FINANCIAL RISK MANAGEMENT 1

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Course Description:

This course provides students concepts techniques and mathematical tools used in finance: quantitative methods for risk management e.g., Value at Risk, Expected shortfall, Portfolio risk management and interest risk.

References

1. A.J. MacNeil, R. Frey and P. Embrechts. Quantitative risk management. Princeton University press. 2015
2. Peter Christoffersen, Elements of Financial Risk Management, Academic Press, 2003.
3. Allan M. Malz, Financial Risk Management: Models, History, and Institutions, Willey, 2011.
4. Materials for Financial Risk Management certificate, Part I.

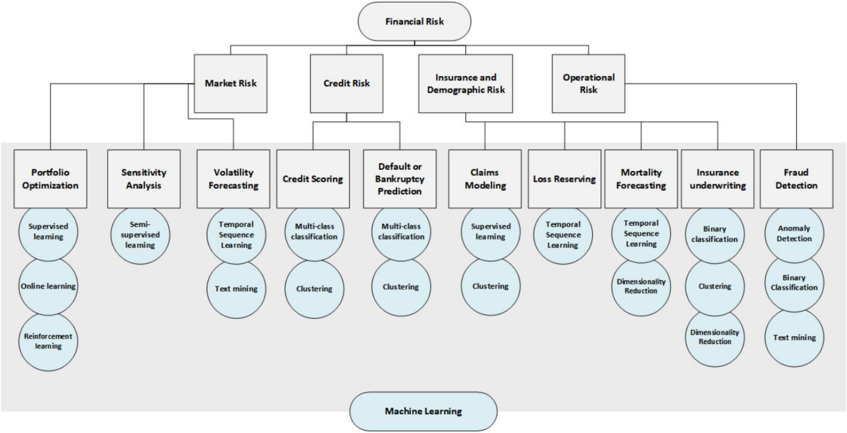
1.1. Some concepts

What is risk and methods for quantifying risks?

- Risk is defined as uncertainty, that is, as the deviation from an expected outcome. Risk=change of loss $\Rightarrow$ Randomness
- Risk is the uncertainty surrounding outcomes. The traditional view of risk is that it is negative only.
- In a business context, risk is usually expressed only the negative deviations from expected values. Investors are generally more concerned about negative outcomes, i.e., unexpected investment losses, than they are about positive surprises (unexpected investment gain).
- Hence, The uncertainty must be quantifiable. Statistical methods allows us to quantify this specific uncertainty by using measures of dispersion.

- Beside statistical methods to quantify financial risks, the machine learning techniques have been widely used for modeling, quantifying and predicting risks

The following Table outlines the major types of machine learning methods that are used in financial risk management



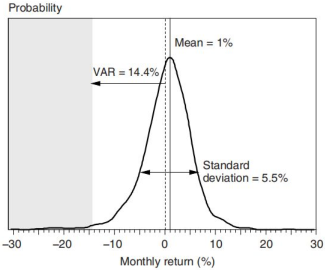
1.2 Financial risks

- Financial risk management is the process by which financial risks are identified, assessed, measured, and managed in order to create economic value.
- Financial risks are referred to possible loss owning to financial market activities. For example: the loss can occur as a result of changes in the value of the underlying assets, such as stock and bond prices, exchange rates, interest-rate movements.
- Some risks can be quantified using statistical tools to generate a probability distribution of profits and losses.

Centralized risk management tools such as Value at Risk (VaR) were developed in the early 1990s. There are two main ideas combined

(i) The first is that risk should be measured at the top level of the institution or the portfolio. It was developed by Harry Markowitz (1952), who emphasized the importance of measuring risk in a total portfolio context.

(ii) The second idea is that risk should be measured on a forward-looking basis, using the current positions.



Distribution of Monthly Returns on U.S. Stocks

Classification of risks (Risk taxonomy)

- *Market risk.* is the risk that declining prices or volatility of prices in the financial markets. It refers to the uncertainties in the value of the company's underlying assets, liabilities, or income due to exposure to a highly dynamic financial market
- *Credit risk.* refers to a loss suffered by a party which fails to meet the contractual obligations, or the risk of not receiving promised payments on outstanding investments such as loans and bonds, because of the default of the borrower.
- *Insurance risk.* refers to the variance in insurance claim experience due to unpredictable events (e.g. catastrophes, car accidents) as well as uncertainties involved with the demographic profile of its policyholders (e.g. mortality).

- *Operational risk* is the risk of loss due to inadequate or failed internal process, people and system, can be further broken down into *business risk and event risk*. Business risk indicates the uncertainty related to business performance (e.g. uncertainty in earnings, demand volatility, customer churn, faulty business operations) and event risk includes uncertainty in events that have an adverse effect in business operations (e.g. fraudulent activities, change in regulations)This risk receives a lot of recent attention.
- *Liquidity risk.* Significant losses due to the inability to sufficiently liquidate at a fair price. Can be thought of as "oxygen for a healthy market"

1.3. Absolute risk and Relative risk

- **Absolute risk** focuses on the volatility of total returns. Example: consider a portfolio with the initial investment P. Denote ΔP as the profit or loss for the portfolio over a fixed horizon (coming month).

The future rate of return R_P. Then the absolute risk is

$$\sigma(\Delta P) = \sigma(\Delta P/P)P = \sigma(R_P)P.$$

- **Relative risk** is referred to tracking error volatility because it is usually measured relative to a benchmark index or portfolio.

Example. Assume that the benchmark index B with return R_B. The deviation $e := R_P - R_B$ is known as the tracking error. So the risk is

$$\sigma(e)P = \sigma(R_P - R_B)P = \omega P$$

where ω is called tracking error volatility (TEV).

Comparing two approaches. Take the concrete case of an active equity portfolio manager who is given the task of beating a benchmark. Assume that in the first year, the active portfolio returns -6%, but the benchmark drops by -10%. Hence, the excess return is

$$e = -6 - (-10) = 4\% > 0$$

We see that in relative terms, the portfolio has done well even thought the absolute performance is negative.
Assume that in the second year, the portfolio returns +6%, which is good for absolute measures. However, not good if the benchmark goes up to +10%.

Risk Management Strategies

Generally, financial institutions apply two types of risk management strategies:

- **Risk decomposition** identifies each risk and handles it separately.
- **Risk aggregation** diversifies the risk exposure to minimize the overall risk exposure.

1.4. Evaluation of the risk management process

A major function of the risk measurement process is to (i) estimate the distribution of future profits and (ii) losses

- The first part of this assignment is easy. For example: in Figure 1, an investment of 1000$ should have a standard deviation $\sigma(\Delta P) = 1000 \times 5.5\% = 55\$$
- The second part of the assignment, consists of constructing the distribution of future rates of return, is not easy. In Figure 1, we have taken the historical distribution and assumed that this provides a good representation of future risks. Because we have a long history of returns over many different cycles, this is a reasonable approach.

Question arises: How can we tell whether this loss is due to bad luck or a flaw in the risk model?

1.5. Risk and Randomness

As define the financial risk above, Risk is defined as uncertainty, risk=change of loss. So

- We can model situations in finance in which an investor holds today an asset with an uncertain value in the future.
- We use probabilistic methods (random variables, random vectors, distributions, stochastic processes) and statistical tools to modeling risk.

Example: an investor holds stock in a company, insurance company sold an insurance policy, an individual holds an option. These situations can be modeled as a random variable X on the probability space.

1.6. Measurement and management

Risk measurement.

- Suppose we hold a portfolio of $d > 1$ investment with weights $\omega_1, \omega_2, ..., \omega_d$. Let X_i denote the change in value of the i-th investment. We have the change in value (*profit and loss (P&L)*)of the portfolio:

$$X = \sum_{i=1}^{d} \omega_i X_i$$

  Measuring the risk now consists of determining *the distribution function* $F(x) := \mathbb{P}(X \leq x)$ or or functionals of it, e.g. mean, variance, α-quantiles

$$q(\alpha) := \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}$$

The important task is to determine a joint model for random vector $(X_1, X_2, ..., X_d)$. Statistical methodology provides us methods to estimate F one of its functionals based on historical observations of this model.

Risk management (RM): What is risk management?

RM is a discipline for living with the possibility that future events may cause adverse effects

- Financial firms are not passive/defensive towards risk, they willingly take risks because they seek a return
- What does managing risks involve?
  - Determine the capital to hold to absorb losses, both for regulatory purposes and economic capital purposes (to survive as a company).
  - Optimizing portfolios according to risk-return considerations

2.1. Risk management for a financial firm

Consider a balance sheet of a bank and an insurer to understand the risks faced by these firms. A balance sheet provides us a financial statement showing assets and liabilities.

- The assets will describe the financial institution's investments
- The liabilities refer to the funds that have been raised and the obligations of institutions.

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A typical bank will raise funds by taking in customer deposits, issuing bonds and borrowing from other banks or center banks. A typical insurance company will sell insurance contracts, collecting premiums in return and raise additional funds by issuing bonds. Its liabilities are obligations to policyholders.

A stylized balance sheet for a bank

Assets Investments of the firm		Liabilities Obligations from fundraising	
Cash	£10M	Customer deposits	£80M
(and central bank balance)			
Securities	£50M	Bonds issued	
- bonds, stocks, derivatives		- senior bond issues	£25M
Loans and mortgages	£100M	- subordinated bond issues	£15M
- corporates		Short-term borrowing	£30M
- retail and smaller clients		Reserves (for losses on loans)	£20M
- government			
Other assets	£20M	Debt (sum of above)	£170M
- property			
- investments in companies		Equity	£30M
Short-term lending	£20M		
Total	£200M	Total	£200M

A stylized balance sheet for a insurance company

Assets		Liabilities	
Investments		Reserves for policies written (technical provisions)	£80M
- bonds	£50M		
- stocks	£5M	Bonds issued	£10M
- property	£5M		
Investments for unit-linked contracts	£30M	Debt (sum of above)	£90M
Other assets - property	£10M	Equity	£10M
Total	£100M	Total	£100M

So we have the balance sheet equation:

Value of Assets = Value of Liabilities = Debt + Equity

Note that: if Equity > 0, then the company is solvent, otherwise insolvent

2.2. Modeling value and value change

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Mapping of risks. We will build a general mathematical model for changes in value caused by financial risks. Consider a risk (or loss) as a random variable, denoted by L or $X : \Omega \to \mathbb{R}$.

- Consider a portfolio of assets and possibly liability. Denote V_t the value of the portfolio at time t. Consider a give time change horizon Δt. Assume that the portfolio composition remain fixed and there are no intermediate payments over Δt.
- We have the change in value of the portfolio

$$\Delta V_{t+1} := V_{t+1} - V_t$$

- The random loss is defined by

$$L_{t+1} := -\Delta V_{t+1}$$

Remark.

(a) The distribution of L_{t+1} is called loss distribution.

(b) In practice, practitioners often use profit and loss $P\&L$ distribution which is distribution of $-L_{t+1} = \Delta V_{t+1}$

(c) For a long time intervals, we have

$$\Delta V_{t+1} = \frac{1}{1+r} V_{t+1} - V_t,$$

where r denote the risk-free interest rate.

