Chapter 5: Numerical Solutions of Differential Equations

Lecture 2: Higher-order Methods

Midpoint (Modified Euler) Method

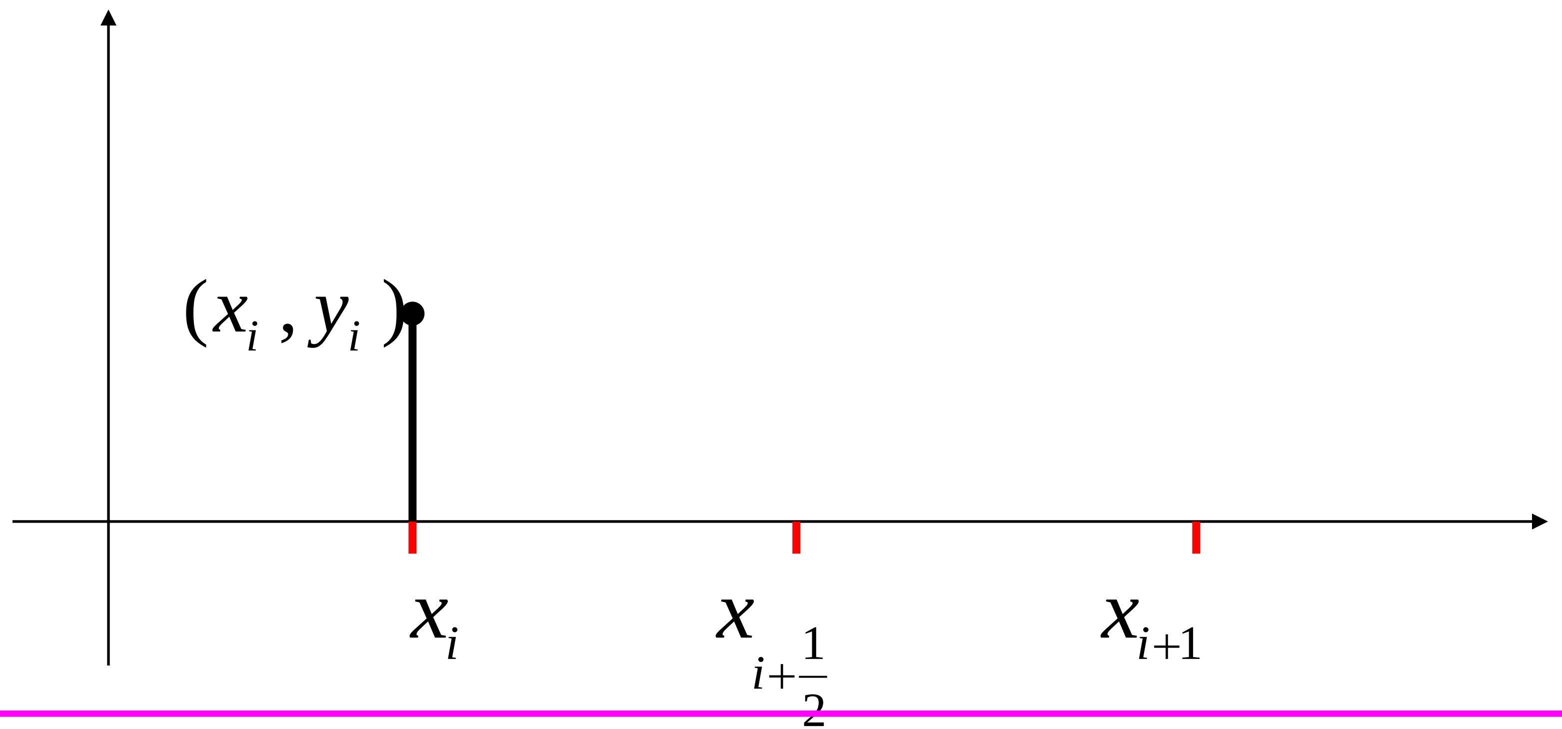
Problem: $y'(x) = f(x, y), \quad y(x_0) = y_0$

$$\begin{cases} y_{i+\frac{1}{2}} = y_i + \dfrac{h}{2} f(x_i, y_i) \\ y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}), \qquad x_{i+\frac{1}{2}} = x_i + \dfrac{h}{2} \end{cases}$$

