Chapter 5: Differential equations

Lecture 3

- ❖ **Systems of ODEs**
- ❖ **Higher-order ODEs**
- ❖ **Multi-step Methods**

Initial-Value problem for a system of ODEs

System of ODEs:

$$
\begin{aligned}
&y_1' = f_1(x, y_1, y_2, \cdots, y_n) \\
&y_2' = f_2(x, y_1, y_2, \cdots, y_n) \\
&\vdots \\
&y_n' = f_n(x, y_1, y_2, \cdots, y_n), \quad x > a
\end{aligned}
$$

Set

$$
y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \cdots \\ y_n(x) \end{bmatrix}, \quad f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \\ \cdots \\ f_n(x, y) \end{bmatrix}, \quad y_0 = \begin{bmatrix} y_1(a) \\ y_2(a) \\ \cdots \\ y_n(a) \end{bmatrix} \qquad y'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \cdots \\ y_n'(x) \end{bmatrix}
$$

$$
\boxed{\begin{aligned} &y'(x) = f(x, y(x)), \quad a \le x \le b \\ &y(a) = y_0 \end{aligned}}
$$

Most methods for a single differential equation work

Example

Use Euler method with h=0.1 to find values at two steps

$$\begin{cases} y_1{}' = y_2 \\ y_2' = 1 - y_1 \end{cases}$$

initial conditions: $y_1(0)$=-1, $y_2(0) = 1$

Solution

$$Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \quad f(x,Y) = \begin{bmatrix} y_2 \\ 1 - y_1 \end{bmatrix}, \quad Y(0) = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$STEP\ 1:$

$$Y_1 = Y_0 + hf(x_0, Y_0)$$

$$\begin{bmatrix} y_{1,1} \\ y_{2,1} \end{bmatrix} = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix} + 0.1 \begin{bmatrix} y_{2,0} \\ 1 - y_{1,0} \end{bmatrix} = \begin{bmatrix} -1 + 0.1 \\ 1 + 0.1(1+1) \end{bmatrix} = \begin{bmatrix} -0.9 \\ 1.2 \end{bmatrix}$$

$STEP\ 2:$

$$Y_2 = Y_1 + hf(x_1, Y_1)$$

$$\begin{bmatrix} y_{1,2} \\ y_{2,2} \end{bmatrix} = \begin{bmatrix} y_{1,1} \\ y_{2,1} \end{bmatrix} + 0.1 \begin{bmatrix} y_{2,1} \\ 1 - y_{1,1} \end{bmatrix} = \begin{bmatrix} -0.9 + 0.12 \\ 1.2 + .1(1+0.9) \end{bmatrix} = \begin{bmatrix} -0.78 \\ 1.39 \end{bmatrix}$$

Exercise

Find approximate solution of problem by

a) Euler's method

b) Midpoint method

c) Heun's method

$$u' = 3u + 2v - (2t^2 + 1)e^{2t}$$

$$v' = 4u + v + (t^2 + 2t - 4)e^{2t}, \quad 0 \le t \le 0.2$$

$$u(0) = 1, \quad v(0) = 1, \qquad h = 0.1$$

Compute the errors, where exact solution is:

$$u(t) = \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} + e^{2t}, \quad v(t) = \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} + t^2e^{2t}$$

Higher-order Differential Equations

Consider nth-order differential equation has the form

$$y^{(n)}(x) = f(x, y(x), y'(x), y''(x), \ldots, y^{(n-1)}(x))$$

Set $y_1 = y, y_2 = y', y_3 = y'', \cdots, y_n = y^{(n-1)}$

$$y_n' = f(x, y_1, y_2, y_3, \ldots, y_n)$$

The nth-order differential equation becomes:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_{n-1}' = y_n \\ y_n' = f(x, y_1, y_2, \ldots, y_n) \end{cases}$$

