INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

MIDTERM EXAMINATION
October 2015
Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Deputy head of Dept. of Mathematics:	Lecturers:
Assoc. Prof. Dr. Pham Huu Anh Ngoc	Assoc. Prof. Dr. Nguyen Ngoc Hai

INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 *(25 marks)* Let c_1, c_2 be positive numbers. For $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, define

$$\|\boldsymbol{x}\| = c_1|x_1| + c_2|x_2|.$$

Show that $\|\cdot\|$ is a norm on $\mathbb{R}^2$. Let ρ be the metric induced by this norm, determine the distance $\rho(\boldsymbol{a}, \boldsymbol{b})$ between $\boldsymbol{a} = (1, 0)$ and $\boldsymbol{b} = (0, 2)$.

Question 2 (a) *(10 marks)* Let a and b be points in the metric space (X, d). Define the function $f : X \to \mathbb{R}$ by $f(x) = d(x, a) + d(x, b)$. Show that f is Lipschitz continuous.

(b) *(15 marks)* Show that the set

$$E = \big\{(x, y) \in \mathbb{R}^2 : -1 \le x \le 3,\ 2 \le y \le 5\big\}$$

is compact in the Euclidean space $\mathbb{R}^2$. Apply part (a) to show that there is a point $(x_0, y_0) \in E$ such that

$$\begin{aligned} &d\big((x_0, y_0), (-1, 1)\big) + d\big((x_0, y_0), (2, 0)\big) \\ &\quad = \sup\{d\big((x, y), (-1, 1)\big) + d\big((x, y), (2, 0)\big) : (x, y) \in E\}, \end{aligned}$$

where d is the Euclidean metric on $\mathbb{R}^2$.

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Question 3 (a) *(10 marks)* Let A and B be subsets of a metric space (X, d). Show that if $A \subset B$, then $\text{int}(A) \subset \text{int}(B)$.

(b) *(15 marks)* Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $A \in \mathcal{M}$ is said to be σ-finite if $A = \cup_{n=1}^{\infty} A_n$, where $A_n \in \mathcal{M}$ and $\mu(A_n) < \infty$ for all n. If A is a σ-finite set, show that there exists a sequence $\{B_n\} \subset \mathcal{M}$ such that

$$A = \bigcup_{n=1}^{\infty} B_n, \quad B_n \subset B_{n+1} \text{ and } \mu(B_n) < \infty \text{ for all } n.$$

Question 4 (a) *(10 marks)* Let μ^* be an outer measure on X and let A, B be subsets of X. Show that if $\mu^*(A) = 0$, then $\mu^*(A \cup B) = \mu^*(B)$.

(b) *(15 marks)* Let $F(x) = x^3$. If μ is the Lebesgue-Stieltjes measure corresponding to F, compute the measure $\mu\big((-1, 0] \cup [2, 4]\big)$.

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SOLUTIONS

Question 1 Let $\alpha \in \mathbb{R}$, $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2$. We have

$$\begin{aligned}
\|\boldsymbol{x}\| &= c_1|x_1| + c_2|x_2| \geq 0 \quad \text{and} \\
\|\boldsymbol{x}\| = 0 &\iff c_1|x_1| + c_2|x_2| = 0 \\
&\iff x_1 = x_2 = 0 \quad (\text{since } c_1, c_2 > 0) \\
&\iff \boldsymbol{x} = \boldsymbol{0}.
\end{aligned}$$

$$\|\alpha\boldsymbol{x}\| = \|(\alpha x_1, \alpha x_2)\| = c_1|\alpha x_1| + c_2|\alpha x_2| = |\alpha|(c_1|x_1| + c_2|x_2|) = |\alpha|\|\boldsymbol{x}\|.$$

Finally,

$$\begin{aligned}
\|\boldsymbol{x} + \boldsymbol{y}\| &= \|(x_1 + y_1, x_2 + y_2)\| = c_1|x_1 + y_1| + c_2|x_2 + y_2| \\
&\leq c_1(|x_1| + |y_1|) + c_2(|x_2| + |y_2|) = \|\boldsymbol{x}\| + \|\boldsymbol{y}\|.
\end{aligned}$$

Thus $\|\cdot\|$ is a norm on $\mathbb{R}^2$.

The metric ρ induced by this norm is

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| = \|(x_1 - y_1, x_2 - y_2)\| = c_1|x_1 - y_1| + c_2|x_2 - y_2|.$$

Therefore $\rho(\boldsymbol{a}, \boldsymbol{b}) = c_1|1 - 0| + c_2|0 - 2| = c_1 + 2c_2$.