or

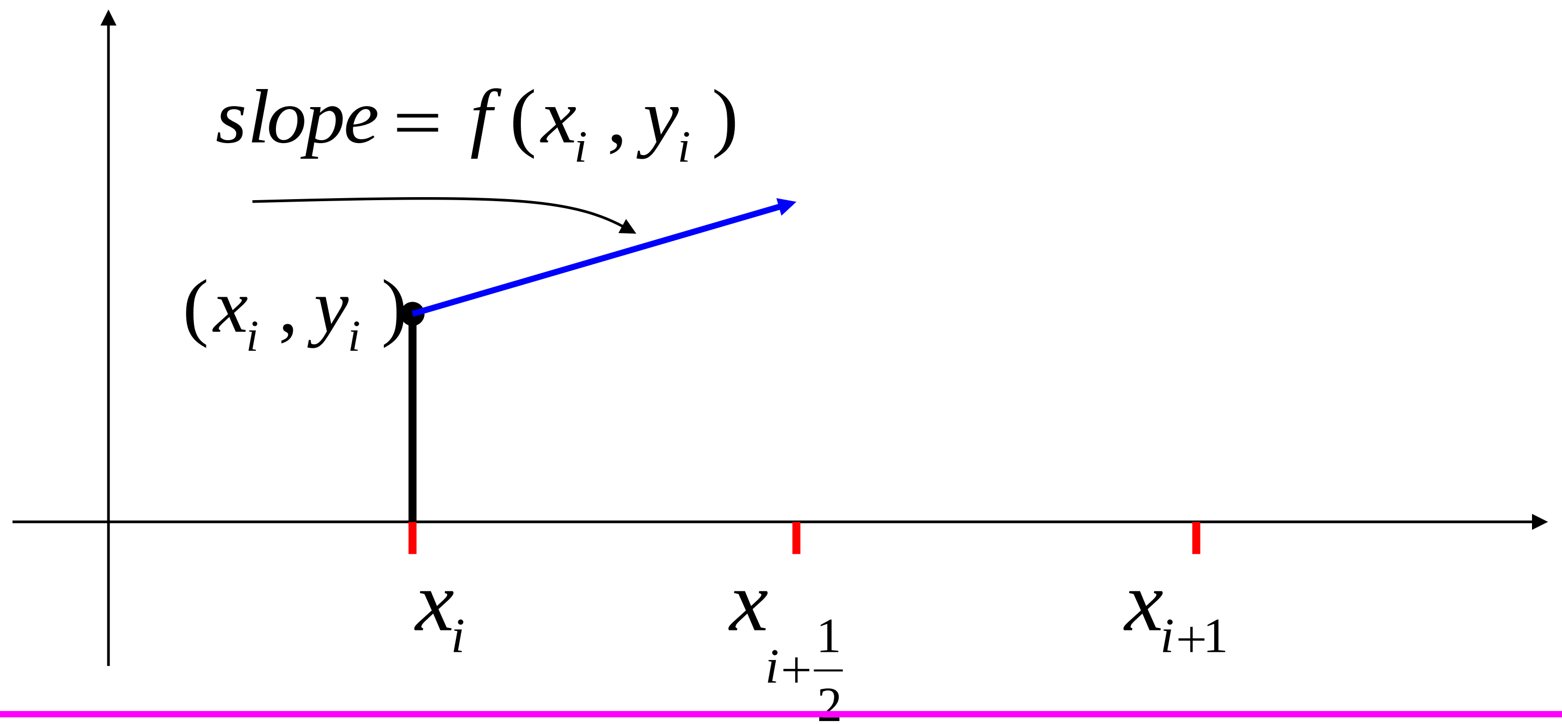
$$\boxed{y_{i+1} = y_i + h\, f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right)}$$

Midpoint (Modified Euler) Method



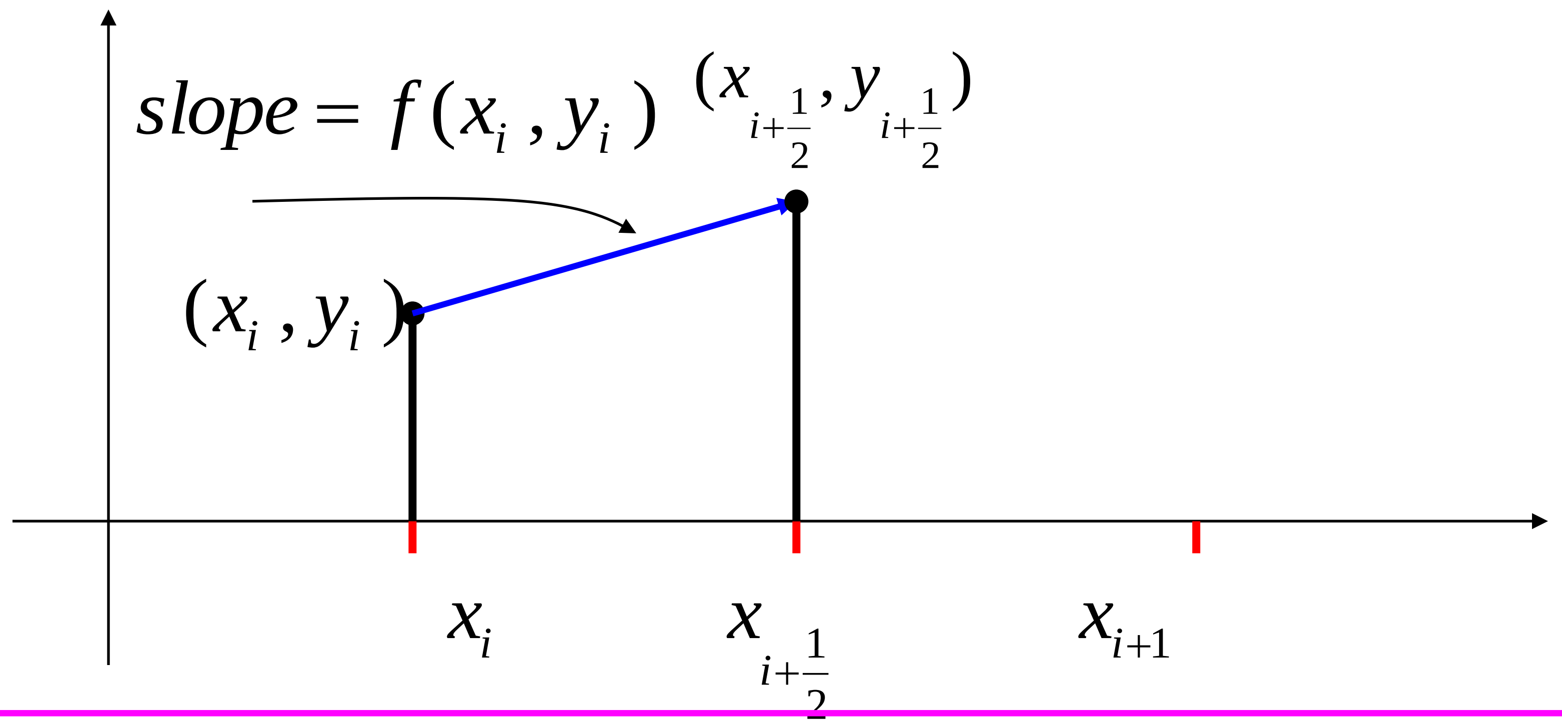
$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint (Modified Euler) Method



