VIETNAM NATIONAL UNIVERSITY-HCMC
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# Chapter 2. Determinants

## Linear Algebra

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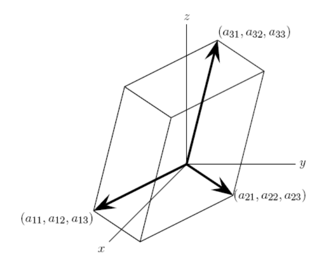
# Contents

Section 1

# DETERMINANT

# Introduction

- Reference: Chapter 3 in the textbook by Kolman-Hill.
- The determinant of $A$ equals the volume of a box in n-dimensional space. The edges of the box come from the rows of $A$.




- They can also be used to compute $A^{-1}$ in terms of the entries of A.

# Determinant functions

## Definition: Permutation

Let $S = \{1, 2, ..., n\}$ be the set of integers from 1 to $n$, arranged in ascending order. A rearrangement $j_1, j_2, ...., j_n$ of the elements of $S$ is called a permutation of $S$. We can consider a permutation of $S$ to be a one-to-one mapping of $S$ onto itself.

There are $n!$ permutations of $S$; we denote the set of all permutations of $S$ by $S_n$.

## Example

Let $S = \{1, 2, 3\}$. Then 231 is a permutation of $S$. It corresponds to the function $f : S \to S$ defined by

$$f(1) = 2, f(2) = 3, f(3) = 1$$

There are $3! = 6$ permutations of $S_3$.

# Determinant functions

## Inversion

A permutation $j_1 j_2 \ldots j_n$ is said to have an inversion if a larger integer, $j_r$ precedes a smaller one $j_s$. A permutation is called even if the total number of inversions in it is even, or odd if the total number of inversions in it is odd.

## Example

In the permutation 231 in $S_3$, 2 precedes 1, 3 precedes 1, no other inversions. Thus the total number of inversions in this permutation is 2.

In the permutation 4312 in $S_4$, 4 precedes 3, 4 precedes 1, 4 precedes 2, 3 precedes 1, and 3 precedes 2. Thus the total number of inversions in this permutation is 5, and the permutation 4312 is odd.

Q: What is the number of inversions of the permutation 4132?A: 4.

# Determinant functions

## Definition: Determinant functions

The determinant function, denoted by *det*, is defined by

$$\det(A) = \sum (\pm) a_{1j_1} a_{2j_2} .. a_{nj_n}$$

where the summation is over all permutations $j_1 j_2 .. j_n$ of the set $S = \{1, 2, ..., n\}$. The sign is taken as $+$ or $-$ according to whether the permutation $j_1 j_2 .. j_n$ is even or odd.

# Determinant functions

## Determinants of $2 \times 2$ matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

We have $\det(A) = \sum (\pm) a_{1j_1} a_{2j_2}$. There are 2 permutations: $j_1 j_2 = 12$ (even) and $j_1 j_2 = 21$ (odd). Thus,

$$\det(A) \quad = \quad |A| \quad = \quad a_{11}a_{22} \quad - \quad a_{12}a_{21}$$

Note:

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example:

$$\begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix} = 2 \times 5 - (-3) \times 4 = 22$$

# Determinants of a matrix of order 3

## Determinant of a 3 × 3 matrix

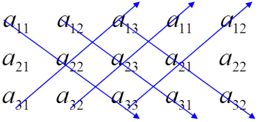
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

There are 6 permutations. The even permutations are 123, 231, 312. The odd permutations are 321, 132, 213.

$$\det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Subtract these three products.




Add these three products.

# Determinants of a matrix of order 3

## Example

$$\begin{array}{ccc} & & \quad\quad\quad\quad\quad -4 \quad 0 \quad 6 \\ A = & \left[\begin{array}{ccc}0 & 2 & 1\\ 3 & -1 & 2\\ 4 & -4 & 1\end{array}\right] & \begin{array}{cc}0 & 2\\ 3 & -1\\ 4 & -4\end{array} \\ & & \quad\quad\quad\quad\quad 0 \quad 16 \quad -12 \end{array}$$

So

$$\det(A) = |A| = 0 + 16 - 12 - (-4) - 0 - 6 = 2$$

# Determinants of a matrix of order 3

## Example

Compute $det(A)$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Answer: $det(A) = 6$

## Exercises

Find the determinant

$$\begin{vmatrix} a & 0 & 0 \\ 1 & b & 0 \\ 3 & d & c \end{vmatrix}$$

Answer: $abc$.

