Chapter 1: Roots of Nonlinear Equations

Lecture 2: Bracketing Methods

- **Bisection Method**
- **False Position Method**

Outline

- **Problems of Finding Roots**
- **Bisection Method**
- **False Position Method**

A ball with mass m is projected upward with initial velocity v_0. Forces acting on the ball are:

- Force of gravity
- Retarding force of air resistance $-pv(t)$

$p>0$: constant, $v(t)$: velocity of the ball at time t.

Newton's Second Law yields

Problem

$$mv'(t) = -pv(t) - mg$$

$$v(t) = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}$$

$y = y(t)$: height of ball at time $t, v(t) = y'(t)$

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}(1 - e^{-pt/m}) - \frac{mgt}{p}$$

$$m = 1kg, v_0 = 20m/s, p = 1/10$$

Find the time at which the ball hits the ground

1. Problems of Finding Roots

Many problems in Science and Engineering are expressed as:

$$\text{Given a continuous function } f(x),$$

$$\text{find the value } r \text{ such that } f(r) = 0$$

These problems are called root finding problems.

r is called a **root** of the equation $f(x)=0$

r is also called a **zero** of the function $f(x)$

Convergence Notation

Let $x_1, x_2, \ldots.,$ converge to x.

Linear Convergence: $$\frac{|x_{n+1} - x|}{|x_n - x|} \le C$$

Quadratic Convergence: $$\frac{|x_{n+1} - x|}{|x_n - x|^2} \le C$$

Convergence of order P: $$\frac{|x_{n+1} - x|}{|x_n - x|^p} \le C$$

2. Bisection Method

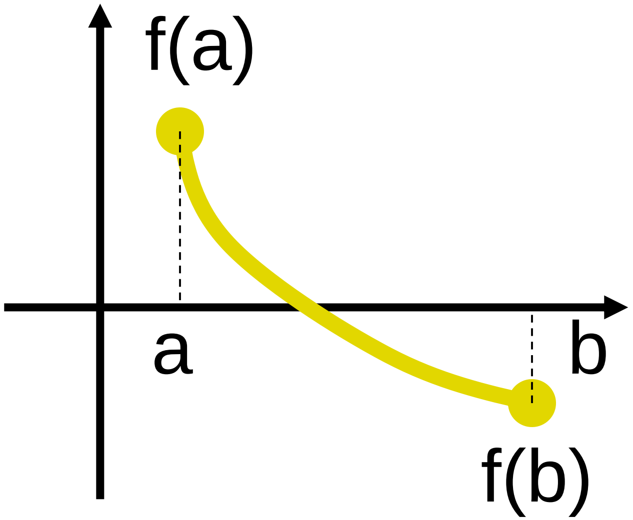
- The **Bisection method** is one of the simplest methods to find a zero of a nonlinear function.
- It is also called **interval halving** method.
- To use the Bisection method, one needs an initial interval that is known to contain a zero of the function. Midpoint of the interval is used as an approximate solution

Intermediate Value Theorem

- Let f(x) be defined on the interval [a,b].

- Intermediate value theorem:

