# Itô-Deoblin formula

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# Outline

- Texbook: Section 4.4 Shreve II
- Content: a rule to differentiate $f(B_t)$
  - Itô formula for Brownian motion
  - Itô process and Itô formula for Itô process

# History of Itô-Doeblin formula

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- When it was opened, the document was found to contain a construction of the stochastic integral slightly different from Ito and a clear statement of the change-of-variable formula.
- Because of this remarkable development, the change of variable formula is called Itô-Doeblin formula.

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How to differentiate $f(B_t)$ when $B_t$ is non differentiable and has non zero quadratic variation?

# Itô - Doeblin formula for Brownian motion

Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$ and $f_{xx}(t,x)$ are defined and continuous and $(B_t)_{t\geq 0}$ be a Brownian motion. Then for every $T>0$,

$$
\begin{aligned}
f(T,B_T) =& f(0,B_0) + \underbrace{\int_0^T f_t(t,B_t)dt}_{\text{Riemann integral}} + \underbrace{\int_0^T f_x(t,B_t)dB_t}_{\text{Itô integral}} \\
&+ \frac{1}{2}\underbrace{\int_0^T f_{xx}(t,B_t)dt}_{\text{Riemann integral}}
\end{aligned}
$$

$dB_t$: the change in $B_t$ when $t$ change a little bit $dt$

## Proof for $f(x) = \frac{1}{2}x^2$

- Taylor's formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

  for $x \approx x_0$
- Remark that $f'(x) = x$, $f''(x) = 1$
- Fix $T > 0$ and let $\Pi = [t_0, t_1, ..., t_n]$ be a partition of $[0, T]$ then

$$f(B_T) - f(B_0) = \sum_{i=0}^{n-1} f(B_{t_{i+1}}) - f(B_{t_i}))$$

$$= \sum_{i=0}^{n-1} f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) + \frac{1}{2}\sum_{i=0}^{n-1} f''(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2$$

$$= \sum_{i=0}^{n-1} B_{t_i}(B_{t_{i+1}} - B_{t_i}) + \frac{1}{2}\sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2$$

Let $\|\Pi\| \to 0$, then the first time converges to an Itô integral

$$\sum_{i=0}^{n-1} B_{t_i}(B_{t_{i+1}} - B_{t_i}) \to \int_0^T B_t dB_t$$

while the second term

$$\frac{1}{2}\sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2 \to \frac{1}{2}\langle B\rangle(T) = \frac{1}{2}T$$

Hence

$$f(B_T) - f(B_0) = \int_0^T B_t dB_t + \frac{1}{2}T$$

The Itô - Doeblin holds for $f(x) = \frac{1}{2}x^2$

$$B_T^2 - B_0^2 = \int_0^T B_t dB_t + \frac{1}{2}T$$

# Proof for general $f(t, x)$

Taylor's expansion for $f(t, x)$

$$\begin{aligned}
&f(t_{i+1}, x_{i+1}) - f(t_i, x_i) \\
&= f_x(t_i, x_i)(x_{i+1} - x_i) + f_t(t_i, x_i)(t_{i+1} - t_i) \\
&+ \frac{1}{2} f_{xx}(t_i, x_i)(x_{i+1} - x_i)^2 + f_{tx}(t_i, x_i)(t_{i+1} - t_i)(x_{i+1} - x_i) \\
&+ \frac{1}{2} f_{tt}(t_i, x_i)(t_{i+1} - t_i)^2 + \text{ higher - order terms}
\end{aligned}$$

$$f(T,B_T)-f(0,B_0)=\sum_{i=0}^{n-1}\big(f(t_{i+1},B_{t_{i+1}})-f(t_i,B_{t_i})\big)$$

$$=\sum_{i=0}^{n-1} f_x(t_i,B_{t_i})(B_{t_{i+1}}-B_{t_i})+\sum_{i=0}^{n-1} f_t(t_i,B_{t_i})(t_{i+1}-t_i)$$

$$+\frac{1}{2}\sum_{i=0}^{n-1} f_{xx}(t_i,B_{t_i})(B_{t_{i+1}}-B_{t_i})^2$$

$$+\sum_{i=0}^{n-1} f_{tx}(t_i,B_{t_i})(t_{i+1}-t_i)(B_{t_{i+1}}-B_{t_i})$$

$$+\sum_{i=0}^{n-1}\frac{1}{2}f_{tt}(t_i,B_{t_i})(t_{i+1}-t_i)^2+\text{ higher - order terms}$$

