HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

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4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

Definition 1.1

An ***n*th order linear differential equation** is an equation of the form

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x) = b(x). \qquad (0.1)$$

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When $a_0, a_1, ..., a_n$ are constants, we say that Equation (0.1) has **constant coefficients**.

If $b(x) \equiv 0$, Equation (0.1) is called **homogeneous**; otherwise it is **nonhomogeneous**.

4.1 GENERAL THEORY OF *n*th ORDER LINEAR EQUATIONS

Assume that $a_0(x), a_1(x), ..., a_n(x)$, and $b(x)$ are continuous on an interval I and $a_0(x)$ is nowhere zero on I.

Then, on dividing by $a_0(x)$, we can rewrite (0.1) in the **standard form**

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x), \tag{0.2}$$

where $p_1(x), ..., p_n(x)$, and $g(x)$ are continuous on I.

4.1 GENERAL THEORY OF *n*th ORDER LINEAR EQUATIONS

Theorem 4.1 (Existence and uniqueness of solutions)

Suppose that $p_1(x), \ldots, p_n(x)$, and $g(x)$ are each continuous on an interval (a, b) that contains the point x_0. Then for any choice of the initial values $y_0, y_1, \ldots, y_{n-1}$, there exists a unique solution $y(x)$ on the whole interval (a, b) of the initial value problem

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x),$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

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$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x),$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

Remark: An initial value problem of n-th order differential equations contains n initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

The Homogeneous Equations

The operator

$$L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y,$$

is *linear* and we can express Equation (0.2) in the form

$$L[y] = g(x). \tag{0.3}$$

- If $y_1, y_2, ..., y_m$ are solutions of the *homogeneous* equation

$$L[y] = 0, \tag{0.4}$$

then any linear combination of these functions,

$$y = C_1y_1 + C_2y_2 + \cdots + C_my_m,$$

is also a solution of (0.4).

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The determinant

$$\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$

is called the **Wronskian** of $y_1, y_2, \ldots, y_n$ and denoted by $W[y_1, y_2, \ldots, y_n](x)$.

4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

Theorem 4.2 **(Representation of solutions (Homogeneous case))**

Let $y_1, y_2, ..., y_n$ be n solutions on (a, b) of

$$L[y] = y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0, \qquad (0.5)$$

where $p_1(x), ..., p_n(x)$ are continuous on (a, b). If the Wronskian

$$W[y_1, y_2, ..., y_n](x) \neq 0$$

for at least one point in (a, b), then every solution of (0.5) can be expressed in the form

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x), \qquad (0.6)$$

where $C_1, C_2, ..., C_n$ are constants.

4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

If $y_1, y_2, ..., y_n$ are solutions of the equation $L[y] = 0$, then

> *$W[y_1, y_2, ..., y_n](x)$ either is zero for every x in the interval (a, b) or else is never zero there.*

• A set of solutions $y_1, y_2, ..., y_n$ of equation $L[y] = 0$ whose Wronskian is nonzero is referred to as a **fundamental set of solutions**.

• We use the term **general solution** to refer to an arbitrary linear combination of any fundamental set of solutions of equation $L[y] = 0$.

4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

Example 1.1 Given that $y_1(x) = x, y_2(x) = x^2$, and $y_3(x) = x^{-1}$ are solutions to

$$x^3y''' + x^2y'' - 2xy' + 2y = 0, \quad x > 0,$$

find the general solution.

4.1 GENERAL THEORY OF *n*th ORDER LINEAR EQUATIONS

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find the general solution.

Solution: Since

$$\begin{vmatrix} x & x^2 & \frac{1}{x} \\ 1 & 2x & -\frac{1}{x^2} \\ 0 & 2 & \frac{2}{x^3} \end{vmatrix} = \frac{6}{x} \neq 0, \quad \forall x \neq 0,$$

$\{y_1(x) = x; y_2(x) = x^2; y_3(x) = \frac{1}{x}\}$ forms a fundamental solution set of the given differential equation. Thus the general solution is

$$y(x) = c_1x + c_2x^2 + c_3\frac{1}{x}, \quad x \in \mathbb{R} \setminus \{0\}.$$

4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

Definition 1.2

The m functions $f_1, f_2, ..., f_m$ are said to be **linearly dependent** on an interval I if there exists constants $c_1, c_2, ..., c_n$ not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0$$

for all x in I. If the functions $f_1, f_2, ..., f_m$ are not linearly dependent on I, they are said to be **linearly independent** on I.

