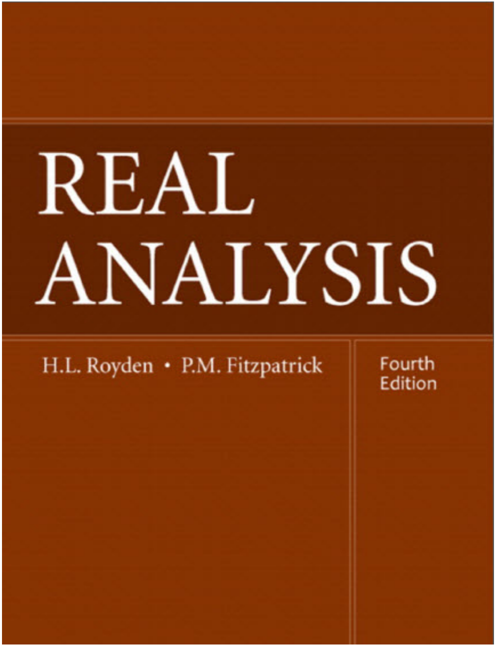
REAL ANALYSIS

References

Textbooks:

1. H. L. Royden, P. M. Fitzpatrick, *Real Analysis*, 4th ed. Pearson Education, 2010
2. G. B. Folland, *Real Analysis. Modern Techniques and Their Applications*, 2nd ed. John Wiley & Sons, New York, 1999


REAL
ANALYSIS
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Fourth
Edition

Karl Theodor Wilhelm Weierstrass (1815–1897), one of the most prominent mathematicians of the nineteenth century, believed that

> "The highest aim of science is to achieve general results."

How to Learn this Subject?

- Read the text. You will need to read relevant passages in your textbook and work through examples step by step. You are reading and searching for detail in a step-by-step logical fashion. This kind of reading, required by any deep and technical content, takes attention, patience, and practice.
- Do the homework, keeping the following principles in mind.
  (a) Sketch diagrams whenever possible.
  (b) Write your solutions in a connected step-by-step logical fashion, as if you were explaining to someone else.

- Finally, try on your own to write short descriptions of the key points each time you complete a section of the text.

(G. B. Thomas, Jr., R. L. Finney, *Calculus and Analytic Geometry*, 9th ed., Addison-Wesley, 1998)

The primary purpose of this chapter is to review a number of topics from analysis that will be called upon in the following chapters.

An excellent reference to this chapter is

R. G. Bartle, D. R. Sherbert, *Introduction to Real Analysis*, 4th ed. John Wiley & Sons, 2011 (pp. 1–44, 348–359)

A **statement** (or **proposition**) is a sentence that is either true (T) or false (F) but not both.

There are no other possibilities, and no statement can be both true and false.

Example 1.1 Some examples of statements are

(a) "4 is an even number."

(b) "$\sqrt{2}$ is rational."

(c) "It is raining."

To make complicated mathematical relationships clear it is convenient to use the notation of symbolic logic.

Symbolic logic is about statements which one can meaningfully claim to be true or false.

Given two statements P and Q.

(a) The **negation** of P denoted $\sim P$, is the statement "not P." The statement $\sim P$ is true exactly when P is false.

(b) The **conjunction** of P and Q, denoted $P \wedge Q$, is the statement "P and Q." The statement $P \wedge Q$ is true exactly when both P and Q are true.

(c) The **disjunction** of P and Q, denoted $P \vee Q$, is the statement "P or Q." The statement $P \vee Q$ is true exactly when at least one of P or Q is true.

The truth values of the above connectives can be summarized by means of the following truth table.

P	Q	$\sim P$	$P \wedge Q$	$P \vee Q$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

- Note that "P or Q" is true if P is true, if Q is true, or if both P and Q are true.
- To prove P is true, we sometimes prove $\sim P$ is false.

A positive integer p greater than 1 is called **prime** if the only positive factors of p are 1 and p. A positive integer that is greater than 1 and is not prime is called **composite**.

The Fundamental Theorem of Arithmetic

Every positive integer greater than 1 *can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.*

Example 1.2 Show that there are infinitely many primes.

Bettmann/CORBIS

Euclid
(fl. 300 B.C.E.)

Example 1.3 Suppose that the 10 integers

$$1,\ 2, \ldots,\ 10$$

are randomly positioned around a circular wheel. Show that the sum of some set of 3 consecutively positioned numbers is at least 17.

If $E(x)$ is an expression which becomes a statement when x is replaced by an object (member, thing) of a specified class (collection, universe) of objects, then E is a **property**. The sentence "x has property E" means "$E(x)$ is true".

If x belongs to a class X, that is, x is an element of X, then we write $x \in X$, otherwise $x \notin X$. Then

$$\{x \in X : E(x)\}$$

is the class of all elements x of the class X which have property E.

We write $\exists$ for the **quantifier** "there exists". The expression

$$\exists x \in X : E(x)$$

has the meaning "There is (at least) one object x in (the class) X which has property E".

We write

$$\exists! x \in X : E(x)$$

when exactly one such object exists.

We use the symbol $\forall$ for the quantifier "for all".

In normal language statements containing $\forall$ can be expressed in various ways. For example,

$$\forall x \in X : E(x) \tag{1}$$

means that "For each (object) x in (the class) X, the statement $E(x)$ is true", or "Every x in X has the property E".

The statement (1) can also be written as

$$E(x), \quad \forall x \in X, \tag{2}$$

that is, "Property E is true for all x in X".

In a statement such as (2) we usually leave out the quantifier $\forall$ and write simply

$$E(x), \quad x \in X. \tag{3}$$

Finally, we use the symbol $:=$ to mean "is defined by". Thus

$$a := b$$

means that the object (or symbol) a is defined by the object (or expression) b.

Of course $a = b$ means that objects a and b are equal, that is, a and b are simply different representations of the same object (statement, etc.).

Example 1.4 Let P and Q be statements, X a class of objects, and E a property. Then, using truth tables or other methods, one can easily verify the following statements:

(a) $\sim(\sim P) = P$.

(b) $\sim(P \wedge Q) = (\sim P) \vee (\sim Q)$.

(c) $\sim(P \vee Q) = (\sim P) \wedge (\sim Q)$.

(d) $\sim\big(\forall x \in X : E(x)\big) = \big(\exists x \in X : \sim E(x)\big)$.

