

Special continuous random variables

January 14, 2021



X is an exponential RV with parameter λ if its pdf is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Denote $X \sim \mathcal{E}(\lambda)$



model arrival time of something, e.g. the amount of time until a piece of equipment breaks down, until a light bulb burns out, or until an accident occurs ...



$X \sim \mathcal{E}(\lambda)$ then

- $E(X) = \frac{1}{\lambda}$
- $\text{Var}(X) = \frac{1}{\lambda^2}$



The time until a small meteorite first lands anywhere in the Sahara desert is modeled as an exponential random variable with a mean of 10 days. The time is currently midnight. What is the probability that a meteorite first lands some time between 6am and 6pm of the first day?



- $X \sim \lambda$: time until a first lands
- $E(X) = 10 \Rightarrow \frac{1}{\lambda} = 10$ or $\lambda = \frac{1}{10}$
- pdf of X

$$f(x) = \begin{cases} \frac{1}{10}e^{-\frac{1}{10}x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



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$$f(x) = \begin{cases} \frac{1}{10}e^{-\frac{1}{10}x}, & x \geq 0 \\ 0, & \textit{otherwise} \end{cases}$$



- between 6am to 6pm of the first day = $\{\frac{1}{4} \leq X \leq \frac{3}{4}\}$ (day)
-

$$P(\frac{1}{4} \leq X \leq \frac{3}{4}) = \int_{1/4}^{3/4} \frac{1}{10} e^{-\frac{1}{10}x} dx$$
$$\approx .0476$$

Continuous RV X is said to be normally distributed or Gaussian with parameter μ and σ^2 if its pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

for $-\infty < x < \infty$

Denote $X \sim \mathcal{N}(\mu, \sigma^2)$.



Mean and variance of $\mathcal{N}(\mu, \sigma^2)$

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

- $E(X) = \mu$
- $\text{Var}(X) = \sigma^2$



Bell shape, symmetric about the mean

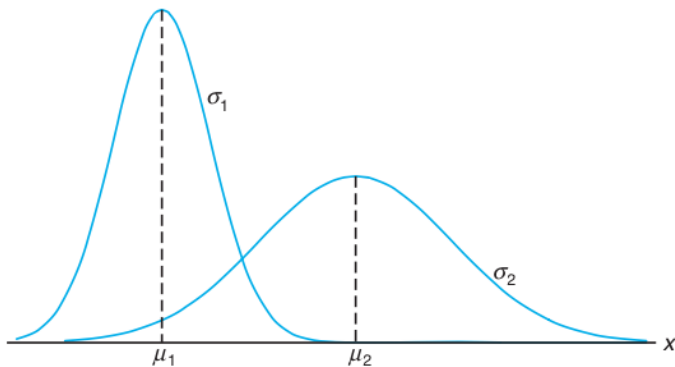


Figure: $\mu_1 < \mu_2, \sigma_1 < \sigma_2$

- Normal distribution is the most widely used distribution
- Many random phenomena obey a normal distribution
- Ex: the height and weight of a person, accuracy of shots from a gun... [▶ Link](#)



- to approximate Binomial (n, p) when n is large
- *limiting distribution* of sample mean ...**broad base** for statistic inference (estimation and hypothesis testing), analysis of variance



- $Z \sim \mathcal{N}(0, 1)$ is standard normal distribution
- pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

