Optimization 2

(for FERM program)

Prof. DrSc. Nguyen Dinh

Department of mathematics
INTERNATIONAL UNIVERSITY, VNU-HCM

April 1, 2022

Contents

1. Transportation problems
    1.1. Statement of the transportation problem
    1.2. Properties of transportation problems
2. Initial BF solutions for Transportation problems
3. Straemlined simplex method
4. Assignment problem

1.1. Statement of the transportation problem.

The Transportation problem is special case of the network flow problems with the network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, where

$$\mathcal{N} = \mathcal{S} \cup \mathcal{D}, \quad \mathcal{S} \cap \mathcal{D} = \emptyset,$$

and with the following properties (see Figure 1):

• Nodes in $\mathcal{S}$ are called supply nodes (or sources). The amount of supply at source i is $s_i \geq 0$, for all $i \in \mathcal{S}$.

• Nodes in $\mathcal{D}$ are called demand nodes (or destinations). The amount of demand at destination j is $d_j \geq 0$, for all $j \in \mathcal{D}$.

1.1. Statement of the transportation problem.

The Transportation problem is special case of the network flow problems with the network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, where

$$\mathcal{N} = \mathcal{S} \cup \mathcal{D}, \quad \mathcal{S} \cap \mathcal{D} = \emptyset,$$

and with the following properties (see Figure 1):

- Nodes in $\mathcal{S}$ are called supply nodes (or sources). The amount of supply at source i is $s_i \geq 0$, for all $i \in \mathcal{S}$.
- Nodes in $\mathcal{D}$ are called demand nodes (or destinations). The amount of demand at destination j is $d_j \geq 0$, for all $j \in \mathcal{D}$.
- Every arcs in $\mathcal{A}$ has its tail in $\mathcal{S}$ and its head in $\mathcal{D}$.
- Each supply node has arcs connecting to every destination.

- Every arcs in $\mathcal{A}$ has its tail in $\mathcal{S}$ and its head in $\mathcal{D}$.
- Each supply node has arcs connecting to every destination.
- Arc (i, j) joins source i to destination j.

c_{ij}: Transportation cost per unit,

x_{ij}: the amount shipped along (i, j).

Statement of the transportation problem

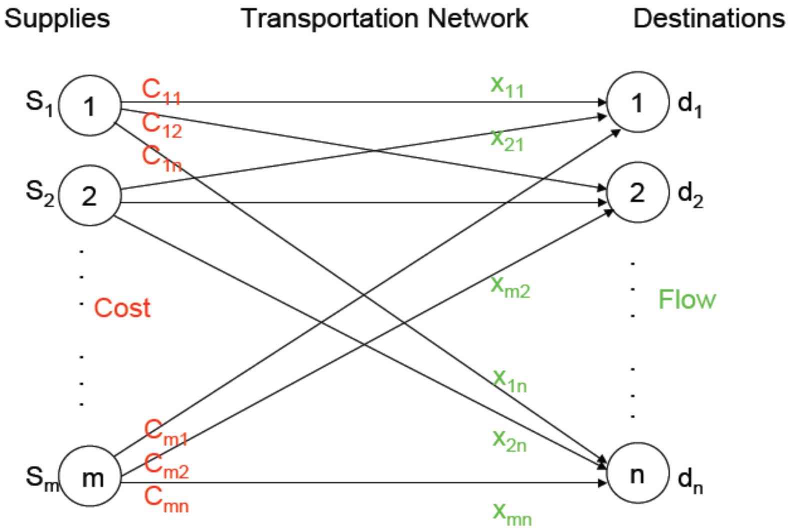

Figure 1

Prototype example: P & T COMPANY - Three canneries (near Bellingham, Washington; Eugene, Oregon; and Albert Lea, Minnesota) and then shipped by truck to four distributing warehouses in the western United States (Sacramento, California; Salt Lake City, Utah; Rapid City, South Dakota; and Albuquerque, New Mexico), as shown in Fig. 8.1.

