ANALYSIS 2

1. Integrals
(Chapter 5)

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Section 1

Indefinite Integrals

Antiderivatives

Antiderivatives

Let f and F be functions defined on (a, b), such that $F'(x) = f(x)$, for all $x \in (a, b)$. Then F is called an *antiderivative* of f on (a, b).

Remarks: If $F(x)$ is an antiderivative of $f(x)$ then $F(x) + C$ is also an antiderivative of $f(x)$. These are all antiderivatives of f.

Example

- The antiderivative of $f(x) = x^2$ is $x^3/3 + C$.
- A particle moves on a straight line with its acceleration given by $a(t) = 6t + 4$. If the initial velocity is $v(0) = 6$ cm/s and the initial position is $s(0) = 9$ cm. Find its position function.

Lemma

Let F be a differentiable function on (a, b) such that $F'(x) = 0$ for all $x \in (a, b)$. Then $F(x)$ is constant on (a, b).

Proof. Take any two points $x_1 \neq x_2$ in (a, b). Then by the Mean Value Theorem for derivatives, there exists $c \in (a, b)$ such that

$$F'(c) = \frac{F(x_2) - F(x_1)}{x_2 - x_1}.$$

Since by hypothesis, $F'(c) = 0$, we obtain $F(x_1) = F(x_2)$.

Theorem

Suppose F and G are two antiderivatives of f on (a, b). Then there is constant C so that $G(x) = F(x) + C$ on (a, b).

Proof. Let $H(x) = G(x) - F(x)$, then
$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0$ on (a, b). By the previous lemma $H(x) = C$, that is $G(x) = F(x) + C$ on (a, b).

Indefinite Integrals

Definition

We call the collection of all antiderivatives of f the *indefinite integral* of f, denoted by

$$\int f(x)dx.$$

If F is an antiderivative of f then

$$\int f(x)dx = F(x) + C.$$

Indefinite Integrals

- Powers: For $a \neq -1$

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C$$

$a = -1$:

$$\int \frac{1}{x} dx = \ln|x| + C$$

- Exponentials: Suppose $a \neq 1$, then

$$\int a^x dx = \frac{a^x}{\ln a} + C.$$

In particular,

$$\int e^x dx = e^x + C.$$

Indefinite Integrals of Trigonometric Functions

$$\int \sin x dx = -\cos x + C, \qquad \int \cos x dx = \sin x + C,$$

$$\int \tan x dx = -\ln|\cos x| + C, \qquad \int \cot x dx = \ln|\sin x| + C,$$

$$\int \sec^2 x dx = \tan x + C, \qquad \int \csc^2 x dx = -\cot x + C,$$

$$\int \sec x \tan x dx = \sec x + C, \qquad \int \csc x \cot x dx = -\csc x + C,$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C, \qquad \int \frac{dx}{1+x^2} = \arctan x + C$$

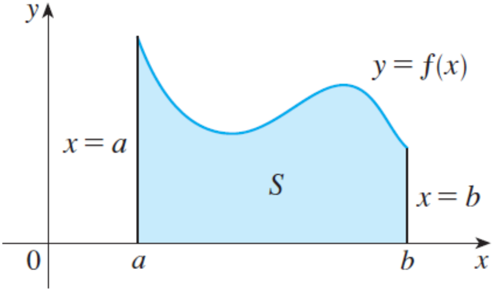
$$\int \csc x dx = -\ln|\csc x + \cot x| + C, \qquad \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

Section 2

Definite Integrals

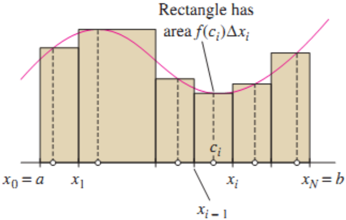
Definite Integrals

- Motivation: Find area of the region S lying under the graph of a nonnegative continuous function $f(x)$, above the x-axis and between the vertical lines $x = a$ and $x = b$.



- Main idea: approximate S by unions of rectangles whose areas are easy to calculate.

Partitions



- We use $n + 1$ points

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$

  to divide $[a, b]$ into n closed subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ with width $\Delta x_i = x_i - x_{i-1}$.
- The set $P = \{x_0, x_1, ..., x_{n-1}, x_n\}$ is called a partition of $[a, b]$.
- Most commonly, we use **regular partitions** with $\Delta x_i = (b - a)/n$, for all $i = 1, .., n$.

Definite Integrals - Definition

- The norm of partition P is

$$\|P\| = \max_{1 \le i \le n} \Delta x_i$$

- Partition Q is a refinement of partition P if $P \subset Q$.

Riemann sums

- For each $i = 1, \ldots, n$ choose $c_i \in [x_{i-1}, x_i]$. Let $c = \{c_1, \ldots, c_n\}$.
- The Riemann sum of f with respect to P and c is

$$R(f, P, c) = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

- For any choice of c,

$$L(f, P) \leq R(f, P, c) \leq U(f, P),$$

where

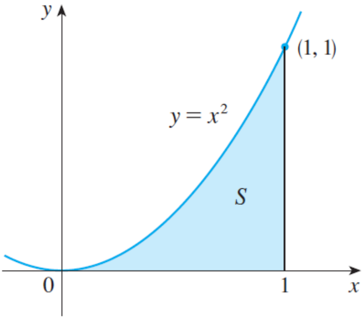
$$U(f, P) = \sum_{i=1}^{n} \Delta x_i \sup_{[x_{i-1}, x_i]} f(x),$$

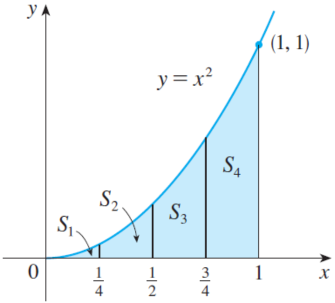
$$L(f, P) = \sum_{i=1}^{n} \Delta x_i \inf_{[x_{i-1}, x_i]} f(x).$$