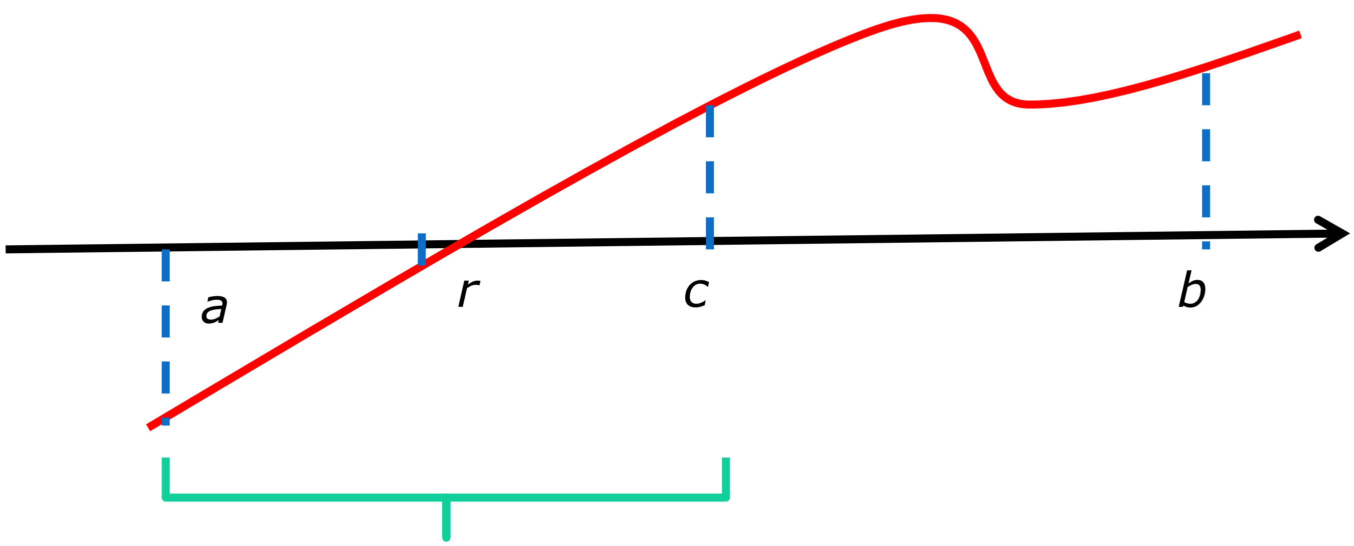
  if a function is continuous and f(a) and f(b) have different signs then the function has at least one zero in the interval [a,b].



Bisection method

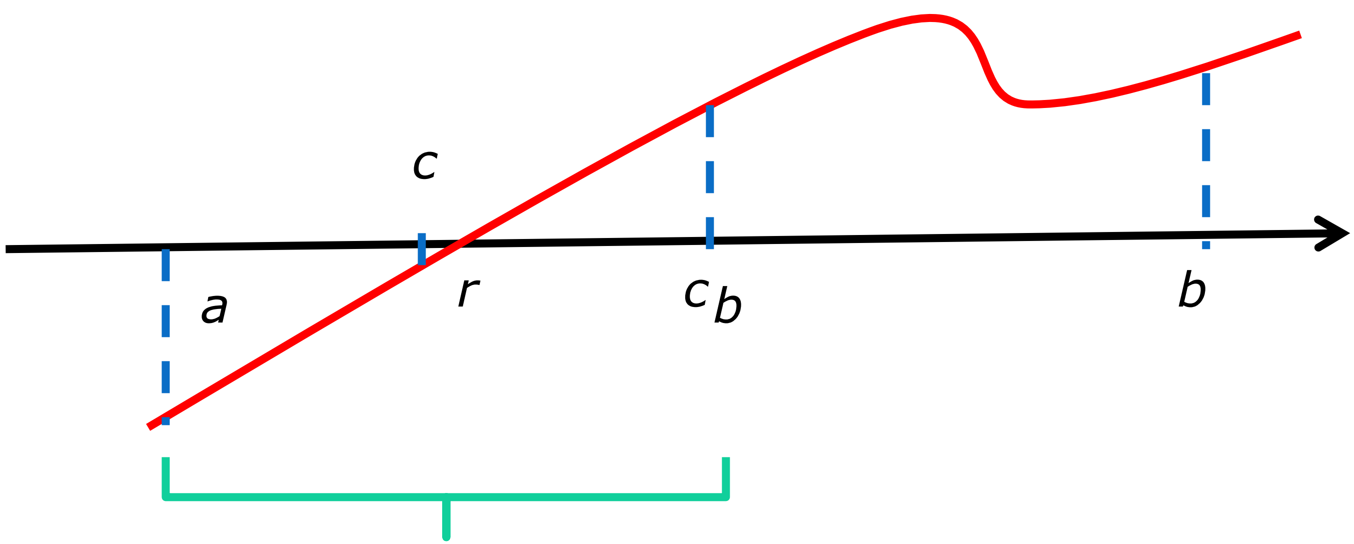
Test for values at Endpoints of each subinterval

- The midpoint c=(a+b)/2 halving [a, b]
- Root is in a subinterval : either [a, c] or [c, b]
- Having this subinterval again and again until the midpoint is sufficiently closed to the root