- The value V_t is typically modeled as a function of time t and a d-dimensional random vector $Z_t = (Z_{t,1}, Z_{t,2}, ..., Z_{t,d})$ of risk factors, more precisely

$$V_t = f(t, Z_t)$$

Denote the vector

$$X_{t+1} := Z_{t+1} - Z_t$$

the risk-factor changes. So we have

$$\begin{aligned} L_{t+1} = -(V_{t+1} - V_t) &= -\Big(f(t+1, Z_{t+1}) - f(t, Z_t)\Big) \\ &= -\Big(f(t+1, Z_t + X_{t+1}) - f(t, Z_t)\Big) \end{aligned}$$

and, hence, the loss df is determined the loss df of X_{t+1} So we also write

$$L_{t+1} = g(X_{t+1})$$

Example 1

Consider a portfolio P of d stocks $S_{t,1}, S_{t,2}, .., S_{t,d}$, where $S_{t,i}$ denote the value of stock i at time t. Denoted by λ the number shares of stock i in the portfolio. In finance and risk management we usually use logarithmic prices as risk factors, so $Z_{t,i} = \log(S_{t,i})$. We have

$$V_t = f(t, Z_t) = \sum_{i=1}^{d} \lambda_i S_{t,i} = \sum_{i=1}^{d} \lambda_i e^{Z_{t,i}}$$

Hence, one period ahead loss is given

$$\begin{aligned} L_{t+1} = -(V_{t+1} - V_t) &= -\sum_{i=1}^{d} \lambda_i (e^{Z_{t,i}+X_{t+1,i}} - e^{Z_{t,i}}) \\ &= -\sum_{i=1}^{d} \lambda_i e^{Z_{t,i}} (e^{X_{t+1,i}} - 1) \end{aligned}$$

Note that $e^x - 1 \approx x$, then we have the linear approximation of L_{t+1}

$$\hat{L}_{t+1} = -\sum_{i=1}^{d} \lambda_i e^{Z_{t,i}} X_{t+1,i} = -\sum_{i=1}^{d} \lambda_i S_{t,i} X_{t+1,i}$$

Putting $\omega_{t,i} = \lambda_i S_{t,i}/V_t$, the term $\omega_{t,i}$ is called i^{th} portfolio weight. We get

$$\hat{L}_{t+1} = -V_t \sum_{i=1}^{d} \omega_{t,i} X_{t+1,i}.$$

So if $\mathbb{E}(X_{t+1,i}) = \mu_i$ and $Cov(X_{t+1,i}) = \Sigma$ then

$$\mathbb{E}_t(\hat{L}_{t+1}) = -V_t \sum_{i=1}^{d} \omega_{t,i} \mathbb{E}(X_{t+1,i}) = -V_t \omega^T \mu$$

and

$$Var_t(\hat{L}_{t+1}) = V_t^2 \omega^T \Sigma \omega$$

Example 2 (European call option)

Consider a portfolio consisting of a European call option on a non-dividend paying stock price S_t with maturity T, strike (exercise) price K. From Black-Scholes formula, the today value of European call option is

$$V_t = C^{BS}(t, S_t, T, K, r, \sigma) = S_t\Phi(d_1) - Ke^{-r(T-t)\Phi(d_2)}$$

where

- Φ is distribution function of $N(0, 1)$
- r is the continuously compounded risk-free interest rate;
- $d_1 = \frac{\log(S_t/K)+(r+\sigma^2/2}{)}(T-t)\sigma\sqrt{T-t}$; and $d_2 := d_1 - \sigma\sqrt{T-t}$
- σ is the the annualized volatility (standard deviation) of the return $\log(S_t/S_{t-1})$

Remark. *In BS formula, we assume that r, σ are constants, this assumption is often not true in the real markets and in practice. Therefore, $\log(S_t)$ and r_t, σ_t can be seen as risk factors.*
We now consider vector

$$
\begin{aligned}
&Z_t = (\log(S_t), r_t, \sigma_t) \\
&\Rightarrow X_{t+1} = Z_{t+1} - Z_t = (\log(S_{t+1}/\log(S_t)), r_{t+1} - r_t, \sigma_{t+1} - \sigma_t) \\
&\Rightarrow V_t = C^{BS}(t, e^{Z_{t,1}}, Z_{t,2}, Z_{t,3}, K, T) =: f(t, Z_t)
\end{aligned}
$$

Assume there is a portfolio contains d different options with a position of λ_i in the i^{th} option, then we have

$$
\begin{aligned}
L_{t+1} &= -\lambda_0(S_{t+1} - S_t) \\
&= -\sum_{i=1}^{d} \lambda_i \Big(C(S_{t+1}, t+1, \sigma(K_i, T_i, t+1)) - C(S_t, t, \sigma(K_i, T_i, t)) \Big)
\end{aligned}
$$

where λ_0 is the position in the underlying security.

2.3 Fundamentals of Probability

2.3.1 Characterizing random variables

- A random variable X is characterized by a distribution function or cumulative distribution function

$$F(x) = \mathbb{P}(X \leq x)$$

  which is the probability that the realization of the random variable X ends up less than or equal to the given number x.

- When the variable X takes discrete values, this distribution is obtained by summing the step values less than or equal to x, i.e.,

$$F(x) = \sum_{x_i \leq x} f(x_i),$$

  where $f(x)$ is called the frequency function or the probability density function (p.d.f.).

Here, $f(x)$ is the probability of observing x. When the variable is continuous, the distribution is given by

$$F(x) = \int_{-\infty}^{x} f(u)du$$

Example

A gambler wants to characterize the probability density function of the outcomes from a pair of dice. Let X denote the sum of two dies. Then we have

Outcome x_i	Frequency $n(x)$	Probability $f(x)$		Cumulative Probability $F(x)$
2	1	1/36	0.0278	0.0278
3	2	2/36	0.0556	0.0833
4	3	3/36	0.0833	0.1667
5	4	4/36	0.1111	0.2778
6	5	5/36	0.1389	0.4167
7	6	6/36	0.1667	0.5833
8	5	5/36	0.1389	0.7222
9	4	4/36	0.1111	0.8333
10	3	3/36	0.0833	0.9167
11	2	2/36	0.0556	0.9722
12	1	1/36	0.0278	1.0000
Sum	36	1	1.0000	

2.3.2 Covariances and Correlations

- Joint distribution of two random variables X_1, X_2

$$F(x_1, x_2) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(u_1, u_2) du_1 du_2$$

where $f(x_1, x_2)$ is the joint density function. The marginal density is

$$f_1(x_1) = \int_{-\infty}^{x_1} f(x_1, u_2) du_2$$

- When dealing with two random variables, the comovement can be described by the covariance

$$\begin{aligned} Cov(X_1, X_2) = \sigma_{12} &= \mathbb{E}(X - \mathbb{E}(X_1))(X_2 - \mathbb{E}(X_2)) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \mathbb{E}(X_1))(x_2 - \mathbb{E}(X_2)) f(x_1, x_2) dx_1 dx_2 \end{aligned}$$

The correlation coefficient

$$\rho_{12}(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2}$$

2.3.3 Sum of Random Variables

- Another useful transformation is the summation of two random variables. A portfolio, for instance, could contain one share of Intel plus one share of Microsoft. The rate of return on each stock behaves as a random variable.
- The expectation of the sum $Y = X_1 + X_2$ can be written as

$$\mathbb{E}(Y) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2)$$

- Its variance is

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$$

Conditional distribution and conditional expectation

- Let f denote the join distribution of X_1, X_2, assume $f_1(x) > 0$, then we have

$$\begin{aligned}\mathbb{P}(X_2 \le x_2 \mid x_1 \le X_1 \le x_1 + dx_1) &= \frac{\mathbb{P}(X_2 \le x_2, x_1 \le X_1 \le x_1 + dx_1)}{\mathbb{P}(x_1 \le X_1 \le x_1 + dx_1)} \\ &\approx \frac{\int_{u_2=-\infty}^{x_2} f(x_1, u_2) dx_1 du_2}{f_1(x_1) dx_1} \\ &= \int_{u_2=-\infty}^{x_2} \frac{f(x_1, u_2)}{f_1(x_1)} du_2\end{aligned}$$

As dx_1 goes to 0, then the conditional distribution of X_2 given $X_1 = x_1$

$$F_{X_2|X_1}(x_2 \mid x_1) = \int_{-\infty}^{x_2} \frac{f(x_1, u_2)}{f_1(x_1)} du_2$$

- We have the conditional density function

$$f_{X_2|X_1}(x_2 \mid x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}.$$

- Conditional expectation of X_2 given $X_1 = x_1$ is

$$\psi(x_1) := \mathbb{E}(X_2 \mid X_1 = x_1) = \int_{\infty}^{\infty} x_2 f_{X_2|X_1}(x_2 \mid x_1) dx_2$$

Property:

$$\mathbb{E}(X_2) = \mathbb{E}(\psi(X_1)) = \int_{\infty}^{\infty} \mathbb{E}(X_2 \mid X_1 = x_1) f_1(x_1) dx_1$$

- Note that, let A be an event. Denote I_A the indicator function then for a random variable X we have

$$\mathbb{P}(X \in A) = \mathbb{E}(I_A(X))$$

Therefore in the special case when X_1 is continuous and X_2 is the discrete random variable, then for an event B we have

$$\mathbb{P}(Y \in B) = \mathbb{E}(\psi(X_1)) = \int_{\infty}^{\infty} \mathbb{P}(Y \in B \mid X_1 = x_1) f_1(x_1) dx_1$$

Special case: Le X be a random variable with density function f_X. Then

$$\mathbb{E}(X \mid X \in A) = \frac{\mathbb{E}(X I_A(X))}{\mathbb{E}(I_A(X))} = \frac{\int_A x f_X(x) dx}{\mathbb{P}(X \in A)}$$

The Question Being Asked in VaR

"What loss level is such that we are α% confident it will not be exceeded in N business days?"

VaR and Regulatory Capital

- Regulators have traditionally used VaR to calculate the capital they require banks to keep
- The market-risk capital has been based on a 10-day VaR estimated where the confidence level is 99%
- Credit risk and operational risk capital are based on a one-year 99.9% VaR

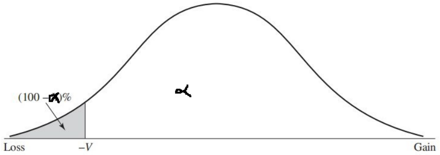
Advantages of VaR

- It captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: "How bad can things get?"

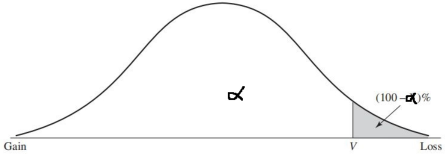
When using the value at risk measure, we are interested in making a statement of the following form:

- We have α percent certain that we will not lose more than V dollars in time T."
- The variable V is the VaR of the portfolio. It is a function of two parameters: the time horizon, T, and the confidence level, α percent. It is the loss level during a time period of length T that we are $\alpha\%$ certain will not be exceeded.

More generally, when the distribution of gains is used, VaR is equal to minus the gain at the $(100 - \alpha)th$ percentile of the distribution, as illustrated in Figure 1 below.



When the distribution of losses is used, VaR is equal to the loss at the αth percentile of the distribution, as indicated in Figure 2 below.



Example

Suppose that the gain from a portfolio during six months is normally distributed with a mean of \$2 million and a standard deviation of \$10 million. From the properties of the normal distribution, we have the one-percentile point of this distribution

$$2 - 10 \times 2.326 = -21.3(???)$$

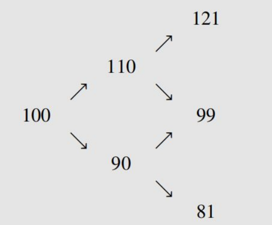
The VaR for the portfolio with a time horizon of six months and confidence level of 99% is therefore \$21.3 million.

3.1. Quantiles of a Distribution

An investor holding an asset whose future value is uncertain may wish to determine whether his discounted gain X on an investment has at least 95% probability of remaining above a certain (usually negative) level. Value at Risk at 5% answers this question by specifying the minimum loss incurred in the worst 5% of possible outcomes.

Example

Consider a two step binomial model with stock prices



Assume that the probability p of the price going up in a single step is $p = 0.8$. In this example we neglect the time value of money and compute the gain after the second step of buying a single share of stock as

$$X = S(2) - S(0)$$

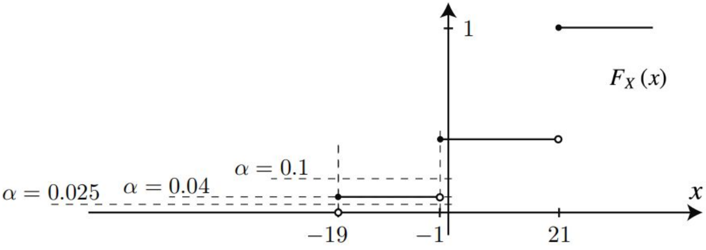
which gives

$$X = \begin{cases} 12 & \text{with probability} \quad p^2 = 0.64 \\ -1 & \text{with probability} \quad 2p(1-p) = 0.32 \\ -19 & \text{with probability} \quad (1-p)^2 = 0.04 \end{cases}$$

the probability that our investment will lead to a loss $L = -X < 19$ is

$$\mathbb{P}(L < 19) = \mathbb{P}(X > -19) = 0.96$$

We have the graph of distribution function of X



Definition

For $\alpha \in (0,1)$, $F(x)$ is the distribution function of X, the number

$$q^{\alpha}(X) := \inf\{x : \alpha < F(x)\}$$

is called the upper α-quantile of X

The number

$$q_\alpha(X) := \inf\{x : \alpha \leq F(x)\}$$

is called the lower α-quantile of X. Any

$$q \in [q_\alpha(X), q^\alpha(X)]$$

is called α quantile of X.

