Analysis 1 - Course Review

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Contents

1. Preliminaries

We first review some basic concepts, formally.

Definition 1.1 (Sets)

A set is a collection of objects, usually satisfying some common properties. These objects are referred to as members (or elements) of the set.

We write $x \in A$ if x is a member of A and $x \notin A$ otherwise.

Example 1.1

Some familiar sets:

(a) The emptyset $\emptyset$, which has no element;

(b) The singleton $\{x\}$, which has exactly one element;

(c) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or any interval in $\mathbb{R}$.

A set with at least one element is called nonempty.

Definition 1.2 (Set Operations)

Let A and B be any two sets.

(a) We say that A is a subset of B, in symbol $A \subset B$ or $B \supset A$, if each member of A is also a member of B. Otherwise, we write $A \not\subset B$;

(b) We say that A equals B, i.e. $A = B$, if $A \subset B$ and $B \subset A$;

(c) We define the union, intersection and difference of A and B as

$$\begin{aligned} A \cup B &= \{x : x \in A \text{ or } x \in B\}, \\ A \cap B &= \{x : x \in A \text{ and } x \in B\}, \\ A \setminus B &= \{x : x \in A \text{ and } x \notin B\}. \end{aligned}$$

If $A \subset B$, we sometimes refer to $B \setminus A$ as the complement of A, i.e. A^c.

1. Preliminaries

Definition 1.3 (Functions & Sequences)

Let X and Y be nonempty sets.

(a) A mapping $f : X \to Y$ is is a rule assigning to each member x in X a unique member $f(x)$ in Y;

(b) If $Y = \mathbb{R}$, then f is called a function;

(c) If $X = \mathbb{N}$, then f is called a sequence.

- X and Y are called the domain and range of f;
- For a function f, we usually concern on cases where X is an interval;
- A sequence has a representation $\{x_n\}_{n=1}^{\infty}$, where $x_n = f(n)$.

Exercise. Try to provide examples of functions & sequences by yourself.

2. Supremum & Infimum

Definition 2.1

Let $X \subset \mathbb{R}$ be nonempty and $\alpha \in \mathbb{R}$ be arbitrary.

(a) α is called a lower bound of X if all members of X are not smaller than α. X is called bounded below if it has a lower bound;

(b) α is called an upper bound of X if all members of X are not greater than α. X is called bounded above if it has an upper bound;

(c) X is called bounded if it is bounded above and bounded below.

Example 2.1

(a) 2 and 3 are lower bounds of $[4, \infty)$, so $[4, \infty)$ is bounded below;

(b) 5 is an upper bound of $(0, 1)$, so $(0, 1)$ is bounded above;

(c) $[-3, 3)$ is bounded but $(7, \infty)$ and $\mathbb{N}$ are not bounded.

2. Supremum & Infimum

Definition 2.2 (Supremum & Infimum)

Let $X \subset \mathbb{R}$ be nonempty and $\alpha \in \mathbb{R}$ be arbitrary.

(a) If α is an upper bound of X, then we call α the supremum of X, in symbol $\alpha = \sup X$ if it is the smallest upper bound;

(b) If α is a lower bound of X, then we call α the infimum of X, in symbol $\alpha = \inf X$ if it is the largest lower bound.

A bounded below (above) set must have an infimum (a supremum).

Example 2.2

$$\inf[0,1] = \sup(-\infty, 0) = \inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \sup\left\{-|z| : z \in \mathbb{Z}\right\} = 0.$$

Question. How to determine supremum/infimum of an arbitrary set?

Theorem 2.1

Let $X \subset \mathbb{R}$ be nonempty and $\alpha \in \mathbb{R}$ be arbitrary.

(a) If α is an upper bound of X, then α is the supremum of X if
$$\forall \epsilon > 0, \exists x \in X : x > \alpha - \epsilon.$$

(b) If α is a lower bound of X, then α is the infimum of X if
$$\forall \epsilon > 0, \exists x \in X : x < \alpha + \epsilon.$$

In other words,

- the supremum subtracting ϵ is no longer an upper bound;
- the infimum adding ϵ is no longer a lower bound.

Example 2.3

Any $\epsilon > 0$ is not a lower bound of $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.

3. Convergent of Sequences

Definition 3.1

Let $\{x_n\}$ be a sequence in $\mathbb{R}$. We say that $\{x_n\}$ is

(a) increasing (decreasing) if $x_n \leq x_{n+1}$ ($x_n \geq x_{n+1}$) holds for all $n \in \mathbb{N}$;

(b) strictly increasing if $x_n < x_{n+1}, \forall n \in \mathbb{N}$;

(c) strictly decreasing if $x_n > x_{n+1}, \forall n \in \mathbb{N}$;

(d) bounded below (above) if so is the set $\{x_n : n \in \mathbb{N}\}$.

Definition 3.2 (Convergent)

We say that $\{x_n\}$ converges to $\alpha \in \mathbb{R}$, denoted $x_n \to \alpha$ or $\lim\limits_{n\to\infty} x_n = \alpha$, if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : |x_n - \alpha| < \epsilon, \forall n > N.$$

α, if exists, is called the limit of the convergent sequence $\{x_n\}$.

