Real Analysis, Chapter 2
Worksheet 3: Vitali Set

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- The history of measurements (the concepts of length/area/volume) dates back to the Ancient Greece.
- The foundations of measure theory were laid during 1890-1910, in the works of Emile Borel, Henri Lebesgue, Constantin Caratheodory, and others.
- The intuitive idea behind measure theory is to attach to each set of a given space a (nonnegative) number representing its "size".
- The fathers of measure theory concerned on properties a measure may have: additivity, monotonicity, continuity, etc.

When considering measures on $\mathbb{R}$, a question arised as follows.

Proposition 2.3.1

Does there exist a set function μ satisfying the following properties?

(a) The domain of μ is $\mathcal{P}(\mathbb{R})$, i.e. every subset of $\mathbb{R}$ has a measure.

(b) The range of μ is $[0, \infty]$.

(c) μ generalises the concept of length on $\mathbb{R}$, i.e. $\mu(\emptyset) = 0$ and

$$\mu([a, b]) = b - a \text{ if } a, b \in \mathbb{R} \text{ and } a < b.$$

(d) μ is $\sigma-$ additive.

(e) μ is translation invariant, i.e.

$$\mu(A) = \mu(A + x), \forall A \subset \mathbb{R}, x \in \mathbb{R}.$$

The answer is, unfortunately, no.

- In 1905, the Italian mathematician Giuseppe Vitali constructed an abstract set, called the **Vitali set**.
- He proved that the Vitali set is non-measurable, and therefore Proposition 2.3.1 does not hold.
- This result by Vitali forced mathematicians to restrict the domain of μ onto collections of sets strictly smaller than $\mathcal{P}(\mathbb{R})$.
- There are other non-measurable sets as well, a famous one is from the Banach-Tarski paradox.

Sul problema della misura dei gruppi di punti di una retta

NOTA

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1905

First, we define an equivalence relation $\sim$ on $\mathbb{R}$, as follows.

Definition 2.3.2

Two real numbers x, y are called equivalent, denoted $x \sim y$ if $x - y \in \mathbb{Q}$. For each real number x, the set

$$[x] = \{y : y \sim x\}$$

is called the equivalence class of x. Clearly if $x \sim y$, then $[x] = [y]$.

Denote Λ the collection of all equivalence classes on $\mathbb{R}$.

Lemma 2.3.1

For each $\alpha \in \Lambda$, the set $\alpha \cap (0, 1)$ is nonempty.

Now for each $\alpha \in \Lambda$, select precisely one arbitrary point $x_\alpha \in \alpha \cap (0, 1)$. The set V of all selected points x_α is known as the Vitali set.

Lemma 2.3.2

If $p, q \in \mathbb{Q}$ and $p \neq q$, then $V + p \cap V + q = \emptyset$.

Guidelines:

- If $\exists x \in V + p \cap V + q$, then $x = \alpha + p = \beta + q$ for some $\alpha, \beta \in V$;
- Hence $\alpha - \beta = p - q \in \mathbb{Q} \Rightarrow \alpha \sim \beta \Rightarrow \alpha = \beta \Rightarrow p = q$.

Lemma 2.3.3

Let $A = \mathbb{Q} \cap (-1, 1)$, then one can easily verify the following properties:

(a) $V + r \subset (-1, 2), \forall r \in A$;

(b) $W \subset (-1, 2)$, where $W = \bigcup \{V + r : r \in A\}$;

(c) $\sum \{\mu(V + r) : r \in A\} = \mu(W) \leq \mu((-1, 2)) = 3$.

Note that (c) implies $\mu(V) = 0$, and thus $\mu(W) = 0$.

Theorem 2.3.4

$(0,1) \subset W$, and thus $\mu((0,1)) = 0$, a contradiction.

Guidelines:

- Pick any $x \in (0,1)$, then $\exists y \in [x] \cap V \subset (0,1)$;
- Then $x - y = r \in \mathbb{Q}$ and $-1 < r < 1$;
- Thus $r \in A$ and $x = y + r \in V + r \subset W$.