Chapter 2: Solutions of Linear Systems of Equations

Lecture 2

Gauss-Seidel iteration Method

Gauss-Seidel Method

Description

A set of *n* equations and *n* unknowns:

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n \end{array}}$$

The diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for x_1

Second equation, solve for x_2

Gauss-Seidel Method

Transformation

Rewriting each equation

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$ From Equation 1

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$ From equation 2

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$ From equation n-1

$$x_n = \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$ From equation n

Algorithm for Gauss-Seidel Method

Let $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, ..., x_n^{(k)})$ be a given approximate value,

$k = 0, 1, 2, ...,$ we compute the new approximate value $x^{(k+1)}$:

$$x_1^{(k+1)} = \frac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} \ldots\ldots - a_{1n}x_n^{(k)}}{a_{11}}$$

$\ldots$

$$x_i^{(k+1)} = \frac{b_i - a_{i1}x_1^{(k+1)} - ... - a_{i,i-1}x_{i-1}^{(k+1)} - a_{i,i+1}x_{i+1}^{(k)} \ldots\ldots - a_{1n}x_n^{(k)}}{a_{ii}}$$

$\ldots$

$$x_n^{(k+1)} = \frac{b_n - a_{n1}x_1^{(k+1)} - ... - a_{1,n-1}x_{n-1}^{(k+1)}}{a_{nn}}$$

Repeat this process until a stopping criterion is met

Stopping Criteria

We can use either Approximate Error or Relative Approximate Error to stop the computation process

$$E_a = \max\{|\, x_i^{new} - x_i^{old} \,|, i = 1, 2, \ldots, n\}$$

$$\varepsilon_a = \max\left\{\left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right|, i = 1, 2, \ldots, n\right\}$$

The iterations are stopped when the absolute relative approximate error is less than a prespecified tolerance for all unknowns

.

$$x_1, x_2, \ldots, x_n$$

Example

- Use Gauss-Seidel method, find the root of the system with stopping criterion $|\mathrm{E_a}| \le 0.01$ with starting point $x^{(0)} = (0,0,0)$:

$$\begin{aligned} 6x_1 - x_2 + 2x_3 &= 1 \\ x_1 - 8x_2 + 3x_3 &= -5 \\ -2x_1 + 2x_2 - 9x_3 &= -7 \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} x_1 &= (1 + x_2 - 2x_3)/6 \\ x_2 &= (-5 - x_1 - 3x_3)/(-8) \\ x_3 &= (-7 + 2x_1 - 2x_2)/(-9) \end{aligned}$$

$\Rightarrow$ Iterative Formula:

$$\begin{aligned} x_1^{(k+1)} &= (1 + x_2^{(k)} - 2x_3^{(k)})/6 \\ x_2^{(k+1)} &= (-5 - x_1^{(k+1)} - 3x_3^{(k)})/(-8) \\ x_3^{(k+1)} &= (-7 + 2x_1^{(k+1)} - 2x_2^{(k+1)})/(-9) \end{aligned}$$

Solution

Starting with $x^{(0)} = (0,0,0)$, and using:

$$\boxed{\begin{aligned} x_1^{(k+1)} &= (1 + x_2^{(k)} - 2x_3^{(k)})/6 \\ x_2^{(k+1)} &= (-5 - x_1^{(k+1)} - 3x_3^{(k)})/(-8) \\ x_3^{(k+1)} &= (-7 + 2x_1^{(k+1)} - 2x_2^{(k+1)})/(-9) \end{aligned}}$$

we get

$x^{(k)}$	$\mathbf{x_1}$	$\lvert x_1^{new} - x_1^{old} \rvert$	$\mathbf{x_2}$	$\lvert x_2^{new} - x_2^{old} \rvert$	$\mathbf{x_3}$	$\lvert x_3^{new} - x_3^{old} \rvert$	$\mathbf{E_a}$
1	0.1667		0.6458		0.8843		--
2	-0.0204	0.1871	0.9540	0.3082	0.9943	0.1100	0.3082
3	-0.0058	0.0146	0.9972	0.0432	1.0006	0.0063	0.0432
4	-0.0007	0.0051	1.0002	0.0030	1.0002	0.0004	0.0051

So, approximate solution is x=(-0.0007, 1.0002, 1.0002)

Remark

Gauss-Seidel method: not all systems of equations will converge.

One class of system of equations always converges: One with a *diagonally dominant* coefficient matrix.