Algorithm

- **Step 1**: Choose an interval [*a*, *b*] such that $f(a)\,f(b) < 0$
- **Step 2**: Compute c=(a+b)/2 as approximate root. Verify the stopping criterion from the 2nd iteration
- **Step 3**: If f(a) f(c) ≤ 0, then the root lies in [a, c], set b=c. Otherwise, set a=c. Return to Step 2



Termination Criteria

- A stopping criterion must be used to terminate the computations
- We can use one of the following stopping criteria:
  1. |Approximate error| $< \varepsilon$
  2. |Relative Approximate error ε_a | $< \varepsilon$
  3. $|f(c)| < \varepsilon$
  4. Stop after a fixed number of iterations
  5. Others

Example

- Using Bisection method to determine the root of equation

$$f(x) = \frac{x^3}{2} + x - 6 = 0$$

- with

$$a = 0, b = 3 \text{ until } |\varepsilon_a| \leq 1\%$$

Solution

$$f(0) < 0, f(3) > 0$$

Bisection method can be used

Solution

$f(0) < 0, f(3) > 0$

1st Iteration: The value of midpoint of [a, b]=[0, 3]:

c=(a+b)/2=1.5; f(c)= -2.8125<0 Root belongs [c,b]=[1.5, 3]

So, c replaces a (i.e., we set a=c) [a,b]=[1.5, 3]

2nd Iteration: c= (a+b)/2=2.25; f(c)= 1.9453>0. [a,b]=[1.5 2.25]

So, root belongs to [a, c]. Thus, c replaces b (i.e., we set b=c)

3rd Iteration: c=(a+b)/2= 1.8750, f(c)= -0.8291<0. Root belongs to [c, b]

So, c relaces a (we set a=c) [a, b]=[1.8750, 2.25]

4th Iteration: c=(a+b)/2= 2.0625 , f(c)=0.4493>0. Root belongs to [a, c]. Thus, c replaces b (set b=c) [a, b]=[1.8750, 2.0625]

5fth Iteration: c =(a+b)/2= 1.9688, f(c)=-0.2158<0. Root belongs to [c, b]. Thus, c replaces a (set a=c) [a, b]=[1.9688, 2.0625]

6th iteration: c=(a+b)/2=2.0156, f(c)=0.1101>0. Root belongs to [a, c]. Thus, c replaces b (set b=c) [a, b]=[1.9688, 2.0156]

Solution…

[a, b]=[1.9688, 2.0156]

- **7th Iteration:** c=(a+b)/2= 1.9922, f(c)= -0.0545<0. Root belongs to [c, b]. So, c replaces a (we set a=c)

[a, b]=[1.9922, 2.0156]

- **8th Iteration:** c=(a+b)/2=2.0039.

Relative approximate error is given by

$$\varepsilon_a = (c_{new} - c_{old}) / c_{new} = (2.0039 - 1.9922) / 2.0039 = 0.0058$$

$$|\varepsilon_a| = 0.58\% < 1\%$$

- So, Approximate root is $\boxed{r \approx 2.0039}$

Bisection Method: Error Analysis

c_n : is the midpoint of the interval at the n$^{\text{th}}$ iteration

(c_n is usually used as the estimate of the root).

r : is the zero of the function.

After n iterations:

$$|error| = |r - c_n| \leq \frac{b-a}{2^n} = \frac{\Delta x^0}{2^n}$$

Convergence Analysis

$Given \quad f(x), a, \; b, and \;\; \varepsilon$

How many iterations are needed such that: $|x - r| \le \varepsilon$

where r is the zero of $f(x)$ and x is the bisection estimate (i.e., $x = c_k$)?

$$|\text{Error}| \le \frac{b-a}{2^n} \le \varepsilon \quad \Rightarrow \quad 2^n \ge \frac{b-a}{\varepsilon} \quad \Rightarrow \quad \ln(2^n) \ge \ln(\frac{b-a}{\varepsilon})$$

$$\boxed{n \ge \frac{\ln(b-a) - \ln(\varepsilon)}{\ln(2)}}$$

Remarks on Bisection Method

- Advantages:
  - Always convergent
  - The root bracket gets halved with each iteration - guaranteed.
- Disadvantages:
  - Slow convergence

