Chapter 2: Partial derivatives

Lecture 7

Lagrange Multipliers

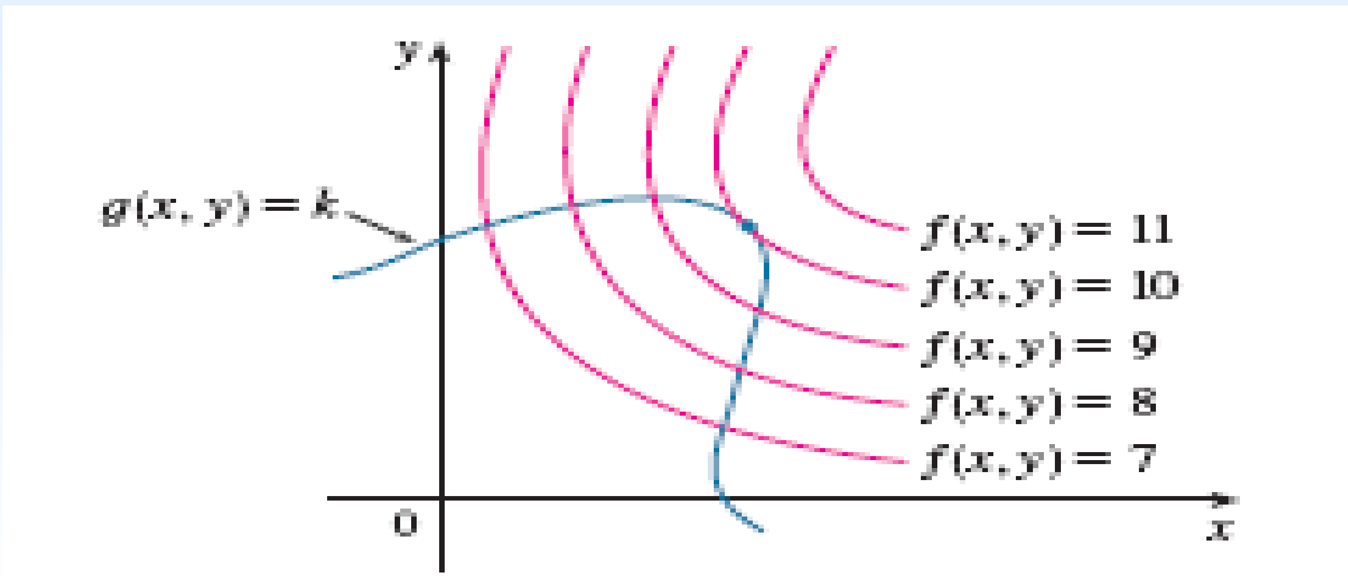
Motivation

You want to find the maximum or minimum values of a function when its variables are constraint to each other

For example: Find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$, where k is a constant

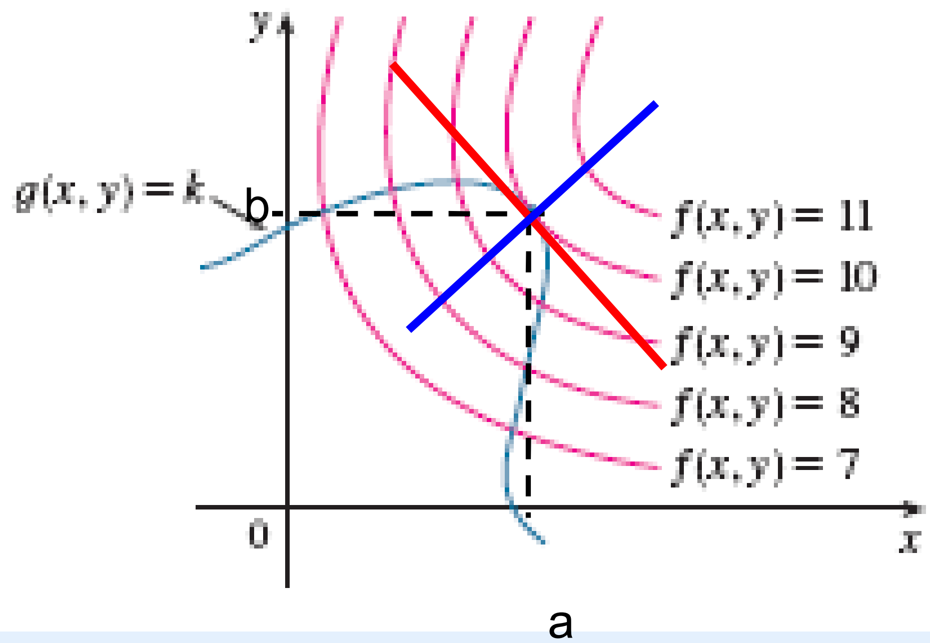
Geometric Explanation in 2D

- **Problem:** Maximize (or minimize) $f(x,y)$ subject to the constraint $g(x,y)=k$



- Find largest value c such that level curve $f(x,y)=c$ intersects $g(x,y)=k$.