Question 2 (a) For $x, y \in X$, we have

$$\begin{aligned}
|f(x) - f(y)| &= |d(x, a) + d(x, b) - d(y, a) - d(y, b)| \\
&\leq |d(x, a) - d(y, a)| + |d(x, b) - d(y, b)| \\
&\leq d(x, y) + d(x, y) = 2d(x, y).
\end{aligned}$$

Thus f is Lipschitz and therefore is continuous on X.

(b) Suppose that $\{(x_k, y_k)\} \subset E$ and $(x_k, y_k) \to (x, y) \in \mathbb{R}^2$. Since $-1 \leq x_k \leq 3$ and $2 \leq y_k \leq 5$ for all k, letting $k \to \infty$ we get $-1 \leq x \leq 3$ and $2 \leq y \leq 5$, showing that $(x, y) \in E$. Thus E is closed. Furthermore, E is bounded, hence it is compact.

Let

$$f(x, y) = d\big((x, y), (-1, 1)\big) + d\big((x, y), (2, 0)\big), \quad (x, y) \in \mathbb{R}^2.$$

Applying part (a) for $a = (-1, 1)$ and $b = (2, 0)$ we find that f is continuous on $\mathbb{R}^2$. As E is compact, there exists $(x_0, y_0) \in E$ such that $f(x_0, y_0) = \sup\{f(x, y) : (x, y) \in E\}$, that is,

$$\begin{aligned}
&d\big((x_0, y_0), (-1, 1)\big) + d\big((x_0, y_0), (2, 0)\big) \\
&\quad = \sup\{d\big((x, y), (-1, 1)\big) + d\big((x, y), (2, 0)\big) : (x, y) \in E\}.
\end{aligned}$$

Question 3 (a) If $x \in \operatorname{int}(A)$, there is $r > 0$ such that $B(x, r) \subset A$. Since $A \subset B$, $B(x, r) \subset B$, which shows that $x \in \operatorname{int}(B)$. Thus $\operatorname{int}(A) \subset \operatorname{int}(B)$.

(b) By definition, there exists a sequence $\{A_n\} \subset \mathcal{M}$ such that $\mu(A_n) < \infty$ for all n and $A = \bigcup_{k=1}^{\infty} A_k$. For each $n \in \mathbb{N}$, set

$$B_n = \bigcup_{k=1}^{n} A_k.$$

As $\mathcal{M}$ is a σ-algebra, $B_n \in \mathcal{M}$ for every n. Moreover,

$$B_n \subset B_{n+1}, \quad \bigcup_{n=1}^{\infty} B_n = \bigcup_{k=1}^{\infty} A_k = A$$

and, by subadditivity,

$$\mu(B_n) = \mu\Bigg(\bigcup_{k=1}^{n} A_k\Bigg) \leq \sum_{k=1}^{n} \mu(A_k) < \infty \quad \text{for all } n.$$

Question 4 (a) Applying monotonicity and subadditivity gives

$$\mu^*(B) \leq \mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B) = \mu^*(B),$$

which implies $\mu^*(A \cup B) = \mu^*(B)$.

(b) Since F is increasing and continuous, it induces a measure μ such that

$$\mu\big((a, b]\big) = F(b) - F(a) = b^3 - a^3, \quad a, b \in \mathbb{R},\ a \leq b.$$

Since $(-1, 0]$ and $[2, 4]$ are disjoint, we have

$$\begin{aligned}\mu\big((-1, 0] \cup [2, 4]\big) &= \mu\big((-1, 0]\big) + \mu\big([2, 4]\big) = F(0) - F(-1) + \mu\big([2, 4]\big) \\ &= 0^3 - (-1)^3 + \mu\big([2, 4]\big) = 1 + \mu\big([2, 4]\big).\end{aligned}$$

Furthermore,

$$\begin{aligned}\mu\big([2, 4]\big) &= \mu\Bigg(\bigcap_{n=1}^{\infty} \Big(2 - \frac{1}{n}, 4\Big]\Bigg) = \lim_{n \to \infty} \mu\Bigg(\Big(2 - \frac{1}{n}, 4\Big]\Bigg) \\ &= \lim_{n \to \infty} \Bigg[F(4) - F\Big(2 - \frac{1}{n}\Big)\Bigg] = \lim_{n \to \infty} \Bigg[4^3 - \Big(2 - \frac{1}{n}\Big)^3\Bigg] \\ &= 64 - 8 = 56.\end{aligned}$$

Thus $\mu\big((-1, 0] \cup [2, 4]\big) = 1 + 56 = 57$.