CHAPTER 3: CONDITIONAL PROBABILITY

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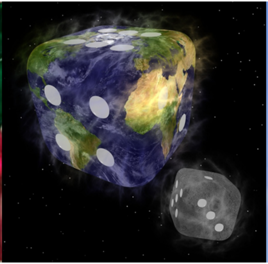



CONTENTS

Conditional probability

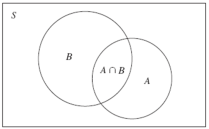
- Unconditional probability refers to the probability of an event regardless of the past or future occurrence of other event.
- Let B be an arbitrary event in a sample space S with $P(B) > 0$. A conditional probability is one where the occurrence of one event affects the probability of the occurrence of another event.
- The conditional probability of A given B is denoted by $P(A|B)$.

Conditional Probability

How to compute conditional probability?

Theorem

If $P(B) > 0$, then $P(A|B) = \frac{P(AB)}{P(B)}$



If B is known to have occurred, then A can occur iff A∩B occurs.

That is,

Let S be a finite equiprobable space with events A and B. Then $P(A|B) = \#$ elements in $A \cap B / \#$ elements in B, or

$P(A|B) = \#$ ways A and B can occur/ $\#$ ways B can occur.

Using conditional probability formula

Example

Rolling a pair of dice. Suppose that we observe that the first die is a 3. Then, given this information, what is the probability that the sum of the 2 dice equals 8?

Solution

Let A be the event that the sum of the 2 dice equals 8, and B be the event that the first die is a 3. The sample space S can be taken to be the following set of 36 outcomes:

$$S = \{(i, j) : i = 1, 2, 3, 4, 5, 6; j = 1, 2, 3, 4, 5, 6\}$$

$AB = \{(3, 5)\}$ and $B = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}$.
Therefore,

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/36}{6/36} = \frac{1}{6}$$

Using conditional probability formula

Example

A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is x/2, for all $0 \leq x \leq 1$. Then, given that the student is still working after .75 hour, what is the conditional probability that the full hour is used?

Solution Let L_x denote the event that the student finishes the exam in less than x hours, $0 \leq x \leq 1$, and let F be the event that the student uses the full hour. We have

$$P(F) = 1 - P(L_1) = 0.5$$

By the conditional probability

$$P(F|\, L_{0.75}^c) = \frac{P(FL_{0.75}^c)}{P(L_{0.75}^c)} = \frac{P(F)}{1 - P(L_{0.75})} = \frac{0.5}{0.625} = 0.8$$

Using conditional probability formula

Example

A total of n balls are sequentially and randomly chosen, without replacement, from an urn containing r red and b blue balls ($n \leq r + b$). Given that k of the n balls are blue, what is the conditional probability that the first ball chosen is blue?

Answer: k/n.

Multiplication Theorem for Conditional Probability

- The multiplication theorem is used to determine the intersection of two events or more (the joint probability of two events or more).

- Theorem: $P(AB) = P(B)\,P(A|B)\,.$

- Theorem: $P(A_1A_2...A_n) =$

$$P(A_1)\,P(A_2|A_1)\,P(A_3|A_1A_2)\,...P(A_n|A_1A_2...A_{n-1})\,.$$

Examples of the Multiplication Theorem

Example

Ms. Van figures that there is a 30 percent chance that her company will set up a branch office in Hanoi. If it does, she is 60 percent certain that she will be made manager of this new operation. What is the probability that Van will be a Hanoi branch office manager?

Answer

If we let B denote the event that the company sets up a branch office in Hanoi and M the event that Van is made the Hanoi manager, then the desired probability is $P(BM)$, which is obtained as follows:

$$P(BM) = P(M|B)P(B) = (0.6)(0.3) = 0.18.$$

Hence, there is an 18% chance that Van will be the Hanoi manager.

Examples of the Multiplication Theorem

Example

Celine is undecided as to whether to take a French course or a chemistry course. She estimates that her probability of receiving an A grade would be $1/2$ in a French course and $2/3$ in a chemistry course. If Celine decides to base her decision on the flip of a fair coin, what is the probability that she gets an A in chemistry?

Answer $\frac{1}{3}$.

Examples of the Multiplication Theorem

Example

Suppose that five good fuses and two defective ones have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and find both of the defective fuses in the first two tests?

