DIFFERENTIAL EQUATIONS

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September, 2012

Chapter 3 SECOND ORDER DIFFERENTIAL EQUATIONS

Contents

INTRODUCTION

A second order differential equation has the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0. \tag{0.1}$$

For example

$$(x^2 - 1)\frac{d^2y}{dx^2} + x^3y^2\frac{dy}{dx} + e^x \sin y + xy + x^2 + 2 = 0.$$

- Newton's second law of motion

$$m\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right)$$

governs the motion of a particle of mass m moving under the influence of a force F.

3.1 Existence and uniqueness of solution of linear second order differential equations

Definition 1.1 A **linear** second-order differential equation is an equation that can be written in the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

- Assume that $a_0(x),\ a_1(x),\ a_2(x)$ and $b(x)$ are continuous functions of x on an open interval I and $a_2(x) \neq 0$ on I.

- The **standard form**

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x).$$

Equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{0.2}$$

is called the **homogeneous** equation associated with

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x). \tag{0.3}$$

- We usually rewrite Equation (0.3) as

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x).$$

Theorem 1.1 (**Existence and uniqueness of solution**)

Suppose $p(x)$, $q(x)$, and $g(x)$ are continuous on an interval (a, b) that contains the point x_0. Then for any choice of the initial values y_0 and y_1, there exists a unique solution $y(x)$ on the whole interval (a, b) to the initial value problem

$$y'' + p(x)y' + q(x)y = g(x),$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1.$$

3.2 FUNDAMENTAL SOLUTIONS OF LINEAR HOMOGENEOUS EQUATIONS

- Consider the expression on the left-hand side of the equation

$$y'' + p(x)y' + q(x)y = g(x).$$

With each function y having two derivatives, we associate another function, denoted $L[y]$, by the relation:

$$L[y](x) = y''(x) + p(x)y'(x) + q(x)y(x).$$

L is called a **differential operator**.

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Example 1.2 Let

$$L[y](x) = y''(x) + xy(x).$$

If $y(x) = \cos x$, then $L[y](x) = (x-1)\cos x$.
If $y(x) = x^3$, then $L[y](x) = 6x + x^4$.

Theorem 2.1 (**Linearity of the differential operator L**) *Let*

$$L[y](x) = y''(x) + p(x)y'(x) + q(x)y(x). \tag{0.4}$$

If y, y_1 and y_2 are any functions with continuous second derivatives on the interval I, and if c is any constant, then

$$L[y_1 + y_2] = L[y_1] + L[y_2], \tag{0.5}$$
$$L[cy] = cL[y]. \tag{0.6}$$

In (0.5) and (0.6) equality is meant in the sense of equal functions on I.

An operator that satisfies properties (0.5) and (0.6) for any constant c and any functions y, y_1, and y_2 in its domain is called a **linear operator**.

Define
$$L[y](x) = y''(x) + p(x)y'(x) + q(x)y(x).$$
Then
$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \iff L[y] = 0.$$

Theorem 2.2 (**Linear combinations of solutions**)
Let y_1 and y_2 be solutions to the homogeneous equation
$$y'' + p(x)y' + q(x)y = 0. \tag{0.7}$$
Then any linear combination $c_1y_1 + c_2y_2$ of y_1 and y_2, where c_1 and c_2 are constants, is also a solution to (0.7).

Example 2.1 Given that e^x and e^{2x} are solutions to the homogeneous equation

$$y'' - 3y' + 2y = 0, \tag{0.8}$$

find a solution to (0.8) that satisfies the initial conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = -1.$$

Definition 2.1 For any two differentiable functions y_1 and y_2, the function

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

is called the **Wronskian determinant** (or shortly, **Wronskian**) of y_1 and y_2.

Definition 2.2 A pair of solutions $\{y_1, y_2\}$ of

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) is called a **fundamental solution set** if

$$W[y_1, y_2](x_0) \neq 0$$

at some x_0 in (a, b).

Theorem 2.3 (**Representation of Solutions (Homogeneous Case)**)
Let y_1 and y_2 be two solutions on (a, b) of

$$y'' + p(x)y' + q(x)y = 0, \tag{0.9}$$

where $p(x)$ and $q(x)$ are continuous on (a, b). If at some point x_0 in (a, b) these solutions satisfy

$$W[y_1, y_2](x_0) = y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0,$$

then every solution of Equation (0.9) on (a, b) can be expressed in the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are constants.

• We call the expression $y(x) = c_1 y_1(x) + c_2 y_2(x)$ with arbitrary constant coefficients the **general solution** of (0.9).

In other words, Theorem 2.3 says that

If $\{y_1, y_2\}$ is a fundamental solution set of equation

$$y'' + p(x)y' + q(x)y = 0, \tag{0.10}$$

then this equation has the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad c_1, c_2 \in \mathbb{R}.$$

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Important Remark: To obtain the general solution of Equation (0.10), we must find a fundamental solution set $\{y_1, y_2\}$ of this equation and then

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad c_1, c_2 \in \mathbb{R}$$

is its general solution.

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Solution: It is easy to check that $y_1(x) := \cos 2x$ and $y_2(x) := \sin 2x$ are solutions to $y'' + 4y = 0$ on $(-\infty, \infty)$. Moreover, we have

$$W[y_1, y_2](x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2((\cos 2x)^2 + (\sin 2x)^2) = 2.$$

Thus, $\{y_1(x) := \cos 2x, y_2(x) := \sin 2x\}$ is a fundamental solution set of the given differential equation. So the general solution is given by

$$y(x) = c_1 \cos 2x + c_2 \sin 2x, \qquad c_2, c_2 \in \mathbb{R}.$$

3.2 FUNDAMENTAL SOLUTIONS OF LINEAR HOMOGENEOUS EQUATIONS

Theorem 2.4 (**Existence of fundamental solution set**) *If $p(x)$ and $q(x)$ are continuous on an open interval, then a fundamental set of solutions for the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0$$

always exists.

3.3 LINEAR INDEPENDENCE AND THE WRONSKIAN

Definition 3.1 Two functions $y_1(x)$ and $y_2(x)$ are said to be **linearly dependent** on an interval I if there exist two constants $\alpha_1, \ \alpha_2$, **not both zero**, such that

$$\alpha_1 y_1(x) + \alpha_2 y_2(x) = 0 \quad \text{for all } x \text{ in } I.$$

If two functions are not linearly dependent, they are said to be **linearly independent**.