Exercise 1

- Using Bisection method, find the approximate root of equation

$$f(x) = x^2\sqrt{x^2+1} - 2x^2 - 3x + 2 = 0$$

in the interval [0, 1] with the stopping condition

$$|\mathrm{E}_a| < 0.01$$

Exercise 2

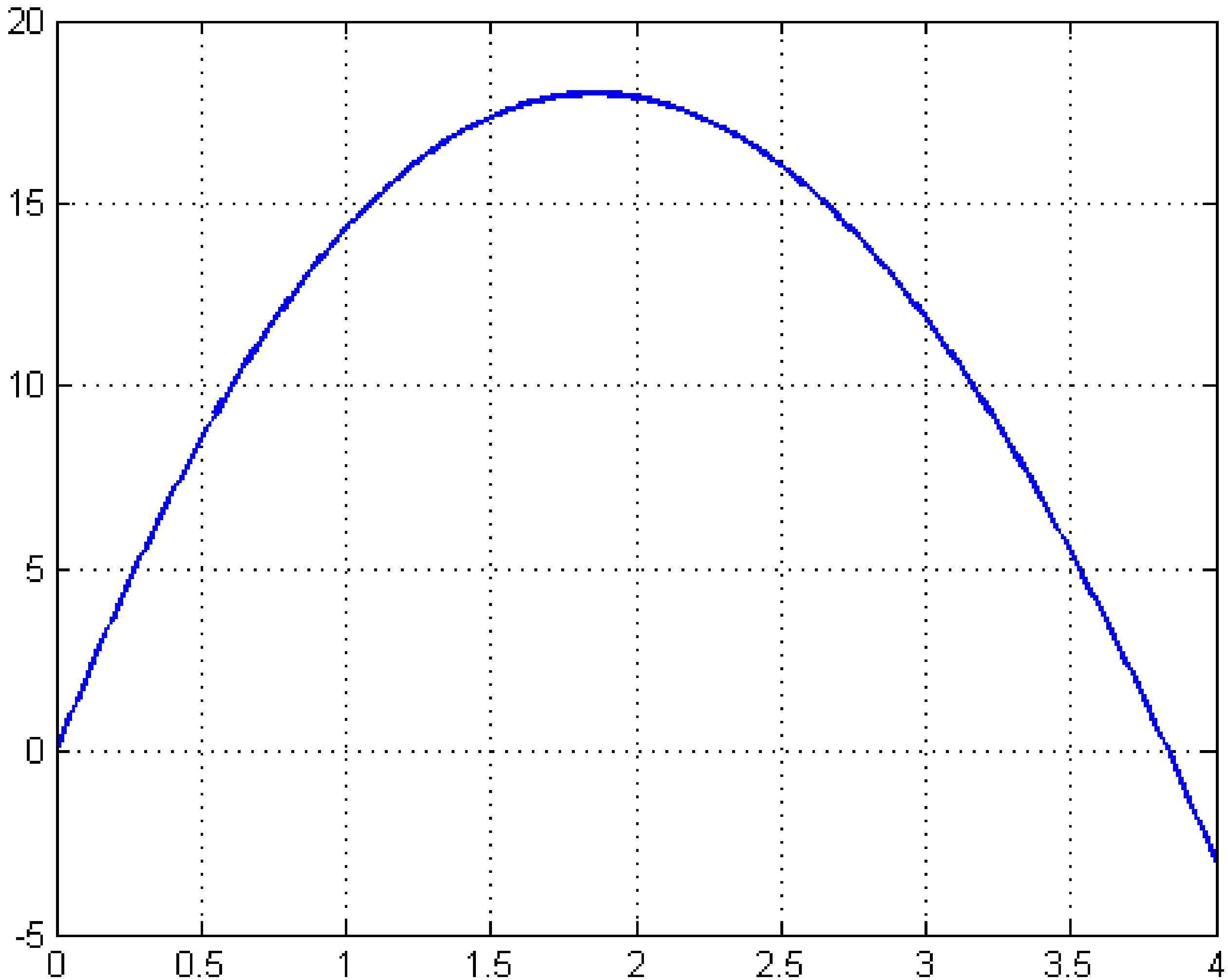
$$y(t)=\left(v_0+\frac{mg}{p}\right)\frac{m}{p}(1-e^{-pt/m})-\frac{mgt}{p}$$

$$m=1kg, v_0=20m/s, p=1/10$$

Find root of equation using bisection method in [3, 4] until $|\varepsilon_a|<1\%$

