Chapter 2
Solutions of Linear Systems of Equations

Lecture 1:

- **Gauss Elimination Method**
- **LU Decomposition Methods**

Linear system of equations

A set of *n* linear equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$

Gaussian Elimination: General Framework

A method to solve simultaneous linear equations

Two phases:

I. **Forward Elimination of Unknowns**: to reduce the set of equations to an upper triangular system

II. **Back Substitution**

Phase I: Forward Elimination

Step 1: Eliminating x_1 from the 2nd through the nth equations

-Multiply 1st equation by $\dfrac{a_{21}}{a_{11}}$

$$\left[\frac{a_{21}}{a_{11}}\right](a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Phase I: Forward Elimination

-Subtract the result from Equation 2:

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or

$$\boxed{a_{22}^{'}x_2 + \ldots + a_{2n}^{'}x_n = b_2^{'}}$$

Phase I: Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b'_3$$

$$\vdots$$

$$a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n = b'_n$$

End of Step 1

Phase I: Forward Elimination

Step 2: Eliminating x_2 from the 3rd through the nth equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

$$\vdots$$

$$a''_{n3}x_3 + \ldots + a''_{nn}x_n = b''_n$$

End of Step 2

Step n-1: Eliminating x_{n-1} from the nth equation

At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
\vdots \qquad \vdots \qquad &\vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

End of Step (n-1)

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n\text{-}1)} \end{bmatrix}$$

Phase II: Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Phase II: Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)} x_{i+1} - a_{i,i+2}^{(i-1)} x_{i+2} - \ldots - a_{i,n}^{(i-1)} x_n}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

Example 1: Solve the following system using Gauss elimination method

$$2x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 5x_2 + 3x_3 = -2$$
$$-3x_1 + 6x_2 - 2x_3 = 4$$

Solution

$$Eq2 - \frac{3}{2}Eq1$$
$$Eq3 - \frac{-3}{2}Eq1$$

1st step gives us:

$$2x_1 - 2x_2 + 3x_3 = 1$$
$$-2x_2 - \frac{3}{2}x_3 = -\frac{7}{2}$$
$$11x_2 - 5x_3 = 6$$

$$Eq3 - \frac{11}{-2}Eq2$$

2nd step gives us:

$$2x_1 - 2x_2 + 3x_3 = 1$$
$$-2x_2 - \frac{3}{2}x_3 = -\frac{7}{2}$$
$$-\frac{53}{4}x_3 = -\frac{53}{4}$$

Backward substitution gives us

$$x_1 = 0, x_2 = 1, x_3 = 1$$

Gauss Elimination Method: Using Augmented Matrix

Augmented of Linear System of Equations

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{aligned}
$$

is the matrix

$$
\overline{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}
$$

Example: Solve the following system by Gauss elimination method

$$
\begin{aligned}
x_1 + 2x_2 - 2x_3 &= -1 \\
-2x_1 - 3x_2 + x_3 &= 1 \\
2x_1 + 5x_2 - 2x_3 &= -3
\end{aligned}
$$

Solution

$$
\overline{A} = \begin{bmatrix} 1 & 2 & -2 & -1 \\ -2 & -3 & 1 & 1 \\ 2 & 5 & -2 & -3 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -3 & -1 \\ 2 & 5 & -2 & -3 \end{bmatrix}
$$

$$
\xrightarrow{R_3-2R_1} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 5 & 0 \end{bmatrix}
$$

$$
\begin{aligned}
x_1 + 2x_2 - 2x_3 &= -1 \\
x_2 - 3x_3 &= -1 \\
5x_3 &= 0
\end{aligned}
\qquad \Rightarrow x_3 = 0, x_2 = -1, x_1 = 1
$$

Exercises

Solve the following systems by Gauss elimination method

1)

$$2x_1 - 3x_2 + 4x_3 = 1$$
$$4x_1 - 3x_2 + 6x_3 = 3$$
$$-3x_1 - 4x_2 + 2x_3 = -2$$

