Chapter 6

Numerical Methods for Partial Differential Equations

Lecture 2: Time-dependent PDEs

- Parabolic Partial Differential Equations
- Hyperbolic Partial Differential Equations

Finite Difference Method for Parabolic Partial Differential Equations

Approximate solution of the heat equation in one-dimensional space

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \qquad 0 < x < l, \quad t > 0$$

with boundary conditions $u(0,t) = g_1(t), \quad u(l,t) = g_2(t), \quad t > 0$

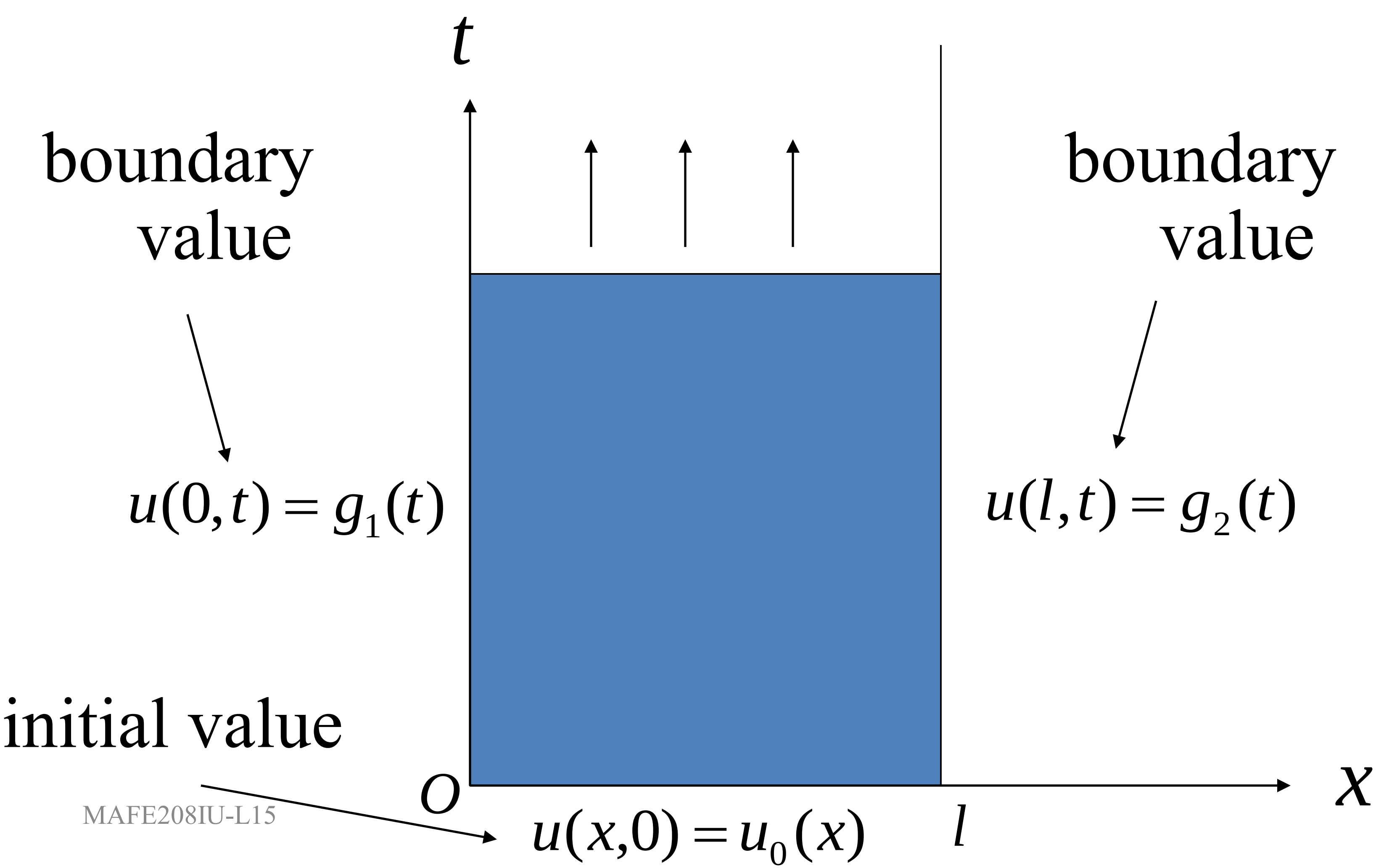
and initial condition $u(x,0) = u_0(x), \qquad 0 \le x \le l$