Example

Consider example above, for $\alpha \in \{0.025, 0.04, 0.1\}$ we have

$$\begin{aligned} q^{0.025}(X) &= -19, \quad q_{0.025}(X) = -19 \\ q^{0.04}(X) &= -1, \quad q_{0.04}(X) = -19 \\ q^{0.15}(X) &= -1, \quad q_{0.01}(X) = -1 \end{aligned}$$

Proposition

Let X, Y be random variables. We have

(i) If $X \geq Y$ then $q^{\alpha}(X) \geq q^{\alpha}(Y)$

(ii) For any $b \in \mathbb{R}$, $q^{\alpha}(X+b) = q^{\alpha}(X) + b$

(iii) For $b > 0$, $q^{\alpha}(bX) = bq^{\alpha}(X)$

(iv) $q^{\alpha}(-X) = -q_{1-\alpha}(X)$.

Proof.

Lemma

If the distribution function $F_X(x)$ of X is continuous and strictly increasing then

$$q^{\alpha}(X) = F_X^{-1}(\alpha).$$

proof. Since F_X is continuous, strictly increasing then there exists continuous inverse function F_X^{-1}. We have

$$q^{\alpha}(X) = \inf\{x : \alpha < F_X(x)\} = \inf\{x : F_X^{-1}(\alpha) < x\} = F_X^{-1}(\alpha).$$

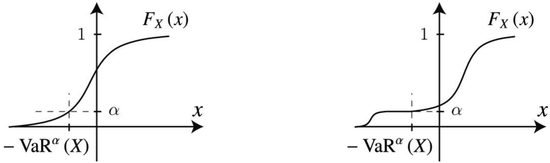
3.2. Value-at-Risk (measuring downside risk)

Assume that we invest at time $t = 0$ and terminate our investment at $t = T$. Denote X by the discounted value of investor position at time $t = T$.

Definition

Given the confidence level $1 - \alpha$, the Value at Risk (VaR) of X is defined by

$$VaR^{\alpha}(X) = -q^{\alpha}(X) = -\inf\{x : \alpha < F_X(x)\} = F_L^{-1}(1-\alpha) \quad (1)$$



Example

Consider example 1, first we find the distribution $F_X(x)$, and, then we have

$$VaR^{0.04}(X) = -q^{0.04}(X) = -\inf\{x : 0.04 < F_X(x)\} = 1$$

Similarly, we have

$$VaR^{0.0025}(X) = -q^{0.025}(X) = -\inf\{x : 0.025 < F_X(x)\} = 19$$

Remark. Note that since X denotes the gain of an investment, $L = -X$ denotes the loss. Then we can express VaR in terms of the loss as follows

$$\begin{aligned} VaR^{\alpha}(X) &= -q^{\alpha}(X) \\ &= q_{1-\alpha}(-X) \\ &= \inf\{x : 1-\alpha \leq \mathbb{P}(-X \leq x)\} \\ &= \inf\{x : \mathbb{P}(L \leq x) \geq 1-\alpha\} = q_{1-\alpha}(L) \end{aligned}$$

- So $VaR^{\alpha}(X)$ is simply the upper $(1-\alpha)$-quantitle of L. Some time, we also denote by $VaR_{1-\alpha}(L)$ or $VaR_{\alpha}(L)$ if we use the following equivalent definition

Alternative Definition

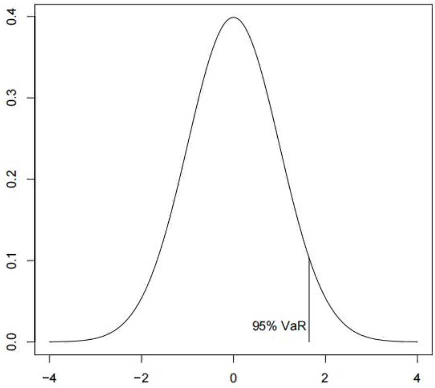
Value at risk (VaR) is a measure of risk which is used to estimate the amount that can potentially be lost on an investment within a certain time range. For a loss L, Value-at-Risk (VaR) at the confident $\alpha \in (0,1)$ is defined by

$$VaR_{\alpha} = VaR_{\alpha}(L) = \inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\} \quad (2)$$

We also have

$$VaR_{\alpha}(L) = \inf\{x \in \mathbb{R} : \bar{F}_L(x) \leq 1-\alpha\}$$

$VaR_{\alpha}(L)$ is the smallest loss which is exceeded with prob. at most $1-\alpha$



- We might think of L as the (potential) loss resulting from holding a portfolio over some fixed time horizon.
- In market risk the time horizon is typically one day or ten days.

- In credit risk the portfolio may consist of loans and the time horizon is often one year.

Definition

Given a nondecreasing function $F : \mathbb{R} \to \mathbb{R}$ *the generalized inverse of* F *is given by*

$$F^{\leftarrow}(y) := \inf\{x \in \mathbb{R} : F(x) \geq y\}$$

with the convention $\inf \emptyset = \infty$

If F is strictly increasing then $F^{\leftarrow} = F^{-1}$, i.e., the usual inverse. So we have the definition of α-quantile of a random variable X with distribution F via the generalized inverse

$$q_\alpha(X) = F^{\leftarrow}(\alpha) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}, \quad , \alpha \in (0, 1)$$

Remark. For aggregate insurance losses, where the risk is that X is high, α is picked high, with values like 0.95, 0.975, 0.99, 0.995. However, for profits or rates of return, where the risk is that X is low, α is picked low, with values like 0.05, 0.025, 0.01, 0.005, and, hence, we have the definition in this case

Example

Profits X have a distribution with the following density function

$$f(x) = \frac{3}{(1+x)^4}, \quad x > 0$$

Calculate VaR of profits at 0.01 level.

We have the distribution of X

$$F(x) = \int_0^x \frac{3}{(1+t)^4} dt = 1 - \frac{1}{(1+x)^3}$$

Since F is continuous and strictly increasing, then there exists the inverse function. So we have

$$F(-VaR^{0.01}(X)) = 0.01 \Rightarrow -VaR^{0.01}(X) = 0.003356.$$

In case we want to calculate VaR of the potential loss, $L = -X$, we have

$$F_L(x) = \mathbb{P}(L \leq x) = 1 - F_X(-x) = \frac{1}{(1-x)^3}$$

Remember that the confidence level in this case is picked high, i.e., $\alpha = 1 - 0.01 = 0.99$. So we have from definition (2)

$$F(VaR_{0.99}(L)) = 0.99 \Rightarrow VaR_{0.99}(L) = 0.003356$$

Remark. In practice, to estimate VaR from a sample, the sample is should be ordered from lowest to highest, and then the $100 \times \alpha$ percentile is selected. This percentile is not well-defined since a sample is a discrete distribution, so some rule for selecting the percentile is needed.

Example. if the sample is size 1000 and $\alpha = 0.05$, then one might set the sample VaR equal to the $50 - th$ order statistic.

Example

Losses on an insurance are distributed as follows:

Greater than	Less than or equal to	Probability
0	1000	0.45
1000	2000	0.25
2000	5000	0.22
5000	10000	0.05
10000	20000	0.03

a) Find the cumulative distribution function F?

(b) Find Value-at-Risk with the confidence level 0.95?

(b) Note that $F_L(x) = \mathbb{P}(L \leq x) = 0.92$ if $2000 \leq L < 5000$ and

$$\mathbb{P}(5000 < L < 10000) = 0.05; \mathbb{P}(10000 < L < 20000) = 0.03$$

For the confidence level $\alpha = 0.95$ we have

$$VaR_{0.95}(L) = \inf\{x : F_L(x) \geq 0.95\}$$

Therefore, in the interval $(5000, 10000)$ with probability 0.05 we need 3% of the subset of $(5000, 10000)$ combining with $F_L(x) = \mathbb{P}(L \leq x) = 0.92$ to get 95% of losses. So we need 2/5 of $(5000, 10000)$ making the point 8000 with 95% of losses, i.e.,

$$\mathbb{P}(L < 8000) = 95\% \Rightarrow VaR_{0.095} = 8000$$

Properties of VaR

Let X, Y be random variables.

(i) If $X \geq Y$ then $VaR^{\alpha}(X) \leq VaR^{\alpha}(Y)$

(ii) For any $a \in \mathbb{R}$, $VaR^{\alpha}(X+a) = VaR^{\alpha}(X) - a$

(iii) For any $a \geq 0$, $VaR^{\alpha}(aX) = aVaR^{\alpha}(X)$

Example (VaR for Normal and t-distributions)

1) Now let $L \sim N(\mu, \sigma^2)$. Since F_L is continuous, strictly increasing then from

$$VaR_{\alpha}(L) = \inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\} \Rightarrow F_L(VaR_{\alpha}(L)) = \alpha.$$

So we have

$$\begin{aligned} F_L(VaR_{\alpha}(L)) = \mathbb{P}(L \leq VaR_{\alpha}(L)) &= \mathbb{P}\Big(\frac{L-\mu}{\sigma} \leq \frac{VaR_{\alpha}(L)-\mu}{\sigma}\Big) \\ &= \Phi\big((VaR_{\alpha}(L)-\mu)/\sigma\big) = \alpha \end{aligned}$$

Hence

$$VaR_\alpha(L) = \mu + \sigma\Phi^{-1}(\alpha)$$

2) Assume that $L \sim t_\nu(\mu, \sigma^2)$, So we have $(L - \mu)/\sigma \sim t_\nu(0, 1)$. Similarly as example 1, we get

$$VaR_\alpha(L) = \mu + \sigma t^{-1}(\alpha)$$

where t^{-1} is the inverse distribution of $t_\nu(0, 1)$-distribution

Example

Consider a \$1000 million portfolio of medium-term bonds. Suppose the confidence interval is 95%, what is the maximum monthly loss under normal markets over any month?

We have $L \sim N(0, 1), \alpha = 0.95$, then we have

$$VaR_{0.95}(L) = 1000 \times \Phi^{-1}(0.95) = \$1.65 \quad \text{milion}$$

To calculate $\Phi^{-1}(0.95)$ we use command $qnorm(0.95, 0, 1)$ in R

Remarks

- In practice, the holding period to define the capital is 10 trading days. For that, banks can compute the one-day VaR and converts it to a ten-day VaR

$$VaR_\alpha(X, \text{ten days}) = \sqrt{10} \times VaR_\alpha(X, \text{one day})$$

- In general, we have

$$\text{T-day VaR} = \text{one-day VaR}$$

Example

The losses for the 500 different scenarios are then ranked in the following table

Scenario number	*Loss ($000s)*
494	477.841
339	345.435
349	282.204
329	277.041
487	253.385
227	217.974
131	202.256
238	201.389
473	191.269
306	191.050
477	185.127
495	184.450
376	182.707
237	180.105
365	172.224
⋮	⋮

We see from the table that the worst scenario is number 494 and the one-day 99% value at risk can be estimated as the fifth worst loss. This is $253,385. So the ten-day 99% VaR is usually calculated as $\sqrt{10}$ times the one-day 99% VaR. More precisely, we have the ten-day VaR

$$\sqrt{10} \times 253,385 = 801,274$$

Although Value-at-Risk has become a very popular risk measure among practitioners it has several limitations. For instance, it does not give any information about how bad losses may be when things go wrong. In other words, what is the size of an “average loss” given that the loss exceeds the 99%-Value-at-Risk?

4.1. Coherent risk measures

Coherent risk measures were introduced in 1998 by Artzner.

- Define a coherent risk measure as the amount of cash that has to be added to a portfolio to make its risk acceptable
- Properties of coherent risk measure
  1. *Subadditivity:* The risk measures for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged
  2. *Homogeneity:* Changing the size of a portfolio by λ should result in the risk measure being multiplied by λ
  3. *Monotonicity:* If one portfolio always produces a worse outcome than another its risk measure should be greater
  4. *Translation invariance:* If we add an amount of cash K to a portfolio its risk measure should go down by K

We can formulate the properties of coherent risk measure above as follows:

Let $\mathcal{R}$ is a risk measure of a portfolio . The risk measure $\mathcal{R}$ is called coherent if it satisfies the following properties

(1) Subadditivity

$$\mathcal{R}(L_1 + L_2) \leq \mathcal{R}(L_1) + \mathcal{R}(L_2)$$

The risk of two portfolios should be less than adding the risk of the two separate portfolios.

(2) Homogeneity: for all $\lambda > 0$

$$\mathcal{R}(\lambda L) = \lambda \mathcal{R}(L)$$

Leveraging or deleveraging of the portfolio increases or decreases the risk measure in the same magnitude.

