Chapter 3: Curve Fitting & Interpolation

Lecture 2: Interpolation

Problem on Flood-tide Levels

The flood-tide levels measured at a station at different times in a certain day are given in the following table

Time (h)	15	16	17	18
Level (m)	0.4	0.6	1.0	1.2

a) Estimate the flood-tide level at 16h30

b) At what time the flood tide level is 0.8m?

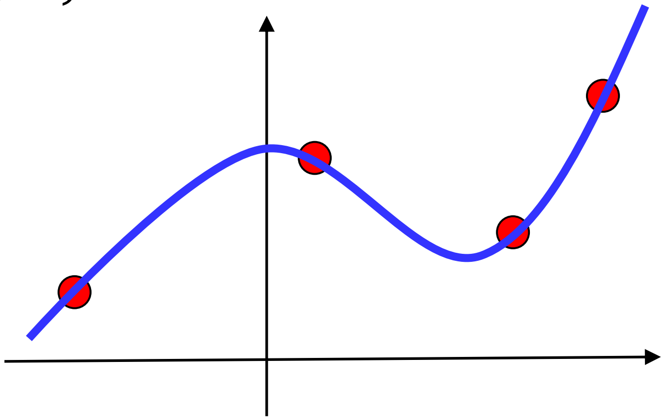
The Interpolation Problem

Given a set of n+1 points

$$\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \ldots., \left(x_n, f(x_n)\right)$$

Find an n[th] order polynomial $f_n(x)$ that passes through all points:

$$f_n(x_i) = f(x_i) \quad for\ i = 0, 1, 2, \ldots, n$$



Existence and Uniqueness

Given a set of n+1 points:

$$\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \ldots., \left(x_n, f(x_n)\right)$$

Assumption: $x_0, x_1, \ldots, x_n$ are distinct

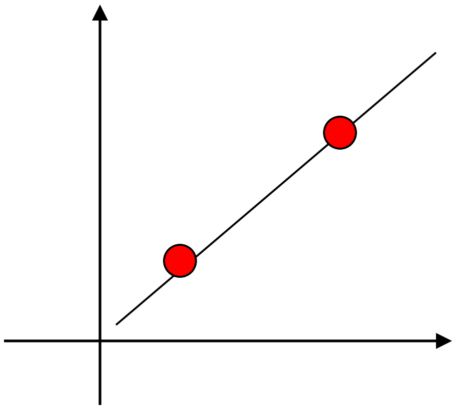
Theorem:

There is a unique polynomial ***$f_n(x)$*** of order ≤ n such that:

$$f_n(x_i) = f(x_i) \qquad for \quad i = 0,1,\ldots,n$$

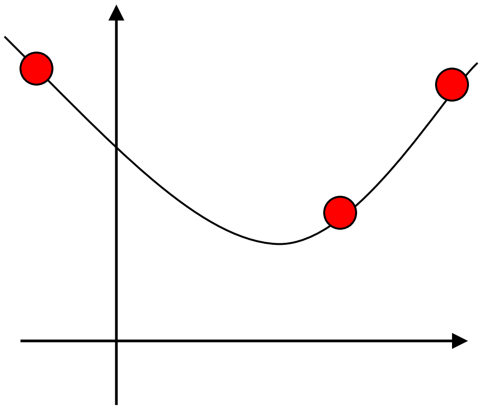
Examples of Polynomial Interpolation

Linear Interpolation



- Given any two points, there is one polynomial of order ≤ 1 that passes through the two points.

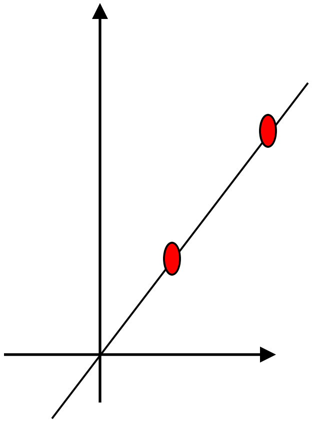
Quadratic Interpolation



Given any three points there is one polynomial of order ≤ 2 that passes through the three points.

