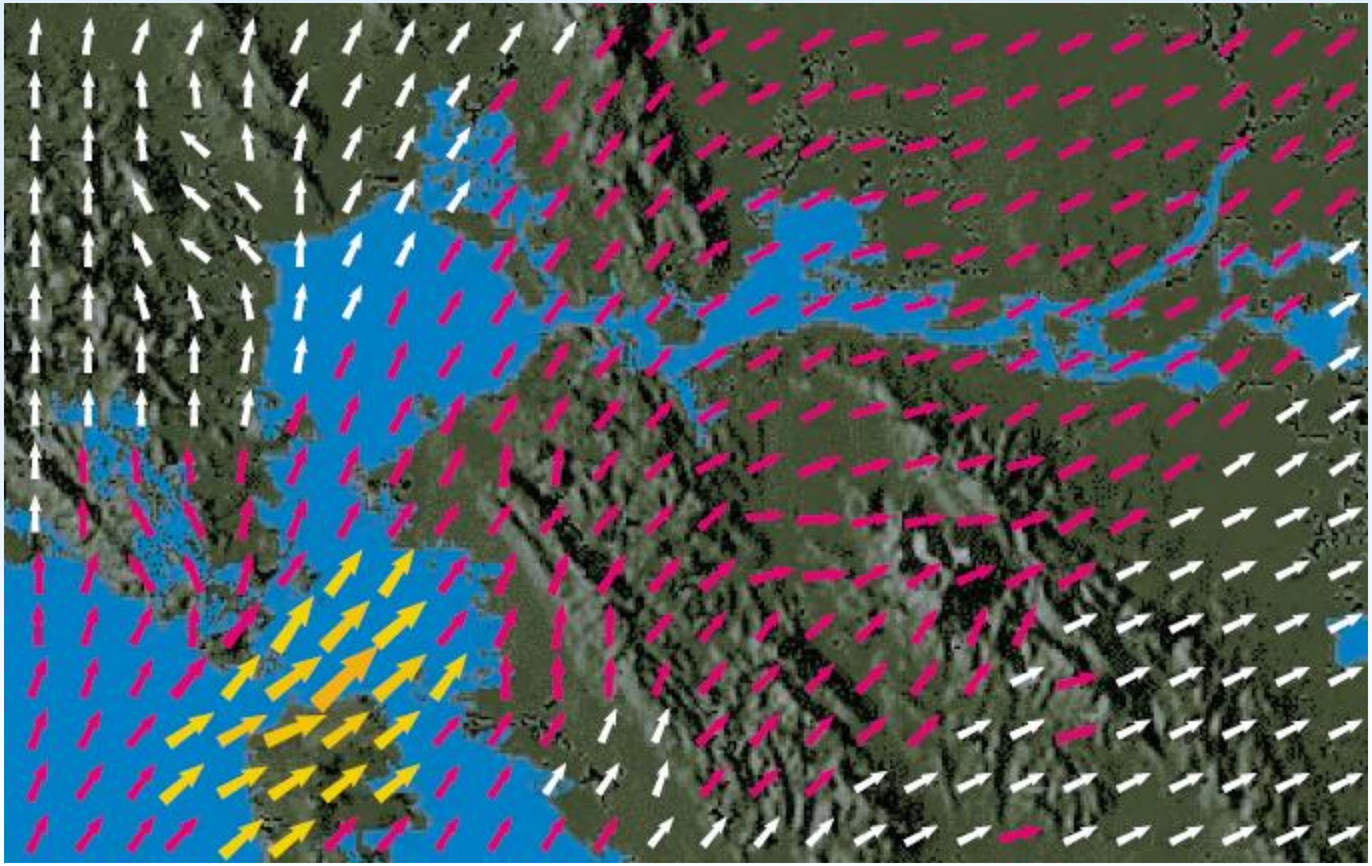
Chapter 4: Vector Calculus

Lecture 12

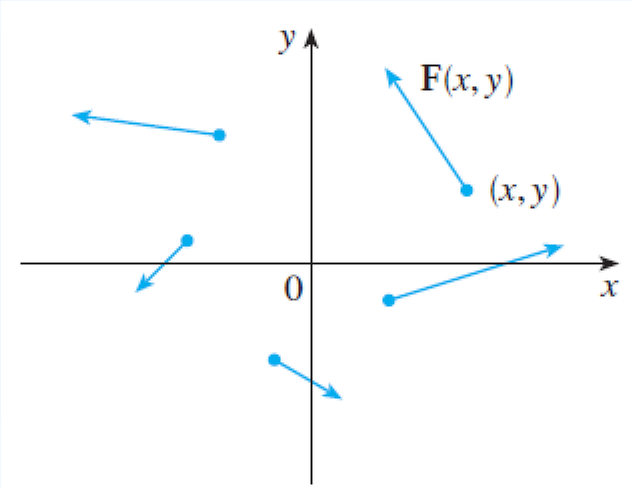
- **Vector Fields**
- **Line Integrals**