Method can be extended for multi-dimensional space

$$\frac{\partial u(x,t)}{\partial t} = \sum_{i=1}^{n} \alpha_i (x,t) \frac{\partial^2 u(x,t)}{\partial x_i^2} + \sum_{i=1}^{n} \beta_i (x,t) \frac{\partial u(x,t)}{\partial x_i} + \gamma(x,t) u(x,t) + f(x,t)$$

for $x = (x_1, x_2, \dots, x_n), \quad t > 0$

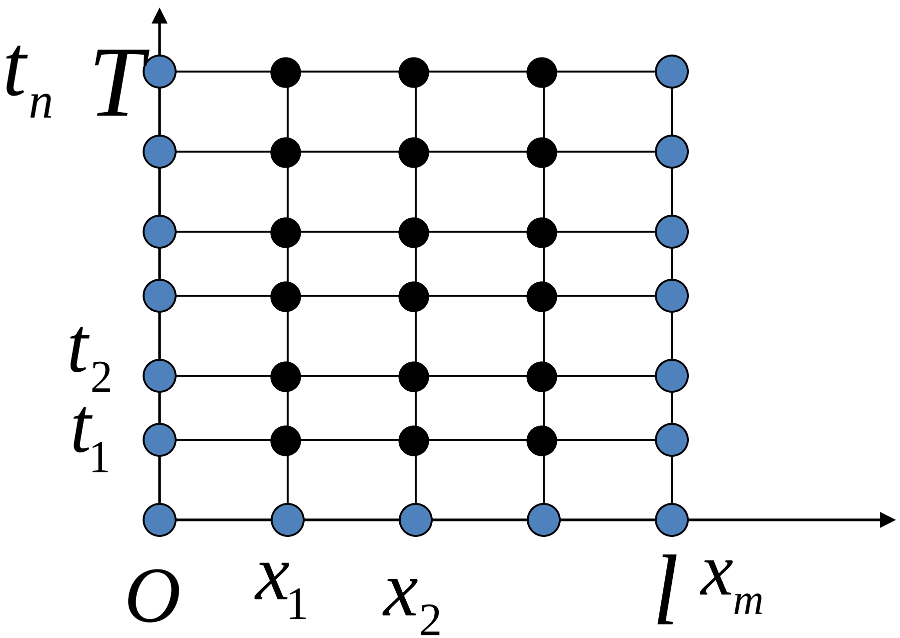
Domain of solutions of Parabolic Partial Differential Equations



Finite Difference Grid

$T =$ maximum time used to compute the solution; $\Omega = [0, l] \times [0, T]$

is divided into m equal parts along $x-$axis and n parts along $t-$axis



boundary points

interior points

Step sizes: $\Delta x = \dfrac{l}{m}, \quad \Delta t = \dfrac{T}{n}$

Coordinates of mesh (grid) points: (x_i, t_k),

where $x_i = i\Delta x,\ t_k = k\Delta t,\ 0 \le i \le m, 0 \le k \le n$

Discretization

$$\frac{\partial u(x_i,t_k)}{\partial t}=\alpha\frac{\partial^2 u(x_i,t_k)}{\partial x^2}+f(x_i,t_k)$$

$$\frac{u(x_i,t_{k+1})-u(x_i,t_k)}{\Delta t}=\alpha\frac{u(x_{i+1},t_k)-2u(x_i,t_k)+u(x_{i-1},t_k)}{(\Delta x)^2}+f(x_i,t_k)$$

Let $f_i^k = f(x_i,t_k),\ \ u_i^k \approx u(x_i,t_k),\ 0\le i\le m, 1\le k\le n$

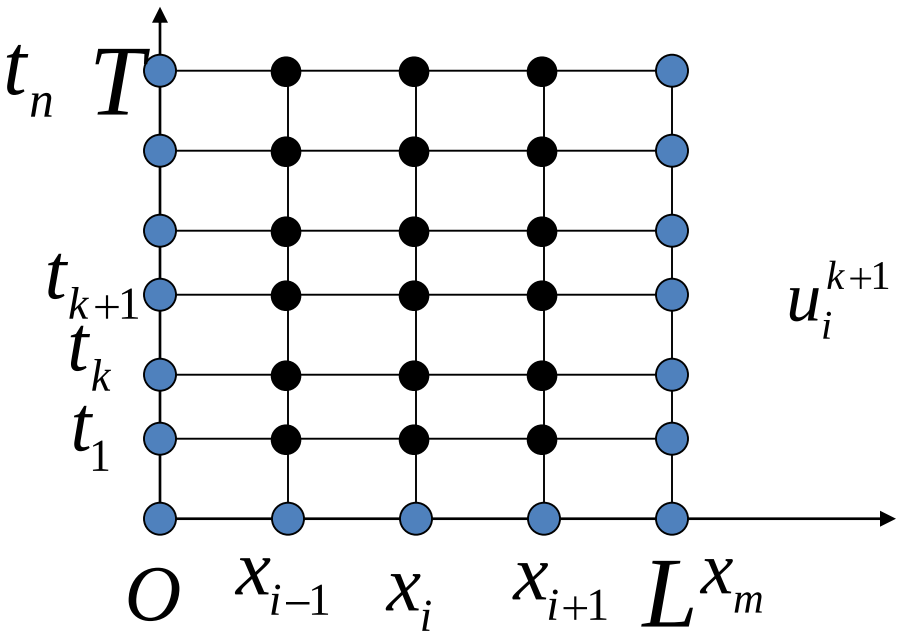
$$\frac{u_i^{k+1}-u_i^k}{\Delta t}=\alpha\frac{u_{i+1}^k-2u_i^k+u_{i-1}^k}{(\Delta x)^2}+f_i^k$$

