Decision Making

(for Financial Engineering & Risk Management program)

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Contents

1. Introduction

In the previous chapter, Decision theory, we consider situations in which the decision maker have to deal with the passive opponents, namely the states of nature. In contrast, game theory deals with decision situations in which two intelligent opponents have conflicting objectives. Such situations appear, for example, in military battles, political campaigns, advertising and marketing campaigns by competing business firms, and so on.

In general, the term "payoff" is used to describe a decision-maker?s gain (or loss), and the term "player" is used to refer to a decision-maker. In some cases, the amount that one player gains equals the amount that the other loses; such a situation is called a zero-sum game. In a mixed-motive game, however, the payoffs to the players are more general. There may be results in which both players gain and others in which they both lose.

A game is a set of n players, n sets of strategies (one set for each player), and n payoff functions (one/each player). Player i will receive a payoff $M_i(s_1, ..., s_n)$ if players 1 to n choose: s_1 to s_n.

In a two-player game, a **strategy pair** (s_1^*, s_2^*) **is an equilibrium point** (or optimal solution of the game) if

$$M_1(s_1^*, s_2^*) \geq M_1(s_1, s_2^*)$$

for any strategy s_1 available to player 1 and

$$M_2(s_1^*, s_2^*) \geq M_2(s_1^*, s_2)$$

for any strategy s_2 available to player 2.
That is, a strategy pair is an equilibrium point if neither player can gain more by switching to another strategy while the other keeps the same strategy. SOLVE a game means to find an optimal solution (equilibrium point) to this game.

In game theory, one studies rather complicated types of competitive situations. However, in this chapter we study only the simplest case, called two-person, zero sum games. The game involves only two players. The players can be firms, armies, teams, politicians, etc.

They are called zero-sum games because for this class of problems, one player wins whatever the other one loses, so that the sum of their net winnings is zero. That is,

$$M_1(s_1, s_2) + M_2(s_1, s_2) = 0.$$

- A primary objective of game theory is the development of rational criteria for solving a game.

- Key assumptions:

(a) Both players are rational (intelligent),

(b) Both players choose their strategies solely to promote their own welfare (no compassion for the opponent).

2. Two-players, zero-sum game

In two-person, zero-sum games, each player will have a (finite or infinite) number of alternatives which are referred to as strategies. Associated with each pair of strategies is a payoff that one player pays/gains to/from the other.

A two-person, zero-sum game is often summarized in terms of the payoff (the gain, positive or negative) to one player described by a so-called payoff matrix (or payoff table) to one player.

		Player B			
		B_1	B_2	$\cdots$	B_n
	A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
Player	A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
A	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Table 1.

Let A and B be two players.
A has n strategies: $A_1, A_2, \cdots, A_n$,
B has m strategies: $B_1, B_2, \cdots, B_m$.

If A uses strategy A_i and B uses strategy B_j then the payoff to A is a_{ij} (which means that the payoff to B is $-a_{ij}$).

The actual play of the game consists the fact that of each player simultaneously choosing a strategy without knowing the opponent's choice.

Recall: • strategy pair (A_i^*, B_J^*) is an equilibrium point/optimal solution of the game if

$$a_{ij} \geq a_{kj}$$

for any strategy A_k available to player 1 and

$$-a_{ij} \geq -a_{i\ell}$$

for any strategy B_ℓ available to player 2.

• Solve a game means to find an optimal solution to this game.

A prototype example. Two politicians are running against each other for the U.S. Senate. Campaign plans must now be made for the final two days, which are expected to be crucial because of the closeness of the race. Therefore, both politicians want to spend these days campaigning in two key cities, Bigtown and Megalopolis.

In order to avoid wasting time, both of them plan to travel at night and spend either 1 full day in each city or 2 full days in just one of the cities.

The politicians have to arrange their schedules in advance and neither of them can learn his (or her) opponent's campaign schedule.

Formulation as a two-person, zero-sum game

We must identify:

- Players: two politicians,

- Strategies for each player. Each player has 3 strategies:

Strategy 1: spend 1 day in each city,

Strategy 2: spend both days in Bigtown,

Strategy 3: spend both days in Megalopolis.