Linear Interpolation

Given any two points, $\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right)$



The line that interpolates the two points is:

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}\left(x - x_0\right)$$

Example :

Find a polynomial that interpolates (1,2) and (2,4).

$$f_1(x) = 2 + \frac{4-2}{2-1}\left(x-1\right) = 2x$$

Quadratic Interpolation

- Given any three points: $(x_0, f(x_0)), (x_1, f(x_1)), and\ (x_2, f(x_2))$
- The polynomial that interpolates the three points is:

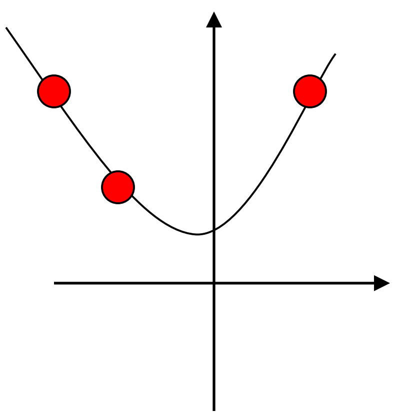
$$\boxed{f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)}$$

$where:$

$$b_0 = f(x_0)$$

$$b_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_0, x_1, x_2] = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Divided Differences

$$f[x_k] = f(x_k)$$ Zeroth order DD

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$ First order DD

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$ Second order DD

....

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}$$ kth-order DD

General nth Order Interpolation

Given any n+1 points: $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$

The polynomial that interpolates all points is:

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \ldots + b_n(x - x_0)\ldots(x - x_{n-1})$$

$$b_0 = f(x_0)$$

$$b_1 = f[x_0, x_1]$$

....

$$b_n = f[x_0, x_1, \ldots, x_n]$$

This is called Newton's divided-difference interpolating polynomial

Divided Difference Table

x	f[ ]	f[,]	f[, ,]	f[, , ,]
x_0	f[x0]	$f[x_0,x_1]$	$f[x_0,x_1,x_2]$	$f[x_0,x_1,x_2,x_3]$
x_1	$f[x_1]$	$f[x_1,x_2]$	$f[x_1,x_2,x_3]$	
x_2	$f[x_2]$	$f[x_2,x_3]$		
x_3	$f[x_3]$			

$$f_n(x) = \sum_{i=0}^{n} \left\{ f[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1} \left(x - x_j \right) \right\}$$

Example

- Find a polynomial to interpolate the data.