Let $\|\Pi\| \to 0$,

- The first term converges to an Itô integral

$$\sum_{i=0}^{n-1} f_x(t_i, B_{t_i})(B_{t_{i+1}} - B_{t_i}) \to \int_0^T f_x(u, B_u) dB_u$$

- The second term converges to a Riemann integral

$$\sum_{i=0}^{n-1} f_t(t_i, B_{t_i})(t_{i+1} - t_i) \to \int_0^T f_t(u, B_u) du$$

- Approximate $(B_{t_{i+1}} - B_{t_i})^2$ by $t_{i+1} - t_i$ in the 3rd term to get

$$\begin{aligned} &\frac{1}{2}\sum_{i=0}^{n-1} f_{xx}(t_i, B_{t_i})(B_{t_{i+1}} - B_{t_i})^2 \\ &\approx \frac{1}{2}\sum_{i=0}^{n-1} f_{xx}(t_i, B_{t_i})(t_{i+1} - t_i) \to \frac{1}{2}\int_0^T f_{xx}(u, B_u) du \end{aligned}$$

- The 4th term contributes to 0

$$
\begin{aligned}
&\left|\sum_{i=0}^{n-1} f_{tx}(t_i, B_{t_i})(t_{i+1} - t_i)(B_{t_{i+1}} - B_{t_i})\right| \\
&\leq \sum_{i=0}^{n-1} \left|f_{tx}(t_i, B_{t_i})(t_{i+1} - t_i)(B_{t_{i+1}} - B_{t_i})\right| \\
&\leq \max_{0 \leq i \leq n-1} |B_{t_{i+1}} - B_{t_i}| \sum_{i=0}^{n-1} |f_{tx}(t_i, B_{t_i})|(t_{i+1} - t_i) \\
&= \max_{0 \leq i \leq n-1} |B_{t_{i+1}} - B_{t_i}| \int_0^T |f_{tx}(u, B_u)| du \\
&\to 0 \int_0^T |f_{tx}(u, B_u)| du = 0
\end{aligned}
$$

- The 5th term contributes to 0

$$
\begin{aligned}
&\left|\sum_{i=0}^{n-1} f_{tt}(t_i, B_{t_i})(t_{i+1}-t_i)^2\right| \\
&\leq \sum_{i=0}^{n-1} \left|f_{tt}(t_i, B_{t_i})(t_{i+1}-t_i)^2\right| \\
&\leq \max_{0\leq i\leq n-1} |t_{i+1}-t_i| \sum_{i=0}^{n-1} |f_{tt}(t_i, B_{t_i})|(t_{i+1}-t_i) \\
&= \max_{0\leq i\leq n-1} |t_{i+1}-t_i| \int_0^T |f_{tt}(u, B_u)|du \\
&\to 0 \int_0^T |f_{tt}(u, B_u)|du = 0
\end{aligned}
$$

- The higher-order terms likewise contribute zero to the final answer

# Differential form of Itô- Doeblin formula

Taylor's expansion

$$df(t,B_t) = f_t(t,B_t)dt + f_x(t,B_t)dB_t + \frac{1}{2}f_{xx}(t,B_t)dB_t dB_t$$
$$+ f_{tx}((t,B_t)dt dB_t + \frac{1}{2}f_{tt}(t,B_t)dt dt$$

but $dB_t dB_t = dt$, $dt dt = 0$ and $dt dB_t = 0$. So

$$\boxed{df(t,B_t) = f_t(t,B_t)dt + f_x(t,B_t)dB_t + \frac{1}{2}f_{xx}(t,B_t)dt}$$

## Example

Find

$$dB_t^2$$

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## Solution

- *Choose $f$ such that $f(t, B_t) = B_t^2$*

## Example

Find

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## Solution

- *Choose* $f$ *such that* $f(t, B_t) = B_t^2$
- $f(t, x) = x^2$

## Example

Find

$$dB_t^2$$

## Solution

- *Choose $f$ such that $f(t, B_t) = B_t^2$*
- $f(t, x) = x^2$
- $f_t = 0,\ f_x = 2x,\ f_{xx} = 2$