(3) Monotonicity: For $L_1 \leq L_2$ almost surely, we have

$$\mathcal{R}(L_1) \leq \mathcal{R}(L_2)$$

if X, Y are returns of portfolios, such that $X < Y$ almost surely, then

$$\mathcal{R}(X) \geq \mathcal{R}(Y)$$

(4) Translation invariance: for $c \in R$

$$\mathcal{R}(X + c) = \mathcal{R}(X) - c$$

Adding a cash position of amount c to the portfolio reduces the risk by c. This implies that we can hedge the risk of the portfolio by considering a capital that is equal to the risk measure:

$$\mathcal{R}(X + \mathcal{R}(X)) = \mathcal{R}(X) - \mathcal{R}(X) = 0$$

For the loss L of the portfolio then

$$\mathcal{R}(L + c) = \mathcal{R}(L) + c$$

it states that by adding or subtracting a deterministic quantity c to a position leading to the loss L we alter our capital requirements by exactly that amount. Adding the amount of capital $\mathcal{R}(L)$ to the position leads to the adjusted loss $\tilde{L} = L - \mathcal{R}(L)$, we have

$$\mathcal{R}(\tilde{L}) = \mathcal{R}(L - \mathcal{R}(L)) = \mathcal{R}(L) - \mathcal{R}(L) = 0$$

Remark

Note that Value-at-Risk is not a coherent risk measure since it fails to be subadditive. Indeed, we consider the following example.

Example 1

Let X, Y be two i.i.d losses of assets. Where

$$X = \epsilon + \eta$$

where, $\epsilon \sim N(0, 1)$, and

$$\mathbb{P}(\eta = 0) = 0.991; \quad \mathbb{P}(\eta = -10) = 0.009$$

is independent of ϵ.
We have

$$\mathbb{P}(X \leq VaR_{0.99}(X)) = 0.99$$

hence,

$$\mathbb{P}(\epsilon + \eta \leq VaR_{0.99}(X)) = 0.99$$

Consequently,

$$\mathbb{P}(\epsilon + 0 \leq VaR_{0.99}(X))\mathbb{P}(\eta = 0) + \mathbb{P}(\epsilon - 10 \leq VaR_{0.99}(X))\mathbb{P}(\eta = 10) = 0.99$$

or

$$0.991 \times \mathbb{P}(\epsilon \leq VaR_{0.99}(X)) + 0.009 \times \mathbb{P}(\epsilon \leq VaR_{0.99}(X) + 10) = 0.99$$

So we get

$$0.991 \times \Phi(VaR_{0.99}(X)) + 0.009 \times \Phi(VaR_{0.99}(X) + 10) = 0.99$$

Solving this equation (by, e.g., Matlab) we obtain

$$VaR_{0.99}(X) = 3.1$$

Since Y has the same distribution to X, then $VaR_{0.99}(Y) = 3.1$

Now consider a portfolio $Z = X + Y$, where $X = \epsilon_1 + \eta_1$, and $Y = \epsilon_2 + \eta_2$. Then

$$Z = (\epsilon_1 + \epsilon_2) + (\eta_1 + \eta_2) = \hat{\epsilon} + \hat{\eta}$$

where, $\hat{\epsilon} = \epsilon_1 + \epsilon_2 \sim N(0, 2)$ and $\hat{\eta} = \eta_1 + \eta_2$ has distribution

$$\begin{aligned}
\mathbb{P}(\hat{\eta} = 0) &= 0.991 * 9.991 = 0.9821 \\
\mathbb{P}(\hat{\eta} = -10) &= 2 * 0.009 * 0.991 = 0.00178 \\
\mathbb{P}(\hat{\eta} = -20) &= 0.009 * 0.009 = 0.000081
\end{aligned}$$

So we have the equation

$$\begin{aligned}
&\mathbb{P}(Z = X + Y \leq VaR_{0.99}(X+Y)) = \mathbb{P}(\hat{\epsilon} + \hat{\eta} \leq VaR_{0.99}(X+Y)) \\
&= 0.982 * \mathbb{P}(\hat{\epsilon} + 0 \leq VaR_{0.99}(X+Y)) + 0.00178 * \mathbb{P}(\hat{\epsilon} - 10 \leq VaR_{0.99}(X+Y) \\
&\quad + 0.000081 * \mathbb{P}(\hat{\epsilon} - 20 \leq VaR_{0.99}(X+Y)) \\
&= 0.99
\end{aligned}$$

or

$$
\begin{aligned}
&0.982 * \mathbb{P}(\hat{\epsilon} \leq VaR_{0.99}(X+Y)) + 0.00178 * \mathbb{P}(\hat{\epsilon} \leq VaR_{0.99}(X+Y) + 10) \\
&\quad + 0.000081 * \mathbb{P}(\hat{\epsilon} \leq VaR_{0.99}(X+Y) + 20) \\
&= 0.99
\end{aligned}
$$

Solving this equation we obtain

$$VaR_{0.99}(X+Y) = 9.8$$

compare to $VaR_{0.99}(X) + VaR_{0.99}(Y) = 3.1 + 3.1 = 6.2$ we see that

$$VaR_{0.99}(X+Y) > VaR_{0.99}(X) + VaR_{0.99}(Y)$$

which does not satisfy the subadditive property, and, hence, VaR is not a coherent measure.

4.2. Expected Shortfall (Average Value at Risk)

Expected shortfall (ES) is also called Average Value at Risk (AVaR), Expected tail loss (ETL) or Conditional Value-at-Risk (CVaR). Expected shortfall is a risk measure to evaluate the market risk or credit risk of a portfolio.

Definition 1

The Average Value at Risk(Expected shortfall) of continuous random variable X is given by

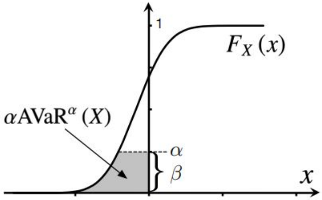
$$AVaR^{\alpha}(X) = ES^{\alpha}(X) := -\mathbb{E}(X \mid X \leq -VaR^{\alpha}(X))$$

This equation implies that AVaR is related to the conditional loss distribution.

Theorem 1. The Average Value at Risk(Expected shortfall) of continuous random variable X is given by

$$AVaR^{\alpha}(X) = ES^{\alpha}(X) := \frac{1}{\alpha}\int_0^{\alpha} VaR^{\beta}(X)d\beta = -\frac{1}{\alpha}\int_0^{\alpha} q^{\beta}(X)d\beta$$

for all $0 < \beta \leq \alpha < 1$



Definition 2

For a loss L with continuous loss distribution function F_L the expected shortfall at confidence level $\alpha \in (0, 1)$ is given by

$$ES_\alpha(L) := \mathbb{E}(L \mid L \geq VaR_\alpha(L))$$

From this we obtain the representation of $ES_\alpha(L)$ (why???)

$$ES_\alpha(L) = \frac{1}{1-\alpha} \int_{q_\alpha(L)}^{\infty} x dF_L(x)$$

or

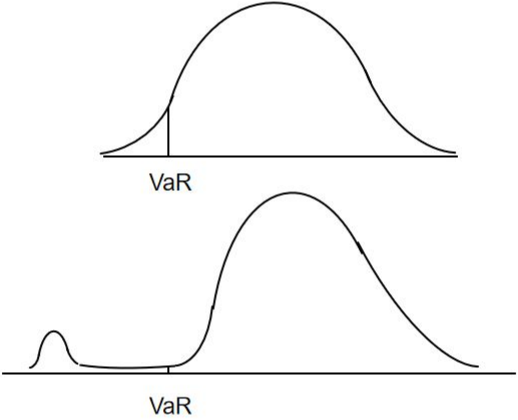
$$ES_\alpha(L) = \frac{1}{1-\alpha} \int_{q_\alpha(L)}^{\infty} x f_L(x) dx$$

where f_L is the density distribution of L.

VaR vs. Expected Shortfall

- VaR is the loss level that will not be exceeded with a specified probability
- Expected shortfall (ES) is the expected loss given that the loss is greater than the VaR level (also called CVaR and Tail Loss)
- Regulators have indicated that they plan to move from using VaR to using ES for determining market risk capital
- Two portfolios with the same VaR can have very different expected shortfalls

We will see that distributions with the Same VaR but they have different Expected Shortfalls



the second distribution is called heavy tail.

Theorem 2. For a loss L with continuous distribution function F_L expected shortfall is given by

$$ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_p(L)dp$$

proof. Let U be uniformly distributed random variable on (0,1), and $F_L^{\leftarrow}$ the strictly decreasing if F_L is continuous. Setting $L = F_L^{\leftarrow}(U)$ we have

$$\begin{aligned}
ES_\alpha(L) &= \frac{1}{1-\alpha}\mathbb{E}(L\mathbb{1}_{[q_\alpha(L),\infty)}(L)) \\
&= \frac{1}{1-\alpha}\mathbb{E}\Big(F_L^{\leftarrow}(U)\mathbb{1}_{[F_L^{\leftarrow}\alpha(L),\infty)}(F_L^{\leftarrow}(U))\Big) \\
&= \frac{1}{1-\alpha}\mathbb{E}\Big(F_L^{\leftarrow}(U)\mathbb{1}_{[\alpha,\infty)}(U)\Big) \\
&= \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du
\end{aligned}$$

Expected shortfall for a discrete distribution

There are different possibilities to define expected shortfall. A useful definition called generalized expected shortfall, which is a so-called coherent risk measure, is given by

$$GES_\alpha(L) := \frac{1}{1-\alpha}\Big(\mathbb{E}(L\mathbb{1}_{[q_\alpha(L),\infty)}(L)) + q_\alpha(L)(1-\alpha-\mathbb{P}(L \geq q_\alpha(L)))\Big)$$

It is seen that when the distribution of L is continuous then the second term is ZERO, and, hence

$$GES_\alpha(L) = ES_\alpha(L)$$

Example

(a) Let $L \sim Exp(\lambda)$, calculate $ES_\alpha(L)$

(b) Let L have distribution function $F(x) = 1 - (1 - \gamma x)^{1/\gamma}, x \geq 0, \gamma \in (0, 1)$. Calculate $ES_\alpha(L)$

Solution

(a) We have $F(x) = 1 - e^{-\lambda x}$, hence

$$1 - e^{-\lambda VaR_p(L))} = p$$

or

$$VaR_p(L) = -\frac{1}{\lambda} \log(1 - p)$$

So we get

$$ES_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_p dp = \lambda^{-1}\Big(1 - \log(1-\alpha)\Big)$$

(b) Similarly we have (proof??)

$$ES_\alpha(L) = \gamma^{-1}[(1-\alpha)^{-1}(1-\gamma)^{-1} - 1]$$

Example

The loss L of an investment has probability distribution

L	0	25	50	75	100	150
prob(%)	96.04	1.96	0.01	1.96	0.02	0.01

Calculate $VaR_{99\%}(L)$ and $ES_{99\%}(L)$.

Solution. We have $VaR_{99\%}(L) = 75???$ and

$$\begin{aligned} ES_{99\%}(L) &= \mathbb{E}(L \mid L \geq VaR_{99\%}(L)) \\ &= \frac{75 \times 1.96\% + 100 \times 0.02\% + 150 \times 0.01\%}{1.96\% + 0.02\% + 0.01\%} = 75.63\$ \end{aligned}$$

Properties

Proposition 1. $ES_\alpha(L)$ dominates $VaR_\alpha(L)$, i.e.,

$$ES_\alpha(L) \geq VaR_\alpha(L)$$

proof. Since, for all $\beta \geq \alpha$ implies $VaR_\beta \geq VaR_\alpha$. We have

$$ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\beta(L)d\beta \geq \frac{1}{1-\alpha}\int_\alpha^1 VaR_\alpha(L)d\beta = VaR_\alpha(L).$$

Proposition 2. For any $X \leq Y$ and any real number m we have

(i) $ES^\alpha(X) \geq ES^\alpha(Y)$

(ii) $ES^\alpha(X+m) = ES^\alpha(X) - m$

(iii) For $\lambda > 0$, $ES^\alpha(\lambda X) = \lambda ES^\alpha(X)$

4.3.1. Historical method

Instead of using a probabilistic model to estimate distribution of loss L, we use can estimate the distribution by using a historical simulation. This method is called non-parametric method. Given a sample losses $L_i, i = 1, 2, .., n$ of independent copies of L. Suppose that the data L_i are ordered by

$$\tilde{L}_{n,n} \leq \tilde{L}_{n-1,n} \leq ... \leq \tilde{L}_{1,n}$$

Then the Value at Risk at confidence level α is

$$VaR_\alpha(L) = \tilde{L}_{[n(1-\alpha)]+1,n}$$

where $[n(1-\alpha)]$ is the largest integer not exceeding $n(1-\alpha)$.
The Expected Shortfall can be estimated by

$$ES_\alpha(L) = (\tilde{L}_{[n(1-\alpha)]+1,n} + ... + \tilde{L}_{1,n})/([n(1-\alpha)] + 1)$$

