Chapter 3: MULTIPLE INTEGRALS
Lecture 10

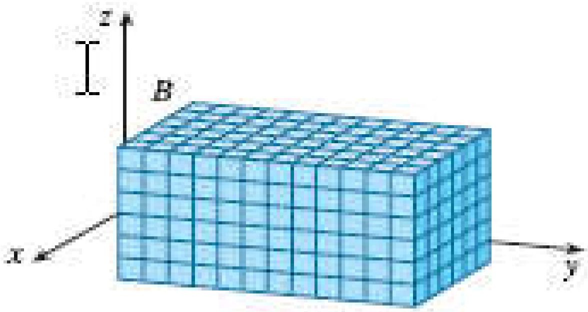
Triple Integrals

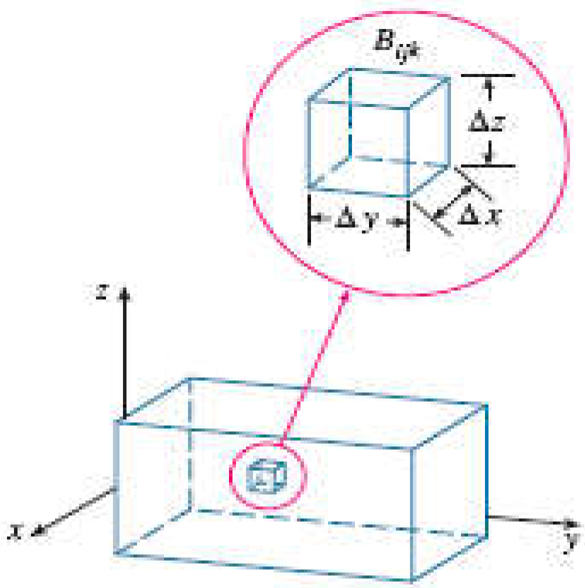
Applications of Triple Integrals

Triple Integrals over a Box

$w = f(x, y, z)$ defined on a box

$$B = \{(x, y, z) \mid a \le x \le b, c \le y \le d, r \le z \le s\}$$





❖ Divide *B into sub-boxes:*

$$[a, b] = \bigcup_{i=1}^{l} [x_{i-1}, x_i], \quad [c, d] = \bigcup_{j=1}^{m} [y_{j-1}, y_j]$$

$$[r, s] = \bigcup_{k=1}^{n} [z_{k-1}, z_k], \quad x_i - x_{i-1} = \Delta x$$

$$y_j - y_{j-1} = \Delta y, z_k - z_{k-1} = \Delta z$$

$$B = \bigcup_{i,j,k} B_{ijk}, \quad B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

$$B_{ijk} \text{ has volume } \Delta V = \Delta x \Delta y \Delta z$$

Triple Integrals over a Box

❖ Then we form the **triple Riemann sum**

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} f(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})\Delta V, \text{ where } (x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*}) \in B_{ijk}$$

<u>Definition</u>: The **triple integral** of *f* over the box B is

$$\iiint_B f(x,y,z)\,dV = \lim_{m,n,l\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} f(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*})\Delta V$$

if this limit exists

Triple integral always exists if f is continuous

Fubini's Theorem for Triple Integrals

Theorem: If f is continuous on the box $B = [a,b]\times[c,d]\times[r,s]$

then

$$\iiint_B f(x,y,z)\,dV = \int_r^s\left(\int_c^d\left(\int_a^b f(x,y,z)\,dx\right)dy\right)dz = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

Remark: There are five other possible orders in which we can integrate, all of which give the same value

Example: Evaluate triple Integral

$$\iiint_B (x+6yz)\,dV, \quad \text{where } B=[-1,1]\times[0,2]\times[0,1]$$

Solution

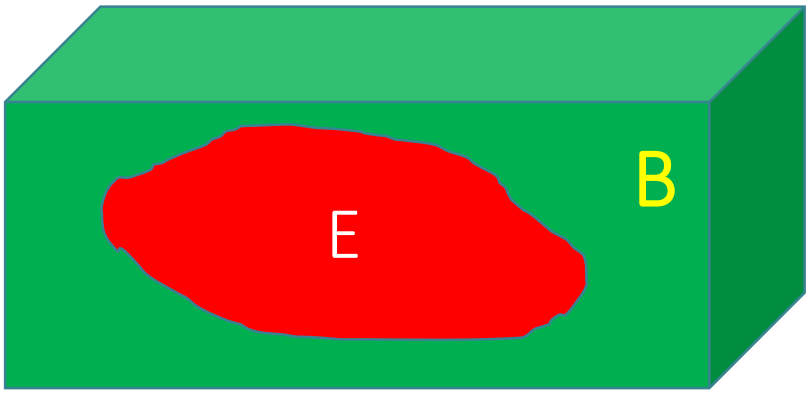
Using Fubini's Theorem, we have:

$$\iiint_B (x+6yz)\,dV = \int_{-1}^{1}\int_{0}^{2}\int_{0}^{1}(x+6yz)\,dz\,dy\,dx = \int_{-1}^{1}\int_{0}^{2}(xz+3yz^2)\Big|_{z=0}^{z=1}\,dy\,dx$$

$$= \int_{-1}^{1}\int_{0}^{2}(x+3y)\,dy\,dx = \int_{-1}^{1}(xy+3y^2/2)\Big|_{y=0}^{y=2}\,dx = \int_{-1}^{1}(2x+6)\,dx$$

$$= (x^2+6x)\Big|_{-1}^{1} = 12$$

Triple Integrals over General Regions



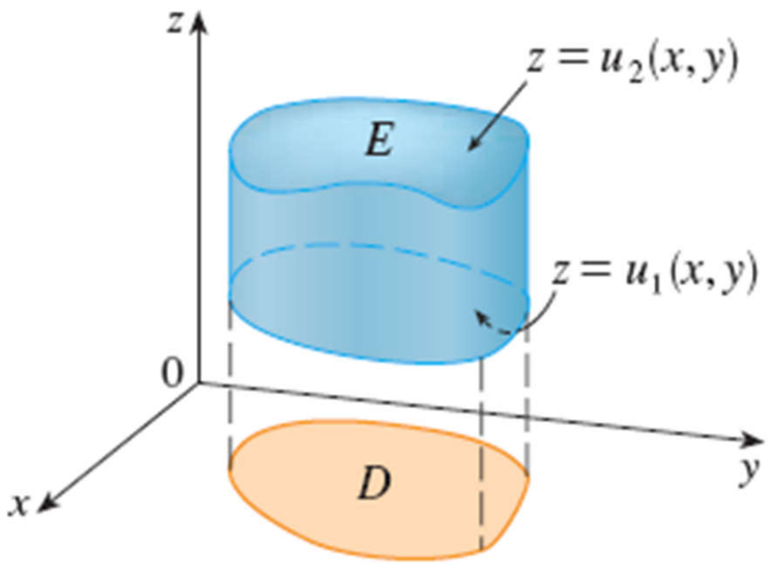
We enclose the region E in a box B

Then we define on B the function:

$$F(x,y,z)=\begin{cases} f(x,y,z), & \text{if } (x,y,z)\in E, \\ 0, & \text{if } (x,y,z)\notin E \end{cases}$$

Definition:

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$



Regions of type 1

A solid region is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

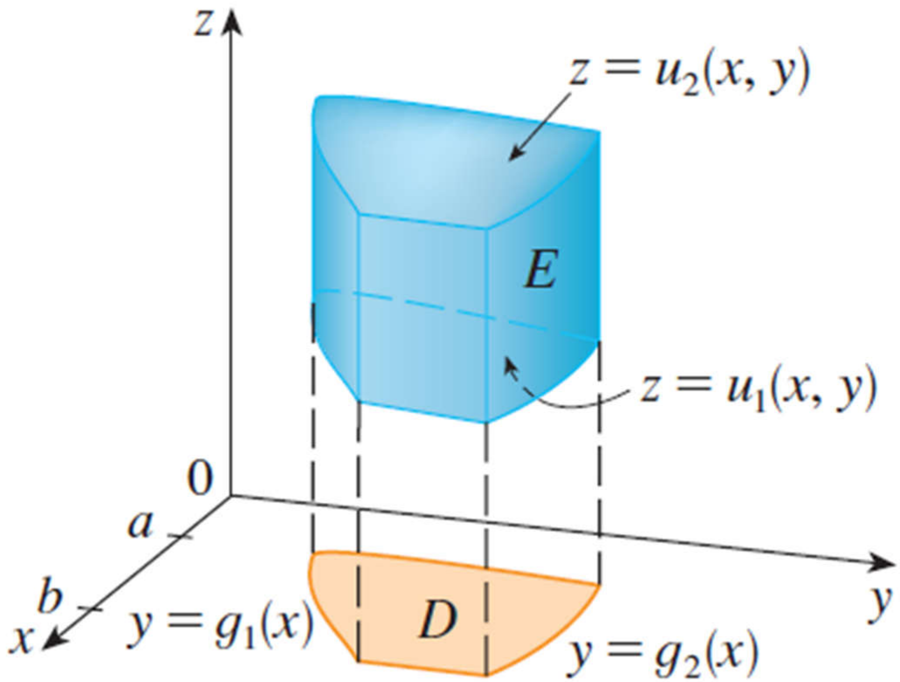
$$E = \{(x, y, z) \mid \ (x, y) \in D, \ \ u_1(x, y) \le z \le u_2(x, y)\}$$