## Example

Find

$$dB_t^2$$

## Solution

- *Choose* $f$ *such that* $f(t, B_t) = B_t^2$
- $f(t, x) = x^2$
- $f_t = 0, \ f_x = 2x, \ f_{xx} = 2$
- $$\begin{aligned} dB_t^2 = df(t, B_t) &= f_t(t, B_t)dt + f_x(t, B_t)dB_t + \frac{1}{2}f_{xx}(t, B_t)dt \\ &= 0dt + 2B_t dB_t + \frac{1}{2}2dt \\ &= 2B_t dB_t + dt \end{aligned}$$
- *Remark that integral form is* $B_T^2 = B_0^2 + 2\int_0^T B_t dB_t + \int_0^T dt$ *or* $\int_0^T B_t dB_t = \frac{1}{2}B_T^2 + \frac{1}{2}T$

# Exercise

Compute

1. $d(tB_t)$
2. $d(B_t^3)$
3. $d\sin(B_t)$
4. $d\left(e^{tB_t}\right)$

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# Itô processes

Let $(B_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$ be an associated filtration. An Itô process is a stochastic process of the integral form:

$$X_T = X_0 + \int_0^T \mu_t dt + \int_0^T \sigma_t dB_t$$

or the differential form:

$$dX_t = \underbrace{\mu_t}_{\text{drift coefficient}} dt + \underbrace{\sigma_t}_{\text{diffusion coefficient}} dB_t.$$

Here $\mu_t$ (drift term) and $\sigma_t$ (diffusion term) are adapted process of the filtration $(\mathcal{F}_t)_{t\geq 0}$.

# Example

- Consider $X_t = B_t^2$.
- Apply Itô formula for $f(X_t)$ with $f(x) = x^2$, we have

$$dX_t = 2B_t dB_t + dt$$

- $(B_t^2)_{t \geq 0}$ is a Itô process with drift coefficient $\mu = 1$ and diffusion coefficient $\sigma = 2B_t$

# Example

- Consider $X_t = \ln B_t$.
- Apply Itô formula for $f(X_t)$ with $f(x) = \ln x$, we have

$$dX_t = \frac{1}{B_t} dB_t - \frac{1}{B_t^2} dt$$

- $(\ln B_t)_{t \geq 0}$ is a Itô process with drift coefficient $\mu = -\frac{1}{B_t^2}$ and diffusion coefficient $\sigma = \frac{1}{B_t}$

To understand the volatilities of Itô process, need to determine the rate at which they accumulate quadratic variation

## Quadratic variation of Itô process

$$\langle X \rangle(T) = \int_0^T \sigma_u^2 du$$

at each time $t$, the process $X$ is accumulating quadratic variation at rate $\sigma_t^2$ per unit time

# Easy way to remember rate of accumulating quadratic variation

- Write in differential form

$$dX_t = \mu_t dt + \sigma_t dB_t$$

- $$dX_t dX_t = \mu_t^2 dt dt + 2\mu_t \sigma_t dB_t dt + \sigma_t^2 dB_t dB_t$$

- Use $dtdt = dtdB_t = 0$ and $dB_t dB_t = dt$ to get

$$dX_t dX_t = \sigma_t^2 dt$$

# Integral with respect to an Itô process

Let $(X_t)_{t\geq 0}$ be an Itô process given by

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s$$

and $(\Gamma_t)_{t\geq 0}$ be an adapted process.
Then

$$\int_0^T \Gamma_s dX_s = \int_0^T \Gamma_s \mu_s ds + \int_0^T \Gamma_s \sigma_s dB_s$$

is an integral with respect to the Itô process $(X_t)_{t\geq 0}$

# Itô-Doeblin formula for an Itô process

Let $(X_t)_{t\geq 0}$ be an Itô process

$$X_t = X_0 + \int_0^t \sigma_u dB_u + \int_0^t \mu_u du$$

and let $f(t,x)$ be a function for which partial derivatives $f_t(t,x)$, $f_x(t,x)$ and $f_{xx}(t,x)$ are defined and continuous. Then for every $T \geq 0$

$$\begin{aligned} f(T, X_T) =& f(0, X_0) + \int_0^T f_t(t, X_t)dt + \int_0^T f_x(t, X_t)dX_t \\ &+ \frac{1}{2}\int_0^T f_{xx}(t, X_t)d\langle X\rangle(t) \\ &= f(0, X_0) + \int_0^T f_t(t, X_t)dt + \int_0^T f_x(t, X_t)\sigma_t dB_t \\ &+ \int_0^T f_x(t, X_t)\mu_t dt + \frac{1}{2}\int_0^T f_{xx}(t, X_t)\sigma_t^2 dt \end{aligned}$$