Explicit method:

$$u_i^{k+1}=u_i^k+\lambda(u_{i+1}^k-2u_i^k+u_{i-1}^k)+\Delta t\cdot f_i^k$$

where $\lambda=\dfrac{\alpha\Delta t}{(\Delta x)^2}, \qquad 1\le i\le m-1, k=0,1,2,\ldots,n-1$

Matrix Form of Explicit Method



$$u_i^{k+1} = u_i^k + \lambda(u_{i+1}^k - 2u_i^k + u_{i-1}^k) + \Delta t \cdot f_i^k$$

$$\boxed{V^{k+1} = AV^k + B_k}$$

$$V^k = \begin{bmatrix} u_1^k \\ u_2^k \\ u_3^k \\ \vdots \\ u_{m-1}^k \end{bmatrix}, A = \begin{bmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & & 0 \\ 0 & \lambda & 1-2\lambda & & 0 \\ \cdots & & & \cdots & \lambda \\ 0 & 0 & 0 & \lambda & 1-2\lambda \end{bmatrix}, B_k = \Delta t \begin{bmatrix} f_1^k \\ f_2^k \\ \vdots \\ \\ f_{m-1}^k \end{bmatrix} + \begin{bmatrix} \lambda g_1^k \\ 0 \\ \vdots \\ 0 \\ \lambda g_2^k \end{bmatrix}$$

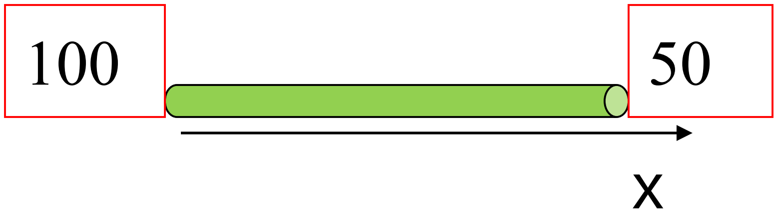
Example 1

Use the explicit method to solve for the temperature distribution of a long, thin rod with a length of 10 cm and the following values

$$\alpha = 0.8 \text{ cm}^2/\text{s}, \qquad \Delta x = 2 \text{ cm}, \qquad \Delta t = 0.1 \text{ s}$$

$$u(x,0) = u_0(x) = 0 \ ^{\circ}\text{C}, \quad 0 < x < 10$$

$$u(0,t) = g_1(t) = 100 \ ^{\circ}\text{C} \qquad u(10,t) = g_2(t) = 50 \ ^{\circ}\text{C}$$



Solution

Heat equation $$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad \alpha = 0.8$$

$$\lambda = \frac{\alpha \Delta t}{(\Delta x)^2} = \frac{0.8(0.1)}{2^2} = 0.02$$

$$\begin{aligned} u_i^{k+1} &= u_i^k + \lambda(u_{i+1}^k - 2u_i^k + u_{i-1}^k) \\ &= u_i^k + 0.02(u_{i+1}^k - 2u_i^k + u_{i-1}^k) \end{aligned}$$

At $t = 0.1$ s

$$\begin{aligned} u_1^1 &= 0 + 0.02[0 - 2(0) + 100] = 2.0 \\ u_2^1 &= 0 + 0.02[0 - 2(0) + 0] = 0 \\ u_3^1 &= 0 + 0.02[0 - 2(0) + 0] = 0 \\ u_4^1 &= 0 + 0.02[50 - 2(0) + 0] = 1.0 \end{aligned}$$

At $t = 0.2$ s

$$\begin{aligned} u_1^2 &= 2.0 + 0.02[0 - 2(2) + 100] = 3.92 \\ u_2^2 &= 0 + 0.02[0 - 2(0) + 2] = 0.04 \\ u_3^2 &= 0 + 0.02[1 - 2(0) + 0] = 0.02 \\ u_4^2 &= 1 + 0.02[50 - 2(1) + 0] = 1.96 \end{aligned}$$

