VIETNAM NATIONAL UNIVERSITY - HOCHIMINH CITY
INTERNATIONAL UNIVERSITY

LESSON 4. Brownian motion

CHAPTER 2. Common stochastic processes

Course: Random processes

Lecturer: Le Nhat Tan, PhD

Lesson content

- Random walk

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- Martingale processes

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- Martingale processes
- Brownian motion

Symmetric Random walk

We toss a fair coin infinitely many times.

- Let X_i be a random variable such that
$$X_i(\omega_i) = \begin{cases} 1 & \text{if} \quad \omega_i = H \\ -1 & \text{if} \quad \omega_i = T. \end{cases}$$

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- X_i and X_j are independent random variables for any $i \neq j$.
- For each $k \in \mathbb{N}$, we denote $M_k = \sum_{i=1}^{k} X_i$.
- $(M_k)_{k \in \mathbb{N}}$: symmetric random walk.

Example

For each toss of a fair coin, you will win \$1 if it lands up heads and lose \$1 if it lands up tails. Suppose that you start the game with \$0 and denote M_k is the amount of money you obtain after k tosses. Then $(M_k)_{k\in\mathbb{N}}$ is a random walk.

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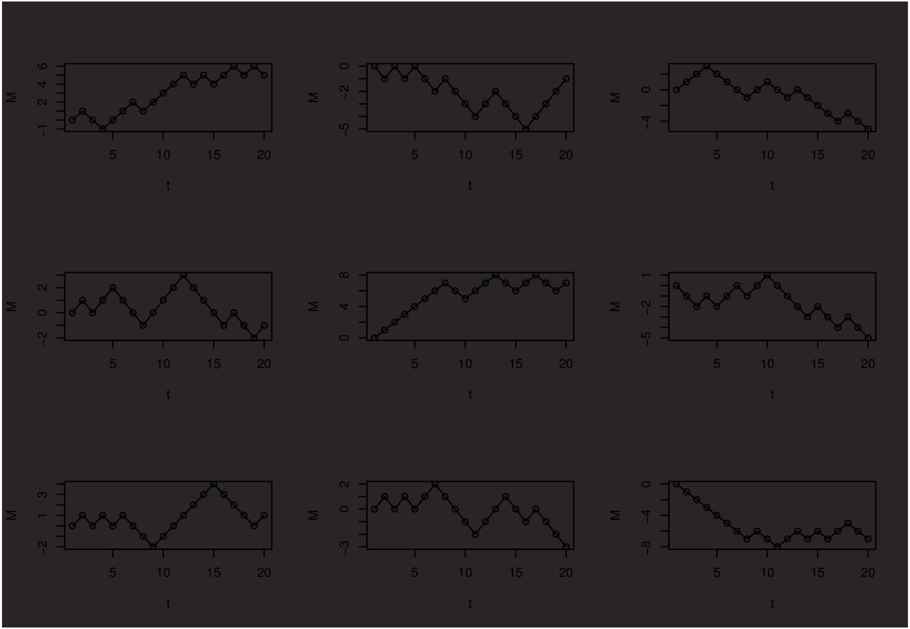
- Play the game with your friend 10 times and plot your accumulated money after every play.
- Write a R code to simulate the game

R code

```
par(mfrow=c(3,3))
N <- 20 ### number of playing game
M <- rep(0,N) ## create a space to store the evolution of M
M[1] <- 0 # the initial amount of money

for (i in 2:N) {
X <- rbinom(n=1,size=1,prob=0.5) # simulate a Bernoulli
    random variable
if (X==1) M[i] =M[i-1]+1 ## get $1 if head. i.e., X==1
else M[i] =M[i-1]-1 ## lose $1 if head. i.e., X==0
}
plot(M, type="o", col="blue",xlab="number of playing
    games",ylab="the accumulated money")
```


M
t

Exercise

For each toss of a unfair coin, you will win \$2 if it lands up heads and lose \$1 if it lands up tails. The probability of landing up heads is 0.3 and the probability of landing up tails is 0.7. Suppose that you start the game with \$0 and denote M_k is the amount of money you obtain after k tosses.

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- Should you play the game?

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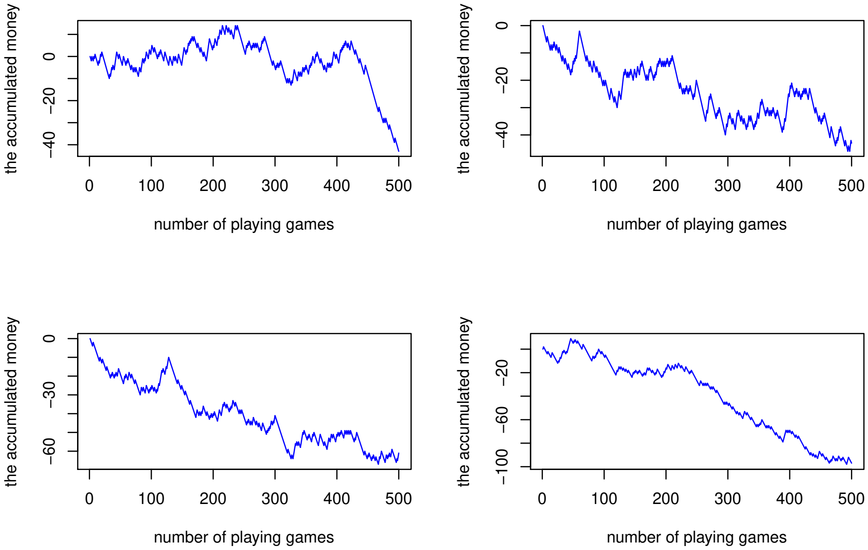
For each toss of a unfair coin, you will win \$2 if it lands up heads and lose \$1 if it lands up tails. The probability of landing up heads is 0.3 and the probability of landing up tails is 0.7. Suppose that you start the game with \$0 and denote M_k is the amount of money you obtain after k tosses.

