Brownian Motions

June 19, 2021

Outline

- Textbook: chapter 3 Shreve II
- Content: basis property of Brownian motion
  - Martingale property
  - Markov property
  - First passage time distribution and reflection principle
  - **Quadratic variation**
- Simulation

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Symmetric random walk

- Start at 0: $M_0 = 0$
- Time scale 1
- Each time, your wealth increases or decreases by 1 with same prob 1/2
- Independent and stationary increment
- Markov property
- Martingale property
- Reflection principle
- Quadratic variation $\langle M_n, M_n \rangle = n$

Scaled symmetric random walk

$$W_t^{(n)} = \frac{1}{\sqrt{n}} M_{nt}, t = 0, \frac{1}{n}, \frac{2}{n}, \dots$$

- Start at 0: $W_0^{(n)} = 0$
- Time scale $\frac{1}{n}$
- Each time, your wealth increases or decreases by $\frac{1}{\sqrt{n}}$ with same prob 1/2
- Independent and stationary increment
- Markov property
- Martingale property
- Reflection principle
- Quadratic variation $\langle W_t^n, W_t^n \rangle = t$

Brownian motion

- Limit of scaled symmetric random walk as $n \to \infty$: continuous time random process process
- $W_t^{(n)} \xrightarrow{d} \mathcal{N}(0, t)$

Brownian motion - Limit of Scaled Symmetric Random Walk

- Limit of scaled symmetric random walk is use to studied the limit of binomial model which allow us to study asset pricing in continuous time
- Limit of scaled symmetric random walk: Brownian motion
- Inherit properties from these random walks

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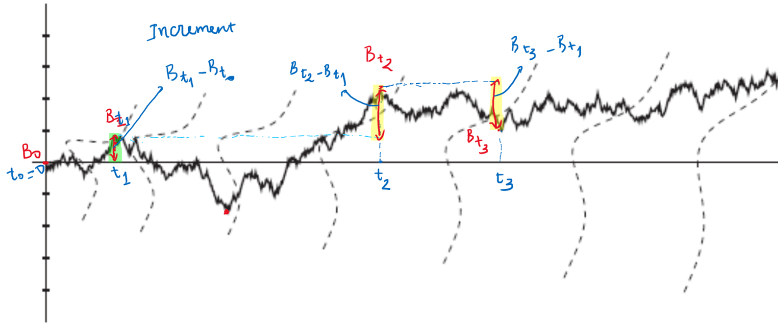


Increment
$B_{t_1} - B_{t_0}$
$B_{t_2} - B_{t_1}$
B_{t_2}
$B_{t_3} - B_{t_1}$
B_{t_1}
B_0
$t_0 = 0$
t_1
t_2
t_3
B_{t_3}

Definition of standard Brownian motion

A random (stochastic) process $(B_t)_{t \geq 0}$ is a standard Brownian motion on probability space $(\Omega, \mathcal{F}, P)$ if for each $\omega \in \Omega$, $B_t(w)$ of $t \geq 0$ is a continuous function of time t and

1. **Starting at 0** $B_0 = 0$
2. At time t, $B_t \sim \mathcal{N}(0, t)$
3. **Increments** for $t_0 = 0 < t_1 < \cdots < t_m$

$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}$$

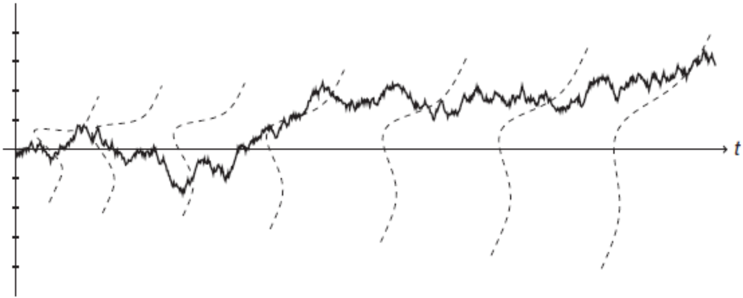
are **independent** and **stationary**

$$B_{t_{i+1} - B_{t_i}} \overset{d}{=} B_{t_{i+1} - B_{t_1}} \sim N(0, t_{i+1} - t_i)$$

Brownian motion is also called by Wiener process and (W_t) is also referred as Brownian motion in many references

Brownian motion

Brownian motion can be thought of as the motion of a particle that diffuses randomly along a line. At each point t, the particle's position B_t is normally distributed about the line with variance t, i.e., $B_t \sim N(0, t)$. As t increases, the particle's position is more diffuse.



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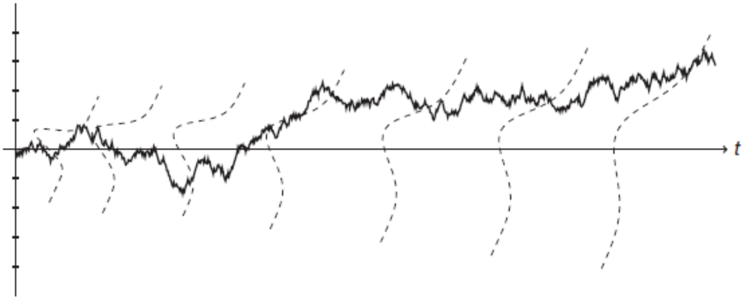

If $(B_t)_{t \geq 0}$ is a standard Brownian motion then so is $(-B_t)_{t \geq 0}$