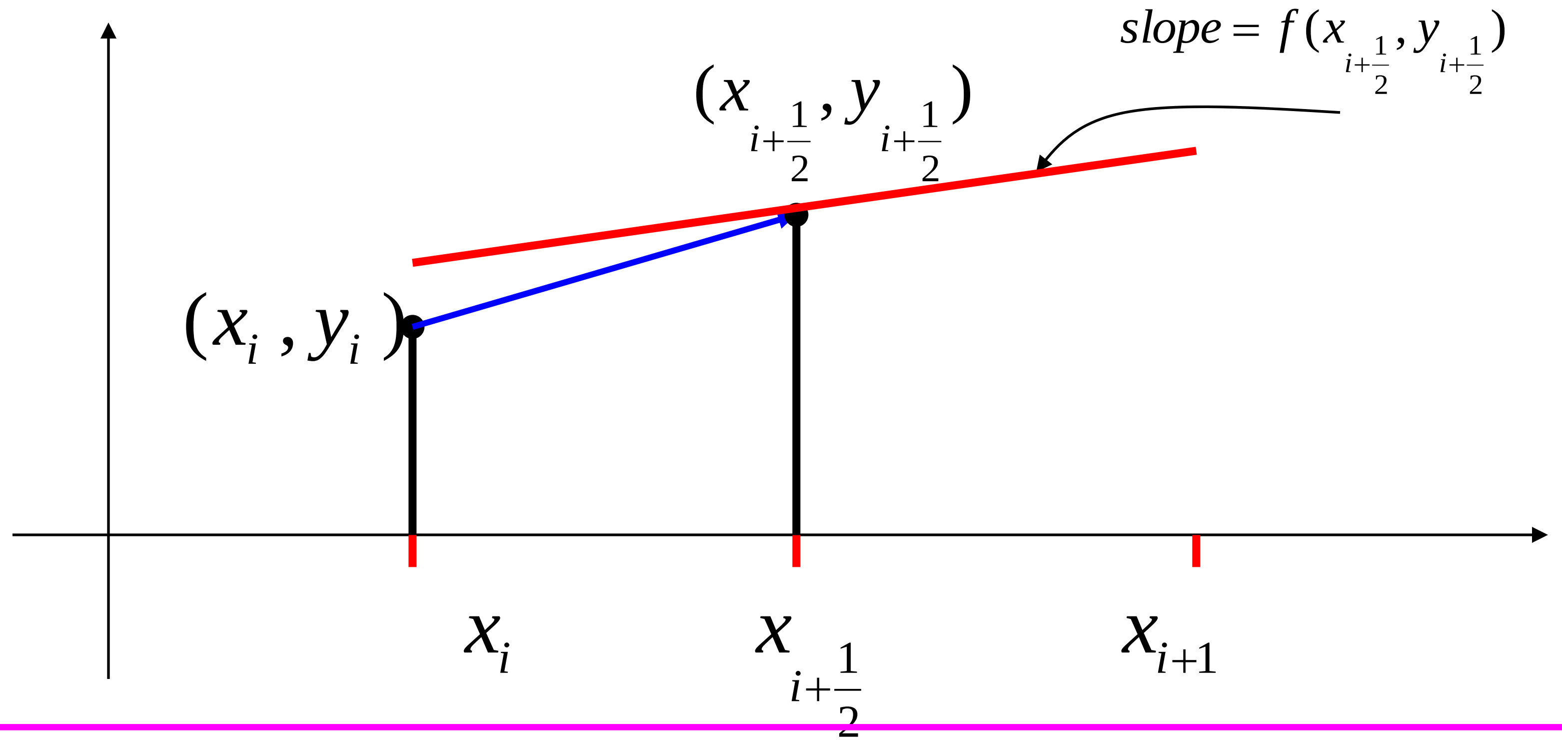
$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint (Modified Euler) Method



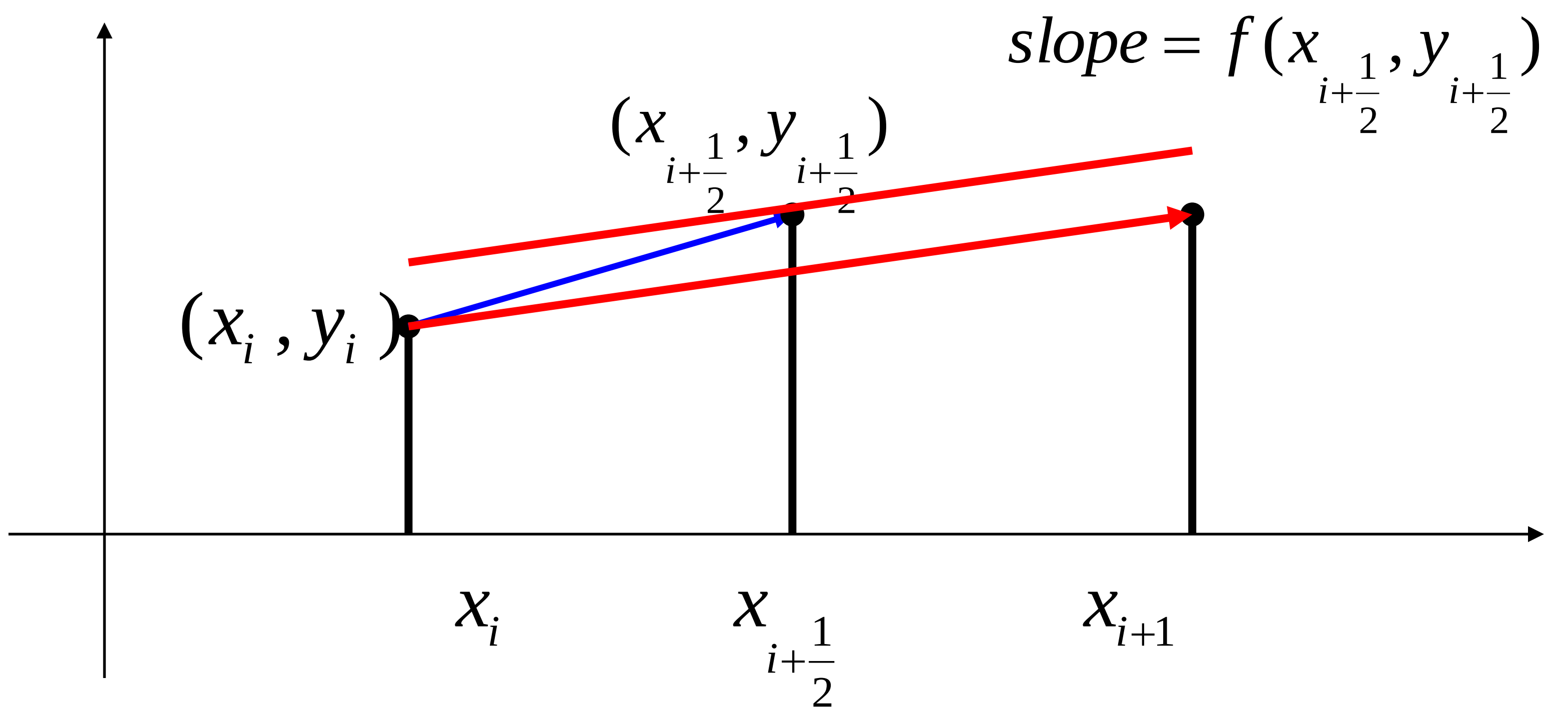
$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2}\, f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint (Modified Euler) Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint (Modified Euler) Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Example 1