If E is a region of type 1, it holds that

$$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz \right] dA$$

If D is a type I plane region, then

$$E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}$$

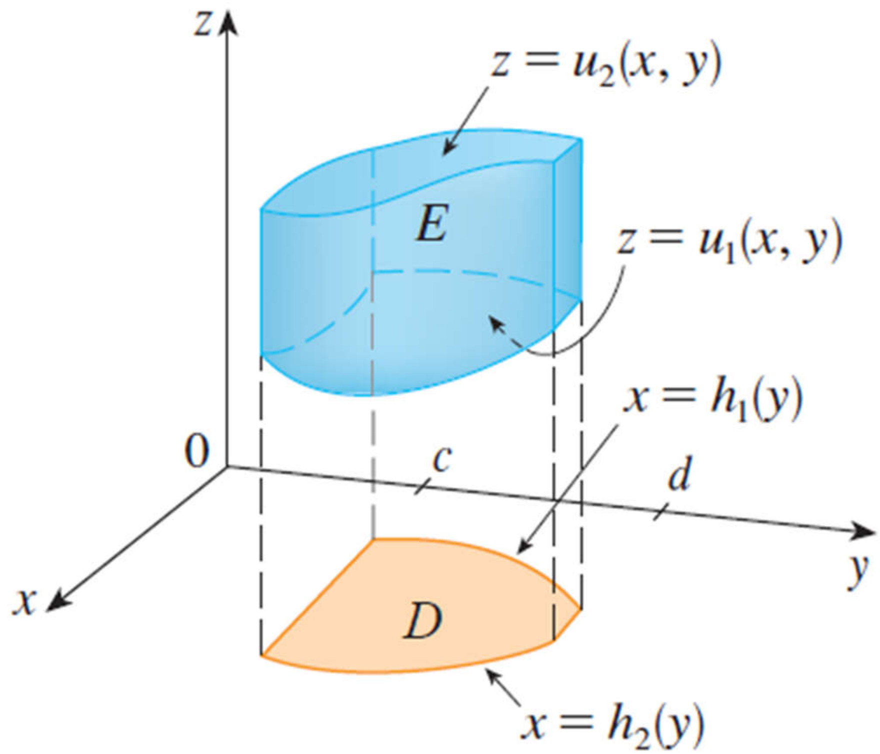


So

$$\iiint_E f(x, y, z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\,dy\,dx$$

If D is a type II plane region, then

$$E = \{(x, y, z) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y),\ u_1(x, y) \le z \le u_2(x, y)\}$$