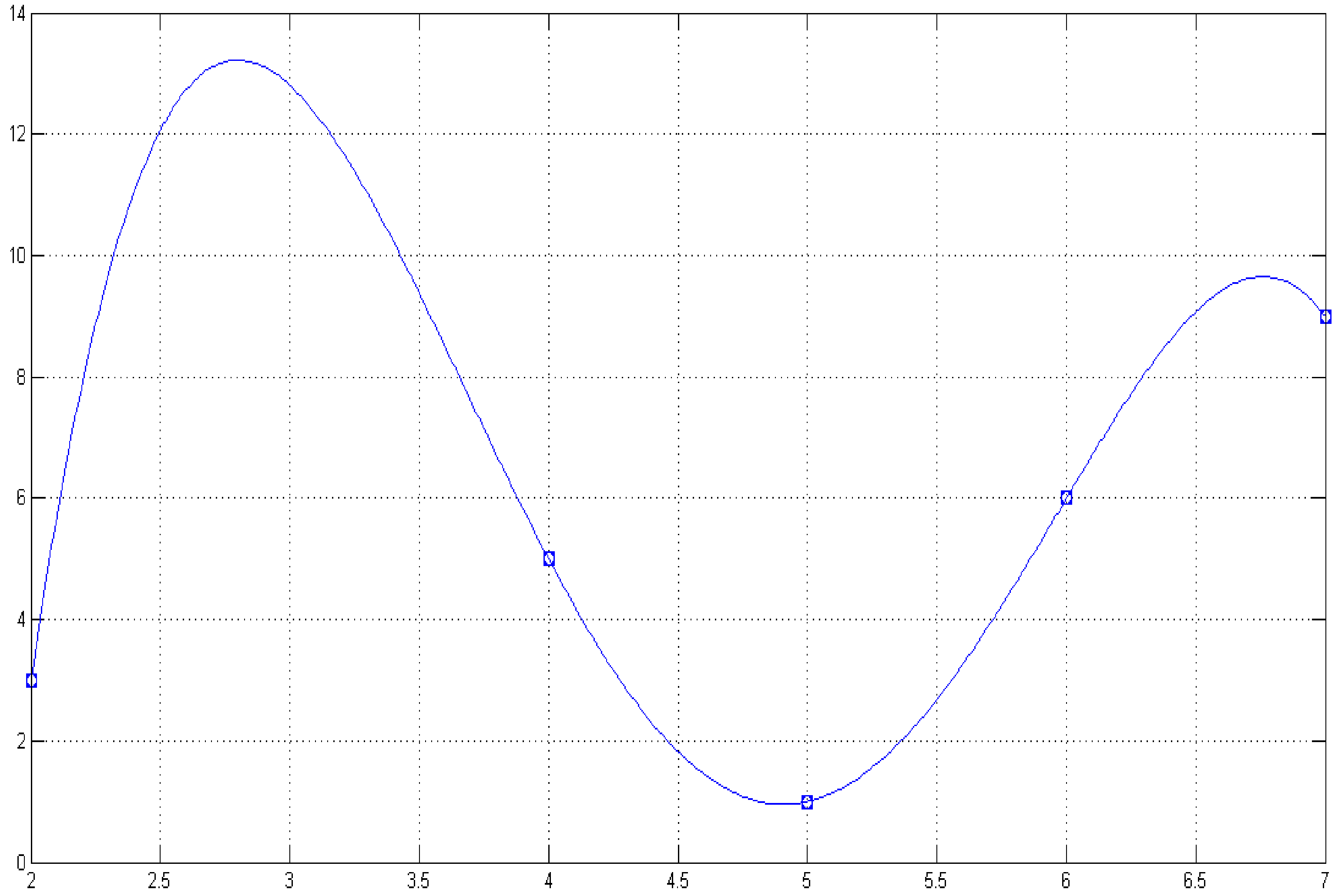
x	f(x)
2	3
4	5
5	1
6	6
7	9

Solution

x	f(x)	f[,]	f[, ,]	f[, , ,]	f[, , , ,]
2	**3**	**1**	**-1.6667**	**1.5417**	**-0.6750**
4	**5**	**-4**	**4.5**	**-1.8333**	
5	**1**	**5**	**-1**		
6	**6**	**3**			
7	**9**				

$$f_4(x) = 3 + 1(x-2) - 1.6667(x-2)(x-4) + 1.5417(x-2)(x-4)(x-5) - 0.6750(x-2)(x-4)(x-5)(x-6)$$

$$f(3) \approx f_4(3) = 3 + 1(3-2) - 1.6667(3-2)(3-4) + 1.5417(3-2)(3-4)(3-5) - 0.6750(3-2)(3-4)(3-5)(3-6) = \ 12.8001$$



14
12
10
8
6
4
2
0
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7

Exercise

The money (in millions USD) donated for a certain event is given in the following table.

Days	**1**	**3**	**5**	**6**
Money	2	4	8	3

Use a third-degree interpolating polynomial to predict the money donated in the 2^{nd} and 4^{th} days

Lagrange Interpolation

The Interpolation Problem

Given a set of n+1 points:

$$\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \ldots., \left(x_n, f(x_n)\right)$$

Find an n[th] order polynomial: $f_n(x)$
that passes through all points, such that:

$$\boxed{f_n(x_i) = f(x_i) \quad for\ i = 0, 1, 2, \ldots, n}$$

Lagrange Interpolation

Problem:

Given

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

Find the polynomial of least order $f_n(x)$ such that:

$$f_n(x_i) = f(x_i) \qquad for \quad i = 0,1,\ldots,n$$

Lagrange Interpolation Formula:

$$f_n(x) = \sum_{i=0}^{n} f\left(x_i\right) \ \ell_i(x)$$

$$\ell_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{\left(x - x_j\right)}{\left(x_i - x_j\right)}$$

Lagrange Interpolating Polynomial

$$\boxed{\begin{aligned} f_n(x) &= \sum_{i=0}^{n} f\left(x_i\right)\ \ell_i(x) \\ \ell_i(x) &= \prod_{j=0,\, j\neq i}^{n} \frac{\left(x - x_j\right)}{\left(x_i - x_j\right)} \end{aligned}}$$

$\ell_i(x)$ are called the cardinals.