Use the Midpoint Method with h=0.1 for the initial-value problem

$$y' = 1 + x^2 + y$$

$$y(0) = 1$$

to approximate y(0.1) and y(0.2)

Solution

$$x_i = a + ih = 0.1i$$

$$x_{i+\frac{1}{2}} = x_i + \frac{h}{2} = a + ih + \frac{h}{2} = a + (i + \frac{1}{2})h = 0.1(i + \frac{1}{2})$$

Problem: $f(x, y) = 1 + x^2 + y,\; y_0 = y(0) = 1, h = 0.1$

Step 1:

$$y_{0+\frac{1}{2}} = y_0 + \frac{h}{2}\; f(x_0, y_0) = 1 + 0.05(1 + 0 + 1) = 1.1$$

$$y_1 = y_0 + h\; f(x_{0+\frac{1}{2}}, y_{0+\frac{1}{2}}) = 1 + 0.1(1 + 0.0025 + 1.1) = 1.2103$$

Step 2:

$$y_{1+\frac{1}{2}} = y_1 + \frac{h}{2}\; f(x_1, y_1) = 1.2103 + .05(1 + 0.01 + 1.2103) = 1.3213$$

$$y_2 = y_1 + h\; f(x_{1+\frac{1}{2}}, y_{1+\frac{1}{2}}) = 1.2103 + 0.1(2.3438) = 1.4446$$

Example 2

Find approximate values and the errors of the solution of the initial-value problem by Midpoint method

$$y' = 3xy, \quad 0 \le x \le 1/2,$$

with step size h=0.1, and initial condition $y(0) = 1$

Solution

- Exact solution

$$y' = 3xy \Rightarrow y'/y = 3x \Rightarrow \int y'/y\,dx = \int 3x\,dx$$

$$\ln(y) = 3x^2/2 + C$$

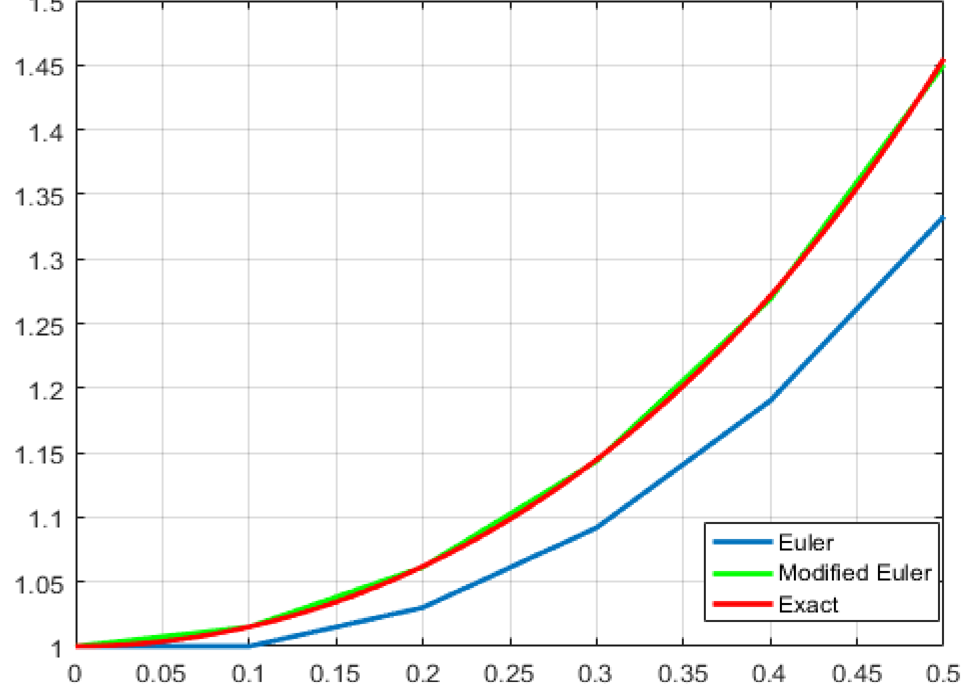
Using initial condition:

$$\ln(y(0)) = \ln(1) = 0 = C$$

$$\Rightarrow \ln(y) = 3x^2/2 \Rightarrow y = e^{3x^2/2}$$

Midpoint method:

$$y_{i+1} = y_i + h\ f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}\ f(x_i, y_i)\right)$$



x	y-exact	y-E	Error	y-ME	Error
0	1	1	0	1	0
0.1	1.015	1	0.01511	1.015	0.0001131
0.2	1.062	1.03	0.03184	1.061	0.0004764
0.3	1.145	1.092	0.05274	1.143	0.001187
0.4	1.271	1.19	0.08119	1.269	0.002445
0.5	1.455	1.333	0.1221	1.45	0.004621