Example

Consider an investment with 500 different scenarios. The losses are ranked as follows

Highest to Lowest for 500 Scenarios

Scenario Number	Loss ($000s)
494	477.841
339	345.435
349	282.204
329	277.041
487	253.385
227	217.974
131	202.256
238	201.389
473	191.269
306	191.050
477	185.127
495	184.450
376	182.707
237	180.105
365	172.224

Let confidence level $\alpha = 99\%$. We have the observations $n = 500$, then $[n(1-\alpha)] = 5$. So we have

$$VaR_{0.99}(L) = \tilde{L}_{6,500} = L_{487} = 217.947\$$$

Expected Shortfall

$$\begin{aligned} ES_{0.99}(L) &= \frac{\tilde{L}_{6,500} + \tilde{L}_{5,500} + \tilde{L}_{4,500} + \tilde{L}_{3,500} + \tilde{L}_{2,500} + \tilde{L}_{1,500}}{5} \\ &= \frac{L_{227} + L_{487} + L_{329} + L_{349} + L_{339} + L_{494}}{6} \\ &= \frac{217.947 + 253.385 + 277.041 + 282.204 + 345.435 + 477.841}{5} \\ &= 308.975\$ \end{aligned}$$

Example

Consider 1000000 $ long position in S&P 500 index. Using two years of price data from August 28, 2011 to August 28, 2013. The profit and Loss X is given by

i	t	S_t	date t	S_{t-1}	$\tilde{r}^{(i)}$	$\tilde{r}^{\text{arith},(i)}$	P&L
1	52	1229.10	09Nov2011	1275.92	-0.03739	-0.03670	-36 695.09
2	18	1129.56	22Sep2011	1166.76	-0.03240	-0.03188	-31 883.16
3	17	1166.76	21Sep2011	1202.09	-0.02983	-0.02939	-29 390.48
4	25	1099.23	03Oct2011	1131.42	-0.02886	-0.02845	-28 450.97
5	46	1218.28	01Nov2011	1253.30	-0.02834	-0.02794	-27 942.23
6	9	1154.23	09Sep2011	1185.90	-0.02707	-0.02671	-26 705.46
7	5	1173.97	02Sep2011	1204.42	-0.02561	-0.02528	-25 281.88
8	455	1588.19	20Jun2013	1628.93	-0.02533	-0.02501	-25 010.28
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
499	64	1192.55	28Nov2011	1158.67	0.02882	0.02924	29 240.42
500	80	1241.31	20Dec2011	1205.35	0.02940	0.02983	29 833.66
501	30	1194.89	10Oct2011	1155.46	0.03356	0.03412	34 124.94
502	43	1284.59	27Oct2011	1242.00	0.03372	0.03429	34 291.47
503	66	1246.96	30Nov2011	1195.19	0.04240	0.04332	43 315.29

Let confidence level $\alpha = 0.99$. The observations are 503. Then we have $[n(1-\alpha)] = [503 * 0.01] = 5$. We see from the table that the profit and loss are ranked. Then we have

$$VaR_{0.99}(L) = \tilde{L}_6 = -\tilde{X}_6 = X_{t=9} = 26705.46\$$$

The Expected Shortfall is

$$\begin{aligned} ES_\alpha(L) = \frac{1}{6}&(36695.09 + 31883.16 + 29390.48 \\ &+ 28450.97 + 27942.23 + 26705.46) \\ &= 30177.898 \end{aligned}$$

4.3.2. Monte Carlo Method

Monte Carlo method is useful method for simulate the stock's returns. Consider a stock with S_t the price at time t. The dynamic of price is modeled by SDE

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Let $Y_t := log(S_t/S_{t-1}) = log(S_t) - log(S_{t-1})$ the log return. We have

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma B_t}$$

Hence,

$$\log(S_t) = \log(S_0) + (\mu - \sigma^2/2)t + \sigma B_t$$

and

$$\log(S_{t-1}) = \log(S_0) + (\mu - \sigma^2/2)(t-1) + \sigma B_{t-1}$$

So we have

$$Y_t = \log(S_t/S_{t-1}) = (\mu - \sigma^2/2) + \sigma(B_t - B_{t-1})$$

Put $Z_{t,t-1} = \sigma(B_t - B_{t-1})$. Then $Z \sim N(0, \sigma^2)$. Note that μ and σ are mean and volatility of the stock. We have

$$Y_t = (\mu - \sigma^2/2) + Z \sim N((\mu - \sigma^2/2), \sigma^2)$$

We have the procedure for obtain VaR by using Monte Carlo method in the following steps

1. Compute the initial value V_t of portfolio (x is the number of shares on stock)

$$V_t = xS_t$$

2. Simulate M one-day normal return $Z_{t,t-1}$ from

$$N(0, 1), i = 1, 2, ..., M$$

3. Calculate the one-day future price by

$$S_{t,i} = S_{t-1} e^{(\mu - \sigma^2/2) + Z_{t,i}}, \quad i = 1, 2, .., M$$

4. Calculate the simulated futures value of the portfolio

$$V_{t,i} = xS_{t,i}$$

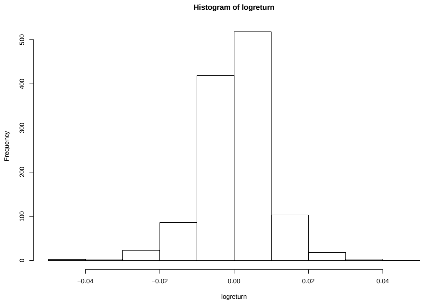
5. The profit and loss value is

$$Q_{t,i} = V_{t,i} - V_t$$

6. The VaR can be directly from the quantile of the sequence $\{Q_{t,i}, i = 1, 2, ..., M\}$

Example

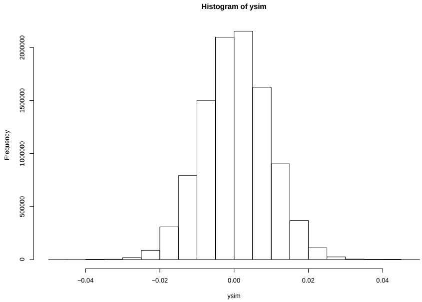
Consider a stock of McDonalds with daily prices from 1.4.2010 to 9.5.2014. We have the histogram of returns



Then Value at Risk at confidence level using historical method is

$$VaR^{\alpha}(r_t) = 0.02029304$$

Using Monte Carlo method we obtain the histogram of returns



Then we have Value at Risk is

$$VaR^{0.01}(r_t) = 0.02346678$$

1. Portfolio VaR

1.1. Return of a portfolio

For simple, we just consider a portfolio of two stock. We consider two stocks S_1, S_2. Denote $S_1(t), S_2(t)$ the prices at time t of stocks S_1, S_2, respectively. Suppose that we buy x_1 shares of stock S_1, and x_2 shares of stock S_2. Then the initial value ($t = 0$) of this portfolio is

$$V(0) = x_1 S_1(0) + x_2 S_2(0)$$

The weights of assets S_1 and S_2 are defined by

$$\omega_1 := x_1 \frac{S_1(0)}{V(0)}; \quad \omega_2 := x_2 \frac{S_2(0)}{V(0)}$$

then the funds allocated to a particular stock are $\omega_1 V(0)$ and $\omega_2 V(0)$, respectively. We have the property: $\omega_1 + \omega_2 = 1$.
The numbers of shares we buy are

$$x_1 = \omega_1 \frac{V(0)}{S_1(0)}; \quad x_2 = \omega_2 \frac{V(0)}{S_2(0)}.$$

At time t, the value of the portfolio is

$$V(t) = x_1 S_1(t) + x_2 S_2(t)$$

The simple returns of the portfolio, stock S_1, and stock S_2, respectively, are defined by

- $R(t) := \frac{V(t)-V(0)}{V(0)}$
- $R_1(t) := \frac{S_1(t)-S_1(0)}{S_1(0)}$
- $R_2(t) := \frac{S_2(t)-S_2(0)}{S_2(0)}$

Proposition

The return $R(t)$ of the portfolio is represented by

$$R(t) = \omega_1 R_1(t) + \omega_2 R_2(t)$$

Remark.

- Note that if the weight is negative, it means that the stock is short sell.
- Since $S_1(t)$ and $S_2(t)$ are random variables, and hence, the simple returns $R(t), R_1(t), R_2(t)$ are also random variables.

Example 1

Consider two stocks S_1, S_2 with the initial prices are $S_1(0) = 200\$$ and $S_2(0) = 300\$$. Assume that after 1 year ($t = 1$), the price $S_1(t)$ goes up to 260 with probability 0.6 or goes down to 180 with probability 0.4, and the price $S_1(t)$ goes down to 270 with probability 0.6 or goes up to 360 with probability 0.4.

Assume that we borrow 3 shares of S_1 and sell 4 shares of S_2. Then our initial value of portfolio is

$$V(0) = -3S_1(0) + 4S_2(0) = 600$$

after 1 year, our portfolio' value is

$$V(t) = -3S_1(t) + 4S_2(t)$$

its takes value: $-3 * 260 + 4 * 270 = 300$ with probability 0.6 and $-3 * 180 + 4 * 360 = 900$ with probability 0.4. We have the expected value of the portfolio is

$$\mathbb{E}(V(t)) = 300 * 0.6 + 900 * 0.4 = 540$$

The weights are

$$\omega_1 = \frac{-3 * 200}{600} = -1; \quad \omega_2 = \frac{4 * 300}{600} = 2$$

The expected returns are (???);

$$\mathbb{E}(R_1(t)) = 0.14; \quad \mathbb{E}(R_2(t)) = 0.02,$$

and, hence

$$\mathbb{E}(R(t)) = -\omega_1\mathbb{E}(R_1(t)) + \omega_2\mathbb{E}(R_2(t)) = -0.1.$$

1.2. Value at Risk of Portfolio

Consider a portfolio consisting of N assets. Denote $R_i(t), i = 1, 2, ..., N$ the returns of assets i from $t-1$ to t. We have the return of the portfolio

$$R_p(t) = \sum_{i=1}^{N} \omega_i R_i(t) = \omega' R(t)$$

where $\omega = (\omega_1, \omega_2, ..., \omega_N)'$ the vector of weights, and $R(t) := (R_1(t), R_2(t), ..., R_N(t))'$ the vector of returns. Denote $\mu_i := \mathbb{E}(R_i(t))$, and $\mu_p = \mathbb{E}(R_p)$, we have

$$\mu_p = \sum_{i=1}^{N} \omega_i \mu_i$$

and

$$\sigma_p^2 = Var(R_p(t)) = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i} \omega_i \omega_j \sigma_{ij}$$

Then we have

$$\sigma_p^2 = \omega' \Sigma \omega$$

where Σ is the covariance matrix of the returns, and ω are weights, which have no units. We can also written in term units (dollar, euro or VND,..) $y_i := x_i S_i(0), i = 1, 2, .., N$

$$\sigma_p^2 V^2(0) = y' \Sigma y.$$

Note that x_i is the number of shares on asset S_i then $y_i = x_i S_i(0)$ the portion of asset i in the portfolio.

Theorem 1

Assume that the returns in the portfolio are normally distributed with $\mu_i = \mathbb{E}(R_i); \sigma_i = Var(R_i), i = 1, 2, ..., N$. Then the Value-at-Risk of the portfolio with the initial value $V(0)$ at confidence level α is

$$VaR^{\alpha}(P) = -V(0)\Big(\mu_p + \sigma_p \Phi^{-1}(\alpha)\Big).$$

where

$$\mu_p = \sum_{i=1}^{N} \omega_i \mu_i \quad \text{and} \quad \sigma_p = \sqrt{\omega' \Sigma \omega}$$

We consider some special cases. For a simple example, the portfolio consists of two assets, which are assumed to be normally distributed with zero mean, assume that there is no short sale (i.e., $\omega_i > 0$), then

$$\sigma_p^2 = \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 + 2\omega_1 \omega_2 \rho_{12} \sigma_1 \sigma_2$$

If $\rho_{12} = 0$, i.e, two asset move independently, then

$$VaR^\alpha(P) = -V(0)\sigma_p \Phi^{-1}(\alpha) = V(0)\Phi^{-1}(\alpha)\sqrt{\omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2}$$

Note that (why???)

$$VaR^\alpha(S_1) = -x_1 S_1(0) \sigma_1 \Phi^{-1}(\alpha)$$

and

$$VaR^\alpha(S_2) = -x_2 S_2(0) \sigma_2 \Phi^{-1}(\alpha)$$

Then we have (why???)

$$VaR^{\alpha}(P) = \sqrt{(VaR^{\alpha}(S_1))^2 + (VaR^{\alpha}(S_2))^2}$$

and, hence,

$$VaR^{\alpha}(P) < VaR^{\alpha}(S_1) + VaR^{\alpha}(S_2).$$

If $\rho_{12} = 1$, two assets are perfectly correlated, then

$$VaR^{\alpha}(P) = V(0)\Phi^{-1}(\alpha)\sqrt{\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2} = VaR^{\alpha}(S_1) + VaR^{\alpha}(S_2)$$

Remark 1

The situation differs when short sales are allowed. Suppose that the portfolio is long asset 1 but short asset 2 (i.e., ω_1 is positive and ω_2 is negative). We hedge fund 1\$ long position in corporate bonds and 1\$ short position in Treasury bonds, if the correlation is 1 then the fund has no risk because any loss in one asset will be offset by a matching gain in the other then VaR is zero.