Remark: *If*

$$\alpha_1 y_1(x) + \alpha_2 y_2(x) = 0, \forall x \in I \qquad \Longrightarrow \qquad \alpha_1 = \alpha_2 = 0$$

then y_1, y_2 are linearly independent.

3.3 LINEAR INDEPENDENCE AND THE WRONSKIAN

Example 3.1 Determine whether the following pairs of functions y_1 and y_2 are linearly dependent on $(-2, 3)$:

$$\begin{array}{lll} \text{(a)} & y_1(x) = e^x, & y_2(x) = x + 1; \\ \text{(b)} & y_1(x) = \sin 2x, & y_2(x) = \cos x \sin x; \\ \text{(c)} & y_1(x) = x, & y_2(x) = |x|. \end{array}$$

Example 3.2 Show that if $\alpha \neq \beta$ then the functions $y_1(x) = e^{\alpha x}$ and $y_2(x) = e^{\beta x}$ are linearly independent on any interval.

3.3 LINEAR INDEPENDENCE AND THE WRONSKIAN

Theorem 3.1 *If f and g are differentiable and linearly dependent on an open interval I, then*

$$W(f, g)(x) = 0 \text{ for every point } x \text{ in } I.$$

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$$W(f,g)(x) = 0 \text{ for every point } x \text{ in } I.$$

So if

$$W(f,g)(x_0) \neq 0 \text{ for some point } x_0 \in I,$$

then f and g are linearly independent on I.

3.3 LINEAR INDEPENDENCE AND THE WRONSKIAN

Theorem 3.2 *Let y_1 and y_2 be solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$, where $p(x)$ and $q(x)$ are continuous on an open interval I. Then the Wronskian $W(y_1, y_2)(x)$ is given by*

$$\boxed{W(y_1, y_2)(x) = C \exp\left(-\int p(x)\, dx\right)}$$

where C is a certain constant that depends on y_1 and y_2, but not on x.

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where C is a certain constant that depends on y_1 and y_2, but not on x.

Corollary 3.1 *If y_1 and y_2 are solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$ on an interval I, then the Wronskian $W(y_1, y_2)(x)$ of the two solutions is either identically zero or never zero on I. Furthermore, the Wronskian of two solutions is identically zero if and only if the solutions are linearly dependent.*

3.3 LINEAR INDEPENDENCE AND THE WRONSKIAN

We summarize the facts in

Theorem 3.3 *Let y_1 and y_2 be solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$ on an interval I. The following four statements are equivalent:*

(a) *The functions y_1 and y_2 are a fundamental set of solutions on I.*

(b) *The functions y_1 and y_2 are linearly independent on I.*

(c) $W(y_1, y_2)(x_0) \neq 0$ *for some x_0 in I.*

(d) $W(y_1, y_2)(x) \neq 0$ *for all x in I.*

3.3 LINEAR INDEPENDENCE AND THE WRONSKIAN

Example 3.3 Show that $y_1(x) = \frac{1}{x}$ and $y_2(x) = x^3$ are solutions to

$$x^2y'' - xy' - 3y = 0$$

on the interval $(0, \infty)$ and give a general solution.

Example 3.4 Show that $y_1(x) = x^3$ and $y_2(x) = x^{-4}$ are solutions of the differential equation

$$x^2y'' + 2xy' - 12y = 0$$

on the interval $(0, \infty)$. Find the solution that satisfies the initial conditions

$$y(1) = 4, \qquad y'(1) = 5.$$

3.4 COMPLEX NUMBERS

The solution of a quadratic equation $ax^2 + bx + c = 0$ is given by the formula

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \qquad (0.11)$$

But if $\Delta = b^2 - 4ac$ is negative, it is impossible to use (0.11) unless we introduce a new kind of numbers.

The symbol i that has the property $i^2 = -1$ is called **the imaginary unit**. We could also call i *the square root of* -1, $i = \sqrt{-1}$. Of course i is NOT a real number.

3.4.1 Definition of Complex Numbers

Definition 4.1 A **complex number** is an expression of the form

$$a + ib \quad \text{or} \quad a + bi$$

where a and b are real numbers, and i is the imaginary unit.

The **real part** of the complex number $z = a + bi$ is the real number a and it is denoted by $\mathrm{Re}(z)$. We call the real number b **the imaginary part** of z and it is denoted by $\mathrm{Im}(z)$. So

$$\boxed{\textbf{a complex number} = \textbf{Real part} + i(\textbf{Imaginary part})}$$

Example 4.1

$$\mathrm{Re}(2+\sqrt{3}i) = 2 \qquad \mathrm{Im}(2+\sqrt{3}i) = \sqrt{3}$$
$$\mathrm{Re}(-4i) = \mathrm{Re}\big(0+(-4)i\big) = 0 \qquad \mathrm{Im}(-4i) = -4$$

If $a = 0$, the complex number $z = bi$ is said to be **purely imaginary**, and if $b = 0$, the complex number $z = a$ is **purely real**.

Equality

Definition 4.2 Two complex numbers are **equal** if their real parts are equal and their imaginary parts are equal.

$$\boxed{a + bi = c + di \Longleftrightarrow a = c \quad \text{and} \quad b = d}$$

For example, if $a - 5i = 7 + bi$ then $a = 7$ and $b = -5$.

Addition and Subtraction

Definition 4.3 If $z_1 = a + bi$ and $z_2 = c + di$, then we define

$$\boxed{\begin{aligned} z_1 + z_2 &= (a+c) + (b+d)i \\ z_1 - z_2 &= (a-c) + (b-d)i \end{aligned}}$$

For instance

$$(5+7i) - (4-3i) = (5-4) + \big(7 - (-3)\big)i = 1 + 10i.$$

Multiplication and Division

The product of complex numbers is defined so that the usual communicative and distributive laws hold:

$$\begin{aligned}(a+bi)(c+di) &= a(c+di)+bi(c+di)\\ &= ac+adi+bci+bdi^2.\end{aligned}$$

Since $i^2=-1$, this becomes

$$\boxed{(a+bi)(c+di) = ac-bd+(ad+bc)i}$$

Example 4.2 Determine
(a) $(4 - 5i)(2 + i)$;
(b) $(3 + 4i)(2 - 5i)(1 - 2i)$.