Statement of the transportation problem

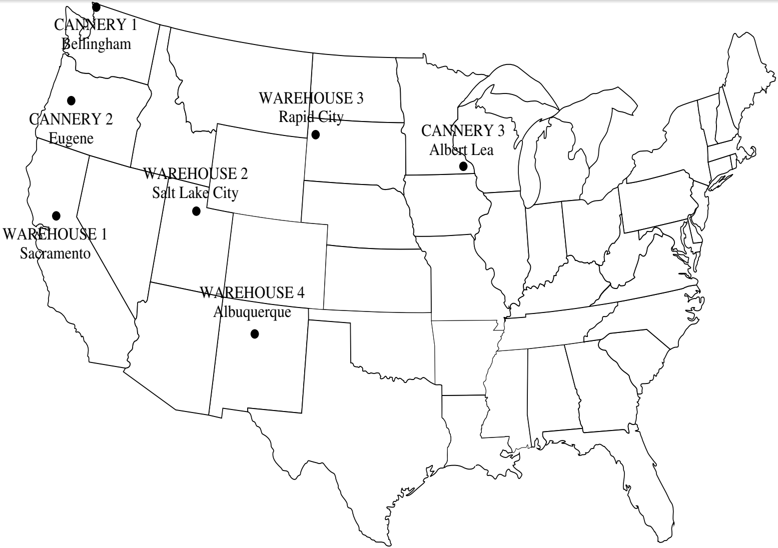

FIGURE 8.1
Location of canneries and warehouses for the P & T Co. problem.

TABLE 8.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

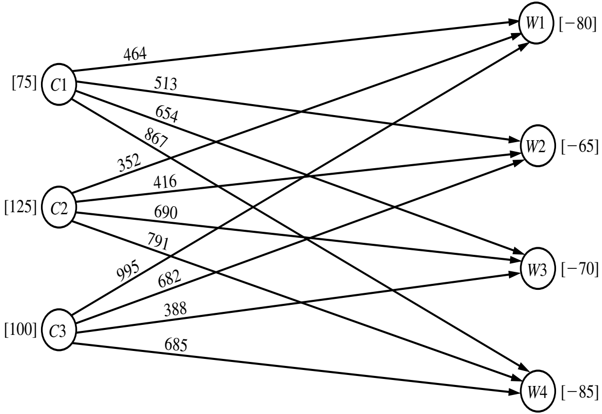

FIGURE 8.2
Network representation of the P & T Co. problem.

Statement of the transportation problem

Assumptions

- Conditions at each node i, j:

$$\begin{aligned} \textstyle\sum_{j\in\mathcal{D}} x_{ij} &= s_i, \quad i \in \mathcal{S}, \\ \textstyle\sum_{i\in\mathcal{S}} x_{ij} &= d_j, \quad j \in \mathcal{D}. \end{aligned}$$

- The total demand equals the total supply

$$\sum_{i\in\mathcal{S}} s_i = \sum_{j\in\mathcal{D}} d_j. \tag{1}$$

A transportation problem satisfying (1) is called a balanced transportation problem.

Coming back to the example of P & T Co.

Statement of the transportation problem

TABLE 8.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	**2**	**3**	**4**	**Output**
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

The constraints are

$$
\begin{array}{lcr}
x_{11} + x_{12} + x_{13} + x_{14} & & = 75 \\
x_{21} + x_{22} + x_{23} + x_{24} & & = 125 \\
x_{31} + x_{32} + x_{33} + x_{34} & & = 100 \\
x_{11} + x_{21} + x_{31} & & = 80 \\
x_{12} + x_{22} + x_{32} & & = 65 \\
x_{13} + x_{23} + x_{33} & & = 70 \\
x_{14} + x_{24} + x_{34} & & = 85
\end{array}
$$

and

$$x_{ij} \geq 0 \qquad (i = 1, 2, 3;\ j = 1, 2, 3, 4).$$

Statement of the transportation problem

TABLE 8.3 Constraint coefficients for P & T Co.

Coefficient of:

$$
\begin{array}{ccccccccccccc}
 & x_{11} & x_{12} & x_{13} & x_{14} & x_{21} & x_{22} & x_{23} & x_{24} & x_{31} & x_{32} & x_{33} & x_{34}
\end{array}
$$

$$
\mathbf{A} = \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 \\
1 & & & & 1 & & & & 1 & & & \\
 & 1 & & & & 1 & & & & 1 & & \\
 & & 1 & & & & 1 & & & & 1 & \\
 & & & 1 & & & & 1 & & & & 1
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{Cannery constraints} \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{Warehouse constraints}
\end{array}
$$

Statement of the transportation problem

TABLE 8.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
Cannery	1	464	513	654	867	75
	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

The objective function of the problem:

$$\begin{aligned}\text{Minimize} \quad Z = {} & 464x_{11} + 513x_{12} + 654x_{13} + 867x_{14} + 352x_{21} + 416x_{22} \\ & + 690x_{23} + 791x_{24} + 995x_{31} + 682x_{32} + 388x_{33} + 685x_{34},\end{aligned}$$

The objective of the model is to determine the unknowns flows x_{ij} that will minimize the total transportation cost while satisfying all the supply and demand restrictions.