- Payoff table. The suitable entries for the payoff table for politician 1 is the total net votes (unit: 1,000 votes) won from the opponent resulting from the two days of campaigning. For an example of such a payoff table, see Table 2.

	Strategy	Player 2 (politician 2) 1	2	3
	1	1	2	4
Player	2	1	0	5
1	3	0	1	-1

Table 2

3.1. Dominated strategies

A strategy is dominated (called dominated strategy) by a second strategy if the second strategy is always at least as good (and sometimes better) regardless of what the opponent does.

A dominated strategy can be eliminate immediately from further consideration.

In Table 2, for the player 1, the strategy 3 is dominated by the strategy 1. So it can be eliminated from the table. We get the reduced payoff table:

	Strategy	Player 2 (politician 2) 1	2	3
Player 1	1	1	2	4
	2	1	0	5

Table 3

In some simple situation (game), by such a way of using the notion of dominated strategies, we can rule out the dominated strategies until only one choice remains. The strategy remained is the solution to the game.

Turning back to the example:

Now, for the player 2 (who is intelligent and knows that Player 1 will eliminate Strategy 3), He sees that (his) Strategy 3 is dominated by Strategy 1 (see Table 3). He eliminates Strategy 3. The resulting payoff table is as follows:

	Strategy	Player 2: 1	Player 2: 2
Player 1	1	1	2
	2	1	0

Table 4

At this point, for Player 1, Strategy 2 is dominated by Strategy 1. Eliminating this yielding

	Strategy	Player 2: 1	Player 2: 2
Player 1	1	1	2

Table 5

In turn, Player 2 recognizes that (looking at Table 5) Strategy 2 is dominated by Strategy 1. He eliminate Strategy 2.

Consequently, both players should choose their strategy 1. The Player 1 then will gain a payoff 1 from the Player 2 (i.e.,the politician 1 will gain 1,000 votes from the politician 2).

In general, the payoff to Player 1 when both players use the optimal strategies (optimal solution) will be called the value of the game.

A game with the value 0 is called a fair game.

In general, we are not always lucky arrive at a solution as in the previous example. However, the concept of dominated strategies is useful in reducing the size of the payoff table.

3.2. Minimax criterion

Minimax criterion is a standard criterion proposed by game theory for selecting a strategy.

We will learn the way how this criterion works by considering again the previous example with the payoff table (for Player 1/politician 1) described in Table 6 (note: different from Table 2).

	Strategy	Player 2 (politician 2): 1	2	3	Minimum
Player 1	1	-3	-2	6	-3
	2	2	0	2	0 (maximin)
	3	5	-2	-4	-4
Maximum		5	0 (minimax)	6	

Table 6

In this table, there is no strategy (of either players) is dominated by the other. Which strategies do the players use?

	Strategy	Player 2 (politician 2) 1	2	3	Minimum
Player 1	1	-3	-2	6	-3
	2	2	0	2	0 (maximin)
	3	5	-2	-4	-4
	Maximum	5	0 (minimax)	6	

Consider Player 1:

Select strategy 1: could win: 6, could lose: 3,

Select strategy 3: could win: 5, could lose: 4,

Select strategy 2: could win 2, no loss.

It seems that for the player 1, strategy 2 provides the best guarantee (a payoff of 0). He will lose nothing and he even win something. So strategy 2 seems to be a “rational” (reasonable) choice for player 1 against his rational opponent.

Note: Strategy 2 for player 1 (second row of the matrix) corresponds to the MAXIMUM value 0 of the MINIMUM values of the three rows.

By a similar reasoning the player 2 should choose strategy 2 for the best guarantee (students should do it in details).

The strategy 2 for Player 2 corresponds to the minimum value 0 of the maximum values of the three columns.

The pair of strategies (2,2) is optimal. Any of the players who use another (while the other player keeping the chosen strategy) will lose something (instead of losing nothing). The value of the game is 0. The game is a fair game.

The end product of this reasoning is that each player should play in such a way as to minimize his maximum losses. So this is called minimax criterion.

For the Player 1: select the strategy whose minimum payoff is largest.

For the Player 2: select the strategy whose maximum payoff to Player 1 is smallest.