1. Vector Fields: Example

Air velocity vectors that indicate the wind speed and direction at points



- Let D be a set in $\mathbf{R}^2$ (a plane region). A **vector field on D** is a function F that assigns to each point (x, y) in D a two-dimensional vector F(x, y)



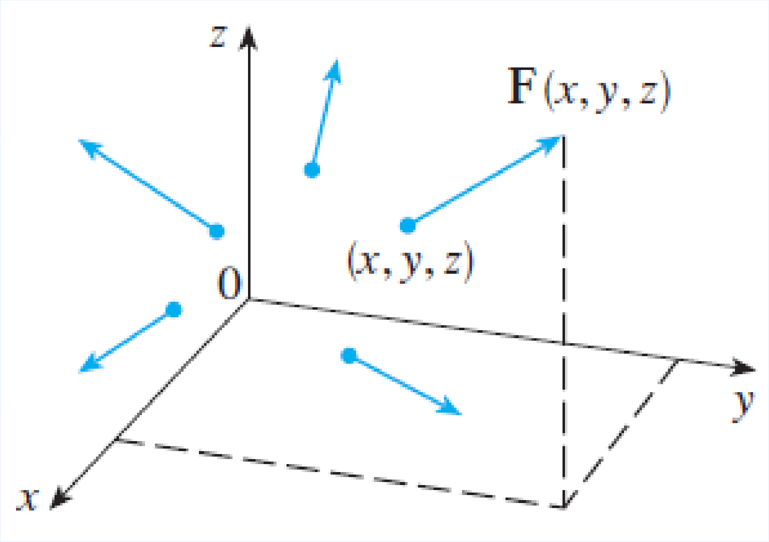
The best way to picture a vector field is to draw the arrow representing the vector F(x,y) starting at the point (x,y)

Since *F(x,y)* is a two-dimensional vector, we can write it in terms of its **component functions:**

$$F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} = < P(x,y), Q(x,y) >$$

$$\text{or} \quad F = P\vec{i} + Q\vec{j}$$

- Let E be a subset of $\mathbf{R}^3$. A **vector field on E** is a function F that assigns to each point (x, y, z) in E a unique vector F(x, y, z) in 3D



$$F(x,y,z) = P(x,y,z)\vec{\mathbf{i}} + Q(x,y,z)\vec{\mathbf{j}} + R(x,y,z)\vec{\mathbf{k}}$$

$$= < P(x,y,z), Q(x,y,z), R(x,y,z) >$$

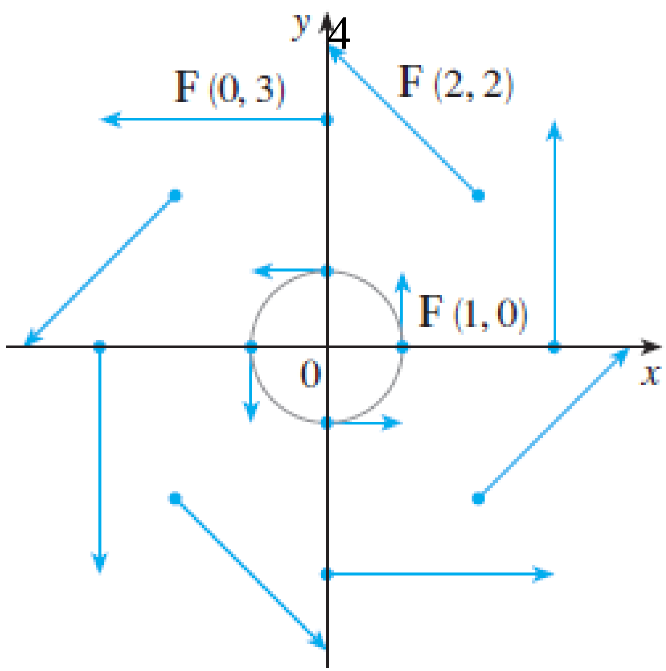
or $F = P\vec{\mathbf{i}} + Q\vec{\mathbf{j}} + R\vec{\mathbf{k}}$

Example: A vector field F(x,y)=-y**i**+x**j**. Describe F by sketching some of the vectors F(x,y)

