FINAL EXAMINATION

Semester 1, Academic year 2014-2015
Duration: 120 minutes

SUBJECT: **Real Analysis**	
Chair of Department of Mathematics	Lecturer:
Signature:	Signature:
Full name: Assoc.Prof. Nguyen Dinh	Full name: Assoc Prof. Mai Duc Thanh

Instructions:

- **All documents and electronic devices, except scientific calculators and dictionaries, are not allowed.**
- **Marks are shown in italics.**

Question 1. (*20 marks*) State the definition and basic properties of measures.

Question 2. (*20 marks*)

(a) State the definition of outer measures.

(b) Prove that any set of outer measure zero is measurable.

Question 3. (*20 marks*) Let $(X, \mathcal{M})$ be a measurable space. Prove that a function $f : X \to \bar{R} = [-\infty, \infty]$ is measurable if and only if for each open set $O \subseteq R$, the set $f^{-1}(O)$ is measurable.

Question 4. (*20 marks*) Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f \geq 0$ be a measurable function on X and $\int_X f d\mu < \infty$. Prove that f is finite almost everywhere on X and the set $\{x \in X | f(x) > 0\}$ is σ-finite.

Question 5. (*20 marks*) Prove Hölder's inequality for integrals: For $1 \leq p < \infty, 1/p + 1/q = 1$, if $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$ then

$$\int_X |f \cdot g| d\mu = \|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q. \tag{1}$$

Moreover, for $f \neq 0$, show that the function

$$f^* = \|f\|_p^{1-p} \cdot \operatorname{sign}(f) \cdot |f|^{p-1} \in L^q(X, \mu),$$

and prove that

$$\int_X f \cdot f^* d\mu = \|f\|_p \quad and \quad \|f^*\|_q = 1.$$

*** END OF QUESTIONS ***