The cardinals are n^{th} order polynomials :

$$\ell_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Example: Find Langrange Interpolating polynomial that pases though the data points

x	1/3	1/4	1
y	2	-1	7

Solution

$$f_2(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + f(x_2)\ell_2(x)$$

$$\ell_0(x) = \frac{(x-x_1)}{(x_0-x_1)}\frac{(x-x_2)}{(x_0-x_2)} = \frac{(x-1/4)}{(1/3-1/4)}\frac{(x-1)}{(1/3-1)}$$

$$\ell_1(x) = \frac{(x-x_0)}{(x_1-x_0)}\frac{(x-x_2)}{(x_1-x_2)} = \frac{(x-1/3)}{(1/4-1/3)}\frac{(x-1)}{(1/4-1)}$$

$$\ell_2(x) = \frac{(x-x_0)}{(x_2-x_0)}\frac{(x-x_1)}{(x_2-x_1)} = \frac{(x-1/3)}{(1-1/3)}\frac{(x-1/4)}{(1-1/4)}$$

$$\boxed{\begin{aligned} f_2(x) = {} & 2\{-18(x-1/4)(x-1)\} - 1\{16(x-1/3)(x-1)\} \\ & + 7\{2(x-1/3)(x-1/4)\} \end{aligned}}$$

Example

Find a polynomial to interpolate:

Both Newton’s interpolation method and Lagrange interpolation method must give the same answer.

x	y
0	1
1	3
2	2
3	5
4	4

Newton's Interpolation Method

0	1	2	-3/2	7/6	-5/8
1	3	-1	2	-4/3	
2	2	3	-2		
3	5	-1			
4	4				

Interpolating Polynomial

$$f_4(x) = 1 + 2(x) - \frac{3}{2}x(x-1) + \frac{7}{6}x(x-1)(x-2)$$
$$-\frac{5}{8}x(x-1)(x-2)(x-3)$$

$$f_4(x) = 1 + \frac{115}{12}x - \frac{95}{8}x^2 + \frac{59}{12}x^3 - \frac{5}{8}x^4$$

Interpolating Polynomial Using Lagrange Interpolation Method

$$f_4(x) = \sum_{i=0}^{4} f(x_i)\ \ell_i = \ell_0 + 3\ell_1 + 2\ell_2 + 5\ell_3 + 4\ell_4$$

$$\ell_0 = \frac{(x-1)}{(0-1)}\ \frac{(x-2)}{(0-2)}\ \frac{(x-3)}{(0-3)}\ \frac{(x-4)}{(0-4)} = \frac{(x-1)(x-2)(x-3)(x-4)}{24}$$

$$\ell_1 = \frac{(x-0)}{(1-0)}\ \frac{(x-2)}{(1-2)}\ \frac{(x-3)}{(1-3)}\ \frac{(x-4)}{(1-4)} = \frac{x(x-2)(x-3)(x-4)}{-6}$$

$$\ell_2 = \frac{(x-0)}{(2-0)}\ \frac{(x-1)}{(2-1)}\ \frac{(x-3)}{(2-3)}\ \frac{(x-4)}{(2-4)} = \frac{x(x-1)(x-3)(x-4)}{4}$$

$$\ell_3 = \frac{(x-0)}{(3-0)}\ \frac{(x-1)}{(3-1)}\ \frac{(x-2)}{(3-2)}\ \frac{(x-4)}{(3-4)} = \frac{x(x-1)(x-2)(x-4)}{-6}$$

$$\ell_4 = \frac{(x-0)}{(4-0)}\ \frac{(x-1)}{(4-1)}\ \frac{(x-2)}{(4-2)}\ \frac{(x-3)}{(4-3)} = \frac{x(x-1)(x-2)(x-3)}{24}$$