- Should you play the game?
- Write a R code to simulate possible results for playing the games 500 times


the accumulated money
0
−20
−40
0 100 200 300 400 500
number of playing games
the accumulated money
0
−20
−40
0 100 200 300 400 500
number of playing games
the accumulated money
0
−30
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Fair games

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- Is this game fair, i.e., the chance of winning a fair game is equal for all players?

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- Compute the probability of winning after playing the game 9 times
- Compute the probability of winning after playing the game 10 times
- Is this game fair, i.e., the chance of winning a fair game is equal for all players?
- A fair game is modeled by a martingale process.

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- If $E[X_t|\mathcal{F}_s] < X_S, \quad \forall 0 \leq s \leq t$: supermartingale (no tendency to rise, might have a tendency to fall).

Martingale processes

To prove a random process $(X_t)_{t \geq 0}$ be a martingale process under the filtration $(\mathcal{F}_t)_{t \geq 0}$, we need to prove three things:

1. $(X_t)_{t \geq 0}$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$
2. $E[|X_t|] < +\infty$.
3. $E[X_t | \mathcal{F}_s] = X_s, \forall 0 < s \leq t$.

To prove the second, we may use the Holder inequality: $E[|X_t|] \leq \sqrt{E[X_t^2]}$ and the monotonicity property of expectation $E[|X + Y|] \leq E[|X|] + E[|Y|]$.

Martingale processes

- If a stochastic process $(X_t)_{t\geq 0}$ is said to be a martingale process, without specifying the corresponding filtration, it is understood that the filtration is the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the process, i.e., $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ (the σ-algebra generated from all random variables $X_s, 0 \leq s \leq t$)

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- If a stochastic process $(X_t)_{t\geq 0}$ is said to be a martingale process with respect to another stochastic process $(Y_t)_{t\geq 0}$, the corresponding filtration is the natural filtration of the process $(Y_t)_{t\geq 0}$.

Some properties of a symmetric random walk

Consider a random walk $(M_k)_{k\geq 0}$.

1. $M_k - M_l$ and $M_t - M_s$ are independent random variables for any $0 \leq k \leq l \leq s \leq t$.
2. $$E[M_k - M_l] = E[\sum_{i=l+1}^{k} X_i] = \sum_{i=l+1}^{k} E[X_i] = 0$$
3. Each increment $M_k - M_l$ has variance $k - l$, for any $l \leq k$.
4. $(M_k)_{k\in\mathbb{N}}$ is a martingale process. More precisely,
$$E[|M_k|] = E[|\sum_{i=1}^{k} X_i|] = E[\sum_{i=1}^{k} |X_i|] = k < +\infty \text{ and}$$
$$E[M_t|\mathcal{F}_s] = E[(M_t - M_s) + M_s|\mathcal{F}_s] = E[M_t - M_s|\mathcal{F}_s] + M_s$$
$$= E[M_t - M_s] + M_s = M_s.$$
5. Quadratic variation up to time t, defined as $[M, M]_k = \sum_{j=1}^{k}(M_j - M_{j-1})^2 = k$.