The following sets of functions are linearly independent on every interval (a, b):

$$\{1, x, x^2, ..., x^n\},$$
$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, ..., \cos nx, \sin nx\}$$
$$\{e^{\alpha_1 x}, e^{\alpha_2 x}, ..., e^{\alpha_n x}\}, \quad (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n \text{ are distinct numbers})$$

4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

Theorem 4.3

Let $y_1, y_2, ..., y_n$ be n solutions of

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0,$$

on (a, b). Then $\{y_1, y_2, ..., y_n\}$ is a fundamental set of solutions on (a, b) if and only if these functions are linearly independent on (a, b).

4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

The Nonhomogeneous Equations

Consider the nonhomogeneous equation

$$L[y] = y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x). \tag{0.7}$$

If y_1 and y_2 are any two solutions of (0.7), then

$$L[y_1 - y_2](x) = L[y_1](x) - L[y_2](x) = g(x) - g(x) = 0.$$

Hence,

> The difference of any two solutions of the nonhomogeneous equation (0.7) is a solution of the corresponding homogeneous equation.

4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

It follows that

Any solution of the nonhomogeneous equation

$$L[y] = y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x). \quad (0.8)$$

can be written as

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) + y_p(x) = y_h(x) + y_p(x),$$

where $y_h(x)$ is the general solution of the corresponding homogeneous equation and $y_p(x)$ is some particular solution of the nonhomogeneous equation (0.8).

4.1 GENERAL THEORY OF nth ORDER LINEAR EQUATIONS

Theorem 4.4 (**Representation of Solution (Nonhomogeneous case)**)

Let $y_p(x)$ be a particular solution of the nonhomogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x) \tag{0.9}$$

on the interval (a, b) and let $\{y_1, y_2, ..., y_n\}$ be a fundamental set of solutions on (a, b) for the corresponding homogeneous equation $y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0$. Then every solutions of the nonhomogeneous equation on (a, b) can be expressed in the form

$$\boxed{y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) + y_p(x)} \tag{0.10}$$

The linear combination (0.10) is called the **general solution** of the nonhomogeneous equation (0.9).

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Example 1.2 Given that $y_p(x) = x^2$ is a particular solution of

$$y''' - 2y'' - y' + 2y = 2x^2 - 2x - 4 \tag{0.11}$$

on $(-\infty, \infty)$ and that $y_1(x) = e^{-x}$, $y_2(x) = e^{x}$, $y_3(x) = e^{2x}$ are solutions to the corresponding homogeneous equation, find the general solution of (0.11).

4.2 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Let's consider the nth order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_n y(x) = 0, \tag{0.12}$$

where $a_0, a_1, ..., a_n$ are constants. Then e^{rx} is a solution of Equation (0.12) if and only if r is a root of the **characteristic equation**

$$\boxed{P(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0}$$

of the differential equation (0.12).

4.2 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Distinct Real Roots

If the roots $r_1, r_2, \ldots, r_n$ of the characteristic equation are real and distinct, then n linearly independent solutions of the equation

$$L[y] = a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_n y(x) = 0 \qquad (0.13)$$

are

$$y_1(x) = e^{r_1 x},\ y_2(x) = e^{r_2 x}, \ldots,\ y_n(x) = e^{r_n x}.$$

Thus the general solution of Equation (0.13) is

$$\boxed{y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x}}$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

4.2 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

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Solution: The characteristic equation of the given differential equation is:

$$k^3 - 2k^2 - 5k + 6 = 0, \quad \text{or equivalently,} \quad (k-1)(k+2)(k-3) = 0.$$

Then $\{e^x, e^{-2x}, e^{3x}\}$ is a fundamental solution set of the given differential equation. Therefore, the general solution is given by

$$y(x) = c_1 e^x + c_2 e^{-2x} + c_3 e^{3x}, \quad x \in \mathbb{R}.$$

Example 2.2 Find the general solution of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0.$$

Also find the solution that satisfies the initial conditions

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = -1$$

and plot its graph.

