INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

MIDTERM EXAMINATION

November 2020

Duration: 90 minutes

SUBJECT: REAL ANALYSIS	
Head of Dept. of Mathematics:	Lecturer:
Prof. Pham Huu Anh Ngoc	Assoc. Prof. Nguyen Ngoc Hai

INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 *(25 marks)* Let $X = (0, \infty)$. Show that

$$\rho(x, y) = |\ln x - \ln y|, \quad x, y \in X,$$

is a distance on X. Determine the closed unit ball $\overline{B}(1, 1)$ in the metric space (X, ρ).

Question 2 (a) *(15 marks)* Let a, b, c be constants such that $a^2 + b^2 > 0$. Let

$$A = \{(x, y) \in \mathbb{R}^2 : ax + by \leq c\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 : ax + by < c\}.$$

Show that A is closed and B is open in the Euclidean space $\mathbb{R}^2$.

(b) *(10 marks)* Let S be a nonempty and closed subset of $\mathbb{R}$. Show that if $a := \sup S < \infty$, then $a \in S$.

Question 3 *(20 marks)* Let (X, d) be a metric space and $f : X \to \mathbb{R}$. Suppose that the sets $\{x \in X : f(x) > \alpha\}$ and $\{x \in X : f(x) < \alpha\}$ are open for every $\alpha \in \mathbb{R}$.

(a) Show that for all $a, b \in \mathbb{R}$ with $a < b$, the sets $\{x \in X : a < f(x) < b\}$ are open.

(b) Show that f is continuous.

Question 4 (a) *(15 marks)* Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$. Show that f is continuous if and only if

$$\overline{f^{-1}(E)} \subset f^{-1}(\overline{E}) \quad \text{for each } E \subset Y.$$

(b) *(15 marks)* Let a, b, c be positive numbers. Show that the set

$$A = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$$

is compact.

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