Properties

1. $E(B_t) = 0$ and $Var(B_t) = t$
2. $Cov(B_{t+s}, B_s) = s$

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Proof

- $B_t \hookrightarrow N(0,t)$ hence $E(B_t) = 0$ and $Var(B_t) = t$
- $$\begin{aligned} Cov(B_{t+s}, B_s) &= Cov(B_{t+s} - B_s + B_s, B_s) \\ &= Cov(\underbrace{B_{t+s} - B_s, B_s}_{independent}) + Cov(B_s, B_s) \\ &= 0 + Var(B_s) = s \end{aligned}$$

Example

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Find

1. $P(B_1 < 2)$
2. $P(B_2 < 3 | B_1 = 1)$
3. $P(B_1 + B_2 > 2)$

Solution

1. $B_1 \hookrightarrow N(0,1)$. So

$$P(B_1 < 2) = \int_{-\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \ldots$$

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2.

$$\begin{aligned}
P(B_2 < 3 | B_1 = 1) &= P(B_2 - B_1 + B_1 < 3 | B_1 = 1) \\
&= P(B_2 - B_1 + 1 < 3 | B_1 = 1) \\
&= P(\underbrace{B_2 - B_1}_{\text{independent of } B_1} < 2 | B_1 = 1) \\
&= P(\underbrace{B_2 - B_1}_{\text{same distribution as } B_{2-1}} < 2) \\
&= P(B_1 < 2)
\end{aligned}$$

3. Need to **find distribution of** $B_1 + B_2$ first.
We have $B_1 \hookrightarrow \mathcal{N}(0,1)$, $B_2 \hookrightarrow \mathcal{N}(0,2)$, $Cov(B_1, B_2) = 1$.
So $B_1 + B_2$ is normally distributed with

$$E(B_1 + B_2) = E(B_1) + E(B_2) = 0$$

and

$$Var(B_1 + B_2) = \underbrace{Var(B_1)}_{=1} + \underbrace{Var(B_2)}_{=2} + 2\underbrace{Cov(B_1, B_2)}_{=1} = 5$$

So

$$B_1 + B_2 \hookrightarrow \mathcal{N}(0,5)$$

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Another approach

$$B_1 + B_2 = B_1 + B_2 - B_1 + B_1 = \overbrace{\underbrace{2B_1}_{\mathcal{N}(0,4)} + \underbrace{(B_2 - B_1)}_{\mathcal{N}(0,1)}}^{independent}$$

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Hence

$$P(B_1 + B_2 > 2) = \int_2^\infty \frac{1}{\sqrt{2\pi}\sqrt{5}} e^{-\frac{x^2}{10}} dx$$

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Find conditional pdf of B_3 given $B_1 = 2$, $f_{B_3|B_1}(x|2)$

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Solution

- Given $B_1 = 2$, we have

$$B_3 = \underbrace{B_3 - B_1}_{\text{indepdendent of } B_1} + B_1 = \underbrace{B_3 - B_1}_{\mathcal{N}(0,2)} + 2 \sim \mathcal{N}(2,2)$$

- conditional pdf

$$f_{B_3|B_1}(x|2) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{(x-2)^2}{2*2}} = \frac{1}{2\sqrt{\pi}} e^{-\frac{(x-2)^2}{4}}$$

Another approach to find conditional distribution of B_t given $B_s = x$

- (B_t, B_s) is bivariate normally distributed with mean

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  and covariance matrix

$$\Sigma = \begin{pmatrix} t & \min(s,t) \\ \min(s,t) & s \end{pmatrix}$$

- Conditional pdf

$$f_{B_t|B_s}(u|x) = \frac{f_{B_t,B_s}(u,x)}{f_{B_s}(x)}$$

- One can verify that

$$B_t|(B_s = x) \sim \mathcal{N}\left(\frac{\min(s,t)x}{s}, t - \frac{\min^2(s,t)}{s}\right)$$

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Proof

- **Starting at 0** $-B_0 = -0 = 0$
- **Independent increments** For $0 = t_0 < t_1 < t_2 < \cdots < t_m$ then

$$\underbrace{-B_{t_1} - (-B_{t_0})}_{-(B_{t_1} - B_{t_0})}, \underbrace{-B_{t_1} - (-B_{t_0})}_{-(B_{t_2} - B_{t_1})}, \ldots, \underbrace{-B_{t_m} - (-B_{t_{m-1}})}_{-(B_{t_m} - B_{t_{m-1}})}$$

  are independent because of independent increments of $(B_t)_{t\geq 0}$
- **Stationary increments**

$$-B_{t_{i+1}} - (-B_{t_i}) = -(B_{t_{i+1}} - B_{t_i}) \hookrightarrow \mathcal{N}(0, t_{i+1} - t_i)$$

  because of symmetric property of centered normal distribution.

Practice - Scaling a Brownian Motion

For $a > 0$, let

$$W_t = \frac{B_{at}}{\sqrt{a}}$$

then $(W_t)_{t\geq 0}$ is also a Brownian motion.

Filtration for Brownian Motion

In addition to the Brownian motion itself, we will need some notation for the amount of information available at each time.

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Let $(B_t)_{t\geq 0}$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. A filtration for the Brownian motion is a collection of σ - algebra $(\mathcal{F}_t)_{t\geq 0}$, satisfying

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2. **(Adaptivity)** For every $t \geq 0$, B_t is $\mathcal{F}_t$ - measurable. In other words, the information available at time t is sufficient to evaluate the Brownian motion B_t at that time.
3. **(Independent of the future increments)** For $0 \leq t < u$, the increment $B_u - B_t$ is independent of $\mathcal{F}_t$.

