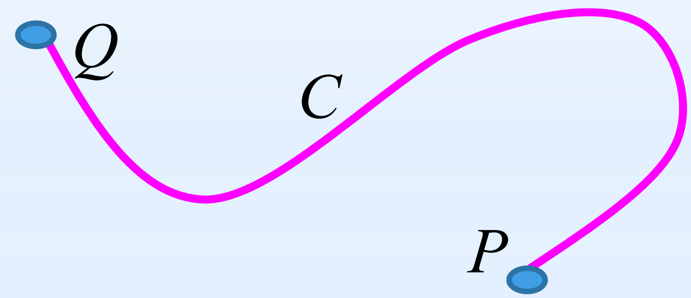
Chapter 4: Vector Calculus

Lecture 13

- **Line Integral of Vector Fields**
- **The Fundamental Theorem for Line Integrals**
- **Green’s Theorem**
- **Curl and Divergence**

1. Line Integrals of Vector Fields

? Work done in moving an object from *P* to *Q* by a force field F along a curve C



Along a line: Work done = Force × Distance

$$W = |\overrightarrow{PS}|.PQ$$

$$W = |\mathbf{F}|\cos\theta.PQ$$

$$\boxed{W = \mathbf{F}\cdot\overrightarrow{PQ}}$$

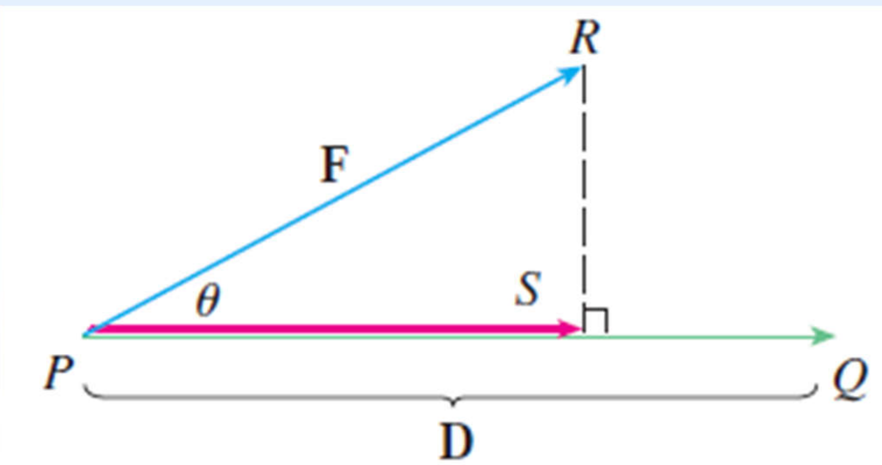
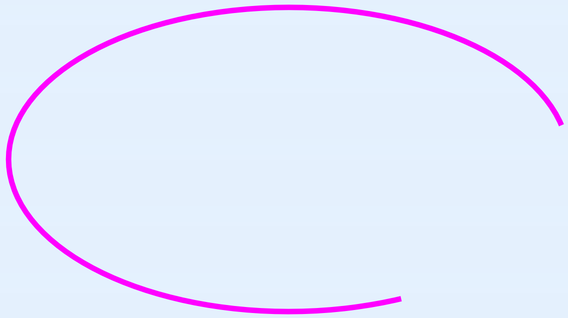
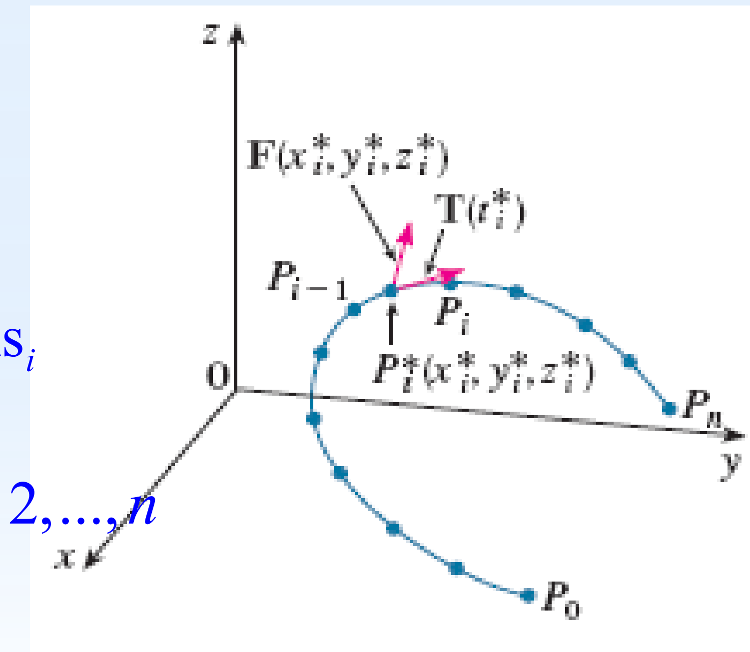


