Basic probability elements

November 16, 2020

Outline

- Introduction
- Sample space
- Event and Operations
- Axioms of Probability on finite space

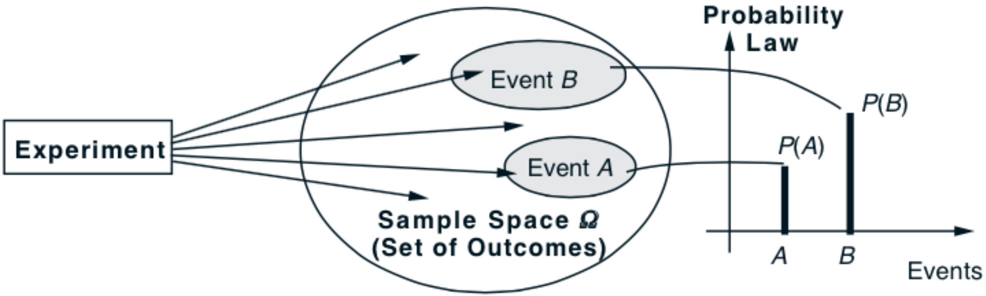
What is randomness?

- Flip 1 coin
- Can you tell exactly that it will show the head or the tail?
- Can you be 100% sure?
- No! - The outcome is random
- Means that you can not be sure

- Can’t know for sure it’ll rain or clear tomorrow
- Can’t be sure the traffic will be bad or not this afternoon
- Can’t be sure you’ll pass the Probability class this semester
- All the things that you can’t be sure to happen : Random

- A probabilistic model is a mathematical description of an uncertain situation.
- Measure randomness or uncertainty with probability
- Based on Combinatorics, Calculus, Measure theory ...

Main Ingredients of Probabilistic Models



Outcome of experiment

- **Random experiment -** a process leading to an uncertain outcome
- Suppose we want to estimate the chance of some **outcome** (or **event**) to happen
- If we toss a coin, how likely it will land on its head?
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- The coin could land on head or tail.
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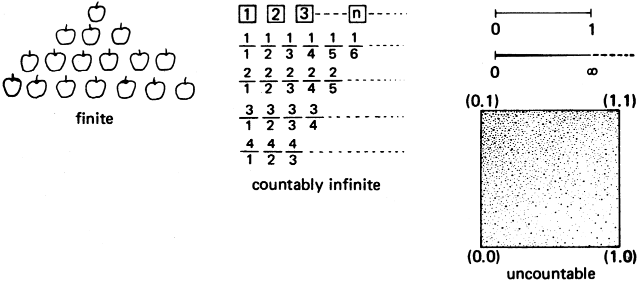
- The set of all possible outcomes is called the **sample space** Ω or S
- Ω could be finite, countable or uncountable



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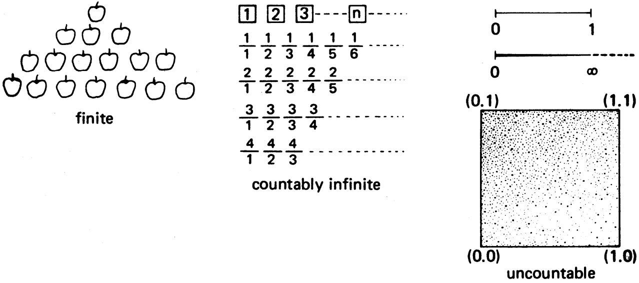

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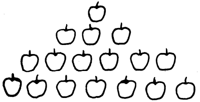
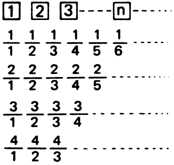
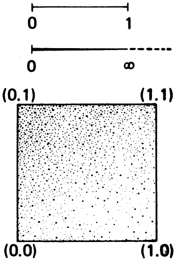


finite



countably infinite



uncountable

Example

- Filpping 2 coins

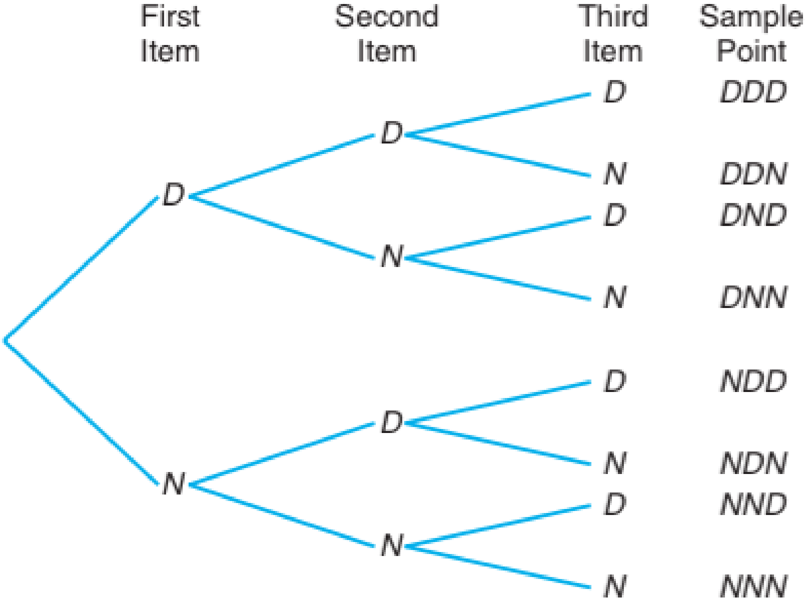
$$\Omega = \{(HH), (HT), (TH), (TT)\}$$

- Measuring the lifetime of a light bulb

$$\Omega = \{x : 0 \leq x < \infty\}$$

Example

Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified defective, D, or nondefective, N. Use tree diagram to list all element in the sample space