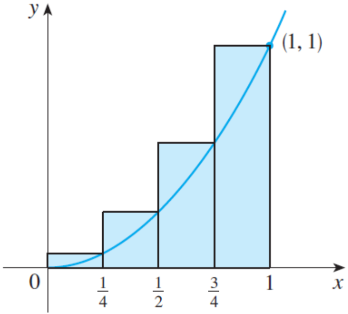
Definite Integrals

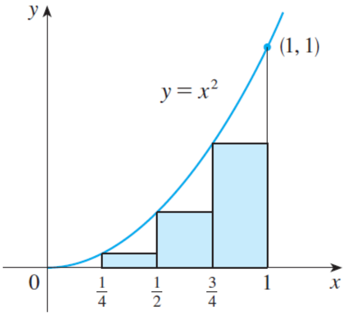
Example

Find the area under $y = x^2$ from 0 to 1



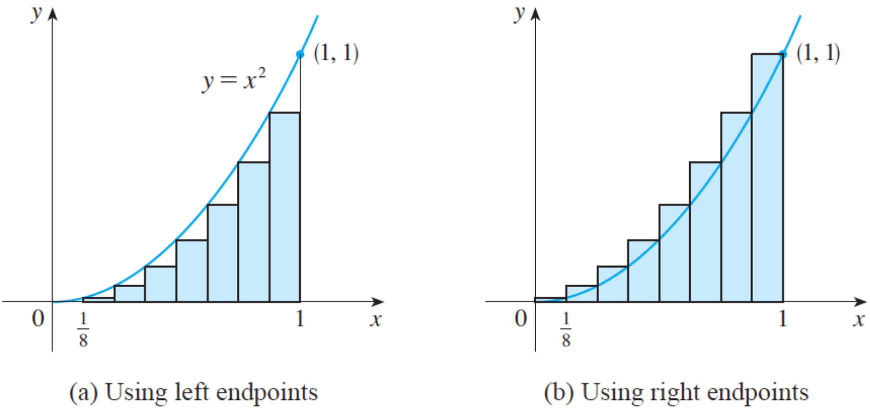






- Width of rectangles $= \frac{1}{4}$
- Heights $= \left(\frac{1}{4}\right)^2, \left(\frac{2}{4}\right)^2, \left(\frac{3}{4}\right)^2, \left(\frac{4}{4}\right)^2$
- Using the right-end points:
  $R_4 = \frac{1}{4}\left[\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{4}\right)^2\right] = \frac{15}{32} = 0.46875$
- Using the left-end points:
  $L_4 = \frac{1}{4}\left[\left(\frac{0}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right] = \frac{7}{32} = 0.21875$

Approximation with 8 rectangles



leads to better estimate

$$L_8 = 0.2734375, \quad R_8 = 0.3984375$$

- In general,

$$R_n = \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right]$$
$$= \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2) = \frac{(n+1)(2n+1)}{6n^2}.$$

$$L_n = \frac{1}{n}\left[\left(\frac{0}{n}\right)^2 + \left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right]$$
$$= \frac{1}{n^3}(1^2 + 2^2 + \cdots + (n-1)^2) = \frac{(n-1)(2n-1)}{6n^2}.$$

- Since R_n and L_n approach $\frac{1}{3}$ as n increase,

$$A = \frac{1}{3}.$$

Here are the approximations corresponding to various n, the number of small intervals.

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

Note that R_{1000} and L_{1000} agree to 2 decimal places and their average $\frac{L_{1000}+R_{1000}}{2} = 0.3333335$ is correct to 6 decimal places.
Remark: Since x^2 is an increasing function,

$$L_n < A < R_n.$$

Properties of Upper/Lower Riemann Sums

Lemma

If Q is a refinement of P then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Corollary

For any partitions P and Q,

$$L(f,P) \leq U(f,Q).$$

Theorem

There exists *at least one* real number I so that

$$L(f,P) \leq I \leq U(f,P)$$

for any P.

Definite Integrals

Definition

If there is *only one* number I so that

$$L(f, P) \leq I \leq U(f, P)$$

for any partition P, then we say f is *integrable* on $[a, b]$.
I is called the *definite integral* of f on $[a, b]$,

$$I = \int_a^b f(x)dx.$$

Theorem

If f is integrable on $[a, b]$ then

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} R(f, P, c)$$

Terminology:

- $\int$: integral sign
- $f(x)$: integrand
- a: lower limit, b: upper limit.

Remarks:

- $\int_a^b f(x)dx$ is a number
- It doesn't depend on x. In fact, we can use any letter in place of x

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

Signed area

- If f positive on $[a, b]$ then I is the area under the graph of f.
- If f negative on $[a, b]$ then I is minus of the area.
- In general, I is called the *signed* area of the region bounded by the graph of f.

Example

Consider

$$\int_0^3 (x^3 - 6x)dx$$

- Find the Riemann sum $R(f, P, c)$ where P is the regular partition and c are the right end points, i.e. $c_i = x_i$.
- Calculate the limit to find the integral

Solution:

- Width $\Delta x = 3/n$
- End points $x_0 = 0/n, x_1 = 3/n, x_2 = 6/n, \ldots,$ so the right end points are:

$$c_i = 3i/n, \ \ i = 0, \ldots, n$$

Thus,

$$
\begin{aligned}
R_n &= \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n} \right)^3 - 6 \left(\frac{3i}{n} \right) \right] \\
&= \frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i \\
&= \frac{81}{n^4} \left(\frac{n(n+1)}{2} \right)^2 - \frac{54}{n^2} \frac{n(n+1)}{2} \\
&= \frac{81(n+1)^2}{4n^2} - \frac{27(n+1)}{n}.
\end{aligned}
$$

Since

$$\lim_{n\to\infty} R_n = \frac{81}{4} - 27 = -\frac{27}{4},$$

we obtain

$$\int_0^3 (x^3 - 6x)dx = -\frac{27}{4}.$$

Example

Use the midpoints, i.e. $c_i = \frac{x_{i-1}+x_i}{2}$, with $n = 5$ to approximate

$$\int_1^2 \frac{1}{x} dx.$$

Solution:

- End points: 1, 1.2, 1.4, 1.6, 1.8, 2
- Mid points: 1.1, 1.3, 1.5, 1.7, 1.9
- Width $\Delta x = 1/5$

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\,[\,f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx 0.691908$$

Section 3

Properties of Definite Integrals

Properties of Definite Integrals

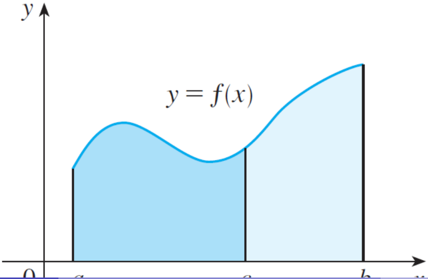
- Reversing limits changes the sign of the integral:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

- Additivity:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

for any a, b, c.