Mathematical model

$$
\begin{array}{lll}
(\mathrm{P}) & \text{Minimize} & \sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{D}} c_{ij}x_{ij} \\
& \text{subject to} & \sum_{j\in\mathcal{D}} x_{ij} = s_i, \quad i \in \mathcal{S}, \\
& & \sum_{i\in\mathcal{S}} x_{ij} = d_j, \quad j \in \mathcal{D}, \\
& & \sum_{i\in\mathcal{S}} s_i = \sum_{j\in\mathcal{D}} d_j, \\
& & x_{ij} \geq 0 \ \forall (i,j) \in \mathcal{A}.
\end{array}
$$

Statement of the transportation problem

Tabular form of Transportation problems

Transportation tableau

destinations / supplies	d_1	…	d_j	…	d_n	
S_1	c_{11} x_{11}		c_{1j} x_{1j}		c_{1n} x_{1n}	flow
$\vdots$						
S_i	c_{i1} x_{i1}		c_{ij} x_{ij}		c_{in} x_{in}	flow
$\vdots$						
S_m	c_{m1} x_{m1}		c_{mj} x_{mj}		c_{mn} x_{mn}	flow
	Cost		Cost		Cost	

Figure 2

Transportation tableau		**network**
cell (i, j)	$\leftrightarrows$	arc (i, j)
path	$\leftrightarrows$	path
cycle	$\leftrightarrows$	cycle

Let $x = (x_{ij})$ be a feasible solution (feasible flow) $(i \in \mathcal{S},\ j \in \mathcal{D})$.
The set

$$G(x) := \{(i, j) \mid x_{ij} > 0\}$$

is called the set of selected cells of the feasible solution x.

Statement of the transportation problem

• Path: A path in a transportation tableau is a sequence of cells satisfying the following conditions:

(a) Two cells next together are in the same row or column,
(b) There are never three cells in the same row or the same column.

• Cycle: A cycle is a path such that the last cell and the first cell are in the same row or column (see Tableau 3).

• Note that the number of cells of a cycle is always an even number.

Statement of the transportation problem

destinations / supplies	d_1	d_2	d_3	d_4	
S_1	x	x			
S_2		x	x		
S_3			x	x	Path
S_4					

Tableau 3

destinations / supplies	d_1	d_2	d_3	d_4	
S_1	x	x			
S_2		x	x		
S_3			x	x	Path
S_4				x	

Statement of the transportation problem

destinations / supplies	d_1	d_2	d_3	d_4	
S_1	x	x			
S_2		x	x		
S_3			x	x	Cycle
S_4	x			x	

Tableau 3

1.2. Properties of transportation problems.

• Existence of optimal solution. A balanced transportation problem possesses at least an optimal solution.

Note: a transportation problem is a linear programming problem.

1.2. Properties of transportation problems.

• Existence of optimal solution. A balanced transportation problem possesses at least an optimal solution.

Note: a transportation problem is a linear programming problem.

• Basic feasible solution (in brief, BF solution). A basic feasible solution is a feasible solution $x = (x_{ij})$ such that the columns of the matrix A corresponding to the components $x_{ij} > 0$ are linear independent.

• For the transportation problem described by a transportation tableau, How can we realize a feasible solution $x = (x_{ij})$ is a basic feasible solution?
The answer is in the next theorem.

Theorem 1.

A feasible solution x of a transportation problem is a basic feasible solution if and only if the set of all selected cells $G(x)$ does not contain any cycle.