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3. **(Independent of the future increments)** For $0 \leq t < u$, the increment $B_u - B_t$ is independent of $\mathcal{F}_t$.In other words, any increment of the Brownian motion after time t is independent of the information available at time t

Interpretation

- Properties 1 and 2 guarantee that the information available at each time t is at least as much as one would learn from observing the Brownian motion up to time t.
- Property 3 says that this information is of no use in predicting future movements of the Brownian motion. In the asset-pricing models we build, property 3 leads to the efficient market hypothesis

Example

1. Natural filtration $\mathcal{F}_t = \sigma(B_s, s \leq t)$
2. The other is to include in $\mathcal{F}_t$ information obtained by observing the Brownian motion and one or more other processes. However, if the information in $\mathcal{F}_t$ includes observations of processes other than the Brownian motion B, this additional information is not allowed to give clues about the future increments of B

Remark

From now, we always consider $(B_t)_{t\geq 0}$ is a Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$

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Discretization Brownian motion

Suppose we want to model a standard Brownian motion B_t in a time interval $[0, T]$

- We divide the interval into n equal subintervals by discrete time points $0 = t_0 < t_1 < \cdots < t_n = T$, with the time step $h = t_i - t_{i-1}$.
- The value of B_{t_i} is determined by a recursive formula:

$$B_{t_i} = B_{t_{i-1}} + dB_{t_i},$$

  where $Z_i \sim N(0, 1)$.
- Increment $dB_{t_i} \sim \mathcal{N}(0, h)$

Simulation algorithm

- Discretize the interval $[0, T]$ into n subinterval with width $h = \frac{T}{n}$ and partition point $t_i = ih$

$$\begin{pmatrix} 0 & h & 2h & \dots & nh = T \end{pmatrix}$$

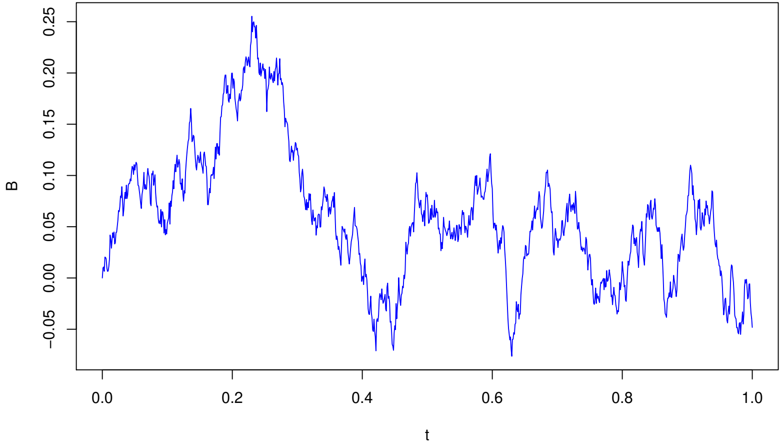
- Simulate a sequence of independent increments $dB_h, dB_{2h}, \dots, dB_{nh} \sim N(0, h)$

$$dB = \begin{pmatrix} dB_h & dB_{2h} & \dots & dB_{nh} \end{pmatrix}$$

- Generate value of a path of Brownian motion $B_0, B_h, B_{2h}, \dots, B_{nh}$

$$\begin{aligned} B_0 &= 0 \\ B_h &= B_0 + dB_h \\ B_{2h} &= B_h + dB_{2h} \\ &\dots \\ B_{nh} &= B_{(n-1)h} + dB_{nh} \end{aligned}$$

A sample of standard Brownian motion



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Martingale property

A standard Brownian motion is a martingale

$$E(B_t|\mathcal{F}_s) = B_s$$

for all $t \geq s$

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Proof.

$$\begin{aligned}
E(B_t|\mathcal{F}_s) &= E(\underbrace{B_t - B_s}_{\text{independent of } \mathcal{F}_s} + \underbrace{B_s}_{\mathcal{F}_s - measurable} |\mathcal{F}_s) \\
&= E(B_t - B_s|\mathcal{F}_s) + E(B_s|\mathcal{F}_s) \\
&= E(B_t - B_s) + B_s \\
&= E(B_{t-s}) + B_s \\
&= 0 + B_s \\
&= B_s
\end{aligned}$$

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Markov process

Let $(X_t)_{t \geq 0}$ be a adapted process with respect to the filtration probability space $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$. X is said to be a Markov process if for any Borel - measurable function f, there is another Borel - measurable function g such that

$$E(f(X_t)|\mathcal{F}_s) = g(X_s), \forall t > s$$

Theorem (Markov Property of Brownian Motion)

Let $(B_t)_{t\geq 0}$ is a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ be a filtration for this Brownian motion. Then $(B_t)_{t\geq 0}$ is a Markov process.

Proof

- $E(f(B_{t+s})|\mathcal{F}_s) = E(f(B_{t+s} - B_s + B_s)|\mathcal{F}_s)$
- $B_{t+s} - B_s$ is independent of $\mathcal{F}_s$. So $f(B_{t+s} - B_s + B_s)$ only depends on B_s, i.e

  $$E(f(B_{t+s})|\mathcal{F}_s) = E(f(B_{t+s} - B_s + B_s)|B_s) = g(B_s)$$

  for some g.
- Find g by replace B_s by a dummy variable x

  $$\begin{aligned} E(f(B_{t+s})|B_s = x) &= E(f(B_{t+s} - B_s + B_s|B_s = x) \\ &= E(f(B_{t+s} - B_s + x|B_s = x) \\ &= E(f(B_{t+s} - B_s + x)) \\ &= \int_{-\infty}^{\infty} f(z+x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dz \end{aligned}$$

- Substituting $y = z + x$, we get

$$E(f(B_{t+s})|B_s = x) = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy$$