Note. The reason for the pair of strategies $(2, 2)$ is optimal is that both the minimax and the maximin values attain at only one entry $a_{22} = 0$. This entry a_{22} is called saddle point of the payoff matrix. In this case, the solution is called a stable solution.

Example with unstable solutions

	Strategy	Player 2: 1	2	3	Minimum
Player 1	1	0	-2	2	-2(maximin)
	2	5	4	-3	-3
	3	2	3	-4	-4
	Maximum	5	4	2 (minimax)	

Table 7

Note that the maximin value (-2) and the minimax value (2) do not coincide in this case. So there is no saddle point. The solution $(1, 3)$ is unstable solution.

What happens in this case? (the students reason out by themselves).

• Even the game is played once, the game with unstable solution leaves the players with a motive to change their strategies.

• The process of tentative choices would make a cycle of pairs of strategies and starts over again and again unlimitedly.

What should the players do in this case? How to improve the situation?

In order to have a solution for the case where the game has no saddle point, one suggest a concept of mixed strategy. Each player does not choose a pure strategy as before. Instead, he assigns a probability distribution over his set of strategies. To express this mathematically, let

- x_i = probability that player 1 will use strategy i $(i = 1, 2, ..., m)$,
- y_j = probability that player 2 will use strategy j $(j = 1, 2, ..., n)$.

Note that

$$x_1 + x_2 + \cdots + x_m = 1,$$

$$y_1 + y_2 + \cdots + y_n = 1.$$

The plans $(x_1, x_2, ..., x_m)$ and $(y_1, y_2, ..., y_n)$ are referred to as mixed strategies.

When a game is actually played, it is necessary for each player to use one of his pure strategies. However, this pure strategy would be chosen by using some "random device" which reflects the probability distribution of the mixed strategy.

For an example, let us turn back to the game described by the payoff table 7. Suppose that the player 1 and 2 select the mixed strategies $(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, 0)$ and $(y_1, y_2, y_3) = (0, \frac{1}{2}, \frac{1}{2})$.

What do these mixed strategies really mean when the game is being played?

- The player 1 is giving an equal chance (probability of $\frac{1}{2}$) of choosing either pure strategy 1 or strategy 2 while he discards the strategy 3.

- The player 2 is randomly choosing between his two last strategies 2 or 3 (discards strategy 1).

- To play the game, each of player could then flip (toss) a coin to determine which of his two acceptable pure strategies he will actually use.

Questions remain unsolved:

Do mixed strategies improve the situation?

How do we understand an optimal mixed solution? (What is the definition of solution in case mixed strategies being used?).

How to find the optimal mixed strategies?

• When the mixed strategies are used, since the probability distribution is involved, the payoffs are random variables which is a_{ij} with a probability $x_i y_j$.

Therefore, the expected payoff for Player 1 is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j.$$

Recalling that a_{ij} is the payoff for Player 1 when he uses pure strategy i and Player 2 uses pure strategy j.

In the last example when Player 1 uses the strategy $(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, 0)$ and Player 2 uses $(y_1, y_2, y_3) = (0, \frac{1}{2}, \frac{1}{2})$, then the expected payoff (for Player 1) is

$$\frac{1}{4}(-2+2+4-3) = \frac{1}{4}.$$

	Strategy	Player 2: 1	2	3	Minimum
Player 1	1	0	-2	2	-2(maximin)
	2	5	4	-3	-3
	3	2	3	-4	-4
Maximum		5	4	2 (minimax)	

Table 7

Explain the meaning of this fact (expected value)?

This quantity (the expected payoff for Player 1) does indicate what the average payoff will tend to be if the game is played many times.

For each player, say Player 1, how to choose the best mixed strategy?

The best way is to use the minimax criterion.
• He (1^{st} player) then choose the mixed strategy $(x_1^*, x_2^*, \cdots, x_m^*)$ satisfying:

$$\max_{x_i} \min_{y_j} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j := \underline{v}.$$

• The Player 2 also wants to use the minimax criterion to choose the mixed strategy $(y_1^*, y_2^*, \cdots, y_n^*)$ satisfying:

$$\min_{y_j} \max_{x_i} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j := \overline{v}.$$

Note that it is always true that $\underline{v} \leq \overline{v}$ and when the game is unstable we have (always)

$$\max_i \min_j a_{ij} \; < \; \min_j \max_i a_{ij}$$

(verify this fact by inspecting the concrete last example and derive a reasoning for general cases).