Exercise

Find approximate values and the errors of the solution of the initial-value problem

$$y' = 3y - 2x, \quad 0 \le x \le 1,$$

with step size h=0.2, and initial condition $y(0) = 2$

by

a) Euler Method

b) Midpoint Method

Heun's Predictor Corrector Method

Problem: $y'(x) = f(x, y), \qquad y(x_0) = y_0$

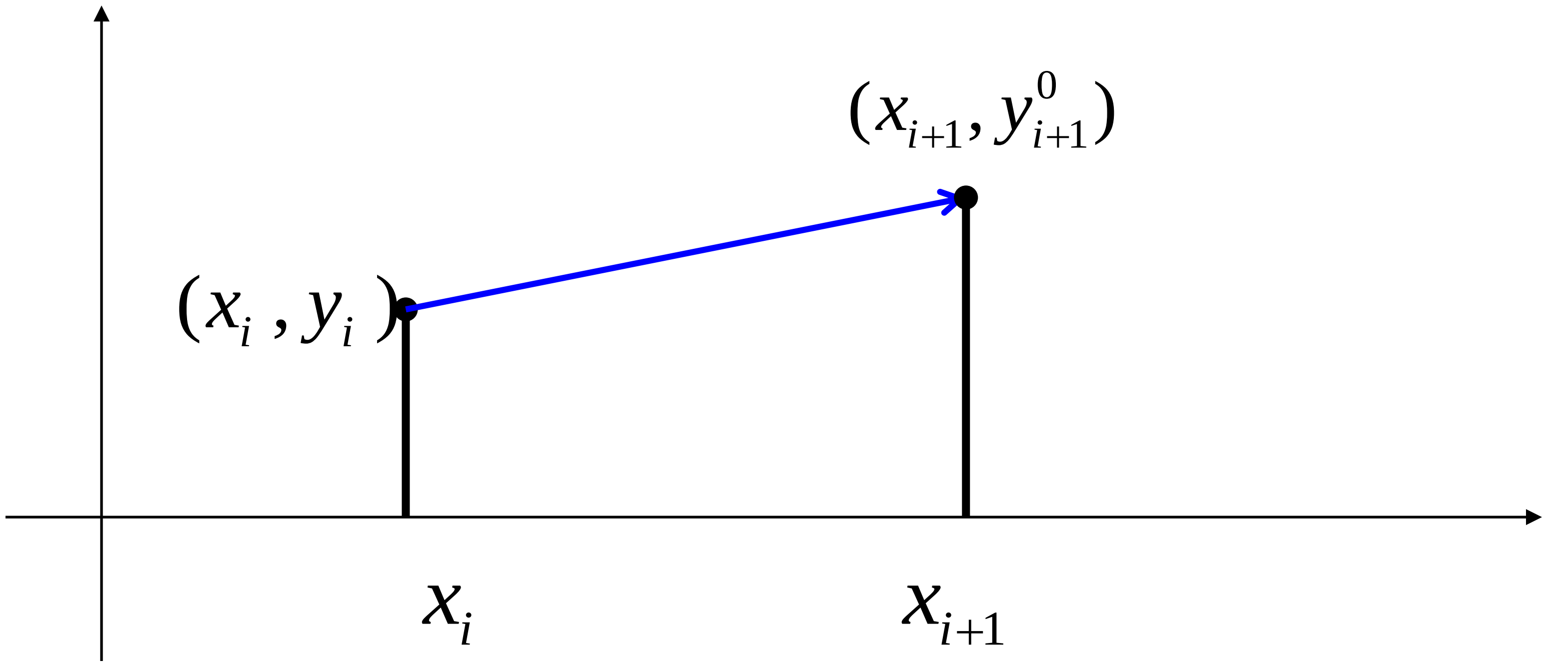
$$\begin{cases} \text{Predictor: } y_{i+1}^0 = y_i + h\ f(x_i, y_i) \\ \text{Corrector: } y_{i+1} = y_i + h\dfrac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2} \end{cases}$$

or

$$\boxed{y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_i + h, y_i + h\ f(x_i, y_i))\right)}$$