(e) $\sim\big(\exists x \in X : E(x)\big) = \big(\forall x \in X : \sim E(x)\big)$.

Given two statements P and Q.

(a) The **implication** $P \implies Q$ (read "P implies Q") is the statement "If P, then Q." The statement $P \implies Q$ is a true statement unless P is true and Q is false, in which case it is a false statement.

(b) The **equivalence** $P \iff Q$ is the statement "P if and only if Q." The sentence $P \iff Q$ is true exactly when P and Q have the same truth values, otherwise it is false.

Thus,

$$P \Longrightarrow Q := (\sim P) \vee Q$$
$$(P \iff Q) := (P \Longrightarrow Q) \wedge (Q \Longrightarrow P).$$

In "$P \Longrightarrow Q$", the statement P is the **hypothesis** and Q is the **conclusion** or **consequent**.

The truth values for $P \implies Q$ and $P \iff Q$ are given in the following table.

P	Q	$P \implies Q$	$P \iff Q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

• $P \Longrightarrow Q$ is true when P and Q are both true, or when P is false (independent of whether Q is true or false). This means that a true statement cannot imply a false statement, and also that a false statement implies any statement–true or false.

It is common to express $P \Longrightarrow Q$ as "To prove Q it suffices to prove P", or "Q is necessary for P to be true". In other words, P is a sufficient condition for Q, and Q is a necessary condition for P.

- The statements P and Q are equivalent when
  - ◇ both $P \Longrightarrow Q$ and its converse $Q \Longrightarrow P$ are true, or when
  - ◇ P is a necessary and sufficient condition for Q (or vice versa).

Another common way of expressing this equivalence is to say "P is true if and only if Q is true".

Two fundamental observations are that

$$(P \Longrightarrow Q) \iff (\sim Q \Longrightarrow \sim P)$$
$$\sim (P \Longrightarrow Q) \iff P \wedge (\sim Q)$$

Example 1.5 Prove that for all integers n, if n^2 is even then n is even.

One of the most important tools in modern mathematics is the theory of sets.

In the present section we define the basic concepts of the theory of sets and state the principal results which will be used later.

A set is a collection of objects called the **members** (or **elements** or **points**) of the set.

Sets will generally be denoted by ordinary capital letters and their elements by lower case letters; the same letter will *not* always refer to the same set or element.

A set is said to *contain* its elements.

To indicate that an object x is an element of a set X, we will write $x \in X$; if x is not an element of X, we will write $x \notin X$. A statement such as $x, y \in X$ will be used as an abbreviation for the two statements $x \in X$ and $y \in X$.

We usually deal with the following sets of numbers.

(a) The set $\mathbb{N}$ of all natural numbers, that is, positive integers.
(b) The set $\mathbb{Z}$ of all integers, positive, zero, or negative.
(c) The set $\mathbb{Q}$ of all rationals.
(d) The set $\mathbb{R}$ of all real numbers.

We have

$$-5, 10 \in \mathbb{Z} \quad \text{but} \quad -5 \notin \mathbb{N}.$$

To determine a set is to determine its members.

To show the elements of a set we always enclose them in braces and give

- either a complete listing (e.g., $\{a, 2, \text{Fred}, \text{New Jersey}\}$ is the set containing the four elements a, 2, Fred, and New Jersey), or
- an indication of a pattern (e.g., $\{1, 2, 3, \ldots\}$ is the set $\mathbb{N}$), or
- a description of a rule of formation following a colon (e.g., $\{x : x \in \mathbb{R},\ x \geq 0\}$ is the set of all nonnegative real numbers).

Frequently, the members of a set X are determined by the possession of some common property.

For example, if $P(x)$ denotes a given statement relating to the object x, then we write

$$X = \{x : \; P(x)\}$$

to state that X is the set of all objects x for which the statement $P(x)$ holds.

- If $k, l \in \mathbb{Z}$, $k < l$, the set

$$\{x \in \mathbb{Z} : k \leq x \leq l\}$$

is usually written as

$$\boldsymbol{\{k, k+1, \ldots, l\}}.$$

A set X is said to be **empty** if it has no members. The empty set will be denoted by the symbol $\emptyset$.

A set X is said to be a **singleton** if it has one and only one member. If the lone member of a singleton X is x then we denote $X = \{x\}$.

A singleton set

Example 2.1

(a) Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$.
What are the elements of A, B, and C? How are A, B, and C related?

(b) Is $\{0\} = 0$?

(c) How many elements are in the set $\{1, \{1\}\}$?

(d) For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1 and U_0.

Subsets

Let A and B be any two sets.

- If each member of A is also a member of B, then we say that A is a **subset** of B; in symbols,

  $$A \subset B, \quad B \supset A.$$

  The phrases A **is contained in** B and B **contains** A are alternative ways of saying that A is a subset of B.

- To say that A is *not* a subset of B means that A has some member which is not a member of B, and if this is the case we write $A \not\subset B$ or $B \not\supset A$.

$A \not\subset B$ means that "There is at least one element x such that $x \in A$ and $x \notin B$".

It is clear that $\emptyset \subset B$ for any set B since $\emptyset$ has no member which fails to be a member of B.

- If $A \subset B$ and $B \subset A$, then we say that A and B are **equal**; in symbols, $A = B$. Thus, $A = B$ means that A and B have precisely the same members.

$$A = B \iff (A \subset B) \wedge (B \subset A).$$

- If $A \subset B$ and $A \neq B$, then A is said to be a **proper subset** of B.

Question: How to show that

- $A \subset B$?
- $A \not\subset B$?
- $A = B$?

Example 2.2 Which of the following are true statements?

(a) $2 \in \{1, 2, 3\}$ (b) $\{2\} \in \{1, 2, 3\}$

(c) $2 \subset \{1, 2, 3\}$ (d) $\{2\} \subset \{1, 2, 3\}$

(e) $2 \subset \{\{1\}, \{2\}\}$ (f) $\{2\} \in \{\{1\}, \{2\}\}$.

Operations on Sets

Let A and B be sets.

