Chapter 1 METRIC SPACES

References

Textbook:

H. L. Royden, P. M. Fitzpatrick, *Real Analysis*, 4th ed. Pearson Education, 2010 (**pp. 183–221**)

Definition 1.1

A **metric** (or a **distance**) d on a nonempty set X is a function $d : X \times X \to \mathbb{R}$ satisfying the three properties:

(a) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$, $x, y \in X$ (symmetry);

(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (the triangle inequality).

The pair (X, d) is called a **metric space**.

When the metric is clear from context, we write simply X for (X, d).

In this context, we call the elements of X **points**, and refer to $d(x, y)$ the **distance between the points x and y**.

Here are simple consequences of Axioms (a)–(c).

1. *In a metric space* (X, d) *the inequality*
$$|d(x,z) - d(y,z)| \leq d(x,y)$$
holds for all points $x, y, z \in X$.
2. *In a metric space* (X, d) *the inequality*
$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$$
holds for arbitrary points $x_1, \ldots, x_n$ *in* X.

Example 1.1 (The usual distance on $\mathbb{R}$)
The set of real numbers $\mathbb{R}$ equipped with the **usual distance**

$$d(x, y) = |x - y| \quad \text{for all} \quad x, y \in \mathbb{R}$$

is a metric space.

Unless otherwise stated, we consider $\mathbb{R}$ to be a metric space with the usual distance.

Example 1.2 **(The Euclidean n-Space $\mathbb{R}^n$)**
The Euclidean space $\mathbb{R}^n$ equipped with the distance

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, is a metric space.

This distance on $\mathbb{R}^n$ is called the **Euclidean distance**.

Example 1.3 **(The Discrete Metric)**
Let X be a nonempty set. Then the function d defined by

$$d(x,y) = \begin{cases} 1 & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y \end{cases}$$

is a distance on X.

This distance is called the **discrete distance** on X, and X with this distance is called a **discrete metric space**.

Example 1.4 **(Metric Subspaces)**
Let (X, d) be a metric space and Y a nonempty subset of X. Then the restriction of d to $Y \times Y$,

$$d_Y(x, y) = d(x, y) \quad \text{for all} \quad x, y \in Y,$$

is a metric on Y, the **induced metric**, and (Y, d_Y) is a metric space, a **metric subspace** of X.

When no misunderstanding is possible, we write d instead of d_Y.

Example 1.5 **(Metric Products)**
For metric spaces (X_1, d_1) and (X_2, d_2), we define the product metric d on the Cartesian product $X := X_1 \times X_2$ by setting, for $(x_1, x_2), (y_1, y_2) \in X$,

$$d\big((x_1, x_2), (y_1, y_2)\big) = \sqrt{\big(d_1(x_1, y_1)\big)^2 + \big(d_2(x_2, y_2)\big)^2}.$$

(X, d) is called the **(metric space) product** of the metric spaces (X_1, d_1) and (X_2, d_2).

Definition 1.2

A real-valued function $\|\cdot\|$ defined on a vector space X is called a **norm** if it satisfies the following three properties:

(i) $\|x\| \geq 0$ for each $x \in X$, and
$\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Property (iii) is called the **triangle inequality** for the norm.

A vector space X equipped with a norm is called a **normed vector space**, or simply a **normed space**.

If the norm is clear from context, we write X instead of $(X, \|\cdot\|)$.

To avoid trivialities, the vector spaces will be tacitly assumed to be different from $\{\mathbf{0}\}$.

A norm $\|\cdot\|$ on a vector space X induces a metric ρ on X by defining

$$\rho(x, y) = \|x - y\|, \quad x, y \in X.$$

We shall call this metric on X the **metric induced by the norm**. Hence,

Any normed space is also a metric space

Example 1.6 The vector space $\mathbb{R}^n$ with the norm

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, is a normed space.

This norm is called the **Euclidean norm**, and it gives the Euclidean metric.

Example 1.7 For $a, b \in \mathbb{R}$, $a < b$, consider the vector space $C([a, b])$ of all continuous real-valued functions on $[a, b]$. The **maximum norm** $\|\cdot\|$ is defined for $f \in C([a, b])$ by

$$\|f\| = \max_{a \leq x \leq b} |f(x)|.$$

This norm is also called the **uniform norm** on $C([a, b])$.