- Note that

$$\boxed{(a + bi)(a - bi) = a^2 + b^2}$$

is entirely real.

3.4.2 ARITHMETICAL OPERATIONS

The division of two complex numbers is defined as follows

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(a+bi)(c-di)}{c^2+d^2}.$$

Example 4.3 Express the number $\frac{1+2i}{3-5i}$ in the form $a+bi$.

Example 4.4 Find

$$\frac{(2+3i)(1-2i)}{3+4i}.$$

Example 4.5 Find the real and imaginary parts of the complex number $z+\frac{1}{z}$ for $z=\frac{2+i}{1-i}$.

Conjugate numbers

Definition 4.4 The **conjugate** or **complex conjugate** of a complex number $z = a + bi$ is the complex number $\bar{z} = a - bi$.

- Some of the properties of the complex conjugate:

$$\boxed{\overline{z+w} = \bar{z} + \bar{w} \qquad \overline{zw} = \bar{z}\,\bar{w} \qquad \overline{z^n} = (\bar{z})^n}$$

Definition 4.5 The **modulus** of a complex number $z = a + bi$ is the nonnegative number

$$\boxed{|z| = \sqrt{a^2 + b^2}}$$

Notice that

$$z\bar{z} = (a+bi)(a-bi) = a^2 - b^2 i^2 = a^2 + b^2$$

and so

$$\boxed{z\bar{z} = |z|^2}$$

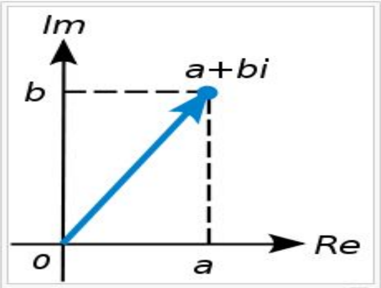
Therefore

$$\boxed{\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}}$$

To divide one complex number by another, we multiply both numerator and denominator by the conjugate of the denominator.

3.4.3 REPRESENTATION OF COMPLEX NUMBERS

We use the point with coordinates (a, b) to represent the complex number $z = a + bi$.



• The representation of complex numbers as points in a plane is called an **Argand diagram**.

• The set of all complex numbers, denoted $\mathbb{C}$, is often refer to as **the complex plane**. The x-axis is called the **real axis**, and the y-axis is called the **imaginary axis**.

Example 4.6 Represent on the complex plane the numbers
(a) $z = 3 + 2i$ (b) $z = -1 + 4i$ (c) $z = -3i$

• Note that the real part corresponds to the x-value and the imaginary part corresponds to the y-value.

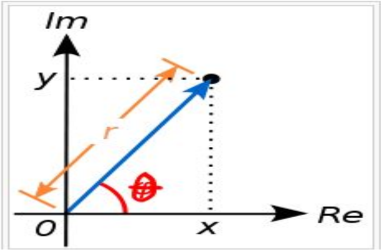
3.4.3 POLAR FORM

Any complex number $z = a + bi$ can be written in the polar form

$$\boxed{z = r(\cos\theta + i\sin\theta)} \tag{0.12}$$

where

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan\theta = \frac{b}{a}.$$



• The expression on the right side of (0.12) is called the **polar representation** or **polar form** of z.

• The angle θ is called the **argument** of z and we write $\theta = \arg(z)$

• $\arg(z)$ is not unique and the argument of the complex number $0 = 0 + 0i$ is NOT defined.

• If we restrict $\theta = \arg(z)$ to an interval of length 2π, say, $[0, 2\pi)$ or $(-\pi, \pi]$, then nonzero complex numbers will have unique arguments.

• We will call the value of $\arg(z)$ in the interval $-\pi < \theta \leq \pi$ the **principal argument** of z and denote it $\mathrm{Arg}(z)$.

3.4.3 POLAR FORM

Principal argument of $z = x + iy$

$$\theta = \operatorname{Arg}(z) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{indeterminate} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

$$\theta \in (-\pi, \pi]$$

Example 4.7 Write the following complex numbers in polar form

(a) $z = 1 + i$ (b) $z = -\sqrt{3} + i$.

Multiplication and Division in Polar Form

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be two complex numbers written in polar form. Then

$$\boxed{z_1 z_2 = |z_1||z_2|\big[\cos\big(\theta_1+\theta_2\big) + i\sin(\theta_1+\theta_2)\big]}$$

- So

  to multiply two complex numbers, we multiply the moduli and add the arguments:

$$|z_1 z_2| = |z_1||z_2| \qquad \text{and} \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

3.4.3 POLAR FORM

Similarly,

$$\boxed{\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right] \quad \text{if} \quad z_2 \neq 0}$$

This formula shows that

to divide two complex numbers, we divide the moduli and subtract the arguments:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \text{and} \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

- In particular,

$$\boxed{\text{If} \quad z = r(\cos\theta + i\sin\theta), \ \text{then} \ \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)}$$

Example 4.8 Find the product of the complex numbers $z_1 = 1 + \sqrt{3}i$ and $z_2 = -1 - i$ in polar form.

Example 4.9 If

$$z_1 = 4\big(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\big) \quad \text{and} \quad z_2 = 5\big(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\big),$$

find

$$\frac{z_1}{z_2} \quad \text{and} \quad \frac{z_2}{z_1}.$$

Powers of Complex Numbers

Theorem 4.4 (De Moivre's theorem) *If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then*

$$\boxed{z^n = \left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)}$$

This says that

to take the nth power of a complex number we take the nth power of the modulus and multiply the argument by n.

Example 4.10 Express $(1+i)^5$ and $(1+i)^{10}$ in the form $a+bi$.

Roots of Complex Numbers

Definition 4.6 An ***n*th root** of the complex number z is a complex number w such that $w^n = z$.

- Since $(1+i)^{10} = 32i$, $w = 1+i$ is a 10th root of $32i$.