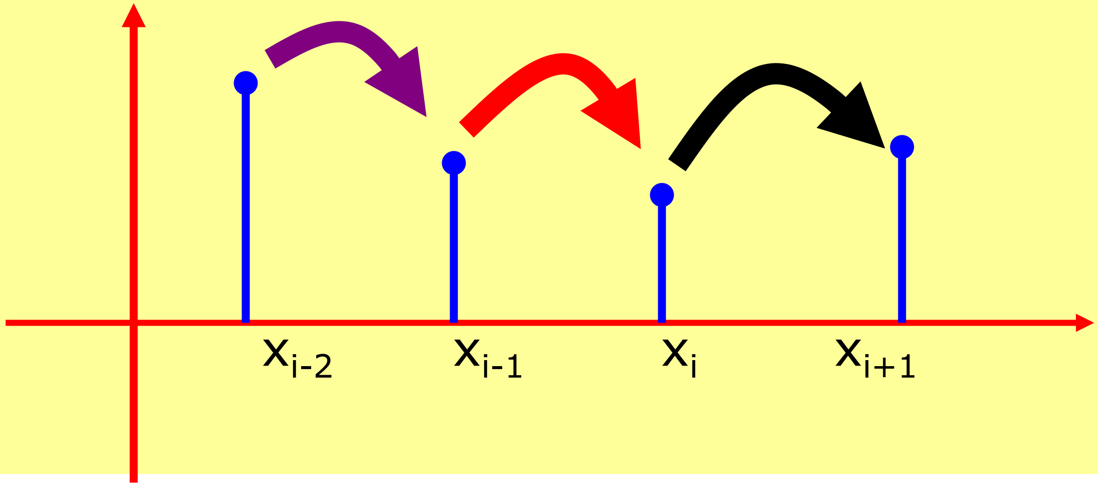
$$\boxed{Y' = F(x, Y), \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}$$

1st-order diff eq

Multiple Step Methods

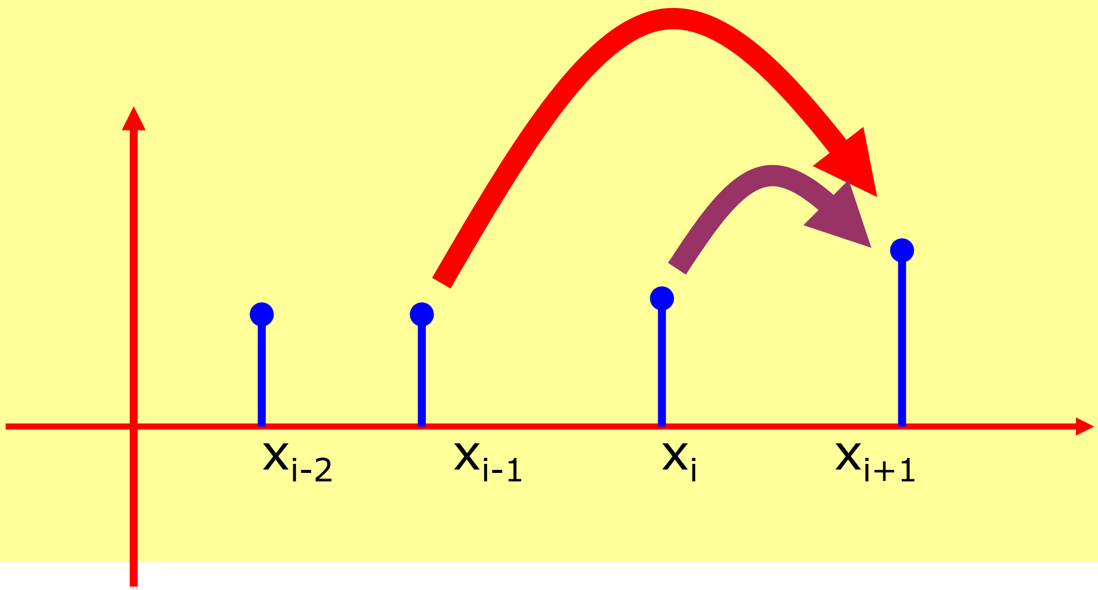
Single Step Methods

- Single Step Methods:
  - Euler and Runge-Kutta are single step methods.
  - Estimates of $\boldsymbol{y_{i+1}}$ depends only on $\boldsymbol{y_i}$ and $\boldsymbol{x_i}$.