Many concepts studied in Euclidean spaces can be naturally and usefully extended to general metric spaces.

1.2 OPEN SETS, CLOSED SETS, INTERIOR, AND CLOSURE

In the metric space (X, d), for $a \in X$ and $r > 0$, the set

$$B(a, r) := \{x \in X : d(x, a) < r\}$$

is called the **open ball** with center at a and radius r, while

$$\overline{B}(a, r) := \{x \in X : d(x, a) \leq r\}$$

is called the **closed ball** with center at a and radius r.

In a normed space X we refer to $B(\mathbf{0}, 1)$ as the **open unit ball** and $\overline{B}(\mathbf{0}, 1)$ as the **closed unit ball**.

Definition 2.1

A subset U of X is called **open** if for every $x \in U$, there exists some $r > 0$ such that $B(x, r) \subset U$.

A set is said to be **closed** if its complement is open.

$$U \text{ open } \iff (\forall x \in U)(\exists r = r(x) > 0)\big(B(x, r) \subset U\big)$$

$$A \text{ is closed } \iff A^c \text{ is open}$$

Example 2.1

(a) Every open ball $B(x, r)$ in a metric space X is an open set.

(b) For any $a \in \mathbb{R}$, the intervals $(-\infty, a)$ and (a, ∞) are open sets in $\mathbb{R}$. If $a, b \in \mathbb{R}$, $a < b$, then the interval (a, b) is an open set in $\mathbb{R}$. These are called **open intervals**.

Theorem 2.1

For a metric space X the following statements hold:

(a) *X and $\emptyset$ are open sets.*

(b) *Arbitrary unions of open sets are open sets.*

(c) *Finite intersections of open sets are open sets.*

Theorem 2.2

For a metric space (X, d) the following statements hold:

(a) *X and $\emptyset$ are closed sets.*

(b) *Arbitrary intersections of closed sets are closed sets.*

(c) *Finite unions of closed sets are closed sets.*

Example 2.2

(a) Every closed ball is closed.

(b) Any one element subset of a metric space is closed.

(c) For any $a \in \mathbb{R}$, the intervals $(-\infty, a]$ and $[a, \infty)$ are closed sets in $\mathbb{R}$. If $a, b \in \mathbb{R}$, $a < b$, then the interval $[a, b]$ is a closed set in $\mathbb{R}$. These are called **closed intervals**.

(d) $(0, 1]$ and $[3, 10)$ are neither open nor closed.

Note Infinite intersections of open sets may _not_ be open sets; infinite unions of closed sets may _not_ be closed sets.

Theorem 2.3

Let Y be a nonempty subset of the metric space X and A a subset of Y. Then

(a) *A is open in the subspace Y if and only if $A = Y \cap G$, where G is open in X;*

(b) *A is closed in the subspace Y if and only if $A = Y \cap F$ for some closed set F in X.*

Theorem 2.4 (Open sets in $\mathbb{R}$)

A subset of $\mathbb{R}$ is open if and only if it is the union of a countable collection of disjoint open intervals.

$$A \subset \mathbb{R} \text{ is open} \iff \begin{cases} A = \cup_n (a_n, b_n) \text{ and} \\ \quad (a_n, b_n) \text{ are disjoint.} \end{cases}$$

Definition 2.2

A point a is called an **interior point** of a subset A if there exists an open ball $B(a, r)$ such that $B(a, r) \subset A$. The set of all interior points of A is denoted by $\operatorname{int} A$ or A° and is called the **interior** of A.

$$a \text{ is an interior point of } A \iff \exists\, r > 0 : B(a, r) \subset A$$

$$\operatorname{int} A = \{a \in X : a \text{ is an interior point of } A\}.$$

Question: Find $\operatorname{int} \mathbb{Z}$ and $\operatorname{int} \mathbb{Q}$.

Theorem 2.5

int A *is the largest open subset of* X *included in* A,

$$\operatorname{int} A = \bigcup \{U \subset X : U \subset A \text{ and } U \text{ is open}\}.$$

Definition 2.2

For a point $a \in X$, an open set that contains a is called a **neighborhood** of a.