2)

$$2x_1 + 2x_2 - 3x_3 + 4x_4 = 1$$
$$4x_1 + 6x_2 - 2x_3 + 5x_4 = 3$$
$$-x_1 + 4x_2 - 7x_3 + 5x_4 = -2$$
$$-4x_1 + 2x_2 - 3x_3 + 2x_4 = -1$$

2. LU Decomposition Methods

Vector Form of Linear systems of equations

Linear systems of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

can be put into the vector form $Ax = b$

where

$$
A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}
$$

LU Decomposition Methods: General Framework

Decompose A as the product of a lower triangular matrix L and an upper triangular matrix U:

$$A = LU$$

where

$$L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

Then

$$b = Ax = (LU)x = L(Ux) = Ly, \text{ where } y = Ux$$

So

$$Ax = b \Leftrightarrow \begin{cases} Ly = b & \text{solved by forward substitution} \\ Ux = y & \text{solved by backward substitution} \end{cases}$$

Remarks

Several methods have been proposed to determine L and U uniquely.

- Doolittle's method : $l_{11} = l_{22} = \ldots = l_{nn} = 1$
- Crout's method : $u_{11} = u_{22} = \ldots = u_{nn} = 1$
- Choleski's method : $u_{11} = l_{11}, u_{22} = l_{22}, \ldots, u_{nn} = l_{nn}$

Formula for Crout's method $A = LU$

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}, \quad i \ge j, i = 1, 2, \ldots, n$$

$$u_{ij} = \frac{1}{l_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \right), \quad i < j, i = 2, 3, \ldots, n$$

$$u_{ii} = 1, \quad i = 1, 2, \ldots, n$$

For the relevant indices i and j, it is convenient to compute in the order: $l_{i1}, u_{1j}; l_{i2}, u_{2j}; l_{i3}, u_{3j}; \ldots; l_{i,n-1}, u_{n-1,j}, l_{nn}$

$$L^1, U_1, L^2, U_2, L^3, U_3 \ldots, L^{n-1}, U_{n-1}, l_{nn}$$

L^i: ith column of L $\quad$ U_j: jth row of U

Remark on Crout's method

- Decompose

$$A = LU$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- The first column of L coincides with first column of A

Example

1. Use Crout's method to solve the system

$$
\begin{aligned}
-2x_1 + 3x_2 - x_3 &= 2 \\
4x_1 - 2x_2 + 3x_3 &= 1 \\
2x_1 - x_2 + 3x_3 &= 2
\end{aligned}
$$

Example: Decomposition A=LU

Decompose A= LU using Crout's method

$$A = \begin{pmatrix} -2 & 3 & -1 \\ 4 & -2 & 3 \\ 2 & -1 & 3 \end{pmatrix}$$

Solution

L1:

$$A = \begin{pmatrix} \boxed{\begin{matrix} -2 \\ 4 \\ 2 \end{matrix}} & \begin{matrix} 3 \\ -2 \\ -1 \end{matrix} & \begin{matrix} -1 \\ 3 \\ 3 \end{matrix} \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} -2 \\ 4 \\ 2 \end{matrix}} & \begin{matrix} 0 \\ a \\ b \end{matrix} & \begin{matrix} 0 \\ 0 \\ c \end{matrix} \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

U_1:

$$A=\begin{pmatrix}-2 & 3 & -1\\ 4 & -2 & 3\\ 2 & -1 & 3\end{pmatrix}=\begin{pmatrix}-2 & 0 & 0\\ 4 & a & 0\\ 2 & b & c\end{pmatrix}\begin{pmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}$$

$$a_{12}=3=-2x\Rightarrow x=-3/2$$

$$a_{13}=-1=-2y\Rightarrow y=1/2$$

$$A=\begin{pmatrix}-2 & 3 & -1\\ 4 & -2 & 3\\ 2 & -1 & 3\end{pmatrix}=\begin{pmatrix}-2 & 0 & 0\\ 4 & a & 0\\ 2 & b & c\end{pmatrix}\begin{pmatrix}1 & -3/2 & 1/2\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}$$

