Student Name: Nguyen Minh Quan

Student ID: MAMAIU19036

Probability, Homework 3.

A/ Elements of Probability:

IV/ Conditional probability - Multiplication rule: (continued)

5/ Consider the following events:

A: a light bulb is produced in city A $\rightarrow A^c$: a light bulb is produced in city B

D: a light bulb is defective $\rightarrow D^c$: a light bulb is not defective.

From the given information, $P(A) = \frac{2}{3}$ and $P(D \mid A^c) = 1\%$.

$\Rightarrow P(A^c) = 1 - P(A) = 1 - \frac{2}{3} = \frac{1}{3}$.

$P(D \mid A^c) = \dfrac{P(DA^c)}{P(A^c)} = 1\% \Rightarrow P(DA^c) = P(A^c) \times 1\% = \frac{1}{3} \times 1\% = \frac{1}{300}$.

$P(D^cA^c) + P(DA^c) = P(A^c)$, since $A^c = (D^c \cap A^c) \cup (D \cap A^c)$ and $(D^c \cap A^c) \cap (D \cap A^c) = \emptyset$.

$\Rightarrow P(D^cA^c) = P(A^c) - P(DA^c) = \frac{1}{3} - \frac{1}{300} = 33\%$

6/ $\Omega = \{(i,j) : i,j \in \mathbb{N} \cap [1,6]\} \Rightarrow |\Omega| = 6 \times 6 = 36$.

Consider the following events:

A: Doubles are rolled. $\Rightarrow A^c$: Two dices land on different numbers.

B: Sum is 4 or less.

C: At least 1 dice roll is 6.

a) $A = \{(1,1); (2,2); (3,3); (4,4); (5,5); (6,6)\} \Rightarrow P(A) = \dfrac{|A|}{|\Omega|} = \dfrac{6}{36} = \dfrac{1}{6}$.

b) $B = \{(1,1), (1,2), (2,1), (1,3), (2,2), (3,1)\} \Rightarrow P(B) = \dfrac{|B|}{|\Omega|} = \dfrac{6}{36} = \dfrac{1}{6}$

$AB = \{(1,1), (2,2)\} \Rightarrow P(AB) = \dfrac{|AB|}{|\Omega|} = \dfrac{2}{36} = \dfrac{1}{18}$.

$\Rightarrow P(A \mid B) = \dfrac{P(AB)}{P(B)} = \dfrac{1/18}{1/6} = \dfrac{1}{3}$.

c) $C^c = \{(i,j) : i,j \in \mathbb{N} \cap [1,5]\} \Rightarrow |C^c| = 5 \times 5 = 25 \Rightarrow |C| = |\Omega| - |C^c| = 36 - 25 = 11$.

$\Rightarrow P(C) = \dfrac{|C|}{|\Omega|} = \dfrac{11}{36}$

d) $A^cC = \{(i,6) : i \in \mathbb{N} \cap [1,5]\} \cup \{(6,i) : i \in \mathbb{N} \cap [1,5]\} \Rightarrow P(A^cC) = \dfrac{|A^cC|}{|\Omega|} = \dfrac{10}{36} = \dfrac{5}{18}$

$\Rightarrow P(C \mid A^c) = \dfrac{P(A^cC)}{P(A^c)} = \dfrac{P(A^cC)}{1 - P(A)} = \dfrac{5/18}{1 - 1/6} = \dfrac{1}{3}$

## V/ Total probability - Bayes's formula :

1/ $P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c) = 0.2 \times 0.8 + 0.3 \times (1-0.8) = 0.22.$

2/ Consider the following events:

A : You win the game.

$B_i$ : Your opponent is type $i$, $i = \overline{1,3}$.

From the given information,

$P(A \mid B_1) = 0.3,\ P(B_1) = \frac{1}{2},\ P(A \mid B_2) = 0.4,\ P(A \mid B_3) = 0.5,\ P(B_2) = P(B_3) = \frac{1}{4}.$

Hence, $P(A) = \sum_{i=1}^{3} P(A \mid B_i)P(B_i) = 0.3 \times \frac{1}{2} + 0.4 \times \frac{1}{4} + 0.5 \times \frac{1}{4} = \frac{3}{8}.$

3/ Consider the following events:

A : Odd box is chosen $\Rightarrow A^c$ : Even box is chosen.

$B_i$ : Ball $i$ is picked, $i = \overline{1,5}$.

From the given information, $P(A) = P(A^c) = \frac{1}{2},\ P(B_3 \mid A) = \frac{1}{3},\ P(B_3 \mid A^c) = 0.$

Hence $P(B_3) = P(B_3 \mid A)P(A) + P(B_3 \mid A^c)P(A^c) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}.$

4/ Consider the following events:

A : A person has the disease $\rightarrow A^c$ : A person doesn't have the disease.