Heun's Predictor Corrector Method

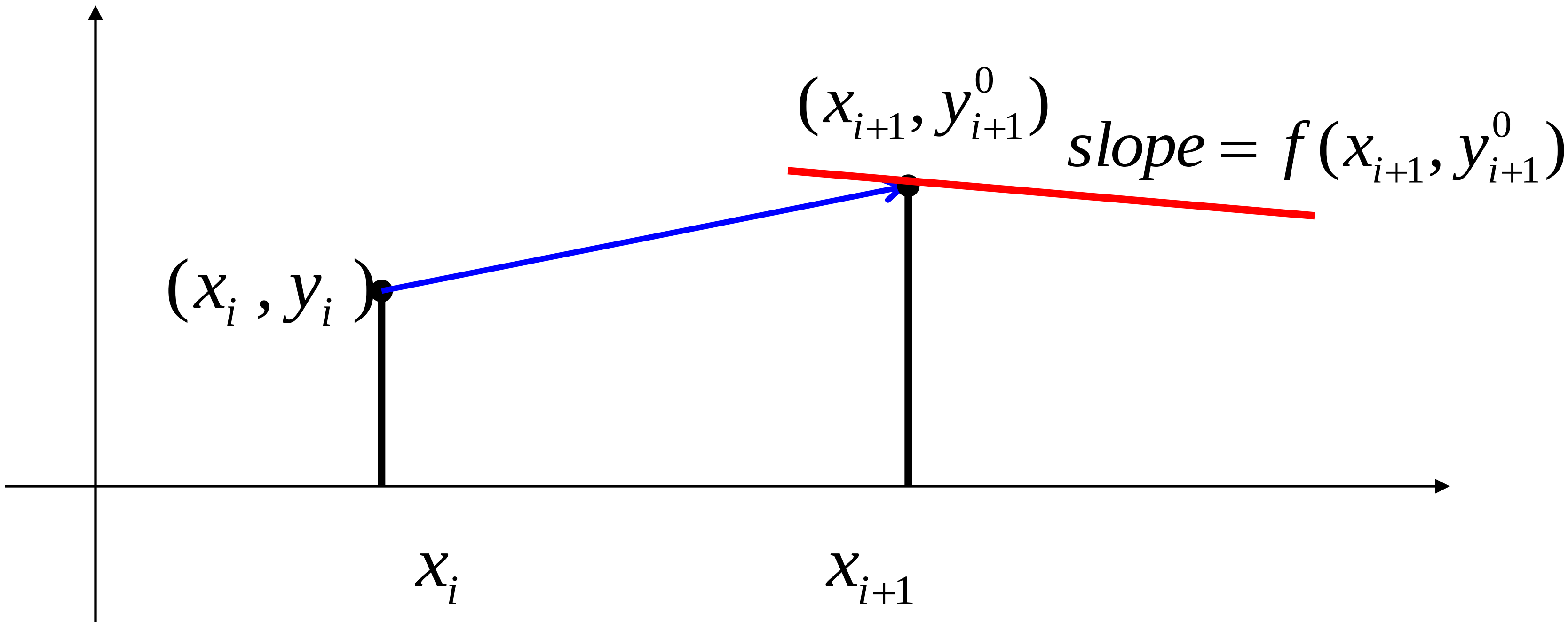
(Prediction)



Prediction $\quad y_{i+1}^{0} = y_i + h \; f(x_i, y_i)$

Heun's Predictor Corrector Method

(Prediction)

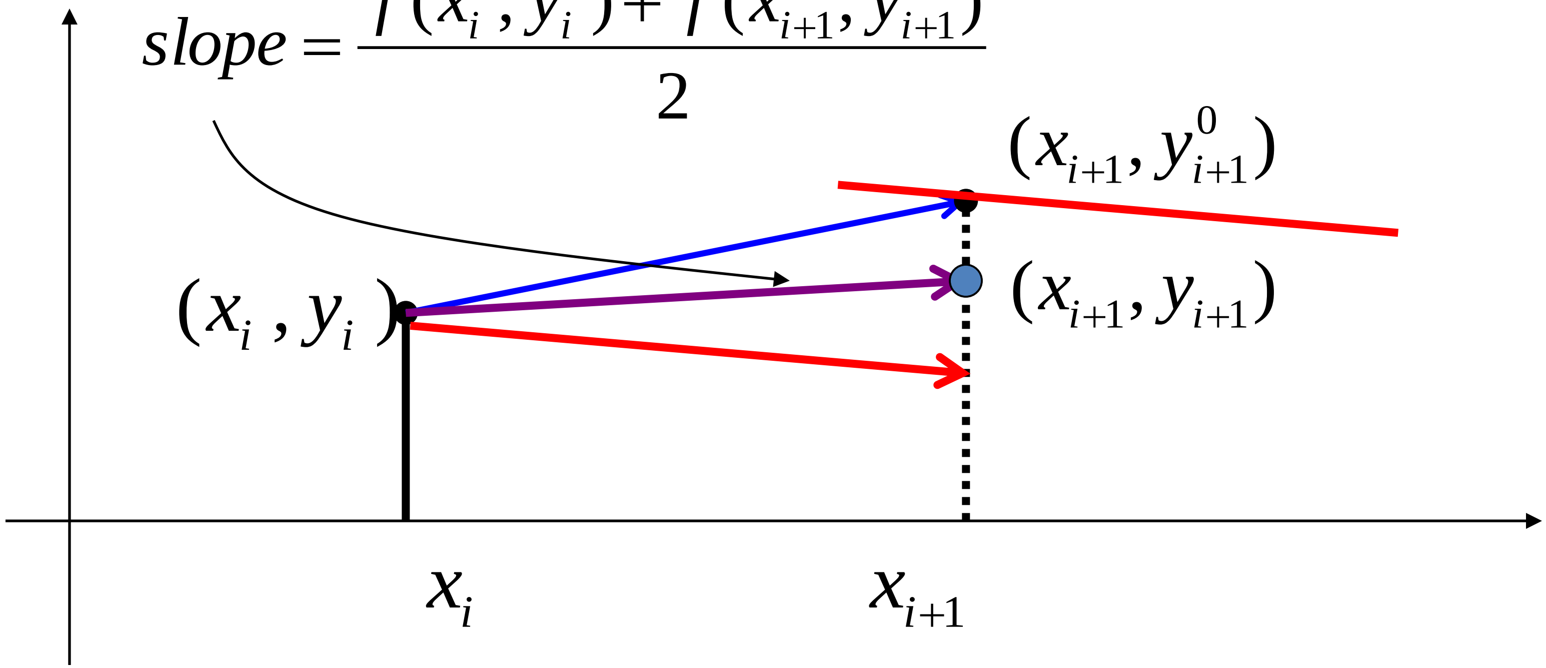


Prediction $\quad y_{i+1}^0 = y_i + h\ f(x_i, y_i)$

Heun's Predictor Corrector Method

(Correction)

$$slope = \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2}$$



$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0) \right)$$

Example 1

Use the Heun's Method with $h = 0.1$ for the initial-value problem

$$y'(x) = 1 + x^2 + y$$

$$y(0) = 1$$

to approximate y(0.1) and y(0.2)

