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* Chap 1:

1/ Limit: $\lim_{(x,y)\to(0,0)} f(x,y)$.

Giả sử $f(x,y) = \dfrac{A(x,y)}{B(x,y)}$.

Nếu bậc $A>B$ thì $\lim f(x,y) = 0$ (?).

Cách làm. tách bậc thừa ra khỏi A.

Nếu bậc $A=B$ thì $\nexists \lim f(x,y)$ (?).

Cách làm: xét $x=y$, $x=0$, $y=0$ (???)

Squeeze Theorem: $0 \le |A| \le |x| \to 0 \Rightarrow |A| \to 0$.

2/ Continuous:

f liên tục tại $(a,b) \Leftrightarrow \lim_{(x,y)\to(a,b)} f = f(a,b)$.

3/ Partial Derivatives:

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$f_{xx}(x,y) = \frac{\partial}{\partial x} f_x, \quad f_{xy}(x,y) = \frac{\partial}{\partial y} f_x.$$

Laplace equation: $f_{xx} + f_{yy} = 0$.

f harmonic $\Leftrightarrow$ f satisfy Laplace equation.

4/ Marginal Analysis:

Suppose $P(x,y)$ at (a,b): $P_x(a,b) > P_y(a,b)$.

Then adding a will increase more than adding b.

5/ Linerization: Consider $f(x,y)$.

Equation of tangent plane to f at (a,b,c):

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

6/ Approximation:

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y.$$

$\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b)$: increment of f.

7/ Differentiable:

If f_x, f_y exist near (a,b) and continuous at (a,b) then f differentiable at (a,b).

Total differential:

$$dz = f_x(x,y)dx + f_y(x,y)dy.$$

8/ Chain rule:

$$f(x(t),y(t)) \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$f(x(s,t),y(s,t)) \Rightarrow \begin{cases} \dfrac{df}{ds} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s} \\ \dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}. \end{cases}$$

9/ Implicit function: $F(x,y) = 0$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(a,b)} = -\frac{F_x(a,b)}{F_y(a,b)} \quad (F_y(a,b) \neq 0)$$

$$z = f(x,y) \Leftrightarrow F(x,y,z) = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}, \quad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

10/ Directional Derivatives:

Direction of f at (x_0,y_0) in the direction of $u = \langle a,b\rangle$:

$$D_u f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

$$D_u f(x,y) = \nabla f(x,y)\, u = f_x \cdot a + f_y \cdot b.$$

Unit vector: $u = \langle \cos u, \sin u\rangle = \dfrac{v}{\|v\|}$.

11/ Gradient:

Gradient vector: $\nabla f = \langle f_x, f_y\rangle$.

Intepretation of Gradient:

Assume $\nabla f(x_0,y_0) \neq 0$. Let u be unit vector and $(\nabla f(x_0,y_0), u) = \theta$. Then

$$D_u f(x_0,y_0) = \|\nabla f(x_0,y_0)\| \cos\theta \quad (= \|\nabla f\|)$$

i) $\nabla f(x_0,y_0)$ $\nearrow$ direction of max rate increase.

ii) $-\nabla f(x_0,y_0)$ $\searrow$ direction of max rate decrease.

iii) $\nabla f(x_0,y_0)$ normal to level curve of f at (x_0,y_0).

Tangent plane at $P(a,b,c)$ on $F(x,y,z) = k$:

$$F_x(P)(x-a) + F_y(P)(y-b) + F_z(P)(z-c) = 0.$$

Normal line of $F(x,y,z) = k$ at P:

$$x = x_0 + F_x(P)t, \quad y = y_0 + F_y(P)t, \quad z = z_0 + F_z(P)t.$$

12/ Local max/min: Consider $f(x,y)$.

$f_x(a,b) = f_y(a,b) = 0 \Rightarrow (a,b)$: critical point

$$D = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2.$$

i) $D>0$, $f_{xx}(a,b)>0 \to$ local min.

ii) $D>0$, $f_{xx}(a,b)<0 \to$ local max.

iii) $D<0 \to$ saddle point.

iv) $D=0 \to$ no conclusion (consider nearby points)

13/ Extreme Value Theorem:

Finding max/min of f on D:

i) Find inside interior with Derivative Test.

ii) Find on boundary of D.

iii) Compare and conclude.

14/ Lagrange Multiplier:

Finding max/min of $f(x,y)$ with constraint $g(x,y) = 0$:

$$F(x,y,\lambda) = f(x,y) - \lambda g(x,y).$$

$$\Rightarrow F_x = F_y = F_\lambda = 0.$$

* Chap 2:

1/ Riemann Sum: $S \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_i)\Delta A$.

2/ Double Integral:

- Type 1: $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$.

  Integrate $y \to$ integrate x.

- Type 2: $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$.

  Integrate $x \to$ integrate y.

- Interchanging Limits of Integration:

  Không nguyên hàm theo x được

  $\to$ nguyên hàm theo y trước.