TABLE 8.16 **Initial BF solution from the Northwest Corner Rule**

		Destination						
		1	**2**	**3**	**4**	**5**	**Supply**	u_i
Source	1	16 30	16 20	13	22	17	50	
	2	14	14 0	13 60	19	15	60	
	3	19	19	20 10	23 30	M 10	50	
	4(D)	M	0	M	0	0 50	50	
Demand		30	20	70	30	60	$Z = 2{,}470 + 10M$	
	v_j							

Theorem 1.2

In a transportation tableau with $m \times n$ cells ($m \geq 2, n \geq 2$), any collection (set) of cells of more than or equal to $m + n$ cells contains at least a cycle.

Theorem 1.3

Given a transportation tableau of $m \times n$ cells ($m \geq 2, n \geq 2$). Let P be a set of $m + n - 1$ cells that contains NO cycle and $(i, j) \notin P$. Then $P \cup \{(i, j)\}$ contains exactly one cycle. This unique cycle is denoted by $C(P, ij)$.

Some methods:

- Northwest-corner method,
- Least-cost method,
- Vogel approximation method.

Northwest-corner method.

The method starts at the northwest corner cell of the transportation tableau (i.e. cell $(1,1)$).

Step 1. Allocate as much as possible to the selected cell. Then adjust the associated amounts of supply and demand by subtracting the allocated amount. Go to Step 2.

Step 2. Cross out the row or column with zero supply or demand. If both row and column are of zero supply and demand, cross out only one of them. Goto Step 3.

Step 3. If exactly one row or one column is left uncrossed, STOP. Otherwise,

• If a column has just been crossed out, move to the cell to the right and go to Step 1.

• If a row has just been crossed out, move to the cell below and go to Step 1.

The process will terminate after realizing $m+n-1$ iterations of the above algorithm and the flow obtained is a BF solution.

NorthWest Corner Method

destinations / supplies	30	60	46	25
50	9	7	12	7
70	5	9	6	1
41	8	2	9	1

NorthWest Corner Method

destinations / supplies	~~30~~	60	46	25
20 ~~50~~	9 30	7	12	7
70	5	9	6	1
41	8	2		1

2. Initial BF solutions for transportation problems

NorthWest Corner Method

supplies \ destinations	~~30~~	40 ~~60~~	46	25
~~20~~ ~~50~~	9 **30**	7 **20**	12	7
70	5	9	6	1
41	8	2	9	1

NorthWest Corner Method

supplies \ destinations	~~30~~	~~40~~ ~~60~~	46	25
~~20~~ ~~50~~	9: 30	7: 20	12	7
30 ~~70~~	5	9: 40	6	1
41	8	2	9	1

NorthWest Corner Method

destinations / supplies	~~30~~	~~40~~ ~~60~~	16 ~~46~~	25
~~20~~ ~~50~~	9 30	7 20	12	7
~~30~~ ~~70~~	5	9 40	6 30	1
41	8	2	9	1

2. Initial BF solutions for transportation problems

NorthWest Corner Method

destinations / supplies	~~30~~	~~40~~ ~~60~~	~~16~~ ~~46~~	25
~~20~~ ~~50~~	9 30	7 20	12	7
~~30~~ ~~70~~	5	9 40	6 30	1
25 ~~41~~	8	2	9 16	1 25

2. Initial BF solutions for transportation problems

NorthWest Corner Method

supplies \ destinations	30	60	46	25
50	9 / 30	7 / 20	12	7
70	5	9 / 40	6 / 30	1
41	8	2	9 / 16	1 / 25

Tableau 4: a BF solution

Cost: 1119

Other methods: Students are required to read the book of H.A. Taha, pages 179-182.

2. Initial BF solutions for transportation problems

Test-cost method

destinations / supplies	30	60	46	~~25~~
50	9	7	12	7
45 ~~70~~	5	9	6	1 25
41	8	2	9	1

Test-cost method

destinations / supplies	30	19 ~~60~~	46	~~25~~
50	9	7	12	7
45 ~~70~~	5	9	6	1 25
~~41~~	8	2 41	9	1

Test-cost method

destinations / supplies	~~30~~	19 ~~60~~	46	~~25~~
50	9	7	12	7
15 ~~45~~ ~~70~~	5 30	9	6	1 25
~~41~~	8	2 41	9	1

Test-cost method

destinations / supplies	~~30~~	19 ~~60~~	31 ~~46~~	~~25~~
50	9	7	12	7
~~15~~ ~~45~~ ~~70~~	5 30	9	6 15	1 25
~~41~~	8	2 41	9	1

