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Probability, Homework 9.

1/

a) Marginal pmf of X_1: $P_{X_1}(x_1) = \sum_{x_2=0}^{3} P_{X_1,X_2}(x_1, x_2)$.

$\Rightarrow P_{X_1}(0) = \frac{3}{16}$, $P_{X_1}(1) = \frac{1}{8}$, $P_{X_1}(2) = \frac{5}{16}$, $P_{X_1}(3) = \frac{3}{8}$.

$$\Rightarrow \text{Marginal cdf of } X_1: \quad F_{X_1}(x_1) = \begin{cases} 0, & x_1 < 0 \\ \frac{3}{16}, & 0 \le x_1 < 1 \\ \frac{5}{16}, & 1 \le x_1 < 2 \\ \frac{5}{8}, & 2 \le x_1 < 3 \\ 1, & x_1 \ge 3. \end{cases}$$

Marginal pmf of X_2: $P_{X_2}(x_2) = \sum_{x_1=1}^{2} p(x_1, x_2) \Rightarrow P_{X_2}(1) = P_{X_2}(2) = \frac{1}{2}$.

$$\Rightarrow \text{Marginal cdf of } X_2: \quad F_{X_2}(x_2) = \begin{cases} 0, & x_2 < 1 \\ \frac{1}{2}, & 1 \le x_2 < 2 \\ 1, & x_2 \ge 2. \end{cases}$$

b) $P_{X_1}(0) \cdot P_{X_2}(1) = \frac{3}{32} \neq \frac{1}{8} = P_{X_1,X_2}(0,1)$, so X_1, X_2 are not independent.

c) $E(X_1) = \sum_{x_1=0}^{3} x_1 P_{X_1}(x_1) = \frac{15}{8}$, $E(X_2) = \sum_{x_2=1}^{2} x_2 P_{X_2}(x_2) = \frac{3}{2}$.

$$Var(X_1) = E(X_1^2) - E(X_1)^2 = \left[\sum_{x_1=0}^{3} x_1^2 P_{X_1}(x_1)\right] - \left(\frac{15}{8}\right)^2 = \frac{79}{64}.$$

$$Var(X_2) = E(X_2^2) - E(X_2)^2 = \left[\sum_{x_2=1}^{2} x_2^2 P_{X_2}(x_2)\right] - \left(\frac{3}{2}\right)^2 = \frac{1}{4}.$$

2/ Marginal pmf of X: $p_X(x) = \sum_{y=-1}^{1} p_{X,Y}(x,y) \Rightarrow p_X(-1) = p_X(1) = \frac{1}{4},\ p_X(0) = \frac{1}{2}$.

Marginal pmf of Y: $p_Y(y) = \sum_{x=-1}^{1} p_{X,Y}(x,y) \Rightarrow p_Y(-1) = p_Y(1) = \frac{1}{4},\ p_Y(0) = \frac{1}{2}$.

$p_X(0) \cdot p_Y(0) = \frac{1}{4} \neq 0 = p_{X,Y}(0,0)$, so X and Y are not independent.

3/ Marginal pmf of X: $p_X(x) = \binom{4}{x} \cdot \left(\frac{1}{6}\right)^x \cdot \left(\frac{5}{6}\right)^{4-x}$, $\forall x = \overline{1,4}$.

Conditional pmf of Y: $p_{Y|X}(y|x) = \binom{4-x}{y}\left(\frac{1}{5}\right)^y \left(\frac{4}{5}\right)^{4-x-y}$, $\forall x,y \in \mathbb{N} \cap [0,4]: x+y \leq 4$.
(0 otherwise)

$\Rightarrow$ Joint pmf of X and Y:

$$p_{X,Y}(x,y) = p_X(x) \cdot p_{Y|X}(y|x) = \binom{4}{x}\left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{4-x} \binom{4-x}{y}\left(\frac{1}{5}\right)^y \left(\frac{4}{5}\right)^{4-x-y}, \ \forall x,y \in \mathbb{N} \cap [0,4]: x+y \leq 4.$$
(0 otherwise).

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a) Since X and Y are independent, $p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)$, $\forall x,y$.