Exercise

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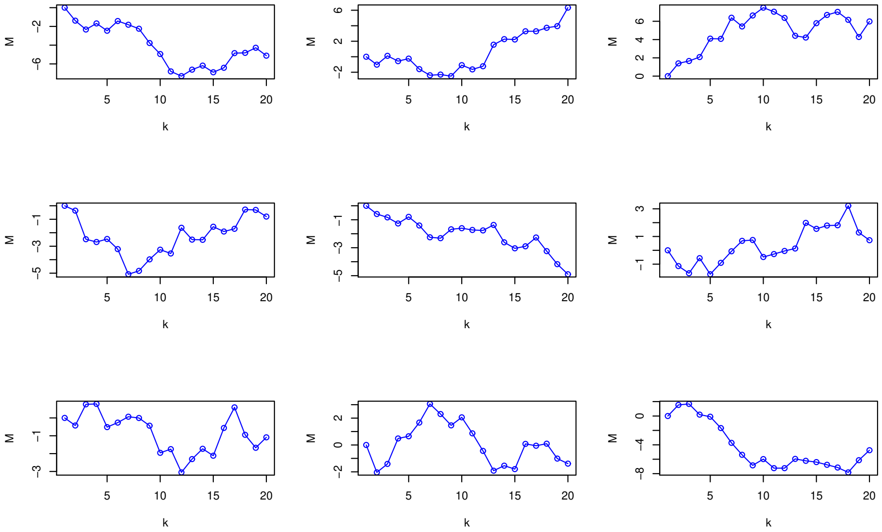
M[1] <- 0 # the initial amount of money
for (i in 2:N) {
X <- rnorm(n=1,mean=0,sd=1) # simulate a Bernoulli random
    variable
M[i] =M[i-1]+X
}
plot(M, type="o", col="blue",xlab="k",ylab="M")
```


M
k
5
10
15
20

Exercise

Consider a game based on a standard normal random variable Z. At each run, the player will receive the money equals to \$0.3 plus the value of Z at that run. Suppose that you start the game with \$0 and denote M_k is the amount of money you obtain after k plays.

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- Compute the probability that $M_{10} > 0$, given that $\int_0^{0.3\sqrt{10}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0.3286.$

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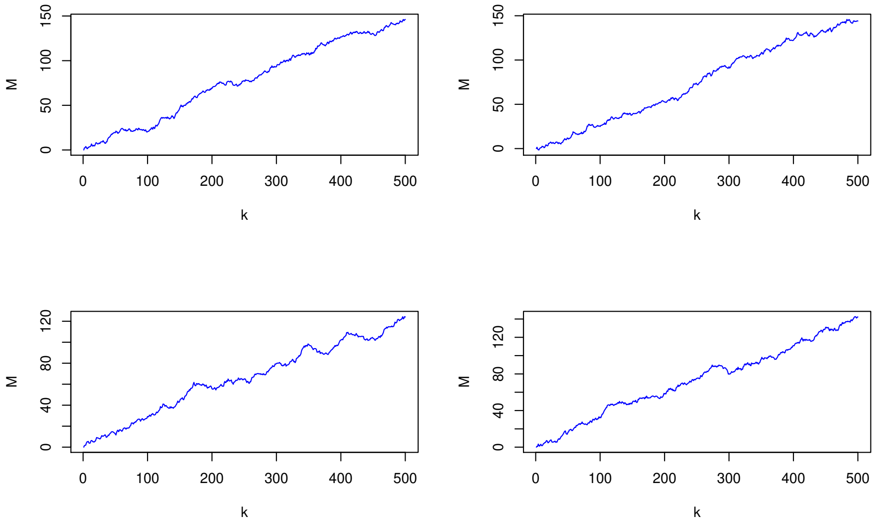
- Write a R code to simulate the obtained result after playing the game 500 times.


M
k
0
50
100
150
0
100
200
300
400
500
M
k
0
40
80
120

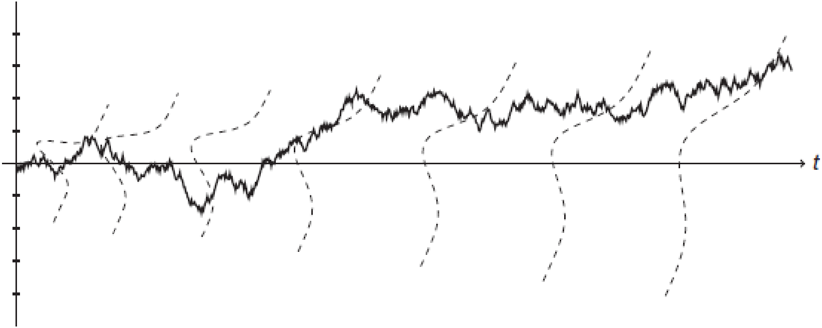
Brownian motion

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{P})$ be a filtered probability space and let $(B_t)_{0\le t<\infty}$ be an adapted process of this space. The process $(B_t)_{0\le t<\infty}$ is called a standard Brownian motion if it satisfies the following properties:

1. $B_0 = 0$
2. Independent increments: $B_t - B_s$ is independent of $\mathcal{F}_s$, $0 \le s < t$. That means
$$P(B_t - B_s \le k|\mathcal{F}_s) = P(B_t - B_s \le k).$$
3. Stationary increments:
$$B_t - B_s \overset{d}{=} B_{t-s} \sim N(0, \sqrt{t-s}),\ \forall\ 0 < s < t.$$
4. Continuous paths: all sample paths of process $(B_t)_{t\ge 0}$ are almost surely continuous, i.e.
$$P(\omega \in \Omega | B_t(\omega) \text{ is a continuous sample path}) = 1.$$

Brownian motion

Brownian motion can be thought of as the motion of a particle that diffuses randomly along a line. At each point t, the particle's position B_t is normally distributed about the line with variance t, i.e., $B_t \sim N(0, t)$. As t increases, the particle's position is more diffuse.



Some interesting notes

Can we predict the value of B_t from time $s < t$? We cannot predict exactly the value of the B_t, but we know the possible range of values B_t.

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Model standard Brownian motion

Suppose we want to model a Brownian motion in a time interval $[0, T]$.

- We divide the interval into n equal subintervals by discrete time points $0 = t_0 < t_1 < \cdots < t_n = T$, with the time step $h = t_i - t_{i-1}$.

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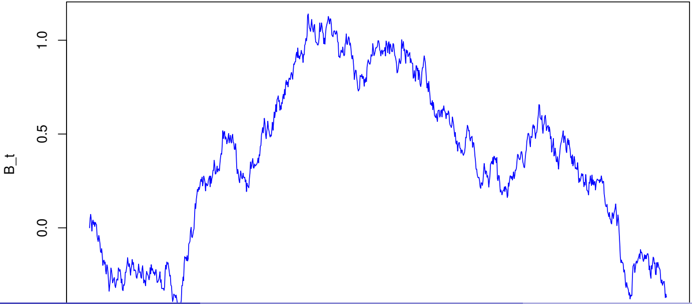
- The value of B_{t_i} is determined by a recursive formula: $B_{t_i} = B_{t_{i-1}} + \sqrt{h} * X$, where $X \sim N(0, 1)$.
- All we need to simulate the Brownian motion is to simulate a standard normal random variable, which can be done by using command "rnorm(1)" in software R.

```
BM=function(T,N) {
# T=1 expiry time
# N=100 number of simulation points
h=T/N # the timestep of the simulation
X=rep(0, (N+1)) # create space to store value of Brownian
   motion
X[1]=0 ## start at 0
for(i in 1:N) { X[i+1]=X[i] +sqrt(h)*rnorm(1)}
return(X)
}
```

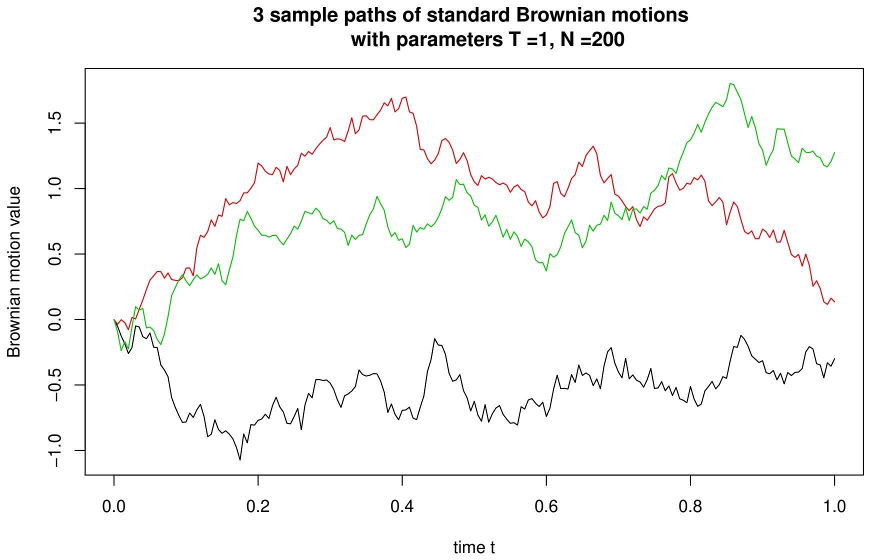
A sample of standard Brownian motion