Test-cost method

destinations / supplies	~~30~~	~~19~~ ~~60~~	31 ~~46~~	~~25~~
31 ~~50~~	9	7 19	12	7
~~15~~ ~~45~~ ~~70~~	5 30	9	6 15	1 25
~~41~~	8	2 41	9	1

Test-cost method

destinations / supplies	~~30~~	~~19~~ ~~60~~	~~31~~ ~~46~~	~~25~~
~~31~~ ~~50~~	9	7 19	12 31	7
~~15~~ ~~45~~ ~~70~~	5 30	9	6 15	1 25
~~41~~	8	2 41	9	1

Cost: 852

Let $x = (x_{ij})$ be a BF solution with $G(x)$ (the set of selected cells) defined as above.

3.1. Leaving and entering variables

For a BF solution $x = (x_{ij})$ of the transportation problem (P), if $(i,j) \notin G(x)$, the cycle formed by $G(x)$ and (i,j) is $C(G(x), ij)$. In this situation, by convention, we mark the cell (i,j) with a plus sign and other cells of the cycle with minus and plus signs successively, starting from the cell (i,j).

Let C^+ and C^- be the sets of all plus cells and minus cells of the cycle $C(G(x), ij)$. The reduced cost of the cycle $C(G(x), ij)$ is:

$$\bar{c}_{ij} := \sum_{(k,l)\in C^+} c_{kl} - \sum_{(k,l)\in C^-} c_{kl}.$$

supplies \ destinations	30	60	46	25
50	9 30	7 20	12	7
70	5	9 "-" 40	6 "+" 30	1
41	8	2 "+" x	9 "-" 16	1 25

Tableau 5 $\overline{C}_{32} = 6+2 - 9 - 9 = -10 < 0$

This means that if we ship one unit of commodity around the circle $C(G(x), ij)$ the total cost will change an amount $\bar{c}_{ij}$.

Suppose that $\bar{c}_{ij} < 0$ [Why consider this case?]

• If we increase $x_{ij} = \theta$ then the total cost will be reduced an amount $\theta \cdot \bar{c}_{ij}$ and the basic variable will change following the formula:

$$\bar{x}_{kl} = \begin{cases} x_{kl} + \theta^* & \text{if } (k,l) \in C^+, \\ x_{kl} - \theta^* & \text{if } (k,l) \in C^-, \\ x_{kl} & \text{otherwise.} \end{cases} \tag{2}$$

Note that, now $x_{ij} = \theta > 0$ (see Tableau 5)

This means that if we ship one unit of commodity around the circle $C(G(x), ij)$ the total cost will change an amount $\bar{c}_{ij}$.

Suppose that $\bar{c}_{ij} < 0$ [Why consider this case?]

• If we increase $x_{ij} = \theta$ then the total cost will be reduced an amount $\theta \cdot \bar{c}_{ij}$ and the basic variable will change following the formula:

$$\bar{x}_{kl} = \begin{cases} x_{kl} + \theta^* & \text{if } (k,l) \in C^+, \\ x_{kl} - \theta^* & \text{if } (k,l) \in C^-, \\ x_{kl} & \text{otherwise.} \end{cases} \tag{2}$$

Note that, now $x_{ij} = \theta > 0$ (see Tableau 5)

• The maximum increment of x_{ij} (i.e., θ) should be chosen as:

$$\theta^* := \min_{(k,l) \in C^-} x_{kl}.$$

- The maximum increment of x_{ij} (i.e., θ) should be chosen as:

$$\theta^* := \min_{(k,l)\in C^-} x_{kl}.$$

- The maximum increment of x_{ij} (i.e., θ) should be chosen as:

$$\theta^* := \min_{(k,l)\in C^-} x_{kl}.$$

- Let $(u, v) \in C^-$ be such that

$$x_{uv} = \theta^* = \min_{(k,l)\in C^-} x_{kl}. \tag{3}$$

Choose x_{uv} as the leaving variable.