L^2:

$$A=\begin{pmatrix}-2 & 3 & -1\\ 4 & -2 & 3\\ 2 & -1 & 3\end{pmatrix}=\begin{pmatrix}-2 & 0 & 0\\ 4 & a & 0\\ 2 & b & c\end{pmatrix}\begin{pmatrix}1 & -3/2 & 1/2\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}$$

$$a_{22}=-2=-6+a\Rightarrow a=4$$

$$a_{32}=-1=-3+b\Rightarrow b=2$$

$$A=\begin{pmatrix}-2 & 3 & -1\\ 4 & -2 & 3\\ 2 & -1 & 3\end{pmatrix}=\begin{pmatrix}-2 & 0 & 0\\ 4 & 4 & 0\\ 2 & 2 & c\end{pmatrix}\begin{pmatrix}1 & -3/2 & 1/2\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}$$

U_2:

$$A=\begin{pmatrix}-2 & 3 & -1\\ 4 & -2 & 3\\ 2 & -1 & 3\end{pmatrix}=\begin{pmatrix}-2 & 0 & 0\\ 4 & 4 & 0\\ 2 & 2 & c\end{pmatrix}\begin{pmatrix}1 & -3/2 & 1/2\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}$$

$$a_{23}=3=2+4z\Rightarrow z=1/4$$

l_{33}:

$$a_{33}=3=1+1/2+c\Rightarrow c=3/2$$

$$A=\begin{pmatrix}-2 & 3 & -1\\ 4 & -2 & 3\\ 2 & -1 & 3\end{pmatrix}=\begin{pmatrix}-2 & 0 & 0\\ 4 & 4 & 0\\ 2 & 2 & 3/2\end{pmatrix}\begin{pmatrix}1 & -3/2 & 1/2\\ 0 & 1 & 1/4\\ 0 & 0 & 1\end{pmatrix}$$

LU-Decomposition of matrix A

$$\Leftrightarrow Ax = b, \text{ where}$$

$$A = \begin{pmatrix} -2 & 3 & -1 \\ 4 & -2 & 3 \\ 2 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 2 & 3/2 \end{pmatrix} \begin{pmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{pmatrix} = LU$$

Solving Ly=b

$$\begin{pmatrix} -2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 2 & 3/2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow -2y_1 = 2 \Rightarrow y_1 = -1$$

$$4y_1 + 4y_2 = 1 \Rightarrow -4 + 4y_2 = 1 \Rightarrow y_2 = 5/4$$

$$2y_1 + 2y_2 + 3y_3/2 = 2 \Rightarrow -2 + 5/2 + 3y_3/2 = 2 \Rightarrow y_3 = 1$$

$$y = \begin{pmatrix} -1 \\ 5/4 \\ 1 \end{pmatrix}$$

Solving Ux=y

$$\begin{pmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5/4 \\ 1 \end{pmatrix}$$

$$x_3 = 1$$

$$x_2 + x_3/4 = 5/4 \Rightarrow x_2 = 1$$

$$x_1 - 3x_2/2 + x_3/2 = -1 \Rightarrow x_1 = 0$$

So, the solution is given by $x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Exercise

Use Crout's method to solve the linear system

a)

$$x_1 - 2x_2 + 4x_3 = 2$$
$$2x_1 - x_2 - x_3 = -2$$
$$-3x_1 + 8x_2 - 14x_3 = -6$$

b)

$$2x_1 + 2x_2 - 3x_3 + 4x_4 = 1$$
$$4x_1 + 8x_2 - 2x_3 + 5x_4 = 3$$
$$-x_1 + 4x_2 - 7x_3 + 5x_4 = -2$$
$$-4x_1 + 2x_2 - 3x_3 + 2x_4 = -1$$