These curves touch each other and so they have a common tangent line. Thus, they have the same normal line at this point: $\nabla f(a,b)= \lambda \nabla g(a,b)$



For functions of three variables, the two level surfaces have the same normal vector: $\nabla f(a,b,c)= \lambda \nabla g(a,b,c)$

Method of Lagrange Multipliers for Constraint Optimization Problems

Problem: Find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$, where k is a constant

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and

$$g(x,y,z) = k$$

(b) Evaluate f at all points found in step (a):
the largest value is maximum, the smallest is minimum

Remarks

- If we write the vector equation

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

in terms of its components, then the equations in step (a) become

$$f_x(x,y,z) = \lambda g_x(x,y,z), \qquad f_y(x,y,z) = \lambda g_y(x,y,z)$$

$$f_z(x,y,z) = \lambda g_z(x,y,z), \qquad g(x,y,z) = k$$

- This is a system of four equations in the four unknowns x, y, z, and λ, but it is not necessary to find explicit values for λ.

Functions of two variables

- The method of Lagrange multipliers is similar to the method just described.
- To find the maximum and minimum values of $f(x,y)$ subject to the constraint $g(x,y)=k$ we look for values of x, y and λ such that
  - $\nabla f(x,y)= \lambda \nabla g(x,y)$ and $g(x,y)=k$
- This gives three equations in three unknowns:

$$f_x(x,y) = \lambda g_x(x,y), \quad f_y(x,y) = \lambda g_y(x,y), \quad g(x,y) = k$$

Example

- The profit from the sale of x units of radiators for automobiles and y units of radiators for generators is given by

$$f(x,y) = -x^2 - y^2 + 4x + 8y$$

- Find values of x and y that lead to a maximum profit if the firm must produce a total of 6 units of radiators.

Solution

- $f(x,y) = -x^2 - y^2 + 4x + 8y$
- Set $g(x,y)=x+y=6$
- Method of Lagrange multipliers yields:

$$f_x(x,y) = -2x + 4 = \lambda g_x(x,y) = \lambda$$

$$f_y(x,y) = -2y + 8 = \lambda g_y(x,y) = \lambda$$

$$\Rightarrow 2(x-y) + 4 = 0 \Rightarrow x - y = -2$$

$$g(x,y) = x + y = 6$$

$$\Rightarrow x = 2, y = 4$$

So,this must be the maximum value

$\boxed{\text{Verify: } f(2,4) = 20 > f(3,3) = 12}$

Method of Lagrange Multipliers: Problems with Two Constraints

Problem: Find the maximum and minimum values of $f(x,y,z)$ subject to two constraints: $g(x,y,z) = k$ and $h(x,y,z) = c$

(a) Find all values of x, y, z, λ and μ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$

and

$$g(x,y,z) = k, \qquad h(x,y,z) = c$$

(b) Evaluate f at all points found in step (a):
the largest value is maximum, the smallest is minimum

Example:

Find extreme values of

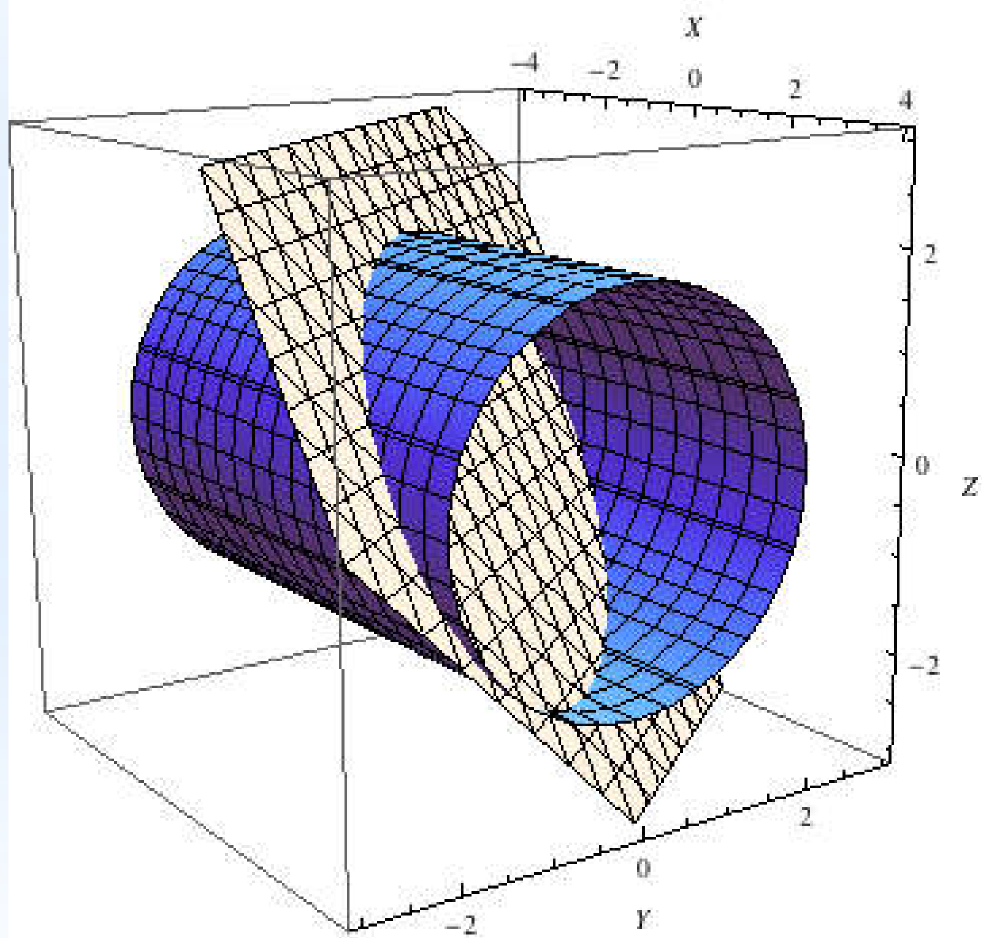
$$f(x,y,z) = x+2y,$$

subject to constraints

$$g(x,y,z)=x+y+z = 1$$

$$h(x,y,z)= y^2 + z^2 = 4$$

(Figure shows the two constraints)



$$f(x,y,z) = x + 2y \Rightarrow \nabla f(x,y,z) = <1,2,0>$$

$$g(x,y,z) = x + y + z \Rightarrow \nabla g(x,y,z) = <1,1,1>$$

$$h(x,y,z) = y^2 + z^2 \Rightarrow \nabla h(x,y,z) = <0,2y,2z>$$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$<1,2,0> = \lambda <1,1,1> + \mu <0,2y,2z> = <\lambda, \lambda + 2\mu y, \lambda + 2\mu z>$$

$$1 = \lambda, \qquad 2 = \lambda + 2\mu y, \qquad 0 = \lambda + 2\mu z$$

$$1 = 2y\mu, \qquad -1 = 2z\mu \Rightarrow -1 = y/z \Rightarrow z = -y$$

$$g(x,y,z) = x + y + z = 1, \qquad h(x,y,z) = y^2 + z^2 = 4$$

$$\Rightarrow x = 1, 2y^2 = 4, y = \pm\sqrt{2} = -z$$

$$f(1,\sqrt{2},-\sqrt{2}) = 1 + 2\sqrt{2}, \quad f(1,-\sqrt{2},\sqrt{2}) = 1 - 2\sqrt{2}$$

$$\Rightarrow \max_{g=1,h=4} f = f(1,\sqrt{2},-\sqrt{2}) = 1 + 2\sqrt{2}$$

$$\min_{g=1,h=4} f = f(1,-\sqrt{2},\sqrt{2}) = 1 - 2\sqrt{2}$$

Remark

- Sometimes, the unconstraint and constraint optimization problems come together
- **Example**: Find the extreme values of $f(x, y) = e^{-xy}$

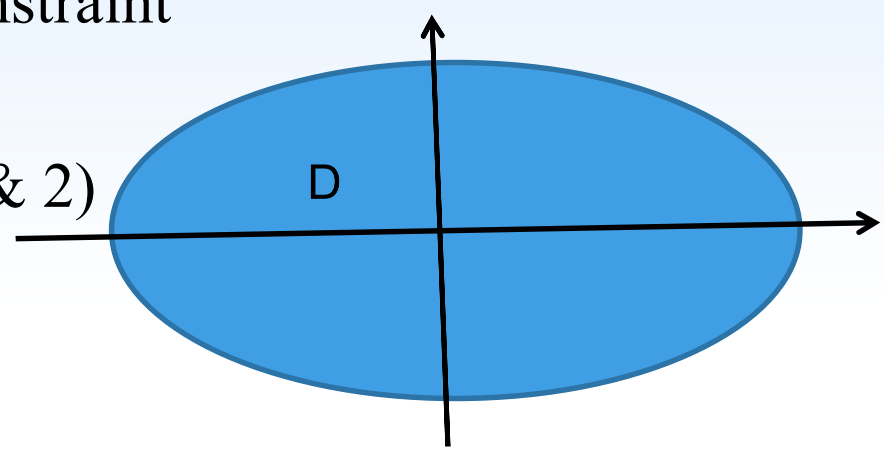
on the region D described by the inequality $D: \quad x^2 + 4y^2 \leq 1$

Solution:

1. Evaluate values of f at critical points inside $D: \quad x^2 + 4y^2 < 1$
2. Find max, min f subject to constraint

$$x^2 + 4y^2 = 1$$

3. Compare values of f from 1) & 2)



Exercises

1. Let $f(x, y) = (2x - y)(1 - 2xy)$
   a) Find the local maximum and local minimum values and saddle point(s) of f
   b) Find absolute maximum and minimum values of f on the triangular region with the vertices (0,4), (-8,0) and (4,0).
2. Using Lagrange Multipliers method, find the maximum volume of a rectangular box whose surface area is 1800cm^2 and whose total edge length is 240cm.