Diagonally dominant: [A] in [A] [X] = [C] is diagonally dominant if:

$$\left|a_{\text{ii}}\right| \geq \sum_{\substack{j=1 \\ j\neq \text{i}}}^{n} \left|a_{ij}\right| \quad \text{for all 'i'} \qquad \text{and} \qquad \left|a_{ii}\right| \rangle \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right| \quad \text{for at least one 'i'}$$

Comments on Gauss-Seidel

- Disadvantages:
  - it may not converge
  - When it converged, it did so very slowly
- Advantages:
  - When matrix is very sparse (most elements are zero), elimination methods waste large amount of computer memory by storing zeros
  - Gauss-Seidel method saves memory because only nonzero coefficients are involved in the structure of equations

Exercise 1

Solve the following system

$$
\begin{aligned}
5x_1 - 2x_2 &= 1 \\
x_1 + 8x_2 - 2x_3 &= 11 \\
-x_2 + 7x_3 - 2x_4 &= 11 \\
-2x_3 + 5x_4 &= 14
\end{aligned}
$$

Using

a) LU decomposition method

b) Gauss-Seidel iteration method

Use $x^{(0)}$ =(1,1,1,1), stopping condition: $|E_a| \le 0.02$

Exercise 4

Using Gauss-Seidel iteration method, solve the following system

$$8x_1 + 3x_2 = -21$$

$$x_1 - 5x_2 + 2x_3 = 10$$

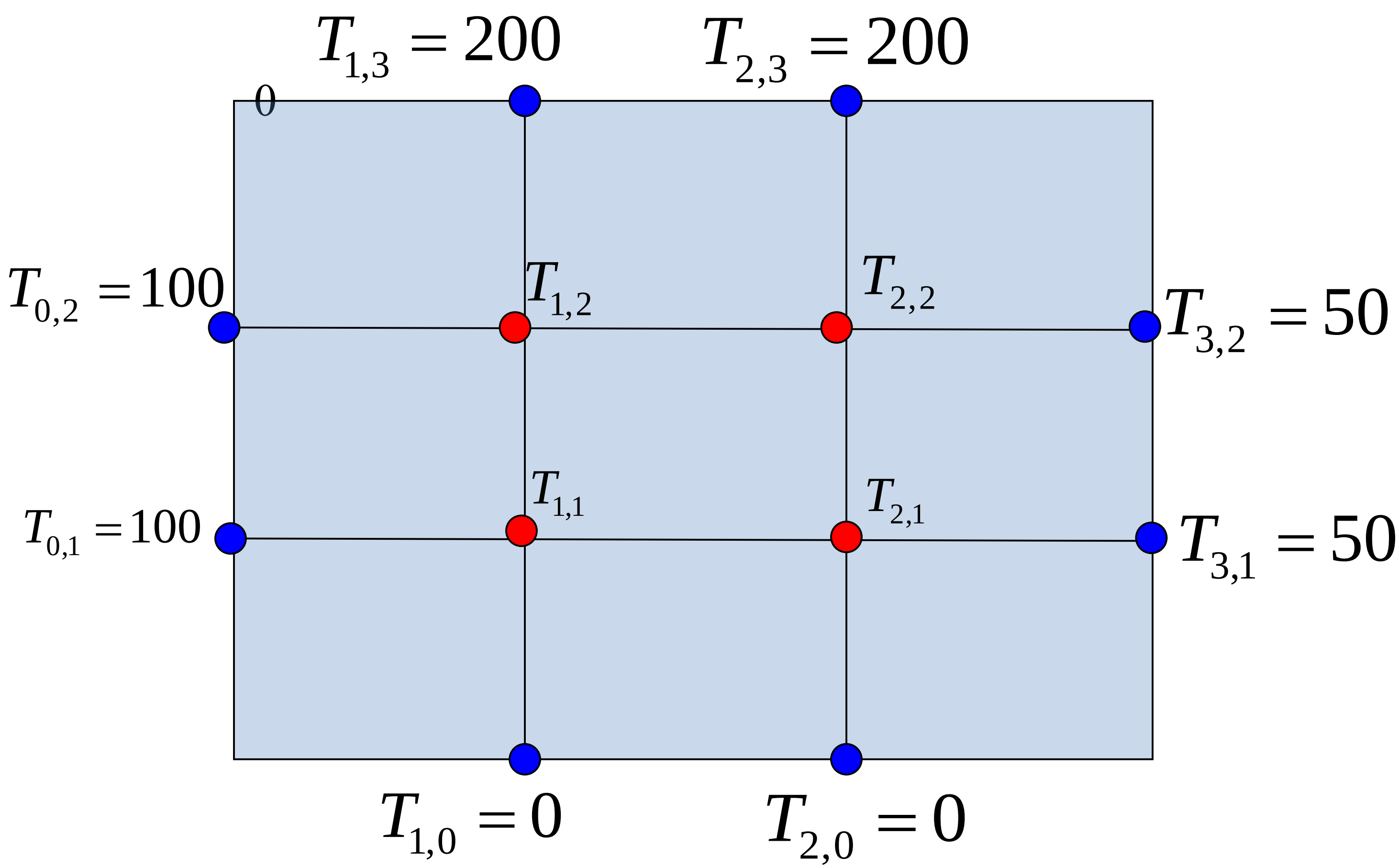
$$-x_2 + 6x_3 + 2x_4 = 71$$

$$3x_3 - 5x_4 + 2x_5 = -2$$

$$-2x_4 + 5x_5 = 22$$

Use $x^{(0)}$ =(1,2,3,4,5), stopping condition: $|E_a| \leq 0.05$

Ex 2: Find temperature in a square sheet of metal. The temperature at the edges of the sheet are kept at : 200, 100, 50, and 0 degrees.