Properties of Definite Integrals

- $\int_a^b c dx = c(b-a)$, where c is a constant
- Linearity:

$$\int_a^b [\alpha f(x) + \beta g(x)]\, c dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx,$$

  where α, β are constants.
- Comparison:
  If $f(x) \geq 0$ for $a \leq x \leq b$ then

$$\int_a^b f(x) dx \geq 0.$$

Properties of Definite Integrals

- If $f(x) \geq g(x)$ for $a \leq x \leq b$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

- If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

- If $a \leq b$ then

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

Even and odd functions

- f is odd, i.e. $f(x) = -f(-x)$ for all x, then

$$\int_{-a}^{a} f(x)dx = 0$$

- f is even, i.e. $f(x) = f(-x)$ for all x, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Section 4

Mean Value Theorem for Integrals

Mean Value Theorem

Definition

If f is integrable on $[a, b]$ then the *average value* or *mean value* of f on $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$

The MVT states that a *continuous* function attains its mean value on the interval.

Theorem

If f is continuous on $[a, b]$ there exists $c \in [a, b]$ so that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Mean Value Theorem

Proof:

- Since f is continuous on $[a, b]$, it attains max and min on $[a, b]$, i.e. there exist $p, q \in [a, b]$ such that

$$f(p) = m = \min_{x \in [a,b]} f(x),$$

$$f(q) = M = \max_{x \in [a,b]} f(x).$$

- We have

$$m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M.$$

- By the Intermediate Value Theorem, the continuous function f takes every values between $f(p) = m$ and $f(q) = M$.
- In particular, there exists $c \in [a, b]$ so that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Section 5

Fundamental Theorem of Calculus

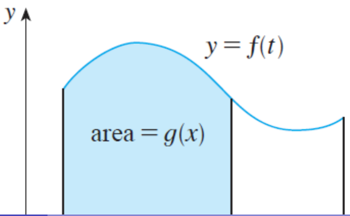
Fundamental Theorem of Calculus

- FTC gives a quick way to compute definite integrals, without using Riemann sums.
- It shows that Integration and Differentiation are inverse operations, they 'cancel' each other.

Area function

Given $f(x)$ integrable on $[a, b]$, the *area function* of f on $[a, b]$ is

$$G(x) = \int_a^x f(t)dt.$$



Fundamental Theorem of Calculus

Part 1

If f is continuous then the area function $G(x)$ is an antiderivative of $f(x)$

$$G'(x) = f(x).$$

In other words,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Part 2

If f is continuous on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is *any* antiderivative of f.

Fundamental Theorem of Calculus

Example

Find derivative of Fresnel function

$$S(x) = \int_0^x \sin(\pi t^2/2)dt.$$

Example

Find the area under graph $y = x^2$ from 0 to 1.

Solution: $F(x) = x^3/3$ is an antiderivative of x^2. By FTC part 2

$$A = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1$$
$$= \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

Proof of FTC part 1

Fixed $\epsilon > 0$. Since f is continuous at c, there exists $\delta > 0$ so that

$$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Thus, if $|x - c| < \delta$

$$\begin{aligned}\left|\frac{G(x) - G(c)}{x - c} - f(c)\right| &= \frac{1}{|x - c|}\left|\int_c^x (f(t) - f(c))dt\right| \\ &\leq \frac{1}{|x - c|}|x - c| \max_{t\in[c,x]} |f(t) - f(c)| \\ &\leq \frac{1}{|x - c|}|x - c|\epsilon dt = \epsilon.\end{aligned}$$

This implies that

$$\lim_{x\to c} \frac{G(x) - G(c)}{x - c} = f(c).$$

It follows that $G'(c) = f(c)$.

Proof of FTC part 2

Let $G(x) = \int_a^x f(t)dt$ be the area function. By FTC part 1, G is an antiderivative of f. We know that two antiderivatives of f differ by a constant, hence

$$F(x) = G(x) + C.$$

Therefore,

$$F(b) - F(a) = G(b) - G(a) = \int_a^b f(x)dx.$$

Examples

1. We have

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 (2 + t^{1/2} - t^{-2})\,dt$$
$$= 2t + \frac{t^{3/2}}{3/2} - \frac{t^{-1}}{-1}\Big|_1^9 = 32\frac{4}{9}.$$

2. What is wrong with this calculation?

$$\int_{-1}^3 \frac{1}{x^2}\,dx = -x^{-1}\Big|_{-1}^3 = -\frac{4}{3}.$$

- The FTC can only be applied to continuous functions
- $1/x^2$ has infinite discontinuity at 0, so the FTC is not applicable.

Generalization of FTC part 1

Theorem

If u and v are differentiable and

$$F(x) = \int_{u(x)}^{v(x)} f(t)dt,$$

then

$$F'(x) = f(v(x))v'(x) - f(u(x))u'(x).$$

Note: The FTC part 1 is a special case where $u(x) = a$ and $v(x) = x$.