Example 2

Consider a portfolio consisting of two assets. Assume that they are uncorrelated and have volatility 5 and 12 percent, respectively. The portfolio has 2 million dollars invested in asset 1, and 1 million dollars invested to asset 2. Assume that the returns of assets are normally distributed with zero mean. Find the Value at Risk of the portfolio's returns with confidence level at 5%.

Solution. We have the covariance matrix

$$\Sigma = \begin{pmatrix} 0.05^2 & 0 \\ 0 & 0.12^2 \end{pmatrix}$$

The vector portion of assets is $y = (2, 1)'$ then we have the variance of the portfolio in dollar unit is

$$\sigma_p^2 V(0) = y'\Sigma y = (2, 1) \begin{pmatrix} 0.05^2 & 0 \\ 0 & 0.12^2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0.0244$$

Then the dollar volatility is

$$\sigma_p V(0) = \sqrt{0.0244} = 0.156025 \quad \text{million} = 156025$$

with $\alpha = 0.05$ then $\Phi^{-1}(0.05) = -1.65$ And, hence, the Value-at-Risk of the portfolio is

$$VaR^\alpha(P) = -(156025)(-1.65) = 257738\$$$

For individual Value at Risk of two assets, we have

$$VaR^\alpha(S_1) = -x_1\sigma_1\Phi^{-1} = -2 \times 0.05 \times (-1.65) = 165000\$$$

and

$$VaR^\alpha(S_2) = -x_2\sigma_2\Phi^{-1} = -1 \times 0.12 \times (-1.65) = 198000\$$$

Then

$$VaR^\alpha(S_1) + VaR^\alpha(S_2) = 363000$$

which is greater that $VaR^\alpha(P) = 257738\$$.

2.1. Marginal VaR

When changing positions (portfolio weights change) on portfolio, the question is how to measure the portfolio risk. Consider a portfolio consisting of N assets with weights $\omega_i, i = 1, 2, ..., N$. Then we have

$$\frac{\partial \sigma_p^2}{\partial \omega_i} = 2\omega_i \sigma_i^2 + 2\sum_{j=1, j\neq i}^{N} \omega_j \sigma_{ij} = 2cov(R_i, \omega_i R_i + \sum_{j\neq i}^{N} \omega_j R_j) = 2cov(R_i, R_p)$$

Note that we have

$$\frac{\partial \sigma_p^2}{\partial \omega_i} = 2\sigma_p \frac{\partial \sigma_p}{\partial \omega_i}$$

and, hence,

$$\frac{\partial \sigma_p}{\partial \omega_i} = \frac{cov(R_i, R_p)}{\sigma_p}$$

So we obtain the marginal Value-at-Risk formula

$$
\begin{aligned}
\Delta VaR_i^{\alpha} = \frac{\partial VaR^{\alpha}(P)}{\partial y_i} &= \frac{\partial VaR^{\alpha}(P)}{\partial \omega_i V(0)} \\
&= \frac{\partial(-V(0)\Phi^{-1}(\alpha)\sigma_p)}{\partial y_i V(0)} \\
&= -\Phi^{-1}(\alpha)\frac{cov(R_i, R_p)}{\sigma_p} = \Phi^{-1}(1-\alpha)\frac{cov(R_i, R_p)}{\sigma_p}
\end{aligned}
$$

So the change in portfolio Value-at-Risk resulting from adding capital of exposure to given a component. It is a partial derivative with respect to the component position.

Putting

$$\beta_i = \frac{cov(R_i, R_p)}{\sigma_p^2} = \frac{\sigma_{ip}}{\sigma_p^2} = \rho_{ip}\frac{\sigma_i}{\sigma_p}$$

this coefficient measures the contribution of one asset to total portfolio risk. Beta is also called the systematic risk of security. It is the slope coefficient in the regression of R_i on R_p

$$R_{i,t} = \alpha_{i,t} + \beta_i R_{p,t} + \epsilon_{i,t}$$

we have

$$\Delta VaR_i^{\alpha} = -\Phi^{-1}(\alpha)(\beta_i \times \sigma_p)$$

Denote β is the vector of the systematic risk, then we have (proof??)

$$\beta = \frac{\Sigma\omega}{\omega'\Sigma\omega}$$

We can also obtain the expression of margin VaR by

$$\Delta VaR_i^{\alpha} = \frac{VaR^{\alpha}(P)}{V(0)}\beta_i$$

Example 3

Consider example 2, if we increase the position of asset 1 by 1 dollar, then we have the marginal VaR of asset 1:

$$\begin{aligned} cov(R_1, R_p) &= cov(R_1, \omega_1 R_1 + \omega_2 R_2) \\ &= \omega_1 cov(R_1, R_1) + \omega_2 cov(R_1, R_2) \\ &= \omega_1 \sigma_1^2 + \omega_2 \sigma_{12} = \frac{2}{3} * 0.05^2 = 0.00167 \end{aligned}$$

and

$$\sigma_p^2 = \omega' \Sigma \omega = \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 = 0.00271$$

Hence,

$$\beta_1 = \frac{cov(R_1, R_p)}{\sigma^2} = \frac{0.00167}{0.00271} = 0.616$$

So we get

$$\Delta VaR_1(P) = -\Phi(0.05) \times 0.616\sqrt{0.00271} = 0.0528$$

2.2. Component measures and incremental VaR

- The component Value at Risk for ith subportfolio is

$$C_i := \frac{\partial VaR_i^{\alpha}(P)}{\partial y_i} \times a_i$$

where a_i is the increase amount added to the component ith.

- if the position of the portfolio is now changed by position $a = (a_1, a_2, ..., a_N)$, i.e., the initial component $y = (y_1, y_2, ..., y_N)$ is now changed to $y + a = (y_1 + a_1, y_2 + a_2, ..., y_N + a_N)$ then we have the incremental VaR

$$\text{incremental VaR} \approx \Delta VaR(P) \times a = \sum_{i=1}^{N} C_i$$

where

$$\Delta VaR(P) = (\frac{\partial VaR_1^{\alpha}(P)}{\partial y_1}, \frac{\partial VaR_2^{\alpha}(P)}{\partial y_2}, ..., \frac{\partial VaR_N^{\alpha}(P)}{\partial y_N})'$$

Example 4

Consider example 2, if we now consider to increase only the asset 1 by 10000 dollars. From example 3, we have

$$\Delta VaR_1 = \frac{\partial VaR_1^{\alpha}(P)}{\partial y_1} = 0.0529$$

then the component VaR is

$$C_1 = 0.0528 \times 10000 = 528\$$$

It is seen that $C_2 = 0$

Therefore, we have the incremental VaR

$$\text{incremental VaR} \approx \Delta VaR(P) \times a = \sum_{i=1}^{N} C_i = 528\$$$

When we add $10000\$ \sim 0.01$ million to the first asset then the new position is $Y := y + a = (2.01, 1)$, we have

$$\sigma_{p+a}^2 V^2(y+a) = Y'\Sigma Y = \begin{pmatrix} 2.01 & 1 \end{pmatrix} \begin{pmatrix} 0.05^2 & 0 \\ 0 & 0.12^2 \end{pmatrix} \begin{pmatrix} 2.01 \\ 1 \end{pmatrix} = 0.02450025$$

we have the Value-at-Risk at this position

$$\begin{aligned} VaR^{\alpha}(P+a) &= -\Phi^{-1}(0.05)\sigma_{p+a}V(y+a) = -(1.65)\sqrt{0.02450025} \\ &= 0.258267 \quad \text{million} \end{aligned}$$

or

$$VaR^{\alpha}(P+a) = 258267\$$$

In example 2, we have the initial $VaR^{\alpha}(P) = 257738\$$. So the true incremental VaR is

$$VaR^{\alpha}(P + a) - VaR^{\alpha}(P) = 258267 - 257738 = 529$$

which is close to the approximation $528\$$.

3. Euler's Theorem and Risk Decompositions

In this section we will define individual asset risk contributions. Euler's theorem provides a general method for decomposing risk into asset specific contributions.

Definition 1

(*homogenous function of degree one*) *Let* $f(x_1, x_2, ..., x_n)$ *be a continuous and differentiable function of the variables* $x_1, x_2, ..., x_N$ *The function* f *is called homogeneous of degree one if for any constant* $c > 0$

$$f(cx_1, cx_2, ..., cx_n) = cf(x_1, x_2, ..., X_n)$$

Example 5

- Function $f(x_1, x_2) = x_1 + x_2$ is homogenous function of degree one.
- Function $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ is not homogenous function of degree one.

- Is the variance of a portfolio $\sigma_p(\omega_1, \omega_2, ..., \omega_n)$ homogenous function of degree one?

Euler's theorem

Let $f(x_1, x_2, ..., x_n)$ be a continuous, differentiable and homogenous of degree one function of the variables $x_1, x_2, .., x_n$. Then

$$f(x_1, x_2, ..., x_n) = \sum_{t=1}^{n} \frac{\partial f}{\partial x_i} x_i$$

Check the Euler's theorem for functions $f(x_1, x_2, x_3) = x_1 + x_2 + x_3$ and $f(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Risk decomposition using Euler's theorem

Denote $RM_p(\omega)$ the portfolio risk measure that is a homogenous function of degree one in the portfolio weight vector $\omega = (\omega_1, \omega_2, ..., \omega_n)$. Then by Euler's theorem, $RM_p(\omega)$ has risk decomposition

$$RM_p(\omega) = \sum_{t=1}^{n} \frac{\partial RM_p(\omega)}{\partial \omega_i} \omega_i$$

The partial derivatives $\frac{\partial RM_p(\omega)}{\partial \omega_i}$ with respect to ω_i are called asset marginal contributions to risk

$$MCR_i := \frac{\partial RM_p(\omega)}{\partial \omega_i} = \text{marginal contribution of asset } i$$

The asset contributions to risk (CR) are defined as the weighted marginal contributions

$$CR_i := \omega_i \frac{\partial RM_p(\omega)}{\partial \omega_i}$$

Then we have the decomposition

$$RM_p(\omega) = \sum_{i=1}^{n} \omega_i \frac{\partial RM_p(\omega)}{\partial \omega_i} = \sum_{i=1}^{n} CR_i$$

an hence,

$$\frac{CR_1}{RM_p(\omega)} + \frac{CR_2}{RM_p(\omega)} + \cdots + \frac{CR_n}{RM_p(\omega)} = 1$$

Putting

$$PCR_i := \frac{CR_i}{RM_p(\omega)}$$

is called percent contribution risk (PCR) of asset i. So we get

$$PCR_1 + PCR_2 + \cdots + PCR_n = 1$$

Risk decomposition for σ_p

Denote $\sigma_p(\omega)$ the standard deviation of a portfolio with weight $\omega = (\omega_1, \omega_2, ..., \omega_n)$. Then $\sigma_p(\omega)$ is homogeneous function of degree one in the portfolio weight. We have

$$RM_p(\omega) = \sigma_p(\omega) = (\omega \Sigma \omega')^{1/2}$$

Hence, by Euler's theorem

$$\sigma_p(\omega) = \sum_{i=1}^{n} \frac{\partial \sigma_p(\omega)}{\partial \omega_i} \omega_i = \omega \frac{\partial \sigma_p(\omega)}{\partial \omega}$$

where,

$$\frac{\partial \sigma_p(\omega)}{\partial \omega} = \frac{\Sigma \omega'}{\sigma_p(\omega)}$$

Then we get the marginal contributions to risk

$$MCR_i = \frac{\partial \sigma_p(\omega)}{\partial \omega_i} = \frac{(\Sigma\omega')_i}{\sigma_p(\omega)}$$