U is a neighborhood of $a \Longleftrightarrow U$ is open and $U \ni a$

Definition 2.3

A point x in a metric space X is called a **closure point** (or **point of closure**) of a subset A of X if every open ball centered at x contains (at least) one element of A; that is, $B(x, r) \cap A \neq \emptyset$ for all $r > 0$.

The set of all closure points of A is denoted by $\overline{A}$, and is called the **closure** of A.

$$\boxed{x \in \overline{A} \iff (\forall r > 0)\big(B(x, r) \cap A \neq \emptyset\big)}$$

Clearly,

$$A \subset \overline{A}.$$

Theorem 2.6

For every subset A of a metric space, $\overline{A}$ is closed. Moreover, $\overline{A}$ is the smallest closed set that includes A,

$$\overline{A} = \bigcap\{B : B \text{ is closed and } B \supset A\}.$$

In particular, a set A is closed if and only if $A = \overline{A}$.

Thus,

$$A \text{ is closed} \iff \Big[(\{x_n\} \subset A) \wedge (x_n \to x) \Longrightarrow x \in A\Big].$$

Definition 2.4

A point $x \in X$ is called a **boundary point** of a set A if every open ball of x contains points from A and $A^c = X \setminus A$, that is, if $B(x,r) \cap A \neq \emptyset$ and $B(x,r) \cap A^c \neq \emptyset$ for all $r > 0$. The set of all boundary points of a set A is denoted by $\mathbf{bd\ A}$ or $\boldsymbol{\partial A}$ and is called the **boundary** of A.

By the symmetry of the definition,

$$\mathrm{bd\ A} = \overline{A} \cap \overline{A^c} = \mathrm{bd\ A^c}.$$

♠ Let $\{a_n\} \subset \mathbb{R}$.

$$a = \lim_{n\to\infty} a_n \overset{def}{\Longleftrightarrow} ?$$

Definition 3.1

A sequence $\{x_n\}$ of points in a metric space (X, d) is said to **converge** to the point $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0,$$

that is, for each $\epsilon > 0$, there is an index N such that for every $n > N$, $d(x_n, x) < \epsilon$. The point x is called the **limit** of the sequence $\{x_n\}$ and we write $\lim x_n = x$, or $x_n \to x$.

$$\begin{aligned} x_n \to x &\iff d(x_n, x) \to 0 \\ &\iff (\forall \epsilon > 0)(\exists N)(\forall n > N)\big(d(x_n, x) < \epsilon\big). \end{aligned}$$

Example 3.1 In $\mathbb{R}$ (with the usual distance) a sequence of real numbers $\{x_n\}$ converges to a (real) limit x if, given any $\epsilon > 0$, there exists an integer $N > 0$ such that $|x_n - x| < \epsilon$ for all $n > N$.

Theorem 3.1

(a) *If a sequence is convergent, then the limit is unique*

(b) *If* $\{x_n\}$ *is a convergent sequence with limit* x*, then each subsequence of* $\{x_n\}$ *is also convergent with limit* x*.*

(c) *If* $\lim x_n = x$ *and* $\lim y_n = y$*, then*

$$\lim d(x_n, y_n) = d(x, y).$$

Theorem 3.2 (Convergence in $\mathbb{R}^n$)

In the Euclidean space $\mathbb{R}^n$, a sequence $\{x_k = (x_{k,1}, \dots, x_{k,n})\}$ converges to a point $x = (x_1, \dots, x_n)$ if and only if

$$\lim_{k\to\infty} x_{k,1} = x_1, \dots, \lim_{k\to\infty} x_{k,n} = x_n,$$

i.e., if and only if each component of $\{x_k\}$ converges to the corresponding component of x.

$$(x_{k,1}, x_{k,2} \ldots, x_{k,n}) \to (x_1, x_2, \ldots, x_n)$$

$$\Updownarrow$$

$$x_{k,1} \to x_1, x_{k,2} \to x_2, \ldots, x_{k,n} \to x_n$$

Theorem 3.3

For a subset A of a metric space X, a point $x \in X$ is a closure point of A if and only if x is the limit of a sequence in A. Therefore, A is closed if and only if it contains the limits of all convergent sequences in A.