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and plot its graph.

Solution:

Assuming that $y = e^{rt}$, we must determine r by solving the polynomial equation

$$r^4 + r^3 - 7r^2 - r + 6 = 0. \tag{8}$$

The roots of this equation are $r_1 = 1$, $r_2 = -1$, $r_3 = 2$, and $r_4 = -3$. Therefore the general solution of Eq. (6) is

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}. \tag{9}$$

The initial conditions [redacted] require that $c_1, \ldots, c_4$ satisfy the four equations

$$
\begin{aligned}
c_1 + c_2 + c_3 + c_4 &= 1, \\
c_1 - c_2 + 2c_3 - 3c_4 &= 0, \\
c_1 + c_2 + 4c_3 + 9c_4 &= -2, \\
c_1 - c_2 + 8c_3 - 27c_4 &= -1.
\end{aligned}
\tag{10}
$$

By solving this system of four linear algebraic equations, we find that

$$c_1 = 11/8, \qquad c_2 = 5/12, \qquad c_3 = -2/3, \qquad c_4 = -1/8.$$

Therefore the solution of the initial value problem is

$$y = \tfrac{11}{8}e^{t} + \tfrac{5}{12}e^{-t} - \tfrac{2}{3}e^{2t} - \tfrac{1}{8}e^{-3t}. \tag{11}$$

4.2 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Complex Roots

If the characteristic equation has a complex root $\alpha + i\beta$, then so is its conjugate $\alpha - i\beta$. Take the real and imaginary parts of the complex root

$$e^{(\alpha+i\beta)x} = e^{\alpha x}\cos\beta x + ie^{\alpha x}\sin\beta x$$

to get two real-valued solutions

$$e^{\alpha x}\cos\beta x \quad \text{and} \quad e^{\alpha x}\sin\beta x.$$

Example 2.3 Find the general solution of the equation

$$y^{(4)} - y = 0 \qquad\qquad (14).$$

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Solution: The characteristic equation is

$$r^4 - 1 = (r^2 - 1)(r^2 + 1) = 0.$$

Therefore the roots are $r = 1, -1, i, -i$, and the general solution of Eq. (14) is

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

Example 2.4 Find the general solution of $y^{(4)} + y = 0$.

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Solution:

The characteristic equation is

$$r^4 + 1 = 0.$$

To solve the equation we must compute the fourth roots of -1. Now -1, thought of as a complex number, is $-1 + 0i$. It has magnitude 1 and polar angle π. Thus

$$-1 = \cos\pi + i\sin\pi = e^{i\pi}.$$

Moreover, the angle is determined only up to a multiple of 2π. Thus

$$-1 = \cos(\pi + 2m\pi) + i\sin(\pi + 2m\pi) = e^{i(\pi+2m\pi)},$$

where m is zero or any positive or negative integer. Thus

$$(-1)^{1/4} = e^{i(\pi/4+m\pi/2)} = \cos\left(\frac{\pi}{4} + \frac{m\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{m\pi}{2}\right).$$

The four fourth roots of -1 are obtained by setting $m = 0, 1, 2,$ and 3; they are

$$\frac{1+i}{\sqrt{2}}, \quad \frac{-1+i}{\sqrt{2}}, \quad \frac{-1-i}{\sqrt{2}}, \quad \frac{1-i}{\sqrt{2}}.$$

The general solution is

$$y = e^{t/\sqrt{2}}\left(c_1 \cos\frac{t}{\sqrt{2}} + c_2 \sin\frac{t}{\sqrt{2}}\right) + e^{-t/\sqrt{2}}\left(c_3 \cos\frac{t}{\sqrt{2}} + c_4 \sin\frac{t}{\sqrt{2}}\right).$$

4.2 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Repeated Roots

- If r is a root of multiplicity m of the characteristic equation, then

$$e^{rx},\ xe^{rx}, \ldots,\ x^{m-1}e^{rx}$$

are solutions of the differential equation $L[y] = 0$.

4.2 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Repeated Roots

- If r is a root of multiplicity m of the characteristic equation, then

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are solutions of the differential equation $L[y] = 0$.