- cdf

$$\Phi(x) = P(Z \leq x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

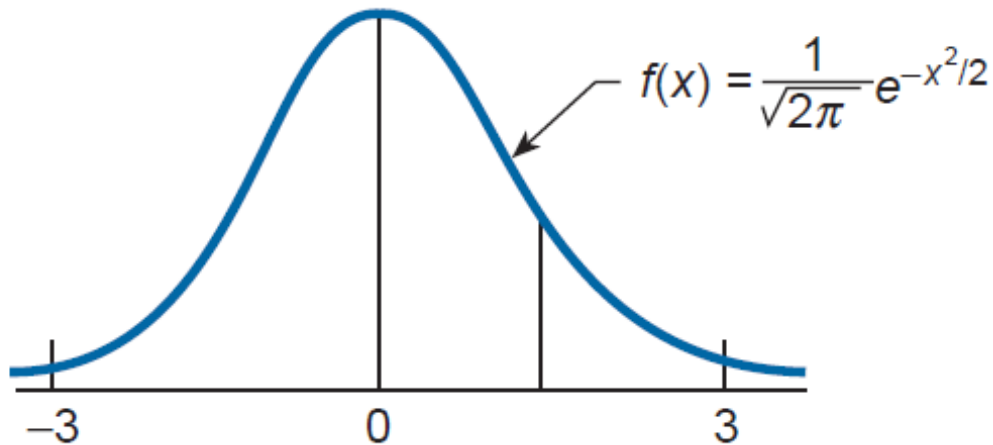


Figure: Pdf of Z



Compute probability of standard normal distribution

- Calculator
- Look up values in Normal Probability Table



Table A.3 Normal Probability Table

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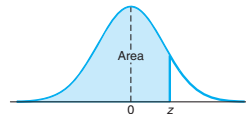