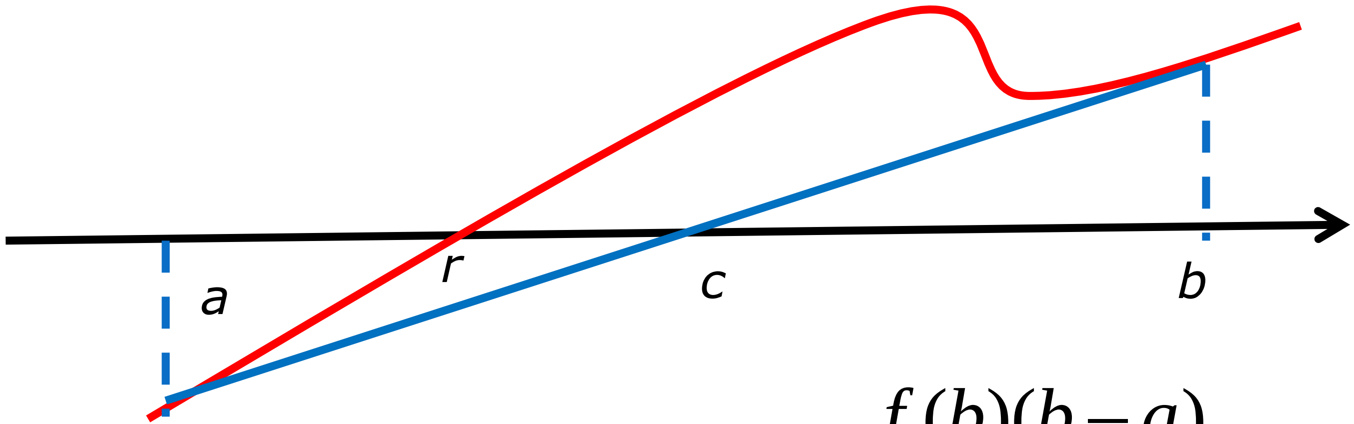
$$y(t)=1180(1-e^{-t/10})-98t=0$$

Does the ball go up faster than it comes down?


20
15
10
5
0
-5
0
0.5
1
1.5
2
2.5
3
3.5
4

3. False Position Method

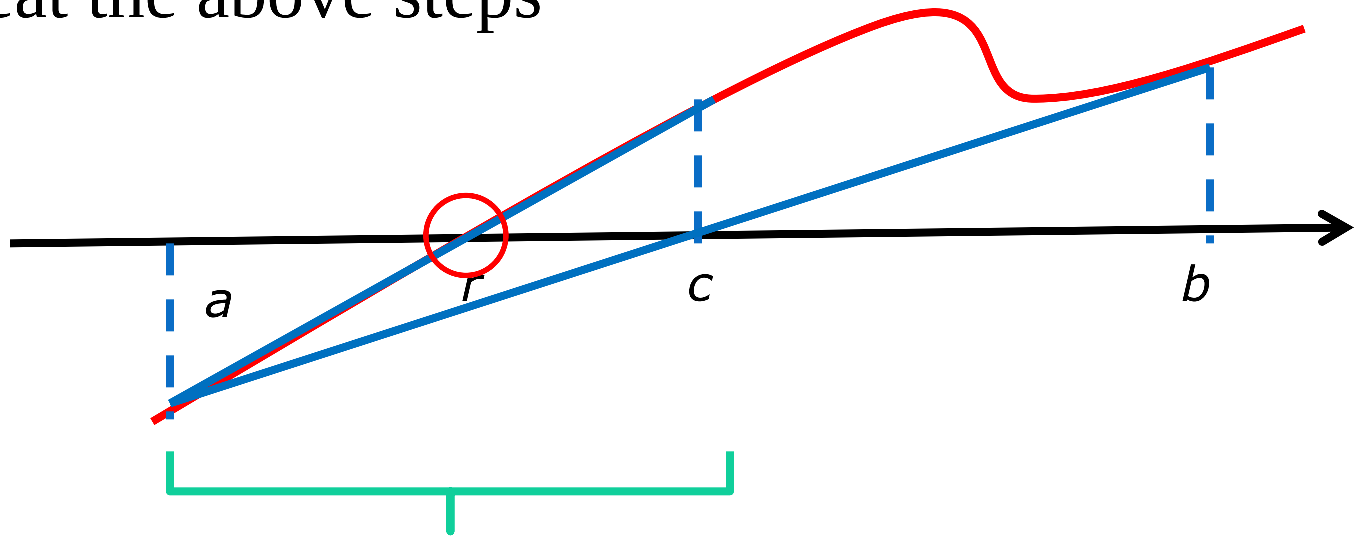
- An improvement of Bisection method: we take into account the magnitude of $f(a)$ and $f(b)$
- The x-intercept c of the straight line between $(a, f(a))$ and $(b, f(b))$ is the new end point instead of the midpoint $(a+b)/2$