Exercise

Approximate solution of the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{16}\frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < 1, \quad t > 0$$

with boundary conditions $u(0,t) = u(1,t) = 0, \quad t > 0$

and initial condition $u(x,0) = 2\sin(2\pi x), \qquad 0 \le x \le 1$

at the time t=0.1 and t=0.2 using explicit method with $\Delta x=0.2$, $\Delta t=0.1$ and find error. Exact solution:

$$u(x,t) = e^{\frac{-\pi^2 t}{4}} \sin(2\pi x)$$

Implicit Methods

Consider CDA of 1st derivative at middle grid point with time step $\Delta t/2$:

$$\frac{\partial u}{\partial t}(x_i, t_{k+1/2}) \approx \frac{u(x_i, t_{k+1}) - u(x_i, t_k)}{\Delta t} \approx \frac{u_i^{k+1} - u_i^k}{\Delta t}$$

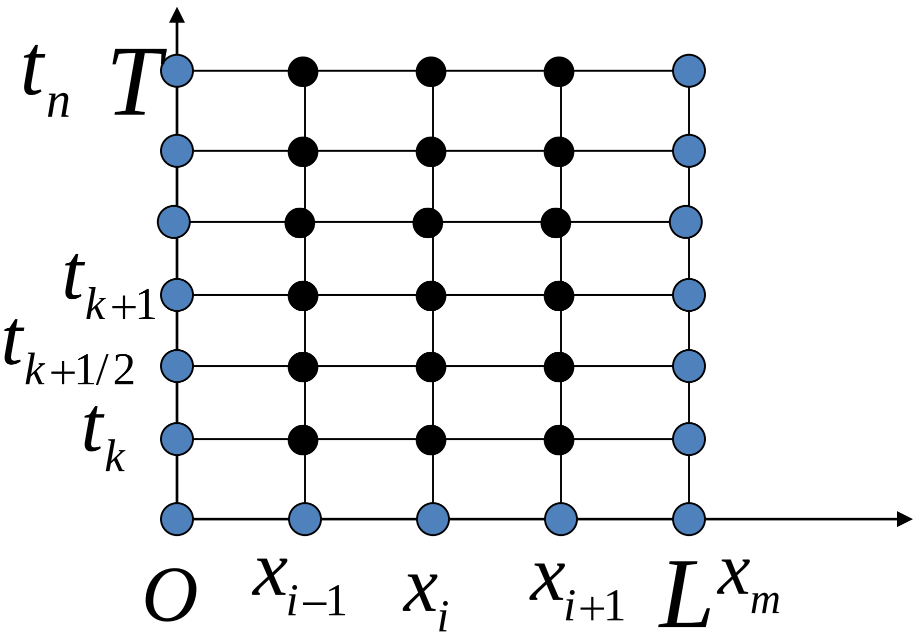
2nd derivative at (i, k+1/2) is computed as weighted average of CDA values at (i, k) and (i, k+1):

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1/2}) \approx (1-\theta)\frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \theta\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1}), \text{ for } 0 \le \theta \le 1$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1/2}) \approx (1-\theta)\frac{u(x_{i+1}, t_k) - 2u(x_i, t_k) + u(x_{i-1}, t_k)}{(\Delta x)^2}$$

$$+\theta\frac{u(x_{i+1}, t_{k+1}) - 2u(x_i, t_{k+1}) + u(x_{i-1}, t_{k+1})}{(\Delta x)^2}$$

Implicit Methods



A set of implicit methods: Variable-weighted implicit formula

$$\frac{\partial u}{\partial t}(x_i, t_{k+1/2}) = \alpha \frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1/2}) \qquad \lambda = \frac{\alpha \Delta t}{(\Delta x)^2}$$

$$u_i^{k+1} - u_i^k = \lambda(1-\theta)(u_{i+1}^k - 2u_i^k + u_{i-1}^k) + \lambda\theta(u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1})$$

Defines the **variable-weighted implicit formula:**

$$-\theta\left(\lambda u_{i+1}^{k+1} + (2\lambda - \frac{1}{\theta})u_i^{k+1} + \lambda u_{i-1}^{k+1}\right) = (1-\theta)\left(\lambda u_{i+1}^k + (\frac{1}{(1-\theta)} - 2\lambda)u_i^k + \lambda u_{i-1}^k\right)$$

where $\lambda = \dfrac{\alpha \Delta t}{(\Delta x)^2}$ $\qquad 1 \le i \le m-1,\ k = 0, 1, 2, \ldots, n-1$

Method is stable for any value of λ

Simple Implicit Method

Taking θ=1 in the **variable-weighted implicit formula** defines backward implicit scheme, or simple implicit method:

$$-\lambda u_{i-1}^{k+1} + (1+2\lambda)u_i^{k+1} - \lambda u_{i+1}^{k+1} = u_i^k$$