FIGURE 6

Work done by a Force Field

- **Problem:** Find the work done in moving an object from P to Q along a curve C by a Force Field F =<P,Q,R>





Divide C into subarcs $\overset{\frown}{P_{i-1}P_i}$ with lengths Δs_i

Choose any $P_i^*(x_i^*, y_i^*, z_i^*) \in \overset{\frown}{P_{i-1}P_i}$, $i = 1, 2, \ldots, n$

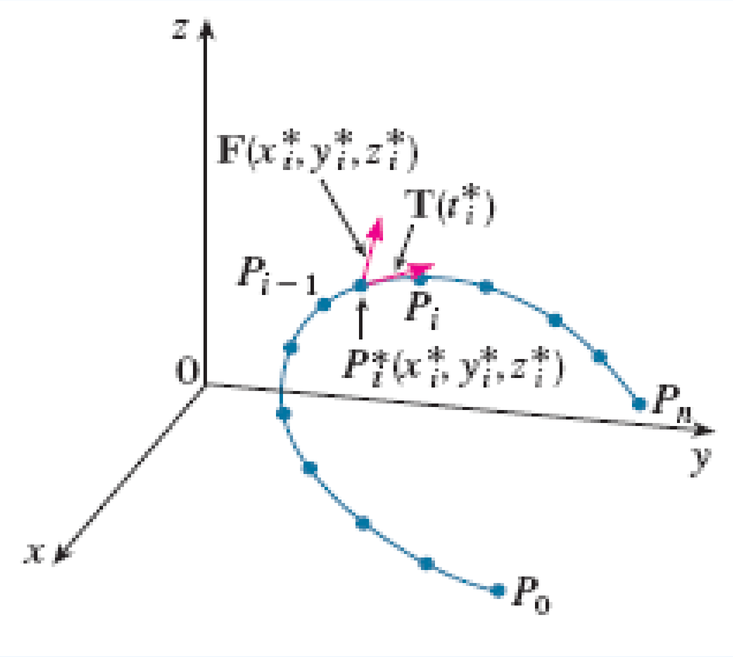
where $(x_i^*, y_i^*, z_i^*) = (x(t_i^*),\ y(t_i^*), z(t_i^*))$

If Δs_i is small, then $\overset{\frown}{P_{i-1}P_i} \approx \Delta s_i T(t_i^*)$,

for unit tangent vector $T(t_i^*)$ at P_i^*

Work done by force F in moving particle from P_{i-1} to $P_i \approx$

$$F(x_i^*, y_i^*, z_i^*).[\Delta s_i T(t_i^*)] = [F(x_i^*, y_i^*, z_i^*).T(t_i^*)]\Delta s_i$$



- The total work done in moving the particle along C is approximately

$$\sum_{i=1}^{n} [F(x_i^*, y_i^*, z_i^*).T(x_i^*, y_i^*, z_i^*)]\Delta s_i$$

- where T(x,y,z) is the unit tangent vector at the point (x,y,z) on C

Work done by Force F:

$$\boxed{W = \int_C F(x,y,z).T(x,y,z)ds = \int_C F.Tds}$$

Evaluation of work done by force F in moving a particle along a curve C

C is given by: $\mathbf{r}(t) = < x(t), y(t), z(t) >, \quad a \le t \le b$

Unit tangent vector along C: $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

$$\mathrm{W} = \int_C (\mathbf{F}.\mathbf{T})\,ds = \int_a^b \left(\mathbf{F}\,(\mathbf{r}(\mathrm{t})).\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right) |\mathbf{r}'(t)|\,dt$$

$$= \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big).\mathbf{r}'(t)\,dt \quad := \int_C \mathbf{F}.\,\mathrm{d}\mathbf{r}$$