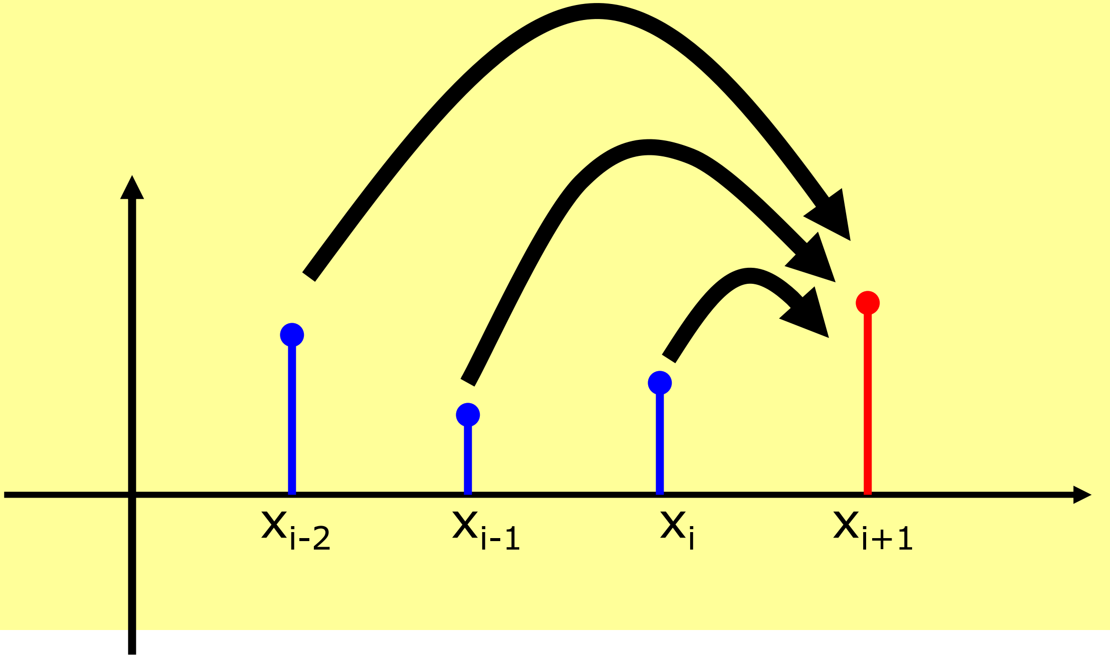
Multi-Step Methods

- 2-Step Methods
  - In a two-step method, estimates of y_{i+1} depends on y_i, y_{i-1}, x_i, and x_{i-1}



Multi-Step Methods

- 3-Step Methods
  - In an 3-step method, estimates of y_{i+1} depends on y_i ,y_{i-1} ,y_{i-2}, x_i , x_{i-1}, and x_{i-2}



Adams-Bashforth Formulas

Taylor series expansion of solution y at x_i

$$y_{i+1} = y_i + f_i h + \frac{f_i'}{2} h^2 + \frac{f_i''}{3} h^3 + \ldots \qquad f_i = f(x_i, y_i)$$

where $f_i = y'(x_i),\ f_i' = y''(x_i),\ f_i'' = y'''(x_i), \ldots$

Substitute backward difference $f_i' = \dfrac{f_i - f_{i-1}}{h} + \dfrac{f_i''}{2} h + O(h^2)$

$$y_{i+1} = y_i + h\left(\frac{3}{2} f_i - \frac{1}{2} f_{i-1}\right) + \frac{5}{12} h^3 f_i'' + O(h^4)$$

yields 2nd-order Adams-Bashforth formula

$$\boxed{y_{i+1} = y_i + \frac{h}{2}\left(3 f_i - f_{i-1}\right)}$$

Higher-order Adams-Bashforth formulas

Substituting higher difference approximations in

$$y_{i+1} = y_i + h\left(f_i + \frac{f_i'}{2} h + \frac{f_i''}{3} h^2 + \ldots \right)$$

we get

$$\boxed{y_{i+1} = y_i + h\sum_{k=0}^{n-1} \beta_k f_{i-k} + O(h^{n+1})}$$

β_0	β_1	β_2	β_3
1			
3/2	−1/2		
23/12	−16/12	5/12	
55/24	−59/24	37/24	−9/24

4th-order Adams-Bashforth formulas

Taking n=4 results 4th-order Adams-Bashforth method:

$$\boxed{y_{i+1} = y_i + \frac{h}{24}h\left(55 f_i - 59 f_{i-1} + 37 f_{i-2} - 9 f_{i-3}\right)} \qquad f_i = f(x_i, y_i)$$

Often, a single-step method of the same order of accuracy is used to compute the first n-1 values for the nth-order Adams-Bashforth method

For example, 4th-order Runge-Kutta method is used to compute the first 3 values y_1, y_2, and y_3 for the 4th-order Adams-Bashforth method

Quiz

1. Approximate the first two values of the solution of the following problem and compute the errors by

a) Heun's method

b) 4^{th}-order Runge-Kutta method

$$y' = 2xy, y(0) = 1, h = 0.2$$

2. Approximate the first two values of the solution of the following problem and compute the errors by

a) Euler's method

b) Midpoint method

$$y'' - 2y' + 2y = e^{2t}\sin t, \quad 0 \le t \le 1/2$$

$$y(0) = -0.4, \quad y'(0) = -0.6, \text{using } h = 1/4$$

$$\text{Exact solution: } y(t) = 0.2e^{2t}(\sin t - 2\cos t)$$

$$y_1 = y, y_2 = y'$$

$$y_1' = y_2$$

$$y_2' = 2y_2 - 2y_1 + e^{2t}\sin t$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$