Nguyen Minh Quan - MAMAIU19036

Random Processes, Final Exam.

1/ $I_1 = \int_0^1 e^s dB_s$, so the integrand $\delta_s = e^s$ is deterministic.

Therefore, I_1 is normally distributed with mean 0 and

$$Var(I_1) = \int_0^1 \delta_s^2 ds = \int_0^1 e^{2s} ds = \frac{e^2-1}{2}$$

thus $I_1 \sim \mathcal{N}(0, (e^2-1)/2)$.

Nguyen Minh Quan - MAMAIU19036

Random Processes, Final Exam.

2/ Let $f(t, x) = \frac{x}{t^2+1}$, then

$f_t = \frac{-2xt}{(t^2+1)^2}$, $f_x = \frac{1}{t^2+1}$, $f_{xx} = 0$ and thus by Ito-Doeblin formula,

$$d\left(\frac{B_t}{t^2+1}\right) = df(t, B_t) = f_t(t, B_t)dt + f_x(t, B_t)dB_t + \frac{f_{xx}(t, B_t)dt}{2}$$

$$= \frac{-2tB_t}{(t^2+1)^2}dt + \frac{1}{t^2+1}dB_t.$$

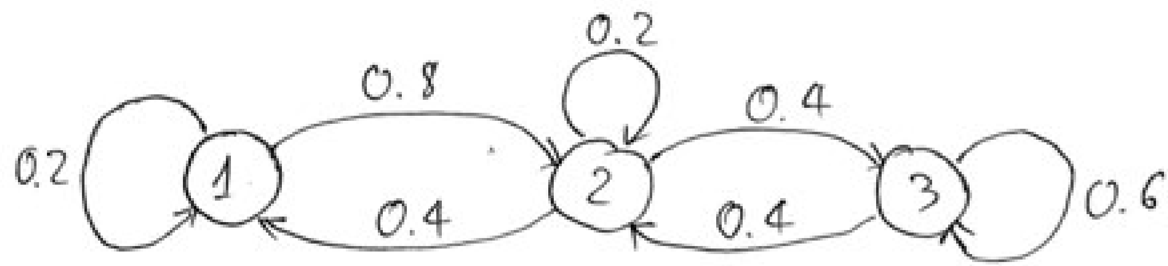
Nguyen Minh Quan - MAMAIU19036

Random Processes, Final Exam.

3/ The transition matrix is

From \ To	State 1	State 2	State 3
State 1	0.2	0.8	0
State 2	0.4	0.2	0.4
State 3	0	0.4	0.6

so the model graph is given below.



From the graph, we can see that there are no absorbing state, and we can move with positive probability between 1 and 2, and between 2 and 3 in either directions. Therefore, all 3 states of the given Markov chain are recurrent states, and they form a recurrent class.

Nguyen Minh Quan - MAMAIU19036

Random Processes, Final Exam.

5/ $dX_t + \frac{2}{\sqrt{t}} X_t dt = dB_t$, $X_0 = 1$.

Let $a(t) = \int \frac{2}{\sqrt{t}} dt = 4\sqrt{t}$ and $Y_t = e^{a(t)} X_t = e^{4\sqrt{t}} X_t$ (so $Y_0 = X_0 = 1$).

Let $f(t, x) = e^{4\sqrt{t}} x$, then

$f_t = \frac{2xe^{4\sqrt{t}}}{\sqrt{t}}$, $f_x = e^{4\sqrt{t}}$, $f_{xx} = 0$ and by Ito - Doeblin formula,

$$dY_t = df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{f_{xx}(t, X_t) \cdot 1^2 dt}{2}$$

$$= \frac{2X_t e^{4\sqrt{t}}}{\sqrt{t}} dt + e^{4\sqrt{t}} dX_t = \frac{2X_t e^{4\sqrt{t}}}{\sqrt{t}} dt + e^{4\sqrt{t}}\left(dB_t - \frac{2X_t}{\sqrt{t}} dt\right)$$

so $dY_t = e^{4\sqrt{t}} dB_t$, integrating both sides gives

$$Y_T - 1 = Y_T - Y_0 = \int_0^T e^{4\sqrt{t}} dB_t \text{ or } Y_T = 1 + I_T,\ I_T = \int_0^T e^{4\sqrt{t}} dB_t.$$