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0075	0.0073	0.0071	0.0069	0.0067	0.0065	0.0063	0.0061	0.0059	0.0057
−2.3	0.0090	0.0088	0.0086	0.0084	0.0082	0.0080	0.0078	0.0076	0.0074	0.0072
−2.2	0.0103	0.0101	0.0099	0.0097	0.0095	0.0093	0.0091	0.0089	0.0087	0.0085
−2.1	0.0125	0.0123	0.0121	0.0119	0.0117	0.0115	0.0113	0.0111	0.0109	0.0107
−2.0	0.0143	0.0141	0.0139	0.0137	0.0135	0.0133	0.0131	0.0129	0.0127	0.0125
−1.9	0.0160	0.0158	0.0156	0.0154	0.0152	0.0150	0.0148	0.0146	0.0144	0.0143
−1.8	0.0199	0.0196	0.0194	0.0192	0.0190	0.0188	0.0186	0.0184	0.0183	0.0181
−1.7	0.0255	0.0252	0.0250	0.0248	0.0246	0.0244	0.0242	0.0240	0.0238	0.0236
−1.6	0.0324	0.0320	0.0318	0.0315	0.0313	0.0311	0.0309	0.0307	0.0305	0.0303
−1.5	0.0398	0.0394	0.0392	0.0389	0.0387	0.0385	0.0383	0.0381	0.0379	0.0377
−1.4	0.0478	0.0474	0.0471	0.0468	0.0466	0.0464	0.0462	0.0460	0.0458	0.0456
−1.3	0.0570	0.0566	0.0563	0.0560	0.0558	0.0556	0.0554	0.0552	0.0550	0.0548
−1.2	0.0675	0.0671	0.0668	0.0665	0.0663	0.0661	0.0659	0.0657	0.0655	0.0653
−1.1	0.0808	0.0804	0.0801	0.0798	0.0796	0.0794	0.0792	0.0790	0.0788	0.0786
−1.0	0.1038	0.1034	0.1031	0.1028	0.1026	0.1024	0.1022	0.1020	0.1018	0.1016
−0.9	0.1255	0.1251	0.1248	0.1245	0.1243	0.1241	0.1239	0.1237	0.1235	0.1233
−0.8	0.1539	0.1534	0.1531	0.1528	0.1526	0.1524	0.1522	0.1520	0.1518	0.1516
−0.7	0.1854	0.1849	0.1846	0.1843	0.1841	0.1839	0.1837	0.1835	0.1833	0.1831
−0.6	0.2188	0.2183	0.2180	0.2177	0.2175	0.2173	0.2171	0.2169	0.2167	0.2165
−0.5	0.2578	0.2573	0.2570	0.2567	0.2565	0.2563	0.2561	0.2559	0.2557	0.2555
−0.4	0.3085	0.3080	0.3077	0.3074	0.3072	0.3070	0.3068	0.3066	0.3064	0.3062
−0.3	0.3693	0.3688	0.3685	0.3682	0.3680	0.3678	0.3676	0.3674	0.3672	0.3670
−0.2	0.4309	0.4304	0.4301	0.4298	0.4296	0.4294	0.4292	0.4290	0.4288	0.4286
−0.1	0.4903	0.4898	0.4895	0.4892	0.4890	0.4888	0.4886	0.4884	0.4882	0.4880
0.0	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
0.1	0.5398	0.5394	0.5391	0.5388	0.5386	0.5384	0.5382	0.5380	0.5378	0.5376
0.2	0.5793	0.5788	0.5785	0.5782	0.5780	0.5778	0.5776	0.5774	0.5772	0.5770
0.3	0.6179	0.6174	0.6171	0.6168	0.6166	0.6164	0.6162	0.6160	0.6158	0.6156
0.4	0.6574	0.6569	0.6566	0.6563	0.6561	0.6559	0.6557	0.6555	0.6553	0.6551
0.5	0.6950	0.6945	0.6942	0.6939	0.6937	0.6935	0.6933	0.6931	0.6929	0.6927
0.6	0.7324	0.7319	0.7316	0.7313	0.7311	0.7309	0.7307	0.7305	0.7303	0.7301
0.7	0.7704	0.7699	0.7696	0.7693	0.7691	0.7689	0.7687	0.7685	0.7683	0.7681
0.8	0.8090	0.8085	0.8082	0.8079	0.8077	0.8075	0.8073	0.8071	0.8069	0.8067
0.9	0.8478	0.8473	0.8470	0.8467	0.8465	0.8463	0.8461	0.8459	0.8457	0.8455
1.0	0.8849	0.8844	0.8841	0.8838	0.8836	0.8834	0.8832	0.8830	0.8828	0.8826
1.1	0.9236	0.9231	0.9228	0.9225	0.9223	0.9221	0.9219	0.9217	0.9215	0.9213
1.2	0.9608	0.9603	0.9601	0.9598	0.9596	0.9594	0.9592	0.9590	0.9588	0.9586
1.3	0.9772	0.9767	0.9764	0.9762	0.9760	0.9758	0.9756	0.9754	0.9752	0.9750
1.4	0.9844	0.9839	0.9836	0.9834	0.9832	0.9830	0.9828	0.9826	0.9824	0.9822
1.5	0.9901	0.9896	0.9893	0.9891	0.9889	0.9887	0.9885	0.9883	0.9881	0.9879
1.6	0.9944	0.9939	0.9936	0.9934	0.9932	0.9930	0.9928	0.9926	0.9924	0.9922
1.7	0.9970	0.9965	0.9962	0.9960	0.9958	0.9956	0.9954	0.9952	0.9950	0.9948
1.8	0.9984	0.9979	0.9976	0.9974	0.9972	0.9970	0.9968	0.9966	0.9964	0.9962
1.9	0.9990	0.9985	0.9982	0.9980	0.9978	0.9976	0.9974	0.9972	0.9970	0.9968
2.0	0.9993	0.9988	0.9985	0.9983	0.9981	0.9979	0.9977	0.9975	0.9973	0.9971
2.1	0.9995	0.9990	0.9987	0.9985	0.9983	0.9981	0.9979	0.9977	0.9975	0.9973
2.2	0.9996	0.9991	0.9988	0.9986	0.9984	0.9982	0.9980	0.9978	0.9976	0.9974
2.3	0.9997	0.9992	0.9989	0.9987	0.9985	0.9983	0.9981	0.9979	0.9977	0.9975
2.4	0.9998	0.9993	0.9990	0.9988	0.9986	0.9984	0.9982	0.9980	0.9978	0.9976
2.5	0.9999	0.9994	0.9991	0.9989	0.9987	0.9985	0.9983	0.9981	0.9979	0.9977
2.6	0.9999	0.9995	0.9992	0.9990	0.9988	0.9986	0.9984	0.9982	0.9980	0.9978
2.7	0.9999	0.9996	0.9993	0.9991	0.9989	0.9987	0.9985	0.9983	0.9981	0.9979
2.8	0.9999	0.9997	0.9994	0.9992	0.9990	0.9988	0.9986	0.9984	0.9982	0.9980
2.9	0.9999	0.9998	0.9995	0.9993	0.9991	0.9989	0.9987	0.9985	0.9983	0.9981
3.0	0.9999	0.9999	0.9996	0.9994	0.9992	0.9990	0.9988	0.9986	0.9984	0.9982



$$P(Z \leq -2.54)$$



1

$$\begin{aligned} P(Z \leq -2.54) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2.54} e^{-\frac{x^2}{2}} dx \\ &= \lim_{a \rightarrow -\infty} \frac{1}{\sqrt{2\pi}} \int_a^{-2.54} e^{-\frac{x^2}{2}} dx \end{aligned}$$