- Writing z and w in polar form as

$$z = r(\cos\theta + i\sin\theta) \qquad \text{and} \qquad w = s(\cos\phi + i\sin\phi)$$

and using De Moivre's theorem, we have

$$s^n = r, \quad \cos n\phi = \cos\theta, \quad \text{and} \quad \sin n\phi = \sin\theta.$$

Thus, $s = r^{1/n}$ and $n\phi = \theta + 2k\pi$. Therefore,

$$w = r^{1/n}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right]$$

This expression gives exactly n different roots, corresponding to $k = 0, 1, ..., n-1$.

Theorem 4.5 *Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then z has the n distinct nth roots*

$$\boxed{w_k = r^{1/n}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right], \;\; k = 0, 1, ..., n-1}$$

The root

$$w_0 = |z|^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

is called the **principal** nth root of z.

Example 4.11 Find the 4th roots of $z = -4$.
Solution: We rewrite z in the polar form $z = 4(\cos\pi + i\sin\pi)$.

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$$w_1 = 4^{1/4}(\cos(\frac{\pi}{4} + \frac{\pi}{2}) + i\sin(\frac{\pi}{4} + \frac{\pi}{2})) = \sqrt{2}(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = -1 + i.$$

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$$w_0 = 4^{1/4}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = 1 + i.$$

$$w_1 = 4^{1/4}(\cos(\frac{\pi}{4} + \frac{\pi}{2}) + i\sin(\frac{\pi}{4} + \frac{\pi}{2})) = \sqrt{2}(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = -1 + i.$$

$$w_2 = 4^{1/4}(\cos(\frac{\pi}{4} + \pi) + i\sin(\frac{\pi}{4} + \pi)) = \sqrt{2}(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = -1 - i.$$

Example 4.11 Find the 4th roots of $z = -4$.
Solution: We rewrite z in the polar form $z = 4(\cos\pi + i\sin\pi)$.Then the 4th roots of $z = -4$ are given by

$$w_0 = 4^{1/4}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = 1 + i.$$

$$w_1 = 4^{1/4}(\cos(\frac{\pi}{4} + \frac{\pi}{2}) + i\sin(\frac{\pi}{4} + \frac{\pi}{2})) = \sqrt{2}(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = -1 + i.$$

$$w_2 = 4^{1/4}(\cos(\frac{\pi}{4} + \pi) + i\sin(\frac{\pi}{4} + \pi)) = \sqrt{2}(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = -1 - i.$$

$$w_3 = 4^{1/4}(\cos(\frac{\pi}{4} + \frac{3\pi}{2}) + i\sin(\frac{\pi}{4} + \frac{3\pi}{2})) = \sqrt{2}(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = 1 - i.$$

3.4.5 EXPONENTIAL FORM OF A COMPLEX NUMBER

Recall that for any real number x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

(exponential function with real variable)
If we now define

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

then this complex exponential function has the same properties as the real exponential function e^x. In particular,

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}.$$

3.4.5 EXPONENTIAL FORM OF A COMPLEX NUMBER

When θ is a real number, we have **Euler's formula**:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Therefore $z = r(\cos\theta + i\sin\theta)$ can be written as $z = re^{i\theta}$. This is called the **exponential form** of the complex number.

- Note that in the exponential form, the angle must be in radians.

3.4.5 EXPONENTIAL FORM OF A COMPLEX NUMBER

For instance, $2(\cos 60^\circ + i\sin 60^\circ) = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 2e^{\frac{i\pi}{3}}$.

- Since $e^{\alpha+i\beta} = e^{\alpha}e^{i\beta}$,

$$\boxed{e^{\alpha+i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)}$$

Example 4.12 Evaluate (a) $e^{\frac{3\pi i}{4}}$ (b) $e^{2+\frac{3\pi i}{2}}$

The three ways of expressing a complex number are:

- (a) $z = a + bi$
- (b) $z = r(\cos\theta + i\sin\theta)$ Polar form
- (c) $z = re^{i\theta}$ Exponential form

Solving equations with constant coefficients

3.5 REAL AND COMPLEX ROOTS OF THE CHARACTERISTIC EQUATION

Consider the homogeneous equation with constant coefficients

$$ay'' + by' + cy = 0, \tag{0.13}$$

where a, b, and c are real constants and $a \neq 0$.

We see that $y = e^{rx}$ is a solution of (0.13) if and only if

$$\boxed{ar^2 + br + c = 0} \tag{0.14}$$

Equation (0.14) is called the **characteristic equation** of (0.13). Its roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

3.5.1 DISTINCT REAL ROOTS

If the characteristic equation has distinct real roots r_1 and r_2, then the general solution is

$$\boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

where c_1 and c_2 are arbitrary constants.

Example 5.1 Find the general solution of the equation

$$y'' - 3y' + 2y = 0.$$

Example 5.2 Solve the initial value problem

$$y'' + 4y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

3.5.2 COMPLEX ROOTS

If $\Delta = b^2 - 4ac < 0$, then the characteristic equation

$$ar^2 + br + c = 0$$

has two conjugate complex roots

$$r = \alpha \pm i\beta \quad \text{where} \quad \alpha = -\frac{b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

and the differential equation

$$ay'' + by' + cy = 0$$

has two complex solutions

$$z_1 = e^{(\alpha + i\beta)x} \quad \text{and} \quad z_2(x) = e^{(\alpha - i\beta)x}.$$

3.5.2 COMPLEX ROOTS

Lemma 5.1 *Let $z(x) = u(x) + iv(x)$ be a solution to equation*

$$ay'' + by' + cy = 0, \tag{0.15}$$

where a, b, and c are real constants. Then the real part $u(x)$ and the imaginary part $v(x)$ are real-value solutions of (0.15).

It follows from Lemma 5.1 that

If the characteristic equation has complex conjugate roots $\alpha \pm i\beta$, then two linearly independent solutions of the differential equation are

$$e^{\alpha x} \cos \beta x \quad \text{and} \quad e^{\alpha x} \sin \beta x$$

and the general solution is

$$\boxed{y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x}$$

where c_1 and c_2 are arbitrary constants.