# Determinants of triangular matrices

## Determinants of upper triangular matrices

All the entries above the main diagonal are zeros.

$$\begin{vmatrix} a & 0 & 0 \\ 1 & b & 0 \\ 3 & d & c \end{vmatrix} = abc$$

## Determinants of lower triangular matrix

All the entries below the main diagonal are zeros.

$$\begin{vmatrix} a & 3 & 4 \\ 0 & b & 1 \\ 0 & 0 & c \end{vmatrix} = abc$$

Section 2

# Properties of Determinants

# Triangular matrices

## Theorem

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}$$

## Example

If

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 10 & 4 & 0 & 0 & 0 \\ 5 & 0 & 2 & 0 & 0 \\ 7 & 6 & 0 & 5 & 0 \\ 9 & 2 & 4 & 1 & -1 \end{bmatrix}$$

then $\det(A) = (-2)(4)(2)(5)(-1) = 80$

# Triangular matrices

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# Determinants: Properties

**Theorem (Determinant of a transpose)**

If A is a $n \times n$ matrix A then $\det(A^T) = \det(A)$.

Proof: Let

$$A = [a_{ij}], A^T = [b_{ij}]$$

$$\det\left(A^T\right) = \sum (\pm) b_{1j_1} b_{2j_2} ... b_{nj_n} = \sum (\pm) a_{j_1 1} a_{j_2 2} ... a_{j_n n}$$

We can then write

$$b_{1j_1} b_{2j_2} ... b_{nj_n} = a_{j_1 1} a_{j_2 2} ... a_{j_n n} = a_{1k_1} a_{2k_2} ... a_{nk_n}$$

which is a term of $det(A)$. Thus the terms in $det(A^T)$ and $det(A)$ are identical.

# Determinants: Properties

## Example

If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

then

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}$$

and

$$|A| = 6 = \left| A^T \right|$$

# Elementary operations

## Theorem (Row operations)

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B then $\det(B) = \det(A)$.

b. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$.

c. If one row of A is multiplied by $k$ to produce B, then $\det(B) = k\det(A)$.

## Corollary

a. If two rows (columns) of $A$ are equal, then $det(A) = 0$.

b. If a row (column) of $A$ consists entirely of zeros, then $det(A) = 0$.

# Find the determinant using elementary operations

## Example

$$\text{Given} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad \det(A) = -2.$$

$$\text{Let } A_1 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix} \quad (R_2 - 2 \times R_1),$$

then $\det(A_1) = \det(A) = -2$.

$$\text{Let } A_2 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

then $\quad \det(A_2) = 4\det(A) = (4)(-2) = -8$.

# Find the determinant using elementary operations

## Example

Evaluate

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \overset{R_2-2R_1}{=} \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} \overset{I_{23}}{=} - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = 15$$

# Find the determinant using elementary operations

## Example

Compute $det(A)$ where

$$A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

Solution:

Add 2 times row 1 to row 3 to obtain

$$\det(A) = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix} = 0$$

# Find the determinant using elementary operations

## Exercises

$$A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{pmatrix}$$

Evaluate $\det(A)$.
Answer: $-30$.

## Example

Evaluate

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{vmatrix}$$

## Example

Compute

$$D_n = \begin{vmatrix} 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 0 & 1 & 1 & .. & 1 \\ 1 & 1 & 0 & 1 & .. & 1 \\ . & . & . & . & . & . \\ 1 & 1 & .. & 1 & 0 & 1 \\ 1 & 1 & 1 & .. & 1 & 0 \end{vmatrix}$$

Solution: Employ the row operations: $R_k - R_1 \to R_k$, for $k = 2, 3, .., n$.

$$D_n = \begin{vmatrix} 1 & 1 & 1 & 1 & .. & 1 \\ 0 & -1 & 0 & 0 & .. & 0 \\ 0 & 0 & -1 & 0 & .. & 0 \\ . & . & . & . & . & . \\ 0 & 0 & .. & 0 & -1 & 0 \\ 0 & 0 & 0 & .. & 0 & -1 \end{vmatrix} = (-1)^{n-1}$$

## Example

Compute

$$E_n = \begin{vmatrix} 0 & 1 & 1 & 1 & .. & 1 \\ 1 & 0 & 1 & 1 & .. & 1 \\ 1 & 1 & 0 & 1 & .. & 1 \\ . & . & . & . & . & . \\ 1 & 1 & .. & 1 & 0 & 1 \\ 1 & 1 & 1 & .. & 1 & 0 \end{vmatrix}$$

Hint:

$$R_1 + R_2 + ... + R_n \to R_1$$

Thus, $E_n = (n-1)D_n = (n-1)(-1)^{n-1}$.