Crank-Nicholson Method

Taking θ=1/2 defines Crank-Nicholson **implicit method:**

$$-\lambda u_{i-1}^{k+1} + 2(1+\lambda)u_i^{k+1} - \lambda u_{i+1}^{k+1} = \lambda u_{i-1}^{k} + 2(1-\lambda)u_i^{k} + \lambda u_{i+1}^{k}$$

$$\text{where } \lambda = \frac{\alpha \Delta t}{(\Delta x)^2}, \qquad 1 \le i \le m-1, \quad k = 0,1,2,\ldots,n-1$$

$$\text{Accuracy: } O(\Delta x^2 + \Delta t^2)$$

Using boundary conditions, first and last equations in C-N method are

$$2(1+\lambda)u_1^{k+1} - \lambda u_2^{k+1} = \lambda(g_1^k + g_1^{k+1}) + 2(1-\lambda)u_1^k + \lambda u_2^k$$

$$-\lambda u_{m-2}^{k+1} + 2(1+\lambda)u_{m-1}^{k+1} = \lambda u_{m-2}^{k} + 2(1-\lambda)u_{m-1}^{k} + \lambda(g_2^k + g_2^{k+1})$$

Crank-Nicholson Method

Solve

$$\boxed{AV^{k+1} = BV^k + C_k}$$

$$A = \begin{bmatrix} 2(1+\lambda) & -\lambda & 0 & \cdots & 0 \\ -\lambda & 2(1+\lambda) & -\lambda & & 0 \\ 0 & -\lambda & 2(1+\lambda) & & 0 \\ \cdots & & & \cdots & -\lambda \\ 0 & 0 & 0 & -\lambda & 2(1+\lambda) \end{bmatrix}, \qquad V^k = \begin{bmatrix} u_1^k \\ u_2^k \\ u_3^k \\ \vdots \\ u_{m-1}^k \end{bmatrix}$$

$$B = \begin{bmatrix} 2(1-\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & 2(1-\lambda) & \lambda & & 0 \\ 0 & \lambda & 2(1-\lambda) & & 0 \\ \cdots & & & \cdots & \lambda \\ 0 & 0 & 0 & \lambda & 2(1-\lambda) \end{bmatrix}, \text{ and } \quad C_k = \begin{bmatrix} \lambda(g_1^k + g_1^{k+1}) \\ 0 \\ \vdots \\ 0 \\ \lambda(g_2^k + g_2^{k+1}) \end{bmatrix}$$

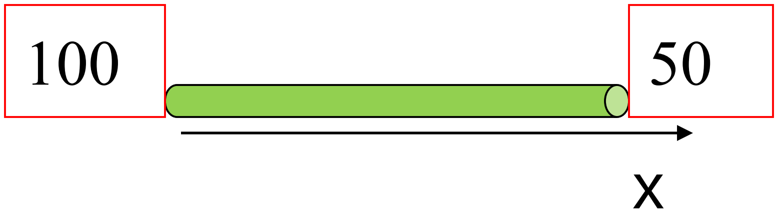
Example 2

Use the Crank-Nicholson method to solve for the temperature distribution of a long, thin rod with a length of 10 cm at time 2 s and the following values

$$\alpha = 1 \text{ cm}^2/\text{s}, \qquad \Delta x = 2 \text{ cm}, \qquad \Delta t = 1 \text{ s}$$

$$u(x,0) = u_0(x) = 0 \ ^\circ\text{C}, \quad 0 < x < 10$$

$$u(0,t) = g_1(t) = 100 \ ^\circ\text{C} \qquad u(10,t) = g_2(t) = 50 \ ^\circ\text{C}$$



Solution:

$$2(1+\lambda)u_1^{k+1} - \lambda u_2^{k+1} = 200\lambda + 2(1-\lambda)u_1^k + \lambda u_2^k$$

System

$$-\lambda u_{i-1}^{k+1} + 2(1+\lambda)u_i^{k+1} - \lambda u_{i+1}^{k+1} = \lambda u_{i-1}^k + 2(1-\lambda)u_i^k + \lambda u_{i+1}^k, \quad i = 2,3$$

$$-\lambda u_{m-2}^{k+1} + 2(1+\lambda)u_{m-1}^{k+1} = \lambda u_{m-2}^k + 2(1-\lambda)u_{m-1}^k + 100\lambda,$$

$$\lambda = \frac{\alpha\Delta t}{(\Delta x)^2} = \frac{1}{4}$$

can be expressed as a Tridiagonal system of equations:

$$\boxed{AV^{k+1} = BV^k + C_k}$$

where

$$A = \begin{bmatrix} 2(1+\lambda) & -\lambda & 0 & 0 \\ -\lambda & 2(1+\lambda) & -\lambda & 0 \\ 0 & -\lambda & 2(1+\lambda) & -\lambda \\ 0 & 0 & -\lambda & 2(1+\lambda) \end{bmatrix} = \begin{bmatrix} 10/4 & -1/4 & 0 & 0 \\ -1/4 & 10/4 & -1/4 & 0 \\ 0 & -1/4 & 10/4 & -1/4 \\ 0 & 0 & -1/4 & 10/4 \end{bmatrix}$$

$$B = \begin{bmatrix} 2(1-\lambda) & \lambda & 0 & 0 \\ \lambda & 2(1-\lambda) & \lambda & 0 \\ 0 & \lambda & 2(1-\lambda) & \lambda \\ 0 & 0 & \lambda & 2(1-\lambda) \end{bmatrix} = \begin{bmatrix} 6/4 & 1/4 & 0 & 0 \\ 1/4 & 6/4 & 1/4 & 0 \\ 0 & 1/4 & 6/4 & 1/4 \\ 0 & 0 & 1/4 & 6/4 \end{bmatrix}$$

$$V^k = \begin{bmatrix} u_1^k \\ u_2^k \\ u_3^k \\ u_4^k \end{bmatrix}, \text{ and } \quad C_k = \begin{bmatrix} \lambda(g_1^k + g_1^{k+1}) \\ 0 \\ 0 \\ \lambda(g_2^k + g_2^{k+1}) \end{bmatrix} = \begin{bmatrix} 200\lambda \\ 0 \\ 0 \\ 100\lambda \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ 25 \end{bmatrix}$$

$$\begin{bmatrix} 10/4 & -1/4 & 0 & 0 \\ -1/4 & 10/4 & -1/4 & 0 \\ 0 & -1/4 & 10/4 & 0 \\ 0 & 0 & -1/4 & 10/4 \end{bmatrix} \xrightarrow{R2-(-1/10)R1} \begin{bmatrix} 10/4 & -1/4 & 0 & 0 \\ 0 & 99/40 & -1/4 & 0 \\ 0 & -1/4 & 10/4 & 0 \\ 0 & 0 & -1/4 & 10/4 \end{bmatrix}$$

$$\xrightarrow{R3-(-10/99)R2} \begin{bmatrix} 10/4 & -1/4 & 0 & 0 \\ 0 & 99/40 & -1/4 & 0 \\ 0 & 0 & 245/99 & 0 \\ 0 & 0 & -1/4 & 10/4 \end{bmatrix}$$

$$\xrightarrow{R4-(-99/980)R3} \begin{bmatrix} 10/4 & -1/4 & 0 & 0 \\ 0 & 99/40 & -1/4 & 0 \\ 0 & 0 & 245/99 & 0 \\ 0 & 0 & 0 & 10/4 \end{bmatrix} = U$$

LU Decomposition of A:

$$A = LU, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/10 & 1 & 0 & 0 \\ 0 & -10/99 & 1 & 0 \\ 0 & 0 & -99/980 & 1 \end{bmatrix}$$

$$A = LU, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/10 & 1 & 0 & 0 \\ 0 & -10/99 & 1 & 0 \\ 0 & 0 & -99/980 & 1 \end{bmatrix}, U = \begin{bmatrix} 10/4 & -1/4 & 0 & 0 \\ 0 & 99/40 & -1/4 & 0 \\ 0 & 0 & 245/99 & 0 \\ 0 & 0 & 0 & 10/4 \end{bmatrix}$$

$$C_k = C = const$$

k=0: temperature at 1 second is computed as follows

$$AV^1 = BV^0 + C = C$$

$$Ly = C \Leftrightarrow y = (50,\ 5,\ 50/99,\ \ 2455/98)^T$$

$$UV^1 = y \Rightarrow V^1 = (20.2041, \quad 2.0408, \quad 0.2041, \quad 10.0204)^T$$

k=1: temperature at 2 seconds is computed as follows

$$AV^2 = BV^1 + C = D = (3960/49,\ \ 400/49,\ \ 40/49,\ \ 1964/49)^T$$

$$Lz = D \Leftrightarrow z = (\ 3960/49,\ \ 796/49,\ 833/339,\ \ 3912/97\ )^T$$

$$\boxed{UV^2 = z \Rightarrow V^2 = (32.9929, \quad 6.6639, \quad 0.9929, \quad 16.1319\ )^T}$$

Quiz

Approximate solution of the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{16}\frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < 1, \quad t > 0$$

with boundary conditions $u(0,t) = u(1,t) = 0, \quad t > 0$

and initial condition $u(x,0) = \sin(2\pi x), \qquad 0 \le x \le 1$

at the time t=0.1 and t=0.2 using implicit Crank-Nicholson method with Δx=0.2, Δt=0.1 and find the error. Exact solution:

$$u(x,t) = e^{\frac{-\pi^2 t}{4}} \sin(2\pi x)$$

Finite difference method for Hyperbolic Partial Differential Equations

Approximate solution of the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < l, \quad t > 0$$

with boundary conditions $u(0,t) = u(l,t) = 0, \quad t > 0$

and initial conditions $u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), \quad 0 \le x \le l$