• When a game is unstable, a saddle point does not exists. Fortunately, when mixed strategies are allowed, a saddle point does exist as the following minimax principle claims.

Theorem (minimax) If mixed strategies are allowed, the pair of mixed strategies that is optimal according to the minimax criterion provides a stable solution with $\underline{v} = \overline{v} = v$ (the value of the game). So that neither of players can do better by unilaterally changing his (or her) strategy.

Note: When the mixed strategies are allowed, the number of (mixed) strategies are infinite and hence, we can not use the table to find an optimal solution! Another method should be suggested.

Used only when one of the players has only two pure strategies.

Suppose that (just to keep our mind fixed) the player 1 has only two pure strategies. His mixed strategies are of the forms (x_1, x_2) and $x_2 = 1 - x_1$.

It is necessary for him to solve only for the optimal value of x_1. For this, the procedure is as follows:

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- Plot the expected payoff as a function of x_1 for each of his opponent's pure strategy.

5. Graphical solution procedure

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Suppose that (just to keep our mind fixed) the player 1 has only two pure strategies. His mixed strategies are of the forms (x_1, x_2) and $x_2 = 1 - x_1$.

It is necessary for him to solve only for the optimal value of x_1. For this, the procedure is as follows:

• Plot the expected payoff as a function of x_1 for each of his opponent's pure strategy.

• These graphs can be used to identify the point that maximizes the minimum expected payoffs (for player 1, remind!). The opponent's minimax mixed strategy can also be identified from the graphs.

Illustrating example

Coming back to the last unstable example.

	Strategy	Player 2: 1	2	3
	1	0	-2	2
Player 1	2	5	4	-3
	3	2	3	-4

It is worth observing that for Player 1, the third pure strategy is dominated by the second. So this pure strategy will be eliminated from the payoff table. The reduced payoff table is given as:

	Strategy	Player 2: 1	2	3
	1	0	-2	2
Player 1	2	5	4	-3

Table 8

	Strategy	Player 2: 1	2	3
	1 (x_1)	0	-2	2
Player 1	2 $(1-x_1)$	5	4	-3

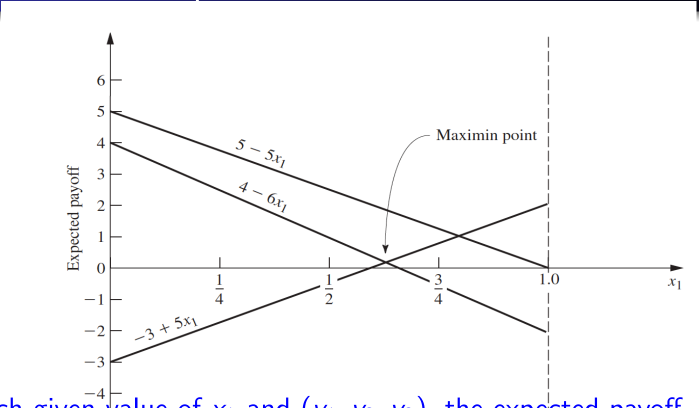
- The expected payoffs for Player 1 corresponding to the pure strategies of Player 2 are given in the following table:

(y_1, y_2, y_3)	Expected payoff
(1,0,0)	$0x_1 + 5(1-x_1) \quad = 5 - 5x_1$
(0,1,0)	$-2x_1 + 4(1-x_1) = 4 - 6x_1$
(0,0,1)	$2x_1 - 3(1-x_1) \quad = -3 + 5x_1$

Table 9

- Plot the expected payoff lines on a graph (Figure 1).

Figure 1



- For each given value of x_1 and (y_1, y_2, y_3), the expected payoff will be the appropriate weighted average of the corresponding points on the three lines (see Figure 1).