Example

An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

Finite Stochastic Processes and Tree Diagrams

- A (finite) sequence of experiments in which each experiment has a finite number of outcomes with given probabilities is called a (finite) stochastic process.
- A convenient way of describing such a process and computing the probability of any event is by a tree diagram.
- A general framework called a tree diagram is used to show the probabilities of various outcome.

Example

An urn contains two black balls and three white balls. Two balls are selected at random from the urn without replacement and the sequence of colors is noted. Find the probability that both balls are black.

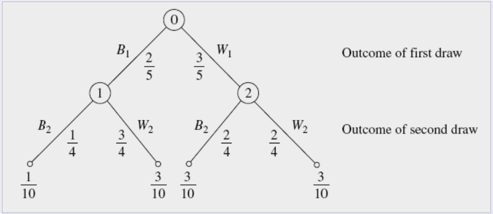
Tree Diagrams

Solution

Let B_1 and B_2 be the events that the outcome is a black ball in the first and second draw, respectively. We have,

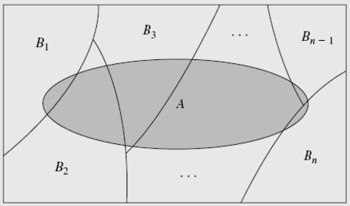
$$P(B_1 \cap B_2) = P(B_1) P(B_2|B_1) = \frac{2}{5}\frac{1}{4} = \frac{1}{10}.$$

Note: In general, the probability of any sequence of colors is obtained by multiplying the probabilities corresponding to the node transitions in the tree.



Total probability

Let $B_1, B_2, ..., B_n$ be mutually exclusive events whose union equals the sample space S as shown in the following figure.



A partition of S into n disjoint sets.

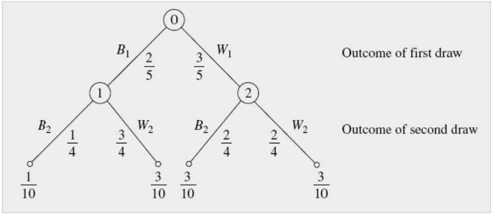
Then, for any event A, we have the theorem on total probability:

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) ... + P(A|B_n)P(B_n)$$

Total probability

Example

An urn contains two black balls and three white balls. Two balls are selected at random from the urn without replacement and the sequence of colors is noted. Find the probability that the second ball is white.



$$P(W_2) = P(W_2|B_1)\,P(B_1) + P(W_2|B_2)\,P(B_2) = \frac{3}{4}\frac{2}{5} + \frac{1}{2}\frac{3}{5} = \frac{3}{5}$$

Total probability

Example

A semiconductor manufacturer which makes microchips implanted into animals has the following data regarding the effect of contaminants on the probability that microchips fail.

Probability of Failure	Level of Contamination
0.1	High
0.01	Medium
0.001	Low

In a particular production run 20% of the microchips have high-level, 30% have medium-level, and 50% have low-level contamination. What is the probability that one of the resulting microchips fails?

Bayes' theorem

Bayes' Theorem

- Special case:

$$P(A|B) = \frac{P(A)\,P(B|A)}{P(B)} = \frac{P(A)\,P(B|A)}{P(A)\,P(B|A) + P(A^c)\,P(B|A^c)}$$

- General case: Suppose the events $A_1, A_2, ..., A_n$ form a partition of a sample space S, and B is an arbitrary event, then:

$$P(A_i|B) = \frac{P(A_i)\,P(B|A_i)}{P(A_1)\,P(B|A_1) + ... + P(A_n)\,P(B|A_n)}$$

Key proof: Use conditional probability and $P(AB) = P(A)P(B|A)$.

Bayes' theorem

Example

Consider two urns. The first contains two white and seven black balls, and the second contains five white and six black balls. We flip a fair coin and then draw a ball from the first urn or the second urn depending on whether the outcome was heads or tails. What is the conditional probability that the outcome of the toss was heads given that a white ball was selected?