Solution

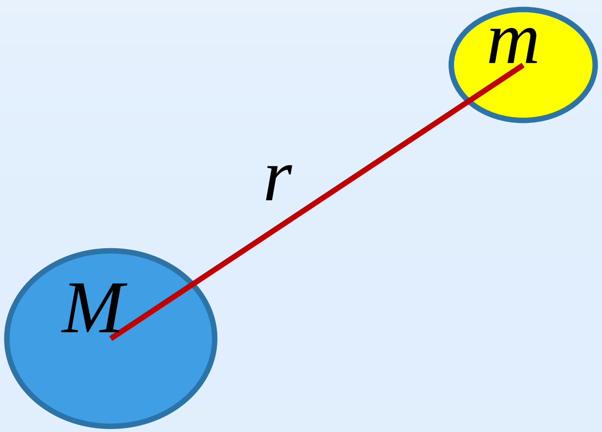
Since $\mathbf{F}(1, 0) = \mathbf{j}$, we draw the vector $\mathbf{j} = \langle 0, 1 \rangle$ starting at the point $(1, 0)$

Since $\mathbf{F}(0, 1) = -\mathbf{i}$, we draw the vector $\langle -1, 0 \rangle$ with starting point $(0, 1)$.



(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1 \rangle$	$(-1, 0)$	$\langle 0, -1 \rangle$
$(2, 2)$	$\langle -2, 2 \rangle$	$(-2, -2)$	$\langle 2, -2 \rangle$
$(3, 0)$	$\langle 0, 3 \rangle$	$(-3, 0)$	$\langle 0, -3 \rangle$
$(0, 1)$	$\langle -1, 0 \rangle$	$(0, -1)$	$\langle 1, 0 \rangle$
$(-2, 2)$	$\langle -2, -2 \rangle$	$(2, -2)$	$\langle 2, 2 \rangle$
$(0, 3)$	$\langle -3, 0 \rangle$	$(0, -3)$	$\langle 3, 0 \rangle$

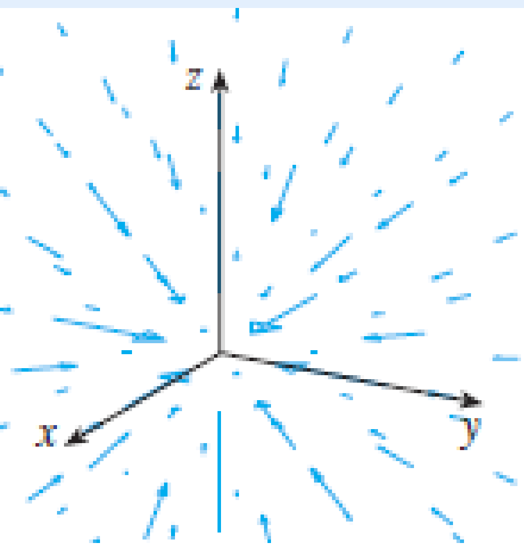
Example 2: Gravitational field



Newton's Law of Gravitation: gravitational force between m and M is $|F| = \dfrac{gmM}{r^2}$

r: distance between m & M

g : gravitational constant.



Assume: Center of M = origin in R^3

Position vector of m is **u**=<x,y,z>. Then r=|**u**|.
The unit vector in this direction is –**u**/|**u**|
Gravitational force acting on m at **u**=<x,y,z> is

$$\boxed{F(u) = -\frac{gmM}{|u|^3}u}$$

Gradient fields

- Recall: gradient of z=f(x, y) is defined by

$$\boxed{\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}}$$

- Gradient of w=f(x,y,z), the gradient is

$$\boxed{\nabla f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}}$$

- Gradient is a vector field, called **gradient field**

Conservative Fields

- A vector field F is called a **conservative vector field** if it is the gradient of some scalar function f: $F = \nabla f$
- The function f is called a **potential function for** F
- **<u>Note:</u>** Not all vector fields are conservative, but such fields do arise frequently in physics.

Example

Gravitational field **F is conservative:**

$$f(x,y,z)=\frac{gmM}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla f(x,y,z)=<\frac{-gmMx}{(x^2+y^2+z^2)^{3/2}},\frac{-gmMy}{(x^2+y^2+z^2)^{3/2}},\frac{-gmMz}{(x^2+y^2+z^2)^{3/2}}>$$