Definition of line Integral of Vector Field

- Let F be a continuous vector field defined on a smooth curve C given by a vector function

$$\mathbf{r}(t) = < x(t), y(t), z(t) >, \quad a \le t \le b$$

- **Definition:** **The line integral of F along C is**

$$\int_C \mathbf{F}.\,\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)).\,\mathbf{r}'(t)dt = \int_C (\mathbf{F}.\,\mathbf{T})ds$$

where $T(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ is unit tangent vector along C

Example

Calculate $\int_C \mathbf{F}.\,\mathrm{d}\mathbf{r}$

$$F(x,y) = < xy, 3y^2 >$$

$$\mathbf{r}(t) = < 11t^4, t^3 >, \quad 0 \le t \le 1$$

$$\mathbf{r}'(t) = < 44t^3, 3t^2 >$$

$$F(\mathbf{r}(t)) = < (11t^4)t^3, 3(t^3)^2 > = < 11t^7, 3t^6 >$$

$$F(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = < 11t^7, 3t^6 > \bullet < 44t^3, 3t^2 >$$

$$= (2\times 11)^2\, \mathrm{t}^{10} + 9t^8$$

$$\int_C F \bullet d\mathbf{r} = \int_0^1 F(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt = \int_0^1 ((2\times 11)^2\, \mathrm{t}^{10} + 9t^8)dt$$

$$= 4\times 11\,\mathrm{t}^{11} + t^9 \Big|_0^1 = 44 + 1 - 0 = 45$$

Remark

Connection between line integrals of vector fields and line integrals of scalar fields: Let F=<P,Q,R>. Then

$$\int_C \mathbf{F}.\,\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big).\,\mathbf{r}'(t)dt$$

$$= \int_a^b \langle P, Q, R\rangle.\,\langle x'(t), y'(t), z'(t)\rangle dt$$

$$= \int_a^b (Px'(t) + Qy'(t) + Rz'(t))dt$$

$$= \int_C Pdx + Qdy + Rdz$$

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.

SOLUTION Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\,\mathbf{i} - \cos t \sin t\,\mathbf{j}$$

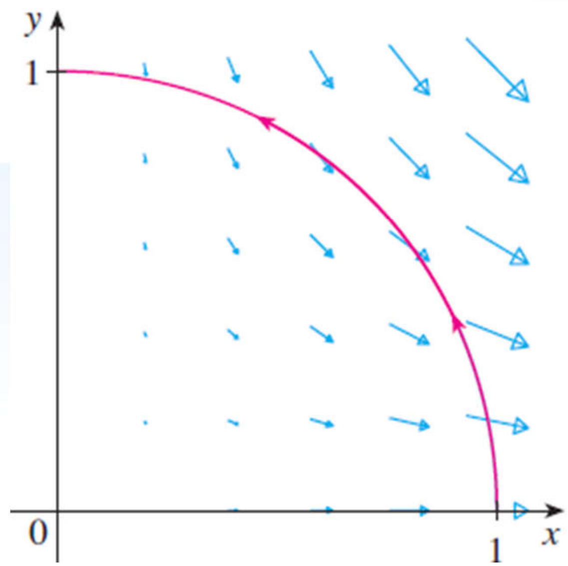
and

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Therefore the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{\pi/2} (-2\cos^2 t \sin t)\,dt$$

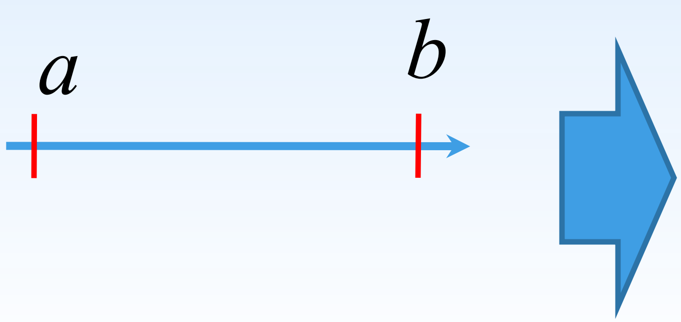
$$= 2\,\frac{\cos^3 t}{3}\bigg]_0^{\pi/2} = -\frac{2}{3}$$