- The **union** of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x : \ x \in A \quad \text{or} \quad x \in B\}.$$

- The **intersection** of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x : \ x \in A \quad \text{and} \quad x \in B\}.$$

- Two sets are called **disjoint** if their intersection is the empty set.
- Let A and B be sets. The **difference** of A and B, denoted by $A \setminus B$, is the set containing those elements that are in A but not in B.
$$A \setminus B = \{x : \; x \in A \quad \text{and} \quad x \notin B\}.$$
The difference of A and B is also called the **complement** of B in A.
- We use the notation $A \Delta B$ for the **symmetric difference** of two sets defined by
$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

- If X is a set and $A \subset X$, then the set $X \setminus A$ is called the **complement** of A in X and is denoted by A^c.

$$A^c = \{x \in X : \ x \notin A\}.$$

The following equations are true for all subsets of a set X.

$$A \cup \emptyset = A, \qquad A \cap \emptyset = \emptyset$$
$$A \cup A^c = X, \qquad A \cap A^c = \emptyset$$
$$(A^c)^c = A, \qquad X^c = \emptyset, \quad \emptyset^c = X.$$

If $A \subset B$, then
$$A \cap B = A \quad \text{and} \quad A \cup B = B.$$

For two sets A and B in X we have
$$A \setminus B = A \setminus (A \cap B) = A \cap B^c,$$
$$A \subset B \iff A^c \supset B^c.$$

In order to avoid the confusion that might arise when considering sets of sets, we often use the words "**collection**", "**class**", "**family**", and "**system**" as synonyms for the word "set."

The same notations and terminology will be used for classes as for sets.

Thus, if E is a set and $\mathcal{E}$ is a class of sets, then

$$E \in \mathcal{E}$$

means that the set E belongs to (is a member of, is an element of) the class $\mathcal{E}$.

If $\mathcal{E}$ and $\mathcal{F}$ are classes, then

$$\mathcal{E} \subset \mathcal{F}$$

means that every set of $\mathcal{E}$ belongs also to $\mathcal{F}$, i.e., $\mathcal{E}$ is a subclass of $\mathcal{F}$.

- Given a set X, the collection of all subsets of X is called the **power set** of X, and is denoted by $\boldsymbol{\mathcal{P}(X)}$ or $\mathbf{2^X}$.

Note that $\emptyset$ and X are members of $\mathcal{P}(X)$.

Let $\mathcal{F}$ be a collection of sets.

We define the **union of $\mathcal{F}$** to be the set

$$\{x : \ x \in A \text{ for some } A \in \mathcal{F}\}$$

and is denoted by

$$\bigcup_{A \in \mathcal{F}} A \quad \text{or} \quad \bigcup \{A : A \in \mathcal{F}\}.$$

Thus $\cup_{A\in\mathcal{F}} A$ is the set of all points which belong to at least one set of the class $\mathcal{F}$.

We define the **intersection of $\mathcal{F}$** to be the set

$$\{x : \ x \in A \text{ for every } A \in \mathcal{F}\}.$$

and is denoted by

$$\bigcap_{A \in \mathcal{F}} A \quad \text{or} \quad \bigcap \{A : A \in \mathcal{F}\}.$$

Thus $\bigcap_{A \in \mathcal{F}} A$ is the set of all points which belong to every set of the class $\mathcal{F}$.

Usually it is more convenient to consider indexed families of sets.

If I is a set and if to each $i \in I$ there is assigned a set A_i, then the family $\{x : \ x = A_i \text{ for some } i \in I\}$ is more commonly denoted by

$$\{A_i : \ i \in I\} \quad \text{or} \quad \{A_i\}_{i \in I}$$

and this family is called an **indexed family of sets**.

In this case, I is called the **index set** for the family and the members of I are called **indices**.

We write $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ for the union and the intersection, respectively, of this indexed family.

We use the notations

$$A_1 \cup A_2 \cup \cdots \cup A_n \quad \text{or} \quad \bigcup_{k=1}^{n} A_k$$

to denote the union of the sets $A_1, A_2, \ldots, A_n$ and

$$A_1 \cap A_2 \cap \cdots \cap A_n \quad \text{or} \quad \bigcap_{k=1}^{n} A_k$$

to denote the intersection of the sets $A_1, A_2, \ldots, A_n$.

When considering families of sets indexed by $\mathbb{N}$, our usual notation will be

$$\{A_n : n \in \mathbb{N}\}, \quad \{A_n\}_{n=1}^{\infty} \quad \text{or} \quad \{A_n\},$$

and likewise we use the notations

$$\bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} A_n$$

to denote the union (the intersection, respectively) of the sets $A_1, A_2, \ldots, A_n, \ldots$.

If $\{A_n\}_{n \in I}$, where $I = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$ or $I = \mathbb{N}$, the corresponding union and intersection will be written $\bigcup_n A_n$ and $\bigcap_n A_n$.

We say that a family $\mathcal{F}$ is **pairwise disjoint** (or simply **disjoint**) if

$$A \cap B = \emptyset \quad \text{whenever} \quad A, B \in \mathcal{F} \text{ and } A \neq B.$$

The terms "disjoint collection of sets" and "collection of disjoint sets" are used interchangeably, as are "disjoint union of sets" and "union of disjoint sets."

Finally, if to every element i of a certain index set I there corresponds a statement $P_i(x)$ concerning x, then we shall denote the set of all those points x for which the statement $P_i(x)$ is true for every i in I by

$$\{x : P_i(x),\ i \in I\}.$$

Set Identities

De Morgan's identities

For a family $\{A_i\}_{i\in I}$ of subsets of a set X, the following identities hold:

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c \quad \text{and} \quad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$$

The distributive laws for general families of subsets of X take the following form.

Distributive laws

$$\left(\bigcup_{i\in I} A_i\right) \cap B = \bigcup_{i\in I}(A_i \cap B),$$
$$\left(\bigcap_{i\in I} A_i\right) \cup B = \bigcap_{i\in I}(A_i \cup B).$$

- Let X and Y be nonempty sets.
  A **mapping** $f : X \to Y$ is a rule which assigns to each member x of X a unique member $f(x)$ of Y.

- For each point $x \in X$, the point $f(x) \in Y$ which is assigned to x by the mapping f is called the **image of x under f**. Sometimes, $f(x)$ is called the **value** of the mapping f at the point x.

- We will also use the notation

$$\boldsymbol{x \mapsto f(x)},$$

which indicates that x is mapped to $f(x)$ by f.

- The terms **map**, **function**, **transformation** are frequently used interchangeably with mapping.
- In the case Y is a set of real numbers $\mathbb{R}$ we usually use the word “function".