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- Hence

$$g(x) = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy$$

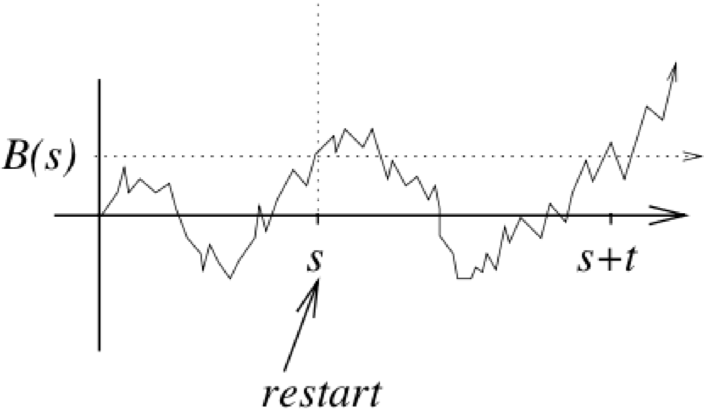

Figure: Markov property of Brownian motion

Transition density

The the probability (density) that a Browninan motion change from x to y in time t

$$p(t,x,y) = f_{B_{t+s}|B_s}(y|x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$$

is called to be *transition density*.

Interpretation: Conditioned on information in $\mathcal{F}_t$ (which contains information obtained by observing the Brownian motion up to and including time s), the conditional density of B_{t+s} is $p(t, B_s, y)$

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We have

$$E(f(B_{t+s})|\mathcal{F}_s) = \int_{-\infty}^{\infty} f(y)p(t, B_s, y)dy$$

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First passage time

Let $(B_t)_{t\geq 0}$ be a Brownian motion and let m be a real number.

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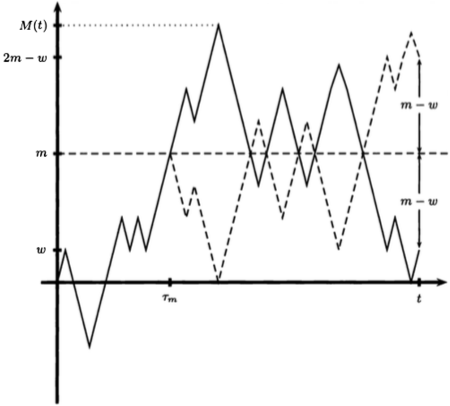
- Denote the first time $(B_t)_{t \geq 0}$ hits level m is $\tau_m = \inf\{t \geq 0 | B_t = m\}$: the first passage time.
- If $(B_t)_{t \geq 0}$ never reaches the level m, we set $\tau_m = \infty$.

Reflection principle of Brownian motion

$P(\tau_m \leq t, B_t \leq w) = P(B_t > 2m - w)$, $m \geq w$



Explain

- "count" the Brownian motion paths that reach level m at or before time t
  - reach level m prior to t but at time t are at some level w below m
  - reach level m at time t: probability is 0 because B_t is a continuous random variable. *Ignore this possibility*
- for each Brownian motion path that reaches level m prior to time t but is at a level w below m at time t, there is a "reflected path" that is at level $2m - w$ at time t. This reflected path is constructed by switching the up and down moves of the Brownian motion from time τ_m onward

$$P(\tau_m \leq t, B_t \leq w) = P(\tau_m \leq t, B_t > 2m - w)$$

- But $m > s$ so $B_t > 2m - w > m$. It implies that Brownian motion hits level m before t. Then we get *reflection equality*

$$P(\tau_m \leq t, B_t \leq w) = P(B_t > 2m - w)$$

First passage time τ_m, $m > 0$

- Starting from 0, if $B_t > m$ then we are guaranteed that $\tau_m \leq t$
$$P(\tau_m \leq t, B_t \geq m) = P(B_t \geq m)$$
- Substitute w by m in reflection formula
$$P(\tau_m \leq t, B_t \leq m) = P(B_t > m)$$
- cdf of τ_m
$$\begin{aligned} F_{\tau_m}(t) = P(\tau_m \leq t) \\ &= P(\tau_m \leq t, B_t \geq m) + P(\tau_m \leq t, B_t < m) \\ &= 2P(B_t > m) = 2P(\underbrace{Z}_{\mathcal{N}(0,1)} > \frac{m}{\sqrt{t}}) \end{aligned}$$

because $\frac{B_t}{\sqrt{t}} \sim \mathcal{N}(0, 1)$

First passage time distribution

cdf of τ_m if given by

$$F_{\tau_m}(t) = 2\int_{\frac{|m|}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

pdf of τ_m is given by

$$f_{\tau_m}(t) = \frac{d}{dt} F_{\tau_m}(t) = \frac{|m|}{\sqrt{2\pi t^3}} e^{-\frac{m^2}{2t}}$$

Proof

For $m \geq 0$, the cdf of τ_m is given by

$$F_{\tau_m}(t) = P(\tau_m \leq t) = 2P(B_t > m) \quad = 2\int_{\frac{|m|}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

Hence the pdf of τ_m is

$$f_{\tau_m}(t) = \frac{d}{dt} F(t) = \frac{m}{\sqrt{2\pi t^3}} e^{-\frac{m^2}{2t}}$$

For $m < 0$??

Proof

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Hence the pdf of τ_m is

$$f_{\tau_m}(t) = \frac{d}{dt}F(t) = \frac{m}{\sqrt{2\pi t^3}} e^{-\frac{m^2}{2t}}$$

For $m < 0$??
$(-B_t)$ is also a standard BM. Working on $(-B_t)$ and hitting time of $-m$

Properties

- $P(\tau_m < \infty) = 1$
- $E(\tau_m) = \infty$

Example

A particle moves according to standard Brownian motion (BM) started. Find the probability that the particle reaches level 2 sometime in 1 hour.