A sequence is called divergent if it is not convergent.

3. Convergent of Sequences

Question. How to verify convergence of an arbitrary sequence?

Theorem 3.1 (Monotone Convergence Theorem)

(a) An increasing, bounded above sequence is convergent;

(b) A decreasing, bounded below sequence is convergent.

Theorem 3.2 (Squeeze Theorem)

Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences in $\mathbb{R}$ and $\alpha \in \mathbb{R}$. If $x_n \leq y_n \leq y_n$ for all $n \in \mathbb{N}$ and both $\{x_n\}, \{z_n\}$ converges to α, then so is $\{y_n\}$.

Example 3.1

Determine convergence and limits (if exist) of the following sequences:

(a) $a_n = (-1)^n/n, \quad b_n = n/2^n, \quad c_n = 0, \quad d_n = \ln(n+1);$

(b) $e_1 = 1$ and $e_{n+1} = n \cdot e_n/(n+1)$.

Theorem 3.3

Assume that $x_n \to \alpha$.

(a) If β is another limit of x_n, then $\beta = \alpha$;

(b) Every subsequence of $\{x_n\}$ also converges to α.

Example. Why the sequence $a_n = (-1)^n$ diverges?

Theorem 3.4

Let $X \subset \mathbb{R}$ be nonempty and $\alpha \in \mathbb{R}$ s.t. $\exists \{x_n\} \subset X : x_n \to \alpha$. Then:

(a) If α is an upper bound of X, then α is the supremum of X;

(b) If α is a lower bound of X, then α is the infimum of X.

Example 3.2

Determine $\inf X$ and $\sup X$, where $X = \mathbb{Q}^c \cap [0, 1]$.

In the remaining sections, assume $f, g : \mathcal{I} \to \mathbb{R}$, $x_0 \in \mathcal{I}$ and $\alpha \in \mathbb{R}$.

Definition 4.1 (Limits & Continuity of Functions)

(a) α is called the left limit of f at x_0, in symbol $\lim\limits_{x \to x_0^-} f(x) = \alpha$, if

$$\forall \epsilon > 0, \exists \delta > 0 : x \in \mathcal{I} \cap (x_0 - \delta, x_0) \text{ implies } |f(x) - f(x_0)| < \epsilon;$$

(b) α is called the right limit of f at x_0, in symbol $\lim\limits_{x \to x_0^+} f(x) = \alpha$, if

$$\forall \epsilon > 0, \exists \delta > 0 : x \in \mathcal{I} \cap (x_0, x_0 + \delta) \text{ implies } |f(x) - f(x_0)| < \epsilon;$$

(c) α is called the limit of f at x_0, in symbol $\lim\limits_{x \to x_0} f(x) = \alpha$, if it is both the left limit and right limit of f at x_0. We say that f is continuous at x_0;

(d) We say that f is continuous on $\mathcal{I}$ if f is continuous at every point in $\mathcal{I}$.

Note that if α is the limit of f at x_0, then $f(x_0) = \alpha$.

Definition 4.2 (Derivative & Differentiability)

We say that α is the derivative of f at x_0, in symbol $f'(x_0) = \alpha$, if

$$\alpha = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We say that f is differentiable if f' exists and is continuous on $\mathcal{I}$.

Note. $f'(x_0)$ is the slope of the tangent line to the graph of f at x_0.

Derivative Rules

(a) $(f(g(x)))' = f'(g(x)) \cdot g'(x)$;

(b) $(u \pm v)' = u' \pm v'$, $\quad (u/v)' = (u'v - v'u)/v^2$;

(c) Derivative of familiar functions...

Theorem 5.1 (L'Hospital Rule)

Assume f and g are differentiable on $\mathcal{I}$ except at x_0, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

provided the latter limit exists and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) \in \{0, \pm\infty\}$.

We say in such cases that the former limit is in **indeterminate form**.

Example 5.1

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

Definition 5.1

We say that x_0 is:

(a) a maximum of f on $\mathcal{I}$ if $f(x_0) = \sup\{f(x) : x \in \mathcal{I}\}$;

(b) a minimum of f on $\mathcal{I}$ if $f(x_0) = \inf\{f(x) : x \in \mathcal{I}\}$;

(c) an extremum of f if it is either a maximum or a minimum;

(d) a stationary point of f if $f'(x_0) = 0$.

Theorem 5.2 (Extreme Value Theorem)

If f is continuous on $\mathcal{I}$, then f has a minimum and a maximum.

Theorem 5.3

If f is differentiable, then every extremum is a stationary point.

Theorem 5.4 (Intermediate Value Theorem)

If $\mathcal{I} = [a, b]$ and f is continuous, then $\exists c, d \in \mathbb{R} : f(\mathcal{I}) = [c, d]$.

Theorem 5.5 (Mean Value Theorem)

If $\mathcal{I} = [a, b]$ and f is differentiable on (a, b), then

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 5.6 (Rolle's Theorem)

If $\mathcal{I} = [a, b]$, f is differentiable on (a, b) and $f(a) = f(b)$, then

$$\exists c \in (a, b) : f'(c) = 0.$$