Thus, $P(\{X+Y=n\}) = \sum_{k=0}^{n} P(\{X=k\} \cap \{Y=n-k\})$

$= \sum_{k=0}^{n} p_{X,Y}(k, n-k) = \sum_{k=0}^{n} [p_X(k) \cdot p_Y(n-k)] = \sum_{k=0}^{n} [P(\{X=k\}) \cdot P(\{Y=n-k\})]$.

b) $P(\{X+Y=8\}) = \sum_{k=0}^{8} [P(\{X=k\}) \cdot P(\{Y=8-k\})] = \sum_{k=2}^{6} \left(\frac{k-1}{36} \cdot \frac{7-k}{36}\right) = \frac{35}{1296}$.

5/

a) Table of $p_{X,Y}(x,y)$:

x \ y	-3	-2	-1	0	1	2	3	4	5	$p_X(x)$
-2	1/21	1/21	1/21	0	0	0	0	0	0	1/7
-1	0	1/21	1/21	1/21	0	0	0	0	0	1/7
0	0	0	1/21	1/21	1/21	0	0	0	0	1/7
1	0	0	0	1/21	1/21	1/21	0	0	0	1/7
2	0	0	0	0	1/21	1/21	1/21	0	0	1/7
3	0	0	0	0	0	1/21	1/21	1/21	0	1/7
4	0	0	0	0	0	0	1/21	1/21	1/21	1/7
$p_Y(y)$	1/21	2/21	1/7	1/7	1/7	1/7	1/7	2/21	1/21	

$E(X) = \sum_{x=-2}^{4} x\, p_X(x) = 1, \quad E(Y) = \sum_{y=-3}^{5} y\, p_Y(y) = 1.$

b) $E(100X + 200Y) = 100E(X) + 200E(Y) = 300.$

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a) Marginal pmf of Y: $p_Y(y) = \sum_{x=1}^{3} p_{X,Y}(x,y) \Rightarrow p_Y(1) = \frac{5}{9},\ p_Y(2) = \frac{1}{6},\ p_Y(3) = \frac{5}{18}.$

Conditional pmf of X: $p_{X|Y}(x|y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)},\ \forall x, y = \overline{1,3}.$

$\Rightarrow p_{X|Y}(1|1) = \frac{1}{5},\ p_{X|Y}(2|1) = \frac{3}{5},\ p_{X|Y}(3|1) = \frac{1}{5},$

$p_{X|Y}(1|2) = \frac{2}{3},\ p_{X|Y}(2|2) = 0,\ p_{X|Y}(3|2) = \frac{1}{3},$

$p_{X|Y}(1|3) = 0,\ p_{X|Y}(2|3) = \frac{3}{5},\ p_{X|Y}(3|3) = \frac{2}{5}.$

b) $p_{X|Y}(1|1) \neq p_{X|Y}(2|1)$, so X and Y are not independent.

8/ Assume that $\text{range}(X) = \{x_i\}_{i=1}^{m}$, $\text{range}(Y) = \{y_j\}_{j=1}^{n}$.

Let $\alpha = \sum_{i=1}^{m} f(x_i),\ \beta = \sum_{j=1}^{n} g(y_j).$

a) $P(\{X = x\}) = p_X(x) = \sum_{j=1}^{n} p_{X,Y}(x, y_j) = \sum_{j=1}^{n} f(x) g(y_j) = \beta f(x),\ \forall x \in \text{range}(X).$

$P(\{Y = y\}) = p_Y(y) = \sum_{i=1}^{m} p_{X,Y}(x_i, y) = \sum_{i=1}^{m} f(x_i) g(y) = \alpha g(y),\ \forall y \in \text{range}(Y).$

b) Note that $1 = \sum_{i=1}^{m} p_X(x_i) = \sum_{i=1}^{m} \beta f(x_i) = \alpha\beta$, thus

$p_X(x) \cdot p_Y(y) = \alpha\beta f(x) g(y) = f(x) g(y) = p_{X,Y}(x,y),\ \forall x, y$, i.e. X and Y are independent.

9/ Marginal pmf of X: $p_X(0) = 0.34,\ p_X(1) = 0.32,\ p_X(2) = 0.34.$

Marginal pmf of Y: $p_Y(0) = 0.05,\ p_Y(1) = 0.18,\ p_Y(2) = 0.15,$

$p_Y(3) = 0.27,\ p_Y(4) = 0.19,\ p_Y(5) = 0.16.$

$E(X) = \sum_{x=0}^{2} x\, p_X(x) = 1, \quad E(Y) = \sum_{y=0}^{5} y\, p_Y(y) = 2.85.$

Expected Profit: $E(8X + 3Y) - 10 = 8E(X) + 3E(Y) - 10 = \$6.55.$