$$x \in \overline{A} \iff (\exists \{x_k\} \subset A,\ x_k \to x).$$

$$A \text{ is closed} \iff \Big[(\{x_k\} \subset A \text{ and } x_k \to x) \Longrightarrow (x \in A)\Big].$$

Example 3.2 Let A be a nonempty subset of $\mathbb{R}$. Then

$$\alpha = \inf A \iff (\alpha \le a \ \forall a \in A) \text{ and } (\exists \{a_n\} \subset A : a_n \to \alpha);$$

$$\beta = \sup A \iff (a \le \beta \ \forall a \in A) \text{ and } (\exists \{a_n\} \subset A : a_n \to \beta).$$

♠

$$f : (a, b) \to \mathbb{R} \text{ is continuous at } x_0 \in (a, b) \overset{???}{\iff} \quad .$$

Definition 4.1

A mapping f from a metric space (X, d) to a metric space (Y, ρ) is said to be **continuous at a point $a \in X$** if for every $\epsilon > 0$ there exists $\delta > 0$ (depending on a and ϵ) such that

$$\rho(f(x), f(a)) < \epsilon \quad \text{whenever} \quad d(x, a) < \delta.$$

The mapping f is said to be **continuous on X** (or simply **continuous**) if f is continuous at every point of X.

If f is not continuous at $a \in X$, we say that f has a **discontinuity** at a, or that f is **discontinuous** at a.

Example 4.1 Let (X, d) and (Y, ρ) be metric spaces.

(a) Any constant mapping $\varphi : X \to Y$, $\varphi(x) = y_0$, is continuous.

(b) The **identity mapping** $\mathrm{id} : X \to X$, $\mathrm{id}(x) = x$, is continuous.

Example 4.2 If $A \subset X$ and $f : X \to Y$ is continuous at $a \in A$, then $f_{|A} : A \to Y$ is continuous at a. Thus if f is continuous on X, then $f_{|A}$ is continuous on A. Here A has the metric induced from X.

f is continuous $\Longrightarrow$ $f_{|A}$ is continuous

Example 4.3 The **Dirichlet function** $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases}$$

is nowhere continuous, that is, it is discontinuous at every $x_0 \in \mathbb{R}$.

- Note that $f_{|\mathbb{Q}}$ is continuous.

Uniform Continuity

Definition 4.2

A mapping f from a metric space (X, d) to a metric space (Y, ρ) is said to be **uniformly continuous** if for every $\epsilon > 0$, there exists some $\delta > 0$ (depending only on ϵ) such that for $x, x' \in X$,

$$\text{if} \quad d(x, x') < \delta \quad \text{then} \quad \rho\big(f(x), f(x')\big) < \epsilon.$$

- Note that

A uniformly continuous mapping is continuous.

Example 4.4 Let (X, d) and (Y, ρ) be metric spaces. A mapping $f : X \to Y$ is **Lipschitz** with **Lipschitz constant** $K > 0$ if

$$\rho\big(f(x), f(x')\big) \leq Kd(x, x') \quad \text{for all } x, x' \in X.$$

Every Lipschitz mapping is uniformly continuous.

Example 4.5 Let $(X, \|\cdot\|)$ be a normed space. The norm $\|\cdot\| : X \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant 1 because

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| = d(x, y) \quad \forall x, y \in X.$$

More general, if $x_0 \in X$ is fixed, then the function $f(x) = \|x - x_0\|$ is Lipschitz continuous with Lipschitz constant 1.

Example 4.6 **(Projections)**

For each $k = 1, \ldots, n$, the ***k*-th projection**

$$\pi_k : \mathbb{R}^n \to \mathbb{R}$$

defined by

$$\pi_k(x_1, \ldots, x_n) = x_k$$

is Lipschitz continuous with Lipschitz constant 1.

Thus,

Every projection is continuous.

Example 4.7 **(Distance from a set)** Let A be a nonempty subset of X. For each $x \in X$, the number

$$d(x, A) := \inf_{a \in A} d(x, a)$$

is called the **distance from x to A**. The **distance function**

$$d(\cdot, A) : X \to \mathbb{R}, \quad x \mapsto d(x, A)$$

is Lipschitz continuous with Lipschitz constant 1. Thus $d(\cdot, A)$ is continuous.