- The entering variable in this case will be x_{ij}.

destinations / supplies	30	60	46	25
50	9 30	7 20	12	7
70	5	9 24	6 46	1
41	8	2 16	9 0	1 25

Tableau 5 Cost = 959

3. Streamlined simplex method for transportation problems

supplies \ destinations	30	60	46	25
50	9 30	7 20	12	7
70	5	9 "-" 40	6 "+" 30	1
41	8	2 "+" x	9 "-" 16	1 25

Tableau 5 $\overline{C}_{32} = 6+2 - 9 - 9 = -10 < 0$

supplies \ destinations	30	60	46	25
50	9 30	7 20	12	7
70	5	9 24	6 46	1
41	8	2 16	9 0	1 25

Tableau 5 Cost = 959

3.2. Computing the reduced cost $\bar{c}_{ij}$

For the transportation problem, the dual variables are u_i and v_j (see the simplex method for network flow problems - there they are y_i, y_j).

• For $(i,j) \notin G(x)$, the reduced cost and u_i, v_j are related by

$$\bar{c}_{ij} = c_{ij} - u_i - v_j, \quad (4)$$

$$c_{ij} = u_i + v_j, \quad \text{for all} \quad (i,j) \in G(x), \quad (5)$$

and[1]

$$u_1 = 0. \quad (6)$$

From (4), (5), and (6) we can compute $\bar{c}_{ij}$ for any $(i,j) \notin G(x)$.

[1] So, if $(i,j) \in G(x)$, $c_{ij} - u_i - v_j = 0$.

3.3. Streamlined simplex method (for a TP)

Let x be a BF solution (found by northwest-corner method, for example) with $G(x)$ be the set of all selected cells of x.

Algorithm

Step 1. Compute the dual variables u_i and v_j from (5 and (6)). Go to Step 2.

Step 2. Compute reduced costs $\bar{c}_{ij} = c_{ij} - u_i - v_j$ for all $(i, j) \notin G(x)$, (see (4)).

3.3. Streamlined simplex method (for a TP)

Let x be a BF solution (found by northwest-corner method, for example) with $G(x)$ be the set of all selected cells of x.

Algorithm

Step 1. Compute the dual variables u_i and v_j from (5 and (6)). Go to Step 2.

Step 2. Compute reduced costs $\bar{c}_{ij} = c_{ij} - u_i - v_j$ for all $(i,j) \notin G(x)$, (see (4)).

- If for all $(i,j) \notin G(x)$, $\bar{c}_{ij} \geq 0$ then the BF solution x is optimal. STOP.

3.3. Streamlined simplex method (for a TP)

Let x be a BF solution (found by northwest-corner method, for example) with $G(x)$ be the set of all selected cells of x.

Algorithm

Step 1. Compute the dual variables u_i and v_j from (5 and (6)). Go to Step 2.

Step 2. Compute reduced costs $\bar{c}_{ij} = c_{ij} - u_i - v_j$ for all $(i,j) \notin G(x)$, (see (4)).

- If for all $(i,j) \notin G(x)$, $\bar{c}_{ij} \geq 0$ then the BF solution x is optimal. STOP.

- Otherwise, Choose a cell $(i,j) \notin G(x)$ with (smallest) $\bar{c}_{ij} < 0$. Choose x_{ij} as entering variable. Go to Step 3.

Step 3. Construct the cycle $C(G(x), ij)$ and determine the set C^+ and C^- (note that (i, j) is the starting plus sign cell of the cycle).

Let

$$\theta^* := \min\{x_{kl} \ : \ (k, l) \in C^-\}$$

and choose (u, v) with $x_{uv} = \theta^*$ to be the leaving variable. Go to Step 4.

Step 4. Update the BF solution.

• The New allocations (BF solution) will be determined by (see (2)): $\bar{x} = (\bar{x}_{kl})$ with

$$\bar{x}_{kl} = \begin{cases} x_{kl} + \theta^* & \text{if } (k,l) \in C^+, \\ x_{kl} - \theta^* & \text{if } (k,l) \in C^-, \\ x_{kl} & \text{otherwise.} \end{cases}$$

• The new set of selected cells of $\bar{x}$ is :

$$G(\bar{x}) = (G(x) \setminus \{(u,v)\}) \bigcup \{(i,j)\}.$$

Go to Step 1.

Remark. If the transportation problem has supplies s_i and demands d_j are integer numbers then it has an optimal solution x with integer components x_{ij}.

