

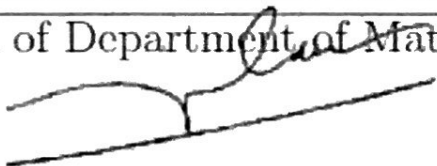
MIDTERM EXAMINATION

April 2022

Duration: 100 minutes

SUBJECT: REAL ANALYSIS

Head of Department of Mathematics:



Prof. Pham Huu Anh Ngoc

Lecturer:



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INSTRUCTIONS: Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.

Question 1 For $x = (x_1, x_2) \in \mathbb{R}^2$, define

$$\varphi(x) = \max \{|x_1|, |x_1 + x_2|\}, \quad \xi(x) = |x_2 - x_1|.$$

(a) (35 marks) Show that φ is a norm on \mathbb{R}^2 . Sketch the closed unit ball $B = \{x \in \mathbb{R}^2 : \varphi(x) \leq 1\}$.

(b) (5 marks) Is ξ a norm on \mathbb{R}^2 ? Explain.

Question 2 (30 marks) Which of the following subsets of \mathbb{R}^2 are compact? Explain.

$$A = (-\infty, 1] \times [0, 1], \quad B = [0, 1] \times [0, 1), \quad C = [0, 1] \times [0, 1].$$

Question 3 (20 marks) Let $X = C([0, 1])$ be equipped with the norm

$$\|x\| = \max_{0 \leq t \leq 1} |x(t)|.$$

Show that the function $\Phi : X \rightarrow \mathbb{R}$ defined by

$$\Phi(x) = x\left(\frac{1}{2}\right) + \int_0^1 |x(t)| dt$$

is Lipschitz continuous.

(Hint: Show that $|\Phi(x) - \Phi(y)| \leq 2\|x - y\|$ for all $x, y \in X$.)

Question 4 (10 marks) Let A be an infinite subset of a metric space (X, d) such that $d(x, y) \geq 1$ when x and y are different points in A . Prove that A is not compact.

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