An application: Finding Inverse Matrix

- Matrix multiplication has the Associative property:

$$(AB)C = A(BC)$$

- Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Then, the product Ae_k = kth column of A

Finding Inverse Matrix

- Therefore

$$AA^{-1} = I \Rightarrow (AA^{-1})e_k = Ie_k = e_k$$

$$\Rightarrow A(A^{-1}e_k) = e_k \Leftrightarrow Ax = e_k, x = A^{-1}e_k = k\text{th column of } A^{-1}$$

- Thus, the solution of the equation

$$Ax = e_k$$

gives the kth column of the inverse matrix A^{-1}

- *These equations can be solved by Crout's method!*

Example

- Find the inverse of the matrix

$$A = \begin{pmatrix} -2 & 3 & -1 \\ 4 & -2 & 3 \\ 2 & -1 & 3 \end{pmatrix}$$

Solution

- We have obtained the LU decomposition of A as

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 2 & 3/2 \end{pmatrix} \begin{pmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

- The first column of A^{-1} is the solution of the equation $Ax = e_1,$ where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Solution

- We solve this equation using Crout's method:

$$Ly = e_1 \Leftrightarrow \begin{pmatrix} -2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 2 & 3/2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow y_1 = -1/2,$$

$$4y_1 + 4y_2 = 0 \Rightarrow y_2 = 1/2$$

$$2y_1 + 2y_2 + 3y_3/2 = 0 \Rightarrow y_3 = 0$$

$$\Rightarrow \quad y = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

Solution

- Solving

$$Ux = y \Leftrightarrow \begin{pmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_3 = 0$$

$$x_2 + x_3/4 = 1/2 \Rightarrow x_2 = 1/2$$

$$x_1 - 3x_2/2 + x_3/2 = -1/2 \Rightarrow x_1 = 1/4$$

- Thus, the first column of inverse matrix is

$$x = \begin{pmatrix} 1/4 \\ 1/2 \\ 0 \end{pmatrix}$$

Solution

- The second column of the inverse matrix is the solution of $Ax=e_2$

$$Ly = e_2 \Leftrightarrow \begin{pmatrix} -2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 2 & 3/2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow y_1 = 0,$$

$$4y_1 + 4y_2 = 1 \Rightarrow y_2 = 1/4$$

$$2y_1 + 2y_2 + 3y_3/2 = 0 \Rightarrow y_3 = -1/3$$

$$\Rightarrow \quad y = \begin{pmatrix} 0 \\ 1/4 \\ -1/3 \end{pmatrix}$$

Solution

- Solving
$$Ux = y \Leftrightarrow \begin{pmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/4 \\ -1/3 \end{pmatrix}$$
$$\Rightarrow x_3 = -1/3$$
$$x_2 + x_3/4 = 1/4 \Rightarrow x_2 = 1/3$$
$$x_1 - 3x_2/2 + x_3/2 = 0 \Rightarrow x_1 - 1/2 - 1/6 = 0 \Rightarrow x_1 = 2/3$$

- Thus, the second column of inverse matrix is
$$x = \begin{pmatrix} 2/3 \\ 1/3 \\ -1/3 \end{pmatrix}$$

Solution

- The 3rd column of the inverse matrix is the solution of $Ax=e_3$

$$Ly = e_3 \Leftrightarrow \begin{pmatrix} -2 & 0 & 0 \\ 4 & 4 & 0 \\ 2 & 2 & 3/2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow y_1 = 0,$$

$$4y_1 + 4y_2 = 0 \Rightarrow y_2 = 0$$

$$2y_1 + 2y_2 + 3y_3/2 = 1 \Rightarrow y_3 = 2/3$$

$$y = \begin{pmatrix} 0 \\ 0 \\ 2/3 \end{pmatrix}$$

Solution

- Solving

$$Ux = y \Leftrightarrow \begin{pmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2/3 \end{pmatrix}$$

$$\Rightarrow x_3 = 2/3$$

$$x_2 + x_3/4 = 0 \Rightarrow x_2 = -1/6$$

$$x_1 - 3x_2/2 + x_3/2 = 0 \Rightarrow x_1 + 1/4 + 1/3 = 0 \Rightarrow x_1 = -7/12$$