Solution

Problem: $f(x,y)=1+y+x^2,\ y_0=y(x_0)=1, h=0.1$

Step 1:

Predictor: $y_1^0=y_0+h\ f(x_0,y_0)=1+0.1(2)=1.2$

Corrector: $y_1=y_0+\dfrac{h}{2}\left(f(x_0,y_0)+f(x_1,y_1^0)\right)=1.215$

Step 2:

Predictor: $y_2^0=y_1+h\ f(x_1,y_1)=1.4326$

Corrector: $y_2=y_1+\dfrac{h}{2}\left(f(x_1,y_1)+f(x_2,y_2^0)\right)=1.4452$

Example 2

Find approximate values and the errors of the solution of the initial-value problem by Heun's method

$$y' = 3xy, \quad 0 < x < 1/2,$$

with step size h=0.1, and initial condition $y(0) = 1$

Solution

- Exact solution

$$y' = 3xy \Rightarrow y'/\,y = 3x \Rightarrow \int y'/\,y dx = \int 3x dx$$

$$\ln(y) = 3x^2/2 + C$$

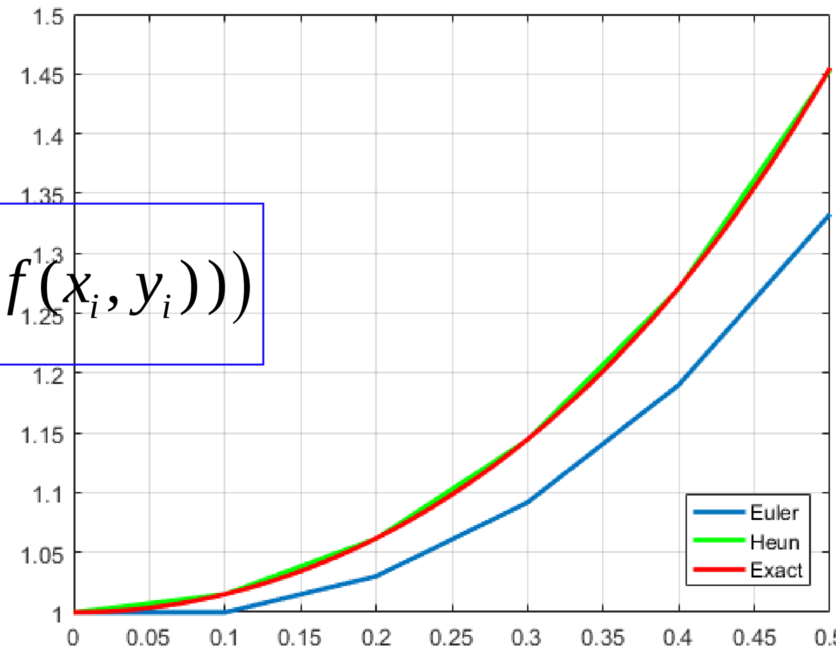
Using initial condition:

$$\ln(y(0)) = \ln(1) = 0 = C$$

$$\Rightarrow \ln(y) = 3x^2/2 \Rightarrow y = e^{3x^2/2}$$

Heun method:

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_i + h, y_i + h \ f(x_i, y_i))\right)$$



x	y-exact	y-Euler	Error-E	y-Heun	Error-H
0	1	1	0	1	0
0.1	1.015	1	0.01511	1.015	0.0001131
0.2	1.062	1.03	0.03184	1.062	0.000248
0.3	1.145	1.092	0.05274	1.144	0.0004629
0.4	1.271	1.19	0.08119	1.27	0.0008695
0.5	1.455	1.333	0.1221	1.453	0.001677

Exercise

Find approximate values and the errors of the solution of the initial-value problem

$$\frac{dy}{dt} = yt - 1.1t, \quad 0 \le t \le 2,$$

with step size h=0.5, and initial condition $y(0) = 1$

by

a) Modified Euler Method

b) Heun's Method

Runge-Kutta methods

$$\boxed{y_{i+1} = y_i + h\phi(x_i, y_i, h)}$$

where $\phi(x_i, y_i, h)$ is called the increment function:

$$\phi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$k_3 = f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$\ldots$$

$$k_n = f(x_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \ldots + q_{n-1,n-1} k_{n-1} h)$$

Second-order Runge-Kutta methods

- The second-order version of Runge-Kutta methods is

$$\boxed{\begin{aligned} y_{i+1} &= y_i + h(a_1 k_1 + a_2 k_2) \\ k_1 &= f(x_i, y_i) \\ k_2 &= f(x_i + \alpha h, y_i + \beta k_1 h) \end{aligned}}$$

$$\alpha = p_1, \beta = q_{11}$$

where

$$\boxed{a_1 + a_2 = 1, \quad a_2\alpha = \frac{1}{2}, \quad a_2\beta = \frac{1}{2}}$$

Derivation of 2nd-order Runge-Kutta Methods

$$y_{i+1} = y_i + h(a_1 k_1 + a_2 k_2)$$

$$k_1 = f(x_i, y_i), k_2 = f(x_i + \alpha h, y_i + \beta k_1 h)$$

$$y_{i+1} = y_i + h\big(a_1 f(x_i, y_i) + a_2 f(x_i + \alpha h, y_i + \beta h f(x_i, y_i))\big)$$

$$x = x_i :$$

$$f(x + \alpha h, y + \beta h \; f) = f + \alpha h \; f_x + \beta h f(x, y) f_y + O(h^2)$$

$$y_{i+1} = y(x) + \big(a_1 + a_2\big) h \; f(x, y) + h^2 (\alpha a_2 f_x + \beta a_2 \; f_y f) + O(h^3)$$

