Optimization 1, Bonus Question

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October 27, 2021

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1 Problem

Give an example of a linear programming of the form

$$\begin{array}{ll} \text{minimize} & -c_1x_1 - c_2x_2 \\ \\ \text{subject to} & \begin{array}{c} a_1x_1 + b_1x_2 \le d_1 \\ a_2x_1 + b_2x_2 \le d_2 \\ \dots \\ a_mx_1 + b_mx_2 \le d_m \end{array} \end{array}$$

satisfying the following constraints:

(a) $c_1, c_2 \in \mathbb{N}$ and $c_1 > c_2$;

(b) For each $i = \overline{1,m}, a_i, b_i \in \mathbb{Z}$ and $d_i \in \mathbb{N}$;

(c) The feasible solution set is an $n-$sided polygon $A_1A_2...A_n$ $(n \ge 6)$ where

$$A_i \in \mathbb{Z}^2, \forall i = \overline{1,n},$$

i.e. all vertices have integer components;

(d) In the initial simplex tableau, if we choose the column vector corresponding to x_1 as the pivot column, then the simplex method terminates after $n-1$ steps;

(e) On the other hand, if we choose the column vector corresponding to x_2 as the pivot column, then the simplex method terminates after 1 step only;

(f) All entries of the simplex tableau during the execution must be integers or fractions with denominators not exceeding 10.

2 Solution

Consider the following linear program

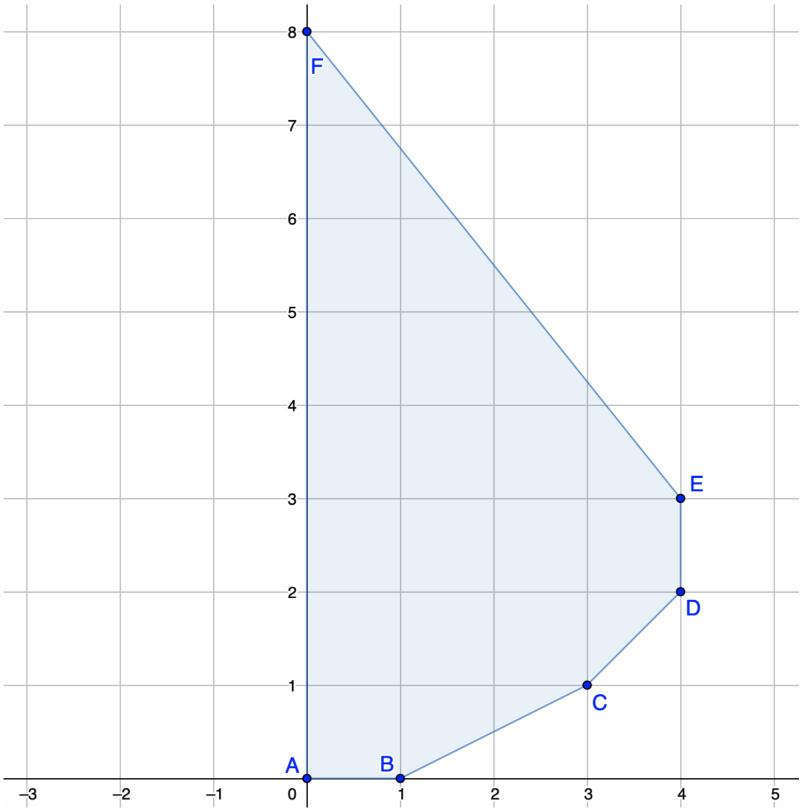
$$\begin{array}{ll} \text{minimize} & -6x_1 - 5x_2 \\ \\ \text{subject to} & \begin{array}{c} x_1 - 2x_2 \le 1 \\ x_1 - x_2 \le 2 \\ x_1 \le 4 \\ 5x_1 + 4x_2 \le 32 \\ x_1, x_2 \ge 0 \end{array} \end{array}$$

We verify that this linear program satisfies the given constraints.