(i-th row)
and asset contributions to risk (CR)

$$CR_i = \omega_i \times \frac{(\Sigma\omega')_i}{\sigma_p(\omega)}$$

and percent contribution risk (PCR)

$$PCR_i = \omega_i \times \frac{(\Sigma\omega')_i}{\sigma_p^2(\omega)}$$

Risk decomposition for $VaR_{p,\alpha}$

Denote $VaR_{p,\alpha}(\omega)$ the Value-at-Risk of a portfolio with weight ω. Then $VaR_{p,\alpha}$ is the homogenous of degree 1 in ω. By using Euler's theorem we get

$$VaR_{p,\alpha}(\omega) = \sum_{i=1}^{n} \frac{\partial VaR_{p,\alpha}(\omega)}{\partial \omega_i} \omega_i = \omega \frac{\partial VaR_{p,\alpha}(\omega)}{\partial \omega}$$

We have

$$VaR_{p,\alpha}(\omega) = -V(0)\Big(\omega\mu' + \Phi^{-1}(\alpha)\sigma_p(\omega)\Big)$$

where $\mu = (\mu_1, \mu_2, .., \mu_n)$ Hence,

$$\frac{\partial VaR_{p,\alpha}(\omega)}{\partial \omega} = -V(0)\Big(\mu' + \frac{\Sigma\omega'}{\sigma_p(\omega)}\Phi^{-1}(\alpha)\Big)$$

So we obtain

$$MCR_i = -V(0)\Big(\mu_i + \frac{(\Sigma\omega')_i}{\sigma_p(\omega)}\Phi^{-1}(\alpha)\Big)$$

and

$$CR_i = -\omega_i \times V(0)\Big(\mu_i + \frac{(\Sigma\omega')_i}{\sigma_p(\omega)}\Phi^{-1}(\alpha)\Big)$$

and

$$PCR_i = -\omega_i \times V(0)\Big(\mu_i + \frac{(\Sigma\omega')_i}{\sigma_p(\omega)}\Phi^{-1}(\alpha)\Big)/VaR_{p,\alpha}(\omega)$$

By Euler's theorem we have

$$VaR_{\alpha,p}(\omega) = \sum_{i=1}^{n} CR_i$$

Similarly,

$$ES_{\alpha,p}(\omega) \sum_{i=1}^{n} \frac{\partial ES}{\omega_i}\omega_i$$

4. Portfolio risk reports

A portfolio risk report summarizes asset and portfolio risk measures as well as risk budgets. The following table shows a typical portfolio risk report. The total amount invested in the portfolio is $V(0)$ and the amounts invested in each asset are $y_i = x_i \times S_i(0) = \omega_i \times V(0)$

Asset	y_i	ω_i	MCR_i	CR_i	PCR_i
Asset 1	y_1	ω_1	MCR_1	CR_1	PCR_1
Asset 2	y_2	ω_2	MCR_2	CR_2	PCR_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Asset n	y_n	ω_n	MCR_n	CR_n	PCR_n

Example

Consider three assets. Assume that their expected returns are equal to zero whereas their volatilities are equal to 30%, 20% and 15%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1 & & \\ 0.8 & 1 & \\ 0.5 & 0.3 & 1 \end{pmatrix}$$

Let us consider the portfolio (50%, 20%, 30%). Then we have the covariance matrix

$$\Sigma = \begin{pmatrix} 9.00 & 4.80 & 2.25 \\ 4.80 & 4.00 & 0.90 \\ 2.25 & 0.90 & 2.25 \end{pmatrix} \times 10^{-2}$$

It follows that the variance of the portfolio w.r.t the given weight is:

$$\begin{aligned}\sigma_p^2 =&0.5^2 \times 0.09 + 0.2^2 \times 0.04 + 0.3^2 \times 0.0225 \\ &+ 2 \times 0.5 \times 0.2 \times 0.0480 + 2 \times 0.5 \times 0.3 \times 0.0225 \\ &+ 2 \times 0.2 \times 0.3 \times 0.0090 \\ &= 4.356\%\end{aligned}$$

Hence, $\sigma_p = 20.87\%$ The marginal contribution to risk is

$$MCR = \frac{\Sigma\omega}{\sigma_p} = \begin{pmatrix} 29.4\% \\ 16.63\% \\ 9.49\% \end{pmatrix}$$

The contribution to risk is

$$CR = \frac{\omega^T \Sigma\omega}{\sigma_p} = \begin{pmatrix} 14.7\% \\ 3.33\% \\ 2.85\% \end{pmatrix}$$

and the percent contribution risk (PCR)

$$PCR = \frac{CR}{\sigma_p} = \begin{pmatrix} 70.43\% \\ 15.93\% \\ 13.64 \end{pmatrix}$$

So the risk report is presented in the table

Asset	ω_i	MCR_i	CR_i	PCR_i
1	50	29.40	14.70	70.43
2	20	16.63	3.33	15.93
3	30	9.49	2.85	13.64
RM_p			20.87	

1. Zero-coupon Bonds

- Zero-coupon bonds, also called pure discount bonds and sometimes known as "zeros," pay no principal or interest until maturity.
- A "zero" has a par value or face value, which is the payment made to the bondholder at maturity.

Definition 2

A T-maturity zero-coupon bond (pure discount bond) is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments.

A dollar today is worth more than a dollar tomorrow. The time t value of a dollar at time $T \geq t$ is expressed by the zero-coupon bond with maturity T, $P(t, T)$, briefly also T-bond. This is a contract which guarantees the holder one dollar to be paid at the maturity date T. The zero sells for less than the par value, which is the reason it is a discount bond.

Example 6

Consider a 20-year zero with par value of $\$1,000$ and 6% interest rate compounded annually. The market price is the present value of $\$1,000$ corresponding to an annual interest rate of 6 % with annual discounting is:

$$\frac{1000}{(1.06)^{20}} = \$311.80$$

- In case the annual interest rate is 6 % but compounded every 6 months, then the price is

$$\frac{1000}{(1.03)^{40}} = \$306.56$$

- If the annual rate is 6 % compounded continuously, then the price is

$$\frac{1000}{e^{0.06\times 20}} = \$301.19.$$

Price and Returns Fluctuate with the Interest Rate

We will see how the interest rate effects to the price and return of bonds. Consider the previous example, assume semiannual compounding is 6%. The price of the zero-coupon bond is \$ 306.56. Assume that 6 months later, the interest rate increase is 7%. The new price of the bond now is

$$\frac{1000}{(1.035)^{39}} = \$261.41$$

So our investment would drop by $306.56 - 261.41 = \$45.15$
Note that at maturity we will still get the face value of \$1000 if we keep the bond for 20 years. The difference is that if we sold it now we will lose \$45.15. This is equivalent to investment's return

$$\frac{261.41 - 306.56}{306.56} = -14.73\%$$

for a half year or -29.46% per year. Now in case the interest rate drops to 5% after 6 months, then the bond price now is

$$\frac{1000}{(1.025)^{39}} = \$381.74$$

Hence the rate of return for one year is

$$2\frac{381.74 - 306.56}{306.56} = 49.05\%$$

If the interest rate unchanged at 6% after 6 months then the bond price is

$$\frac{1000}{(1.03)^{39}} = \$315.75$$

and, hence, the annual rate of return is

$$2\frac{315.75 - 306.56}{306.56} = 6\%$$

This is exactly is the interest rate of the bond.

Remark 2

From this example we see that the interest rate effects the price of the bond. If the interest rate increase then the price of bond is decrease or vice versa.

General formula

Denote B_T the par value or face value of a bond and $P(0, T)$ the price of bond at time $t = 0$.

- The price of a zero-coupon bond is given by

$$P(0, T) = B_T(1 + r)^{-T}$$

  where T is the time to maturity in years, and r the annual rate of interest with annual compounding.

- If we assume semiannual compounding, then the price is

$$P(0, T) = B_T(1 + r/2)^{-2T}$$

Question. What is the price of a zero-coupon bond at time t?

2. Coupon Bonds

- A coupon bond is a type of bond that includes attached coupons and pays periodic (typically annual or semi-annual) interest payments during its lifetime and its par value at maturity.
- These bonds come with a coupon rate, which refers to the bond's yield at the date of issuance. Bonds that have higher coupon rates offer investors higher yields on their investment.
- Let r be the interest rate applicable to the cash flows arising from a fixed coupon bond, giving constant coupon c paid at times $1, 2, ..., T$ and par amount B_T paid at maturity T. Then the fair price of bond is

$$P(0,T) = \frac{c}{(1+r)} + \frac{c}{(1+r)^2} + \cdots + \frac{c}{(1+r)^T} + \frac{B_T}{(1+r)^T}$$
$$= \sum_{t=1}^{T} \frac{c}{(1+r)^t} + \frac{B_T}{(1+r)^T}$$

Hence, it is straightforward algebra (proof???)

$$P(0,T) = \frac{c}{r}\left(1 - \frac{1}{(1+r)^T}\right) + \frac{B_T}{(1+r)^T}$$

In case the bond with par value matures in T years and makes semiannual coupon payments of c and the yield (rate of interest) is r per half-year, then the value of the bond when it is issued is

$$
\begin{aligned}
P(0,T) &= \sum_{t=1}^{2T} \frac{c}{(1+r)^t} + \frac{B_T}{(1+r)^{2T}} \\
&= \frac{c}{r}\left(1 - \frac{1}{(1+r)^{-2T}}\right) + \frac{B_T}{(1+r)^{2T}} \\
&= \frac{c}{r} + \left(B_T - \frac{c}{r}\right)(1+r)^{-2T}
\end{aligned}
$$

Example 7

Consider a 20-year coupon bond with a par value of \$1,000 and 6 % annual coupon rate with semiannual coupon payments, so effectively the 6% is compounded semiannually. Each coupon payment will be \$30.

Since the coupon rate is paid semiannually and each coupon payment is \$ 30. So the bondholder will receive 40 payments of \$30. The annual rate is 6% equivalently 3% semiannual. Therefore we have the present value of all payments ($B_T = 1000, c = 30, T = 40$)

$$\sum_{t=1}^{40} \frac{30}{(1+0.03)^t} + \frac{1000}{(1+0.03)^{40}} = 1000$$

Now after 6 month, if the interest rate unchanged,(we receive immediately \$30), then therefore, we have the present value of all payments

$$P(1,T) = \sum_{t=0}^{39} \frac{30}{(1+0.03)^t} + \frac{1000}{(1+0.03)^{39}} = 1030$$

- If after 6 month, the interest rate now increases to 7%, then the price of the bond is

$$P(1,T) = \sum_{t=0}^{39} \frac{30}{(1+0.035)^t} + \frac{1000}{(1+0.035)^{39}} = 924.49$$

Hence, the rate of return is

$$2\frac{(924.49 - 1000)}{1000} = -15.1\%$$

it means that we lose 15.1% per year.

- If the interest now goes down to 5% after 6 months, then the investment is worth

$$P(1,T) = \sum_{t=0}^{39} \frac{30}{(1+0.025)^t} + \frac{1000}{(1+0.025)^{39}} = 1,153.70$$

Hence the annual return is

$$2\frac{(1,153.70 - 1000)}{1000} = 30.72\%$$

it means that we gain 30.72% per year

3. Yield to Maturity

From the formula of pricing bond

$$P(0,T) = \frac{c}{r}\left(1 - \frac{1}{(1+r)^T}\right) + \frac{B_T}{(1+r)^T}$$

or for semiannual payments

$$P(0,T) = \frac{c}{r} + \left(B_T - \frac{c}{r}\right)(1+r)^{-2T}$$

Remark 3

if the coupon $c = rB_T$ then the price of bond is exactly equals to par value B_T.

Example 8

Suppose that a bond has the maturity $T = 30$, semiannual coupon payments $c = 40$, par value $B_T = 1000$. It the bond were selling at the par value, then the rate $r = c/B_T = 0.04 \sim 4\%$ per half year or 8% per year is called coupon rate.

- If it is selling for price $P(0, T) = \$1200$, i.e, \$200 above par value, then from the formula by solving the equation with variable r, we get the interest $r \approx 0.0324 \sim 3.24\%$ per half year or 6.48% per year. r is called yield to maturity.
- The coupon payments are \$40 which correspond to $40/1200 = 3.333\%$ per half year or 6.67% per year for \$1,200 investment. The interest rate 6.67% is called *the current yield.*

From the example we have the following relationships

(i) price $>$ par value $\Rightarrow$ coupon rate $>$ current yield $>$ yield to maturity.

(ii) price $<$ par value $\Rightarrow$ coupon rate $<$ current yield $<$ yield to maturity.

Exercise. To check the conclusion (ii), in the previous example, calculate the current yield and yield to maturity if the price of the bond is now \$900.