Example 3.1 $f : X \to Y$ is called a **constant mapping** from X into Y if there is $y_0 \in Y$ such that $f(x) = y_0$ for all $x \in X$.

Example 3.2 If $f : X \to Y$ is a mapping and $\emptyset \neq E \subset X$, we define the **restriction of f to E** to be the mapping

$$f_{|E} : E \to Y, \quad f_{|E}(x) = f(x).$$

The mapping f is said to be an **extension** of the mapping $g := f_{|E}$ over the set X.

Example 3.3 Let X be a given set. For an arbitrary subset A of X, define a function $\chi_A : X \to \mathbb{R}$ by taking

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X \setminus A. \end{cases}$$

This function χ_A is called the **characteristic** or **indicator** function of the set A.

All properties of sets and set operations may be expressed by means of characteristic functions.

Example 3.4 A mapping $f : \mathbb{N} \to X$ from the set $\mathbb{N}$ of natural numbers into a given set X is called a **sequence** (of points) in X. For each $n \in \mathbb{N}$, the image $x_n = f(n)$ is called the ***n*th term** of the sequence f.

Customarily, the sequence f can be written in the form

$$\{x_n\}_{n=1}^{\infty}, \quad \{x_n\}_{n\geq 1}, \quad \{x_n\}, \quad \text{or} \quad x_1, x_2, x_3, \ldots.$$

We abuse notation and write $\boldsymbol{\{x_n\}_{n=1}^{\infty} \subset X}$ to indicate that the terms of the sequence are in the set X.

• We will sometimes consider functions whose domains of definition are of the form $\{n \in \mathbb{Z} : n \geq p\}$ for some fixed $p \in \mathbb{Z}$.

Such functions are also called **sequences** and are denoted as $\{x_n\}_{n=p}^{\infty}$.

There is a change of notation to regard such a sequence as being defined on $\mathbb{N}$:

$$\{x_n\}_{n=p}^{\infty} = \{x_{n+p-1}\}_{n=1}^{\infty}.$$

The first term is x_p, etc.

• We also use the term **finite sequence** to mean a mapping from $\{1, 2, \ldots, N\}$ into X where $N \in \mathbb{N}$.

Two mappings $f : X \to Y$ and $g : X \to Y$ are said to be **equal**, in symbols $f = g$, if $f(x) = g(x)$ holds true for each $x \in X$.

$$f = g \overset{def}{\iff} f(x) = g(x) \quad \forall x \in X.$$

For instance, if $f, g : \mathbb{R} \to \mathbb{R}$ are defined by

$$f(x) = |\cos x|, \qquad g(x) = \sqrt{\frac{1 + \cos 2x}{2}},$$

then $f = g$.

Let $f : X \to Y$ be a mapping. If $A \subset X$, we define the **image** of A under the mapping f by

$$f(A) = \{f(x) : \ x \in A\}.$$

The set X is called the **domain** of the mapping f and $f(X)$ is called the **range** of f and is denoted by $\operatorname{Im}(f)$.

We say that **f maps X into Y**. If $\mathrm{Im}(f) = Y$ we say that **f maps X onto Y** or that f is **surjective**.

$$f : X \to Y \text{ is surjective}$$

$$\Updownarrow$$

$$Y \subset \mathrm{Im}(f) \text{ and } \mathrm{Im}(f) \subset Y$$

$$\Updownarrow$$

$$Y \subset \mathrm{Im}(f)$$

$$\Updownarrow$$

$$(\forall y \in Y)(\exists x \in X)\big(f(x) = y\big)$$

Let $f : X \to Y$ be a mapping. For any subset B of Y, the set

$$f^{-1}(B) = \{x : \ f(x) \in B\}$$

is called the **inverse image** of B under the mapping f.

In particular, if B is a singleton, say $B = \{y\}$, then $f^{-1}(B)$ is called the **inverse image of the point y under f** and is denoted by $f^{-1}(y)$.

$$f^{-1}(y) = \left\{x \in X : f(x) = y\right\}.$$

$f : X \to Y$ is said to be **one-to-one** (or **injective**) if $f(x_1) = f(x_2)$ only when $x_1 = x_2$.

$$f : X \to Y \text{ is injective}$$
$$\Updownarrow$$
$$(\forall x_1, x_2 \in X)\big(f(x_1) = f(x_2) \Longrightarrow x_1 = x_2\big)$$
$$\Updownarrow$$
$$(\forall x_1, x_2 \in X)\big(x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)\big)$$

If f is both surjective and injective, it is called a **bijective** mapping.

In this case, for every $y \in Y$, the inverse image $f^{-1}(y)$ is always a singleton, i.e., a point in X; the assignment $y \to f^{-1}(y)$ defines a mapping $g : Y \to X$, which is called the **inverse** mapping of f and is denoted by $f^{-1} : Y \to X$.

$$x = f^{-1}(y) \overset{def}{\Longleftrightarrow} f(x) = y.$$

It is easily seen that f^{-1} is also bijective.

Example 3.5 (a) Suppose that $A \subset X$ and $A \neq \emptyset$. The mapping $i : A \to X$ defined by

$$i(x) = x \quad \text{for every } x \in A$$

is called the **inclusion** (or **embedding**) mapping of A into X.

Obviously, every inclusion mapping is injective.

(b) The mapping $i : X \to X$ defined by

$$i(x) = x, \quad x \in X$$

is called the **identity** mapping on X. Clearly, i is bijective and $i^{-1} = i$.

Remark 3.1.

(a) Note that if $f : X \to Y$ is a bijective mapping and B is a subset of Y, then the preimage of B under f, $f^{-1}(B)$, is also equal to the image of B under f^{-1}, that is,

$$\{x \in X : f(x) \in B\} = \left\{f^{-1}(y) : y \in B\right\}.$$

(b) If $f : X \to Y$ is an injective mapping, then the mapping $g : X \to f(X)$ defined by

$$g(x) = f(x)$$

is a bijective mapping.

Let $\{A_i\}_{i\in I}$ be a family of subsets of X and $\{B_j\}_{j\in J}$ a family of subsets of Y and let $f : X \to Y$ be a mapping. The following relationships hold:

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i), \qquad f\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} f(A_i)$$

$$f^{-1}\left(\bigcup_{j\in J} B_j\right) = \bigcup_{j\in J} f^{-1}(B_j), \quad f^{-1}\left(\bigcap_{j\in J} B_j\right) = \bigcap_{j\in J} f^{-1}(B_j).$$

Moreover, if $B \subset Y$, then

$$f^{-1}\left(B^c\right) = f^{-1}(B)^c.$$

In words, f^{-1} commutes with union, intersections, and complements.

