Question 1. (a) *(25 marks)* Let (X, d), (Y, ρ), and (Z, σ) be metric spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous mappings. Show that $h = g \circ f : X \to Z$ is uniformly continuous.

(b) *(25 marks)* Let (X, d) be a metric space. Prove that X is complete if and only if for each $r > 0$, the closed ball $\overline{B}(x, r)$ is complete.

Question 2. Let (X, d) and (Y, ρ) be metric spaces.

(a) *(25 marks)* Define $\sigma : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$\sigma\big((x, y), (x', y')\big) = \max\big\{d(x, x'), \rho(y, y')\big\}.$$

Show that σ is a metric on $X \times Y$.

(b) *(10 marks)* Let $(x_0, y_0), (x_n, y_n) \in X \times Y$, $n \in \mathbb{N}$. Show that

$$\lim_{n\to\infty} (x_n, y_n) = (x_0, y_0) \iff \lim_{n\to\infty} x_n = x_0 \text{ and } \lim_{n\to\infty} y_n = y_0.$$

(c) *(15 marks)* Let $f : X \to Y$ be a mapping. The **graph** of the mapping f is the set of ordered pairs

$$\mathrm{Gr}\,(f) := \big\{(x, y) \in X \times Y : \; x \in X \text{ and } f(x) = y\big\}.$$

Show that if f is continuous and X is compact, then $\mathrm{Gr}\,(f)$ is a compact set in the metric space $(X \times Y, \sigma)$.