1/

x	1	2
$P(X=x \mid Y=2, Z=1)$	1	0

$\Rightarrow E(X \mid Y=2, Z=1) = 1\cdot 1 + 2\cdot 0 = 1.$

2/

a) $X \sim Geo(p)$ where $p = \frac{1}{6} \Rightarrow E(X) = \frac{1}{p} = 6.$

b) We will generally calculate $E[X \mid Y=m]$, for $m \in \mathbb{N}$.

- We are given that it needed m tosses to obtain a 5, thus for each of the first $m-1$ tosses the chance of obtaining a 6 is $\frac{1}{5}$.
- Also, $P(X=m \mid Y=m)=0$: we cannot obtain both 6 and 5 on the same toss.
- After toss m, things proceed normally.

Thus we have the conditional pmf:

$$p_{X|Y}(x,m) = P(X=x \mid Y=m) = \begin{cases} \left(\frac{4}{5}\right)^{x-1}\cdot\frac{1}{5}, & x<m \\ 0, & x=m \\ \left(\frac{4}{5}\right)^{m-1}\cdot\left(\frac{5}{6}\right)^{x-m-1}\cdot\frac{1}{6}, & x>m \end{cases}$$

$$\Rightarrow E(X \mid Y=m) = \sum_x x\, p_{X|Y}(x,m) = \sum_{x=1}^{m-1}\left(\frac{4}{5}\right)^{x-1}\cdot\frac{x}{5} + \sum_{x=m+1}^{\infty}\left(\frac{4}{5}\right)^{m-1}\left(\frac{5}{6}\right)^{x-m-1}\cdot\frac{x}{6}$$

$$= \left(\frac{1}{5}\cdot\frac{5}{4}\right)\sum_{x=1}^{m-1} x\left(\frac{4}{5}\right)^x + \left(\frac{4}{5}\right)^{m-1}\cdot\frac{1}{6}\cdot\left(\frac{6}{5}\right)^{m+1}\sum_{x=m+1}^{\infty} x\left(\frac{5}{6}\right)^x$$

$$= \frac{1}{4}\cdot 5\left(4-\left(\frac{4}{5}\right)^m(m+4)\right) + \frac{1}{6}\cdot\left(\frac{4}{5}\right)^{m-1}\cdot\left(\frac{6}{5}\right)^{m+1}\cdot 5\left(\frac{5}{6}\right)^m(m+6).$$

$$= 5 - \left(\frac{4}{5}\right)^{m-1}(m+4) + \left(\frac{4}{5}\right)^{m-1}(m+6) = 5 + 2\left(\frac{4}{5}\right)^{m-1}.$$

$$\Rightarrow E(X \mid Y=1) = 5 + 2\left(\frac{4}{5}\right)^0 = 7.$$

c) $E(X \mid Y=5) = 5 + 2\left(\frac{4}{5}\right)^4 = \frac{3637}{625} \approx 5.8192.$

3/ Let $Z = X+Y$, then since X and Y are independent,

$$p_Z(z) = \sum_{x=0}^{z} p_X(x) + p_Y(z-x) = \sum_{x=0}^{z} \frac{e^{-\lambda_1}\lambda_1^x}{x!} \cdot \frac{e^{-\lambda_2}\lambda_2^{z-x}}{(z-x)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda_1^x \lambda_2^{z-x} = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^z}{z!} \Rightarrow Z \sim \text{Poisson}(\lambda_1+\lambda_2).$$

Then we have the conditional pmf:

$$p_{X|Z}(x,n) = \frac{P(X=x, Y=n-x)}{P(Z=n)} = \frac{P(X=x)\cdot P(Y=n-x)}{P(Z=n)}$$

$$= \binom{n}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-x} \Rightarrow X|Z=n \sim \text{Bino}(n,p) \text{ where } p = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$\Rightarrow E(X|Z=n) = np = \frac{n\lambda_1}{\lambda_1+\lambda_2}.$$

4/ Marginal pdf of Y:

$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dx = \int_0^1 6xy(2-x-y)\,dx = -3y^2+4y, \ \forall y \in [0,1].$$

Conditional pdf: $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{6x(2-x-y)}{-3y+4}, \ \forall x,y \in [0,1].$

$$\Rightarrow E(X|Y=y) = \int_{\mathbb{R}} x f_{X|Y}(x|y)\,dx = \int_0^1 \frac{6x^2(2-x-y)}{-3y+4}\,dx = \frac{4y-5}{6y-8}.$$

5/ Marginal pdf of Y:

$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dx = \int_0^\infty 4y(x-y)e^{-x-y}\,dx = -4y(y-1)e^{-y}, \ \forall y \in [0,x].$$

Conditional pdf: $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{y-x}{e^x(y-1)}, \ \forall x \in (0,\infty), y \in [0,x].$

$$\Rightarrow E(X|Y=y) = \int_{\mathbb{R}} x f_{X|Y}(x|y)\,dx = \int_0^\infty \frac{xy-x^2}{e^x(y-1)}\,dx = 1 - \frac{1}{y-1}.$$

7/ Since X_i and N are independent,

$$E(T|N=n) = E\left(\sum_{i=1}^{N} X_i \,\middle|\, N=n\right) = E\left(\sum_{i=1}^{n} X_i \,\middle|\, N=n\right) = E\left(\sum_{i=1}^{n} X_i\right) = n E(X).$$

$$\Rightarrow E(T) = \sum_n E(T|N=n) P(N=n) = \sum_n p_N(n) \cdot n E(X) = E(N) E(X).$$

Similarly, $E(T^2|N=n) = n E(X^2) + n^2 E(X)^2 - n E(X)^2$ and thus

$$E(T^2) = \sum_n E(T^2|N=n) P(N=n) = E(N) E(X^2) + E(N^2) E(X)^2 - E(N) E(X)^2$$

$$\Rightarrow Var(T) = E(T^2) - E(T)^2 = E(N)\left[E(X^2) - E(X)^2\right] + E(N^2) E(X)^2 - E(N)^2 E(X)^2$$

$$= E(N) Var(X) + E(X)^2 \left[E(N^2) - E(N)^2\right] = E(N) Var(X) + E(X)^2 Var(N).$$