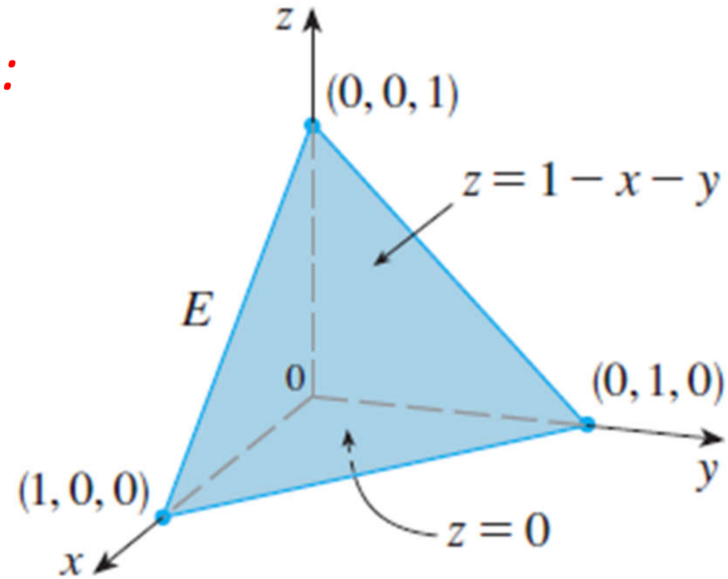
So

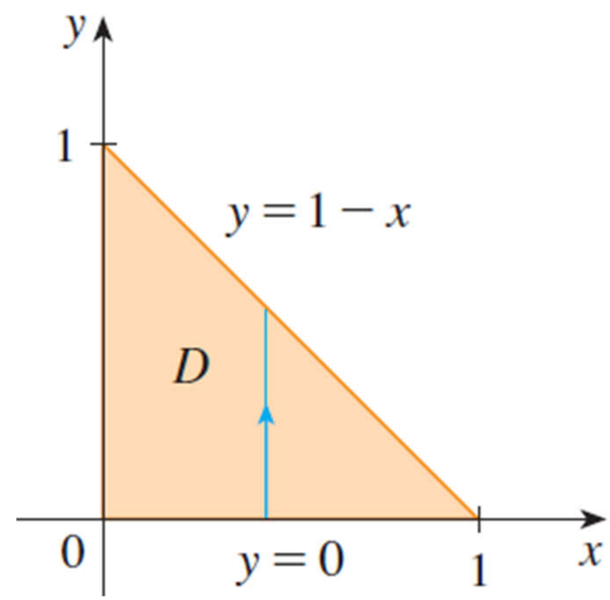
$$\iiint_E f(x, y, z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\,dx\,dy$$

Example: Evaluate triple integral $$\iiint_E 6z\, dV$$

where E is the solid tetrahedron bounded by the four planes x=0, y=0, z=0, and x+y+z=1

Solution:





$$E = \{(x, y, z) \mid \ (x, y) \in D,\ \ 0 \le z \le 1 - x - y\}$$

$$D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$$

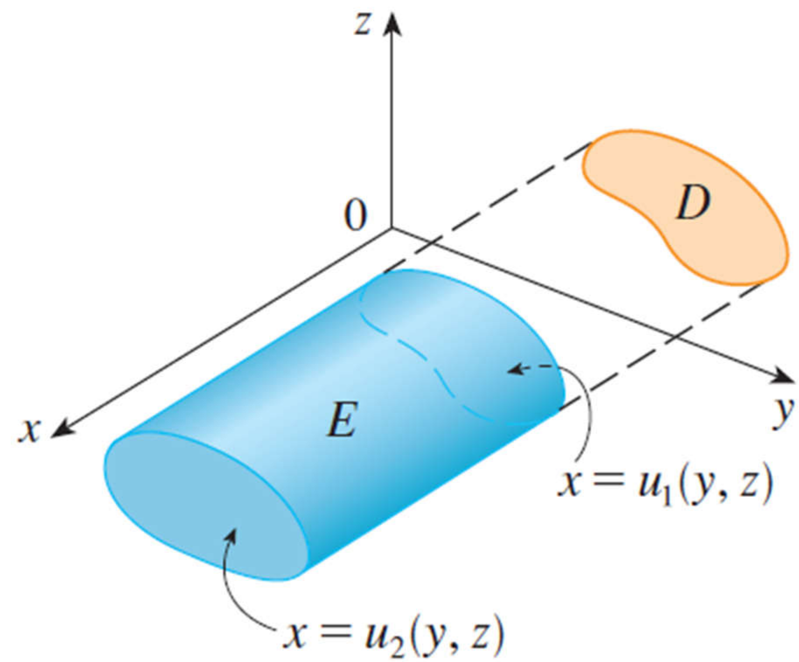
So

$$E=\{(x,y,z)\,|\,0\le x\le 1,0\le y\le 1-x,0\le z\le 1-(x+y)\}$$

$$\iiint\limits_E 6z\,dV=\int_0^1\int_0^{1-x}\int_0^{1-x-y}6z\;dzdydx=\int_0^1\int_0^{1-x}3z^2\Big|_{z=0}^{z=1-x-y}dydx$$

$$=\int_0^1\int_0^{1-x}3(1-x-y)^2dydx=\int_0^1\Big[(x+y-1)^3\Big]_{y=0}^{y=1-x}$$