- If a complex root $\alpha + i\beta$ is repeated m times, then $\alpha - i\beta$ is also a root of multiplicity m. We can find $2m$ linearly independent real solutions

$$e^{\alpha x}\cos\beta x, \qquad xe^{\alpha x}\cos\beta x, \dots, \quad x^{m-1}e^{\alpha x}\cos\beta x,$$

$$e^{\alpha x}\sin\beta x, \quad xe^{\alpha x}\sin\beta x, \dots, x^{m-1}e^{\alpha x}\sin\beta x.$$

4.2 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Example 2.5 Find the general solution of

$$y^{(4)} + 2y'' + y = 0 \qquad (19).$$

Solution:

The characteristic equation is

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0.$$

The roots are $r = i, i, -i, -i$, and the general solution of Eq. (19) is

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

4.2 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Example: Suppose that a 14-th order homogeneous linear differential equation with constant coefficients has characteristic roots:

$$-3, 1, 0, 0, 2, 2, 2, 2, 3+4i, 3+4i, 3+4i, 3-4i, 3-4i, 3-4i.$$

What is the general solution of the differential equation?

Solution: The general solution of the differential equation is given by

$y(x) =$

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What is the general solution of the differential equation?

Solution: The general solution of the differential equation is given by

$$y(x) = C_1 e^{-3x} + C_2 e^{x} + C_3 + xC_4 + C_5 e^{2x} + C_6 x e^{2x} + C_7 x^2 e^{2x} + C_8 x^3 e^{2x} +$$

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$$C_9 e^{3x}\cos 4x + C_{10} e^{3x}\sin 4x + C_{11} x e^{3x}\cos 4x + C_{12} x e^{3x}\sin 4x$$
$$+ C_{13} x^2 e^{3x}\cos 4x + C_{14} x^2 e^{3x}\sin 4x.$$

4.2 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Example 2.6 Find the general solution of

$$y^{(4)} - 3y''' + 3y'' - y' = 0.$$

Example 2.7 Find the general solution of the equation

$$y^{(5)} - y^{(4)} + 8y''' - 8y'' + 16y' - 16y = 0.$$

The Method of Undetermined Coefficients

The method of undetermined coefficients can be applied to linear n-th order differential equations

$$y^{(n)}(x) + p_1 y^{(n-1)}(x) + \cdots + p_n y(x) = g(x).$$

If g(t) is a sum of polynomials, exponentials, sines, and cosines, or products of such functions, it is possible to find a particular solution $y_p(t)$ by choosing a suitable combination of polynomials, exponentials, and so forth, multiplied by a number of undetermined constants.

The main difference in using this method for higher order equations stems from the fact that roots of the characteristic equation may have multiplicity greater than 2.

Example 3.1 Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t.$$

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Solution: The homogeneous equation corresponding to the given equation is

$$y''' - 3y'' + 3y' - y = 0.$$

The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0.$$

So the general solution of the homogeneous equation is

$$\boxed{y_c(t) = c_1 e^t + c_2 te^t + c_3 t^2 e^t}.$$

Since $r = 1$ is a root of multiplicity 3 of the characteristic equation, we find a particular $y_p(t)$ of $y''' - 3y'' + 3y' - y = 4e^t$ in the form

$$y_p(t) = At^3e^t.$$

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Evaluate y_p''', y_p'', y_p' and substitute them into the equation

$$y''' - 3y'' + 3y' - y = 4e^t,$$

to get

$$6Ae^t = 4e^t.$$

Thus $A = \frac{2}{3}$ and $y_p(t) = \frac{2}{3}t^3e^t$. Finally the general solution of $y''' - 3y'' + 3y' - y = 4e^t$ is

$$\boxed{y = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t}.$$

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Solution: The homogeneous equation corresponding to the given equation is $y^{(4)} + 2y'' + y = 0$. The characteristic equation is $r^4 + 2r^2 + 1 = 0$. So the general solution of the homogeneous equation is

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$$y_c(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

Since $r = i$ is a double root of the characteristic equation, we find a particular $y_p(t)$ of $y^{(4)} + 2y'' + y = \mathbf{3}\sin\mathbf{t} - \mathbf{5}\cos\mathbf{t}$ in the form $y_p(t) = t^2(\mathbf{A}\sin\mathbf{t} + \mathbf{B}\cos\mathbf{t})$.