General Method for Yield to Maturity

The yield to maturity (on a semiannual basis) of a coupon bond is the value of r that solves

$$P(0,T) = \frac{c}{r} + \left(B_T - \frac{c}{r}\right)(1+r)^{-2T}$$

or

$$P(0,T) = \frac{c}{r}\left(1 - \frac{1}{(1+r)^T}\right) + \frac{B_T}{(1+r)^T}$$

Spot Rates

- *A spot rate is the rate on a spot loan, an agreement in which a lender gives money to the borrower at the time of the agreement to be repaid at some single, specified time in the future.*
- The yield to maturity of a zero-coupon bond of maturity n years is called the n-year spot rate and is denoted by r_n
- Consider example of one-year coupon bond with semiannual payments of \$ 40 and par value is \$ 1,000. Suppose that the one-half-year spot rate is 2.5 % and the 1-year spot rate is 3% per half year. Then the price of the bond is

$$\frac{40}{(1+0.025)} + \frac{40+1000}{(1+0.03)^2} = 1019.32$$

Now the yield to the maturity r on the coupon bond corresponding to the price of \$ 1019.32 is

$$\frac{40}{(1+r)} + \frac{40+1000}{(1+r)^2} = 1019.32$$

Hence, the solution $r = 0.0299 = 2.99\%$ per half-year or 5.98% per year.

General formula

Let $r_1, r_2, .., r_{2T}$ be the half-year spot rates for zero-coupon bonds of maturities $1/2, 1, 3/2, .., T$ years with the par value B_T Then the price of coupon bond is

$$P(0,T) = \sum_{t=1}^{2T-1} \frac{c}{(1+r_t)^t} + \frac{c+B_T}{(1+r_T)^{2T}}$$

and, then to guarantee that there is no arbitrage, the yield to maturity of the bond is the solution of the equation

$$\sum_{t=1}^{2T-1} \frac{c}{(1+r)^t} + \frac{c+B_T}{(1+r)^{2T}} = \sum_{t=1}^{2T-1} \frac{c}{(1+r_t)^t} + \frac{c+B_T}{(1+r_T)^{2T}}$$

4. Term Structure

Forward rate

Assume that there are load contracts which begin at some specified time in the future, say S years from now Interest rate required by lenders on such a loan (with maturity T) is called the forward rate from $T-S$ to T. If the markets are arbitrage free e (meaning that there are no possibilities to make certain excess returns over and above the risk-free rate of return)

- Denote $P(1), P(2), .., P(n)$ the prices of zero-coupon bonds of maturities 1-year, 2-year,..., n-year
- Denote $y_1, y_2, .., y_n$ the spot rates (yields of maturity of zero-coupon bonds) of maturities 1-year, 2-year,..., n-year
- Denote $f_1, f_2, ..., f_n$ the forward rates in the ith future year ($i = 1$ for the next year)

If the markets are arbitrage free, if the present value of paid \$ 1 paid k periods, $k = 1, 2, .., n$, then

- the yield to maturity for 1-year is

$$P(1) = \frac{1}{(1+y_1)} = \frac{1}{1+f_2} \Rightarrow y_1 = f_1$$

- the yield to maturity for 2-year is

$$P(2) = \frac{1}{(1+y_2)^2} = \frac{1}{(1+f_1)(1+f_2)} \Rightarrow y_2 = \sqrt{(1+f_1)(1+f_2)}$$

- the yield to maturity for n-year is

$$P(n) = \frac{1}{(1+y_n)^n} = \frac{1}{(1+f_1)(1+f_2)...(1+f_n)}$$
$$\Rightarrow y_n = \{(1+f_1)(1+f_2)...(1+f_n)\}^{1/n} - 1$$

To determine forward rates from yield to maturity, we have

$$f_1 = y_1$$

and

$$f_n = \frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}} - 1, n = 2, 3, ..$$

Example 9

Suppose that loans have the forward interest rates listed in the following table. The par value is $ 1000.

Year (i)	interest rate f_i
1	6
2	7
3	8

We have

$$P(1) = \frac{1000}{1+0.06} = 943.40$$

hence the yield $y_1 = f_1 = 0.06$. A par value $ 1000 for 2-year should sells the price

$$P(2) = \frac{1000}{(1+0.06)(1+0.07)} = 881.68$$

hence, the yield to 2-year is

$$\frac{1000}{(1+y_2)^2} = 881.68 \Rightarrow y_2 = 0.0650$$

A par value \$ 1000 for 3-year should sells the price

$$P(3) = \frac{1000}{(1+0.06)(1+0.07)(1+0.08)} = 816.37$$

hence, the yield to 2-year is

$$\frac{1000}{(1+y_3)^3} = 816.37 \Rightarrow y_3 = 0.06997$$

We have the table

Year (i)	interest rate f_i (%)	Yield y_i
1	6	6
2	7	6.5
3	8	6.997

Example 10

Suppose that one-, two-, and three-year par $1,000 zeros are priced as given in the following table

Maturity	Price
1 year	$920
2 years	$830
3 years	$760

Calculate yields to maturities and the forward rates from the prices.

Answer. $y_1 = 0.087; y_2 = 0.0976; y_3 = 0.096$ and $r_1 = 0.087; r_2 = 0.108; r_3 = 0.096$

Continuous compounding

Assume continuous compounding with forward interest rates $f_1, f_2, .., f_n$. If $P(n)$ is the price of zero-coupon bond with par value B_n then

$$P(n) = \frac{B_n}{\exp(f_1 + f_2 + ... + f_n)}$$

Hence

$$\frac{P(n)}{P(n-1)} = \exp(f_n) \Rightarrow f_n = \log(\frac{P(n-1)}{P(n)})$$

But

$$P(n) = \frac{B_n}{\exp(ny_n)}$$

So we obtain

$$y_n = \frac{f_1 + f_2 + ... + f_n}{n}$$

and

$$f_n = ny_n - (n-1)y_n, \quad n \geq 1$$

Example 11

Consider Example 10. We have

$$f_1 = \log(1000/920) = 0.083$$

$$f_2 = \log(920/830) = 0.103$$

$$f_3 = \log(830/760) = 0.088$$

Hence,

$$y_1 = f_1 = 0.083, y_2 = (f_1 + f_2)/2 = 0.093; y_3 = (f_1 + f_2 + f_3)/3 = 0.091$$

Continuous Forward Rates

We have assumed that forward interest rates vary from year to year but are constant within each year. This assumption is, of course, unrealistic and was made only to simplify the introduction of forward rates. Forward rates should be modeled as a function varying continuously in time. To specify the term structure in a realistic way, we assume that the forward interest rate is a continuous function of time t, say $r(t)$. Then the price of the zero-coupon bond with maturity T is

$$P(0,T) = B_T \times \exp\{-\int_0^T r(s)ds\}$$

Denote $d(T) := \exp\{-\int_0^T r(s)ds\}$ the discount function and the price of any zero-coupon bond. The Yield to Maturity

$$y_T = \frac{1}{T}\int_0^T r(s)ds = -\frac{1}{T}\log(d(T))$$

Example 12

Suppose the forward rate is the linear function

$$r(t) = 0.03 + 0.0005t$$

Find $r(15), y_{15}$ and $D(15)$.

We have $r(15) = 0.03 + 0.0005 \times 15 = 0.0375$ and

$$y_{15} = \frac{1}{15}\int_0^{15}(0.03 + 0.0005s)ds = 0.03375$$

and

$$d(15) = e^{-15y_{15}} = 0.603.$$

The problem is that we could be interested in how the discount function and forward rate function change over time. Denote $d(s,T)$ the discount function of the price at time s of the zero-coupon bond with par value B_T with maturity T, $r(s,t)$ the forward interest rate at time s, $t \geq s$

$$P(s, T) = B_T \times \exp\{-\int_s^T r(s,t)dt\}$$

and

$$d(s, T) = \exp\{-\int_s^T r(s,t)dt\}$$

and the yield to maturity

$$y(s, T) = \frac{1}{(T-s)} \int_s^T r(s,t)dt$$

5. Sensitivity of Price to Yield (Measure of interest rate risk)

We see that bonds are risky because bond prices are sensitive to interest rates. This problem is called interest-rate risk. In this section we will study the traditional method of quantifying interest-rate risk.

I) Duration of a coupon bond

The duration of a bond is weighted average of the times of payment of all the cash flows generated by the bond. The weights are the proportional shares of the bond's cash flows in the bond's price.
Recall the price of a bond

$$P(0,T) = \sum_{t=1}^{T} \frac{c_t}{(1+y)^t}$$

where c_t is the cash flow at time t and note that $c_T := c + B_T$, y this the yield to maturity T of the bond.

Macauley's duration. Denote $P := P(0,T)$ the price of the bond for short

$$D = 1\frac{c/P}{(1+y)} + 2\frac{c/P}{(1+y)^2} + \cdots + T\frac{(c+B_T)/P}{(1+y)^T} = \frac{1}{P}\sum_{t=1}^{T}\frac{tc_t}{(1+y)^t}$$

We now measure bond's sensitivity when the interest rate changes. From formula of bond pricing we have

$$\frac{dP}{dy} = \sum_{t=1}^{T}(-t)c_t(1+y)^{-t-1} = -\frac{1}{(1+y)}\sum_{t=1}^{T}tc_t(1+y)^{-t}$$

Hence

$$\frac{1}{P}\frac{dP}{dy} = -\frac{D}{1+y} = -D_m$$

where $D_m = \frac{D}{1+y}$ is called modified duration.

We have

$$\frac{\Delta P}{P} \approx \frac{dP}{P} = -D_m dy$$

and

$$var(\frac{\Delta P}{P}) = D_m^2 var(dy)$$

We see that the standard deviation of the relative changes in the bond price is a linear function of the standard deviation of the changes in yield to maturity or interest rate.

Example 13

Suppose a bond is at par value, its coupon bond is 9%, hence, yield to maturity $y = 9\%$. The duration is 6.99. Suppose that the interest rate (yield) increase by 1%, then we have $D = 6.99, y = 0.09, dy = 1$. Hence

$$\frac{\Delta P}{P} \approx \frac{dP}{P} = -\frac{D}{1+y}dy = -\frac{6.99}{1+0.09} \times 1\% = -6.4\%$$

Example 14

A bond with annual coupon 70, par 1000, and interest rate 5%. Hence duration is found to be $D = 7.7$ years (check!!). Therefore, modified duration is

$$D_m = \frac{7.7}{1.05} = 7.33$$

.

Now if the yield changes from 5% to 6% then the relative change in the bond price

$$\frac{\Delta P}{P} \approx \frac{dP}{P} = -D_m dy = -7.33 \times 1\% = -7.33\%$$

If the yield changes from 5% to 4%, then

$$\frac{\Delta P}{P} \approx \frac{dP}{P} = -D_m dy = -7.33 \times (-1\%) = +7.33\%$$

Example 15

Consider a bond with coupon rate 7%, par value is \$ 1000, maturity $T = 10$ years, and yield to maturity 5%. Then the duration is calculated as in the following table

(1) Time of payment t	(2) Cash flow in current value	(3) Cash flows in present value (i 5%)	(4) Share of cash flows in present value in bond's price	(5) Weighted time of payment (col. 1 X col. 4)
1	70	66.67	0.0577	0.0577
2	70	63.49	0.0550	0.1100
3	70	60.47	0.0524	0.1571
4	70	57.59	0.0499	0.1995
5	70	54.85	0.0475	0.2375
6	70	52.24	0.0452	0.2715
7	70	49.75	0.0431	0.3016
8	70	47.38	0.0410	0.3283
9	70	45.12	0.0391	0.3517
10	1070	656.89	0.5690	5.6901
Total	1700	1154.44	1	7.705 = duration

II) Convexity

We see that

- For small yield changes, then pricing by using Duration is accurate.
- However, for large yield changes pricing by using Duration is not accurate.
- Because bond price is not a linear function of the yield. For large yield changes, the effect of curvature (i.e., non-linearity) becomes important.

By using higher-order Taylor expansion we have the approximation

$$\Delta P = \frac{dP}{dy}(\Delta y) + \frac{1}{2}\frac{d^2P}{dy^2}(\Delta y)^2 + \cdots$$

Hence,

$$\frac{\Delta P}{P} \approx -D_m \times \Delta y + CX \times (\Delta y)^2$$

where,

$$CX := \frac{1}{2}\frac{1}{P}\frac{d^2P}{dy^2} = \frac{1}{P(1+y)^2}\sum_{t=1}^{T} t(t+1)c_t(1+y)^{-t}$$

Example 16

Consider a bond with 15-year zero, par value is \$ 100 and yield is 8%. Assume that we have small change in y: $\Delta y = -0.01\%$

a) Calculate directly the relative change in bond price
b) Using the Duration to approximate the relative change in bond price
c) Using the Complexity to approximate the relative change in bond price