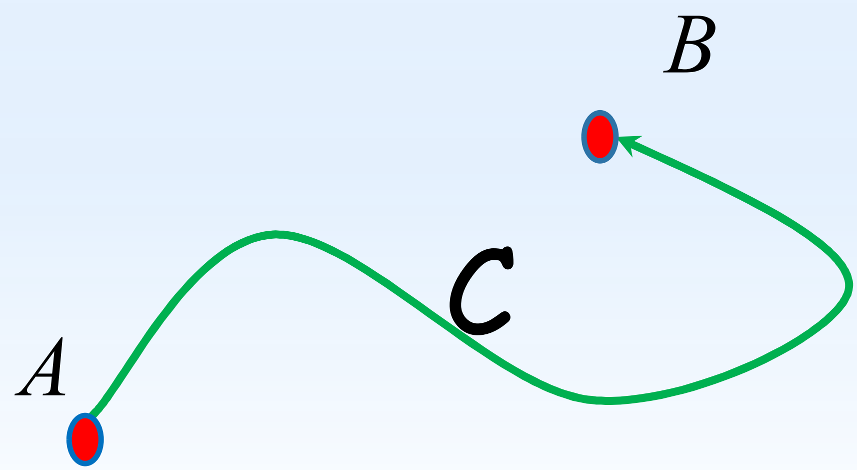


2. Fundamental Theorem for Line Integrals:

Recall that the Fundamental Theorem of Calculus can be written as

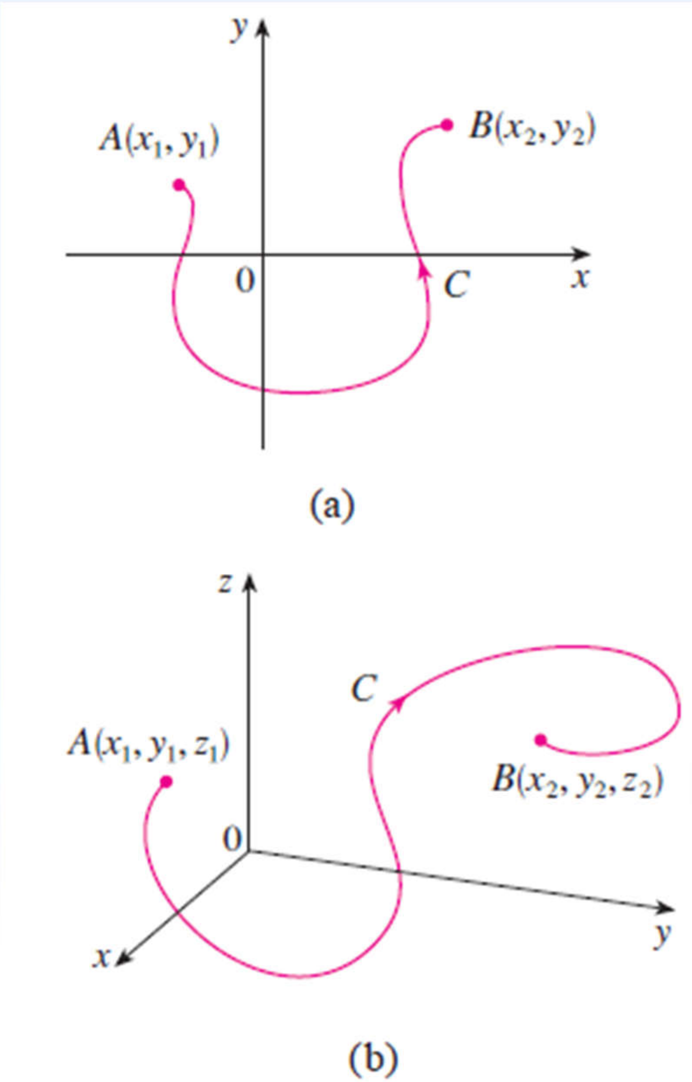
$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)}$$





$$\boxed{\int_C \nabla f \bullet d\mathbf{r} = f(B) - f(A)}$$

Fundamental Theorem for Line Integrals



Theorem : For smooth curve C :

$$\mathbf{r} = \mathbf{r}(t) =< x(t), y(t), z(t) >, \quad a \le t \le b,$$

$$\int_C \nabla f \bullet d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\int_C \nabla f \bullet d\mathbf{r} = f(B) - f(A)$$

$$A = \mathbf{r}(a), B = \mathbf{r}(b)$$

If $\mathbf{F} = \nabla f$: $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{r} = f(B) - f(A)$