$$=-\frac{gmM}{|u|^3}u=F(x,y,z)$$

2. Line Integrals

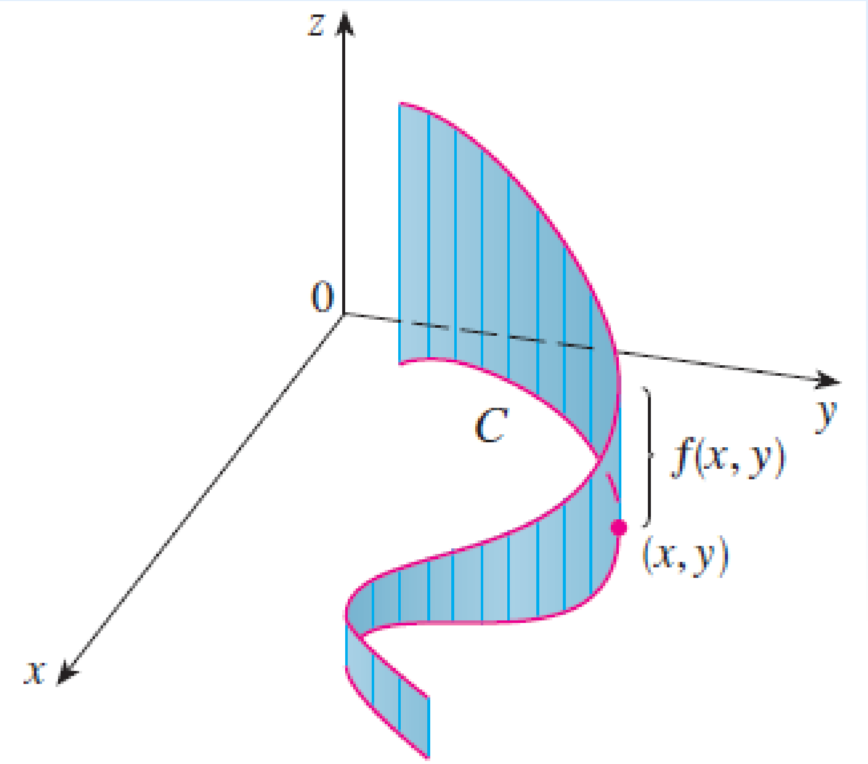
- Line Integrals of a (real-valued) Function
- Line Integrals of a Vector Field

2.1 Line Integrals of a function

A curve C is given by parametric equations:

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

or by vector equation: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$



$z=f(x,y)$ is a function defined on C

Divide $[a,\ b]$ into subintervals

$[t_{i-1}, t_i]$

of equal width, $i = 1, 2, ..., n$

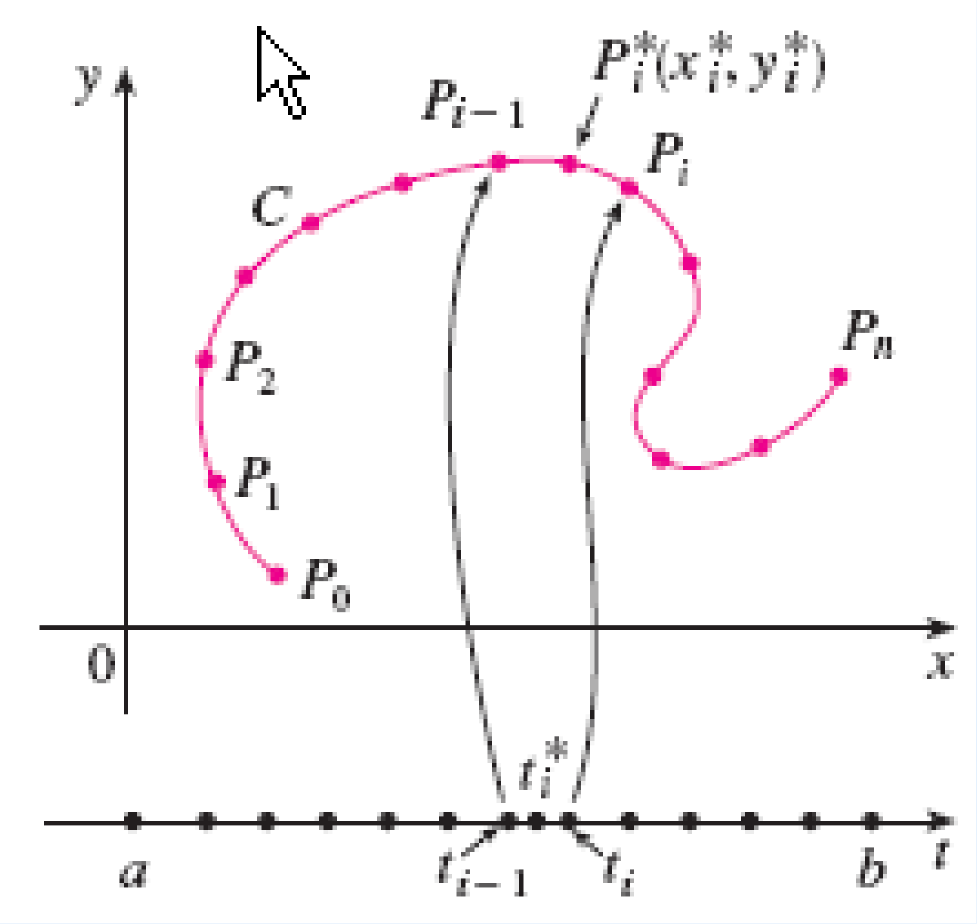
Let $x_i = x(t_i)$, and $y_i = y(t_i)$

$P_i(x_i, y_i)$ divide C into

n arcs $P_{i-1}P_i$,

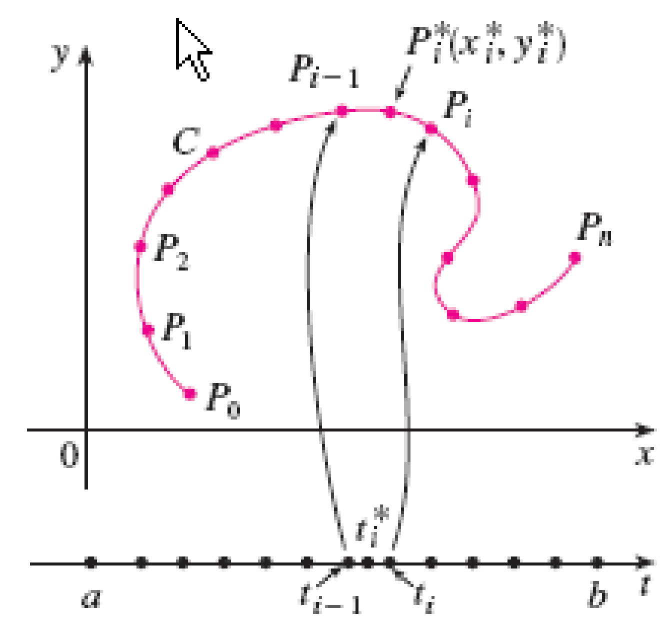
$i = 1, 2, ..., n$,

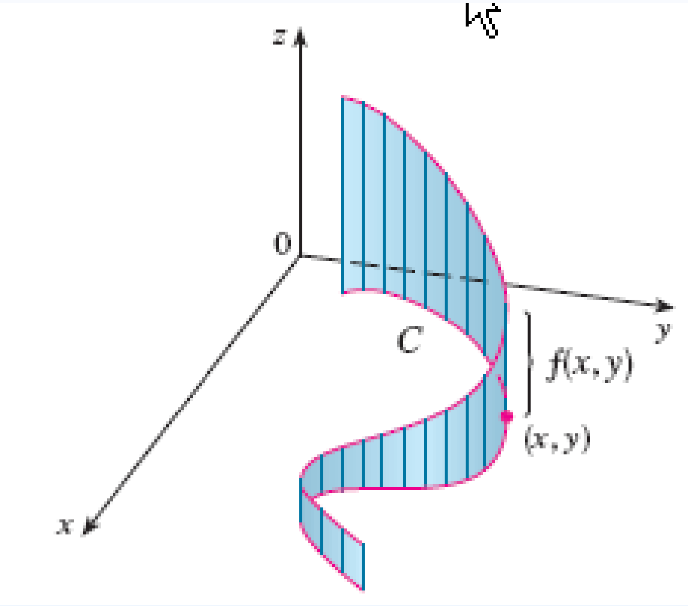
with lengths $\Delta s_1, \Delta s_2, ..., \Delta s_n$



Choose any $P_i^*(x_i^*, y_i^*) \in P_{i-1}P_i$, $\quad i = 1, 2, ..., n$

where $(x_i^*, y_i^*) = (x(t_i^*),\ y(t_i^*))$





Calculate $\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$

Definition. If f is defined on a smooth curve C, the **line integral of f along C** is the limit, if it exists:

$$\int_C f(x,y)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

Evaluate Line Integrals

$$\Delta s_i \approx |P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$\text{where } \Delta x_i = x_i - x_{i-1}, \Delta y_i = y_i - y_{i-1}$$

$$\Delta x_i \approx x'(t_i^*)\Delta t,\ \ \Delta y_i \approx y'(t_i^*)\Delta t$$

$$\Delta s_i \approx |P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$\approx \sqrt{[x'(t_i^*)\Delta t]^2 + [y'(t_i^*)\Delta t]^2} = \sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2}\,\Delta t$$