Example

A particle moves according to standard Brownian motion (BM) started. Find the probability that the particle reaches level 2 sometime in 1 hour.

Solution

- τ_2: first time that BM reaches level 2
- Need to find $P(\tau_2 \leq 1)$

Example

A particle moves according to standard Brownian motion (BM) started. Find the probability that the particle reaches level 2 sometime in 1 hour.

Solution

- τ_2: first time that BM reaches level 2
- Need to find $P(\tau_2 \leq 1)$
- First approach (using cdf)

$$P(\tau_2 \leq 1) = 2P(B_1 > 2) = 2\int_2^\infty f_{B_1}(x)dx$$

- $B_1 \sim \mathcal{N}(0,1)$ with pdf

$$f_{B_1}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

-

$$P(\tau_2 \leq 1) = 2\int_2^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = \dots$$

2nd approach (using pdf)

- pdf of τ_2 (for $m = 2$) is

$$f_{\tau_2}(t) = \frac{2}{\sqrt{2\pi t^3}} e^{-\frac{2^2}{2t}} = \frac{2}{\sqrt{2\pi t^3}} e^{-\frac{2}{t}}$$

2nd approach (using pdf)

- pdf of τ_2 (for $m = 2$) is

$$f_{\tau_2}(t) = \frac{2}{\sqrt{2\pi t^3}} e^{-\frac{2^2}{2t}} = \frac{2}{\sqrt{2\pi t^3}} e^{-\frac{2}{t}}$$

- $$P(\tau_2 \leq 1) = \int_0^1 f_{\tau_2}(t) dt = \int_0^1 \frac{2}{\sqrt{2\pi t^3}} e^{-\frac{2}{t}} dt = \ldots$$

Maximum of Brownian motion

Maximum to date for Brownian motion

$$M_t = \max_{0 \le s \le t} B_s$$

Maximum of Brownian motion

Maximum to date for Brownian motion

$$M_t = \max_{0 \leq s \leq t} B_s$$

Property

$M_t \geq m$ if and only if $\tau_m \leq t$.

Distribution M_t

Survival distribution function

$$F_{M_t}(x) = P(M_t > x) = P(\tau_x < t) = 2P(B_t > x) = \int_x^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}}\,du$$

and pdf

$$f_{M_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,du$$

for $x \geq 0$

Example

A laboratory instrument takes annual temperature measurements. Measurement errors are assumed to be independent and normally distributed. As precision decreases over time, errors are modeled as standard Brownian motion. For how many years can the lab be guaranteed that there is at least 90% probability that all errors are less than 4 degrees, given that the 95th percentile of the standard normal random distribution is 1.645?

Solution

Find t *such that* $P(M_t \leq 4) \geq .9$ *or* $P(M_t > 4) \leq .1$. *It implies that*

$$2P(B_t > 4) \leq .1 \Leftrightarrow P(B_t > 4) \leq .05$$

$B_t \hookrightarrow N(0,t) \Rightarrow B_t = \sqrt{t}Z$ *where* $Z \hookrightarrow N(0,1)$. *Hence*

$$P(B_t > 4) = P(\sqrt{t}Z > 4) = P\left(Z > \frac{4}{\sqrt{t}}\right) \leq .05$$

So $\frac{4}{\sqrt{t}} = z_{.05} = 1.645$ *and then* $t = \frac{4}{1.645}$

Joint distribution of M_t and B_t

Theorem

The joint pdf of (M_t, B_t) is given by

$$f_{M_t,B_t}(m,w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}$$

for $m > 0, b \leq m$

Proof

- Remark that $\tau_m \leq t \Leftrightarrow M_t \geq m$ for $m > 0$. So the reflection equality
$$P(\tau_m \leq t, B_t \leq w) = P(B_t > 2m - w)$$
can be rewritten as
$$P(M_t \geq m, B_t \leq w) = P(B_t > 2m - w)$$
or
$$\int_m^\infty \int_{-\infty}^w f_{M_t,B_t}(x,y)dydx = \int_{2m-w}^\infty f_{B_t}(y)dy$$
- Because $B_t \sim \mathcal{N}(0,t)$, $f_{B_t}(y) = c1\sqrt{2\pi t}e^{-\frac{y^2}{2t}}$. Hence
$$\int_m^\infty \int_{-\infty}^w f_{M_t,B_t}(x,y)dydx \int_{2m-w}^\infty \frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}dy$$

- Differentiate first with respect to m

$$-\int_{-\infty}^{w} f_{M_t,B_t}(m,y)dy = -\frac{1}{\sqrt{2\pi t}}e^{-\frac{m^2}{2t}}$$

- next differentiate with respect to w

$$-f_{M_t,B_t}(m,w) = -\frac{2(2m-w)}{t\sqrt{2\pi t}}e^{-\frac{(2m-w)^2}{2t}}$$

Conditional pdf of M_t given B_t

Remind that

$$f_{M_t|B_t}(m|w) = \frac{f_{M_t,B_t}(m,w)}{f_{B_t}(w)}$$

Corollary

$$f_{M_t|B_t}(m|w) = \frac{2(2m-2)}{t} e^{-\frac{2m(m-w)}{t}}$$

for $m > 0$ and $w \leq m$.