Choose step size in space Δx>0, in time Δt>0. CDA of 2nd derivatives:

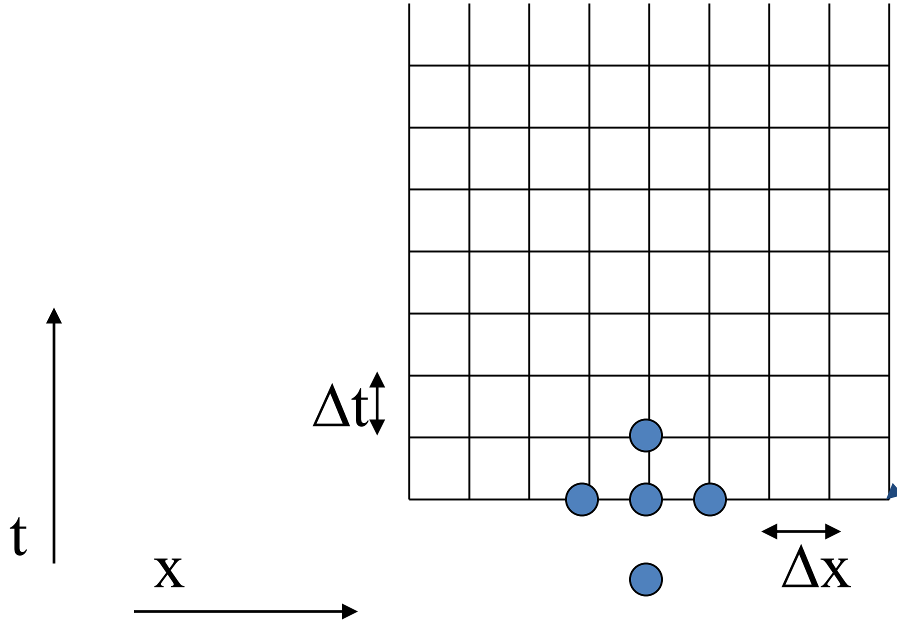
$$\frac{\partial^2 u(x_i,t_k)}{\partial t^2} \approx \frac{1}{\Delta t^2}(u(x_i,t_k-\Delta t) - 2u(x_i,t_k) + u(x_i,t_k+\Delta t)) = \frac{1}{\Delta t^2}(u_i^{k-1} - 2u_i^k + u_i^{k+1})$$

$$\frac{\partial^2 u(x_i,t_k)}{\partial x^2} \approx \frac{1}{\Delta x^2}(u(x_i-\Delta x,t_k) - 2u(x_i,t_k) + u(x_i+h,t_k)) = \frac{1}{\Delta x^2}(u_{i-1}^k - 2u_i^k + u_{i+1}^k)$$

Finite difference method for wave equation

$$\boxed{u_i^{k+1} = 2u_i^k - u_i^{k-1} + \frac{c^2\Delta t^2}{\Delta x^2}(u_{i-1}^k - 2u_i^k + u_{i+1}^k)}$$

Can't use this for first time step.