$$=\int_0^1(1-x)^3dx=\frac{-(1-x)^4}{4}\Bigg|_0^1=\frac{1}{4}$$



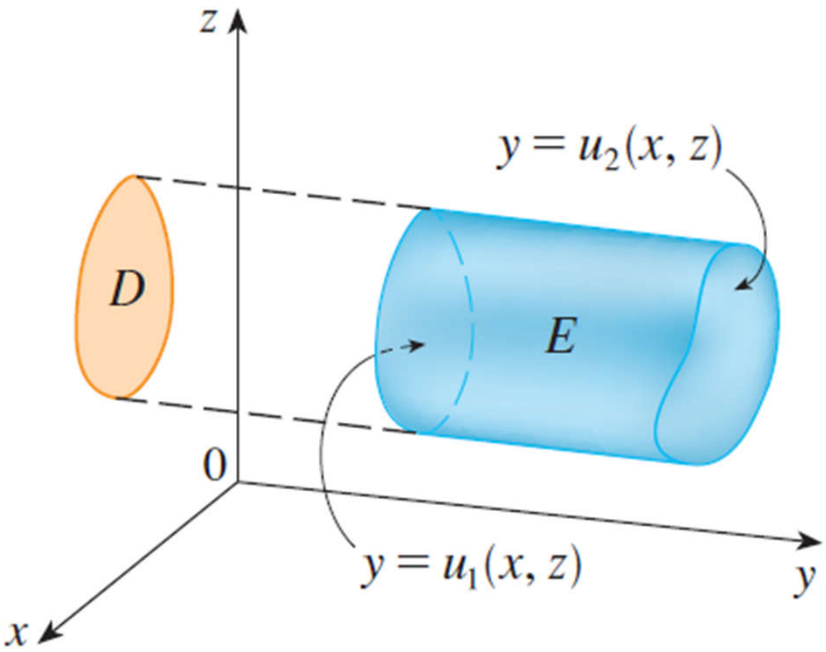
Regions of type 2

A solid region is said to be of **type 2 if**

$$E = \{(x, y, z) \mid (y, z) \in D, \ u_1(y, z) \le x \le u_2(y, z)\}$$

If E is a region of type 2, it holds that

$$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\,dx \right] dA$$



Regions of type 3

A solid region is said to be of **type 3 if**

$$E = \{(x, y, z) \mid (x, z) \in D, \ \ u_1(x, z) \le y \le u_2(x, z)\}$$

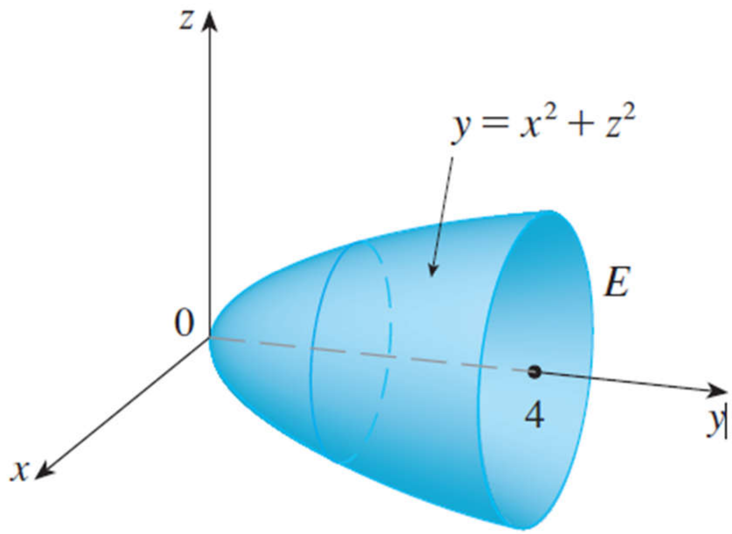
If E is a region of type 3, it holds that

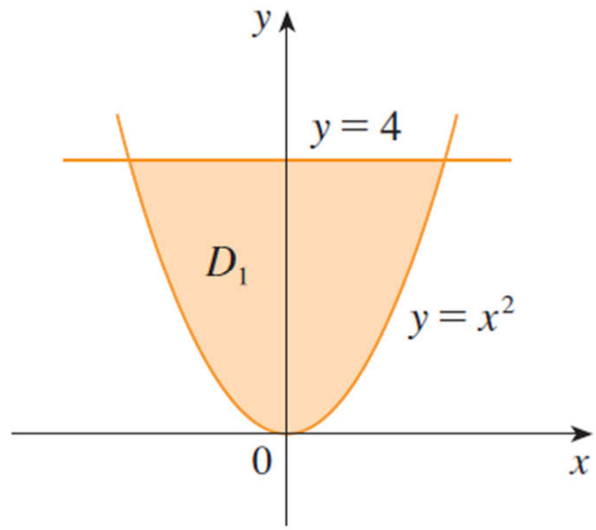
$$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\,dy \right] dA$$