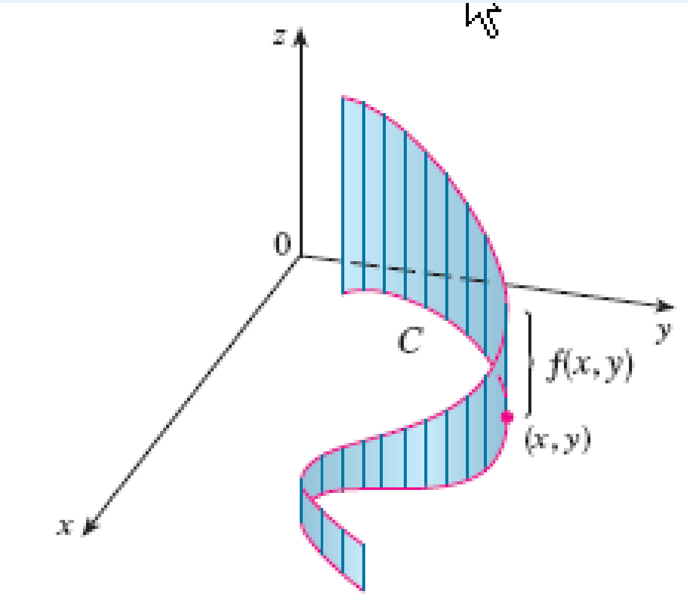
$$\int_C f(x,y)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x(t_i^*), y(t_i^*))\sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2}\,\Delta t$$

=Integral of $g(t) = f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}$ on $[a,b]$

