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Probability, Homework #2

III/ Additive Rules:

1/ a) $P(A^c) = 1 - P(A) = 1 - 0.3 = 0.7$.

b) $P(AB^c) = P(A) - P(AB) = 0.3 - 0.1 = 0.2$ since $A = (A \cap B) \cup (A \cap B^c)$ and $(A \cap B) \cap (A \cap B^c) = \emptyset$.

c) $P(A^cB) = P(B) - P(AB) = 0.2 - 0.1 = 0.1$ since $B = (B \cap A) \cup (B \cap A^c)$ and $(B \cap A) \cap (B \cap A^c) = \emptyset$.

d) By the Inclusion - Exclusion rule, $P[(A \cup B)^c] = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(AB))$

$$= 1 - (0.3 + 0.2 - 0.1) = 0.6.$$

e) By De Morgan's law, $P(A \cup B^c) = 1 - P[(A \cup B^c)^c] = 1 - P(A^cB) = 1 - 0.1 = 0.9$.

2/ Since A and B are mutually exclusive events, by definition, $A \cap B = \emptyset$.

a) By the Inclusion - Exclusion rule, $P(A \cup B) = P(A) + P(B) - P(AB) = 0.3 + 0.5 - 0 = 0.8$.

b) By exercise 1b, $P(AB^c) = P(A) - P(AB) = 0.3 - 0 = 0.3$.

c) $P(AB) = P(\emptyset) = 0$.

3/ Consider the following events:

A: the person is rich $\Rightarrow A^c$: the person is not rich.

B: the person is famous $\Rightarrow B^c$: the person is not famous.

From the given information, $P(A) = 0.1$, $P(B) = 0.05$, $P(AB) = 0.03$.

a) $P(A^c) = 1 - P(A) = 1 - 0.1 = 0.9$.

b) $P(AB^c) = P(A) - P(AB) = 0.1 - 0.03 = 0.07$, by exercise 1b.

c) $P(A \cup B) = P(A) + P(B) - P(AB) = 0.1 + 0.05 - 0.03 = 0.12$, by the Inclusion - Exclusion rule.

4/

a) By Exercise 1c, $P(B) = P(BA) + P(BA^c) \geq P(BA)$, since $P(BA^c) \geq 0$.

Thus if $A \subset B$ then $A \cap B = A$ and hence $P(B) \geq P(BA) = P(A)$.

Therefore the given statement is true.

b) By the Inclusion - Exclusion rule, $P(A \cup B) = P(A) + P(B) - P(AB) \leq P(A) + P(B)$, since $P(AB) \geq 0$.

Therefore the given statement is true.

c) The given statement is false. Consider the following counterexample:

Let any event A satisfying $P(A) = \frac{1}{2}$ and $B = A$, then $P(A \cap B) = P(A) = \frac{1}{2} > \frac{1}{4} = P(A)\, P(B)$.

d) Exactly one of A, B occurs $=$ (A occurs but B does not) $\cup$ (B occurs but A does not).

By Exercise 1, $P(AB^c \cup BA^c) = P(AB^c) + P(BA^c) = P(A) + P(B) - 2P(AB)$, since $AB^c \cap BA^c = \emptyset$.

Thus the given statement is true.

5/ Consider the following events:

A: the fund had low 1-year return $\rightarrow A^c$: the fund had high 1-year return.

B: the fund had low 5-year return $\rightarrow B^c$: the fund had high 5-year return.

a) From the given information, $P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{9}{30} = \frac{3}{10}$, $P(B^c) = \frac{|B^c|}{|\Omega|} = \frac{7}{30}$.

b) From the given information, $P(A^c B^c) = \frac{|A^c B^c|}{|\Omega|} = \frac{5}{30} = \frac{1}{6}$.

c) By De Morgan's Law and the Inclusion - Exclusion rule,

$P[(A^c \cup B^c)^c] = 1 - P(A^c \cup B^c) = 1 - (P(A^c) + P(B^c) - P(A^c B^c)) = 1 - \left(\frac{3}{10} + \frac{7}{30} - \frac{1}{6}\right) = \frac{19}{30}$.

6/ Let x be the number of customers purchase an electric oven among the given 6.

Then $0 \leq x \leq 6$. Consider the following events:

$\{x \leq 2\}$: at most 2 of them purchase the electric oven.

$\{x \geq 3\}$: at least 3 of them purchase the electric oven.

$\{x = 6\}$: all 6 customers purchase the electric oven.

$\{x = 0\}$: all 6 customers purchase the gas oven.

$\{1 \leq x \leq 5\}$: at least 1 of each oven type is purchased by these 6 customers.

a) From the given information, $P(\{x \leq 2\}) = 0.4$

Since $\{x \leq 2\}^c = \{x \geq 3\}$, $P(\{x \geq 3\}) = 1 - P(\{x \leq 2\}) = 1 - 0.4 = 0.6$.

b) From the given information, $P(\{x = 6\}) = 0.007$ and $P(\{x = 0\}) = 0.104$.