Proof

$$\int_C \nabla f \bullet d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt$$

$$= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\,dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Example

- Evaluate line integral of vector field

$$\int_C \mathbf{F} \bullet d\mathbf{r}$$

$$\mathbf{F}(x, y) = < 3 + 2xy, x^2 - 3y^2 >$$

$$C : x = e^t \sin t, y = e^t \cos t, 0 \leq t \leq \pi$$

1st way

$$\mathbf{F}(x,y) =< 3+2xy, x^2-3y^2 >$$

$$C: x = e^t \sin t, y = e^t \cos t, 0 \le t \le \pi$$

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle 3+2e^t \sin t e^t \cos t, (e^t \sin t)^2 - 3(e^t \cos t)^2 \right\rangle$$

$$\mathbf{r}'(t) = \left\langle e^t \cos t + e^t \sin t, -e^t \sin t + e^t \cos t \right\rangle$$

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^\pi \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt \quad \Rightarrow \quad \textit{very}\ \text{complicated!}$$

2nd way

We need to find a function f such that

$$\nabla f(x,y) = \mathbf{F}(x,y) \Leftrightarrow < f_x(x,y), f_y(x,y) > = < 3 + 2xy, x^2 - 3y^2 >$$

$$f_x(x,y) = 3 + 2xy, \quad \text{and } f_y(x,y) = x^2 - 3y^2 \quad (1)$$

$$f(x,y) = \int (3 + 2xy)dx = 3x + x^2 y + g(y) \quad (2)$$

$$f_y(x,y) = x^2 + g'(y) \quad (3)$$

Comparing (1) and (3) gives us: $g'(y) = -3y^2$

$$g(y) = -y^3 + C, \quad C = \text{const.} \quad (4)$$

$$f(x,y) = 3x + x^2 y + g(y) = 3x + x^2 y - y^3 + C$$

Solution...

$$\mathbf{F}(x,y) =< 3+2xy, x^2-3y^2 >$$

$$C: x=e^t \sin t, y=e^t \cos t, 0 \le t \le \pi$$

$$A=(x(0),y(0))=(0,1),\ B=(x(\pi),y(\pi))=(0,-e^{-\pi})$$

Apply Fundamental Theorem for Line Integral:

$$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_C \nabla f \bullet d\mathbf{r} = f(B)-f(A)$$

$$= f(0,-e^{-\pi})-f(0,1) = -e^{-3\pi}+1$$

$$f(x,y)=3x+x^2y-y^3+C$$

Technique for finding potential function in R^3 is the same

V **EXAMPLE 5** If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

SOLUTION If there is such a function f, then

11 $$f_x(x, y, z) = y^2$$

12 $$f_y(x, y, z) = 2xy + e^{3z}$$

13 $$f_z(x, y, z) = 3ye^{3z}$$

Integrating $\boxed{11}$ with respect to x, we get

14 $$f(x, y, z) = xy^2 + g(y, z)$$

where $g(y, z)$ is a constant with respect to x. Then differentiating $\boxed{14}$ with respect to y, we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

V EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

and comparison with $\boxed{12}$ gives

$$g_y(y, z) = e^{3z}$$

Thus $g(y, z) = ye^{3z} + h(z)$ and we rewrite $\boxed{14}$ as

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to z and comparing with $\boxed{13}$, we obtain $h'(z) = 0$ therefore $h(z) = K$, a constant. The desired function is

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that $\nabla f = \mathbf{F}$.

Independence of Paths

The line integral $\int_C \mathbf{F}\bullet d\mathbf{r}$ is independent of path if $\int_{C_1} \mathbf{F}\bullet d\mathbf{r} = \int_{C_2} \mathbf{F}\bullet d\mathbf{r}$ for any two paths C_1 and C_2 that have the same initial and terminal points

Region D is **connected**: any two points in D can be joined by a path that lies in D

Closed curve: terminal point coincides with initial point

Theorem: $\int_C \mathbf{F}\bullet d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F}\bullet d\mathbf{r} = 0$ for every closed path C in D