Example

Example

Find derivative of

$$F(x) = \int_3^{x^2} \ln(t^3 + 4)dt$$

and

$$G(x) = \int_{2x}^{5} \tan(\cos(u) - 4)du$$

Solution:

$$F'(x) = 2x \ln(x^6 + 4)$$

and

$$G'(x) = -2\tan(\cos(2x) - 4)$$

Section 6

Substitution (Change of Variables)

Substitution

Theorem

If u is differentiable and f is continuous then

$$\int f(u(x))u'(x)dx = \int f(u)du.$$

Proof: We need to show if $F(u)$ is an antiderivative of $f(u)$ then

$$\int f(u(x))u'(x)dx = F(u(x)) + C.$$

By the Chain Rule

$$\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x).$$

Hence,

$$\int f(u(x))u'(x)dx = F(u(x)) + C.$$

Example

$$\int 2x\sqrt{x^2+1}dx.$$

- Apply the theorem to $f(u) = \sqrt{u}$ and $u(x) = x^2 + 1$.
- We have $f(u(x))u'(x) = 2x\sqrt{x^2+1}$.
- So
$$\int 2x\sqrt{x^2+1}dx = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + C.$$
- Substitute u by $x^2 + 1$ we get
$$\int 2x\sqrt{x^2+1}dx = \frac{2}{3}(x^2+1)^{3/2} + C.$$

Example

Find $\int x^3 \cos(x^4 + 2)dx$

- $u = x^4 + 2$,
- $du = 4x^3 dx \implies dx = \frac{1}{4x^3} du$
- Hence,

$$\begin{aligned}
\int x^3 \cos(x^4 + 2)dx &= \int x^3 \cos(u) \frac{1}{4x^3} du \\
&= \frac{1}{4} \int \cos(u) du \quad (\text{ No } x \text{ any more}) \\
&= \frac{1}{4} \sin u + C \\
&= \frac{1}{4} \sin(x^4 + 2) + C
\end{aligned}$$

The steps

- Find an u that can simplify the formula of the integrand.
- Write dx in terms of du and possibly x.
- Get rid of the remaining appearances of x.

Example

Compute $\int \sqrt{2x+1}dx$

- Find u: $u = 2x + 1$
- $du = 2dx \to dx = \frac{1}{2}du$
- $$\begin{aligned} \int \sqrt{2x+1}dx &= \int \sqrt{u} \cdot \frac{1}{2}du \\ &= \frac{1}{3}u^{3/2} + C \\ &= \frac{1}{3}(2x+1)^{3/2} + C \end{aligned}$$

Example

Compute $\int \sqrt{x^2+1}x^5 dx$

- $u = x^2 + 1$
- $du = 2xdx \to dx = \frac{1}{2x}du$
- Thus,

$$\int \sqrt{x^2+1}x^5 dx = \int \sqrt{u}x^5 \frac{1}{2x} du$$
$$= \frac{1}{2}\int x^4\sqrt{u}du$$

- Write x^4 in terms of u

$$x^4 = (x^2+1-1)^2 = (u-1)^2$$

Thus,

$$\begin{aligned}
\tfrac{1}{2}\int \sqrt{u}(u-1)^2 du &= \tfrac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2}) du \\
&= \tfrac{1}{7}u^{7/2} - \tfrac{2}{5}u^{5/2} + \tfrac{1}{3}u^{3/2} + C \\
&= \tfrac{1}{7}(x^2+1)^{7/2} - \tfrac{2}{5}(x^2+1)^{5/2} \\
&\quad + \tfrac{1}{3}(x^2+1)^{3/2} + C
\end{aligned}$$

Substitution for Definite Integrals

Theorem

If u' is continuous on $[a, b]$ and f is continuous, then

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

Note: The limits change from (a, b) to $(u(a), u(b))$

Example

Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$

- $u = 3 - 5x$ and $du = -5dx \to dx = -\frac{1}{5}du$
- $x = 1 \to u = -2$, $x = 2 \to u = -7$
- We obtain

$$\int_1^2 \frac{dx}{(3-5x)^2} = -\tfrac{1}{5}\int_{-2}^{-7} \frac{du}{u^2} = -\frac{1}{5u}\Big|_{-2}^{-7} = \frac{1}{14}$$

Section 7

Integration by Parts

Integration by Parts

Theorem

If u and v are differentiable then

$$\int u dv = uv - \int v du.$$

- Recall that $du = u'(x)dx$ and $dv = v'(x)dx$.
- Example: if $u(x) = x^2$, $v(x) = \sin x$ then we get

$$\int x^2 \cos x dx = x^2 \sin x - \int 2x \sin x dx.$$

Integration by Parts - Proof

By the product rule,

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x).$$

Taking integration gives

$$u(x)v(x) + C = \int [u'(x)v(x) + u(x)v'(x)]dx.$$

Rearranging terms, we get

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx.$$

Example

$$\int x \sin x dx$$

- $u = x, \; dv = \sin x dx$
- $du = dx, \; v = -\cos x$
- $$\begin{aligned} \int x \sin x dx &= x(-\cos x) - \int (-\cos x) dx \\ &= -x \cos x + \int \cos x dx \\ &= -x \cos x + \sin x + C \end{aligned}$$

Example

Find $\int \ln x dx$

- $u = \ln x,\ dv = dx$
- $du = \frac{1}{x} dx,\ v = x$
-
$$\begin{aligned}\int \ln x dx &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - x + C\end{aligned}$$

Example

Evaluate $I = \int e^x \sin x dx$

- $u = \sin x,\ dv = e^x dx$
- $du = \cos x dx,\ v = e^x$

$$I = e^x \sin x - \int e^x \cos x dx$$

Integral by parts once more

- $u = \cos x,\ dv = e^x dx$
- $du = -\sin x dx,\ v = e^x$

$$
\begin{aligned}
I &= e^x \sin x - [e^x \cos x + \int e^x \sin x dx] \\
&= e^x(\sin x - \cos x) - I
\end{aligned}
$$

So

$$2I = e^x(\sin x - \cos x) + C$$

or

$$I = \tfrac{1}{2} e^x(\sin x - \cos x) + C$$

Example

Calculate $I_n = \int x^n e^{-x} dx$

- $u = x^n, \; dv = e^{-x} dx$
- $du = nx^{n-1} dx, \; v = -e^{-x}$

Integrating by parts, we get

$$\begin{aligned} I_n &= -x^n e^{-x} + n \int x^{n-1} e^{-x} dx \\ &= -x^n e^{-x} + nI_{n-1}. \end{aligned}$$

We can use this repeatedly to calculate I_n for any n. Note that

$$I_0 = -e^{-x} + C.$$

Change of Variable for Definite Integrals

If u and v are differentiable then

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)dx.$$

- Example: if $u(x) = x^2$, $v(x) = \sin x$ then

$$\int_0^\pi x^2 \cos x dx = x^2 \sin x\Big]_0^\pi - \int_0^\pi 2x \sin x dx.$$

Example

Calculate $I = \int_0^1 \arctan x dx$

- Let $u = \arctan x$, then $dv = dx$ hence $v = x$. As $du = \frac{dx}{1+x^2}$, we get

$$I = x \arctan x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$
$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx$$

- Let $t = 1 + x^2$ then $dt = 2xdx$. Thus,

$$\int_0^1 \frac{x}{1+x^2} dx = \tfrac{1}{2} \int_1^2 \frac{dt}{t} = \tfrac{1}{2} \ln |t| \Big]_1^2 = \tfrac{\ln 2}{2}$$

.