U at initial time given by initial condition.
$u_{i,0} = f(x_i)$

$u(x,0) = f(x)$

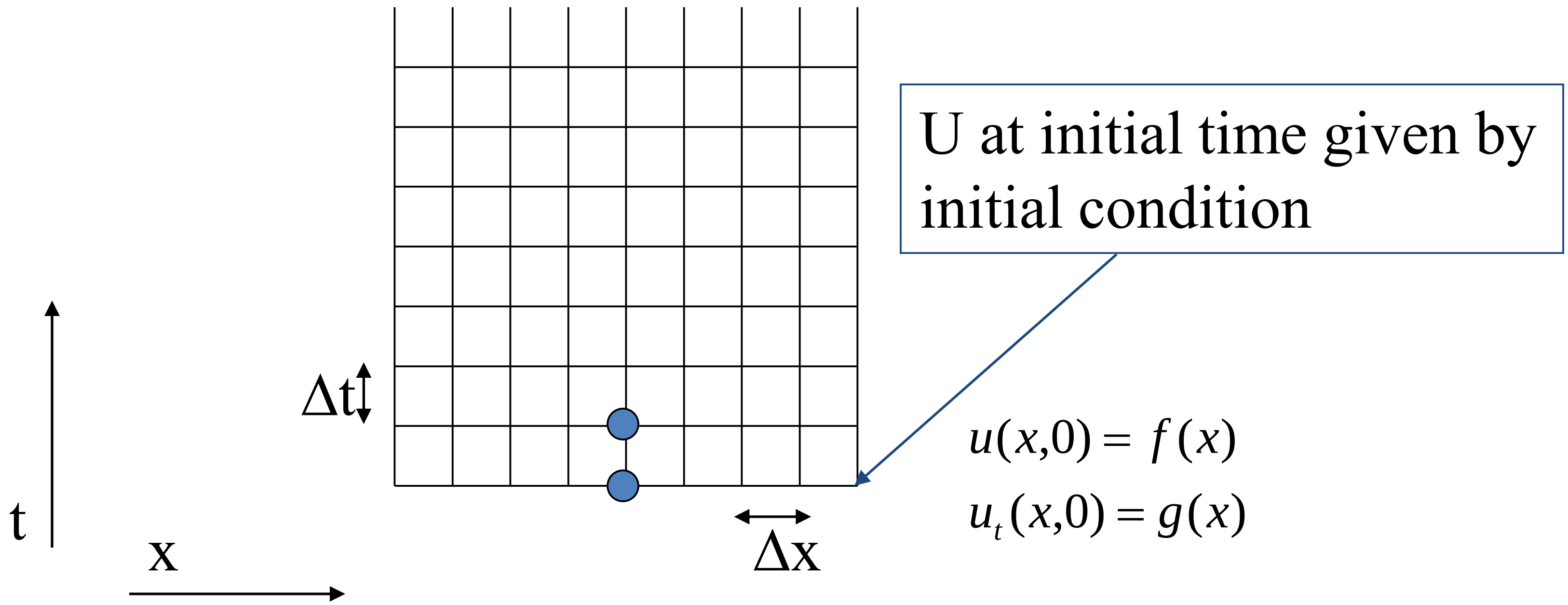
$u_t(x,0) = g(x)$

Finite difference method for wave equation

Use initial derivative to make first time step.

$$g(x_i) = u_t(x_i, 0) \approx \frac{u(x_i, \Delta t) - u(x_i, 0)}{\Delta t} \approx \frac{u_i^1 - f(x_i)}{\Delta t}$$

$$\Rightarrow u_i^1 = g_i \Delta t + f_i$$



$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

Example

Approximate solution of the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 4\frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < 1, \quad t > 0$$

with boundary conditions $u(0,t) = u(1,t) = 0, \quad t > 0$

and initial conditions $u(x,0) = \sin(\pi x), \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 \le x \le 1$

at the time t=0.1 and t=0.2 using Δx=0.2, Δt=0.1 and compare with exact solution

$$u(x,t) = \sin(\pi x)\cos(2\pi t)$$

Solution

$$x_i = i\Delta x = 0.2i,\ \ i = 0,1,2,3,4,5$$

$$t_k = k\Delta t = 0.1k,\ \ \ k = 1,2$$

$$u(0,t) = u(1,t) = 0,\ \ \ t > 0 \Rightarrow u_0^k = u_5^k = 0,\ \ k = 1,2$$

$$f(x) = \sin(\pi x),\ \ \ g(x) = 0$$

$$u_i^1 = g_i \Delta t + f_i = f_i = \sin(\pi x_i) = \sin(0.2i\pi),\ \ \ \ i = 1,2,3,4$$

$$c = 2 \Rightarrow \frac{c^2 \Delta t^2}{\Delta x^2} = 1$$

$$u_i^2 = 2u_i^1 - u_i^0 + \frac{c^2 \Delta t^2}{\Delta x^2}(u_{i-1}^1 - 2u_i^1 + u_{i+1}^1) = -u_i^0 + (u_{i-1}^1 + u_{i+1}^1)$$

Solution

$$x = [\ 0 \quad 0.2000 \quad 0.4000 \quad 0.6000 \quad 0.8000 \quad 1]$$

At $t = 0.1:$

$$u(0.1) = [0 \quad 0.4755 \quad 0.7694 \quad 0.7694 \quad 0.4755 \quad 0]$$

$$u^1 = [0 \quad 0.5878 \quad 0.9511 \quad 0.9511 \quad 0.5878 \quad 0]$$

$$error1 = [\ 0 \quad -0.1123 \quad -0.1816 \quad -0.1816 \quad -0.1123 \quad 0]$$

At $t = 0.2:$

$$u(0.2) = [0 \quad 0.1816 \quad 0.2939 \quad 0.2939 \quad 0.1816 \quad 0]$$

$$u^2 = [0 \quad 0.3633 \quad 0.5878 \quad 0.5878 \quad 0.3633 \quad 0]$$

$$error2 = [\ 0 \quad -0.1816 \quad -0.2939 \quad -0.2939 \quad -0.1816 \quad 0]$$