Inverse Interpolation
Error in Polynomial Interpolation

Inverse Interpolation

Problem: Given a table of values

Find x such that: $f(x) = \bar{y}$, where $\bar{y}$ is given

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

One approach:

Use polynomial interpolation to obtain $f_n(x)$ to interpolate the data then use $\boxed{\text{Newton's method}}$ to find a solution to x

$$f_n(x) = \bar{y}$$

Inverse Interpolation

Inverse interpolation:

1. Exchange the roles of x and y.

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

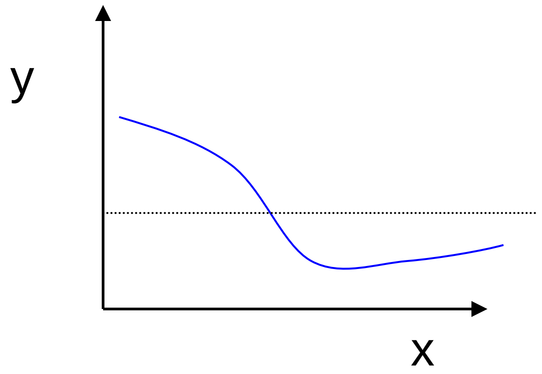
2. Perform polynomial Interpolation on the new table.

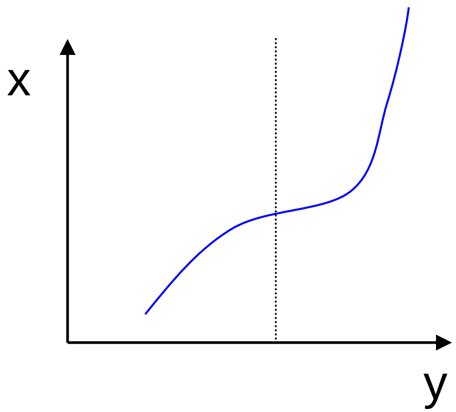
y_i	y_0	y_1	....	y_n
x_i	x_0	x_1	....	x_n

3. Evaluate

$$x = f_n(\overline{y})$$

Inverse Interpolation





Inverse Interpolation

<u>Question:</u>

What is the limitation of inverse interpolation?

- The original function has an inverse.
- $y_1, y_2, \ldots, y_n$ must be distinct.

Inverse Interpolation

Example

x	1	2	3
y	3.2	2.0	1.6

Problem :