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$$y_c(t) = c_1\cos t + c_2\sin t + c_3 t\cos t + c_4 t\sin t.$$

Since $r = i$ is a double root of the characteristic equation, we find a particular $y_p(t)$ of $y^{(4)} + 2y'' + y = \mathbf{3\sin t - 5\cos t}$ in the form $y_p(t) = t^2(\mathbf{A\sin t + B\cos t})$.

Evaluate $y_p^{(iv)}, y_p''', y_p'', y_p'$ and substitute them into the given equation, to get

$$-8A\sin t - 8B\cos t = 3\sin t - 5\cos t.$$

Thus, $A = -\frac{3}{8}$, $B = \frac{5}{8}$. Hence, the particular solution is given by

$$\boxed{y_p(t) = -\frac{3}{8}t^2 sint + \frac{5}{8}t^2 cost.}$$

Example 3.3 Find the general solution of

$$y''' - 3y'' + 4y = xe^{2x}.$$

Example 3.4 Find a particular solution of

$$y''' - 4y' = t + 3\cos t + e^{-2t}.$$

Example 3.5 Solve the initial value problem

$$y''' - y' = 4e^{-x} + 3e^{2x},$$
$$y(0) = 0, \quad y'(0) = -1, \quad \text{and} \quad y''(0) = 2.$$

The method of variation of parameters

The method of variation of parameters for determining a particular solution of the nonhomogeneous n-th order linear differential equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x).$$

is a DIRECT EXTENSION of the method for second order differential equations.

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is a DIRECT EXTENSION of the method for second order differential equations.

Suppose that we know a fundamental set of solutions $y_1, y_2, ..., y_n$ of the homogeneous equation. We find a particular solution of the nonhomogeneous n-th order linear differential equation in the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \; + u_n(t)y_n(t).$$

To obtain $u_1(t), u_2(t), ., u_n(t)$, we solve the following linear system for $u_1', u_2', ..., u_n'$

$$
\begin{aligned}
y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0, \\
y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0, \\
y_1'' u_1' + y_2'' u_2' + \cdots + y_n'' u_n' &= 0, \\
&\vdots \\
y_1^{(n-1)} u_1' + \cdots + y_n^{(n-1)} u_n' &= g.
\end{aligned}
$$

Integrate $u_1', u_2', ..., u_n'$, to get $u_1, u_2, ..., u_n$.

Example Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^x.$$

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Solution: The homogeneous equation corresponding to the given equation is

$$y''' - 3y'' + 3y' - y = 0.$$

The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0.$$

So the general solution of the homogeneous equation is

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We find a particular solution of the differential equation $y''' - 3y'' + 3y' - y = 4e^x$ as

$$y_p(x) = u_1(x) e^x + u_2(x) x e^x + u_3(x) x^2 e^x.$$

Solving the linear system

$$\begin{pmatrix} e^x & xe^x & x^2e^x \\ e^x & (1+x)e^x & (2x+x^2)e^x \\ e^x & (2+x)e^x & (2+4x+x^2)e^x \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4e^x \end{pmatrix},$$

we have

$$u_1'(x) = 2x^2, \quad u_2'(x) = -4x, \quad u_3'(x) = 2.$$

Thus $u_1(x) = \frac{2}{3}x^3, \quad u_2(x) = -2x^2, \quad u_3(x) = 2x$ and

$$y_p(x) = u_1(x)e^x + u_2(x)xe^x + u_3(x)x^2e^x = \frac{2}{3}x^3e^x - 2x^3e^x + 2x^3e^x = \frac{2}{3}x^3e^x.$$

So the general solution of the nonhomogeneous equation is

$$y(x) = c_1e^x + c_2xe^x + c_3x^2e^x + \frac{2}{3}x^3e^x.$$

Exercises and Assignments

Pages	Exercises	Assignments
222-224	7, 11	8, 10, 12, 16, 17, 19
235-237	3, 9, 12, 15	6, 7, 10, 11, 16, 19