```
h <- 1/1000
t=seq(0,1,by=h)
plot(t,BM(1,N=1000), type="l",
    col="blue",xlab="t",ylab="B")
```

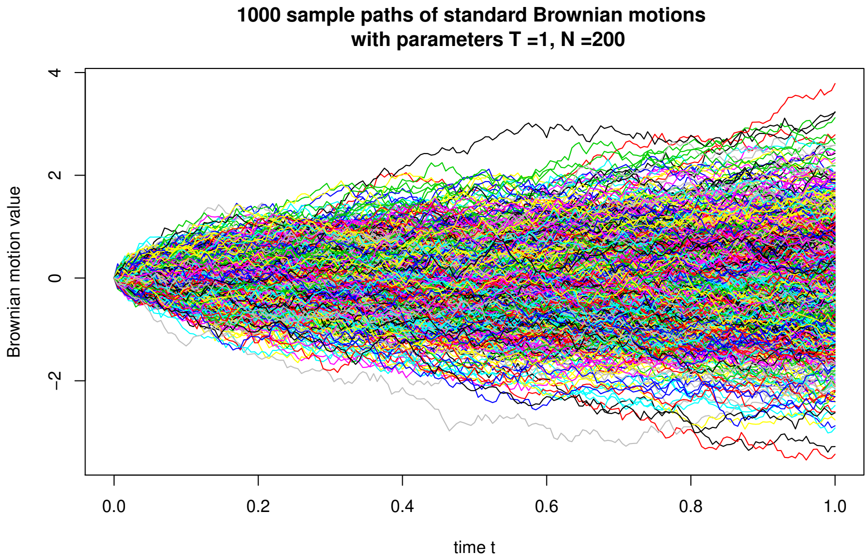


Plot multiple samples of Brownian motion