Given the table. Find x such that $f(x) = 2.5$

3.2	1	-.8333	1.0417
2.0	2	-2.5	
1.6	3		

$$x = f_2(y) = 1 - 0.8333(y - 3.2) + 1.0417(y - 3.2)(y - 2)$$

$$x = f_2(2.5) = 1 - 0.8333(-0.7) + 1.0417(-0.7)(0.5) = 1.2187$$

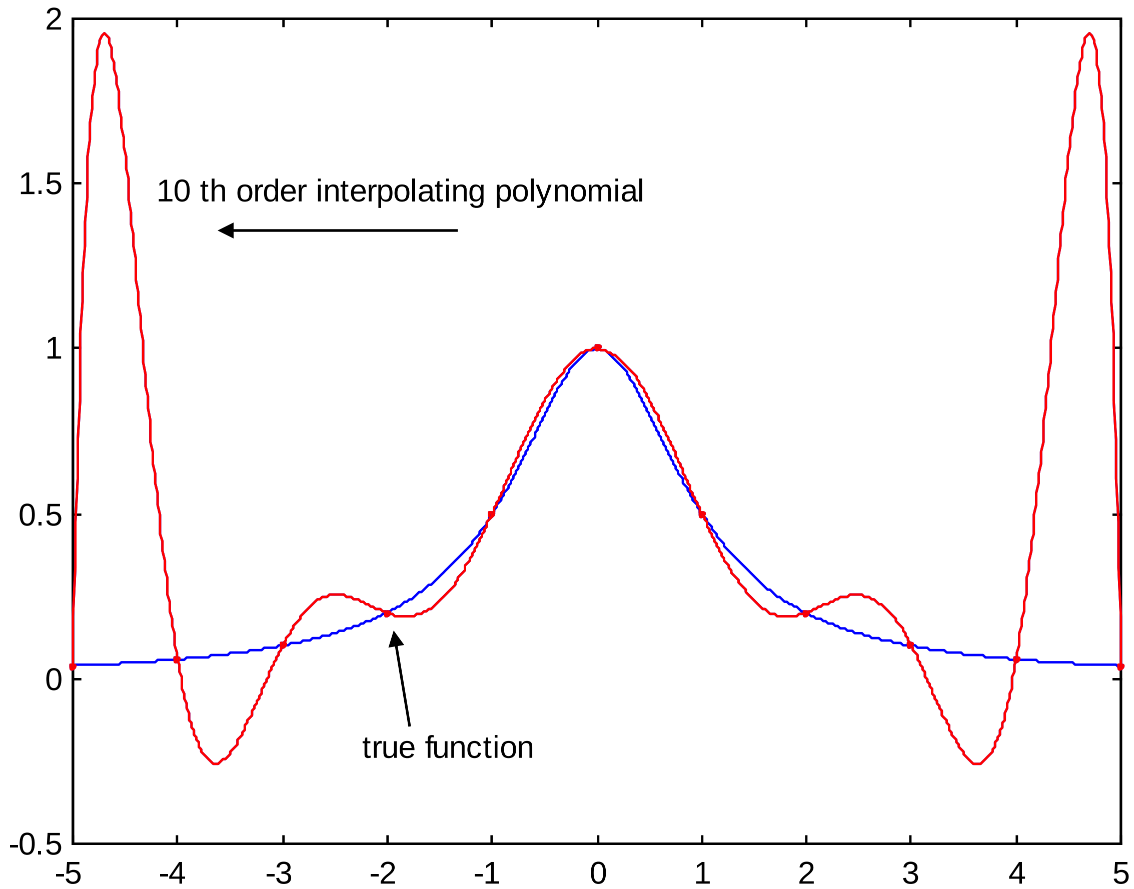
Errors in polynomial Interpolation

- Polynomial interpolation may lead to large errors (especially for high order polynomials).

BE CAREFUL

- When an n^{th} order interpolating polynomial is used, the error is related to the $(n+1)^{th}$ order derivative.

10th Order Polynomial Interpolation



Errors in polynomial Interpolation

Theorem

Suppose the distinct $x_0, x_1, ..., x_n \in [a,b]$ and $f \in C^{n+1}[a,b]$.

Then, $\forall x \in [a,b], \exists \xi \in (a,b)$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)...(x-x_n)$$

where $P(x)$ is the interpolating polynomial

Exercise

The flood-tide levels measured at a station at different times in a certain day are given in the following table

Time (h)	15	16	17	18
Level (m)	0.4	0.6	1.0	1.2

a) Develop Newton’s interpolating polynomial for these data. Use it to to estimate the flood-tide level at 16h30

b) Develop Lagrange interpolating polynomial for these data. Use it to to estimate the flood-tide level at 17h30

c) At what time the flood tide level is 0.9m?

Life expectancy improved dramatically in the 20th century. The table gives the life expectancy at birth (in years) of males born in the US. Predict the life span of a male born in the year 2010 using least squares regression with

a) Linear model, b) Quadratic model, c) Exponential model with Linearization

Which one is the best? Why?

Birth year	Life expectancy	Birth year	Life expectancy
1900	48.3	1960	66.6
1910	51.1	1970	67.1
1920	55.2	1980	70.0
1930	57.4	1990	71.8
1940	62.5	2000	73.0
1950	65.6		

QUIZ 4

1. The money flow rate into a bank in a morning is given in the following table

Time (h)	8	9	10	11
Rate (millions S/h)	18	16	12	5

a) Use 3rd-degree Newton interpolating polynomial to estimate the flow rate at 9h30

b) Use 3rd-degree Lagrange interpolating polynomial to estimate the flow rate at 10h30

c) Use inverse interpolation to determine the time the flow rate reaches 10 millions/h