$$c = b - \frac{f(b)(b-a)}{f(b) - f(a)}$$

False-position method

- Calculate $$\boxed{c = b - \frac{f(b)(b-a)}{f(b)-f(a)}}$$
- Choose the subinterval : either [a, c] or [c, b] that contains the root
- Repeat the above steps



Algorithm

- **Step 1**: Choose an interval [a, b] s.t. $f(a)\,f(b) < 0$
- **Step 2**: Calculate the x-intercept:

$$\boxed{c = b - \frac{f(b)(b-a)}{f(b) - f(a)}}$$

Verify the stopping criterion from the 2nd iteration

- **Step 3**: If $f(a)\,f(c) \le 0$, then the root lies in [a, c], set b=c. Otherwise, set a=c. Return to Step 2

Remarks

- The false-position method sometimes called the regula falsi method. It always converges.
- Sometimes the method converges slowly. For example in the cases that one of the end points does not change.

Example

The height (in m) at the time t (in seconds) of a ball projected vertically into the air is given by

$$f(t) = 1180(1 - e^{-t/10}) - 98t$$

Find the time at which the ball hits the ground using False Position method and stopping criterion

$$|\varepsilon_a| < 0.1\%$$

Solution

$$c = b - \frac{f(b)(b-a)}{f(b) - f(a)}$$

a	b	c	$f(a)$	$f(b)$	$f(c)$	$\lvert \varepsilon_a \rvert$
3	4	3.79897	11.834	-2.97765	0.66132	
3.79897	4	3.83551	0.66132	-2.97765	0.02406	0.95%
3.83551	4	3.83682	0.02406	-2.97765	0.00086	0.034%

Exercise

- Using false-position method, find the approximate root of equation

$$f(x) = \frac{x^3}{2} + x\sqrt{x-1} - 6 = 0$$

with a=1.5, b=3, and the stopping condition

$$|\varepsilon_a| < 0.5\%$$

- Given exact root x=2, find the error and relative error

Exercise 2

Using false-position method, find the approximate root of equation $f(x) = x^{4/3} + x^2 - 2 = 0$

with stopping condition $|\varepsilon_a| < 0.5\%$

Homework N1: Problem 1

- The velocity at the time t of a falling parachutist is given by

$$v(t) = \frac{gm}{c}(1 - e^{-ct/m}), \quad \text{where } g = 9.8, \quad c = (10 + \frac{k}{k+1})\,\mathrm{kg/s}$$

- where k is your two last digits of student ID. Compute the mass m so that the velocity is v=40m/s at t=10s using
- (a) Bisection Method, (b) False Position method,
- (c) Newton-Raphson method, and (d) Secant method
- In all cases use the stopping criterion $|\varepsilon_a| < 0.1\%$
- **Deadline: 4 weeks**

Homework N1.

- Problem 2: Use Bisection, False Position, Newton-Raphson and Secant methods to find the root of the equation

  with $|\varepsilon_a| \le \varepsilon = 0.1\%$

  mn: Your last two digits of student ID number

  $$a)\quad f(x) = x^3 + 4x - (m+2)/(n+1) = 0$$

  $$b)\quad g(x) = e^{-5x^2} + 8x^3 + 6x - 10(1 + m/(n+1)) = 0$$

S. Chapra & R.P. Canale, Numerical Methods for Engineers, McGraw-Hill, 7th ed., 2015:
Page 217-220
Problems: 8.15, 8.16, 8.17, 8.19, 8.22, 8.23