```
BMSamplepaths <- function(T,N,nt)
{h=T/N ## time step
#nt: number of samples
t=seq(0,T,by=h) # produce a sequence of time points
X=matrix(rep(0,length(t)*nt), nrow=nt)
## each row of the matrix stores one sample
# #return(X)
for (i in 1:nt) {X[i,]= BM(T=T,N=N)}
# # ##Plot
ymax=max(X); ymin=min(X) #bounds for simulated prices
plot(t,X[1,],t='l',main='3 sample paths of standard
    Brownian motions
with parameters T =1, N =3',ylim=c(ymin, ymax), col=1,
ylab="Price P(t)",xlab="time t")
for(i in 2:nt){lines(t,X[i,], t='l',ylim=c(ymin,
    ymax), col=i)}
```



3 sample paths of standard Brownian motions
with parameters T =1, N =200
Brownian motion value
−1.0
−0.5
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0
time t


1000 sample paths of standard Brownian motions
with parameters T =1, N =200
Brownian motion value
time t
4
2
0
−2
0.0
0.2
0.4
0.6
0.8
1.0

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& = P(B_5 - B_2 \leq 2 | B_2 = 1) = P(B_5 - B_2 \leq 2) \\
= \quad & P(B_3 \leq 2) = \int_{-\infty}^{2} \frac{1}{\sqrt{3.2\pi}} e^{-x^2/(2.3)} dx \approx 0.875893.
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- Use software R with the command "pnorm(2,0,sqrt(3)) to obtain the result.

Exercise

A particle's position is modelled by a standard Brownian motion. Find the probability that the particle's position at time $t = 3$ is above the level 1.5.

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- Since the difference quotient takes arbitrarily large values, the limit, and hence the derivative, does not exist.

Example

For $0 < s < t$, find the distribution of $B_s + B_t$.

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- $E(B_s + B_t) = E(2B_s + B_t - B_s) = E(2B_s) + E(B_t - B_s) = 0$ and

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- $B_s + B_t \sim N(0, 3s + t)$.

Exercise

Find the covariance of B_s and B_t.

Markov property

Brownian motion is a Markov process. Let $(B_t)_{t\geq 0}$ be a Brownian motion under the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$. Then B_t is also a Markov process.

Exercises

Using Holder's inequality $E[|X_t|] \leq \sqrt{E[X_t^2]}$ to prove:

1. $E[|B_t|] \leq \sqrt{t}$.
2. $E[|B_t^2 - t|] < 2t$

Martingale property

Prove the following random processes are martingale:

1. Brownian motion (B_t) is a martingale process.
2. Let $Y_t = B_t^2 - t$, for $t \geq 0$. Show that $(Y_t)_{t\geq 0}$ is a martingale with respect to Brownian motion.
3. $(Z_t)_{t\geq 0}$, with $Z_t = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$ (σ is a positive constant), is a martingale with respect to the standard Brownian motion.

Solution

- Because B_t is completely determined under the information available in σ-algebra $\mathcal{F}_t$, is Z_t so. In other words, $(Z_t)_{t\geq 0}$ is an adapted process with the filtration $(\mathcal{F}_t)_{t\leq 0}$.

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- For $0 \leq s \leq t$, we have

$$
\begin{aligned}
E[Z_t|\mathcal{F}_s] &= E[\exp(\sigma B_t - \frac{1}{2}\sigma^2 t)|\mathcal{F}_s] \\
&= \exp(-\frac{1}{2}\sigma^2 t)E[\exp(\sigma(B_t - B_s + B_s))|\mathcal{F}_s] \\
&= \exp(\sigma B_s - \frac{1}{2}\sigma^2 t)E[\exp(\sigma(B_t - B_s))|\mathcal{F}_s] \\
&= \exp(\sigma B_s - \frac{1}{2}\sigma^2 t)E[\exp(\sigma(B_t - B_s))]
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Solution

- As $\sigma(B_t - B_s) \sim N(0, \sigma^2(t-s))$,
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- (Z_t) is a martingale process.

Example

Let $S_t = S_0 e^{X_t}$, with $X_t = \mu t + \sigma B_t$, and $(B_t)_{t \geq 0}$ is the standard Brownian motion. Let $r = \mu + \sigma^2/2$. Show that $e^{-rt} S_t$ is a martingale with respect to standard Brownian motion.

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- We have $e^{-rt} S_t = e^{-(\mu + \sigma^2/2)t} S_0 e^{\mu t + \sigma B_t} = S_0 e^{\sigma B_t - (\sigma^2/2)t}$.
- As $e^{\sigma B_t - (\sigma^2/2)t}$ is a martingale with respect to the standard Brownian motion, so is $e^{-rt} S_t$.

Quadratic variation of Brownian motion

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- It can be proved that

$$[W, W](T) = \lim_{\Pi \to 0} \sum_{j=0}^{n-1} [W(t_{j+1}) - W(t_j)]^2 = T, \forall T \geq 0 \text{ almost}$$

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surely (there might be some paths of the Brownian motion for which the assertion $[W, W](T) = T$ is not true, but the set of such paths has zero probability).
- Brownian motion accumulates quadratic variation at rate one per unit time: $dB_t^2 = dt$

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- The probability density function of X_t is given by $K_t(x, y) = \dfrac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/(2t)}$.

Exercise

1. Write a R code to simulate a Brownian motion start at $x = 5$.
2. A particle's position is modelled by a Brownian motion start at $x = 1$. Find the probability that its position is at most 3 at time $t = 3$.

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- $P(X_2 > 0) = P(B_t + 3 > 0) = P(B_t > -3) = \int_{-3}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-x^2/4} dx = 0.983.$

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- In R, type 1-pnorm(-3,0,sqrt(2)) will help us compute the integral.

Conditional distribution of Brownian motion

Compute $E[B_4|B_2]$ and $Var[B_4|B_2]$.

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- $Var[B_4|B_2] = Var[B_4 - B_2 + B_2|B_2] = Var[B_4 - B_2|B_2] + Var[B_2|B_2] = Var[B_4 - B_2] + 0 = 2.$

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- $f_Y(y) = \dfrac{1}{\sqrt{10\pi}} e^{-x^2/10}$
- $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{f_{X,U}(x, y-x)}{f_Y(y)} = \dfrac{f_X(x) f_U(y-x)}{f_Y(y)} =$

$$\frac{1}{\sqrt{2\pi\frac{6}{5}}} \exp\left(-\frac{(x - \frac{2y}{5})^2}{2\frac{6}{5}}\right)$$

Conditional distribution of Brownian motion

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Conditional distribution of Brownian motion

Compute $E[B_2|B_5]$ and $Var[B_2|B_5]$.

- The conditional distribution of B_2 given B_5 is normal with mean $2y/5$ and variance $6/5$.

Conditional distribution of Brownian motion

Compute $E[B_2|B_5]$ and $Var[B_2|B_5]$.

- The conditional distribution of B_2 given B_5 is normal with mean $2y/5$ and variance $6/5$.
- $E[B_2|B_5 = y] = 2y/5$ and $Var[B_2|B_5 = y] = 6/5$

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- $E[B_2|B_5] = 2B_5/5$ and $Var[B_2|B_5] = 6/5$

Stopping time

For a continuous-time stochastic process $(X_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$, a stopping time τ with respect to $(X_t)_{t\geq 0}$ is a non-negative random variable such that the event $\tau \leq t$ depends only on the information available up to time t, but not after time t. In other words, $\{\tau \leq t\} \in \mathcal{F}_t$.

Stopping time

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- τ is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$ if at any time t, it can be decided whether the event $\tau \leq t$ has already occurred or not, just based on the information contained in $\mathcal{F}_t$ generated by $(X_s)_{0\leq \leq t}$.

Stopping time:example

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- The day you will buy your last car is not a stopping time.

Stopping time: Proposition

Let τ and θ be stopping times with respect to the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$.
i) Every constant time is a stopping time.
ii) The minimum $\tau \wedge \theta = \min(\tau, \theta)$ is also a stopping time.
iii) The maximum $\tau \vee \theta = \max(\tau, \theta)$ is also a stopping time.

Proof

i) Let a is a constant time. The event $\{a \leq t\}$ is clearly belong to $\mathcal{F}_t$.
ii) $\{\tau \wedge \theta > t\} = \{\min(\tau, \theta) > t\} = \{\tau > t\} \cap \{\theta > t\} \in \mathcal{F}_t$. Thus $\{\tau \wedge \theta \leq t\} = \{\tau \wedge \theta > t\}^C \in \mathcal{F}_t$.
iii) $\{\tau \vee \theta \leq t\} = \{\max(\tau, \theta) \leq t\} = \{\tau \leq t\} \cap \{\theta \leq t\} \in \mathcal{F}_t$.

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- If $(B_t)_{t\geq 0}$ never reaches the level m, we set $\tau_m = \infty$.
- τ_m is a stopping time: $\{\tau_m \leq t\}$ is equivalent to $\max\limits_{0\leq s\leq t} B_s \leq m$, which is in $\mathcal{F}_t$.

Strong Markov property

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and let τ be a stopping time with respect to $(B_t)_{t\geq 0}$. Define $Z(t) = B_{t+\tau} - B_\tau$ then

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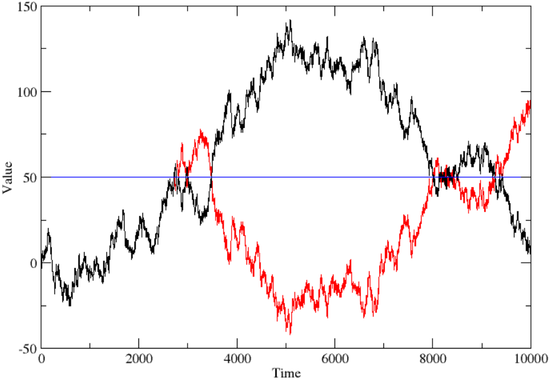
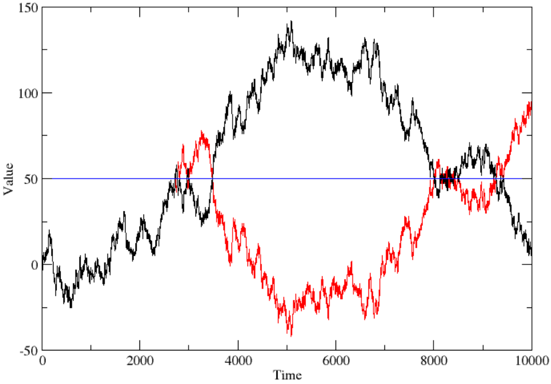
Reflection principle of Brownian motion



Reflection principle of Brownian motion