B : A person has positive result $\rightarrow B^c$ : A person has negative result.

From the given information, $P(B \mid A) = 95\%,\ P(B \mid A^c) = 2\%,\ P(A) = 1\%.$

Then $P(B) = P(B \mid A)P(A) + P(B \mid A^c)P(A^c) = 95\% \times 1\% + 2\% \times 99\% = 2.93\%$

Thus $P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B)} = \frac{95\% \times 1\%}{2.93\%} = \frac{95}{293}.$

## VI/ Independence :

1/ If A is independent of itself, then $P(A) \cdot P(A) = P(A \cap A) = P(A) \Leftrightarrow \left[\begin{array}{l} P(A) = 0 \\ P(A) = 1 \end{array}\right.$

Then A occurs almost surely or $A^c$ occurs almost surely.

2/ Consider the events $A_i$ : A player defeats opponent $i$, $i = \overline{1,4}$. $\Rightarrow P(A_i) = 80\%,\ \forall i = \overline{1,4}.$

a) $P(A_1A_2A_3A_4) = P(A_1)P(A_2)P(A_3)P(A_4) = 80\%^4 = 40.96\%.$ ($A_i$ are pairwise independent)

b) When the player is defeated by any opponent, the game ends.

Thus a player defeats at least 2 opponents in a game if and only if that player defeats the first 2 opponents of that game.

$P(A_1 A_2) = P(A_1)\,P(A_2) = 80\%^2 = 64\%$.

c) Consider the events $B_i$: the player wins game $i$, $i = \overline{1,3}$.

Then $B_i^c$ are pair-wise independent and $P(B_i) = 40.96\%$, $\forall i = \overline{1,3}$.

$\Rightarrow P(B_1 \cup B_2 \cup B_3) = 1 - P(B_1^c B_2^c B_3^c)$ (De Morgan's law) $= 1 - P(B_1^c)\,P(B_2^c)\,P(B_3^c)$

$= 1 - (1 - 40.96\%)^3 \approx 79.42\%$.

3/ b) Consider the events $A_i$: toss $i$ lands on tails., $i \in \mathbb{N}$.

Then $A_i$ are pairwise independent and $P(A_i) = 1 - p$, $\forall i \in \mathbb{N}$.

Hence $P(A_1 A_2 \dots A_{n-1} A_n^c) = P(A_1) \cdot P(A_2) \cdot \dots \cdot P(A_{n-1})\,P(A_n^c) = (1-p)^{n-1} p$.

a) $P(A_1 A_2 A_3^c) = p(1-p)^{3-1} = p(1-p)^2$.

4/ Consider the following events:

$A_k$: the gambler starts with \$$k$ and end up with \$$n$, $k \in \mathbb{N} \cup \{0\}$.

$W$: the gambler wins the current bet $\rightarrow$ $W^c$: the gambler loses the current bet.

From the given information, $P(W) = p$, $P(W^c) = 1 - p$. Of course $P(A_0) = 0$ and $P(A_n) = 1$.

Assume that the gambler starts a bet when he has \$$k$. He has chance $p$ to win the bet, after which he will have \$$k+1$ and a chance $P(A_{k+1})$ to end up with \$$n$. On the other hand, he has chance $1-p$ to lose the bet, after which he will have \$$k-1$ and a chance $P(A_{k-1})$ to end up with \$$n$. By the Total Probability formula, $P(A_k) = p \cdot P(A_{k+1}) + (1-p) \cdot P(A_{k-1})$.

Letting $P(A_k) = u_k$, $\forall k \in \mathbb{N} \cup \{0\}$ yields $p u_{k+1} - u_k + (1-p) u_{k-1} = 0$, $\forall k \in \mathbb{N}$.

Letting $v_k = u_k - u_{k-1}$, $\forall k \in \mathbb{N}$ yields $p v_k = (1-p) v_{k-1} \Leftrightarrow v_k = \frac{1-p}{p} v_{k-1}$, $\forall k \in \mathbb{N}$.

Hence $v_k = \left(\frac{1-p}{p}\right)^{n-1} v_1$, $\forall k \in \mathbb{N}$. Since $v_1 = u_1 - u_0 = u_1$ ($u_0 = P(A_0) = 0$), $v_k = \left(\frac{1-p}{p}\right)^{n-1} u_1$, $\forall n \in \mathbb{N}$.

We consider the following two cases:

Case 1: $p = 1 - p = \frac{1}{2}$.

Then $u_k - u_{k-1} = v_k = v_1$, $\forall k \in \mathbb{N}$ i.e. $u_k = u_{k-1} + v_1$, $\forall k \in \mathbb{N} \Rightarrow u_k = k v_1$, $\forall k \in \mathbb{N}$.