Substitute a by $-10, -30, -50 \dots$ and find the limit



Solution 2 - Look up the table value of normal probability

- ① look up -2 in the first column
- ② look up 54 in the first row
- ③ Intersection of the corresponding row and column

$$P(Z < -2.54) = .0055$$



Find

- ① $P(Z > 2.33)$
- ② $P(-1.65 < Z < 1.65)$
- ③ z such that $P(Z > z) = .95$



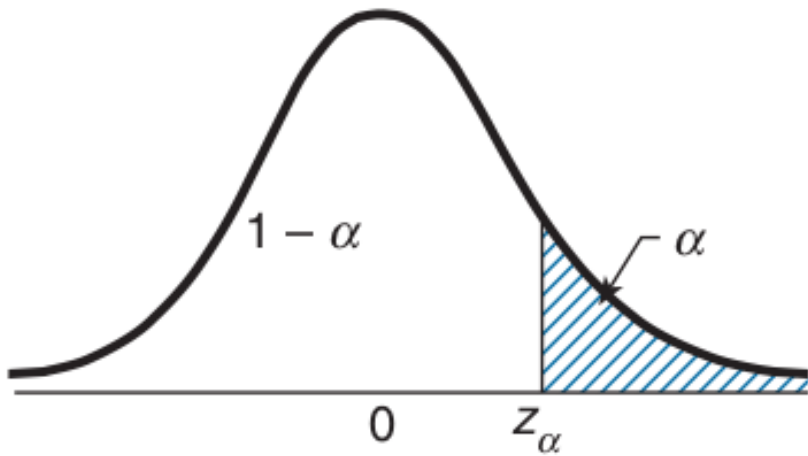
z_α is the $100(1 - \alpha)$ percentile if

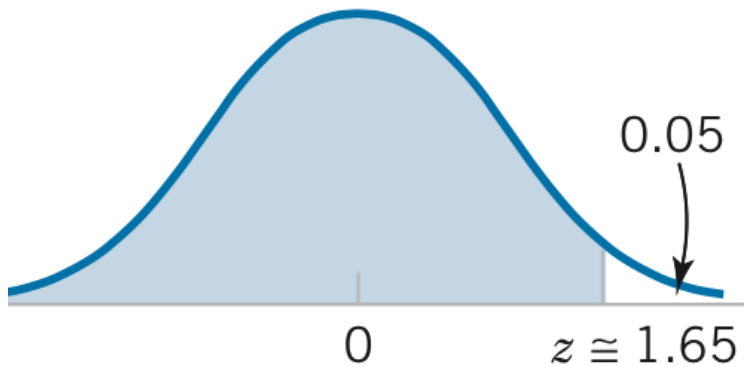
$$P(Z \leq z_\alpha) = 1 - \alpha$$

or

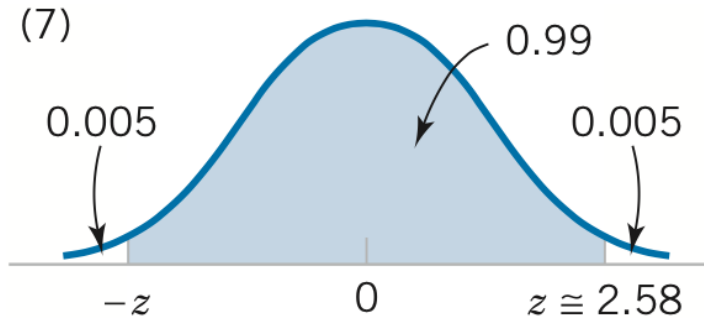
$$P(Z > z_\alpha) = \alpha$$







$$z_{.05} = 1.65$$



$$z_{.005} = 2.58$$

$$z_{.995} = -z_{.005} = -2.58$$



$$z_{1-\alpha} = -z_{\alpha}$$

because of the symmetry about y- axis of standard normal distribution



Normality is Preserved by Linear Transformations

If $X \sim \mathcal{N}(\mu, \sigma^2)$ then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$



Standardize a normal distribution

If $X \sim \mathcal{N}(\mu, \sigma^2)$ then

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$



If $X \sim \mathcal{N}(\mu, \sigma^2)$ then

$$\begin{aligned} P(X \leq x) &= P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \\ &= P\left(Z \leq \frac{x - \mu}{\sigma}\right) \end{aligned}$$

and

$$P(a \leq X \leq b) = P\left(Z < \frac{b - \mu}{\sigma}\right) - P\left(Z < \frac{a - \mu}{\sigma}\right)$$



The annual snowfall at a particular geographic location is modeled as a normal random variable with a mean of $\mu = 60$ inches, and a standard deviation of $\sigma = 20$. What is the probability that this year's snowfall will be at least 80 inches?



