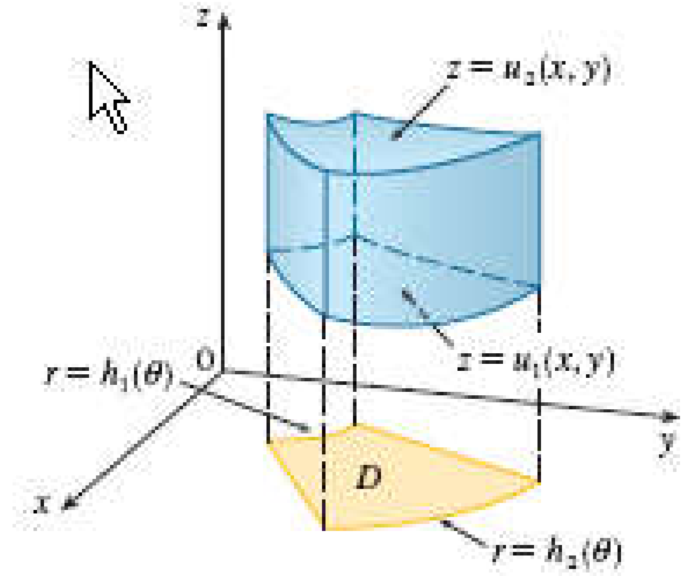


Remarks

- Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry.
- The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point

Triple integration in cylindrical coordinates

- E is a type 1 region whose projection on the xy-plane is conveniently described in polar coordinates



$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$
$$\text{where } D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

Formula for Triple integration in cylindrical coordinates

- We know that

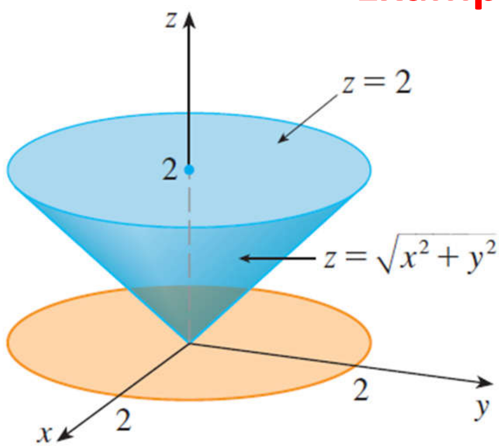
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

- But we also know how to evaluate double integrals in polar coordinates

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta}$$

Example: Evaluate

$$I=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$$



Solution

$$I=\iiint_E\left(x^2+y^2\right)dV$$

$$E=\{(x,y,z)\,|\,(x,y)\in D,\sqrt{x^2+y^2}\le z\le 2\}$$

$$D=\{(x,y)\,|\,-2\le x\le 2,-\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\}=\{(x,y)\,|\,x^2+y^2\le 4\}$$

Change into cylindrical coordinates:

$$x=r\sin\theta,\ y=r\cos\theta$$

$$E=\{(r,\theta,z)\,|\,0\le r\le 2,0\le\theta\le 2\pi,r\le z\le 2\}$$

$$I=\int_0^2\int_0^{2\pi}\int_r^2 r^2 r\,dz\,d\theta\,dr=\int_0^2\int_0^{2\pi}r^3(2-r)\,d\theta\,dr=\frac{16\pi}{5}$$

Formula for Triple integration in spherical coordinates

- It holds that

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

- where

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

Example: Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}} dV$, *where*

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$$

SOLUTION Since the boundary of B is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \phi) \mid 0 \leqslant \rho \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi\}$$

In addition, spherical coordinates are appropriate because

$$x^2 + y^2 + z^2 = \rho^2$$

$$\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} e^{(\rho^2)^{3/2}} \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^{1} \rho^2 e^{\rho^3}\, d\rho$$

$$= \Big[-\cos\phi\Big]_0^{\pi} (2\pi) \Big[\tfrac{1}{3} e^{\rho^3}\Big]_0^1 = \tfrac{4}{3}\pi(e - 1)$$

Triple Integrals

There is a similar change of variables formula for triple integrals. Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

The **Jacobian** of T is the following 3×3 determinant:

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$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

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$$\iiint_R f(x, y, z)\, dV = \iiint_S f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

V EXAMPLE 4 Use Formula 13 to derive the formula for triple integration in spherical coordinates.

SOLUTION Here the change of variables is given by

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\boldsymbol{\theta} \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\,(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta)$$

$$- \rho\sin\phi\,(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta)$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi = -\rho^2\sin\phi$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin \phi \geqslant 0$. Therefore

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = \left| -\rho^2 \sin \phi \right| = \rho^2 \sin \phi$$

and Formula 13 gives

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\, \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

which is equivalent to Formula 15.9.3.