INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

FINAL EXAMINATION
January 2017
Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Deputy Head of Dept. of Mathematics:	Lecturer:
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INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 *(30 marks)* Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) Let f be a measurable function on X and $A \in \mathcal{M}$. Show that if f is integrable on both A and $X \setminus A$, then f is integrable on X.

(b) Suppose that f and g are nonnegative integrable functions on X for which $g \leq f$ a.e. on X. Show that if $\int_X f d\mu = \int_X g d\mu$, then $f = g$ a.e. on X.

Question 2 *(20 marks)* Let $\{f_n\}$ be a decreasing sequence of measurable functions on X and $f = \lim_{n\to\infty} f_n$. If there is an integrable function g on X such that $f_n \leq g$ for all n, show that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Question 3 *(15 marks)* Suppose that f is measurable on X and $a \in \mathbb{R}$. Consider the function

$$g(x) = \begin{cases} a & \text{if } f(x) > a \\ f(x) & \text{if } f(x) \leq a. \end{cases}$$

Let $A = \{x \in X : f(x) > a\}$ and $B = \{x \in X : f(x) \leq a\}$. Show that $g = a\chi_A + f\chi_B$ and that g is also measurable.

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Question 4 *(25 marks)*

(a) Let λ, μ be finite signed measures and ν a positive measure such that $\lambda \ll \nu$ and $\mu \ll \nu$. Show that $\lambda + \mu \ll \nu$.

(b) Let

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } x < 1 \\ (x-1)^2 & \text{if } x \geq 1. \end{cases}$$

Define

$$\mu(E) = \int_E f(x)dx \quad \text{and} \quad \nu(E) = \int_E g(x)dx, \quad E \in \mathcal{L},$$

where $\mathcal{L}$ is the collection of Lebesgue measurable sets in $\mathbb{R}$. Show that $\mu \perp \nu$.

Question 5 *(10 marks)* Suppose that f is increasing on $[a,b]$. Show that there exists a pair of increasing functions g and h on $[a,b]$ that satisfy the following conditions:

(i) $f(x) = g(x) + h(x)$ for all $x \in [a,b]$,

(ii) g is absolutely continuous on $[a,b]$,

(iii) $h' = 0$ a.e. on $[a,b]$,

(iv) $g(a) = 0$.

[*Hint:* Define $g(x) = \int_a^x f'(t)dt$.]

*** END OF QUESTION PAPER ***

SOLUTIONS
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Question 1
(a) Since f is integrable on both A and $X \setminus A$, $\int_A d\mu$ and $\int_{X\setminus A} d\mu$ are both finite. Hence by additivity,

$$\int_X f d\mu = \int_A f d\mu + \int_{X\setminus A} f d\mu \in \mathbb{R}.$$

Thus f is integrable on X.

(b) As g is integrable on X, it is finite a.e. Moreover, since $g \leq f$ a.e., $f - g$ is defined and nonnegative a.e. It follows that

$$\int_X (f-g) d\mu = \int_X f d\mu - \int_X g d\mu = 0.$$

Hence $f - g = 0$ a.e., that is $f = g$ a.e. on X.

Question 2 Since g is integrable on X, it is finite a.e. and hence, $g - f_n$ and $g - f$ are defined a.e. As $f_n \leq g$ and $\{f_n\}$ is decreasing, $0 \leq g - f_n \nearrow g - f$. By the Monotone Convergence Theorem,

$$\int_X g d\mu - \lim_{n\to\infty} \int_X f_n d\mu = \lim_{n\to\infty} \int_X (g-f_n) d\mu = \int_X (g-f) d\mu = \int_X g d\mu - \int_X f d\mu.$$

It follows that $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

Question 3 Clearly A and B are disjoint sets and $A \cup B = X$. Let $x \in X$. If $x \in A$, then $x \notin B$ and $(a\chi_A + f\chi_B)(x) = a\chi_A(x) + f(x)\chi_B(x) = a = g(x)$; if $x \notin A$, then $x \in B$ and $(a\chi_A + f\chi_B)(x) = a \cdot 0 + f(x) \cdot 1 = f(x) = g(x)$. Thus $g = a\chi_A + f\chi_B$. Since f is integrable, A and B are measurable sets, so χ_A and χ_B are measurable functions. Hence $a\chi_A$ and $f\chi_B$ are measurable functions and so is g.

Question 4
(a) Since λ and μ are finite, their sum $\lambda + \mu$ is defined. Let $\nu(A) = 0$. As $\lambda \ll \nu$ and $\mu \ll \nu$, $\lambda(A) = \mu(A) = 0$. Hence $(\lambda + \mu)(A) = \lambda(A) + \mu(A) = 0$. Therefore $\lambda + \mu \ll \nu$.

(b) Since f and g are nonnegative and continuous function on $\mathbb{R}$, μ and ν are measures on $\mathcal{L}$. Let $A = (0, \infty)$ and $B = A^c = (-\infty, 0]$. We have

$$\mu(A) = \int_0^\infty f(x) dx = \int_0^\infty 0 dx = 0$$

and

$$\nu(B) = \int_{-\infty}^{0} g(x)dx = \int_{-\infty}^{0} 0dx = 0.$$

Thus $\mu \perp \nu$.

Question 5 Since f is increasing, f' exists, nonnegative a.e., and integrable on $[a, b]$. Define

$$g(x) = \int_a^x f'(t)dt, \ x \in [a, b], \quad \text{and} \quad h = f - g.$$

Then g is absolutely continuous on $[a, b]$, $g(a) = 0$, and $f = g + h$. Moreover, $h' = f' - g' = f' - f' = 0$ a.e.

If $x, y \in [a, b]$, $x < y$, then

$$g(y) - g(x) = \int_x^y f'(t)dt \geq 0 \quad \text{and}$$
$$h(y) - h(x) = f(y) - f(x) - \int_x^y f'(t)dt \geq 0.$$

Thus both g and h are nondecreasing on $[a, b]$.