(Note that the value of the function $d(\cdot, A)$ at x is the number $d(x, A)$.)

Remarks

Let $f : (X, d) \to (Y, \rho)$ be a mapping. The following are equivalent conditions.

(a) f is continuous at a.

(b) For each $\epsilon > 0$, there is $\delta > 0$ for which $f\big(B(a, \delta)\big) \subset B\big(f(a), \epsilon\big)$.

(c) For every neighborhood V of the point $f(a)$ in Y, there exists a neighborhood U of the point a in X such that $f(U) \subset V$.

Theorem 4.1 (Sequential Criterion for Continuity)

A mapping $f:(X,d)\to(Y,\rho)$ *is continuous at the point* $a\in X$ *if and only if for every sequence* $\{x_k\}$ *in* X *such that* $\lim x_k = a$*, we have* $\lim f(x_k) = f(a)$.

$$f \text{ is continuous at } a \iff \left[x_k \to a \text{ implies } f(x_k)\to f(a)\right]$$

Example 4.8 Suppose f and g are real-valued functions on X that are continuous at a.

(a) The functions

$$f+g, \quad f-g, \quad fg, \quad \text{and } |f|$$

are all continuous at a.

(b) If $g(x) \neq 0$ for all x near a, then f/g is also continuous at a.

(c) If c is a constant, then cf is continuous at a.

Example 4.9 Consider the continuity of the following functions:

(a) $f, g : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} a & \text{if } x = 0 \\ \frac{1}{x} & \text{if } 0 < x \leq 1, \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin\frac{\pi}{x} & \text{if } 0 < x \leq 1. \end{cases}$$

(b) $h(x) = \chi_A(x)$, where A is a subset of a metric space (X, d).

Theorem 4.2

Let $f : X \to Y$ be a mapping between metric spaces X and Y. Then the following are equivalent:

(a) *f is continuous.*

(b) *The inverse image $f^{-1}(U)$ of each open set U in Y is open in X.*

(c) *The inverse image $f^{-1}(F)$ of each closed set F in Y is closed in X.*

Example 4.10

(a) Let X and Y be metric spaces, and $f : X \to Y$ continuous. Then, for each $y \in Y$, the solution set $f^{-1}(y)$ of the equation $f(x) = y$ is closed.

(b) Let $f : X \to \mathbb{R}$ be continuous and $\alpha \in \mathbb{R}$. Then

$$\{x \in X : f(x) \leq \alpha\}, \quad \{x \in X : f(x) \geq \alpha\}$$

are closed and

$$\{x \in X : f(x) < \alpha\}, \quad \{x \in X : f(x) > \alpha\}$$

are open in X.

(c) In a normed space $(X, \|\cdot\|)$, every sphere $S = \{x \in X : \|x - x_0\| = r\}$ is closed.

Theorem 4.3 (Continuity of Compositions)

Let X, Y and Z be metric spaces. Suppose that $f : X \to Y$ is continuous at $x \in X$, and $g : Y \to Z$ is continuous at $f(x) \in Y$. Then the composition $g \circ f : X \to Z$ is continuous at x. Thus the composition of continuous mappings, when defined, is continuous.

Composition of continuous mappings is continuous

Complete Spaces and Examples

•Convergence Tests of Sequences in $\mathbb{R}$

Definition 5.1

A sequence $\{x_n\}$ in a metric space (X, d) is said to be a **Cauchy sequence** if, for each $\epsilon > 0$, there is an index N such that $d(x_m, x_n) < \epsilon$ for all $m, n > N$.

Remarks

1. *Every convergent sequence is a Cauchy sequence.*
2. *If a Cauchy sequence* $\{x_n\}$ *has a subsequence* $\{x_{n_k}\}_k$ *that converges to* x, *then* $\{x_n\}$ *converges to* x.

Definition 5.2

We say that a subset E of a metric space (X, d) is **bounded** if it is contained in some ball. The **diameter** of E is given by

$$\operatorname{diam} E := \sup\left\{d(x, y) : x, y \in E\right\}.$$

$$E \text{ bounded } \overset{def}{\Longleftrightarrow} \big(\exists B(x_0, r)\big)\big(E \subset B(x_0, r)\big)$$

Remarks

(i) E is bounded if and only if $\operatorname{diam} E < \infty$.