Example. Consider the transportation problem given by the tableau in the next slide.

x is a BF solution found by the northwest-corner method.

destinations / supplies	80	140	100	80	u
100	2 80	5 20	4	6	0
200	8	4 120	3 80	8	-1
100	5	1	4 20	5 80	0
v	2	5	4	5	

Cost: 1460

Iteration 1

- u_i, v_j are given in the tableau.

Note: u_i, v_j satisfy $c_{ij} = u_i + v_j$ for all $(i,j) \in G(x)$.

- For $(i,j) \notin G(x)$, we compute $\bar{c}_{ij} := c_{ij} - u_i - v_j$. The result is as follows

$$\begin{aligned} &\bar{c}_{13} = 4 - 0 - 4 = 0, && \bar{c}_{14} = 6 - 0 - 6 = 1, \\ &\bar{c}_{21} = 8 - (-1) - 2 = 7, && \bar{c}_{24} = 8 - (-1) - 5 = 4, \\ &\bar{c}_{31} = 5 - 0 - 2 = 3, && \bar{c}_{32} = 1 - 0 - 5 = -4 < 0, \\ &\bar{c}_{34} = 5 - 0 - 5 = 0. && \end{aligned}$$

- Entering variable: x_{32}.

- The cycle $C(G(x), 32)$ formed by $G(x)$ and the cell $(3,2)$ is shown in Tableau 6.

Tableau 6

destinations / supplies	80	140	100	80	u
100	2 — 80	5 — 20	4 — $\bar{c}_{13} = 0$	6 — $\bar{c}_{14} = 1$	0
200	8 — $\bar{c}_{21} = 7$	4 — 120	3 — 80	8 — $\bar{c}_{24} = 4$	-1
100	5 — $\bar{c}_{31} = 3$	1 — $\bar{c}_{32} = -4$	4 — 20	5 — 80	0
v	2	5	4	5	

Entering variable x_{32}

- $\theta^* = \min\{x_{22}, x_{33}\} = x_{33} = 20$.
- Leaving variable: x_{33}.
- For updated BF solution (Step 4), see Tableau 7.

Tableau 7

destinations / supplies	80	140	100	80	u
100	2 / 80	5 / 20	4	6	0
200	8	4 / $120-\theta$	3 / $80+\theta$	8	-1
100	5	1 / $+\theta$	4 / $20-\theta$	5 / 80	0
v	2	5	4	5	

Entering variable x_{32} $\theta = 20$

destinations / supplies	80	140	100	80	u
100	2 / 80	5 / 20	4	6	0
200	8	4 / 100	3 / 100	8	-1
100	5	1 / 20	4	5 / 80	-4
v	2	5	4	9	

Cost: 1380

Iteration 2

- u_i, v_j are given in the Tableau 7.
- For $(i,j) \notin G(x)$, $\bar{c}_{ij}$ are given by:

$$\begin{array}{lll} \bar{c}_{13} = 0, & \bar{c}_{14} = -3 < 0, & \bar{c}_{21} = 7, \\ \bar{c}_{24} = 0, & \bar{c}_{31} = 7, & \bar{c}_{33} = 4. \end{array}$$

- Entering variable: x_{14}.
- $\theta^* = \min\{x_{12}, x_{34}\} = x_{12} = 20$.
- Leaving variable: x_{12}.
- updated BF solution (Step 4) is given in Tableau 8.

3. Streamlined simplex method for transportation problems

Tableau 8 a

destinations / supplies	80	140	100	80	u
100	2 / 80	5 / 20	4 / 0	6 / -3	0 / C_{ij}
200	8 / 7	4 / 100	3 / 100	8 / 0	-1
100	5 / 7	1 / 20	4 / 4	5 / 80	-4
v	2	5	4	9	

Entering variable x_{42}

Tableau 8b

destinations / supplies	80	140	100	80	u
100	2 80	5 $20 - \theta$	4	6 $+ \theta$	0
200	8	4 100	3 100	8	-1
100	5	1 $20 + \theta$	4	5 $80 - \theta$	-4
v	2	5	4	9	

Entering variable x_{14} $\theta = 20$

Tableau 8c (updated tableau)

destinations / supplies	80	140	100	80	u
100	2 80	5	4	6 20	0
200	8	4 100	3 100	8	2
100	5	1 40	4	5 60	-1
v	2	2	1	6	