3.5.2 COMPLEX ROOTS

Example 5.3 Find a general solution to

$$y'' + y' + y = 0.$$

Example 5.4 Find the solution of the initial value problem

$$y'' + 2y' + 4y = 0, \qquad y(0) = 1, \ \ y'(0) = 2.$$

3.5.3 EQUAL ROOTS

• If $\Delta = b^2 - 4ac = 0$, then the characteristic equation $ar^2 + br + c = 0$ has real equal roots $r_1 = r_2 = r = -b/2a$ so we obtain only one solution

$$y_1(x) = e^{rx} = e^{-\frac{b}{2a}x}.$$

3.5.3 EQUAL ROOTS

- If $\Delta = b^2 - 4ac = 0$, then the characteristic equation $ar^2 + br + c = 0$ has real equal roots $r_1 = r_2 = r = -b/2a$ so we obtain only one solution

$$y_1(x) = e^{rx} = e^{-\frac{b}{2a}x}.$$

We find a solution $y_2(x)$ such that $y_1(x)$ and $y_2(x)$ are linearly independent.

-

Expressing

$$y_2(x) = u(x)y_1(x) = u(x)e^{rx},$$

where the nonconstant function u is to be determined. Choosing $u(x) = x$, it is easy to check that $y_2(x) = xe^{rx}$ is an expected solution to $ay'' + by' + c = 0$.

3.5.3 EQUAL ROOTS

If the characteristic equation has a repeat root r, then the general solution of the differential equation is

$$\boxed{y = c_1 e^{rx} + c_2 x e^{rx} = (c_1 + c_2 x)e^{rx}}$$

where c_1 and c_2 are arbitrary constants.

Example 5.5 Find the general solution to

$$y'' + 2y' + y = 0.$$

3.5.4 EQUATIONS WITH NONCONSTANT COEFFICIENTS

We have no general method for solving the homogeneous equation with nonconstant coefficients

$$L[y] = y'' + p(x)y' + q(x)y = 0. \tag{0.16}$$

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However, if we already know one nonzero solution $y_1(x)$ of Equation (0.16), then we can find its general solution by using the Wronskian.

- The Wronskian of y_1 and any solution y of Equation (0.16) is

$$W[y_1, y] = y'y_1 - y_1'y = c_1 e^{-\int p(x)dx},$$

where c_1 is a constant.

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$$W[y_1, y] = y'y_1 - y_1'y = c_1 e^{-\int p(x)dx},$$

where c_1 is a constant.Dividing by $y_1^2 \neq 0$ we get

$$\frac{d}{dx}\left(\frac{y}{y_1}\right) = \frac{c_1}{y_1^2} e^{-\int p(x)dx}.$$

Thus

$$\boxed{y = y_1 \left\{ \int c_1 \frac{\exp\left(-\int p(x)dx\right)}{y_1^2(x)}\, dx + c_2 \right\}}$$

Important remark: If y_1 is a solution of (0.16) then y_1 and

$$y_2 := y_1 \left\{ \int \frac{\exp\left(- \int p(x)dx \right)}{y_1^2(x)} dx \right\}$$

are linearly independent solutions. So the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad c_1, c_2 \in \mathbb{R}.$$

Example 5.6 Given that $y_1(t) = t$ is a solution of

$$(1 - t^2)y'' + 2ty' - 2y = 0,$$

find the solution of the initial value problem

$$(1 - t^2)y'' + 2ty' - 2y = 0, \qquad y(0) = 3, \quad y'(0) = -4.$$

Example 5.7 Given the equation

$$x^2y'' + 3xy' + y = 0; \qquad x > 0,$$

(a) Find a solution of the form $y = x^\alpha$, where α is a real number.
(b) Find the general solution of the differential equation.

3.6 NONHOMOGENEOUS EQUATIONS; METHOD OF UNDETERMINED COEFFICIENTS

3.6.1 THE GENERAL SOLUTION OF NONHOMOGENEOUS EQUATIONS

Consider the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x), \tag{0.17}$$

where p, q, and g are given continuous function on an interval I. The equation

$$y'' + p(x)y' + q(x)y = 0 \tag{0.18}$$

is called the **homogeneous equation** corresponding to Equation (0.17).

3.6.1 THE GENERAL SOLUTION OF NONHOMOGENEOUS EQUATIONS

Lemma 6.1

The difference of any two solutions of the nonhomogeneous equation (0.17) *is a solution of the homogeneous equation* (0.18).

3.6.1 THE GENERAL SOLUTION OF NONHOMOGENEOUS EQUATIONS

Theorem 6.2

Let $y_p(x)$ be a particular solution to the nonhomogeneous equation

$$L[y] = y'' + p(x)y' + q(x)y = g(x), \tag{0.19}$$

on the interval (a, b) and let $y_1(x)$, $y_2(x)$ be linearly independent solutions on (a, b) of the corresponding homogeneous equation

$$L[y] = y'' + p(x)y' + q(x)y = 0.$$

Then every solution of (0.19) on the interval (a, b) can be expressed in the form

$$\boxed{y = c_1y_1(x) + c_2y_2(x) + y_p(x)} \tag{0.20}$$

- Expression (0.20) is called the **general solution** of (0.19).

3.6.1 THE GENERAL SOLUTION OF NONHOMOGENEOUS EQUATIONS

Example 6.1 Given that $y_p(x) = x$ is a particular solution of the equation

$$y'' + y = x,$$

find the general solution of this equation.

3.6.1 THE GENERAL SOLUTION OF NONHOMOGENEOUS EQUATIONS

Theorem 6.3 (**Superposition Principle**)

Let y_1 *be a solution of the differential equation*

$$L[y](x) = g_1(x)$$

and let y_2 *be a solution of*

$$L[y](x) = g_2(x),$$

where L *is a linear differential operator. Then* $y_1 + y_2$ *is a solution of the differential equation*

$$L[y](x) = g_1(x) + g_2(x).$$

3.6.1 THE GENERAL SOLUTION OF NONHOMOGENEOUS EQUATIONS

Example 6.2 Given that $y_1(x) = 5xe^x$ is a solution to

$$y'' - y' = 5e^x,$$

and $y_2(x) = -(1/10)\cos 2x + (1/5)\sin 2x$ is a solution to

$$y'' - y' = -\sin 2x,$$

find a solution to

$$y'' - y' = 5e^x - \sin 2x.$$

3.6.2 METHOD OF UNDETERMINED COEFFICIENTS

Consider a nonhomogeneous equation

$$L[y] = ay'' + by' + cy = g(x),$$

where a, b, and c are constants.

