MIDTERM EXAMINATION
November 2017
Duration: 90 minutes

SUBJECT: REAL ANALYSIS	
Deputy head of Dept. of Mathematics:	Lecturer:
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INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 Determine whether the following sets are open in $\mathbb{R}^2$, closed in $\mathbb{R}^2$, or neither.

(a) *(15 marks)* $A = \big\{(x, y) \in \mathbb{R}^2 : ax + by \geq c\big\}$ where $a, b, c \in \mathbb{R}$ are constants;

(b) *(15 marks)* $B = \big\{(x, y) \in \mathbb{R}^2 : 0 < x + y \leq 1\big\}$.

Question 2

(a) *(20 marks)* Let (X, d) be a metric space, $f : X \to \mathbb{R}$ a continuous function, and $c \in \mathbb{R}$ a constant. Show that the function $cf : X \to \mathbb{R}$ defined by $(cf)(x) = c \cdot f(x)$ is continuous.

(b) *(5 marks)* Is the following statement true or false:

"If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, so is f^2?"

Explain your answer.

Question 3 *(25 marks)* Let $(X, \mathcal{M})$ be a measurable space. Show that

$$\mathcal{A} = \{A \subset X : A \cap E \in \mathcal{M} \text{ for every } E \in \mathcal{M}\}$$

is a σ-algebra which contains $\mathcal{M}$.

Question 4 Let μ^* be an outer measure on X.

(a) *(10 marks)* Use the definition of μ^*-measurable sets to show that X is μ^*-measurable.

(b) *(10 marks)* Show that if E is a μ^*-measurable set and $\mu^*(E) < \infty$, then $\mu^*(F \setminus E) = \mu^*(F) - \mu^*(E)$ for every set F in X with $F \supset E$.

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SOLUTIONS

Question 1 (a) Suppose $(x_n, y_n) \in A$, $(x_n, y_n) \to (x_0, y_0)$. We have $x_n \to x_0$ and $y_n \to y_0$, so

$$ax_0 + by_0 = \lim_{n\to\infty} ax_n + by_n \geq c \quad \text{as} \quad n \to \infty.$$

Thus $(x_0, y_0) \in A$ and therefore A is closed.

(b) Consider the sequence $\{(x_n, y_n) = (\frac{1}{2n}, \frac{1}{2n})\}$. As $0 < x_n + y_n = \frac{1}{n} \leq 1$, $(x_n, y_n) \in B$ for all n. However, $(x_n, y_n) \to (0, 0) \notin B$, hence B is not closed. Furthermore, $(0, 1) \in B$ and for all $r > 0$, $(0, 1 + \frac{r}{2}) \in B\big((0, 1), r\big)$ but $(0, 1 + \frac{r}{2}) \notin B$. Hence $B\big((0, 1), r\big) \notin B$ and consequently, B is not open.

Question 2 (a) Let $x \in X$ and $\{x_n\} \subset X$, $x_n \to x$. Since f is continuous at x, $f(x_n) \to f(x)$ and hence, $cf(x_n) \to cf(x)$. Thus cf is continuous at x. Since $x \in X$ is arbitrary, cf is continuous on X.

(b) The answer is no. The function $f(x) = x$ is Lipschitz continuous on $\mathbb{R}$ since $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbb{R}$. However, $f^2(x) = x^2$ is not is Lipschitz continuous because for any $\delta > 0$,

$$|f(n + \delta) - f(n)| = |(n + \delta)^2 - n^2| = 2n\delta + \delta^2 > 2n\delta \to \infty \text{ as } n \to \infty.$$

Question 3 Let $A \in \mathcal{M}$. Since $\mathcal{M}$ is a σ-algebra, for every $E \in \mathcal{M}$, we have $A \cap E \in \mathcal{M}$, so $A \in \mathcal{A}$, that is, $\mathcal{M} \subset \mathcal{A}$. In particular, $X \in \mathcal{A}$.

Suppose $A \in \mathcal{A}$ and $E \in \mathcal{M}$. As $A \cap E \in \mathcal{M}$ we get

$$A^c \cap E = E \setminus A = E \setminus (A \cap E) \in \mathcal{M},$$

implying $A^c \in \mathcal{A}$.

If $\{A_n\} \subset \mathcal{A}$ and $E \in \mathcal{M}$, then $A_n \cap E \in \mathcal{M}$ for all n. Since $\mathcal{M}$ is closed under countable union,

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \cap E = \bigcup_{n=1}^{\infty} (A_n \cap E) \in \mathcal{M}.$$

Thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Therefore $\mathcal{A}$ is a σ-algebra.

Question 4 (a) For every $E \subset X$,

$$\mu^*(E) = \mu^*(E) + \mu^*(\emptyset) = \mu^*(E \cap X) + \mu^*(E \cap X^c).$$

Thus X is μ^*-measurable.

(b) Since E is a μ^*-measurable and $F \supset E$,

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c) = \mu^*(E) + \mu^*(F \setminus E).$$

As $\mu^*(E) < \infty$, $\mu^*(F \setminus E) = \mu^*(F) - \mu^*(E)$.