Conservative Field in Plane

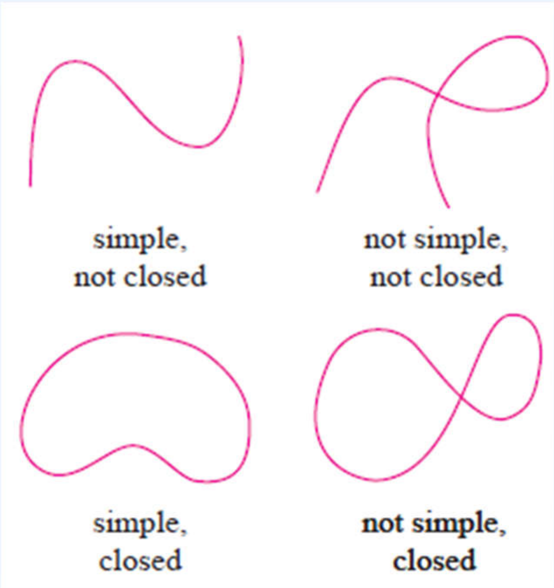
Theorem 1. $\mathbf{F}:$ continuous vector field on open connected region D

If $\int_C \mathbf{F}.d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D

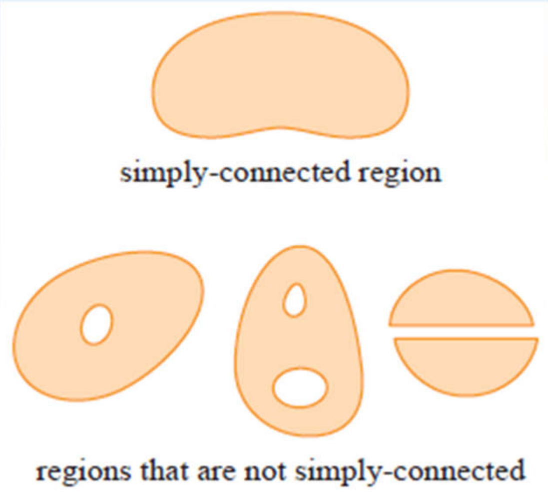
Theorem 2. If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a conservative vector field

P and Q have continuous first-order partial derivatives on D, then $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

Simply-connected regions



A **simple** curve is a curve that doesn't intersect itself anywhere between its endpoints



A **simply-connected region** in the plane is a connected region such that every simple closed curve in D encloses only points that are in D

Theorem : $\mathbf{F} = \langle P, Q \rangle$ on open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ throughout D. Then **F** is conservative

V EXAMPLE 3 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

is conservative.

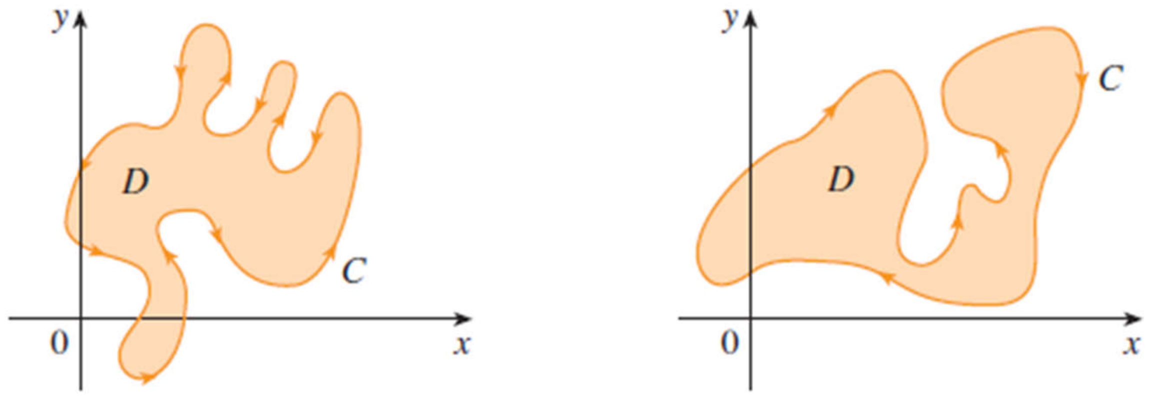
SOLUTION Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

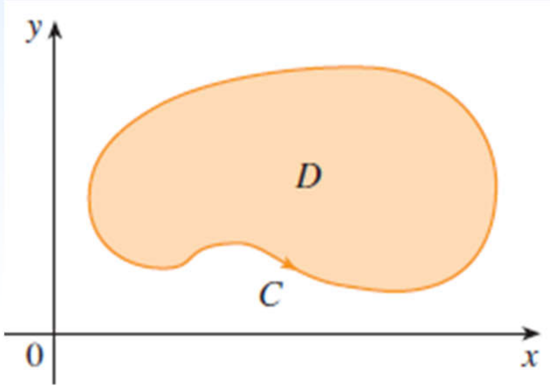
Also, the domain of **F** is the entire plane ($D = \mathbb{R}^2$), which is open and simply-connected. Therefore we can apply Theorem 6 and conclude that **F** is conservative.