- $P(\tau_m \le t, B_t \ge m) = P(\tau_m \le t, B_t \le m)$



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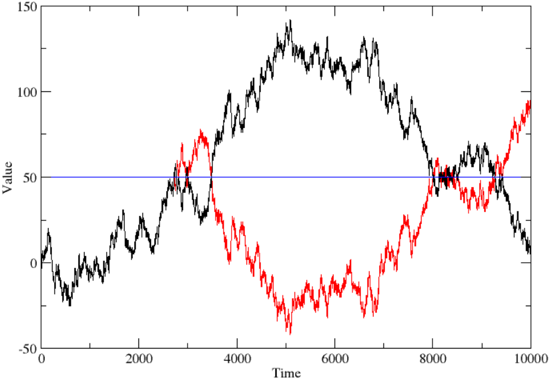

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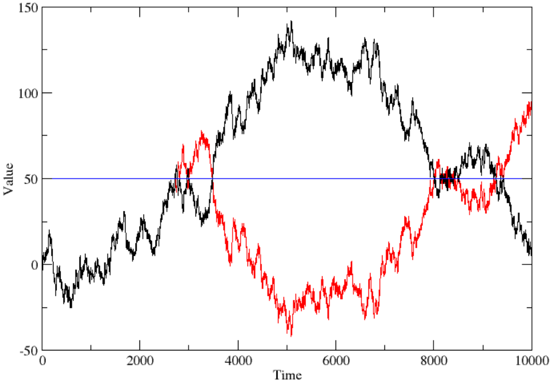

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- $f_{B_t}(x) = \dfrac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$.
- $P(\tau_7 \leq t) = \dfrac{2}{\sqrt{2\pi t}} \displaystyle\int_7^{\infty} e^{-x^2/(2t)} dx =$
  $\dfrac{2}{\sqrt{2\pi}} \displaystyle\int_{7/\sqrt{t}}^{\infty} e^{-y^2/2} dy$, with $y = x/\sqrt{t}$.

First passage time distribution

The density function: $f_{\tau_7}(t) = \dfrac{d}{dt} P(\tau_7 \leq t) = \dfrac{7}{\sqrt{2\pi t^3}} e^{-\frac{49}{2t}}, t \geq 0.$

$$P(\tau_7 < \infty) = \lim_{t \to \infty} \frac{2}{\sqrt{2\pi}} \int_{7/\sqrt{t}}^{\infty} e^{-y^2/2} dy = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-y^2/2} dy = 1$$

First passage time distribution

Let $\tau_{-3} = \inf\{t \geq 0 | W_t = -3\}$. Find the cumulative distribution function and probability density function of τ_{-3}.

Example

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- The desired probability is
$$P(\tau_{0.5} \leq 1) = \int_0^1 \frac{0.5}{\sqrt{2\pi t^3}} e^{-0.5^2/(2t)} dt = 0.617.$$

Example

A particle moves according to Brownian motion started at $x = 2$. After $t = 2$ hours, the particle is at level 1. Find the probability that the particle reaches level -2 sometime in the next two hours.

First passage time

For all $m \neq 0$, the random variable τ_m has cumulative distribution function

$$P(\tau_m \leq t) = \frac{2}{\sqrt{2\pi}} \int_{|m|/\sqrt{t}}^{\infty} e^{-y^2/2} dy$$

and the density function:

$$f_{\tau_m}(t) = \frac{d}{dt} P(\tau_m \leq t) = \frac{|m|}{t\sqrt{2\pi t}} e^{-m^2/(2t)}, t \geq 0.$$

First passage time

The first passage time distribution has some surprising properties. Consider

$$P(\tau_m < \infty) = \lim_{t\to\infty} P(\tau_m < t) = \lim_{t\to\infty} 2\int_{|m|/\sqrt{t}}^{+\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx$$
$$= 2\int_0^{+\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = 1.$$

This means the Brownian motion will hit the level m in a finite period of time, with probability 1, for all m, no matter how large the value of m.

Stopped process

Given $(Z_t)_{t\geq 0}$ a stochastic process and τ is a stopping time. The stopped process $(Z_{t\wedge\tau})_{t\geq 0}$ is defined as

$$(Z_{t\wedge\tau})_{t\geq 0} = \begin{cases} Z_t & \text{if} \quad t < \tau \\ Z_\tau & \text{if} \quad t \geq \tau \end{cases}.$$

Optional stopping theorem

Assume that $(M_t)_{t\geq 0}$ is a martingale and τ is a stopping time, with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Then the stopped process $(M_{t\wedge\tau})_{t\geq 0}$ is also a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

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- If $P(\tau < \infty) = 1$ then

$$E[M_\tau] = E[\lim_{t\to\infty} M_{\tau\wedge t}] = \lim_{t\to\infty} E[M_{\tau\wedge t}] = E[M_0].$$

Exercise

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion, $\tau_{-2} = \inf\{t \geq 0 | W_t = -2\}$, $\tau_3 = \inf\{t \geq 0 | W_t = 3\}$. Compute the probability that $\tau_3 < \tau_{-2}$.

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- Let $\tau = \inf\{t \geq 0 | W_t \notin (-2, 3)\}$
- $$\begin{aligned}\{\tau \leq t\} &= \{\max_{0\leq u\leq t} W_u \geq 3\} \cup \{\min_{0\leq u\leq t} W_u \leq -2\} \\ &= \{\tau_3 \leq t\} \cup \{\tau_{-2} \leq t\} \in \mathcal{F}_t\end{aligned}$$

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- $(W_t)_{t\geq 0}$ is a martingale

Exercise

- From the optional stopping theorem, $(W_{\tau\wedge t})_{t\geq 0}$ is a martingale and

$$E[W_\tau] = E[\lim_{t\to\infty} W_{\tau\wedge t}] = \lim_{t\to\infty} E[W_{\tau\wedge t}]$$
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Exercise

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- $P(W_\tau = 3) = \dfrac{2}{5}, P(W_\tau = -2) = \dfrac{3}{5}.$

Exercise

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- $$\begin{aligned} E[\tau] &= E[W_\tau^2] = 3^2 P(W_\tau = 3) + (-2)^2 P(W_\tau = -2) \\ &= 9\frac{2}{5} + 4\frac{3}{5} = 6. \end{aligned}$$

Maximum of Brownian motion

- Let (B_t) be the standard Brownian motion. Then $(M_t)_{t\geq 0}$, where $M_t = \max\limits_{0\leq s\leq t} B_s$, forms the maximum process of Brownian motion

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- $P(M_t > a) = P(\tau_a < t) = \int_0^t \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/(2s)} ds =$

$\int_a^{+\infty} \sqrt{\frac{2}{\pi t}} e^{-x^2/(2t)} dx, \quad a^2/s = x^2/t.$

Example

A laboratory instrument takes annual temperature measurements. Measurement errors are assumed to be independent and normally distributed. As precision decreases over time, errors are modeled as standard Brownian motion. For how many years can the lab be guaranteed that there is at least 90% probability that all errors are less than 4 degrees, given that the 95th percentile of the standard normal random distribution is 1.645?

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- The 95th percentile of the standard normal random distribution is 1.645. Thus the desired value t should satisfy $4/\sqrt{t} \geq 1.645 \Leftrightarrow t \leq 5.91$ years.

Example

A laboratory instrument takes annual temperature measurements. Measurement errors are assumed to be independent and normally distributed. As precision decreases over time, errors are modeled as standard Brownian motion. For how many years can the lab be guaranteed that there is at least 80% probability that all errors are less than 5 degrees, given that the 90th percentile of the standard normal random distribution is 1.28?

Zeros of Brownian motion

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- The times when the process crosses the t-axis are the zeros of Brownian motion.
- For $0 \leq r < t$, let $z_{r,t}$ be the probability that standard Brownian motion has at least one zero in (r, t). Then $z_{r,t} = \frac{2}{\pi} \arccos(\sqrt{\frac{r}{t}})$.

Zeros of Brownian motion

- $z_{r,t} = P(B_s = 0), \text{for some } s \in (r,t) = \int_{-\infty}^{+\infty} P(B_s = 0 | B_r = x) \frac{1}{\sqrt{2\pi r}} e^{-x^2/(2r)} dx, \text{for some } s \in (r,t)$

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- For $x < 0$, $P(B_s = 0 | B_r = x) = P(M_t \geq 0 | B_r = x) = P(M_t \geq -x | B_r = 0) = P(M_{t-r} > -x | B_0 = 0) = P(M_{t-r} > -x)$.

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- For $x \geq 0$, we consider the reflected Brownian motion $P(B_s = 0|B_r = x) = P(B_s = 0|B_r = -x) = P(M_t \geq 0|B_r = -x) = P(M_t \geq x|B_r = 0) = P(M_{t-r} > x)$. Thus we can express $P(B_s = 0|B_r = x) = P(M_{t-r} > |x|)$

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- $z_{r,t} = \int_{-\infty}^{+\infty} P(M_{t-r} > |x|)\frac{1}{\sqrt{2\pi r}}e^{-x^2/(2r)}dx = \frac{2}{\pi}\arccos(\sqrt{\frac{r}{t}})$.

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- By the strong Markov property, for Brownian motion restarted at t, there is at least one zero in (t, ϵ), with probability 1.
- Continuing this way, there are infinitely many zeros in $(0, \epsilon)$.

Brownian motion with drift

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- $$P(W_t - W_s \leq k|\mathcal{F}_s) = P(\mu(t-s) + \sigma(B_t - B_s) \leq k|\mathcal{F}_s)$$
  $$= P(B_t - B_s \leq \frac{k - \mu(t-s)}{\sigma}|\mathcal{F}_s) = P(B_t - B_s \leq \frac{k - \mu(t-s)}{\sigma})$$
  $$= P(\mu(t-s) + \sigma(B_t - B_s) \leq k) = P(W_t - W_s \leq k).$$

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- $$\begin{aligned} & P(W_t - W_s \leq k|\mathcal{F}_s) = P(\mu(t-s) + \sigma(B_t - B_s) \leq k|\mathcal{F}_s) \\ & = P(B_t - B_s \leq \frac{k - \mu(t-s)}{\sigma}|\mathcal{F}_s) = P(B_t - B_s \leq \frac{k - \mu(t-s)}{\sigma}) \\ & = P(\mu(t-s) + \sigma(B_t - B_s) \leq k) = P(W_t - W_s \leq k). \end{aligned}$$
- $$\begin{aligned} W_t - W_s &= \mu(t-s) + \sigma(B_t - B_s) \\ &\overset{d}{=} \mu(t-s) + \sigma B_{t-s} \overset{d}{=} W_{t-s}, \ \forall\, 0 < s < t. \end{aligned}$$

Model Brownian motion with drift

Suppose we want to model a Brownian motion with drift: $dW_t = \mu dt + \sigma dB_t$ in a time interval $[0, T]$, where $(B_t)_{t \geq 0}$ be a standard Brownian motion.

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$$W_{t_i} = W_{t_{i-1}} + \mu * h + \sigma\sqrt{h} * X,$$

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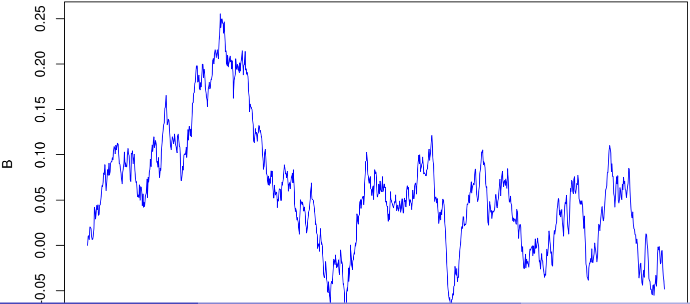
  where $X \sim N(0, 1)$.
- All we need to simulate a generalized Brownian motion is to simulate a standard normal random variable, which can be done by using command "rnorm(1)" in software R.

R code for simulating Brownian motion with drift