- Thus, the third column of inverse matrix is $\begin{pmatrix} -7/12 \\ -1/6 \\ 2/3 \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} 1/4 & 2/3 & -7/12 \\ 1/2 & 1/3 & -1/6 \\ 0 & -1/3 & 2/3 \end{pmatrix}$$

Exercises

1. Find the inverse matrix of the given matrix

a) $$A = \begin{pmatrix} 1 & -2 & 4 \\ 2 & 6 & 1 \\ -3 & 8 & -7 \end{pmatrix}$$

b) $$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -6 & 5 \\ -1 & 4 & -2 \end{pmatrix}$$

Exercise

Use LU decomposition to find the inverse A^{-1} of matrix of coefficients of the following linear system, and then use it to get the solution of the form $x = A^{-1}b$

$$2x_1 - 3x_2 + 4x_3 + x_4 = 1$$

$$x_1 - 2x_2 + 5x_3 - 3x_4 = 3$$

$$-2x_1 + 5x_2 - 6x_3 + 2x_4 = -1$$

$$x_1 + 3x_2 - 2x_3 + 8x_4 = 1$$

Homework N2: Part I

- In S. Chapra & R.P. Canale, Numerical Methods for Engineers: with software and Programming Appl, McGraw-Hill, 7th ed., 2015
- Page 276: Problems 9.8, 9.9, 9.11, 9.14
- Page 297: Problems 10.3, 10.4, 10.7, 10.9
- Page 317: Problems 11.9, 11.11

Deadline: 2 weeks

Homework Chapter 2: Part II

Ex 1. Solve the following systems by Gauss elimination method

1)

$$mx_1 - 3x_2 + 4x_3 = 1$$
$$x_1 - 3x_2 + 6x_3 = 3$$
$$nx_1 - 4x_2 + 2x_3 = -2$$

2)

$$mx_1 + nx_2 - 3x_3 + 4x_4 = 1$$
$$nx_1 + mx_2 - 2x_3 + 5x_4 = 3$$
$$-mx_1 + 3nx_2 - 6x_3 + 2x_4 = -4$$
$$-2mx_1 - 3nx_2 - 3x_3 + 2x_4 = -1$$

$\overline{mn}$ = The last two digits of your student ID number

Part II: Exercise 2

a) Solve the following system

$$mx_1 + 4x_2 - mx_3 = 4 - m$$
$$nx_1 - 5x_2 + 3x_3 = -2$$
$$mx_1 - x_2 + nx_3 = n - 1$$

using Crout's method

b) Find Inverse matrix of matrix of coefficients of the above system using LU decomposition method

Part II: Exercise 3

Using Gauss-Seidel iteration method, solve the following system of linear equations starting at (2,2,2,2,2)

