INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

# FINAL EXAMINATION
**January 2018**
**Duration: 120 minutes**

| **SUBJECT: REAL ANALYSIS** | |
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| Deputy Head of Dept. of Mathematics: | Lecturer: |
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**INSTRUCTIONS:** *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

**Question 1** *(20 marks)* Suppose that $f$ is measurable on $E$ and $a, b \in \mathbb{R}$, $a < b$. Show that the sets

$$A = \{x \in E : a \leq f(x) \leq b\} \quad \text{and} \quad B = \{x \in E : a < f(x) \leq b\}$$

are measurable.

**Question 2** *(20 marks)* Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \sin x & \text{if } x \leq 1 \\ x^2 + e^x & \text{if } x > 1. \end{cases}$$

Show that $g(x) = (\sin x)\chi_{(-\infty,1]}(x) + (x^2 + e^x)\chi_{(1,\infty)}(x)$ and that $g$ is Borel measurable on $\mathbb{R}$.

**Question 3** *(20 marks)* Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a *nonnegative* measurable function that takes countably distinct values $a_1, a_2, \ldots$.

(a) For each $n \in \mathbb{N}$, set $A_n = \{x \in X : f(x) = a_n\}$. Show that $\{A_n\}$ is a sequence of disjoint measurable sets with $\bigcup_{n=1}^{\infty} A_n = X$.

(b) Show that

$$\int_X f d\mu = \sum_{n=1}^{\infty} a_n \mu(A_n).$$

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**Question 4** *(20 marks)* Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose that $f, f_n$ are nonnegative *integrable* measurable functions on $X$ such that

$$\int_X f_n d\mu = \int_X f d\mu \quad \forall n \in \mathbb{N}.$$

(a) Show that for each $n$,

$$\int_X (f - f_n)^+ d\mu = \int_X (f - f_n)^- d\mu$$

and

$$\int_X |f - f_n| d\mu = 2 \int_X (f - f_n)^+ d\mu.$$

(b) If $f_n \to f$ a.e., show that

$$\int_X |f_n - f| d\mu \to 0 \quad \text{as } n \to \infty.$$

(*Hint:* Use the Dominated Convergence Theorem.)

**Question 5** *(20 marks)*

(a) For each Lebesgue measurable set $E$, set

$$\mu(E) = \int_E (x^2 - 1) dx.$$

Find a negative set for the signed measure $\mu$.

(b) Let $(X, \mathcal{A})$ be a measurable space. Show that for every signed measure $\nu$ we have $\nu \ll |\nu|$.

*** END OF QUESTION PAPER ***

**SOLUTIONS**
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**Question 1** We have

$$A = \{x \in E : a \leq f(x)\} \cap \{x \in E : f(x) \leq b\}$$
$$B = \{x \in E : a < f(x)\} \cap \{x \in E : f(x) \leq b\}.$$

Since $f$ is measurable, four sets to the right are measurable. Hence $A$ and $B$ are measurable.

**Question 2** (a) If $x \leq 1$, then

$$(\sin x)\chi_{(-\infty,1]}(x) + (x^2 + e^x)\chi_{(1,\infty)}(x) = \sin x \cdot 1 + (x^2 + e^x) \cdot 0 = \sin x = g(x);$$

if $x > 1$, then

$$(\sin x)\chi_{(-\infty,1]}(x) + (x^2 + e^x)\chi_{(1,\infty)}(x) = \sin x \cdot 0 + (x^2 + e^x) \cdot 1 = x^2 + e^x = g(x).$$

Thus $g(x) = (\sin x)\chi_{(-\infty,1]}(x) + (x^2 + e^x)\chi_{(1,\infty)}(x)$.