## Differential form of Itô-Doeblin formula for an Itô process

$$df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)dX_t dX_t$$

with $dX_t dX_t = \sigma_t^2 dt$

# Example

Find $d(tX_t^2)$ if

$$dX_t = 2dt + dB_t.$$

# Example

Find $d(tX_t^2)$ if

$$dX_t = 2dt + dB_t.$$

## Solution

- Find $f(t, x)$ such that $f(t, X_t) = tX_t^2$

# Example

Find $d(tX_t^2)$ if

$$dX_t = 2dt + dB_t.$$

## Solution

- Find $f(t,x)$ such that $f(t,X_t) = tX_t^2$
- $f(t,x) = tx^2$

# Example

Find $d(tX_t^2)$ if

$$dX_t = 2dt + dB_t.$$

## Solution

- Find $f(t,x)$ such that $f(t,X_t) = tX_t^2$
- $f(t,x) = tx^2$
- $f_t = x^2,\ f_x = 2tx,\ f_{xx} = 2t$
- Apply Itô formula for $f(t,X_t)$ we have

$$\begin{aligned} d(tX_t^2) &= f_x(t,X_t)dX_t + f_t(t,X_t)dt + \frac{1}{2}dX_t dX_t \\ &= X_t^2 dX_t + X_t^2 dt + \frac{1}{2}2t dB_t dB_t \\ &= X_t^2(2dt + dB_t) + X_t^2 dt + dt = (3X_t^2 + 1)dt + X_t^2 dB_t \end{aligned}$$

# Example - Generalized Geometric Brownian Motion

- Asset price

$$S_t = S_0 e^{X_t}$$

with

$$X_t = \int_0^t \sigma_s dB_s + \int_0^t (\alpha_s - \frac{1}{2}\sigma_s^2) ds$$

- $dS_t = ?$

# Solution

- Let $f(x) = S_0 e^x$ then $f_t = 0$, $f_x = f_{xx} = S_0 e^x$
- Apply Itô - Doeblin formula

$$
\begin{aligned}
dS_t &= S_0 e^{X_t} dX_t + \frac{1}{2} S_0 e^{X_t} dX_t dX_t \\
&= S_t(\sigma_t dB_t + (\alpha_t - \frac{1}{2}\sigma_t^2)dt) + \frac{1}{2} S_t \sigma_t^2 dt
\end{aligned}
$$

-

$$
dS_t = \sigma_t S_t dB_t + \alpha_t dt
$$

The asset price $S_t$ has instantaneous mean rate of return $\alpha_t$ and volatility $\sigma_t$. Both the instantaneous mean rate of return and the volatility are allowed to be time-varying and random. In the case of time-varying and random $\alpha_t$ , we will call this the instantaneous mean rate of return since it depends on the time (and the sample path) where it is evaluated.

## Example

Let

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Find $dY_t$ if $Y_t = \ln S_t$

## Example

Let

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Find $dY_t$ if $Y_t = \ln S_t$

### Solution

- $Y_t = f(t, S_t) = \ln S_t$

## Example

Let

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Find $dY_t$ if $Y_t = \ln S_t$

### Solution

- $Y_t = f(t, S_t) = \ln S_t$
- $f(t, x) = \ln x$
- $f_t = 0,\ f_x = \frac{1}{x},\ f_{xx} = -\frac{1}{x^2}$

# Example

Let

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Find $dY_t$ if $Y_t = \ln S_t$

## Solution

- $Y_t = f(t, S_t) = \ln S_t$
- $f(t, x) = \ln x$
- $f_t = 0,\ f_x = \frac{1}{x},\ f_{xx} = -\frac{1}{x^2}$
- $$\begin{aligned} dY_t = df(t, S_t) &= f_t(t, S_t)dt + f_x(t, S_t)dS_t + \frac{1}{2} f_{xx}(t, S_t)dS_t dS_t \\ &= 0 + \frac{1}{S_t}(\mu S_t dt + \sigma S_t dB_t) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(\underbrace{\sigma S_t}_{\sigma_t})^2 dt \\ &= \mu dt + \sigma dB_t - \frac{1}{2}\sigma^2 dt = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \end{aligned}$$

Integral form

$$Y_T = Y_0 + \int_0^T \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t = \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B_T$$

And hence

$$S_T = e^{Y_T} = e^{Y_0} e^{\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B_T}$$

Remark that $e^{-\frac{1}{2}\sigma^2 T + \sigma B_T}$ is a martingale and so $\mu$ is the rate of return.

# Practice

Find $d(e^{-t}X_t)$ if

$$dX_t = X_t dt + dB_t$$