(ii) A set E in a normed space $(X, \|\cdot\|)$ is bounded if and only if there is a constant $K > 0$ such that $\|x\| \leq K$ for all $x \in E$.

(iii) A set E in $\mathbb{R}^n$ is bounded if and only if there is a constant $K > 0$ such that $|x_i| \leq K$ for all $x = (x_1, x_2, \ldots, x_n) \in E$ and $i = 1, 2, \ldots, n$.

(iv) Any Cauchy sequence is bounded.

Hence

A convergent sequence is bounded.

Definition 5.3

A metric space X is called **complete** if every Cauchy sequence in X converges to a point in X.

Example 5.1

(a) The real line $\mathbb{R}$ with its usual metric is complete.

(b) The Euclidean space $\mathbb{R}^n$ is complete.

(c) $C([a, b])$ with the uniform metric is complete.

(d) $[0, 1)$ and $\mathbb{Q}$ with induced metrics are not complete.

Theorem 5.1

(a) *A complete subspace of a metric space is closed.*

(b) *A closed subspace of a complete metric space is complete.*

Dense Sets and Separable Spaces

Definition 5.4

A subset A of a metric space X is called **dense** in X if $\overline{A} = X$. A metric space is called **separable** if it contains a countable dense subset.

Example 5.2 $\mathbb{R}$ and $\mathbb{R}^n$ are separable.

Remarks *The following are equivalent:*

(a) *A is dense in X.*

(b) *For each $x \in X$, there is a sequence $\{x_n\}$ of points in A such that $x_n \to x$.*

(c) *The complement of A has empty interior, that is, $\operatorname{int}(A^c) = \emptyset$.*

Remark

Every subspace of a separable metric space is separable.

The Baire Category Theorem

Definition 5.5

A subset A of X is said to be **nowhere dense** if its closure $\overline{A}$ has no interior point. A subset Y of a metric space is said to be of **first category** if there exists a sequence $\{A_n\}$ of nowhere dense subsets such that $Y = \cup_{i=1}^{\infty} A_n$. All subsets of X that are not of first category in X are said to be of **second category** in X.

Example 5.3 Any countable set in $\mathbb{R}$ is of first category. In particular, $\mathbb{Q}$ is of first category.

Theorem 5.2 (The Baire Category Theorem)

Let X be a complete metric space.

(a) *Let $\{G_n\}_{n=1}^{\infty}$ be a countable collection of open dense subsets of X. Then the intersection $\bigcap_{n=1}^{\infty} G_n$ also is dense.*

(b) *Let $\{F_n\}_{n=1}^{\infty}$ be a countable collection of closed nowhere dense subsets of X. Then the union $\bigcup_{n=1}^{\infty} F_n$ has empty interior.*

Corollary 5.3

Let X be a complete metric space and $\{F_n\}_{n=1}^{\infty}$ a countable collection of closed subsets of X.

(a) *If $\bigcup_{n=1}^{\infty} F_n$ has nonempty interior, then at least one of the F_n's has nonempty interior.*

(a) *If $X = \bigcup_{n=1}^{\infty} F_n$, then at least one of the F_n's has nonempty interior.*

The Baire Category Theorem may also be rephrased as follows:

> *A nonempty open subset of a complete metric space is of the second category. In particular, a complete metric space is of the second category.*

Definition 5.6

Let (X, d) be a metric space. A mapping $f : X \to X$ is called a **contraction mapping** if there is some $\alpha \in (0, 1)$ such that

$$d\big(f(x), f(x')\big) \leq \alpha d(x, x'), \quad x, x' \in X.$$

The number α is called a **contraction constant** for f. A point $x \in X$ for which $f(x) = x$ is called a **fixed point** for f.

The following Banach Contraction Principle *provides an algorithm for approximating the fixed point* of a contraction mapping in a complete space.