# The Vandermonde Matrix

Find the determinant of the following Vandermonde matrix
**1.**

$$V_3 = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}$$

**2*.**

$$V_n = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ . & . & . & . & . \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{bmatrix}$$

# Determinants: Properties

## Definition

An $n \times n$ elementary matrix of type I, type II, or type III is a matrix obtained from the identity matrix $I_n$ by performing a single elementary row (or elementary column) operation of type I, type II or type III, respectively.

## Example

Let

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E_1$ is an elementary matrix of type I and $E_2$ is an elementary matrix of type II.

# Determinants: Properties

### Theorem

If $E$ is an elementary matrix, then $det(EA) = det(E)det(A)$, and $det(AE) = det(A)det(E)$.

### Theorem (Multiplicative property)

If A and B are $n \times n$ matrices then

a. $\det(AB) = \det(A)\det(B)$,

b. $\det(kA) = k^n \det(A)$, where $k$ is constant.

# Determinants: Properties

## Theorem

A square matrix A is invertible (nonsingular) if and only if $\det(A) \neq 0$ and

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

## Equivalent conditions for invertibility

If A is an n $\times$ n matrix, then the following statements are equivalent

- A is invertible.
- Ax = b has a unique solution for every n $\times$ 1 matrix b.
- Ax = 0 has only the trivial solution.
- $det(A) \neq 0$.

## Corollary

If A is a nxn matrix, then $Ax = 0$ has a nontrivial solution if and only if $det(A) = 0$.

# Examples

Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

Compute $\det(A^{-1})$, $\det(2A)$.

Solution

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4.$$

Thus

$$\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{4}$$

and

$$|2A| = 2^3|A| = 32$$

# Exercises

1. Let $A$ and $B$ be $4 \times 4$ matrices, with $\det(A) = -1$ and $\det(B) = 2$. Compute
a. $\det(AB)$, b. $\det(B^5)$, c. $\det(2A)$, d. $\det(A^TA)$.

2. Let $A$ and $P$ be square matrices, with $P$ invertible. Show that $\det(PAP^{-1}) = \det(A)$.

3. Suppose that $A$ is a square matrix such that $det(A^4) = 0$. Explain why $A$ can not be invertible.

# Cofactor expansions

Let $A_{ij}$ be the matrix formed by removing the ith row and jth column of the matrix $A$

## $A_{ij}$

$$
|A_{ij}| = \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1j} & \cdots & a_{1n} \\
\vdots & & & \vdots & \vdots & & \\
a_{(i-1)1} & & \cdots & a_{(i-1)(j-1)} & a_{(i-1)j} & \cdots & a_{(i-1)n} \\
a_{i1} & & \cdots & a_{i(j-1)} & a_{ij} & \cdots & a_{in} \\
\vdots & & & \vdots & \vdots & & \vdots \\
a_{n1} & & \cdots & a_{n(j-1)} & a_{nj} & \cdots & a_{nn}
\end{vmatrix}
$$

# Cofactor expansions

## Determinant of an $n \times n$ matrix A

Let $C_{ij} := (-1)^{i+j}\,|A_{ij}|$, $C_{ij}$ is called the cofactor of $a_{ij}$. Then

$$\det(A) = |A| = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}$$

where $A_{ij}$ is the matrix formed by removing the ith row and jth column of the matrix $A$.
(Cofactor expansion along the i-th row)

Or

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(Cofactor expansion along the j-th column)

# Cofactor expansions

## Determinants of $3 \times 3$ matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

## Example

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 1 \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = -2$$

# Cofactor expansions

## Example

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+3}\, 2 \begin{vmatrix} 0 & 3 & 4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$

$$= 2\,(-1)^{2+1}\,(-5) \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 20$$

# Cofactor expansions

## Example

Find the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}$$

Solution

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43}) = 3C_{13}$$

$$= 3(-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = 3 \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

# Cofactor expansions

## Solution (Cont.)

$$\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = (0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix}$$

$$+(3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= 0 + (2)(1)(-4) + (3)(-1)(-7) = 13$$

Thus, $\det(A) = 39$.

Section 2

# Cramer's rule

# Cramer's Rule

Consider a linear system of n equations in n unknowns:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n
\end{aligned}
$$

or in matrix notation $AX = b$, where $A = (a_{ij})_{n \times n}$.