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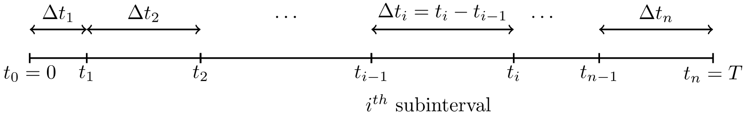
Partition



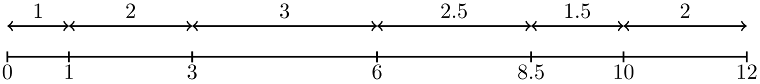
- $\Pi = \{t_0, t_1, ..., t_n\}$ be a **partition** of $[0, T]$, i.e

$$0 = t_0 < t_1 < ... < t_n = T$$

- $\Delta t_i = t_i - t_{i-1}$: length of the i^{th} subinterval $[t_{i+1} - t_i]$
- **Mesh** or **Norm** of Π

$$\|\Pi\| = \max_{1 \leq i \leq n} |t_i - t_{i-1}| = \max_{0 \leq i \leq n-1} \Delta t_i$$

Example



$\Pi = \{0, 1, 3, 6, 8.5, 20, 12\}$ is a partition of $[0, 12]$ with mesh $\|\Pi\| = 3$

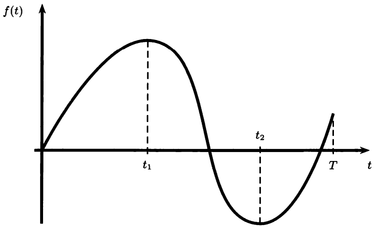
First - order variation

Amount of up and down oscillation undergone by a function f between times 0 and T

First - order variation

Amount of up and down oscillation undergone by a function f between times 0 and T

Example



$$FV_T(f) = (f(t_1) - f(0)) - (f(t_2) - f(t_1)) + (f(T) - f(t_2))$$

Remark that

$$-(f(t_2) - f(t_1)) = f(t_1) - f(t_2)$$

guarantees that the magnitude of the down move of the function $f(t)$ between times t_1 and t_2 is added to rather than subtracted from the total

Definition of First - order Variation

The first-order variation of a function f up to time T is given by

$$FV_T(f) = \lim_{\|\Pi\| \to 0} \sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)|$$

where Π is a partition of $[0, T]$.
The limit is taken as the number n of partition points goes to infinity and the length of the longest subinterval $t_{i+1} - t_i$ goes to zero

Example

Let's compute the FV of the function $f(t) = t$ up to time $T = 1$.

- Divide interval $[0, 1]$ into n subinterval by partition $\pi = \{t_0, t_1, \dots, t_n\}$, $t_i = \frac{i}{n}$
- Every subinterval has the same length of $\frac{1}{n}$. So $\|\Pi\| = \frac{1}{n}$
- Magnitude of increment during each subinterval

$$|f(t_1) - f(t_0)| = |t_1 - t_0| = \left|\frac{1}{n} - 0\right| = \frac{1}{n}$$

$$|f(t_2) - f(t_1)| = |t_2 - t_1| = \left|\frac{2}{n} - \frac{1}{n}\right| = \frac{1}{n}$$

$$\dots$$

$$|f(t_n) - f(t_{n-1})| = |t_n - t_{n-1}| = \left|\frac{n}{n} - \frac{n-1}{n}\right| = \frac{1}{n}$$

- Sum of magnitude of all increment

$$S_n = \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = n \times \frac{1}{n} = 1$$

- $\|\Pi\| \to 0$ if $n \to \infty$
- $$FV_1(f) = \lim_{\|\Pi\| \to 0} \sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)| = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 = 1$$

Practice

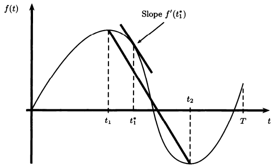
Find the first - order of variation of the function $f(t) = t^2$ up to time 1.

Proposition

If f is differentiable everywhere then

$$FV_T(f) = \int_0^T |f'(t)|dt$$

Proof



By mean value theorem, in each subinterval $[t_i, t_{i+1}]$, there exists t_i^* such that

$$f'(t_i^*) = \frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}$$

or

$$f(t_{i+1}) - f(t_i) = f'(t_i^*)(t_{i+1} - t_i) = f'(t_i^*)\Delta t_i$$

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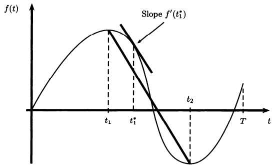

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or

$$f(t_{i+1}) - f(t_i) = f'(t_i^*)(t_{i+1} - t_i) = f'(t_i^*)\Delta t_i$$

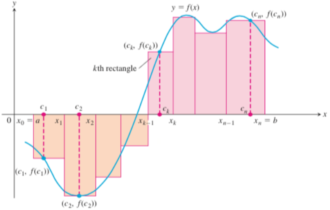
Thus

$$\sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)| = \sum_{i=1}^{n} |f'(t_i^*)|\Delta t_i \xrightarrow{\|\Pi\|\to 0} \int_0^T |f'(t)|dt$$

Remind Riemann Integral

- $\Pi = \{x_0 = a, x_1, \ldots, x_n = b\}$: a partition of $[a, b]$
- $c_k \in [x_{k-1}, x_k]$
- Riemann sum

$$f(c_1)\Delta x_1 + \cdots + f(c_n)\Delta x_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$



Riemann integral

$$\int_a^b f(x)dx = \lim_{\|\Pi\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Quadratic variation

of a function f up to time T is

$$\langle f, f\rangle(T) = \lim_{\|Pi\|\to 0} \sum_{i=0}^{n-1} (f_{i+1} - f_i)^2$$

where $\Pi = \{t_0 = 0, t_1, \ldots, t_n = T\}$ is a partition of $[0, T]$

Example

Let's compute the FV of the function $f(t) = t$ up to time $T = 1$.