Let $f : X \to Y$ and $g : Y \to Z$ be mappings. We define a mapping $h : X \to Z$ by assigning to each point x of X the point $h(x) = g\big(f(x)\big)$ of the set Z. This mapping h is called the **composition** of f and g, denoted by $h = g \circ f$.

For instance, if $f : X \to Y$ is a mapping and $A \subset X$, $A \neq \emptyset$, then $f_{|A} = f \circ h$ where h is the inclusion mapping of A into X.

If $f : X \to Y$ is bijective, then $f^{-1} \circ f$ and $f \circ f^{-1}$ are the identity mappings on X and Y, respectively.

Example 3.6 If $f:\mathbb{N}\to X$ is a sequence and $g:\mathbb{N}\to\mathbb{N}$ satisfies $g(n)<g(n+1)$ for all $n\in\mathbb{N}$, the composition $f\circ g$ is called a **subsequence** of f.

Thus, a subsequence of a sequence $\{x_n\}$ is a sequence $\{y_k\}$ for which there exists a strictly increasing sequence $\{n_k\}$ of natural numbers (that is, $1\le n_1<n_2<n_3<\cdots$) such that $y_k=x_{n_k}$ holds for each k.

Remark 3.2

(a) The composition of any two surjective mappings is surjective.

(b) The composition of any two injective mappings is injective.

(c) The composition of any two bijective mappings is bijective.

Products of Sets

• The **Cartesian product** $X \times Y$ of two sets X and Y is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$,

$$X \times Y = \{(x, y) : \ x \in X, \ \ y \in Y\}.$$

Note that $(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

• Let f be a mapping from the set X to the set Y. The **graph** of the mapping f is the set

$$\Big\{(x, y) \in X \times Y : y = f(x)\Big\}.$$

The ordered ***n*-tuple** $(a_1, a_2, \dots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element,..., and a_n as its nth element.

We say that two ordered n-tuples are **equal** if and only if each corresponding pair of their elements is equal. In other words,

$$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$$

if and only if $a_i = b_i$, for $i = 1, 2, \dots, n$.

The **Cartesian product** of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n$$
$$= \{(a_1, a_2, \ldots, a_n) : \ a_i \in A_i \text{ for } i = 1, 2, ..., n\}.$$

If $A_1 = A_2 = \cdots = A_n = A$, then it is standard to write $A_1 \times A_2 \times \cdots \times A_n$ as A^n.

Supremum and Infimum of a Set in $\mathbb{R}$

Definition 4.1

Let S be a nonempty subset of $\mathbb{R}$.

(a) S is said to be **bounded above** if there exists a number $\beta \in \mathbb{R}$ such that $x \leq \beta$ for all $x \in S$. Each such number β is called an **upper bound** of S.

(b) S is said to be **bounded below** if there exists a number $\alpha \in \mathbb{R}$ such that $\alpha \leq x$ for all $x \in S$. Each such number α is called a **lower bound** of S.

(c) A set is said to be **bounded** if it is both bounded above and bounded below. A set is said to be **unbounded** if it is not bounded.

Definition 4.2

Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above, then a number β is said to be the **supremum** (or the **least upper bound**) of S if it satisfies the conditions:

(i) β is an upper bound of S, and

(ii) if β' is any upper bound of S, then $\beta \leq \beta'$.

Definition 4.2 (cont'd)

Let S be a nonempty subset of $\mathbb{R}$.

(b) A number α is said to be the **infimum** (or the **greatest lower bound**) of S if it satisfies the conditions:

(i') α is a lower bound of S, and

(ii') if α' is any lower bound of S, then $\alpha' \leq \alpha$.

If the supremum or the infimum of a set S exists, we will denote them by

$$\sup S \quad \text{and} \quad \inf S.$$

A real number β is the supremum of a non empty subset S of $\mathbb{R}$ if and only if β satisfies the conditions:

(i) $x \leq \beta$ *for all* $x \in S$,

(ii) *if* $\gamma < \beta$, *then there exists* $x \in S$ *such that* $\gamma < x$.

$$\beta = \sup S \iff \begin{cases} (\forall x \in S)(x \leq \beta) \\ (\forall \gamma < \beta)(\exists x \in S)(x > \gamma). \end{cases}$$

The above condition can be expressed in terms of $\epsilon > 0$ as follows.

An upper bound β of a nonempty set $S \subset \mathbb{R}$ is the supremum of S if and only if for every $\epsilon > 0$ there exists an $x \in S$ such that $\beta - \epsilon < x$.

$$\beta = \sup S \iff \begin{cases} (\forall x \in S)(x \le \beta) \\ (\forall \epsilon > 0)(\exists x \in S)(x > \beta - \epsilon). \end{cases}$$

The Completeness Property of $\mathbb{R}$

Theorem 4.1

Every nonempty set of real numbers that is bounded above has a supremum in $\mathbb{R}$.

Corollary 4.2

Every nonempty set of real numbers that is bounded below has an infimum in $\mathbb{R}$.

Well-Ordering Property

Corollary 4.3

(a) *Every nonempty subset of* $\mathbb{N}$ *has a smallest element. That is, if* A *is a nonempty subset of* $\mathbb{N}$*, then* $\inf A \in A$.

(b) *If* A *is a nonempty subset of* $\mathbb{Z}$ *that is bounded from below, then* A *has a smallest element; that is,* $\inf A \in A$.

(c) *If* A *is a nonempty subset of* $\mathbb{Z}$ *that is bounded from above, then* A *has a greatest element; that is,* $\sup A \in A$.

Let $x \in \mathbb{R}$. According to part (c) of Corollary 4.3, the number

$$\lfloor \boldsymbol{x} \rfloor := \sup\{k \in \mathbb{Z} : k \leq x\}$$

is an integer and called the **integer part** of x.

The integer part of x is the largest integer not exceeding x.

We have

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \quad \text{for all } x \in \mathbb{R}.$$

Theorem 4.4

If $x, y \in \mathbb{R}$ with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

The Principle of Mathematical Induction

Mathematical induction is a powerful method of proof that is frequently used to establish the validity of statements that are given in terms of the natural numbers.