Formula for Evaluating line Integrals

$$C: \quad x = x(t), y = y(t), \quad a \le t \le b$$

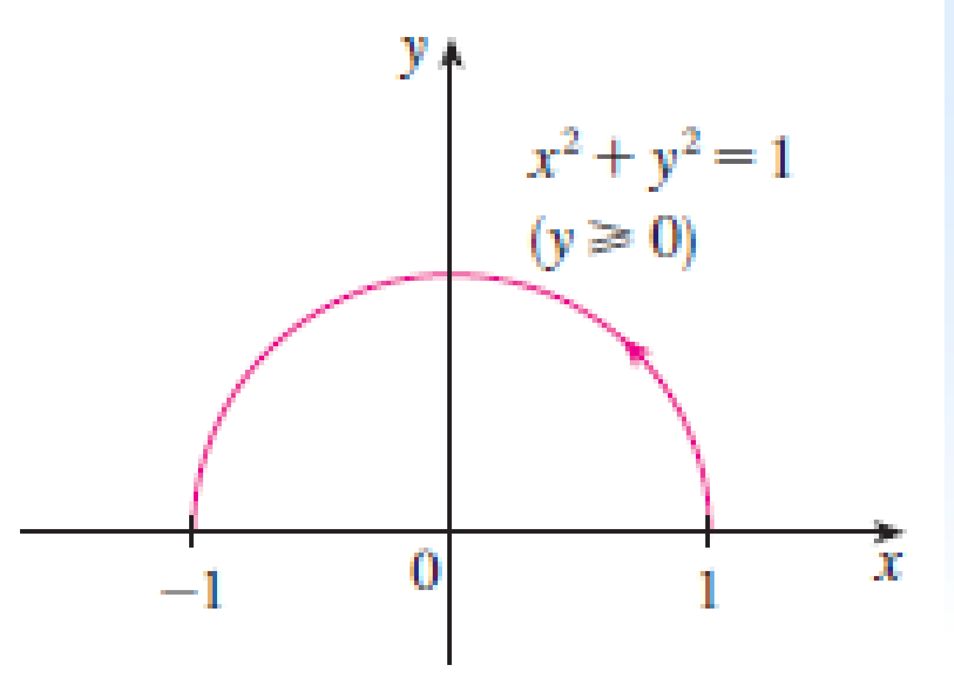


If $z=f(x,y)$ is continuous, then the line integral of f along C is defined by

$$\boxed{\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt}$$

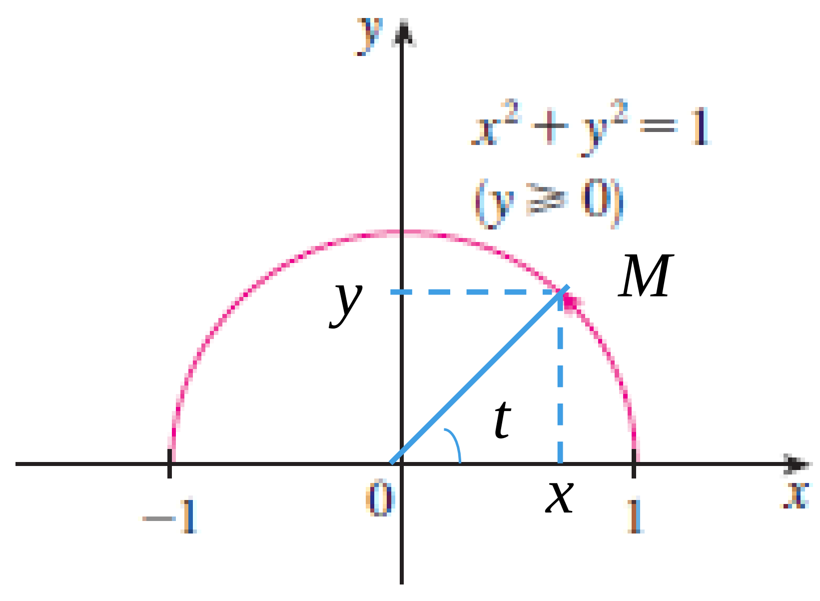
Example 1

Evaluate $\int_C (1+6x^2y)\,ds$, where C is the upper half of the unit circle $x^2+y^2=1$



Solution

$$M(x,y)\in C: \quad x = OM\cos t$$
$$y = OM\sin t$$



$$C: \quad x=\cos t,\ y=\sin t, \quad 0\le t\le \pi$$

$$x'(t) = -\sin t,\ y'(t)=\cos t$$

$$\int_C (1+6x^2y)ds = \int_0^{\pi}(1+6\cos^2 t\sin t)\sqrt{\sin^2 t+\cos^2 t}\,dt =$$

$$= \int_0^{\pi}(1+6\cos^2 t\sin t)dt = t\Big|_0^{\pi} + \int_0^{\pi} 6\cos^2 t\sin t\,dt$$

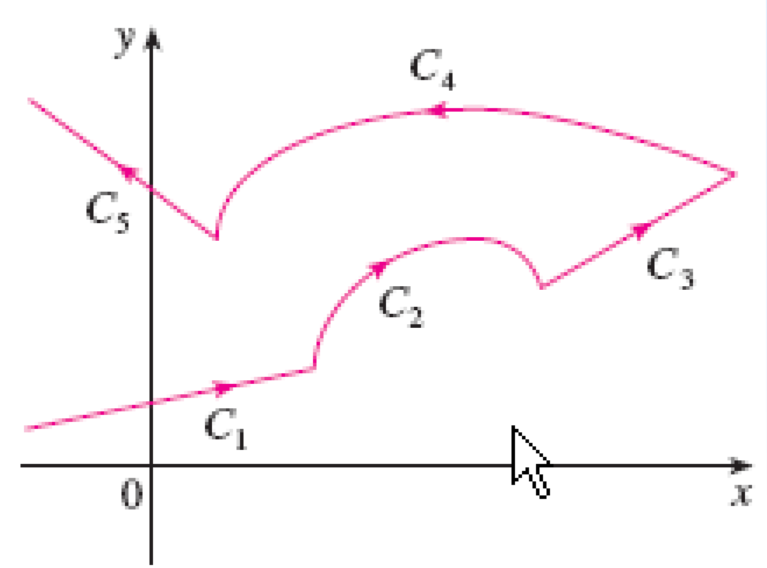
$$= \pi + \int_0^{\pi} -6\cos^2 t\,d(\cos t) = \pi - 2\cos^3 t\Big|_0^{\pi} = \pi + 4$$

Remarks

- If C is a **piecewise-smooth curve: =** a union of curves C_1, C_2, …, C_n so that the initial point of C_{i+1} is the terminal point of C_i. Then

$$\boxed{\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \ldots + \int_{C_n} f(x,y)\,ds}$$

- The line integral $\int_C f(x,y)\,ds$ is called **line integral with respect to arc length.**



Mass and Mass Center of a Wire

- A thin wire has the shape of a curve C.