First Item
Second Item
Third Item
Sample Point
D
N
D
N
D
N
D
N
D
N
D
N
D
N
DDD
DDN
DND
DNN
NDD
NDN
NND
NNN

A subset E of Ω is called an event.

- Flipping 2 coins
- Sample space

$$\Omega = \{(HH), (HT), (TH), (TT)\}$$

- Event: get all heads

$$A = \{HH\}$$

- Event: get a least 1 tail

$$B = \{HT, TH, TT\}$$

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$$B = \{HT, TH, TT\}$$

- Flipping 2 coins: E= no head

$$E = \{(TT)\}$$

- Light bulb lifetime: E=lifetime less than or equal 5 hours

$$E = \{x : 0 \leq x \leq 5\}$$

Exercise

Joe will continue to flip a coin until heads appears. Identify the sample space and the event that it will take Joe at least three coin flips to get a head.

Containment

- A is contained in B, denoted by $A \subset B$, if every element of the set A also belongs to the set B.
- Interpretation of $A \subset B$: if A occurs then so does B

Example

- Roll a dice
- A: an even number is obtained
- B: a number greater than 1 is obtained
- $A = \{2, 4, 6\}$
- $B = \{2, 3, 4, 5, 6\}$
- $A \subset B$

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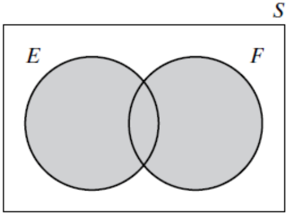
Empty set $\emptyset$

- The subset of sample space that contains no elements is called the *empty set*, or *null set*, denoted by $\emptyset$
- Interpretation: the empty set is any event that cannot occur

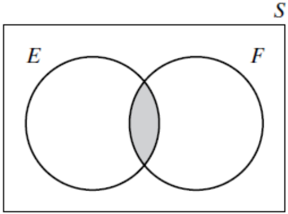
Let E and F be 2 events of Ω

- **Union**: $E \cup F$= either E or F or both occurs
- **Intersection**: EF= both E and F occurs
- **Complement**: E^c= everything not in E

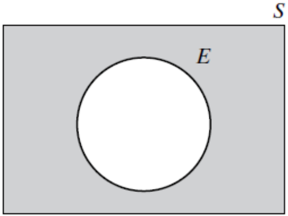
Venn diagram



(a) Shaded region: $E \cup F$.



(b) Shaded region: EF.



(c) Shaded region: E^c.

Light bulb lifetime:
E=bulb last more than 3 hours,
F= bulb last less than or equal 3 hours

$$F = E^c$$

Example

- the sample space Ω = {book, cell phone, mp3, paper, stationery, laptop}
- A = {book, stationery, laptop, paper}
- the complement of A is A^c = {cell phone, mp3}

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Example

- Toss a dice
- $E = \{4, 5, 6\}$
- $F = \{2, 4, 6\}$
- $E \cap F = \{4, 6\}$
- $E \cup F = \{2, 4, 5, 6\}$

- E: a person selected at random in a classroom is majoring in engineering
- F: the person is female
- EF: female engineering students in the classroom.

- If $EF = \emptyset$ then say E and F are mutually exculsive or disjoint
- E is partitioned into $E_1, E_2, \ldots, E_k$ if
  - $E_1, E_2, \ldots, E_k$ mutually exclusive
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- Commutative laws

$$E \cup F = F \cup E \qquad\qquad EF = FE$$

- Associative laws

$$(E \cup F) \cup G = E \cup (F \cup G)$$
$$(EF)G = E(FG)$$

- Distributive laws

$$(E \cup F)G = (EG) \cup (FG)$$
$$(EF) \cup GG = (EG) \cup (FG)$$

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$$

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- Sample space $\Omega = \{w_1, w_2, \ldots, w_n\}$
- Probability (measure)

$$\begin{aligned} P :& \Omega \longrightarrow [0,1] \\ & w \mapsto P(w) \end{aligned}$$

so that

$$\sum_{k=1}^{n} P(w_k) = 1$$

- For every even A

$$P(A) = \sum_{w \in A} P(w)$$

If Ω is finite and has equally likely outcomes then

$$P(E) = \frac{|E|}{|\Omega|}$$

where

- $|E|$: number of element in E
- $|\Omega|$: number of element in Ω

- Roll a fair dice
- Sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$
- $P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$
- $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$

- Toss a biased coin
- a head were twice as likely to appear as a tail

$$P(H) = 2P(T)$$

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