Question 1 Let $E \subset [0,1]$.

(a) *(5 marks)* Show that $\text{int}(E)$ and $\overline{E}$ are Lebesgue measurable.

(b) *(15 marks)* Suppose that $m(\text{int}(E)) = m(\overline{E})$. Show that

$$m(\overline{E} \setminus \text{int}(E)) = 0$$

and that E is Lebesgue measurable.

Question 2 *(20 marks)* Consider the function $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x < 0, \\ \sin x & \text{if } 0 \leq x \leq \frac{\pi}{2}, \\ 1 & \text{if } x > \frac{\pi}{2}. \end{cases}$$

Then g is continuous and increasing. Let μ_g be the Lebesgue-Stieltjes measure associated to g. Evaluate $\mu_g((-5,-1])$, $\mu_g((0,\pi/6])$ and show that μ_g is a probability measure.

Question 3 Let $(X, \mathcal{M})$ be a measurable space and let $A \subset X$.

(a) *(10 marks)* Let $f = \chi_A - \chi_{A^c}$. Show that $|f(x)| = f^2(x) = 1$ for all $x \in X$ and that both $|f|$ and f^2 are $\mathcal{M}$-measurable.

(b) *(10 marks)* Show that if $A \notin \mathcal{M}$, then f is not $\mathcal{M}$-measurable.

(*Hint:* Observe that $A = \{f = 1\}$.)

(c) *(10 marks)* Suppose that $A \in \mathcal{M}$. Show that $g := 2\chi_A + 3\chi_{A^c}$ is integrable over X if and only if μ is a finite measure.

Question 4 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f be integrable over X and $B = \{0 < |f| < \infty\}$, $B_n = \{1/n < |f| \leq n\}$, $n = 1, 2, \ldots$.

(a) *(10 marks)* Show that $\int_X f d\mu = \int_B f d\mu$. (*Hint:* Observe that $X = \{f = 0\} \cup B \cup \{|f| = \infty\}$ and $\mu(\{|f| = \infty\}) = 0$.)

(b) *(10 marks)* Show that $|f|\chi_{B_n} \nearrow |f|\chi_B$ and $\lim_{n\to\infty} \int_{B_n} |f| d\mu = \int_X |f| d\mu$.

(c) *(10 marks)* Show that for each $\epsilon > 0$, there is a subset A of X such that $\mu(A) < \infty$, f is bounded on A, and $\int_{X \setminus A} |f| d\mu < \epsilon$.

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