FINAL EXAMINATION
January 2021
Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Head of Dept. of Mathematics:	Lecturer:
Prof. Pham Huu Anh Ngoc	Assoc. Prof. Nguyen Ngoc Hai

INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\mathcal{B} \subset \mathcal{A}$ be a sub-σ-algebra. Denote $\nu = \mu_{|\mathcal{B}}$, the *restriction* of μ to $\mathcal{B}$.

(a) *(15 marks)* Show that ν is again a measure.

(b) *(5 marks)* Assume that μ is a finite measure. Is ν still a finite measure?

(c) *(10 marks)* Assume that μ is a σ-finite measure. Is ν still a σ-finite measure?

Question 2 (a) *(10 marks)* Let f be a measurable function such that $f \geq g$ where g is μ-integrable over X. Show that $\int_X f d\mu$ exists.

(b) *(15 marks)* Let $f, f_1, f_2, \ldots$ be real-valued integrable functions on the measure space $(X, \mathcal{M}, \mu)$ such that $|f_n| \leq f$ for all $n = 1, 2, \ldots$. Define $F : (0, \infty) \to \overline{\mathbb{R}}$ by

$$F(\alpha) := \sup\left\{ \int_{\{|f_n| \geq \alpha\}} |f_n| d\mu : \quad n \in \mathbb{N} \right\}. \quad \leq \int_{\{|f_n|\geq\alpha\}} f \, d\mu \leq \int_X f d\mu < \infty$$

Show that $F(\alpha) < \infty$ for all $0 < \alpha < \infty$, F is decreasing, ~~and $\lim_{\alpha\to\infty} F(\alpha) = 0$~~. (*Hint.* Use Chebyshev's inequality.)

$$\int_X f = \int f^+ - \int f^-$$

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g int. $\Rightarrow \int g^+ , \int g^-$ finite.

$\Rightarrow \int f^- \leq \int g^- < \infty$

$(f \geq g)$

$\Rightarrow f^- = -\min\{f, 0\} \leq -\min\{g, 0\} = g^-$

Question 3 Let f be a measurable function on the measure space $(X, \mathcal{M}, \mu)$.

(a) *(10 marks)* Show that if $\int_X f d\mu$ is defined, then $\int_X f d\mu = \int_A f d\mu$, where $A = \{f \neq 0\}$.

(b) *(15 marks)* Show that if f is integrable over X, then

$$\lim_{n\to\infty} \int_{A_n} f d\mu = \int_X f d\mu,$$

where $A_n = \{|f| \leq n\}$.

Question 4 *(10 marks)* Show that the Dirac measure δ_a on $\mathbb{R}$ has no density with respect to the Lebesgue m on $\mathcal{L}(\mathbb{R})$.

(b) *(10 marks)* Let μ and ν be two measures on the measurable space $(X, \mathcal{M})$. Suppose that $\nu(E) \leq \mu(E)$ for all E in $\mathcal{M}$. Show that ν is absolutely continuous with respect to μ.

$\delta_a(\{a\}) = 1$
$m(\{a\}) = 0$ $\Rightarrow \delta_a \not\ll m.$

----------END OF QUESTION PAPER----------