$\rho(x, y)$: density at point $(x, y) \in C$

$$\boxed{m = \int_C \rho(x, y)\,ds : \text{ mass of } C}$$

Center of mass $(\bar{x}, \bar{y})$, where

$$\boxed{\bar{x} = \frac{1}{m}\int_C x\rho(x, y)\,ds, \qquad \bar{y} = \frac{1}{m}\int_C y\rho(x, y)\,ds}$$

Example

- Find the mass and center of mass of a thin wire in the shape of a quarter-circle $x^2+y^2=r^2,\ \ x\ge 0, y\ge 0,$ and $\rho(x,y)=x+y$

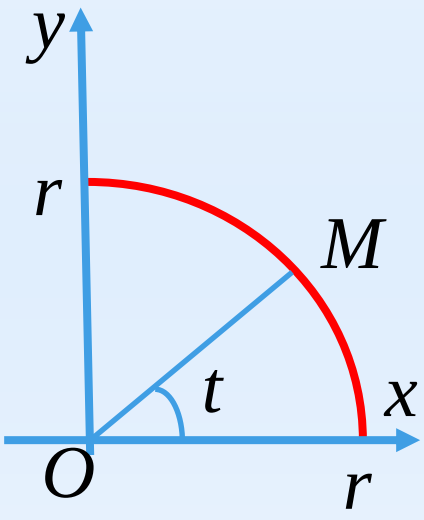
<u>**Solution**</u>

$$x=r\cos t, y=r\sin t,\quad 0\le t\le \pi/2$$

$$x'=-r\sin t, y'=r\cos t \Rightarrow x'^2+y'^2=r^2$$

$$m=\int_C (x+y)ds=\int_0^{\pi/2} r(\cos t+\sin t)\sqrt{x'^2+y'^2}dt$$

$$=\int_0^{\pi/2} r(\cos t+\sin t)rdt=r^2(\sin t-\cos t)\Big|_0^{\pi/2}=2r^2$$



Center of mass:

$$x = r\cos t, y = r\sin t, \quad 0 \le t \le \pi/2$$

$$x' = -r\sin t, y' = r\cos t \Rightarrow x'^2 + y'^2 = r^2$$

$$\bar{x} = \frac{1}{m}\int_C x(x+y)ds = \frac{1}{2r^2}\int_0^{\pi/2}(r\cos t)r(\cos t + \sin t)\sqrt{x'^2 + y'^2}dt$$

$$= \frac{r}{2}\int_0^{\pi/2}\cos t(\cos t + \sin t)dt = \frac{r}{2}\int_0^{\pi/2}(\cos^2 t + \cos t\sin t)dt$$

$$= \frac{r}{2}\int_0^{\pi/2}((1+\cos 2t)/2 - \cos t(\cos t)')dt = \frac{r}{2}(t/2 + \sin 2t/4 - \cos^2 t/2)\Big|_0^{\pi/2}$$

$$= \frac{r}{4}(\pi + 1)$$

$$\bar{y} = \frac{1}{m}\int_C y(x+y)ds = \frac{1}{2r^2}\int_0^{\pi/2}(r\sin t)r(\cos t + \sin t)\sqrt{x'^2 + y'^2}dt = \ldots$$

Line Integrals with respect to x and y

- **Line integrals of *f* along C with respect to x and y** are defined by

$$\int_C f(x,y)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i$$

$$\int_C f(x,y)\,dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i$$

Evaluation

- **Line integrals of *f* along C with respect to x and y** are evaluated by

$$C: \quad x = x(t), y = y(t), \quad a \le t \le b$$

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))x'(t)dt$$

$$\int_C f(x,y)dy = \int_a^b f(x(t), y(t))y'(t)dt$$

Remark

- It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to write

$$\boxed{\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy}$$

Line Integrals in Space

- Let C be a smooth curve in space given by the parametric equations

$$C: \quad x = x(t), \;\; y = y(t), \;\; z = z(t), \;\; a \le t \le b$$

- We define the **line integral of *f* along *C*** (with respect to arc length) in a manner similar to that for plane curves

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$

Evaluating line integrals in space

- Line integrals in space can be evaluated by

$$C: \quad x = x(t), y = y(t), z = z(t), \quad a \le t \le b$$

$$\mathbf{r}(t) = < x(t), y(t), z(t) >$$

$$\int_C f(x, y, z)ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

$$\boxed{\int_C f(x, y, z)ds = \int_a^b f(\mathbf{r}(t))\,|\,\mathbf{r}'(t)\,|\,dt}$$

Line integrals with respect to x, y, z

- Line integrals with respect to *x*, *y*, *z* can also be defined. For example,

$$\int_C f(x,y,z)\,dz = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta z_i$$

$$= \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt$$

Line integrals with respect to x, y, z

- Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\boxed{\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz}$$

- By expressing everything (x, y, z, dx, dy, dz) in terms of the parameter *t*

Homework Chapter 4

- Section 16.1: 1, 2, 4
- Section 16.2: 2, 3, 6, 20, 34
- Section 16.3: 5, 6, 12, 16
- Section 16.5: 6, 8, 9, 12, 15, 18
- Section 16.7: 5, 8, 10, 12, 16, 18, 25, 26