- We obtain

$$I = \frac{\pi}{4} - \tfrac{\ln 2}{2}$$

Section 8

Partial Fractions

Partial Fractions

Consider an integral of the form

$$\int \frac{P(x)}{Q(x)} dx,$$

where $P(x), Q(x)$ are polynomials.

- Long division reduces it to the case degree of $P <$ degree of Q.
- Then one can write $\frac{P(x)}{Q(x)}$ as sum of simpler terms, call partial fractions.
- The exact form of partial fractions depend on $Q(x)$.

Example

Evaluate

$$\int \frac{x^3 + 3x^2}{x^2 + 1} dx$$

Divide

$$\frac{x^3 + 3x^2}{x^2 + 1} = x + 3 - \frac{x + 3}{x^2 + 1}$$

$$\begin{aligned}\int \frac{x^3 + 3x^2}{x^2 + 1} dx &= \int (x + 3) dx - \int \frac{x + 3}{x^2 + 1} dx \\ &= \frac{1}{2}x^2 + 3x - \frac{1}{2}\ln(x^2 + 1) \\ &- 3\arctan x + C\end{aligned}$$

Factorization of $Q(x)$

Any polynomial $Q(x)$ can be factored as product of

- linear factors of the form $(ax + b)$
- irreducible quadratic factors

$$(ax^2 + bx + c) \text{ where } b^2 - 4ac < 0$$

I.e.

$$Q(x) = a \prod_{i=1}^{k} (x + b_i)^{n_i} \prod_{j=1}^{l} (x^2 + c_j x + d_j)^{m_j}.$$

Here the b_i's are distinct, i.e. $b_i \neq b_j$ for $i \neq j$. Similarly, the quadratic factors $x^2 + c_j x + d_j$'s are distinct.

Linear factors of Q

If $Q(x)$ has a linear factor $(ax+b)^n$ then the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ contains the terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

Here $A_1, \dots, A_n$ are real numbers.

Example: The partial fraction decomposition of $\frac{x^4+1}{(x-1)^3(x+1)^2}$ has the form

$$\frac{x^4+1}{(x-1)^3(x+1)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2}.$$

Example

Write as sum of partial fractions

$$\frac{x+5}{x^2+x-2}$$

Solution: Factorization of Q: $x^2+x-2=(x-1)(x+2)$ Form of partial fraction decomposition

$$\frac{x+5}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}.$$

Multiplying both sides with Q to get

$$x+5=A(x+2)+B(x-1)$$

Put in suitable values of x;

- $x=1 \implies 6=3A$ so $A=2$
- $x=-2 \implies 3=-3B$ so $B=-1$

Example

Find

$$I = \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$$

Solution:

- Long division gives

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

- Factorzing Q: $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$
- Form of partial fraction decomposition

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

- Multipling with Q to get

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

for all x.

- Substituting special values of x:
  - $x = 1 \implies B = 2$
  - $x = -1 \implies C = -1$
  - $x = 0 \implies A = 1$
- We obtain

$$\begin{aligned} I &= \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx \\ &= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C. \end{aligned}$$

Irreducible quadratic factors of Q

If $Q(x)$ has an irreducible quadratic factor

$$ax^2 + bx + c \quad \text{where} \quad b^2 - 4ac < 0$$

then the partial fraction decomposition contains the term

$$\frac{Ax + B}{ax^2 + bx + c}$$

Example: The partial fraction decomposition of $\frac{x^3+1}{(x-1)^2(4x^2+4x+3)}$ has the form

$$\frac{x^3 + 1}{(x-1)^2(4x^2+4x+3)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B_1 x + B_2}{(4x^2+4x+3)}.$$

Example

Find

$$I = \int \frac{3x+2}{x^3+2x^2+2x}dx$$

Solution:

- Factorizing $Q(x) = x(x^2+2x+2)$
- Form of partial fraction decomposition:

$$\frac{3x+2}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}$$

- Multiplying with Q to get

$$3x+2 = A(x^2+2x+2) + (Bx+C)x.$$

- Choose $x = 0 \implies A = 1$.
- Comparing coefficients of $x^2 \implies B = -1$.
- Comparing coefficients of $x \implies C = 1$.

We get

$$
\begin{aligned}
I &= \int \left[\frac{1}{x} + \frac{-x+1}{x^2+2x+2}\right] dx \\
&= \ln|x| + \int \frac{-(x+1)+2}{(x+1)^2+1} dx \\
&= \ln|x| - \tfrac{1}{2}\ln(x^2+2x+2) + 2\arctan(x+1) + C
\end{aligned}
$$

The last equality is easy to check using the substitution $u = x + 1$.

Section 9

Trigonometric Integrals

Trigonometric Integrals

We will consider the following two types of integrals

$$\int \sin^n x \cos^m x dx \text{ and } \int \tan^n x \sec^m x dx.$$

The following will be used frequently

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1 \\ \cos(2x) &= 2\cos^2 x - 1 = 1 - 2\sin^2 x \\ \sin(2x) &= 2 \sin x \cos x.\end{aligned}$$

$$\begin{aligned}\sec^2 x &= \tan^2 x + 1 \\ \sec' x &= \sec x \tan x \\ \tan' x &= \sec^2 x = \tan^2 x + 1\end{aligned}$$

$\int \sin^n x \cos^m x dx$

- If $m = 2k + 1$ is odd: Let $u = \sin x$, then

$$\int \sin^n x \cos^{2k+1} x dx = \int u^n (1 - u^2)^k du.$$

- If $n = 2k + 1$ is odd: Let $u = \cos x$, then

$$\int \sin^{2k+1} x \cos^m x dx = -\int (1 - u^2)^k u^m du.$$

- If both $m = 2k$ and $n = 2l$ are even: use half-angle formula to reduce the powers

$$\int \sin^{2k} x \cos^{2l} x dx = \frac{1}{2^{k+l}} \int (1 - \cos(2x))^k (1 + \cos(2x))^l dx.$$