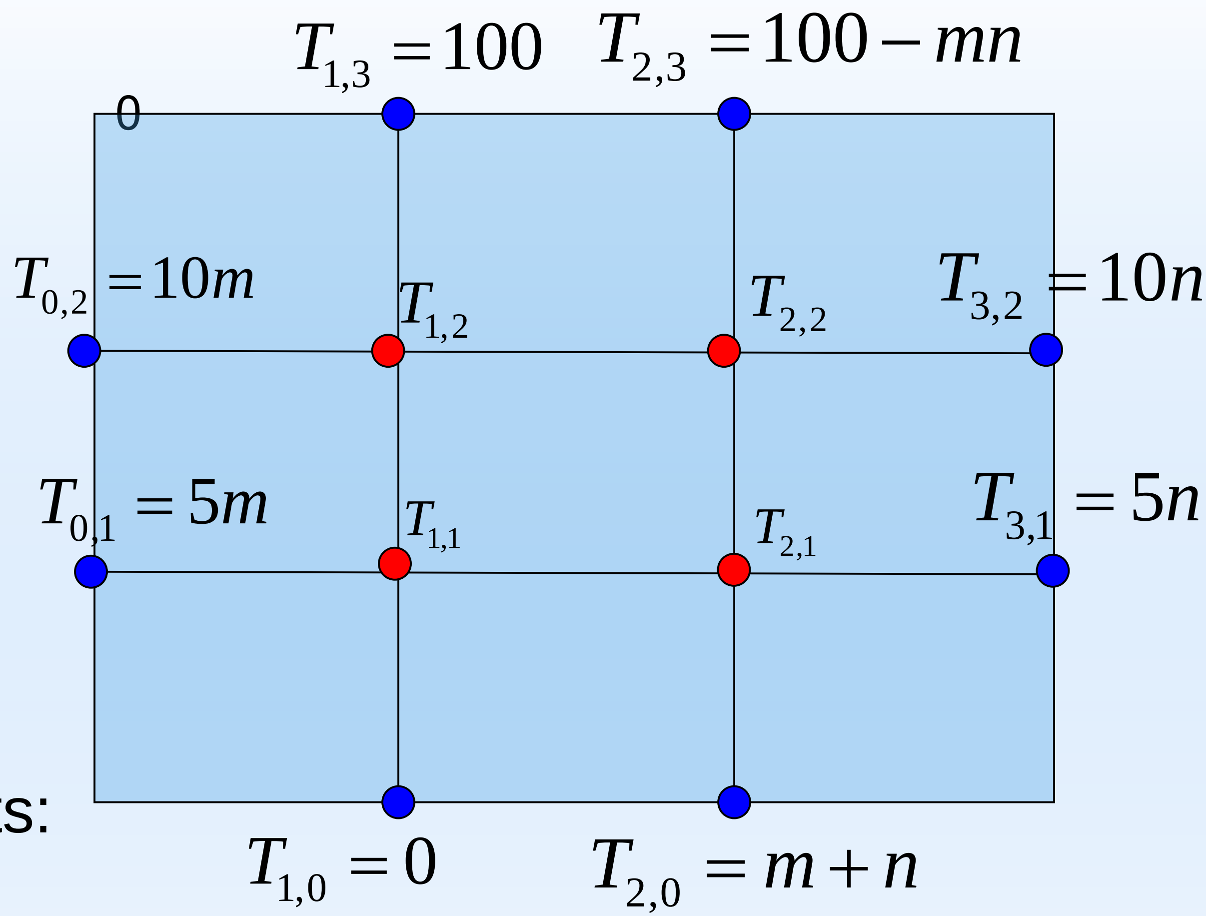
$$
\begin{aligned}
(m+5)x_1 - 3x_2 &= m-2 \\
3x_1 - (n+4)x_2 + 2x_3 &= -n-1 \\
-3x_2 + (m+6)x_3 - 2x_4 &= -3 \\
x_3 - (m+8)x_4 + 2x_5 &= 0 \\
-2x_1 - x_4 + (n+7)x_5 &= -2
\end{aligned}
$$

$\overline{mn}$ = The last two digits of your student ID number

Stopping condition: $|E_a| \le 0.01$

Quiz: Find temperature in a square sheet of metal. The temperature at the edges of the sheet are kept as in Figure

Approximate T at grid points



Temperature at grid points:

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

$\overline{mn}$ = The last two digits of your student ID number

Solve by a) Gauss elimination

b) LU decomposition method

c) Gauss-Seidel method, use $|E_a| \leq 0.5$