It is used extensively to prove results about a large variety of discrete objects.

Mathematical induction is a method for proving that a property defined for integers n is true for all values of n that are greater than or equal to some initial integer.

Principle of Mathematical Induction

For each $n = 1, 2, \ldots$, let $P(n)$ be a statement about n. Suppose that:

(i) *$P(1)$ is true.*

(ii) *For every $k = 1, 2, \ldots$, if $P(k)$ is true, then $P(k+1)$ is true.*

Then $P(n)$ is true for all $n = 1, 2, \ldots$.

To prove that $P(n)$ is true for all positive integers n, we complete two steps:

Basis step: We verify that $P(1)$ is true.

Inductive step: We show that if $P(k)$ is true, then $P(k+1)$ is true.

In the inductive step, the assumption "if $P(k)$ is true" is called the **induction hypothesis**.

Example 4.1 Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

It may happen that statements $P(n)$ are false for certain natural numbers but then are true for all $n \geq m$ for some particular m.

Principle of Mathematical Induction (second version)

Let m be a fixed integer and let $P(n)$ be a statement for each integer $n \geq m$. Suppose that:

(i) *The statement $P(m)$ is true, and*

(ii) *for all $k \geq m$, if $P(k)$ is true, then $P(k+1)$ is true.*

Then $P(n)$ is true for all $n \geq m$.

Example 4.2 Show that any whole number of cents of at least 8 cents can be obtained using 3 and 5 cent coins.

The extended real numbers

For certain applications it is convenient to extend the set of real numbers.

Let ∞ and $-\infty$ be two distinct fixed objects neither of which is an element of $\mathbb{R}$.

- The symbol ∞ is called **infinity** and the symbol $-\infty$ is called **minus infinity**.
- The symbol ∞ is sometimes written as $+\infty$ and read **plus infinity**.

Definition 4.4

The **extended real number system** is the union $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, with the following properties:

(i) For $x, y \in \mathbb{R}$, $x \pm y$, xy, $\dfrac{x}{y}$ have their usual meanings.

(ii) $-\infty < \infty$ and for all $x \in \mathbb{R}$, $-\infty < x$, $x < \infty$.

(iii) For all $x \in \mathbb{R}$,
$$x + \infty = \infty + x = x - (-\infty) = \infty;$$
$$x + (-\infty) = (-\infty) + x = x - \infty = -\infty.$$

Definition 4.4 (cont'd)

(iv) If $x \in \overline{\mathbb{R}}$ and $x > 0$, then

$$x \cdot \infty = \infty \cdot x = \infty \quad \text{and}$$
$$x \cdot (-\infty) = (-\infty) \cdot x = -\infty.$$

(v) If $x \in \overline{\mathbb{R}}$ and $x < 0$, then

$$x \cdot \infty = \infty \cdot x = -\infty \quad \text{and}$$
$$x \cdot (-\infty) = (-\infty) \cdot x = \infty.$$

(vi) $\infty + \infty = \infty - (-\infty) = \infty$,
$(-\infty) + (-\infty) = (-\infty) - \infty = -\infty$;

- By convention, we also define

$$0 \cdot \infty = \infty \cdot 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0 = 0,$$

$$\frac{x}{\pm\infty} = 0 \quad \text{for all } x \in \mathbb{R}.$$

- The operations $(-\infty) - (-\infty)$, $\infty - \infty$, $\dfrac{\pm\infty}{\pm\infty}$, and $\dfrac{x}{0}$ are undefined.

- For $x \in \overline{\mathbb{R}}$, we define the **absolute value** of x by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x \leq 0. \end{cases}$$

- $\overline{\mathbb{R}}$ is customarily written as

$$\overline{\mathbb{R}} = [-\infty, \infty].$$

- To distinguish the real numbers from the extended real numbers, we call real numbers **finite**.

- A function with values in $\overline{\mathbb{R}} = [-\infty, \infty]$ is called an **extended real-valued** function.

For $-\infty \leq a \leq b \leq \infty$, we define the four intervals having **left endpoint a** and **right endpoint b** to be the sets

$$
\begin{aligned}
[a,b] &= \{x \in \overline{\mathbb{R}} : \ a \leq x \leq b\}, \\
[a,b) &= \{x \in \overline{\mathbb{R}} : \ a \leq x < b\}, \\
(a,b] &= \{x \in \overline{\mathbb{R}} : \ a < x \leq b\}, \\
(a,b) &= \{x \in \overline{\mathbb{R}} : \ a < x < b\}.
\end{aligned}
$$

Note that if $a = b$, then $[a,b] = \{a\}$ while the other three are $\emptyset$.

In order to avoid a continual exclusion of undefined operations when statements are made about $\overline{\mathbb{R}}$, we adopt the following convention.

Convention

If P is an assertion about extended-real numbers, then "P holds" is understood to mean "P is true provided that every operation appearing in P is defined."

Supremum and Infimum of a Set in $\overline{\mathbb{R}}$

Let $S \subset \overline{\mathbb{R}}$.

If for each $M \in \mathbb{R}$, there exists $x \in S$ with $x > M$, we say that $+\infty$ is the supremum of S and write

$$\sup S = +\infty.$$

If for each $M \in \mathbb{R}$, there exists $x \in S$ with $x < M$, we say that $-\infty$ is the infimum of S and write

$$\inf S = -\infty.$$

We define

$$\sup \emptyset = -\infty \quad \text{and} \quad \inf \emptyset = +\infty.$$

Therefore

Every subset of $\overline{\mathbb{R}}$ has both supremum and infimum in $\overline{\mathbb{R}}$.

Remark 4.1

Let S, A and B be nonempty subsets of $\overline{\mathbb{R}}$.

(a) Set $-S = \{-x : x \in S\}$. Then

$$\sup S = -\inf(-S) \quad \text{and} \quad \inf S = -\sup(-S).$$

(b) Let a be any finite number in $\overline{\mathbb{R}}$. Define

$$a + S := \{a + x : x \in S\}.$$

Then

$$\sup(a + S) = a + \sup S, \quad \inf(a + S) = a + \inf S.$$

(c) If $x \leq y$ for all $x \in A$ and $y \in B$, then

$$\sup A \leq \inf B.$$

- If $f : X \to \overline{\mathbb{R}}$ and $E \subset X$, it is customary to write
$$\sup_{x\in E} f(x) \quad \text{and} \quad \inf_{x\in E} f(x)$$
rather than $\sup f(E)$ and $\inf f(E)$.