Solution

Let W be the event that a white ball is drawn, and let H be the event that the coin comes up heads. The desired probability $P(H|W)$ can be calculated as follows:

$$P(H|W) = \frac{P(HW)}{P(W)} = \frac{P(W|H)\,P(H)}{P(W|H)\,P(H) + P(W|H^c)\,P(H^c)}$$

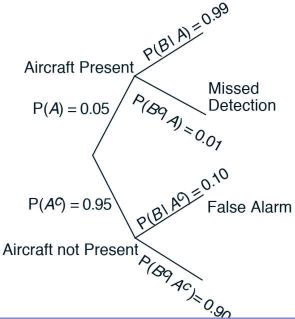
$$= \frac{\frac{2}{9}\frac{1}{2}}{\frac{2}{9}\frac{1}{2} + \frac{5}{11}\frac{1}{2}} = \frac{22}{67}$$

Bayes' theorem

Example

If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05. If the radar registers an aircraft presence, what is the probability that an aircraft is actually present?

Hint: Make a tree diagram. Answer: 0.3426.



Bayes' theorem

Example

At a certain stage of a criminal investigation, the inspector in charge is 60 percent convinced of the guilt of a certain suspect. Suppose, however, that a new piece of evidence which shows that the criminal has a certain characteristic (such as left-handedness, baldness, or brown hair) is uncovered. Given that 20% of the unguilty population possesses this characteristic. Find the probability that **the suspect** is actual guity if the suspect has the characteristic?

Answer Letting G denote the event that the suspect is guilty and C the event that he possesses the characteristic of the criminal, we have $P(G) = 0.6$, $P(C|G) = 1$, $P(C|G^c) = 0.2$.

$$P(G|C) = \frac{P(G)\,P(C|G)}{P(C)} = \frac{1\,(0.6)}{1\,(0.6) + (0.2)\,(0.4)} \simeq 0.882$$

Bayes' theorem

Example

An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability .4, whereas this probability decreases to .2 for a person who is not accident prone. We assume that 30 percent of the population is accident prone.
(a) what is the probability that a new policyholder will have an accident within a year of purchasing a policy?
(b) Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?

Answer (a) Use total probability: 0.26 ;(b) Use Bayes'theorem: $6/13$.

Bayes' theorem

Exercises

A particular hypothetical human disease occurs with a probability of 0.1 in males and with a probability of 0.4 in females.

(a) Assuming that the frequency of males is 0.5 and females 0.5 in a very large population, what is the probability that an individual selected at random from this population will have the disease?

(b) What is the probability that an individual will be male and have the disease?

Hint: The problem is better solved with a tree diagram.

Bayes' theorem

Exercises

A laboratory blood test is 99% effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1% of the healthy persons tested. (That is, if a healthy person is tested, then, with probability 0.01, the test result will imply he or she has the disease.) If 0.5% percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive?

Answer 0.3322

Exercise (Use Bayes' theorem)

A book club classifies members as heavy, medium, or light purchasers, and separate mailings are prepared for each of these groups. Overall, 20% of the members are heavy purchasers, 30% medium, and 50% light. A member is not classified into a group until 18 months after joining the club, but a test is made of the feasibility of using the first 3 months' purchases to classify members. The following percentages are obtained from existing records of individuals classified as heavy, medium, or light purchasers (below Table).
If a member purchases no books in the first 3 months, what is the probability that the member is a light purchaser?

First 3 Months' Purchases	Group (%)		
	Heavy	Medium	Light
0	5	15	60
1	10	30	20
2	30	40	15
3+	55	15	5

Answer $P(light|0) = 0.845$

The odds of an event

Definition: The odds of an event

The odds of an event A are defined by

$$\frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)}$$

That is, the odds of an event A tell how much more likely it is that the event A occurs than it is that it does not occur.

Consider now a hypothesis H that is true with probability P(H), and suppose that new evidence E is introduced. By conditional probability, we have

$$\frac{P(H|E)}{P(H^c|E)} = \frac{P(H)}{P(H^c)} \frac{P(E|H)}{P(E|H^c)}$$

The odds of an event

Example

An urn contains two type A coins and one type B coin. When a type A coin is flipped, it comes up heads with probability $1/4$, whereas when a type B coin is flipped, it comes up heads with probability $3/4$. A coin is randomly chosen from the urn and flipped. Given that the flip landed on heads, what is the probability that it was a type A coin?