- The expected payoff for player 1:

$$E = y_1(5 - 5x_1) + y_2(4 - 6x_1) + y_3(-3 + 5x_1).$$

- Player 1 wants to choose (mixed str.) $(x_1^*, 1-x_1^*)$ satisfying:

$$\max_{(x_1,1-x_1)} \min_{(y_1,y_2,y_3)} \left[y_1(5-5x_1) + y_2(4-6x_1) + y_3(-3+5x_1) \right]$$

He should find the minimum first and then the maximum among these minimum values. That is x_1^* should be the value satisfying ($y_1 = 0$ why?),

$$\max_{0\le x_1\le 1} \min\{4-6x_1, -3+5x_1\}.$$

Thus, $x_1^* = \frac{7}{11}$ and $(x_1^*, 1-x_1^*) = (\frac{7}{11}, \frac{4}{11})$ is the optimal mixed strategy for Player 1. Moreover,

$$\underline{v} = 4 - 6\frac{7}{11} = \frac{2}{11}$$

is the value of the game. Thus,

$$\underline{v} = v = \max_{0\le x_1\le 1}\{\min(4-6x_1,\ -3+5x_1)\} = \frac{2}{11}.$$

- We now find the optimal mixed strategy for Player 2. According to the definition of the minimax value and minimax theorem, with the optimal strategy $(y_1, y_2, y_3) = (y_1^*, y_2^*, y_3^*)$, we have

$$y_1^*(5 - 5x_1) + y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) \leq \overline{v} = \underline{v} = v = \frac{2}{11}$$

for all $x_1 \in [0, 1]$. Furthermore, when the Player 1 is playing optimally, that is, with $x_1^* = \frac{7}{11}$, the inequality becomes equality. Namely,

$$\frac{20}{11}y_1^* + \frac{2}{11}y_2^* + \frac{2}{11}y_3^* = v = \frac{2}{11}.$$

Note that $y_1^* + y_2^* + y_3^* = 1$ (or $y_2^* + y_3^* = 1 - y_1^*$). Substitute this to the last equality we get $y_1^* = 0$ (we also can argue as follows: if $y_1^* > 0$ then in the last equality, the left hand side will be greater than $\frac{2}{11}$).

Hence,

$$y_2^*(4-6x_1)+y_3^*(-3+5x_1)\begin{cases} \leq \frac{2}{11} & \text{for all } 0\leq x_1\leq 1,\\ = \frac{2}{11} & \text{for } \ x_1=\frac{7}{11}. \end{cases}$$

Since y_2^*, y_3^* are numbers, the left-hand side is the equation of a straight line. This line is necessarily horizontal (why?). Therefore,

$$y_2^*(4-6x_1)+y_3^*(-3+5x_1)=\frac{2}{11}, \quad \text{for all } x_1\in[0,1].$$

To solve for y_2^*, y_3^* we can select two values of x_1 (say, 0 and 1). We get

$$\begin{array}{rcl} 4y_2^*-3y_3^* & = & \frac{2}{11}\\ -2y_2^*+2y_3^* & = & \frac{2}{11}. \end{array}$$

This system gives $y_2^* = \frac{5}{11}$ and $y_3^* = \frac{6}{11}$. Thus the optimal mixed strategy for Player 2 is $(y_1^*, y_2^*, y_3^*) = (0, \frac{5}{11}, \frac{6}{11})$.

Remarks:

• Note that graphical solution procedure only can apply to a special case when one of the players has TWO pure strategies.

• How to solve general two-person, zero-sum games? there is at least a way: Use Linear programming (see the text book [Hillier - Leiberman], Chapter 16).

• Many more general kinds of games: Games with "nonzero-game", n-person game, cooperative game, infinite game, etc. These are not considered in our course.

Extra example

Two prisoners are separated into individual rooms and cannot communicate with each other. The normal game is shown below:

Prisoner B Prisoner A	**Prisoner B stays silent** **(*cooperates*)**	**Prisoner B betrays** **(*defects*)**
Prisoner A stays silent **(*cooperates*)**	Each serves 1 year	Prisoner A: 3 years Prisoner B: goes free
Prisoner A betrays **(*defects*)**	Prisoner A: goes free Prisoner B: 3 years	Each serves 2 years

It is assumed that both prisoners understand the nature of the game, have no loyalty to each other, and will have no opportunity for retribution or reward outside the game.

What is the equilibrium (Nash) solution?