Basic probability elements

November 13, 2020

- Sample space $\Omega = \{w_1, w_2, \ldots, w_n\}$
- Probability (measure)

$$\begin{aligned} P : &\Omega \longrightarrow [0,1] \\ &w \mapsto P(w) \end{aligned}$$

so that

$$\sum_{k=1}^{n} P(w_k) = 1$$

- For every even A

$$P(A) = \sum_{w \in A} P(w)$$

Infinite and Uncountable Sample Space

- There are infinitely many events
- Some of events may have such complicated description
- Hopeless to try to give a formula for every subset for uncountable sample space
- give a formula for the probability of certain sample events and then appeal to the properties of probability measure to determine the probability of more complicated events

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Let $\mathcal{F}$ be a σ algebra on Ω:

- $\mathcal{F}$ is a collection of subsets of Ω
- $\Omega \in \mathcal{F}$
- $\mathcal{F}$ is closed under complement

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$$

- $\mathcal{F}$ is closed under countable union and intersection

$$A_i \in \mathcal{F},\ \forall i = 1, 2 \cdots \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F} \text{ and } \cap_{i=1}^{\infty} A_i \in \mathcal{F}$$

Toss a coin 3 times.

1. $\mathcal{F}_0 = \{\emptyset, \Omega\}$: trial $\sigma - algebra$ - contains **no information**. Knowing whether the outcome w of the three tosses is in $\emptyset$ and whether it is in Ω tells you nothing about w.

2

$$
\begin{aligned}
\mathcal{F}_1 &= \{0, \Omega, \{HHH, \{HHT, HTH, HTT\}, \{THH, THT, TTH, TTT\}\} \\
&= \{0, \Omega, A_H, H_T\}
\end{aligned}
$$

where

$$A_H = \{HHH, HHT, HTH, HTT\} = \{\text{ H on first toss }\}$$

$$A_T = \{THH, THT, TTH, TTT\} = \{\text{ H on first toss }\}$$

$\mathcal{F}_1$**: information of the first coin or ”information up to time 1”**. For example, you are told that the first coin is H and no more.

3

$$\mathcal{F}_2 = \{\emptyset, \Omega, \{HHH, HHT\}, \{HTH, HTT\}, \{THH, THT\}, \{TTH, TTT\}$$

and all sets which can be built by taking unions of these }

$$= \{\emptyset, \Omega, A_{HH}, A_{HT}, A_{TH}, A_{TT}\}$$

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where

$$A_{HH} = \{HHH, HHT\} = \{\text{HH on the first two tosses}\}$$

$$A_{HT} = \{HTH, HTT\} = \{\text{HT on the first two tosses}\}$$

$$A_{TH} = \{THH, THT\} = \{\text{TH on the first two tosses}\}$$

$$A_{TT} = \{TTH, TTT\} = \{\text{TT on the first two tosses}\}$$

$\mathcal{F}_2$: information of the first two tosses or ”information up to time 2”

4. $\mathcal{F}_3 = \mathcal{G}$ set of all subsets of Ω: **"full information"** about the outcome of all three tosses

Defintion

The σ-algebra generated by A, denoted $\sigma(A)$, is the collection of possible events from the experiment at hand (i.e. all element in A and compliment, countable union, intersection of elements in A)

Example

Come back to the experiment of tossing a coin three times

1. $\mathcal{F}_1 = \sigma(A_T, A_H)$
2. $\mathcal{F}_2 = \sigma(A_{HH}, A_{HT}, A_{TH}, A_{TT})$
3. Which set generates $\mathcal{F}_3$?

Consider a random experiment of picking a random real number $\mathbb{R}$.

- $\Omega = \mathbb{R}$.
- Open set (a, b)
- Borel σ - algebra $\sigma(\mathbb{R}) = \sigma($ all open sets in $\mathbb{R})$

If we can define a σ - algebra $\mathcal{F}$ on Ω then $(\Omega, \mathcal{F})$ is called a measurable space.
Usually $\mathcal{F}$ is the Borel σ - algebra:

- If Ω is finite or countable, $\mathcal{F} = 2^{\Omega}$: set of all subset of Ω
- If Ω is uncountable (interval of real numbers), $\mathcal{F}$ is generated by open sets (open intervals).