Examples

- Find $I = \int \sin^2 x \cos^3 x dx$.
  Solution: Let $u = \sin x$. Then
  $$\begin{aligned} I &= \int u^2(1-u^2)du = \tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 + C \\ &= \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C. \end{aligned}$$
- Find $I = \int \sin^2 x \cos^2 x dx$.
  Solution:
  $$\begin{aligned} I &= \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)dx \\ &= \frac{1}{4}\int (1-\cos^2(2x))dx \\ &= \frac{1}{4}\int (1-\frac{1}{2}(1+\cos(4x)))dx \\ &= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C. \end{aligned}$$

$\int \tan^n x \sec^m x dx$

1. If $n = 2k + 1$ is odd: Let $u = \sec x$, then

$$\int \tan^n x \sec^m x dx = \int \tan^{2k} x \sec^{m-1} x \sec x \tan x dx$$
$$= \int (u^2 - 1)^k u^{m-1} du.$$

2. If $m = 2k$ is even: Let $u = \tan x$, then

$$\int \tan^n x \sec^m x dx = \int \tan^n x \sec^{2k-2} x \sec^2 x dx$$
$$= \int u^n (u^2 + 1)^{k-1} du.$$

$\int \tan^n x \sec^m x dx$

3. If $m = 2k + 1$ is odd and $n = 2l$ is even: Let $u = \sin x$, then

$$\int \tan^n x \sec^m x dx = \int \frac{sin^{2l} x}{\cos^{2k+2l+1} x} = \int \frac{u^{2l}}{(1-u^2)^{k+l+1}} du.$$

Example: Find $I = \int \sec^6 x \tan^4 x dx$

- $I = \int \sec^4 x \tan^4 x \sec^2 x dx$
- Let $u = \tan x$, then $du = \sec^2 x dx$, hence

$$\begin{aligned} I &= \int (\tan^2 x + 1)^2 \tan^4 x \sec^2 x dx \\ &= \int (u^2+1)^2 u^4 du = \frac{u^9}{9} + \frac{2u^7}{7} + \frac{u^5}{5} + C \\ &= \frac{\tan^9 x}{9} + \frac{2\tan^7 x}{7} + \frac{\tan^5 x}{5} + C \end{aligned}$$

Example

Find $I = \int \sec^3 x \tan^5 x dx$

- $I = \int \sec^2 x \tan^4 x \sec x \tan x dx$
- Let $u = \sec x$, then $du = \sec x \tan x dx$, hence

$$\begin{aligned} I &= \int \sec^2 x (\sec^2 x - 1)^2 \sec x \tan x dx \\ &= \int u^2 (u^2 - 1)^2 du \\ &= \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C \\ &= \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C. \end{aligned}$$

Section 10

Trigonometric Substitution

Trigonometric Substitutions

- Used to get rid of the square root $\sqrt{Ax^2 + Bx + C}$ of a quadratic polynomial of x.
- Any quadratic polynomial of x can be put in the form $\pm(ax + b)^2 + c$.
- Thus, by the change of variable $u = ax + b$, we reduce to one of the following three cases
    - $\sqrt{d^2 - u^2}$, let $x = d \sin t$.
    - $\sqrt{d^2 + x^2}$, let $x = d \tan t$.
    - $\sqrt{x^2 - d^2}$, let $x = d \sec t$.

Example

Find $I = \displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

- Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta d\theta$
- Thus,

$$I = \int \frac{2\sec^2\theta d\theta}{4\tan^2\theta 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta$$

- Let $t = \sin\theta$ then

$$I = \frac{1}{4}\int \frac{dt}{t^2} = \frac{1}{4}(-\frac{1}{t}) + C = -\frac{1}{4\sin\theta} + C$$

Since

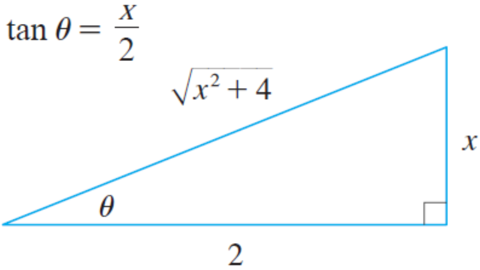
$$\tan\theta = \frac{x}{2}$$

implies

$$\sin\theta = \frac{x}{\sqrt{x^2+4}},$$

we get

$$I = -\frac{\sqrt{x^2+4}}{4x} + C$$



Example

Find $I = \displaystyle\int \frac{xdx}{\sqrt{3-2x-x^2}}$

- Complete the square

$$3 - 2x - x^2 = 4 - (x+1)^2$$

- Let $u = x + 1$ then

$$I = \int \frac{u-1}{\sqrt{4-u^2}} du$$

- Let $u = 2\sin\theta$ then

$$\begin{aligned} I &= \int \frac{2\sin\theta - 1}{2\cos\theta} 2\cos\theta d\theta = -2\cos\theta - \theta + C \\ &= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C. \end{aligned}$$

Here we have used $2\cos\theta = \sqrt{4-u^2}$ and $\theta = \sin^{-1}(u/2)$.

Section 11

Improper Integrals

Improper Integrals of Type 1 (Infinite Intervals)

Integrals over infinite intervals are improper call integrals of Type 1.

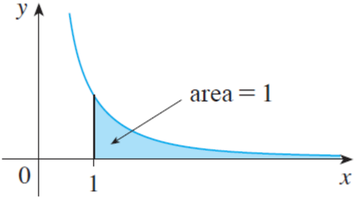
- If f is continuous on $[a, \infty)$, then
$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$
- If f is continuous on $[-\infty, b)$, then
$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$
- If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx.$$

Note: An improper integral is called convergent if it exists as a finite number, and divergent otherwise.