- Snowfall $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 60$, $\sigma = 20$

$$\begin{aligned} P(X \geq 80) &= 1 - P(X < 80) \\ &= 1 - P\left(Z < \frac{80 - \mu}{\sigma}\right) \\ &= 1 - P\left(Z < \frac{80 - 60}{20}\right) \\ &= 1 - P(Z < 1) = 1 - .8413 = .1687 \end{aligned}$$



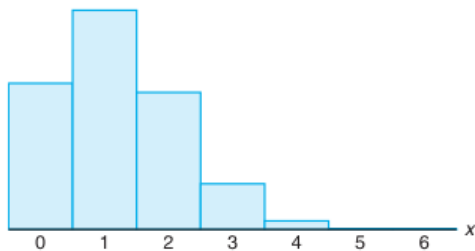


Figure 6.24: Histogram for $b(x; 6, 0.2)$.

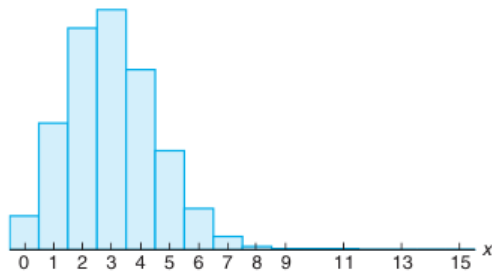


Figure 6.25: Histogram for $b(x; 15, 0.2)$.

- Suppose $Y \sim \text{Bino}(n, p)$ where n is large and np is not too small
- Y can be approximated by $X \sim \mathcal{N}(np, np(1 - p))$
- Y is discrete, X is continuous
- so we have to "fill the gap"



”Fill the gap” - midpoint rule

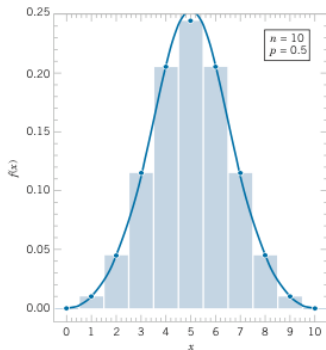


Figure 4-19 Normal approximation to the binomial distribution.

$$P(Y = i) \approx P\left(i - \frac{1}{2} < X < i + \frac{1}{2}\right)$$

To approximate a binomial probability of $X \hookrightarrow \text{Bin}(n, p)$ with a normal distribution, a **continuity correction** is applied as follows:

$$P(X \leq x) = P(X \leq x + 0.5) \approx P\left(Z \leq \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

The approximation is good for $np > 5$ and $n(1-p) > 5$.



Toss a fair coin 40 times. Y is the number of heads. Calculate $P(Y = 20)$ using normal approximation and direct computation.



Approximate Y by $X \sim \mathcal{N}(20, 10)$

$$\begin{aligned} P(Y = 20) &\approx P(19.5 < X < 20.5) \\ &= P\left(\frac{19.5 - 20}{\sqrt{10}} < Z < \frac{20.5 - 20}{\sqrt{10}}\right) \\ &= P(Z < .16) - P(Z < -.16) \\ &= .1272 \end{aligned}$$



Exact value

$$P(Y = 20) = \binom{40}{20} (.5)^{40} = .1254$$



The ideal size of a first-year class at a particular college is 150 students. The college, knowing from past experience that, on the average, only 30 percent of those accepted for admission will actually attend, uses a policy of approving the applications of 450 students. Compute the probability that more than 150 first-year students attend this college.



X : number of attending students $X \sim \text{Bino}(450, .3) \approx \mathcal{N}(135, 94.5)$

$$\begin{aligned} P(X > 150) &= P(X \geq 150.5) \\ &\approx P\left(Z \geq \frac{150.5 - 135}{\sqrt{94.5}}\right) \\ &= 1 - P(Z < 1.59) \\ &= .0559 \approx 5.6\% \end{aligned}$$