Assume for each $E \in \mathcal{F}$ can define a number $P(E)$ satisfying 3 Axioms

- $0 \leq P(E) \leq 1$
- $P(\Omega) = 1$
- If $E_1, E_2, \ldots$ mutually exclusive (i.e $E_i E_j = \emptyset$ for all $i \neq j$) then

$$P\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} P(E_i)$$

Then we call P the probability measure on $(\Omega, \mathcal{F})$, and $P(E)$ the probability of E.

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Example - Uniform (Lebesgue) measure on $[0, 1]$

Construct a model for choosing a number at random from $[0, 1]$

- $$P([a, b]) = b - a, 0 \leq a \leq b \leq 1$$
- If $a = b$ then $[a, b] = \{a\}$ and
$$P(a) = 0$$
- $$P((a, b)) = b - a$$
- This probability defined on $\mathcal{B}([0, 1])$ - σ - algebra generated by closed interval on $[0, 1]$

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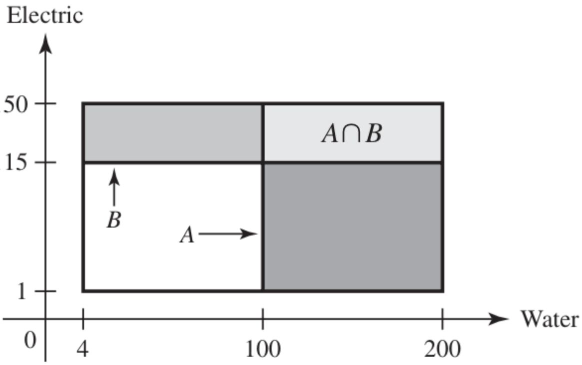
Example

- the demand for electricity will range somewhere between 1 million and 150 million kilowatt-hours per day and water demand will be between 4 and 200 (in thousands of gallons per day)
- Sample space $\Omega = \{(x, y) \in [4, 200] \times [1, 150]\}$
- A: water demand is at least 100
- B: electric demand is at least 115

One simple choice is to make the probability of an event E proportional to the area of E



$$P(A) = \frac{Area(A)}{Area(\Omega)} = \frac{(200-100)(150-1)}{(200-4)(150-1)} = .5102$$

- $0 \leq P(E) \leq 1$
  If $P(E) = 1$, we say that A occurs almost surely
- $P(E^c) = 1 - P(E)$
- If $E \subset F$ then $P(E) \leq P(F)$

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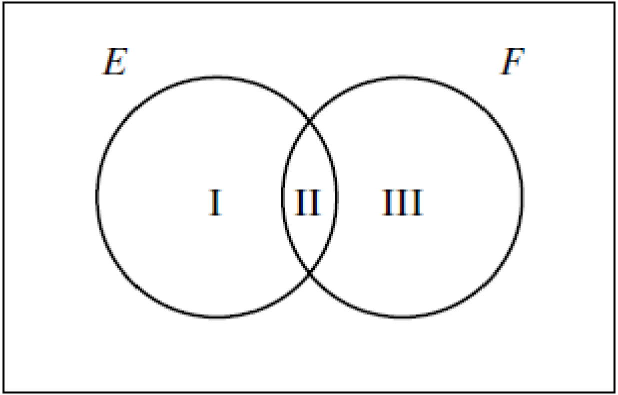
$$P(E \cup F) = P(E) + P(F) - P(EF)$$



A patient arrives at a doctor's office with a sore throat and low- grade fever. After an exam, the doctor decides that the patient has either a bacterial infection or a viral infection or both. The doctor decides that there is a probability of .7 that the patient has a bacterial infection and a probability of .4 that the person has a viral infection. What is the probability that the patient has both infections?

Example

Jane is taking two books along on her holiday vacation. With probability .5, she will like the first book; with probability .4, she will like the second book; and with probability .3, she will like both books. What is the probability that she likes neither book?

Example

What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \cdots$$
$$+ (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r})$$
$$+ \cdots + (-1)^{n+1} P(E_1 E_2 \cdots E_n)$$