Example

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{y\to\infty}\int_1^y \frac{1}{x^2}dx$$
$$= \lim_{y\to\infty}\left(1-\frac{1}{y}\right) = 1$$



p-integrals of Type 1

For $0 < a < \infty$,

$$\int_a^\infty x^{-p}dx = \begin{cases} \frac{a^{1-p}}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \le 1 \end{cases}$$

Examples

Evaluate

1. $I = \int_0^{\infty} e^{-x/2} dx$.
2. $\displaystyle\int_0^{\infty} \frac{2x-1}{e^{3x}} dx$

Solution:

1. By definition

$$I = \lim_{b\to\infty} \int_0^b e^{-x/2} dx = \lim_{b\to\infty} (2 - 2e^{-b/2}) = 2.$$

2. Using integration by parts, we obtain:

$$\begin{aligned}\int_0^{\infty} \frac{2x-1}{e^{3x}} dx &= \lim_{t\to\infty} \int_0^t (2x-1)e^{-3x} dx \\ &= \lim_{t\to\infty} \left(\frac{1-2t}{3e^{3t}} - \frac{2}{9e^{3t}} - \frac{1}{9}\right)\end{aligned}$$

Therefore, $I = -\dfrac{1}{9}$.

Examples

Evaluate

1. $$\int_0^{+\infty} \frac{2\arctan 2x}{1+4x^2}dx$$
2. $$\int_1^{\infty} \frac{\ln x}{x^3}dx.$$
3. $$\int_0^{\infty} xe^{-x}dx.$$
4. $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$
5. $$\int_1^{\infty} \frac{dx}{\sqrt{x}}.$$
6. $\int_0^{\infty} \cos x dx.$

Improper Integrals of Type 2

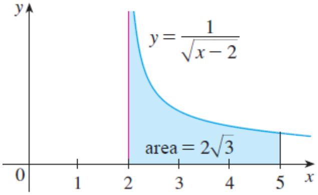
Another type of improper integral arises when the integrand has a vertical asymptote at a point within the interval of integration. These are called improper integrals of Type 2.

- If f is continuous on $(a, b]$ and discontinuous at a, then $\displaystyle\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx.$
- If f is continuous on $[a, b)$ and discontinuous at b, then $\displaystyle\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx.$
- If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Examples

$$\begin{aligned}\int_2^5 \frac{1}{\sqrt{x-2}}dx &= \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}dx\\ &= \lim_{t\to 2^+} 2\sqrt{x-2}\Big]_t^5\\ &= \lim_{t\to 2^+}(2\sqrt{3}-2\sqrt{t-2})\\ &= 2\sqrt{3}.\end{aligned}$$



p-integral of Type 2

For $0 < a < \infty$,

$$\int_0^a x^{-p}dx = \begin{cases} \frac{a^{1-p}}{1-p} & \text{if } p < 1\\ \infty & \text{if } p \geq 1\end{cases}$$

Example

Evaluate

$$I = \int_0^1 \ln x dx.$$

Solution:

$$\begin{aligned}
I &= \lim_{c \to 0^+} \int_c^1 \ln x dx = \lim_{c \to 0^+} (x \ln x - x)\Big|_c^1 \\
&= -1 - \lim_{c \to 0^+} (c \ln c - c) = -1 - \lim_{c \to 0^+} \frac{\ln c}{1/c} \\
&\overset{\text{L'Hôpital}}{=} -1 - \lim_{c \to 0^+} \frac{1/c}{-(1/c^2)} = -1 - 0 = -1
\end{aligned}$$

Example

Example

Evaluate $\int_0^3 \frac{dx}{x-1}$ if it exists.

Solution: The integrand $\frac{1}{x-1}$ has an infinite discontinuity at $x = 1$, so by definition

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

Since

$$\int_0^1 \frac{dx}{x-1} = \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t \to 1^-} \ln|x-1|\Big|_0^t = -\infty.$$

Thus, $\int_0^1 \frac{dx}{x-1}$ diverges and hence $\int_0^3 \frac{dx}{x-1}$ diverges.

Exercises

Evaluate the following integrals

1. $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$.
2. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.
3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$.
4. $\displaystyle\int_0^1 \frac{dx}{x^2}$.
5. $\displaystyle\int_0^1 x(\ln x)^2 dx$.

Comparison Theorems

- Many integrals can not be compute explicitly (e.g. e^{-x^2}, $\frac{\sin x}{x}$, $\cos(x^2)$, etc.)
- To determine the convergence of an integral, most often we compare it to a simpler, explicitly computable integral.

Theorem I

Let f and g be continuous on (a, b), and $0 \leq f(x) \leq g(x)$ for $x \in (a, b)$.

- If $\int_a^b g(x)dx$ converges then $\int_a^b f(x)dx$ also converges, and

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

- If $\int_a^b f(x)dx$ diverges then $\int_a^b g(x)dx$ diverges.

Comparison Theorems

Theorem II

Let f and g be continuous on (a, b), and $|f(x)| \leq g(x)$ for $x \in (a, b)$.

- If $\int_a^b g(x)dx$ converges then $\int_a^b f(x)dx$ also converges, and

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b g(x)dx.$$

- If $\int_a^b f(x)dx$ diverges then $\int_a^b g(x)dx$ diverges.

Comparison Theorems

Example

Show that

$$I = \int_0^{\infty} e^{-x^2} dx$$

converges.

- Try to compare with integral of e^{-x}
- but $e^{-x^2} \leq e^{-x}$ only for $x \geq 1$
- so compare to e^{-x+1}:

$$e^{-x^2} \leq e^{-x+1}, \forall x > 0.$$

Example

Determine whether the following integral converges or diverges

$$I = \int_0^\infty \frac{dx}{\sqrt{x + x^3}}.$$

This is improper integral of both types

$$I = \int_0^1 \frac{dx}{\sqrt{x + x^3}} + \int_1^\infty \frac{dx}{\sqrt{x + x^3}} =: I_1 + I_2$$

On $(0,1]$: $\sqrt{x + x^3} > \sqrt{x}$, so $I_1 < \int_0^1 \frac{dx}{\sqrt{x}} = 2$. On $[1,\infty)$: $\sqrt{x + x^3} > \sqrt{x^3}$, so $I_2 < \int_1^\infty \frac{dx}{x^{3/2}} = 2$.
Thus, I converges.