Example: Evaluate triple integral $\iiint_E \sqrt{x^2+z^2}\,dV$

E is the region bounded by $y = x^2 + z^2$, and the plane $y = 4$

Solution

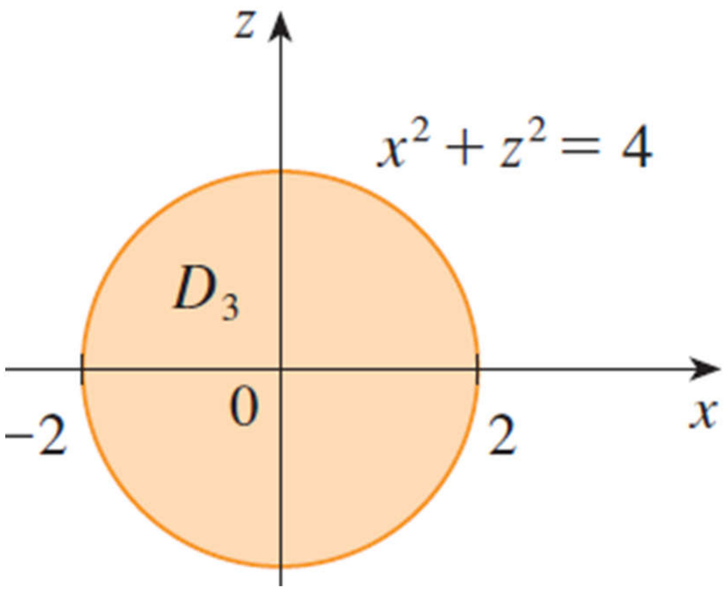




We consider E as region of type 3

$$E = \{(x,y,z) \mid (x,z) \in D, x^2 + z^2 \le y \le 4\}$$

$$D = \{(x,z) \mid x^2 + z^2 \le 4\}$$



$$I=\iiint_E \sqrt{x^2+z^2}\,dV$$

$$=\iint_D \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\,dA$$

$$=\iint_D \sqrt{x^2+z^2}\,y\Big|_{y=x^2+z^2}^{y=4}\,dA$$

$$=\iint_D \sqrt{x^2+z^2}\,(4-(x^2+z^2))\,dA$$

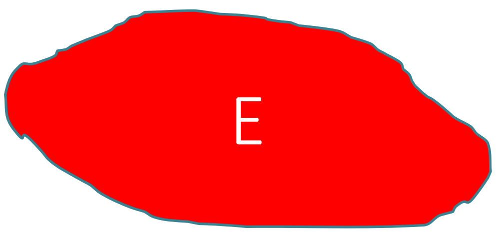
Change of variables into polar coordinates:

$$x=r\cos\theta, z=r\sin\theta \Rightarrow D=\{(r,\theta)\,|\,0\le r\le 2, 0\le\theta\le 2\pi\}$$

$$I=\int_0^2\int_0^{2\pi} r(4-r^2)r\,d\theta\,dr=2\pi\int_0^2(4r^2-r^4)\,dr=\pi(8r^3/3-2r^5/5)\Big|_0^2=\frac{128\pi}{15}$$

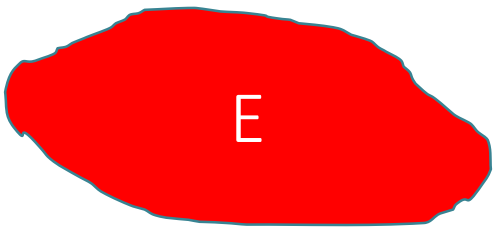
2. Applications of Triple Integrals: Volumes

The volume of the solid occupying the region E is given by



$$V(E) = \iiint_E 1 dV$$

Mass and Center of Mass



If the density function of a solid object that occupies the region E is $\rho(x,y,z)$, in units of mass per unit volume, at any given point *(x,y,z)*, then its **mass is**

$$m = \iiint_E \rho(x,y,z)\,dV$$

Its moments about the coordinate planes are

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV,\; M_{xz} = \iiint_E y\rho(x,y,z)\,dV,\; M_{xy} = \iiint_E z\rho(x,y,z)\,dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$:

$$\bar{x} = \frac{M_{yz}}{m},\quad \bar{y} = \frac{M_{xz}}{m},\quad \bar{z} = \frac{M_{xy}}{m}$$