Then $u_n = P(A_n) = 1 = n v_1 \Rightarrow v_1 = \frac{1}{n} \Rightarrow P(A_k) = u_k = \frac{k}{n}$.

Case 2 $p \neq \frac{1}{2}$.

Then $u_k - u_{k-1} = \left(\frac{1-p}{p}\right)^{k-1} v_1 \Rightarrow u_k = u_{k-1} + \left(\frac{1-p}{p}\right)^{k-1} v_1$, $\forall k \in \mathbb{N}$.

Hence $u_k = u_1 + \sum_{i=1}^{k-1}\left[u_1\left(\frac{1-p}{p}\right)^i\right] = u_1 + u_1\sum_{i=1}^{k-1}\left(\frac{1-p}{p}\right)^i = u_1\sum_{i=0}^{k-1}\left(\frac{1-p}{p}\right)^i, \forall k \in \mathbb{N}.$

$\Rightarrow u_k = u_1\sum_{i=0}^{k-1}\left(\frac{1}{p}-1\right)^i = u_1 \cdot \frac{\left(\frac{1}{p}-1\right)^k - 1}{\left(\frac{1}{p}-1\right)-1}, \forall k \in \mathbb{N}.$

Then $u_n = P(A_n) = 1 = u_1 \cdot \frac{\left(\frac{1}{p}-1\right)^n - 1}{\left(\frac{1}{p}-1\right)-1} \Rightarrow u_1 = \frac{\left(\frac{1}{p}-1\right)-1}{\left(\frac{1}{p}-1\right)^n-1} \Rightarrow P(A_k) = u_k = \frac{\left(\frac{1}{p}-1\right)^k - 1}{\left(\frac{1}{p}-1\right)^n-1}.$

Hence the probability that the gambler will end up with \$n is:

$$P(A_k) = \frac{k}{n} \text{ if } p = \frac{1}{2} \text{ and } P(A_k) = \frac{\left(\frac{1}{p}-1\right)^k - 1}{\left(\frac{1}{p}-1\right)^n - 1} \text{ if } p \neq \frac{1}{2}.$$

B/ Supplementary exercises on Conditional probability:

1/ $\Omega = \mathbb{N} \cap [1, 10]$. Consider the following events:

A : The number on the drawn card is at least 5.

B : The drawn card is 10.

$A = \{5, 6, 7, 8, 9, 10\} \Rightarrow P(A) = \frac{|A|}{|\Omega|} = \frac{6}{10} = \frac{3}{5}.$

$B = \{10\} = AB \Rightarrow P(AB) = \frac{|AB|}{|\Omega|} = \frac{1}{10}.$

$\Rightarrow P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{1/10}{3/5} = \frac{1}{6}.$

2/ Consider the event C: the 1st card is a diamond.

In a standard card deck, there are 52 cards, 13 hearts, 13 diamonds $\Rightarrow P(A) = P(C) = \frac{13}{52} = \frac{1}{4}.$

If the 1st card is red (heart/diamond), there are 25 red cards left among the remaining 51 cards $\Rightarrow P(B \mid A) = P(B \mid C) = \frac{25}{51}$. If the 1st card is black, there are 26 red cards in the remaining 51 cards, i.e. $P(B \mid A^cC^c) = \frac{26}{51}$.

Since there is no card that is both heart and diamond, A and C are mutually exclusive

$\Rightarrow P(AC) = 0$ and $P(A \cup C) = P(A) + P(C) = \frac{1}{2} \Rightarrow P(A^cC^c) = 1 - P(A \cup C) = \frac{1}{2}.$

$\Rightarrow P(B) = P(B|A)P(A) + P(B|C)P(C) + P(B|A^cC^c)P(A^cC^c) = \frac{25}{51}\times\frac{1}{4} + \frac{25}{51}\times\frac{1}{4} + \frac{26}{51}\times\frac{1}{2} = \frac{1}{2}.$

$\Rightarrow P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{25}{51}\times\frac{1}{4} : \frac{1}{2} = \frac{25}{102}.$

3/ Consider the following events:

$A_i$ : coin $i$ is selected, $i = \overline{1,3}$.

$B$ : selected coin lands on head.

From the given information, $P(A_i) = \frac{1}{3}\ \forall i = \overline{1,3}$, $P(B \mid A_1) = 1$, $P(B \mid A_2) = \frac{1}{2}$, $P(B \mid A_3) = 75\%$.

$$\Rightarrow P(B) = \sum_{i=1}^{3} P(B \mid A_i)\, P(A_i) = \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + \frac{3}{4} \times \frac{1}{3} = \frac{3}{4}.$$

$$\Rightarrow P(A_1 \mid B) = \frac{P(A_1 B)}{P(B)} = \frac{P(B \mid A_1)\, P(A_1)}{P(B)} = \frac{1/3}{3/4} = \frac{4}{9}.$$