(a) $c_1 = 6$ and $c_2 = 5$, so $c_1, c_2 \in \mathbb{N}$ and $c_1 > c_2$;

(b) $a_1 = a_2 = a_3 = 1, a_4 = 5, b_1 = -2, b_2 = -1, b_3 = 0, b_4 = 4, d_1 = 1, d_2 = 2, d_3 = 4, d_4 = 32$, so $a_i, b_i \in \mathbb{Z}$ and $d_i \in \mathbb{N}, \forall i = \overline{1, 4}$;

(c) A sketch of the feasible solution set is given below.



The feasible solution set is the hexagon $ABCDEF$ (boundary included) with coordinates

$$A(0,0), B(1,0), C(3,1), D(4,2), E(4,3), F(0,8)$$

i.e. all vertices have integer components;

(d) First, we introduce the slack variables y_1, y_2, y_3, y_4 and convert the given linear program into canonical form as

$$\begin{array}{ll} \text{minimize} & z = -6x_1 - 5x_2 \\ & \\ \text{subject to} & \begin{array}{c} x_1 - 2x_2 + y_1 = 1 \\ x_1 - x_2 + y_2 = 2 \\ x_1 + y_3 = 4 \\ 5x_1 + 4x_2 + y_4 = 32 \\ x_1, x_2, y_1, y_2, y_3, y_4 \geq 0 \end{array} \end{array}$$

or equivalently,

$$\begin{array}{ll} \text{minimize} & z \\ \\ \text{subject to} & \begin{array}{c} x_1 - 2x_2 + y_1 = 1 \\ x_1 - x_2 + y_2 = 2 \\ x_1 + y_3 = 4 \\ 5x_1 + 4x_2 + y_4 = 32 \\ -6x_1 - 5x_2 - z = 0 \\ z \,\text{free},\; x_1, x_2, y_1, y_2, y_3, y_4 \geq 0 \end{array} \end{array}$$

The initial tableau is then given by

x_1	x_2	y_1	y_2	y_3	y_4	rhs
1	-2	1	0	0	0	1
1	-1	0	1	0	0	2
1	0	0	0	1	0	4
5	4	0	0	0	1	32
-6	-5	0	0	0	0	0

Step 1. We choose the column vector corresponding to x_1 as the pivot column. By the Minimum Ratio Test, the ratio at row 1 (which is 1) is smallest, so we pivot on the entry **r1c1** (row 1 - column 1) and obtain the new tableau

x_1	x_2	y_1	y_2	y_3	y_4	rhs
1	-2	1	0	0	0	1
0	**1**	-1	1	0	0	1
0	2	-1	0	1	0	3
0	14	-5	0	0	1	27
0	**-17**	6	0	0	0	6

Step 2. Since the coefficient of x_2 in the objective row is negative, we choose the column vector corresponding to x_2 as the pivot column. By the Minimum Ratio Test, the ratio at row 2 (which is 1) is smallest, so we pivot on the entry **r2c2** and obtain the new tableau

x_1	x_2	y_1	y_2	y_3	y_4	rhs
1	0	-1	2	0	0	3
0	1	-1	1	0	0	1
0	0	**1**	-2	1	0	1
0	0	9	-14	0	1	13
0	0	**-11**	17	0	0	23

Step 3. Since the coefficient of y_1 in the objective row is negative, we choose the column vector corresponding to y_1 as the pivot column. By the Minimum Ratio Test, the ratio at row 3 (which is 1) is smallest, so we pivot on the entry **r3c3** and obtain the new tableau

x_1	x_2	y_1	y_2	y_3	y_4	rhs
1	0	0	0	1	0	4
0	1	0	-1	1	0	2
0	0	1	-2	1	0	1
0	0	0	**4**	-9	1	4
0	0	0	**-5**	11	0	34

Step 4. Since the coefficient of y_2 in the objective row is negative, we choose the column vector corresponding to y_2 as the pivot column. The only positive entry on that column is at row 4, so we pivot on the entry **r4c4** and obtain the new tableau

x_1	x_2	y_1	y_2	y_3	y_4	rhs
1	0	0	0	**1**	0	4
0	1	0	0	-5/4	1/4	3
0	0	1	0	-7/2	1/2	3
0	0	0	1	-9/4	1/4	1
0	0	0	0	**-1/4**	5/4	39