- Divide interval $[0, 1]$ into n subinterval by partition $\pi = \{t_0, t_1, \ldots, t_n\}$, $t_i = \frac{i}{n}$
- Every subinterval has the same length of $\frac{1}{n}$. So $\|\Pi\| = \frac{1}{n}$
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$$\cdots$$

$$|f(t_n) - f(t_{n-1})| = |t_n - t_{n-1}| = \left|\frac{n}{n} - \frac{n-1}{n}\right| = \frac{1}{n}$$

- Sum of square of magnitude of all increment

$$S_n = \left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)^2 = n \times \frac{1}{n^2} = \frac{1}{n}$$

- $\|\Pi\| \to 0$ if $n \to \infty$
- Quadratic variation of f up to time 1 is

$$\langle f \rangle (1) = \lim_{\|\Pi\| \to 0} \sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)| = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

Quadratic Variation of a Continuous Differentiable Function

If f is continuous differentiable on $[0, T]$ then $\langle f, f\rangle(T) = 0$

Proof

- f is differentiable on $[0, T]$, so $\exists t_i^* \in [t_i, t_{i+1}]$ such that
$$f(t_{i+1}) - f(t_i) = f'(t_i^*)(t_{i+1} - t_i)$$
- So $(f(t_{i+1}) - f(t_i))^2 = (f'(t_i^*))^2 (t_{i+1} - t_i)^2 = f'(t_i^*))^2 (\Delta t_i^2)$
- $\|\Pi\| = \max_{i \leq n} \Delta t_i \Rightarrow \Delta t_i \leq \|\Pi\|$ for all i
- $(\Delta t_i^2) \leq \|\Pi\| \Delta t_i$
- $(f(t_{i+1}) - f(t_i))^2 \leq \|\Pi\| (f'(t_i^*))^2 \Delta t_i$

Hence

$$\begin{aligned}
0 \leq & \langle f, f \rangle (T) \leq \lim_{\|\Pi\| \to 0} \|\Pi\| \sum_{i=0}^{n-1} (f'(t_i^*))^2 \Delta t_i \\
& = \left(\lim_{\|\Pi\| \to 0} \|\Pi\| \right) \left(\lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} (f'(t_i^*))^2 \Delta t_i \right) \\
& = 0 * \int_0^T (f'(t))^2 dt = 0
\end{aligned}$$

So $\langle f, f \rangle (T) = 0$

How to compute quadratic variation of Brownian motion

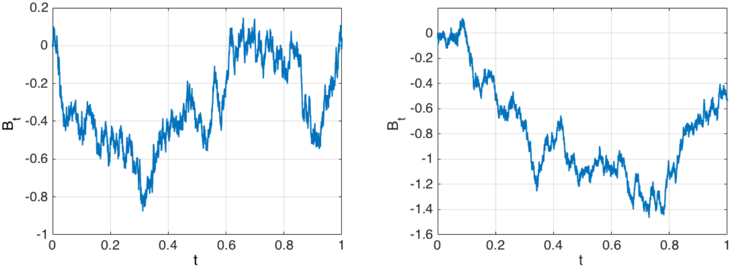
- Value of a standard Brownian motion at each time t is not deterministic

How to compute quadratic variation of Brownian motion

- Value of a standard Brownian motion at each time t is not deterministic but a random variable

How to compute quadratic variation of Brownian motion

- Value of a standard Brownian motion at each time t is not deterministic but a random variable
- Each path of Brownian motion (corresponding to some outcome ω in sample space) is a continuous function in time t
- Need to compute quadratic variation up to time T of each path $(B_t(w))_{t\geq 0}$



Quadractic variation of a standard Brownian motion up to time T

$$\langle B(w) \rangle (T)$$

depends on path $(B_t(w))_{t \geq 0}$

Simulation practice

Approximate quadratic variation of a path of standard Brownian motion up to time 1

- Simulate a sequence value of B_0, B_h,..., $B_{nh} = B_1$ for $h = 10^{-5}$
- Compute the sum of square of all increment along this path

$$(B_h - B_0)^2 + (B_{2h} - B_h)^2 + \cdots + (B_{nh} - B_{(n-1)h})^2$$

Repeat several times to get quadratic variation along different paths of Brownian motion. Compare your obtained values of quadratic variation.

Quadratic Variation of a Brownian Motion

$$P(w : \langle B \rangle(T) = T) = 1$$

or

$$\langle B \rangle(T) = T$$

almost surely.

In particular, almost paths of Brownian motion are not differentiable everywhere.

Quadratic Variation of a Brownian Motion

$$P(w : \langle B\rangle(T) = T) = 1$$

or

$$\langle B\rangle(T) = T$$

almost surely.

In particular, almost paths of Brownian motion are not differentiable everywhere.

Remind important properties used in proof

For $t > s$

- $B_t - B_s \hookrightarrow N(0, t-s)$
- $E((B_t - B_s)^2 = t - s$
- $Var((B_t - B_s)^2 - (t-s)) = 2(t-s)^2$

Outline of proof

- $\Pi = \{t_0, t_1, ..., t_n\}$ is a partition of $[0, T]$
- **Sample quadratic variation** $Q_\Pi = \sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_k})^2$
- Idea: show that $E(Q_\Pi) = T$ and $Var(Q_\Pi) \to 0$. Which implies that $\lim_{\|\Pi\| \to 0} (Q_\Pi - T) = 0$ a.s
- $Q_\Pi - T = \sum_{k=0}^{n-1} ((B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k))$
- $E(Q_\Pi - T) = \sum_{k=0}^{n-1} E((B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)) = 0$
- $Var(Q_\Pi - T) = \sum_{k=0}^{n-1} Var((B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)) =$ $2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \leq 2 \max_{0 \leq k \leq n-1} (t_{k+1} - t_k) \sum_{k=0}^{n-1} (t_{k+1} - t_k) =$ $2\|\Pi\| T \to 0$ as $\|\Pi\| \to 0$