```
BM=function(mu,sigma,T,N) {
# mu: the drift term of Brownian motion
# sigma: the diffusion term of Brownian motion
# T: expiry time
# N: number of simulation points
h=T/N # the timestep of the simulation
X=rep(0, (N+1)) # create space to store values of Brownian
    motion
X[1]=0
for(i in 1:N) { X[i+1]=X[i] +mu*h+sigma*sqrt(h)*rnorm(1)}
return(X)
}
```

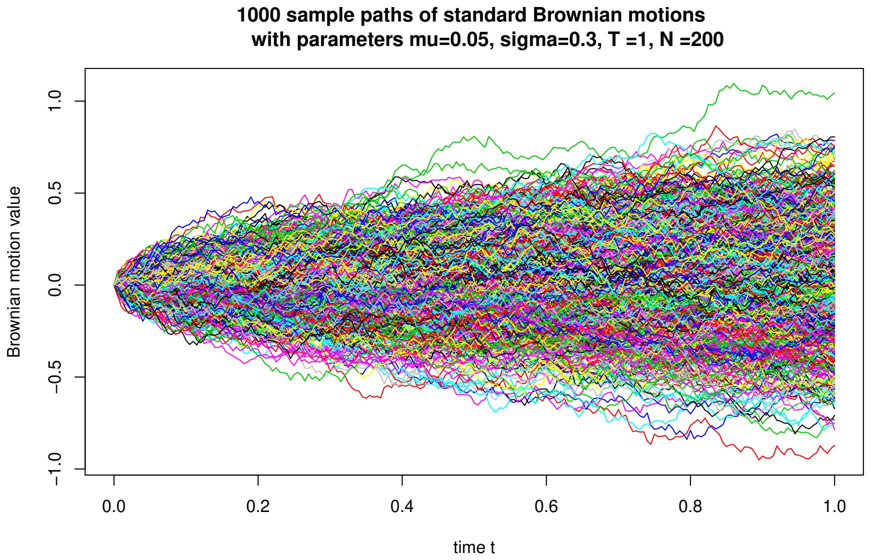
A sample of standard Brownian motion