Case 1: $g(x) = P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

3.6.2 METHOD OF UNDETERMINED COEFFICIENTS

Consider a nonhomogeneous equation

$$L[y] = ay'' + by' + cy = g(x),$$

where a, b, and c are constants.

Case 1: $g(x) = P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

- If $c \neq 0$, we seek a particular solution $y_p(x)$ of the form

$$y_p(x) = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0.$$

- If $c = 0$ and $b \neq 0$, we must take $y_p(x)$ as a polynomial of degree $n+1$:

$$y_p(x) = x\left[A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0\right].$$

If $c = b = 0$, we must take $y_p(x)$ as a polynomial of degree $n+2$:

$$y_p(x) = x^2\left[A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0\right].$$

Example 6.3 Find a particular solution of

$$y'' + y' + y = x^2.$$

Example 6.4 Determine the form of a particular solution of

$$y'' - 7y' = x^3 + 2x + 10.$$

Case 2:

$$g(x) = P_n(x)e^{\alpha x} = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)e^{\alpha x}.$$

Case 2:

$$g(x) = P_n(x)e^{\alpha x} = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)e^{\alpha x}.$$

Then equation $L[y] = P_n(x)e^{\alpha x}$ has a particular solution $y_p(x)$ of the form

(i) $y_p(x) = Q_n(x)e^{\alpha x} = (A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0)e^{\alpha x}$ if α is not a root of the characteristic equation;

(ii) $y_p(x) = xQ_n(x)e^{\alpha x} = x(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0)e^{\alpha x}$ if α is a single root of the characteristic equation;

(iii) $y_p(x) = x^2 Q_n(x)e^{\alpha x} = x^2(\sum_{i=0}^{n} A_i x^i)e^{\alpha x}$ if α is a double root of the characteristic equation.

Example 6.5 Find a particular solution of the equation

$$y'' - y' = (x+1)e^{3x}.$$

Example 6.6 Find a particular solution of

$$y'' - y' = e^{x}(x+1).$$

Example 6.7 Find the general solution of the equation

$$y'' - 4y' + 4y = 6e^{2x}.$$

3.6.2 METHOD OF UNDETERMINED COEFFICIENTS

Case 3: $g(x) = P_n(x)e^{\alpha x} \times \begin{cases} \cos \beta x \\ \sin \beta x \end{cases}$

(i) If the complex number $\alpha + i\beta$ is not a root of the characteristic equation, then

$$\begin{aligned} y_p(x) &= \Big[Q_n(x)\cos \beta x + R_n(x)\sin \beta x\Big] e^{\alpha x} \\ &= \Big[(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0)\cos \beta x \\ &\quad + (B_n x^n + B_{n-1}x^{n-1} + \cdots + B_1 x + B_0)\sin \beta x\Big] e^{\alpha x}. \end{aligned}$$

(ii) If $\alpha + i\beta$ is a root of the characteristic equation, then

$$\begin{aligned} y_p(x) &= x\Big[Q_n(x)\cos \beta x + R_n(x)\sin \beta x\Big] e^{\alpha x} \\ &= x\Big[(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0)\cos \beta x \\ &\quad + (B_n x^n + B_{n-1}x^{n-1} + \cdots + B_1 x + B_0)\sin \beta x\Big] e^{\alpha x}. \end{aligned}$$

Example 6.8 Find a particular solution of

$$y'' + y' - 2y = e^x(\cos x - 7\sin x).$$

Example 6.9 Find a particular solution of

$$y'' + y = 2\sin x.$$

The form of a particular solution $y_p(x)$ of $L[y](x) = g(x)$ when $L[y]$ has constant coefficients

$g(x)$	$y_p(x)$
$P_n(x) = \sum_{k=0}^{n} a_k x^k$	$x^s(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0)$
$P_n(x)e^{\alpha x}$	$x^s(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0)e^{\alpha x}$
$P_n(x)e^{\alpha x} \times \begin{cases} \cos \beta x \\ \sin \beta x \end{cases}$	$x^s\left[Q_n(x)\cos \beta x + R_n(x)\sin \beta x\right]e^{\alpha x}$

Note Here s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation.

Example 6.10 Solve the equation

$$y'' - y' = 5e^{x} - \sin 2x.$$

Example 6.11 Find the general solution of the equation

$$y'' - y = 2e^{-x} - 4xe^{-x} + 10\cos 2x.$$

3.7 VARIATION OF PARAMETERS

There is another method of finding a particular solution of a nonhomogeneous equation, called **variation of parameters**.

- Consider the nonhomogeneous linear second order equation

$$L[y](x) = y'' + p(x)y' + q(x)y = g(x),$$

and let $\{y_1(x), y_2(x)\}$ be a fundamental set of solutions for the corresponding homogeneous equation.

$$L[y](x) = y'' + p(x)y' + q(x)y = 0. \tag{0.21}$$

- The general solution of the homogeneous equation (0.21) is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are constants.

3.7 VARIATION OF PARAMETERS

- We find two functions $u_1(x)$ and $u_2(x)$ such that the expression

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

is a solution of the nonhomogeneous equation $L[y] = g$.

Then u_1' and u_2' satisfy the linear system of equations:

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= g \end{aligned}$$

Using Cramer's rule immediately gives

$$u_1'(x) = \frac{-g(x)y_2(x)}{W[y_1, y_2](x)}, \qquad u_2'(x) = \frac{g(x)y_1(x)}{W[y_1, y_2](x)}.$$

3.7 VARIATION OF PARAMETERS

METHOD OF VARIATION OF PARAMETERS

To determine a particular solution of $y'' + p(x)y' + q(x)y = g(x)$:

Step 1. Find a fundamental set of solutions $\{y_1, y_2\}$ for the corresponding homogeneous equation and take

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Step 2. Determine $u_1(x)$ and $u_2(x)$ by solving the linear system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= g \end{aligned}$$

for $u_1'(x)$ and $u_2'(x)$ and integrating.

Step 3. Substitute $u_1(x)$ and $u_2(x)$ into the expression for $y_p(x)$ to obtain a particular solution.