Since $\{1 \leq x \leq 5\}^c = \{x = 0\} \cup \{x = 6\}$ and $\{x = 0\} \cap \{x = 6\} = \emptyset$,

$$P(\{1 \leq x \leq 5\}) = 1 - P(\{x=0\} \cup \{x=6\}) = 1 - \left(P(\{x=0\}) + P(\{x=6\})\right) = 1 - (0.007 + 0.104)$$

$$= 1 - 0.111 = 0.889.$$

7/

a) By the Inclusion - Exclusion rule, $P(AB) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$, since $P(A \cup B) \leq 1$

b) We will prove by induction that for any n events $A_1, A_2, \ldots, A_n$, $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$.

- For $n = 2$: $P(A_1 A_2) \geq P(A_1) + P(A_2) - 1$, as proved in part a).
- Assume that $P\left(\bigcap_{i=1}^{k} A_i\right) \geq \sum_{i=1}^{k} P(A_i) - (k-1)$. Let $\bigcap_{i=1}^{k} A_i = B$.

By part a), $P(B \cap A_{k+1}) \geq P(B) + P(A_{k+1}) - 1 = P\left(\bigcap_{i=1}^{k} A_i\right) + P(A_{k+1}) - 1$

$$\geq \sum_{i=1}^{k} P(A_i) - (k-1) + P(A_{k+1}) - 1 = \sum_{i=1}^{k+1} P(A_i) - k.$$

By mathematical induction, $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$, for any n events $A_1, \ldots, A_n$, as desired.

IV/ Conditional probability - Multiplication Rule:

1/ Consider the following events:

A: the disk has low scratch resistance $\Rightarrow A^c$: the disk has high scratch resistance.

B: the disk has low shock resistance $\Rightarrow B^c$: the disk has high shock resistance.

a) From the given information, $P(A^c B^c) = \frac{70}{100} = 0.7$.

b) From the given information, $P(B^c) = \frac{70+16}{100} = 0.86$. Hence,

$$P(A^c | B^c) = \frac{P(A^c B^c)}{P(B^c)} = \frac{0.7}{0.86} = \frac{35}{43}.$$

2/ Consider the following events:

A: a person is a Republican $\rightarrow A^c$: a person is a Democrat.

B: a person votes for Republican $\rightarrow B^c$: a person votes for Democrat.

From the given information, $P(A) = \frac{600}{1000} = 0.6$, $P(AB^c) = \frac{60}{1000} = 0.06$, $P(A^c B) = \frac{50}{1000} = 0.05$.

Therefore, $P(A^c | B) = \frac{P(A^c B)}{P(B)} = \frac{P(A^c B)}{P(A^c B) + P(AB)} = \frac{P(A^c B)}{P(A^c B) + P(A) - P(AB^c)}$

$$= \frac{0.05}{0.05 + 0.6 - 0.06} = \frac{5}{59}.$$

3/ $\Omega = \{(X, Y) : X, Y \in \mathbb{N} \cap [1, 6]\}$

$A = \{(X,5) : X \in \mathbb{N} \cap [1,4]\} \cup \{(5,Y) : Y \in \mathbb{N} \cap [1,4]\} \cup \{(5,5)\}.$

$B = \{(X,3) : X \in \mathbb{N} \cap [4,6]\} \cup \{(3,Y) : Y \in \mathbb{N} \cap [4,6]\} \cup \{(3,3)\}.$

$\Rightarrow P(A) = \frac{|A|}{|\Omega|} = \frac{4+4+1}{6 \times 6} = \frac{1}{4}$; $P(B) = \frac{|B|}{|\Omega|} = \frac{3+3+1}{6 \times 6} = \frac{7}{36}$; $A \cap B = \{(3,5); (5,3)\}$

$\Rightarrow P(AB) = \frac{|A \cap B|}{|\Omega|} = \frac{2}{6 \times 6} = \frac{1}{18} \Rightarrow P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/18}{7/36} = \frac{2}{7}.$

4/ From the given information, we have:

a) $P(O) = 0.38 + 0.06 = 0.44.$

b) $P(Rh-) = 0.06 + 0.02 + 0.01 + 0.06 = 0.15.$

c) $P(Rh- \mid O) = \frac{P(ORh-)}{P(O)} = \frac{0.06}{0.44} = \frac{3}{22}.$

d) $P(B \mid Rh+) = \frac{P(BRh+)}{P(Rh+)} = \frac{0.09}{0.34 + 0.09 + 0.04 + 0.38} = \frac{9}{85}.$

e) $[P(Rh-)]^2 = 0.15^2 = 0.0225.$

f) $[P(AB)]^2 = (0.04 + 0.01)^2 = 0.0025.$