```
h <- 1/1000
t=seq(0,1,by=h)
plot(t,BM(mu=0.05,sigma=0.3,T=1,N=1000), type="l",
    col="blue",xlab="t",ylab="B")
```



Plot multiple samples of Brownian motion with drift

```
BMSamplepaths <- function(mu,sigma,T,N,nt)
{h=T/N ## time step
# number of discrete points
#nt: number of samples
t=seq(0,T,by=h) # produce a sequence of time points
X=matrix(rep(0,length(t)*nt), nrow=nt)
## each row of the matrix stores one sample
# #return(X)
for (i in 1:nt) {X[i,]= BM(mu=mu,sigma=sigma,T=T,N=N)}
# # ##Plot
ymax=max(X); ymin=min(X) #bounds for simulated prices
plot(t,X[1,],t='l',main='1000 sample paths of standard
    Brownian motions
with parameters mu=0.05, sigma=0.3, T =1, N
```


1000 sample paths of standard Brownian motions
with parameters mu=0.05, sigma=0.3, T =1, N =200
Brownian motion value
−1.0
−0.5
0.0
0.5
1.0
0.0
0.2
0.4
0.6
0.8
1.0
time t

Example

Find the probability that Brownian motion with drift parameter $\mu = 0.6$ and variance $\sigma^2 = 0.25$ takes values between 1 and 3 at time $t = 4$.

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- Brownian motion with drift parameter $\mu = 0.6$ and variance $\sigma^2 = 0.25$ is $W_t = 0.6t + 0.5B_t$.
- $P(1 \leq W_4 \leq 3) = P(1 \leq 0.6 * 4 + 0.5 * B_4 \leq 3) = P(-2.8 \leq B_4 \leq 1.2) = \int_{-2.8}^{1.2} \frac{e^{-x^2/8}}{\sqrt{8\pi}} dx = 0.645.$

Brownian bridge

- Brownian bridge is a Brownian motion starting at x at time t_0 and passing through some point y at time $T > t_0$.

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- Brownian bridge is a Brownian motion starting at x at time t_0 and passing through some point y at time $T > t_0$.
- $W_{t_0,x}^{T,y}(t) = x + W_{t-t_0} - \dfrac{t-t_0}{T-t_0}(W_{T-t_0} - y + x).$

Exercise: Simulating Brownian bridge

Write a R code to simulate the Brownian motion