Exercises

- Let $p < 1$. Compute
$$\int_0^1 \frac{\ln x}{x^p} dx.$$
- Using the Comparison Test together with integration by parts to show
$$\int_1^\infty \frac{\sin x}{x} dx$$
and
$$\int_1^\infty \cos(x^2) dx$$
converge.

Exercises

Determine whether each integral is convergent or divergent.

1. $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2+1}dx.$

2. $\displaystyle\int_0^{\infty} xe^{-x^2}dx.$

3. $\displaystyle\int_1^{\infty} \frac{x+1}{x^2+2x}dx.$

4. $\displaystyle\int_1^{\infty} \frac{\ln x}{x}dx.$

5. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{4x^2+4x+5}.$

Section 12

Numerical methods

Midpoint Rule

The Riemann sum with

$$c_i = \frac{x_{i-1} + x_i}{2} = a + (i - \tfrac{1}{2})h, i = 1, 2, \dots, n$$

is

$$M_n = h[f(c_1) + f(c_2) + \cdots + f(c_n)].$$

Theorem

If f'' is continuous on $[a, b]$ and $|f''(x)| \leq K$ then

$$\left| \int_a^b f(x)dx - M_n \right| \leq \frac{K(b-a)^3}{24n^2}.$$

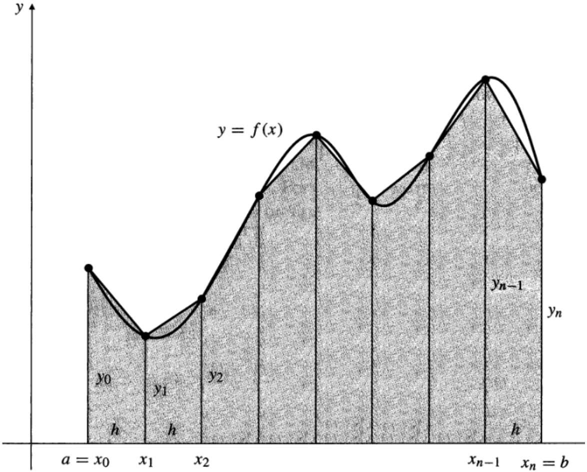
Example

How large should n be to guarantee that the Midpoint Rule approximation for $\int_1^2 \frac{1}{x} dx$ is accurate to within 10^{-4}? *Solution:*

- $|f''(x)| = |\frac{2}{x^3}|$. So $|f''(x)| \leq 2$ on $[1, 2]$
- Error is at most $\frac{2}{24n^2}$
- This is less than 10^{-4} if $n^2 > \frac{2}{24 \times 10^{-4}}$, or $n > 29$
- Take $n = 30$

Trapezoid Rule

Divide $[a, b]$ into n equal subintervals of length $h = \frac{b-a}{n}$ by $x_i = a + ih$, $i = 0, \ldots, n$. Then approximate the graph of f between x_i and x_{i+1} by the straight line connecting $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$.



Trapezoid Rule

- Let $y_i = f(x_i)$
- Area of trapezoid on $[x_i, x_{i+1}]$

$$h\frac{y_i + y_{i+1}}{2}$$

- Sum of areas of trapezoids is

$$\begin{aligned} T_n &= h\Big(\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \cdots + \frac{y_{n-1} + y_n}{2}\Big) \\ &= h(\tfrac{1}{2}y_0 + y_1 + \cdots + y_{n-1} + \tfrac{1}{2}y_n). \end{aligned}$$

Theorem

If f'' is continuous on $[a, b]$ and $|f''(x)| \leq K$ then

$$\left|\int_a^b f(x)dx - T_n\right| \leq \frac{K(b-a)^3}{12n^2}.$$

Example

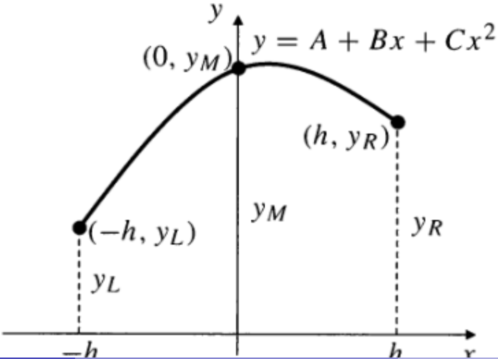
How large should n be to guarantee that the Trapezoidal approximation for $\int_1^2 \frac{1}{x} dx$ is accurate to within 10^{-4}?

- The error is at most $\frac{2}{12n^2}$
- For this to be less than 10^{-4} is equivalent to $n^2 > \frac{1}{6 \times 10^{-4}}$ or $n > 40.8$
- Take $n = 41$

Simpson Rule

- Use quadratic functions through 3 points $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$ to approximate f.
- This leads to

$$\int_{x_{i-1}}^{x_{i+1}} f(x)dx \approx \frac{h}{3}(y_{i-1} + 4y_i + y_{i+1})$$



Simpson Rule

Thus, if n is even, $\int_a^b f(x)dx$ can be approximated by

$$
\begin{aligned}
S_n =& \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n \\
=& \frac{h}{3}(\sum y_{end} + 4\sum y_{odd} + 2\sum y_{even})
\end{aligned}
$$

Theorem

Suppose $|f^{(4)}(x)| \leq K$ on $[a, b]$, then

$$
|\int_a^b f(x)dx - S_n| \leq \frac{K(b-a)^5}{180n^4}.
$$

Simpson Rule

Example

Find n so that the error of approximating $\int_1^2 \frac{1}{x} dx$ using the Simpson rule is less than 10^{-4}.

Solution:

- Since $f^{(4)}(x) = \frac{24}{x^5}$, $K = 24$
- Need $n^4 > \frac{24}{108 \cdot 10^{-4}}$ or $n > 6.866...$.
- Take $n = 8$.

Problem Set 1

- 5.2: 6, 8,10
- 5.3: 10, 14
- 5.4: 10, 14, 22
- 5.5: 16, 22, 36, 48, 50
- 5.6: 6, 9, 12, 14, 16, 27, 30
- 5.7: 4, 6, 8, 10, 12, 16, 18, 32, 26, 28, 36
- 5.10: 7, 8, 14, 19, 27, 31, 32, 62