(b) The sets $(-\infty, 1]$ and $(1, \infty)$ are Borel measurable, hence their characteristic functions $\chi_{(-\infty,1]}$ and $\chi_{(1,\infty)}$ are Borel measurable. The functions $\sin x$ and $x^2 + e^x$ are continuous on $\mathbb{R}$, so they are Borel measurable. Therefore $(\sin x)\chi_{(-\infty,1]}(x)$ and $(x^2 + e^x)\chi_{(1,\infty)}(x)$ are Borel measurable and so is $g$.

**Question 3** (a) Since $f$ is measurable, each set $A_n = \{x \in X : f(x) = a_n\}$ is measurable. If there were $x \in A_m \cap A_n$ for some $m \neq n$, then we would have $f(x) = a_m = a_n$, a contradiction. Thus $A_m \cap A_n = \emptyset$ for all $m \neq n$.

If $x \in X$ then $f(x) = a_n$ for some $n$, so that $X \subset \bigcup_{n=1}^{\infty} A_n$. The in inverse inclusion is obvious. Thus $X = \bigcup_{n=1}^{\infty} A_n$.

(b) By part (a), $X$ is the disjoint union of $A_n$'s so we can apply $\sigma$-additivity to obtain

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_{A_n} f d\mu = \sum_{n=1}^{\infty} \int_{A_n} a_n d\mu = \sum_{n=1}^{\infty} a_n \mu(A_n).$$

*Alternative solution.* For each $x \in X$, there is $i$ (depending on $x$) such that $f(x) = a_i$. Since $x \in A_i$ and $\{A_n\}$ is a disjoint sequence we have $a_i \chi_{A_i}(x) = a_i$ and $a_n \chi_{A_n}(x) = 0$ for $n \neq i$. Thus $f(x) = \sum_{n=1}^{\infty} a_n \chi_{A_n}(x)$ and hence $f = \sum_{n=1}^{\infty} a_n \chi_{A_n}$. As $a_n \chi_{A_n} \geq 0$ for all $n$,

$$\int_X f d\mu = \int_X \left( \sum_{n=1}^{\infty} a_n \chi_{A_n} \right) d\mu = \sum_{n=1}^{\infty} \int_X a_n \chi_{A_n} d\mu = \sum_{n=1}^{\infty} a_n \mu(A_n).$$

**Question 4** (a) Since $f$, $f_n$ are integrable, they are finite a.e., so $g_n := f - f_n$ is defined a.e. for all $n$. By assumption,

$$\int_X g_n d\mu = \int_X f d\mu - \int_X f_n d\mu = 0 = \int_X g_n^+ d\mu - \int_X g_n^- d\mu.$$

Hence $\int_X g_n^+ d\mu = \int_X g_n^- d\mu$. It follows that

$$\int_X |g_n| d\mu = \int_X g_n^+ d\mu + \int_X g_n^- d\mu = 2 \int_X g_n^+ d\mu.$$

(b) Since $f_n \to f$ a.e., $g_n^+ \to 0$ a.e. Further, as $f_n \geq 0$, $g_n = f - f_n \leq f$, implying $0 \leq g_n^+ \leq f$. By hypothesis $f$ is integrable. Thus, we can apply the DCT to deduce that $\int_X g_n^+ d\mu \to 0$. By part (a), $\int_X |g_n| d\mu = 2 \int_X g_n^+ d\mu \to 0$.

**Question 5** (a) We have $x^2 - 1 \leq 0$ if and only if $-1 \leq x \leq 1$. Let $A$ be a (nonempty) Lebesgue measurable subset of $[-1, 1]$. If $E$ is a Lebesgue measurable subset of $A$, then $f \leq 0$ on $E$, implying $\mu(E) = \int_E (x^2 - 1) dx \leq 0$. Thus $A$ is a negative set for $\mu$.

(b) Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of $\nu$. Assume that $|\nu|(E) = 0$. Since $\nu^+$ and $\nu^-$ are measures and $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$, $\nu^+(E) = \nu^-(E) = 0$. Hence $\nu(E) = \nu^+(E) - \nu^-(E) = 0$. Therefore $\nu \ll \nu$.