Hint:

$$\frac{P\left(A \mid heads\right)}{P\left(B \mid heads\right)} = \frac{P\left(A\right)}{P\left(B\right)} \frac{P\left(heads \mid A\right)}{P\left(heads \mid B\right)} = \frac{2/3}{1/3} \frac{1/4}{3/4} = \frac{2}{3}$$

$$P\left(A \mid heads\right) = 2/5$$

Independent events

- Recall that $P(E|F)$, the conditional probability of E given F, is not generally equal to $P(E)$, the unconditional probability of E.
- In the special cases where $P(E|F)$ does in fact equal $P(E)$, we say that E is independent of F. That is, E is independent of F if knowledge that F has occurred does not change the probability that E occurs.
- Since $P(E|F) = \frac{P(EF)}{P(F)}$, we see that E is independent of F if $P(EF) = P(E)P(F)$.
- Whenever E is independent of F so is F of E.
- Two events E and F that are not independent are said to be dependent.

Independent events

Example

Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands on heads and F the event that the second lands on tails, show that E and F are independent.

Hint

$$P(EF) = \frac{1}{4}, P(E) = \frac{1}{2}, P(F) = \frac{1}{2}$$

Independent events

Example

A card is selected at random from an ordinary deck of 52 playing cards. If A is the event that the selected card is an ace and H is the event that it is a heart. Show that A and H are independent.

Hint

$$P(AH) = \frac{1}{52}, P(A) = \frac{4}{52}, P(H) = \frac{13}{52}$$

Independent events

Example

If one die is rolled twice, is the probability that their sum is greater than 5 independent of the probability that the first roll produces a 1?

Hint F_1={the first roll produces a 1}. G_5={sum is greater than 5}.

(1,1)(1,2) (1,3)(1,4)(1,5)(1,6)
(2,1)(2,2)(2,3)(2,4)(2,5)(2,6)
(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)
(4,1)(4,2)(4,3)(4,4)(4,5)(4,6)
(5,1)(5,2)(5,3)(5,4)(5,5)(5,6)
(6,1)(6,2)(6,3)(6,4)(6,5)(6,6)

$$\Rightarrow \quad P[F1 \cap G5] = \frac{2}{36} = \frac{1}{18}. \qquad P[G5] = \frac{26}{36} = \frac{13}{18}. \qquad P[F1] = \frac{1}{6}.$$

$$P[F1 \cap G5] = \frac{1}{18} \neq P[F1]P[G5] = \frac{1}{6} \times \frac{13}{18} = \frac{13}{108}.$$

Answer: NO! Dependent!

Independent events

Definition for n independent events

The events $E_1, E_2, ..., E_n$ are said to be independent if, for every subset $E_{r_1}, E_{r_2}, ..., E_{r_k}$, where $r_j \leq n$, we have

$$P\left(E_{r_1} E_{r_2} ... E_{r_k}\right) = P\left(E_{r_1}\right) P\left(E_{r_2}\right) ... P\left(E_{r_k}\right)$$

Independent events

Example

An infinite sequence of independent trials is to be performed. Each trial results in a success with probability p and a failure with probability 1 - p. What is the probability that

(a) at least 1 success occurs in the first n trials;
(b) exactly k successes occur in the first n trials;
(c) all trials result in successes?

Answer:

(a) $1-(1-p)^n$.

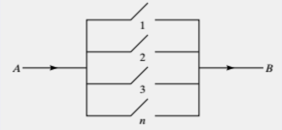
(b) $P\left(exactly\ \ k\ \ successes\right)=\left(\begin{array}{c} n \\ k \end{array}\right)p^k(1-p)^{n-k}$

(c) $\lim_{n\to\infty} p^n$.

Independent events

Exercise

A system composed of n separate components is said to be a parallel system if it functions when at least one of the components functions. For such a system, if component i, which is independent of the other components, functions with probability p_i, i $= 1, \ldots,$ n, what is the probability that the system functions?



$P(.|F)$ is a probability

The following proposition shows that $P(E|F)$ satisfies the three axioms of a probability.

Proposition

(a) $0 \le P(E|F) \le 1$.
(b) $P(S|F) = 1$, where S is the sample space.
(c) If E_i, i = 1, 2,..., are mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{\infty} E_i|F\right) = \sum_{i=1}^{\infty} P\left(E_i|F\right)$$

Key idea: Use the identity equation of conditional probability.

Remark: If we define $Q(E) = P(E|F)$, then $Q(E)$ can be regarded as a probability function on the events of S. Hence, all of the propositions previously proved for probabilities apply to $Q(E)$.

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END OF CHAPTER 3.