- If $f(X)$ is bounded above, f is said to be **bounded above**. Similarly, we define what it means for a function to be bounded below or bounded. Thus,
$$f \text{ is bounded above} \iff \sup_{x\in X} f(x) < \infty$$
$$f \text{ is bounded below} \iff \inf_{x\in X} f(x) > -\infty.$$

Remark 4.2

1. If $\alpha = \inf S$ and $\alpha \in S$, we write $\boldsymbol{\alpha = \min S}$. Similarly, we define what it means for

$$\max S, \quad \min_{x \in E} f(x), \quad \text{and} \quad \max_{x \in E} f(x).$$

2. If $\emptyset \neq A \subset B \subset \overline{\mathbb{R}}$, then

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Example 4.3 Let E be any nonempty subset of a set X and let $f, g : X \to \overline{\mathbb{R}}$ be extended real-valued functions.

(a)

$$\sup_{x\in E} f(x) = -\inf_{x\in E}(-f)(x).$$

(b) If $f(x) \le g(x)$ for all $x \in E$, then

$$\sup_{x\in E} f(x) \le \sup_{x\in E} g(x) \quad \text{and} \quad \inf_{x\in E} f(x) \le \inf_{x\in E} g(x).$$

(c) If $f(x) \le g(y)$ for all $x, y \in E$, then

$$\sup_{x\in E} f(x) \le \inf_{x\in E} g(x).$$

Definition 5.1

Let $X \subset \overline{\mathbb{R}}$ be a set, and let $f : X \to \overline{\mathbb{R}}$ be a function.

(a) The function f is **increasing** if $x < y$ implies $f(x) \leq f(y)$ for all $x, y \in X$.

(b) The function f is **strictly increasing** if $x < y$ implies $f(x) < f(y)$ for all $x, y \in X$.

(c) The function f is **decreasing** if $x < y$ implies $f(x) \geq f(y)$ for all $x, y \in X$.

(d) The function f is **strictly decreasing** if $x < y$ implies $f(x) > f(y)$ for all $x, y \in X$.

(e) The function f is **monotone** if it is either increasing or decreasing.

Definition 5.2

A sequence $\{x_n\}$ of extended real numbers is said to be

(a) **increasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$;

(b) **decreasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$;

(c) **strictly increasing** if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$;

(d) **strictly decreasing** if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.

A sequence is said to be **monotone** provided it is either increasing or decreasing..

Limit Superior and Limit Inferior of Sequences in $\overline{\mathbb{R}}$

Let $\{x_n\}$ be a sequence in $\overline{\mathbb{R}}$.

If for each real number c, there is an index $N \in \mathbb{N}$ such that $x_n \geq c$ for all $n > N$, then we say that $\{x_n\}$ **converges to infinity**, call ∞ the **limit** of $\{x_n\}$ and write

$$\lim x_n = \infty.$$

Similar definitions are made at $-\infty$.

Theorem 5.1

Every monotone sequence in $\overline{\mathbb{R}}$ is convergent in $\overline{\mathbb{R}}$. Furthermore,

$$\lim x_n = \sup\{x_n : n \in \mathbb{N}\} \quad \text{if } \{x_n\} \text{ is increasing}$$

and

$$\lim x_n = \inf\{x_n : n \in \mathbb{N}\} \quad \text{if } \{x_n\} \text{ is decreasing.}$$

Denote

$$\sup_{n\geq k} x_n := \sup\{x_n : n \geq k\} \quad \text{and}$$

$$\inf_{n\geq k} x_n := \inf\{x_n : n \geq k\}.$$

If $\{x_n\}$ is a sequence in $\overline{\mathbb{R}}$ and $k \in \mathbb{N}$, let

$$y_k := \sup_{n \geq k} x_n \quad \text{and} \quad z_k := \inf_{n \geq k} x_n.$$

Then $\{y_k\}$ is a decreasing sequence and $\{z_k\}$ is an increasing sequence in $\overline{\mathbb{R}}$.

Therefore, by Theorem 5.1, these sequences converge in $\overline{\mathbb{R}}$,

$$\lim_{k \to \infty} y_k = \inf_{k \geq 1} y_k \quad \text{and} \quad \lim_{k \to \infty} z_k = \sup_{k \geq 1} z_k.$$

Definition 5.3

(a) Let $\{x_n\}$ be a sequence of extended real numbers. The **limit superior** of $\{x_n\}$, denoted by $\limsup_{n\to\infty} x_n$, is defined by

$$\limsup_{n\to\infty} x_n := \lim_{k\to\infty}\left(\sup_{n\geq k} x_n\right) = \inf_{k\geq 1}\left(\sup_{n\geq k} x_n\right).$$

(b) The **limit inferior** of $\{x_n\}$, denoted by $\liminf_{n\to\infty} x_n$, is defined by

$$\liminf_{n\to\infty} x_n := \lim_{k\to\infty}\left(\inf_{n\geq k} x_n\right) = \sup_{k\geq 1}\left(\inf_{n\geq k} x_n\right).$$

Note

Every sequence in $\overline{\mathbb{R}}$ has limit superior and limit inferior in $\overline{\mathbb{R}}$.

Theorem 5.2

Let $\{x_n\}$ be a sequence in $\overline{\mathbb{R}}$. Then:

(a) $\liminf x_n \leq \limsup x_n$.

(b) *If* $0 \leq c \leq \infty$, *then*

$$\liminf(c \cdot x_n) = c \cdot \liminf x_n \quad \text{and}$$
$$\limsup(c \cdot x_n) = c \cdot \limsup x_n.$$

(c) *If* $-\infty \leq c \leq 0$, *then*

$$\liminf(c \cdot x_n) = c \cdot \limsup x_n \quad \text{and}$$
$$\limsup(c \cdot x_n) = c \cdot \liminf x_n.$$

Theorem 5.2 (cont'd)

(d) $\{x_n\}$ *is convergent in* $\overline{\mathbb{R}}$ *if and only if*

$$\liminf x_n = \limsup x_n,$$

in which case,

$$\lim x_n = \liminf x_n = \limsup x_n.$$

$$\exists \lim x_n \in \overline{\mathbb{R}} \iff \liminf x_n = \limsup x_n.$$

Theorem 5.3

Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\overline{\mathbb{R}}$. Then:

(a) $\liminf x_n + \liminf y_n \le \liminf(x_n + y_n)$.