# Cramer's Rule

1. If $D = |A| \neq 0$, the system of linear equations has a unique solution

$$x_k = \frac{D_k}{D}, k = 1, 2, ..., n$$

where $D_k$ is the determinant of the matrix obtained by substituting the k-th column of the matrix A by the column

$$b = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix}^T$$

For example:

$$D_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

2. If $D = 0$ and at least one of the $D_k$'s is non-zero, then the system has no solution.

# Cramer's Rule

## Example

Use Cramer's Rule to solve the system

$$3x - 2y = 6$$
$$-5x + 4y = 8$$

We have

$$D = \det(A) = \begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix} = 2,$$

$$D_1 = \begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix} = 40, \quad D_2 = \begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix} = 54.$$

Therefore,

$$x = \frac{D_1}{D} = \frac{40}{2} = 20, y = \frac{D_2}{D} = \frac{54}{2} = 27.$$

# Cramer's Rule

## Example

Use Cramer's Rule to solve the system

$$x + y + z = -2$$
$$3x - y + 2z = 4$$
$$4x + 2y + z = -8$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 4 & 2 & 1 \end{vmatrix} = 10, \quad D_1 = \begin{vmatrix} -2 & 1 & 1 \\ 4 & -1 & 2 \\ -8 & 2 & 1 \end{vmatrix} = -10$$

$$D_2 = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 4 & 2 \\ 4 & -8 & 1 \end{vmatrix} = -30, \quad D_3 = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -1 & 4 \\ 4 & 2 & -8 \end{vmatrix} = 20$$

Thus,

$$x = \frac{D_1}{D} = -1, y = \frac{D_2}{D} = -3, z = \frac{D_3}{D} = 2$$

# Cramer's Rule

## Example

Use Cramer's Rule to solve the system

$$-2x_1 + 3x_2 - x_3 = 1$$

$$x_1 + 2x_2 - x_3 = 4$$

$$-2x_1 - x_2 + x_3 = -3$$

$$D = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2, \quad D_1 = \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ 3 & -1 & 1 \end{vmatrix} = -4$$

$$D_2 = \begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & 3 & 1 \end{vmatrix} = -6, \quad D_3 = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & 3 \end{vmatrix} = -8$$

Thus, $x_1 = 2, x_2 = 3, x_3 = 4$.

# Exercises

Use Cramer's Rule to solve the system

$$
\begin{aligned}
2x + 5y - z &= 15 \\
x - y + 3z &= 4 \\
3x + 3y - 5z &= 2
\end{aligned}
$$

Answer: $(x, y, z) = (1, 3, 2)$.

# Exercises

Solve the linear system

$$\begin{aligned} 2x + 3y - z &= 7 \\ x - y + z &= 1 \\ 4x - 5y + 2z &= 3 \end{aligned}$$

Using

a. Gaussian elimination method,

b. Cramer's rule.

# The Adjoint and a Theoretical formula for $A^{-1}$

## Definition: Matrix of cofactors

The **matrix of cofactors** of $A$ has the form

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

## Definition: The Adjoint

If $A$ is $n \times n$ matrix, **the adjoint** of $A$, denoted by $adj(A)$, is the transpose of the matrix of cofactors,

$$adj\,(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

# The Adjoint and a Theoretical formula for $A^{-1}$

The adjoint of $A$ plays the following extremely important role.

## Theoretical formula for $A^{-1}$

If $A$ is $n \times n$ matrix, then

$$Aadj\left(A\right) = \left(\det A\right) I$$

Thus if $\det A \neq 0$ so that $A^{-1}$ exists, then

$$A^{-1} = \frac{1}{\det A} adj\left(A\right)$$

# The Adjoint and a Theoretical formula for $A^{-1}$

## Example

Let

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 0 & -2 \\ 3 & -1 & -3 \end{bmatrix}$$

Thus, $\det A = -26$ and the the matrix of cofactors is

$$(C_{ij}) = \begin{bmatrix} -2 & 6 & -4 \\ -10 & -9 & -7 \\ 6 & 8 & 12 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{\det A} adj\,(A) = \frac{1}{-26} \begin{bmatrix} -2 & -10 & 6 \\ 6 & -9 & 8 \\ -4 & -7 & 12 \end{bmatrix}$$

# Homeworks

Textbook: B. Kolman and David R. Hill, Elementary Linear Algebra with Applications, 9th edition, Prentice Hall, 2008

-Section 3.1: 14, 16

-Section 3.2: 2, 3, 6

-Section 3.3: 1, 3, 11

-Section 3.4: 2, 10

Deadline: April 4th, 2022