The integrand $\delta_t = e^{4\sqrt{t}}$ is deterministic, so I_T is normally distributed with

$$\mathbb{E}(I_T) = 0,\ Var(I_T) = \int_0^T \delta_t^2 dt = \int_0^T e^{8\sqrt{t}} dt = \frac{(8\sqrt{T}-1)e^{8\sqrt{T}} + 1}{32}$$

thus $I_T \sim \mathcal{N}\left(0, \frac{(8\sqrt{T}-1)e^{8\sqrt{T}} + 1}{32}\right) \Rightarrow Y_T \sim \mathcal{N}\left(1, \frac{(8\sqrt{T}-1)e^{8\sqrt{T}} + 1}{32}\right)$

$$\Rightarrow X_T \sim \mathcal{N}\left(\frac{1}{e^{4\sqrt{T}}}, \frac{(8\sqrt{T}-1)e^{8\sqrt{T}} + 1}{32e^{8\sqrt{T}}}\right) \Rightarrow X_1 \sim \mathcal{N}\left(\frac{1}{e^4}, \frac{7e^8 + 1}{32e^8}\right)$$

Also, the solution to the given SDE is

$$X_T = \frac{Y_T}{e^{4\sqrt{T}}} = \frac{1}{e^{4\sqrt{T}}} + \frac{1}{e^{4\sqrt{T}}} \int_0^T e^{4\sqrt{t}} dB_t.$$

Nguyen Minh Quan - MAMAIU19036

Random Processes, Final Exam.

6/ $dX_t = dt + 3\,dB_t$, $X_0 = 1$. Integrating both sides gives

$$X_T - 1 = X_T - X_0 = \int_0^T 1\,dt + \int_0^T 3\,dB_t = T + 3B_T.$$

$$\Rightarrow X_t = 1 + t + 3B_t, \quad \forall t \geq 0.$$ Hence

$$\mathbb{P}(X_2 > 1) = \mathbb{P}(3 + 3B_2 > 1) = \mathbb{P}(B_2 > -2/3)$$

$$= \mathbb{P}\left(Z > \frac{-2/3}{\sqrt{2}}\right) \approx 0.681 \quad (\text{here } Z \sim \mathcal{N}(0,1)).$$

Nguyen Minh Quan - MAMAIU19036

Random Processes, Final Exam.

7/ $\mathbb{P}(B_2 + B_3 > 0 \mid B_1 = 2) = \mathbb{P}((B_3 - B_2) + 2(B_2 - B_1) > -4 \mid B_1 = 2)$

$= \mathbb{P}((B_3 - B_2) + 2(B_2 - B_1) > -4)$ (since $B_3 - B_2$, $B_2 - B_1$ are independent of B_1)

$= \alpha$. Let $X_1 = B_3 - B_2$, $X_2 = B_2 - B_1$ and $X = X_1 + 2X_2$, then

$$\begin{cases} X_1 \sim X_2 \sim B_1 \sim \mathcal{N}(0,1) \\ X_1, X_2 \text{ are independent} \end{cases} \Rightarrow X \sim \mathcal{N}(0,5). \text{ Thus}$$

$\mathbb{P}(B_2 + B_3 > 0 \mid B_1 = 2) = \alpha = \mathbb{P}(X > -4) = \mathbb{P}\left(Z > \frac{-4}{\sqrt{5}}\right) \simeq 0.963$ (here $Z \sim \mathcal{N}(0,1)$)

Nguyen Minh Quan - MAMAIU19036

Random Processes, Final Exam.

8/ $dX_t = -X_t\,dt + dB_t$, $X_0 = 1$.

Let $a(t) = \int 1\,dt = t$ and $Y_t = e^{a(t)} X_t = e^t X_t$ (so $Y_0 = X_0 = 1$).

Let $f(t,x) = e^t x$, then

$f_t = e^t x$, $f_x = e^t$, $f_{xx} = 0$ so by the Ito - Doeblin formula,

$$dY_t = df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{f_{xx}(t, X_t)\cdot 1^2 dt}{2}$$

$$= e^t X_t\,dt + e^t dX_t = e^t X_t\,dt + e^t(dB_t - X_t\,dt) = e^t dB_t,$$

so integrating both sides gives $Y_T - 1 = Y_T - Y_0 = \int_0^T e^t dB_t$.

Hence the solution is: $X_T = \frac{Y_T}{e^T} = \frac{1}{e^T} + \frac{1}{e^T}\cdot\int_0^T e^t dB_t$.

Nguyen Minh Quan - MAMAIU19036

Random Processes, Final Exam.

9/ Let $f(t, x) = x^3$, then

$f_t = 0$, $f_x = 3x^2$, $f_{xx} = 6x$ and thus by the Ito-Doeblin formula,

$$B_T^3 = f(t, B_t) = f(0, B_0) + \int_0^T f_t(t, B_t)\,dt + \int_0^T f_x(t, B_t)\,dB_t + \frac{1}{2}\int_0^T f_{xx}(t, B_t)\,dt$$

so $B_t^3 = \int_0^t 3B_s^2\,dB_s + \frac{1}{2}\int_0^t 6B_s\,ds = 3\int_0^t B_s^2\,dB_s + 3\int_0^t B_s\,ds$.