Note If the given equation is

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y = b(x),$$

then it must be put in the form $y'' + p(x)y' + q(x)y = g(x)$.

3.7 VARIATION OF PARAMETERS

Example 7.1 (a) Find the general solution of the equation

$$y'' + y = \frac{1}{\cos x}. \tag{0.22}$$

(b) Find the solution of (0.22) which satisfies the initial conditions $y(0) = 1$ and $y'(0) = 2$.

Example 7.2 Given the equation

$$(1 - x^2)y'' + 2xy' - 2y = 1 - x^2.$$

(a) Find a solution of the corresponding homogeneous equation of the form $y_1(x) = x^{\alpha}$.

(b) Find the general solution of the homogeneous equation.

(c) Find the general solution of the nonhomogeneous equation.

3.8 APPLICATIONS

8.1 MECHANICAL VIBRATIONS

• Consider the simple mechanical system consisting of a coil spring suspended from a rigid support with a mass m attached to the end of the spring.

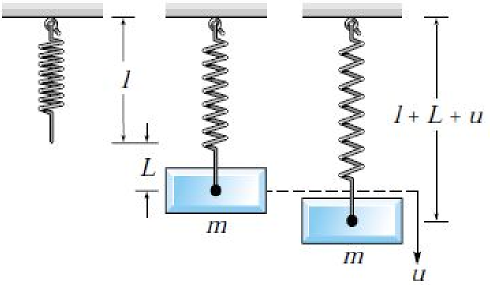

FIGURE 3.8.1 A spring–mass system.

Dynamic problem: Study the motion of the mass when it is acted on by an external force or is initially displaced.

Dynamic problem: Study the motion of the mass when it is acted on by an external force or is initially displaced.

- We need two laws of physics:
- Hooke's law:

> *the spring exerts a restoring force opposite to the direction of elongation of the spring and with a magnitude directly proportional to the amount of elongation.*

That is, the spring exerts a restoring force

$$F = -kx,$$

where x is the amount of elongation and $k > 0$ is the **spring constant**.

- Newton's second law:

$$m\frac{d^2x}{dt^2} = ma = F\left(t, x, \frac{dx}{dt}\right)$$

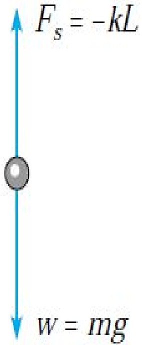

FIGURE 3.8.2 Force diagram for a spring–mass system.

- Choose a vertical coordinate axis passing through the spring, with the origin at the equilibrium position of the mass.

- Let x denote the displacement of the mass from its equilibrium position.

- **Gravity** The force of gravity

$$F_1 = mg.$$

- **Restoring Force** The spring exerts a restoring force

$$F_2 = -kx - mg.$$

- **Damping Force**

There is a damping or frictional force

$$F_3 = -b\frac{dx}{dt}, \qquad b > 0,$$

where b is the **damping constant** given in units of mass/time (or force-time/length).

- **External Forces**

Any external forces acting on the mass will be denoted by $F_4 = f(t)$.

The total force acting on the mass m is $F_1 + F_2 + F_3 + F_4$:

$$\begin{aligned} F\left(t, x, \frac{dx}{dt}\right) &= mg - kx - mg - b\frac{dx}{dt} + f(t) \\ &= -kx - b\frac{dx}{dt} + f(t). \end{aligned}$$

Applying Newton's second law to the system gives

$$\boxed{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t)}$$

• When $b = 0$, the system is said to be **undamped**; otherwise, it is **damped**. When $f(x) \equiv 0$, the motion is said to be **free**; otherwise the motion is **forced**.

Undamped Vibrations (Free Vibrations)

Consider the **undamped, free** case in which $b = 0$ and $f(t) \equiv 0$.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \tag{0.23}$$

where $\omega = \sqrt{k/m}$.
The general solution to (0.23) is

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

or

$$x(t) = A \sin(\omega t + \phi),$$

where

$$A = \sqrt{C_1^2 + C_2^2} \quad \text{and} \quad \tan \phi = \frac{C_1}{C_2}.$$

• The motion of a mass in an undamped, free system is simply a sine wave called **simple harmonic motion**.

• A is the **amplitude** of the motion and ϕ is the **phase angle**. The motion is periodic with period $T = 2\pi/\omega$ and **natural frequency** $\omega/2\pi$, where $\omega = \sqrt{k/m}$.

Example 8.1 A spring is such that it would be stretched 6 inches (in.) by a 12-lb weight. Let the weight be attached to a spring and pulled down 4 in. below the equilibrium point. If the weight is started with an upward velocity of 2 ft/sec, describe the motion. No damping or impressed force is present.

Damped Free Vibrations

Consider the motion of a system that is governed by

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \tag{0.24}$$

The auxiliary equation $m^2 + br + k = 0$ has roots

$$\frac{-b \pm \sqrt{b^2 - 4km}}{2m} = -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4km}$$

- The value $b = 2\sqrt{km}$ is called **critical damping**, while for large values of b the motion is said to be **overdamped**.

- **Underdamped or Oscillatory Motion ($b^2 < 4km$)**

When $b^2 < 4km$, the general solution to (0.24) is

$$x(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$$

or

$$x(t) = Ae^{\alpha t} \sin(\beta t + \phi),$$

where $A = \sqrt{C_1^2 + C_2^2}$ and $\tan \phi = C_1/C_2$.

- The factor $Ae^{\alpha t} = Ae^{-(b/2m)t}$, called the **damping factor**, will approach zero as $t \to \infty$.

- **Critically Damped Motion ($b^2 = 4km$)**

When $b^2 = 4km$, the general solution becomes

$$x(t) = (C_1 + C_2 t)e^{-(b/2m)t}.$$

Since $x(t)$ dies off to zero as $t \to \infty$, $x(t)$ does not oscillate.

- This special case when $b^2 = 4mk$ is called **critically damped motion**, since if b were any smaller, oscillation would occur.

- **Overdamped Motion ($b^2 > 4km$)**

When $b^2 > 4km$, the general solution is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Since both r_1 and r_2 are negative, $x(t) \to 0$ as $t \to \infty$.

- This case is called **overdamped** motion.

