Elements of Probability

January 15, 2021

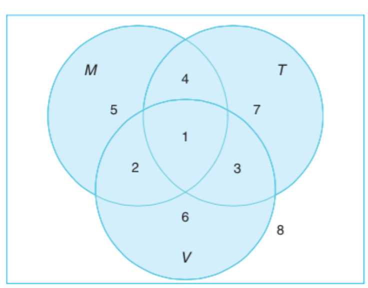
1 Sample space - Events

1. A box contains three marbles — one red, one green, and one blue. Consider an experiment that consists of taking one marble from the box, then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat for the case in which the second marble is drawn without first replacing the first marble.

2. Four students are selected at random from a chemistry class and classified as male or female.
   (a) List the elements of the sample space S_1, using the letter M for male and F for female.
   (b) Define a second sample space S_2 where the elements represent the number of females selected.

3. An experiment involves tossing a pair of dice, one green and one red, and recording the numbers that come up. If x equals the outcome on the green die and y the outcome on the red die
   (a) describe the sample space S
   (b) list the elements corresponding to the event A that the sum is greater than 8;
   (c) list the elements corresponding to the event B that a 2 occurs on either die;
   (d) list the elements corresponding to the event C that a number greater than 4 comes up on the green die;
   (e) list the elements corresponding to the event AC;
   (f) list the elements corresponding to the event AB;
   (g) list the elements corresponding to the event BC;
   (h) construct a Venn diagram to illustrate the intersections and unions of the events A, B, and C.

4. Suppose that a family is leaving on a summer vacation in their camper and that M is the event that they will experience mechanical problems, T is the event that they will receive a ticket for committing a traffic violation, and V is the event that they will arrive at a campsite with no vacancies (Referring to the Venn diagram).



   (a) State in words the events represented by the following regions:
      i. region 5
      ii. region 1 and 2 together

(b) list the numbers of the regions that represent the following events:

   i. The family will experience no mechanical problems and will not receive a ticket for a traffic violation but will arrive at a campsite with no vacancies.

   ii. The family will experience both mechanical problems and trouble in locating a campsite with a vacancy but will not receive a ticket for a traffic violation.

5. The rise time of a reactor is measured in minutes (and fractions of minutes). Let the sample space be positive, real numbers. Define the events A and B as follows: $A = \{x|x < 72.5\}$ and $B = \{x|x > 52.5\}$. Describe the following event

$$a.\, A^c \qquad b.\, B^c \qquad c.\, AB \qquad d.\, A \cup B$$

6. A six-sided die is rolled repeatedly, until the first time (if ever) that an even number is obtained. What is the sample space for this experiment?

7. A, B, and C take turns flipping a coin. The first one to get a head wins. The sample space of this experiment can be defined by

$$S = \{1, 01, 001, 0001, \dots\}$$

   (a) Interpret the sample space.

   (b) Define the following events in terms of S:

      i. A wins = A.

      ii. B wins = B.

      iii. $(A \cup B)^c$.

   Assume that A flips first, then B, then C, then A, and so on

2 Axiom of Probability

1. A decision maker subjectively assigned the following probabilities to the four outcomes of an experiment: $P(E_1) = .10$, $P(E_2) = .15$, $P(E_3) = .40$, and $P(E_4) = .20$. Are these probability assignments valid? Explain.

2. A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.

3. A box contains 500 envelopes, of which 75 contain \$100 in cash, 150 contain \$25, and 275 contain \$10. An envelope may be purchased for \$25. What is the sample space for the different amounts of money? Assign probabilities to the sample points and then find the probability that the first envelope purchased contains less than \$100.

4. If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary, what is the probability that

   (a) the dictionary is selected?

   (b) 2 novels and 1 book of poems are selected?

5. The National Highway Traffic Safety Administration (NHTSA) conducted a survey to learn about how drivers throughout the United States are using seat belts (Associated Press, August 25, 2003). Sample data consistent

with the NHTSA survey are as follows.

	Driver Using Seat Belt?	
Region	**Yes**	**No**
Northeast	148	52
Midwest	162	54
South	296	74
West	252	48
Total	858	228

(a) For the United States, what is the probability that a driver is using a seat belt?

(b) The seat belt usage probability for a U.S. driver a year earlier was .75. NHTSA chief Dr. Jeffrey Runge had hoped for a .78 probability in 2003. Would he have been pleased with the 2003 survey results?

(c) What is the probability of seat belt usage by region of the country? What region has the highest seat belt usage?

(d) What proportion of the drivers in the sample came from each region of the country? What region had the most drivers selected? What region had the second most drivers selected?

3 Additive Rules

1. If $P(A) = 0.3$, $P(B) = 0.2$ and $P(AB) = 0.1$, determine the following probabilities

$$a., P(A^c) \qquad b. P(AB^c) \qquad c. P(A^cB) \qquad d. P[(A \cup B)^c] \qquad e. P(A \cup B^c).$$

2. Suppose that A and B are mutually exclusive events for which $P(A) = .3$ and $P(B) = .5$. What is the probability that

   (a) either A or B or both occurs?

   (b) A occurs but B does not?

   (c) both A and B occur?

3. In a certain population, 10% of the people are rich, 5% are famous, and 3% are rich and famous. For a person picked at random from this population:

   (a) What is the chance that the person is not rich?

   (b) What is the chance that the person is rich but not famous?

   (c) What is the chance that the person is either rich or famous?