Theorem 5.4 (The Banach Contraction Principle)

Let X be a complete metric space and let $f : X \to X$ be a contraction mapping with contraction constant α. Then f has a unique fixed point x_*. Moreover, given any $x_0 \in X$, the sequence $\{x_n\}$ defined recursively $x_{n+1} = f(x_n)$ converges to the fixed-point, and the following estimates hold

$$d(x_{n+1}, x_*) \leq \alpha d(x_n, x_*)$$

$$d(x_n, x_*) \leq \frac{\alpha^n}{1-\alpha} d(x_1, x_0)$$

$$d(x_{n+1}, x_*) \leq \frac{\alpha}{1-\alpha} d(x_{n+1}, x_n).$$

Compact Sets and Characterizations

Definition 6.1

A collection of sets $\{A_i\}_{i\in I}$ is said to be a **cover** of a set A provided $A \subset \cup_{i\in I} A_i$. If a subfamily of $\{A_i\}_{i\in I}$ also covers A, then it is called a **subcover**. If $\{A_i\}_{i\in I}$ is finite, we call it a **finite cover** of A. If A is a subset of a metric space X, by an **open cover** of A we mean a cover of A consisting of open subsets of X.

Definition 6.2

Let X be a metric space. A subset A of X is said to be **compact** if every open cover of A has a finite subcover. If X is itself a compact set, we call X a **compact space**.

Example 6.1

(a) Any finite subset of a metric space X is compact.

(b) The sets $\mathbb{R}$ and $S = (0, 1]$ are not compact.

Properties of Compact sets

Theorem 6.1

Every compact set is closed and bounded.

compactness $\Longrightarrow$ closedness $+$ boundedness

Theorem 6.2

A nonempty compact subset of a metric space is a complete and separable subspace.

Characterizations of Compact sets

Theorem 6.3

For a subset A of a metric space (X, d) the following statements are equivalent:

(a) *A is a compact set.*

(b) *Every sequence in A has a subsequence which converges to a point of A.*

Definition 6.3

A collection $\mathcal{F}$ of sets in X is said to have the **finite intersection property** provided any finite subcollection of $\mathcal{F}$ has a nonempty intersection.

Theorem 6.4

A metric space X is compact if and only if every collection $\mathcal{F}$ of closed subsets of X with the finite intersection property has nonempty intersection.

Theorem 6.5 (The Heine-Borel Theorem)

A subset of a Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

$$\boxed{\text{compactness} \overset{\mathbb{R}^n}{=} \text{closedness} + \text{boundedness}}$$

Corollary 6.6 (The Bolzano-Weierstrass Theorem)

Every bounded sequence of points in $\mathbb{R}^n$ has a convergent subsequence in $\mathbb{R}^n$.

Continuous Mappings on Compact Spaces

Theorem 6.7

If $f : K \to Y$ is continuous and K is a compact subset of a metric space X, then $f(K)$ is a compact subset of Y.

For short,

The continuous image of a compact set is compact.

Theorem 6.8

Let K be a nonempty compact subset of a metric space X and let $f : X \to \mathbb{R}$ be continuous. Then f attains a maximum and a minimum on K, that is, there are $x_0, x_1 \in K$ such that

$$f(x_0) = \min_{x \in K} f(x) \quad \textit{and} \quad f(x_1) = \max_{x \in K} f(x).$$

Example 6.2 Given a nonempty compact subset K of a metric space (X, d) and a point $x_0 \in X$, there exists a point $x_* \in K$ such that

$$d(x_0, x_*) = \min_{x \in K} d(x_0, x).$$

The point x_* is called a **best approximation** in K of the point x_0 in X.

$K \subset X$ is compact and $f : X \to \mathbb{R}$ is continuous

$$\Downarrow$$

$$\exists x_0 \in K : \ f(x_0) = \min_{x \in K} f(x) \quad \text{and}$$
$$\exists x_1 \in K : \ f(x_1) = \max_{x \in K} f(x)$$

Uniform Continuity

Recall that

A mapping from a metric space (X, d) to a metric space (Y, ρ) is said to be **uniformly continuous** if for every $\epsilon > 0$, there exists some $\delta > 0$ (depending only on ϵ) such that for $x, x' \in X$,

$$\text{if} \quad d(x, x') < \delta \quad \text{then} \quad \rho\big(f(x), f(x')\big) < \epsilon.$$

Theorem 6.9

If X is compact, then every continuous mapping $f : X \to Y$ is uniformly continuous.

That is,

Continuous mappings on compact sets are uniformly continuous.