Problem: *Find* α, β, a_1, a_2

to match as many terms of the Taylor series as possible.

$$y(x+h) = y(x) + h y'(x) + \frac{h^2}{2} y''(x) + \frac{h^3}{6} y'''(x) + \ldots$$

Derivation of 2nd-order Runge-Kutta Methods

$$y(x+h) = \boxed{y(x)} + \boxed{hy'(x)} + \boxed{\frac{h^2}{2}y''(x)} + \frac{h^3}{6}y'''(x) + \ldots$$

$$y_{i+1} = \boxed{y(x)} + \boxed{\left(a_1 + a_2\right)h\ f(x,y)} + \boxed{h^2(\alpha\, a_2 f_x + \beta\, a_2\ f_y f)} + O(h^3)$$

$$\Rightarrow a_1 + a_2 = 1, \quad \alpha\, a_2 = \frac{1}{2}, \quad \beta\, a_2 = \frac{1}{2}$$

$$y' = f(x,y)$$

$$y'' = f_x(x,y) + f_y(x,y)y' = f_x(x,y) + f_y(x,y)f(x,y)$$

Derivation of 2nd-order Runge-Kutta Methods

$$y_{i+1} = y_i + h(a_1 k_1 + a_2 k_2)$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + \alpha h, y_i + \beta k_1 h)$$
$$a_1 + a_2 = 1, \quad \alpha a_2 = \frac{1}{2}, \quad \beta a_2 = \frac{1}{2}$$

Heun's method and Midpoint method are special cases of 2nd-order R-K methods:

Heun's method:

$$a_1 = \frac{1}{2}, \; a_2 = \frac{1}{2}, \quad \alpha = 1, \; \beta = 1$$

Modified Euler method:

$$a_1 = 0, \quad a_2 = 1, \quad \alpha = \beta = \frac{1}{2}$$

4^{th}-order Runge-Kutta Method

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1)$$

$$k_3 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2)$$

$$k_4 = f(x_i + h, y_i + hk_3)$$

$$i = 0, 1, 2, \ldots$$

Example

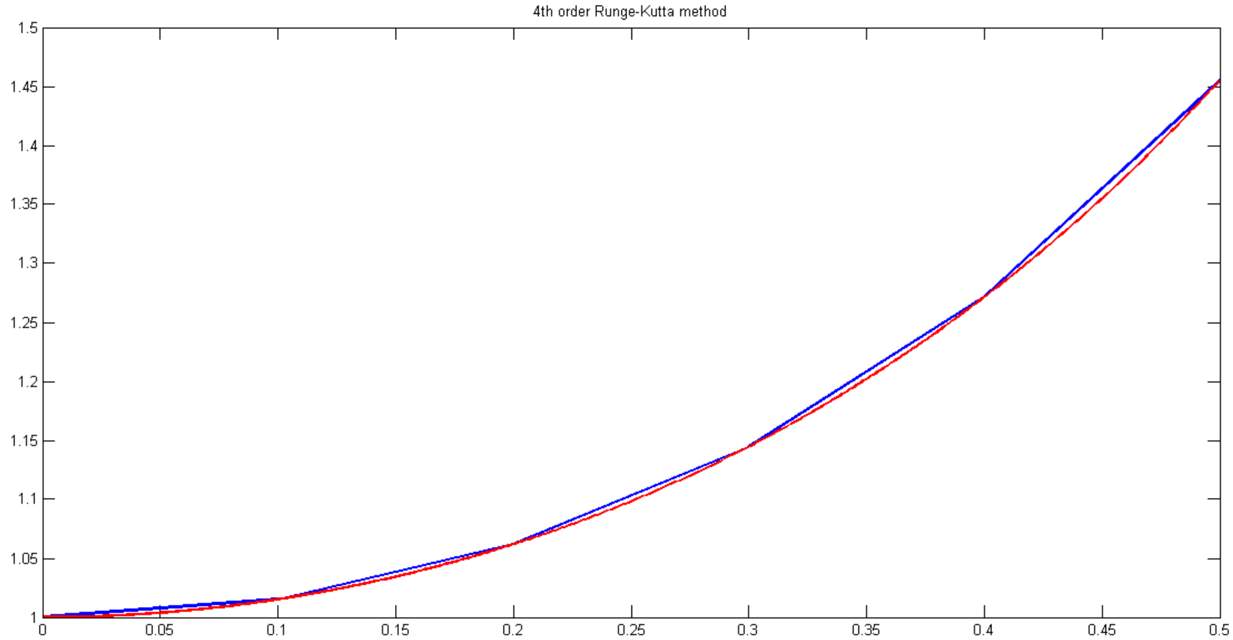
- Find the approximate solution by fourth-order Runge-Kutta method and the error of the initial-value problem

$$y' = 3xy, \quad 0 \le x \le 1/2,$$

with step size $h = 0.1$, and initial condition $y(0) = 1$

Solution

x	y_{exact}	$y_{Approximate}$	Errors
0	1	1	0
0.1	1.01511	1.01511	2.11572e-009
0.2	1.06184	1.06184	2.30571e-008
0.3	1.14454	1.14454	1.02112e-007
0.4	1.27125	1.27125	3.55607e-007
0.5	1.45499	1.45499	1.11013e-006



Exercise

Find the first two approximate values y_i , $i=1,2$, and the errors of the solution of the initial-value problem

$$y' = 3y - 2x, \quad x > 0,$$

with step size h=0.5, and initial condition $y(0) = 1$ by the 4th-order Runge-Kutta method

Exact solution: $y = \frac{1}{9}\left(6x + 7e^{3x} + 2\right)$