4. Not all of the following statements are true. Identify and prove those that are true, and devise examples to demonstrate that the others are false

   (a) If $A \subset B$ then $P(A) \leq P(B)$

   (b) For any events A and B, $P(A \cup B) \leq P(A) + P(B)$

(c) For any events A and B, $P(A \cap B) \leq P(A)P(B)$

(d) For any events A and B, probability that exactly one event occurs is $P(A) + P(B) - 2P(AB)$

5. Data on the 30 largest stock and balanced funds provided one-year and five-year percent- age returns for the period ending March 31, 2000 (The Wall Street Journal, April 10, 2000). Suppose we consider a one-year return in excess of 50% to be high and a five-year return in excess of 300% to be high. Nine of the funds had one-year returns in excess of 50%, seven of the funds had five-year returns in excess of 300%, and five of the funds had both one-year returns in excess of 50% and five-year returns in excess of 300%.

   (a) What is the probability of a high one-year return, and what is the probability of a high five-year return?

   (b) What is the probability of both a high one-year return and a high five-year return?

   (c) What is the probability of neither a high one-year return nor a high five-year return?

6. Interest centers around the nature of an oven purchased at a particular department store. It can be either a gas or an electric oven. Consider the decisions made by six distinct customers.

   (a) Suppose that the probability is 0.40 that at most two of these individuals purchase an electric oven. What is the probability that at least three purchase the electric oven?

   (b) Suppose it is known that the probability that all six purchase the electric oven is 0.007 while 0.104 is the probability that all six purchase the gas oven. What is the probability that at least one of each type is purchased?

7. **Bonferroni's inequality.**

   (a) Prove that for any two events A and B, we have

   $$P(AB) \geq P(A) + P(B) - 1.$$

   (b) Generalize to the case of n events $A_1, A_2, ..., A_n$, by showing that

   $$P(A_1 A_2 \ldots A_n) \geq P(A_1) + P(A_2) + ... + P(A_n) - (n-1)$$

4 Conditional probability - Multiplication rule

1. Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized below:

		Shock resistance	
		high	*low*
scratch	*high*	70	9
resistance	*low*	16	5

   (a) If a disk is selected at random, what is the probability that its scratch resistance is high and its shock resistance is high?

   (b) If a disk is selected at random, what is the probability that its scratch resistance is high given that its shock resistance is high?

2. A total of 600 of the 1,000 people in a retirement community classify themselves as Republicans, while the others classify themselves as Democrats. In a local election in which everyone voted, 60 Republicans voted for the Democratic candidate, and 50 Democrats voted for the Republican candidate. If a randomly chosen community member voted for the Republican, what is the probability that she or he is a Democrat?

3. Toss a fair 6-sided dice twice. Let X and Y be the result of the first and second toss. Find $P(A|B)$ where

   $$A = \{\max(X, Y) = 5\}$$

   and

   $$B = \{\min(X, Y) = 3\}.$$

4. The following table shows the probabilities of blood types in the general population (Hoxworth Blood Center, Cincinnati, Ohio, March 2003)

	A	B	AB	O
Rh+	.34	.09	.04	.38
Rh-	.06	.02	.01	.06

   (a) What is the probability a person will have type O blood?
   (b) What is the probability a person will be Rh-?
   (c) What is the probability a person will be Rh- given he or she has type O blood?
   (d) What is the probability a person will have type B blood given he or she is Rh+?
   (e) What is the probability a married couple will both be Rh-?
   (f) What is the probability a married couple will both have type AB blood?

5. A light bulb company has factories in two cities. The factory in city A produces two thirds of the company's light bulbs. The remainder are produced in city B, and of these, 1% are defective. Among all bulbs manufactured by the company, what proportion are not defective and made in city

6. We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.
   (a) Find the probability that doubles are rolled.
   (b) Given that the roll results in a sum of 4 or less, find the conditional probability that doubles are rolled.
   (c) Find the probability that at least one die roll is a 6.
   (d) Given that the two dice land on different numbers, find the conditional probability that at least one die roll is a 6.

5 Total probability - Bayes's formula

1. Suppose $P(A|B) = 0.2$, $P(A|B^c) = 0.3$ and $P(B) = 0.8$. What is $P(A)$?

2. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?

3. Suppose that there are two boxes, labeled odd and even. The odd box contains three balls numbered 1, 3, 5. The even box contains two balls labeled 2, 4. One of the boxes is picked at random by tossing a fair coin. Then a ball is picked at random from this box. What is the probability that the ball drawn is ball 3?

4. **(False positive)** Suppose that a laboratory test on a blood sample yields one of two results, positive or negative. It is found that 95% of people with a particular disease produce a positive result. But 2% of people without the disease will also produce a positive result (a false positive). Suppose that 1% of the population actually has the disease. What is the probability that a person chosen at random from the population will have the disease, given that the person's blood yields a positive result?

6 Independence

1. What can you say about the event A if it is independent of itself?

2. A player of a video game is confronted with a series of four opponents and an 80% probability of defeating each opponent. Assume that the results from opponents are independent (and that when the player is defeated by an opponent the game ends).

   (a) What is the probability that a player defeats all four opponents in a game?

(b) What is the probability that a player defeats at least two opponents in a game?

(c) If the game is played three times, what is the probability that the player defeats all four opponents at least once?

3. Consider a bias coin with $p(H) = p$ and $p(T) = 1 - p$. Toss this coin until the first Head.

   (a) Find the probability that the first head occurs after 3 tosses.

   (b) Find the probability that the first head occurs after n tosses.

4. **Gambler's ruin.** A gambler makes a sequence of independent bets. In each bet, he wins \$1 with probability p, and loses \$1 with probability $1 - p$. Initially, the gambler has \$k, and plays until he either accumulates \$n or has no money left. What is the probability that the gambler will end up with \$n?