Cost: 1320

Iteration 3

- u_i, v_j are given in the Tableau 9
- For $(i,j) \notin G(x)$, $\bar{c}_{ij}$ are given by:

$$\begin{array}{lll} \bar{c}_{12} = 3, & \bar{c}_{13} = 3, & \bar{c}_{21} = 4, \\ \bar{c}_{24} = 0, & \bar{c}_{31} = 4, & \bar{c}_{33} = 4. \end{array}$$

- For all $(i,j) \notin G(x)$, $\bar{c}_{ij} \geq 0$. The current BF solution is an optimal solution of the transportation problem .
- The minimum total cost of the schedule is

$$Z = 80 \times 2 + 20 \times 6 + 100 \times 4 + 100 \times 3 + 40 \times 1 + 60 \times 5 = 1320.$$

Tableau 9

destinations / supplies	80	140	100	80	u
100	2 / 80	5	4	6 / 20	0
200	8	4 / 100	3 / 100	8	2
100	5	1 / 40	4	5 / 60	-1
v	2	2	1	6	

Cost: 1320

3. Streamlined simplex method for transportation problems

Tableau 10

destinations / supplies	80	140	100	80	u
100	2 / 80	5 / 3	4 / 3	6 / 20	0 / c_{ij}
200	8 / 4	4 / 100	3 / 100	8 / 0	2
100	5 / 4	1 / 40	4 / 4	5 / 60	-1
v	2	2	1	6	

Cost: 1320

Tableau 11

destinations / supplies	80	140	100	80	u
100	2 / 80	5	4	6 / 20	0
200	8	4 / 100	3 / 100	8	2
100	5	1 / 40	4	5 / 60	-1
v	2	2	1	6	

Optimal solution with Cost: 1320

Another optimal solution

Test-cost method

destinations / supplies	80	140	100	80	u
100	2 80	5	4	6 20	0
200	8	4 40	3 100	8 60	2
100	5	1 100	4	5	_1
v	2	2	1	6	

Cost: 1320

Another optimal solution

destinations / supplies	80	140	100	80	u
100	2 80	5 3	4 3	6 20	0
200	8 4	4 40	3 100	8 60	2
100	5 4	1 100	4 4	5 0	-1
V	2	2	1	6	

The assignment problem is a special case of the transportation problem where assignees are being assigned to perform tasks. For the assignment problem $m = n$, each $s_i = 1$ and each $d_j = 1$.

The assignment problem is a special case of the transportation problem where assignees are being assigned to perform tasks. For the assignment problem$m = n$, each $s_i = 1$ and each $d_j = 1$.

Examples:

• Suppose that we have m individuals and m jobs. If individual i is assigned to job j, the cost incurred will be c_{ij}. We wish to find the minimum cost assignment of individuals to jobs. (This is a common application of assignment problem).
Note that for a BF solution $x = (x_{ij})$, either $x_{ij} = 1$ (i.e., individual i is assigned to job j) or $x_{ij} = 0$ (i.e., individual is not assigned to job j).

Assignees need not be people.

• Machines are assigned to locations. In this case, c_{ij} is the cost of shipping/handling machine i to location j. For more details details, see [Hillier and Lieberman, pages 382-383].

Assumptions:

• Number of assignees and number of tasks are the same (i.e., $m = n$),

• Each assignee is assigned to exactly one task,

• Each task is to be performed by exactly one assignee,

• There is a cost c_{ij} associate with assignee i $(i = 1, 2, ..., m)$ performing task j $(j = 1, 2, ..., m)$.

The objective is to determine how all m assignments should be made to minimize the total cost.

Variables:

$$x_{ij} = \begin{cases} 1 & \text{if assignee } i \text{ performs task } j, \\ 0 & \text{if not,} \end{cases}$$

for $i, j = 1, 2, ..., m$. The model of assignment problem is

$$\begin{array}{lll} (\mathrm{P1}) & \text{Minimize} & \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij} \\ & \text{subject to} & \sum_{j=1}^{m} x_{ij} = 1, \quad i = 1, 2, ..., m, \\ & & \sum_{i=1}^{m} x_{ij} = 1, \quad j = 1, 2, ..., m, \\ & & x_{ij} \geq 0, \quad \text{for all } i, j = 1, 2, ..., m. \end{array}$$

Q: The meaning of each constraint?

For more discussions and applications, see the book of Hillier and Lieberman.