Remark

- $E((B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)) = 0$
- $Var((B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)) = 2(t_{k+1} - t_k)^2$

when $t_{k+1} - t_k \simeq 0$

$$(B_{t_{k+1}} - B_{t_k})^2 \simeq t_{k+1} - t_k$$

Differential Representation

$$dB_t dB_t = dt$$

Remark

- $$\sum_{i=1}^{n} |B_{t_{k+1}} - B_{t_k})(t_{k+1} - t_k)|$$

  $$\leq \max_{0 \leq k \leq n-1} |B_{t_{k+1}} - B_{t_k}| \sum_{i=1}^{n} (t_{k+1} - t_k)$$

  $$= \max_{0 \leq k \leq n-1} |B_{t_{k+1}} - B_{t_k}| T \to 0$$

  since B is continuous.
  So we have, when $t_{k+1} - t_k \approx 0$

  $$|B_{t_{k+1}} - B_{t_k})(t_{k+1} - t_k)| \approx 0$$

  $dB_t dt = 0$
- Simular argument leads

  $$\sum_{i=1}^{n} (t_{k+1} - t_k)^2 \to 0$$

  Hence $dtdt = 0$

Important (informal) differential notations

- $dB_t dB_t = dt$
- $dB_t dt = 0$
- $dt dt = 0$

Importance of nonzero quadratic variation of Brownian motion

This makes stochastic calculus different from ordinary calculus and is the source of the volatility term in the Black-Scholes-Merton partial differential equation.

Simulate quadratic variation of a standard Brownian motion

- Discretize the interval $[0, T]$ into n subinterval with width $h = \frac{T}{n}$ and partition point $t_i = ih$

$$\begin{pmatrix} 0 & h & 2h & \dots & nh = T \end{pmatrix}$$

- Simulate a sequence of independent increments $dB_h, dB_{2h}, \dots, dB_{nh} \sim N(0, h)$

$$dB = \begin{pmatrix} dB_h & dB_{2h} & \dots & dB_{nh} \end{pmatrix}$$

- Compute square of increments

$$(dB)^2 = \begin{pmatrix} (dB_h)^2 & (dB_{2h})^2 & \dots & (dB_{nh})^2 \end{pmatrix}$$

- Evaluate quadratic variation at T along this path

$$\langle B, B\rangle(T) = (dB_h)^2 + (dB_{2h})^2 + \cdots + (dB_{nh})^2$$

Quadratic Variation as Absolute Volatility

- $T_1 = t_0 < t_1 < \dots < t_n = T_2$ is a partition of $[T_1, T_2]$
- *Squared sample absolute volatility*

$$\frac{1}{T_2 - T_1} Q_\Pi = \frac{1}{T_2 - T_1} \sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_k})^2$$

is approximated by

$$\frac{1}{T_2 - T_1} (\langle B, B \rangle)(T_2) - \langle B, B \rangle(T_1) = \frac{1}{T_2 - T_1} (T_2 - T_1) = 1$$

- Brownian motion has absolute volatility 1.
- Quadratic variation for Brownian motion accumulates at rate 1 at all times along almost every path.

$$\langle B, B \rangle(T) = \int_0^T 1 dt$$

Volatility of Geometric Brownian Motion (1)

- Geometric Brownian motion

$$S_t = S_0 e^{\sigma B_t + (r - \frac{\sigma^2}{2})t}$$

  is an asset price model used in Black-Scholes-Merton option pricing.
- Observe price on interval time $[T_1, T_2]$ at time $t_0 = T_1 < t_1 < ... < t_n = T_2$ (correspond to a partition Π of $[T_1, T_2]$)
- Log-return

$$\ln \frac{S_{t_{k+1}}}{S_{t_k}} = \sigma(B_{t_{k+1}} - B_{t_k}) + (r - \frac{\sigma^2}{2})(t_{k+1} - t_k)$$

Volatility of Geometric Brownian Motion (2)

- Realized volatility

$$\sum_{k=0}^{n-1}\left(\ln\frac{S_{t_{k+1}}}{S_{t_k}}\right)^2 = \sigma^2\sum_{k=0}^{n-1}(B_{t_{k+1}}-B_{t_k})^2$$

$$+ 2\sigma(r-\frac{\sigma^2}{2})\sum_{k=0}^{n-1}(B_{t_{k+1}}-B_{t_k})(t_{k+1}-t_k)$$

$$+ (r-\frac{\sigma^2}{2})^2\sum_{k=0}^{n-1}(t_{k+1}-t_k)^2$$

- Let $\|\Pi\| \to 0$ then

$$\sum_{k=0}^{n-1}\left(\ln\frac{S_{t_{k+1}}}{S_{t_k}}\right)^2 \to \sigma^2(T_2-T_1)$$

- Approximate the volatility

$$\sigma^2 \approx \frac{1}{T_2-T_1}\sum_{k=0}^{n-1}\left(\ln\frac{S_{t_{k+1}}}{S_{t_k}}\right)^2$$

Practice

- Collect daily data of stock price of Vinamilk or some stock on https://finance.yahoo.com for 1 year (number of day trader $n \approx$)
- Convert price data to log return

$$r_k = \ln \frac{S_{t_{k+1}}}{S_{t_k}}$$

  with $t_k = k$
- under the assumption that the stock price is a Geometric Brownian motion, $r_k = \sigma(B_{k+1} - B_k) + (r - \frac{\sigma^2}{2})$
- Distribution of log - return $\mathcal{N}(r - \frac{\sigma^2}{2}, \sigma^2)$. Use histogram of log - return to check this condition
- Estimate daily volatility σ of the return and interest rate r