(b) $\limsup(x_n + y_n) \le \limsup x_n + \limsup y_n$.

(c) *If $x_n \le y_n$ for all n, then*

$$\liminf x_n \le \liminf y_n \quad \text{and}$$
$$\limsup x_n \le \limsup y_n.$$

(d) *If $x_n \le y_n \le z_n$ for all n, and* $\lim x_n = \lim z_n = a \in \overline{\mathbb{R}}$, *then* $\lim y_n = a$.

Definition 5.4

An extended real number is called a **subsequential** (or **partial**) **limit** of a sequence if the sequence contains a subsequence converging to that number.

Theorem 5.4

For any sequence of extended real numbers, the inferior limit is the smallest of its subsequential limits and the superior limit is the largest of its subsequential limits.

Corollary 5.5

A sequence of extended real numbers is convergent if and only if it has just one subsequential limit.

Corollary 5.6 (The Bolzano-Weierstrass Theorem)

Every bounded sequence of real numbers has a subsequence that converges to a finite number.

Sum of a Series of Nonnegative Extended Real Numbers

By Theorem 5.1, every series of non-negative extended real numbers converges in $\overline{\mathbb{R}}$ since its partial sums form an increasing sequence.

Theorem 5.7

If $\{a_n\}$ is a sequence of non-negative extended real numbers and $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijective function, then

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n = \sup\left\{ \sum_{n\in J} a_n : J \subset \mathbb{N},\ J \text{ finite}\right\}.$$

- We define

$$\sum_{\emptyset} a_n = 0.$$

Consider a double sequence $\{a_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ with $0 \leq a_{n,m} \leq \infty$ for each pair n, m.

For each fixed n, the series $\sum_{m=1}^{\infty} a_{n,m}$ converges in $\overline{\mathbb{R}}$ (possible to $+\infty$) and for each fixed m, the series $\sum_{n=1}^{\infty} a_{n,m}$ converges in $\overline{\mathbb{R}}$.

Theorem 5.8

Suppose that $a_{n,m} \in [0, \infty]$ for all $(n, m) \in \mathbb{N} \times \mathbb{N}$ and that $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijective mapping. Then

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{n,m}\right) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{n,m}\right) = \sum_{k=1}^{\infty} a_{\sigma(k)}.$$

In this section we shall deal with questions concerning the "size" of a set.

Definition 6.1

Two sets A and B are said to be **equivalent** and we write $\boldsymbol{A \sim B}$ if there exists a mapping $f : A \to B$ that is bijective.

Example 6.1

(a) If $a, b \in \mathbb{R}$ and $a < b$, then $(a, b) \sim (0, 1)$.

(b) The interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is equivalent to the set $\mathbb{R}$.

Example 6.2 If X is nonempty, then $\mathcal{P}(X)$ is equivalent to the set of all mappings of X into $\{0, 1\}$.

Theorem 6.1

For any nonempty sets A, B, and C we have

(a) $A \sim A$,

(b) $A \sim B$ *implies* $B \sim A$,

(c) $A \sim B$ *and* $B \sim C$ *implies* $A \sim C$.

Example 6.3

(a) If $a, b, c, d \in \mathbb{R}$, $a < b$ and $c < d$, then

$$(a, b) \sim (c, d).$$

(b) If $a, b \in \mathbb{R}$ and $a < b$, then

$$(a, b) \sim \mathbb{R}.$$

Definition 6.2

A set E is said to be **finite** if either $E = \emptyset$ or there is a natural number n for which E is equivalent to $\{1, \ldots, n\}$. A set that is not finite is called an **infinite** set.

If E is equivalent to $\{1, \ldots, n\}$, f is a bijective mapping from $\{1, \ldots, n\}$ onto E and if $f(k) = x_k$, we can write

$$E = \{x_k : 1 \leq k \leq n\} = \{x_1, x_2, \ldots, x_n\} = \{x_k\}_{k=1}^{n}.$$

In this case , we say that E has ***n* elements**.

By definition, the empty set has zero elements.

Example 6.4 If there is an injective mapping from $\mathbb{N}$ into A, then A is infinite.

Definition 6.3

We say that E is **countably infinite** provided E is equivalent to the set $\mathbb{N}$ of natural numbers.

A set that is either finite or countably infinite is said to be **countable**. A set that is not countable is called **uncountable**.

If A is a countably infinite set and $f : \mathbb{N} \to A$ is a bijective mapping and $a_n = f(n)$, then we usually write

$$A = \{a_1, a_2, \dots\}.$$

Such an f is called an **enumeration** of A.

Example 6.5 The set E of all even natural numbers, $E = \{2n : n \in \mathbb{N}\}$, is countably infinite.

Example 6.6 The set $\mathbb{Z}$ is countably infinite.

Remark 6.1 Suppose $A \sim B$. According to Theorem 6.1,

(a) If A is finite, then so is B;

(b) If A is countably infinite, then so is B;

(c) If A is countable, then so is B;

(d) If A is uncountable, then so is B.

Theorem 6.2

Every subset of a countable set is countable.

Theorem 6.3

Let A be a nonemptyset. The following statements are equivalent.

(a) *A is countable.*

(b) *There exists a surjection of $\mathbb{N}$ onto A.*

(c) *There exists an injection of A into $\mathbb{N}$.*

Corollary 6.4

For a nonempty set A, the following statements are equivalent.

(a) *A is countable.*

(b) *There exist a countable set B and a surjection of B onto A.*

(c) *There exist a countable set C and an injection of A into C.*

Theorem 6.5

Let $A_1, \ldots, A_n$ be a finite collection of sets such that each A_i is countable. Then $A_1 \times \cdots \times A_n$, is countable.

Corollary 6.6

The set of rational numbers $\mathbb{Q}$ is countable.

Theorem 6.7

The union of a countable collection of countable sets is countable. More precisely, if I is a countable nonempty set and A_i is a countable set for each $i \in I$, then the set $A = \cup_i A_i$ is a countable set.

Theorem 6.8

If $a, b \in \mathbb{R}$ and $a < b$, then $[a, b]$ is an uncountable set.

Note: If B contains an uncountable subset A, then B is itself uncountable.