INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

MIDTERM EXAMINATION

April 2021

Duration: 110 minutes

SUBJECT: REAL ANALYSIS	
Head of Dept. of Mathematics:	Lecturer:
Prof. Pham Huu Anh Ngoc	Assoc. Prof. Nguyen Ngoc Hai

INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 Let d_1 and d_2 be metrics on X. Suppose that there are positive constants c_1 and c_2 such that

$$c_1 d_1(x, x') \leq d_2(x, x') \leq c_2 d_1(x, x') \quad \text{for all} \quad x, x' \in X.$$

(a) *(10 marks)* Show that if $x, x_n \in X$, $n = 1, 2, \ldots$, then

$$\lim d_1(x_n, x) = 0 \quad \text{if and only if} \quad \lim d_2(x_n, x) = 0.$$

(b) *(10 marks)* Show that a sequence $\{y_n\}$ is Cauchy in (X, d_1) if and only if $\{y_n\}$ is Cauchy in (X, d_2).

(c) *(5 marks)* Show that (X, d_1) is complete if and only if (X, d_2) is complete

Question 2 (a) *(10 marks)* Let A be a subset of a metric space (X, d). A point $x \in A$ is an **isolated point** of A if $B(x, r) \cap \big(A \setminus \{x\}\big) = \emptyset$ for some $r > 0$. Show that if $x_0 \in A$ is an isolated point of A, then the set $\{x_0\}$ is both open and closed in the subspace A.

(b) *(10 marks)* Suppose that D is a dense set in a metric space (X, d). Prove that D contains all isolated points of X.

(c) *(5 marks)* Show that a separable metric space has at most countably many isolated points.

Question 3 Let E be a subset of a metric space (X, d). Prove each of the following formulas

(a) *(10 marks)* $X = \mathrm{int}(E) \cup \overline{E^c}$.

(b) *(10 marks)* $\overline{E} = E \cup \partial E = \mathrm{int}(E) \cup \partial E$.

(c) *(10 marks)* $X = \mathrm{int}(E) \cup \partial E \cup \mathrm{int}(E^c)$.

Question 4 (a) *(10 marks)* Prove that any mapping f from a discrete metric space (X, d) into a metric space (Y, ρ) is uniformly continuous.

(b) *(10 marks)* Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function. Assume that for each open set B in $\mathbb{R}$ there exists an open set A in $\mathbb{R}$ such that $f^{-1}(B) = A \cap X$. Under such assumptions, prove that f is continuous on the subspace X.