Step 5. Since the coefficient of y_3 in the objective row is negative, we choose the column vector corresponding to y_3 as the pivot column. The only positive entry on that column is at row 1, so we pivot on the entry **r1c5** and obtain the new tableau

x_1	x_2	y_1	y_2	y_3	y_4	rhs
1	0	0	0	1	0	4
5/4	1	0	0	0	1/4	8
7/2	0	1	0	0	1/2	17
9/4	0	0	1	0	1/4	10
1/4	0	0	0	0	5/4	40

Now the objective row has no negative coefficient, thus we conclude that the solution corresponding to this tableau is optimal. This tableau corresponds to the problem

$$\text{minimize} \qquad z = x_1/4 + 5y_4/4 - 40$$

$$\text{subject to} \quad \begin{array}{c} x_1 + y_3 = 4 \\ 5x_1/4 + x_2 + y_4/4 = 8 \\ 7x_1/2 + y_1 + y_4/2 = 17 \\ 9x_1/4 + y_2 + y_4/4 = 10 \\ x_1, x_2, y_1, y_2, y_3, y_4 \geq 0 \end{array}$$

From the tableau,

$$(x_1, x_2, y_1, y_2, y_3, y_4) = (0, 8, 17, 10, 4, 0)$$

is the optimal solution of this problem, with a corresponding optimal value of -40. Hence in this case, the simplex method terminates after 5 steps, with the optimal solution $(x_1, x_2) = (0, 8)$ and the corresponding optimal value of -40.

(e) Reconsider the initial tableau

x_1	x_2	y_1	y_2	y_3	y_4	rhs
1	-2	1	0	0	0	1
1	-1	0	1	0	0	2
1	0	0	0	1	0	4
5	**4**	0	0	0	1	32
-6	-5	0	0	0	0	0

If in **Step 1** we choose the column vector corresponding to x_2 as the pivot column, then the only positive entry on that column is at row 4, so we pivot on the entry **r4c2** and obtain the new tableau

x_1	x_2	y_1	y_2	y_3	y_4	rhs
7/2	0	1	0	0	1/2	17
9/4	0	0	1	0	1/4	10
1	0	0	0	1	0	4
5/4	1	0	0	0	1/4	8
1/4	0	0	0	0	5/4	40

Now the objective row has no negative coefficient, thus we conclude that the solution corresponding to this tableau is optimal. This tableau corresponds to the problem

$$
\begin{array}{ll}
\text{minimize} & z = x_1/4 + 5y_4/4 - 40 \\[1em]
\text{subject to} & \begin{array}{c} 7x_1/2 + y_1 + y_4/2 = 17 \\ 9x_1/4 + y_2 + y_4/4 = 10 \\ x_1 + y_3 = 4 \\ 5x_1/4 + x_2 + y_4/4 = 8 \\ x_1, x_2, y_1, y_2, y_3, y_4 \geq 0 \end{array}
\end{array}
$$

From the tableau,

$$(x_1, x_2, y_1, y_2, y_3, y_4) = (0, 8, 17, 10, 4, 0)$$

is the optimal solution of this problem, with a corresponding optimal value of -40. Hence in this case, the simplex method terminates after 1 step, with the optimal solution $(x_1, x_2) = (0, 8)$ and the corresponding optimal value of -40.

(f) By carefully observing, it can be seen that all entries of the simplex tableau during the execution are integers or fractions with denominators not exceeding 4.

Therefore, the provided linear program, restated below

$$
\begin{array}{ll}
\text{minimize} & -6x_1 - 5x_2 \\[1em]
\text{subject to} & \begin{array}{c} x_1 - 2x_2 \leq 1 \\ x_1 - x_2 \leq 2 \\ x_1 \leq 4 \\ 5x_1 + 4x_2 \leq 32 \\ x_1, x_2 \geq 0 \end{array}
\end{array}
$$

satisfies all the given constraints.