3. Green's Theorem

Positive orientation of a simple closed curve C refers to a single *counterclockwise* traversal of C.



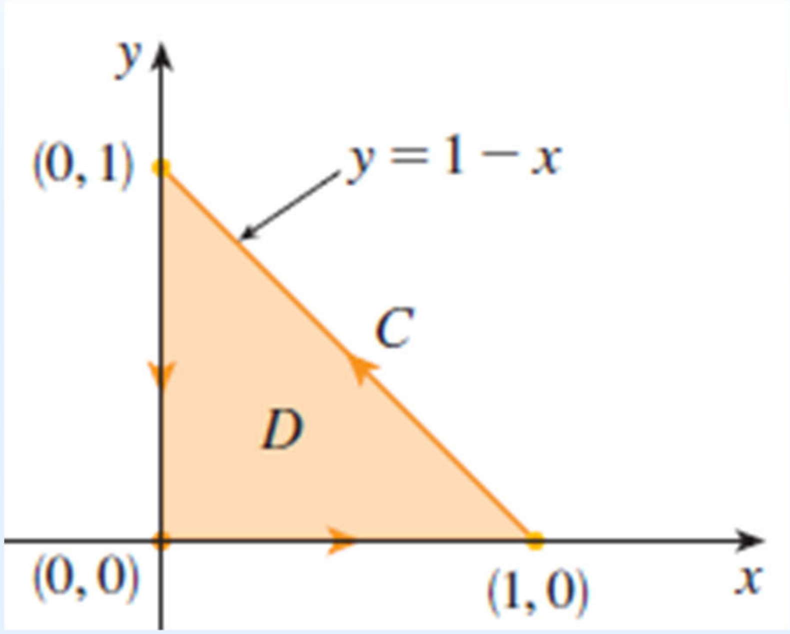
(a) Positive orientation (b) Negative orientation



C: positively oriented, piecewise-smooth, simple Closed curve in the plane and let D be the region bounded by C. Then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Example: $\int_C x^4 dx + xy dy$, where C is the triangular curve connecting the points (0,0), (0,1), and (0,1)



Solution: $P = x^4, Q = xy$

$$\int_C x^4 dx + xy dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_0^{1-x} (y-0) dy dx$$

$$= \int_0^1 \left[\frac{1}{2} y^2 \right]_{y=0}^{y=1-x} \mathrm{dx} = \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{-1}{6} (1-\mathrm{x})^3 \Big|_0^1 = \frac{1}{6}$$

4. Curl and Divergence

Curl

$\mathbf{F} =< P, Q, R >$: vector field

$$\text{curl } \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Use determinant notations:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times \mathbf{F}$$

Example: $\mathbf{F} = < xz, xyz, -y^2 >$. Find curl $\mathbf{F}$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left\langle -2y - xy, -(0 - x), (yz - 0) \right\rangle = \left\langle -y(2 + x), x, yz \right\rangle$$

THEOREM:

a) $\operatorname{curl}(\nabla f) = 0$

b) If $\mathbf{F}$ is defined on $\mathbb{R}^3$, and $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F}$ is conservative

Example: Show that $\mathbf{F} = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle$ is conservative

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= \left\langle 6xyz^2 - 6xyz^2, -(3y^2z^2 - 3y^2z^2), 2yz^3 - 2yz^3 \right\rangle = 0$$

Domain F is R^3, so $\mathbf{F}$ is a conservative vector field

Divergence

Divergence of vector field $\mathbf{F} =< P, Q, R >$ is defined by

$$\text{div}\ \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \bullet \mathbf{F}$$

Theorem: div (curl $\mathbf{F}$) = 0

Example: $\mathbf{F} =< xz, xyz, -y^2 >$

$$\text{div}\ \mathbf{F} = z + xz + 0 = z + zx$$