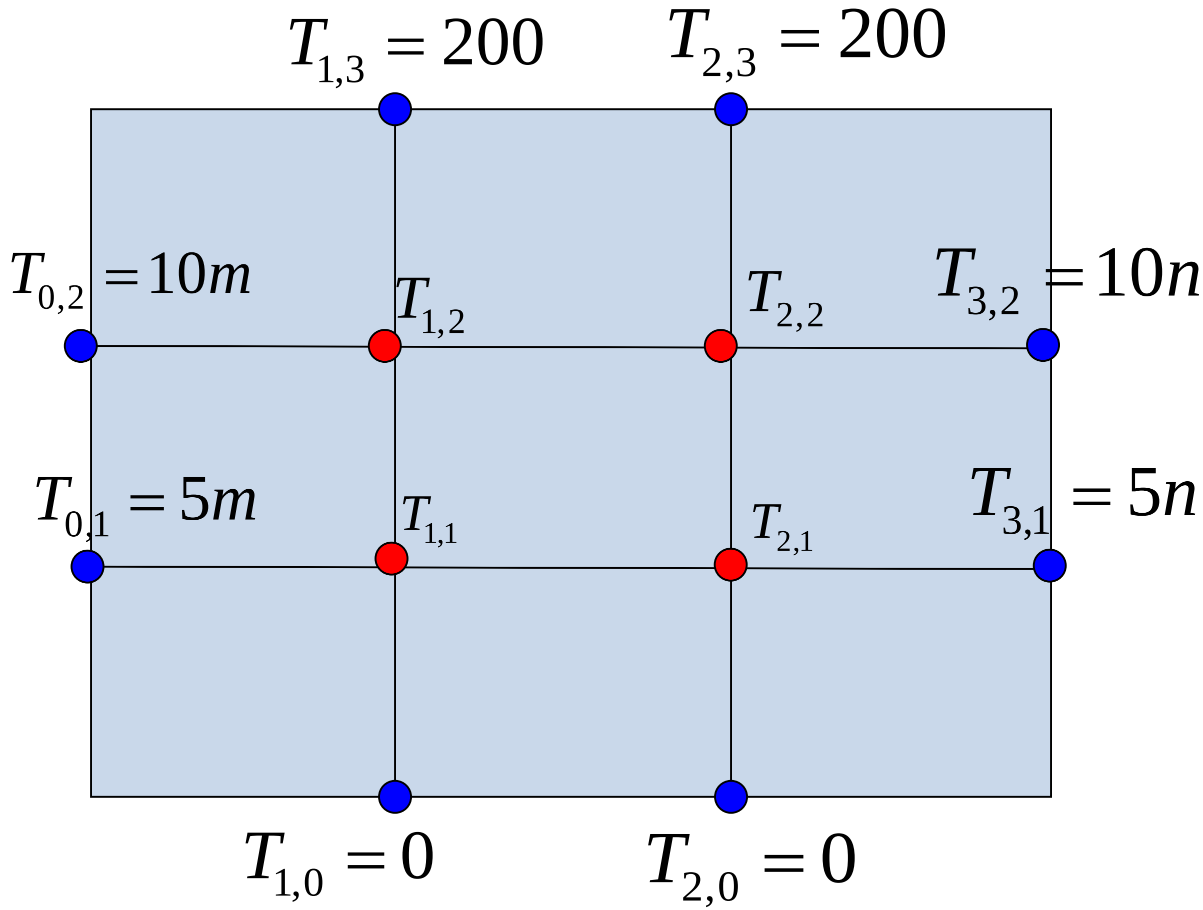
Temperature at grid points: $T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$

Solve by Gauss-Seidel method, use $|E_a| \leq 0.5$

Quiz

Find temperature in a square sheet of metal. The temperature at edges of the sheet are kept as in Figure

Approximate T at grid points



Temperature at grid points:

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

$\overline{mn}$ = The last two digits of your student ID number

Solve by a) Gauss elimination

b) LU decomposition method

c) Gauss-Seidel method, use $|E_a| \leq 0.5$