Example 8.2 Assume that the motion of a spring-mass system with damping is governed by

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + 25x = 0; \qquad x(0) = 1, \quad x'(0) = 0.$$

Find the equation of motion and sketch its graph for the three cases when $b = 8,\ 10$, and 12.

Forced Vibrations

Consider the vibrations of a spring-mass system when an external force is applied:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos \gamma t, \qquad (0.25)$$

where $F_0,\ \gamma$ are nonnegative constants (and $0 < b^2 < 4km$).

- A particular solution of (0.25) is

$$x_p(t) = \frac{F_0}{(k - m\gamma^2)^2 + b^2\gamma^2}\left\{(k - m\gamma^2) \cos \gamma t + b\gamma \sin \gamma t\right\}$$

or

$$x_p(t) = \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} \sin(\gamma t + \theta),$$

where $\tan \theta = (k - m\gamma^2)/(b\gamma)$.

Hence every solution of (0.25) must be of the form

$$x(t) = x_h(t) + x_p(t) = x_h(t) + \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} \sin(\gamma t + \theta),$$

where $x_h(t)$ is a solution of the homogeneous equation.

Thus, the general solution to (0.25) in the case $0 < b^2 < 4km$ is

$$\begin{aligned} x(t) = {} & Ae^{-(b/2m)t} \sin\left(\frac{\sqrt{4km - b^2}}{2m} t + \phi\right) \\ & + \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} \sin(\gamma t + \theta). \end{aligned}$$

- x_h is called a **transient** solution and x_p the **steady-state** solution.

Example 8.3 A 10-kg mass is attached to a spring hanging from the ceiling. This causes the spring to stretch 2 m on coming to rest at equilibrium. At time $t = 0$, an external force $f(t) = 20\cos 4t$ N is applied to the system. The damping constant for the system is 3 N-sec/m. Determine the steady state solution for the system.

- **Forced Free Vibrations**

Consider the undamped system ($b = 0$) with periodic forcing term $F_0 \cos \gamma t$:

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos \gamma t. \tag{0.26}$$

- In the case $\gamma \neq \omega = \sqrt{k/m}$,

$$x(t) = A\sin(\omega t + \phi) + \frac{F_0}{k - m\gamma^2}\sin(\gamma t + \theta)$$

When $\gamma = \omega$ the general solution of (0.26) is

$$x(t) = A\sin(\omega t + \phi) + \frac{F_0}{2m\omega} t \sin \omega t.$$

The first term is a periodic function while the second term $x_p(t)$ oscillates between $\pm(F_0 t)/(2m\omega)$. Hence as $t \to \infty$, the maximum magnitude of $x(t)$ approaches ∞.

If the damping constant b is very small, the system is subject to large oscillations when the forcing function has a frequency near the resonance frequency for the system.

Consider an electromotive force, resistor, inductor, and capacitor in series. These circuits are called **RLC series circuits**.

- The current I (amperes) is a function of time t.

- The resistance R (ohms), the capacitance C (farads), and the inductance L (henrys) are all positive and constants.

- The impressed voltage E (volts) is a given function of time.

- The relation between total charge Q (coulombs) on the capacitor at time t and current I is

$$I = \frac{dQ}{dt}.$$

- Two physical principles governing RLC series circuits are **Kirchhoff's laws**:
  (I) *The current I passing through each of the elements (resistor, inductor, capacitor, or electromotive force) in the series circuit must be the same.*
  (II) *The algebraic sum of the instantaneous change in potential (voltage drop) around a closed circuit must be zero.*

- The voltage drop across each element of the circuit:
  (i) *The voltage drop across a resistance of R ohms equals RI* (Ohm's law).
  (ii) *The voltage drop across an inductance of L henrys equal $L(dI/dt)$.*
  (iii) *The voltage drop across a capacitance of C farads equals Q/C.*

Kirchhoff's second law gives

$$E_L + E_R + E_C = E(t)$$

or

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t). \tag{0.27}$$

Since $I(t) = dQ/dt$, we see that

$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)}$$

The initial conditions are

$$Q(t_0) = Q_0, \qquad Q'(t_0) = I(t_0) = I_0.$$

If we differentiate (0.27) with respect to t and substitute I for dQ/dt, we obtain

$$\boxed{L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}}$$

The initial conditions are

$$I(t_0) = I_0, \qquad I'(t_0) = I_0'.$$

From (0.27) it follows that

$$I_0' = \frac{E(t_0) - RI_0 - (1/C)Q_0}{L}.$$

Example 8.4 An RLC series circuit has an electromotive force given by $E(t) = \sin 100t$ volts, a resistor of 0.02ohms, an inductor of 0.001 henrys, and a capacitor of 2 farads. If the initial current and the initial change on the capacitor are both zero, determine the current in the circuit for $t > 0$.

- In this example,
  - ◇ $I_h(t)$ is a **transient current** that tends to zero as $t \to \infty$ and
  - ◇ $I_p(t)$ is a **steady-state current** that remains.

If we had chosen to solve for the charge $Q(t)$ in this example, we would also have found that there is a **transient charge** Q_h that dies off and a **steady-state charge** $Q(t)$ that remains.
In general, the steady-state solutions $Q(t)$ and $I(t)$ that arise from the electromotive force $E(t) = E_0 \sin \gamma t$ are

$$Q_p(t) = \frac{-E_0 \cos(\gamma t + \theta)}{\sqrt{(1/C - L\gamma^2)^2 + \gamma^2 R^2}},$$

$$I_p(t) = Q'(t) = \frac{E_0 \sin(\gamma t + \theta)}{\sqrt{R^2 + \left[\gamma L - 1/(\gamma C)\right]^2}},$$

where $\tan \theta = (1/C - L\gamma^2)/(\gamma R)$. The quantity $\sqrt{R^2 + \left[\gamma L - 1/(\gamma C)\right]^2}$ is called the **impedance** of the circuit and is a function of the frequency γ of the electromotive force $E(t)$.

Exercises and Assignments

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142-143	6, 8, 12, 15, 21, 23	9, 11, 16, 17, 22, 24, 27, 28
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158-159	1, 8, 9	2, 4, 10, 11, 13, 18, 20, 28
164-165	7, 9, 11, 16, 18, 20, 22	8, 13, 19, 23, 25, 26, 27, 38, 40
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