- Average Value:

$$\bar{f} = \frac{1}{A}\iint_D f(x,y)\,dx\,dy \quad (A: \text{area of region } D)$$

3/ Polar Coordinates:

$R = \{(r,\theta): a \le r \le b;\ \alpha \le \theta \le \beta\}$.

$$\iint_R f\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2+y^2}$$

Đạo hàm: $\sec x = \dfrac{1}{\cos x}$, $\csc = \dfrac{1}{\sin}$

$(uv)' = u'v + uv'$ $\quad \left(\dfrac{u}{v}\right)' = \dfrac{u'v - v'u}{v^2}$

$(u^a)' = a.u^{a-1}.u'$ $\quad (\sqrt[n]{u})' = \dfrac{u'}{n\sqrt[n]{u^{n-1}}}$

$(\sin u)' = u'\cos u$ $\quad (\cos u)' = -u'\sin u$

$(\tan u)' = \dfrac{u'}{\cos^2 u}$ $\quad (\cot u)' = \dfrac{-u'}{\sin^2 u}$

$(a^u)' = u'.a^u \ln a$ $\quad (\log_a u)' = \dfrac{u'}{u \ln a}$.

Tích phân:

$$\int \tan x\,dx = -\ln|\cos x| + C.$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

Partial Fractions: $\dfrac{\cdots}{(ax+b)^n} \to \sum \dfrac{A_i}{(ax+b)^i}$

Half-angle formula:

$$1 - \cos 2x = 2\sin^2 x, \quad 1 + \cos 2x = 2\cos^2 x$$

Integration by Parts. $\int u\,dv = uv - \int v\,du$.

Ex. $A = \int x^2 \cos(mx)\,dx$.

Let $\begin{cases} u = x^2 \to u' = 2x \\ v' = \cos mx \to v = \dfrac{\sin mx}{m} \end{cases}$

$$\Rightarrow A = \frac{x^2 \sin mx}{m} - \int \frac{2x \sin mx}{m}\,dx$$

4/ Change of Variable:

$D = \{(x,y)\} \to D = \{x(u,v), y(u,v)\}$.

$$\iint_D f(x,y)\,dx\,dy = \iint_D f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \mathrm{Jac}(D) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

5/ Cylindrical Coordinates:

$D = \{(r,\theta,z): a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$

$$\iiint_D f\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta, z)\, r\,dz\,d\theta\,dr$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2+y^2}.$$

6/ Spherical Coordinates:

$D = \{(\rho,\theta,\phi): a \le \rho \le b, \alpha \le \theta \le \beta, \gamma_1 \le \phi \le \gamma_2\}$

$$\iiint_D f\,dV = \int_{\gamma_1}^{\gamma_2}\int_\alpha^\beta\int_a^b f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$,

$\rho^2 = x^2+y^2+z^2 = r^2+z^2$, $r = \sqrt{x^2+y^2} = \rho\sin\phi$

7/ Application in Economics / Engineering:

* Mass – center of mass:

Given density $\rho(x,y)$ in $\mathbb{R}^2$.

Mass: $m = \iint_D \rho(x,y)\,dA$.

Center of mass: $(\bar{x}, \bar{y})$ where

$\bar{x} = \frac{1}{m}\iint_D x\rho\,dA$, $\bar{y} = \frac{1}{m}\iint_D y\rho\,dA$.

* Surface area of graph:

Given $f(x,y)$ defined on D,

$$S = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA.$$

* Chap 3:

1/ Vector fields:

– Vector function:

$r(t) = \langle f(t), g(t), h(t)\rangle$

$= f(t)\,i + g(t)\,j + h(t)\,k.$

– Vector fields:

$F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

$= P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k.$

– Gravitational field:

Given point mass m at P_0, position vector r_0.

$F(x,y,z) = F(r) = \dfrac{-km(r - r_0)}{|r - r_0|^3}$ (k: constant, $k > 0$)

F points toward P_0, magnitude: $|F| = \dfrac{km}{|r - r_0|^2}$

– Gradient fields:

Given $f(x,y)$ a scalar fuction of 2 variables,

$\nabla f = \langle f_x, f_y\rangle$ is a gradient vector field.

– Conservative vector fields: (V: potential function)

F is conservative $\Leftrightarrow \exists V$: $F = \nabla V$.

2/ Line Integrals:

Given f on $C = \{(x(t), y(t)) : a \le t \le b\}$.

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

– $\int_C P\,dx + Q\,dy = \int_C P\,dx + \int_C Q\,dy$.

– Line Integral of Vector Fields:

Given F a vector field on $C = r(t)$ $(a \le t \le b)$

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\,dt.$$

– Fundamental Theorem for Line Integrals:

Given f on $C = r(t)$ $(a \le t \le b)$.

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

3/ Green Theorem:

– Conservative vector field:

Given $F = \langle P, Q\rangle$ on D.

F is conservative $\Leftrightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on D.

– Green Theorem:

Let C be a closed curve bound the region D.

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

4/ Curl – Divergence:

– Curl:

Given $F = \langle P, Q, R\rangle$ on $\mathbb{R}^3$.

$$\text{curl } F = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle$$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$\left(\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle\right)$

– Theorem: $\text{curl}(\nabla f) = 0$.

$\text{curl } F = 0 \Leftrightarrow F$ is conservative

– Divergence: given $F = \langle P, Q, R\rangle$ on $\mathbb{R}^3$.

$$\text{div } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot f$$

– Theorem: $\text{div curl } F = 0$.

(F: vector field)