FINAL EXAMINATION SOLUTION

Semester 2, 2014-15 • Date: June 12th, 2015 • Duration: 120 minutes

1. (15 points) Let $\mathcal{R}$ be the region bounded by the curve $y = 2^x$ and the line segment connecting 2 points $(0, 1)$ and $(2, 4)$. Calculate the volume generated by rotating $\mathcal{R}$ about the $x-$ axis.

Solution: Equation of the line: $y = \frac{3}{2}x + 1$
Volume

$$\begin{aligned} V &= \pi \int_0^2 (\tfrac{3}{2}x + 1)^2 - (2^x)^2 \; dx \\ &= \pi(\tfrac{3}{4}x^3 + \tfrac{3}{2}x^2 + x - \tfrac{1}{2}\frac{2^{2x}}{\ln 2})\Big|_0^2 \\ &= \pi(14 - \frac{15}{2\ln 2}) \end{aligned}$$

2. (15 points) Given the sequence defined as follow:

$$a_1 = \sqrt{3}, \quad a_n = \sqrt{1 + 2a_{n-1}} \text{ for } n > 1$$

(a) (5 points) Prove by induction that $\{a_n\}$ is increasing sequence.

- $a_2 > a_1$
- Suppose $a_n > a_{n-1}$. Then $\sqrt{1 + 2a_n} > \sqrt{1 + 2a_{n-1}}$, or $a_{n+1} > a_n$.

(b) (5 points) Prove by induction that $\{a_n\}$ bounded from above by 3.

- $a_1 < 3$
- Suppose $a_n < 3$. Then $\sqrt{1 + 2a_n} < \sqrt{1 + 2 \times 3} < 3$, or $a_{n+1} < 3$.

(c) (5 points) Prove that the sequence converges. Find the limit.
Sequence increasing and bounded from above then it converges. Suppose

$$\lim_{n\to\infty} a_n = L$$

then $L = \sqrt{1 + 2L}$, thus $L = 1 + \sqrt{2}$.

3. (10 points) Calculate the sum $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1} + 1}{\pi^{n-1}}$.

$$
\begin{aligned}
\sum &= \sum_{n=0}^{\infty} \frac{2^{n+2}+1}{\pi^n} \\
&= 4\sum_{n=0}^{\infty}\left(\frac{2}{\pi}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{\pi}\right)^n \\
&= 4\frac{1}{1-\frac{2}{\pi}} + \frac{1}{1-\frac{1}{\pi}} \\
&= \frac{5\pi^2-6\pi}{\pi^2-3\pi+2}
\end{aligned}
$$

4. (30 points) Check if the series converges or diverges

(a) (15 points) $\sum_{n=2}^{\infty} \frac{\sqrt[3]{n}}{2^n \ln n}$

Ratio test:

$$
\frac{a_{n+1}}{a_n} = \frac{1}{2}\frac{\sqrt[3]{n+1}\ln n}{\sqrt[3]{n}\ln(n+1)} \to \frac{1}{2} < 1
$$

(b) (15 points) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2/3}+1}$

Alternating Series test: $\frac{1}{n^{2/3}+1}$ decreases to 0.

5. (15 points)

(a) (10 points) Find the power series representation for $f(x) = \frac{x}{2+x}$ in power of x. On what interval does the series converge?

$$
\begin{aligned}
f(x) &= \frac{x}{2}\frac{1}{1-(-x/2)} \\
&= \frac{x}{2}\sum_{n=0}^{n} \frac{(-1)^n}{2^n} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n} x^n
\end{aligned}
$$

Radius of convergence: $R = 2$, thus interval $(-2, 2)$, no end points.

(b) (5 points) Find the MacLaurin series for $g(x) = \frac{2}{(2+x)^2}$.

Observe that $g(x) = f'(x)$, so

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^n} x^{n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^{n+1}} x^n$$

6 (15 points) Find the equation of the line that goes through the point $P(1, 2, 3)$ and perpendicular to the plane containing 2 vectors

$$\vec{u} = \vec{i} - \vec{j} + \vec{k}, \quad \vec{v} = 2\vec{i} + \vec{k}$$

The line will parallel to the cross product of $\vec{u}$ and $\vec{v}$

$$\vec{u} \times \vec{v} = -\vec{i} + \vec{j} + 2\vec{k}$$

Equation

$$\frac{x-1}{-1} = \frac{y-2}{1} = \frac{z-3}{2}$$

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