Chapter 2: Partial derivatives

Lecture 3

- Functions of Several Variables
- Limit and Continuity

Functions of Two Variables

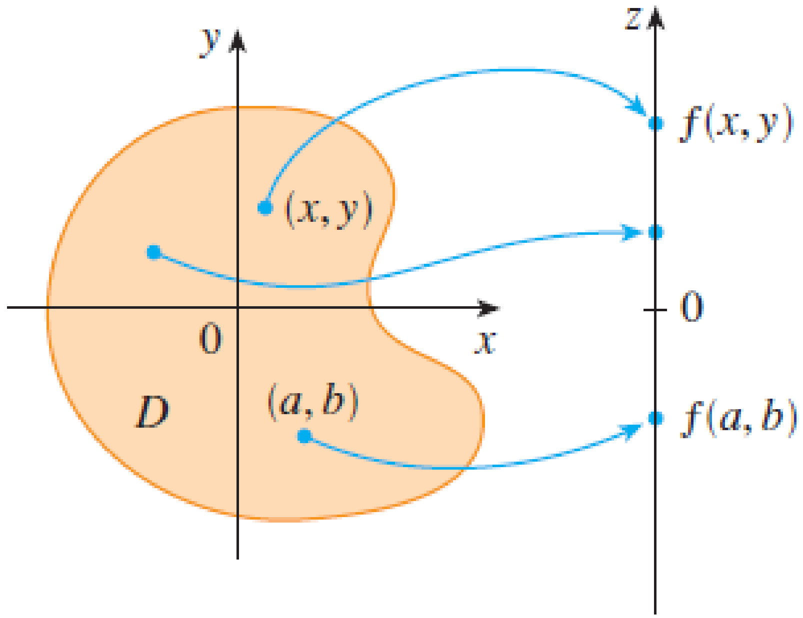


FIGURE 1

Definition. A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x,y) in a set D a unique real number denoted by $f(x,y)$.

D is the **domain** of f

range = $\{f(x,y) \mid (x,y) \in D\}$

We often write $z=f(x,y)$

Usually, domain is the set of all pairs (x,y) for which the expression for f is well-defined

Example 1

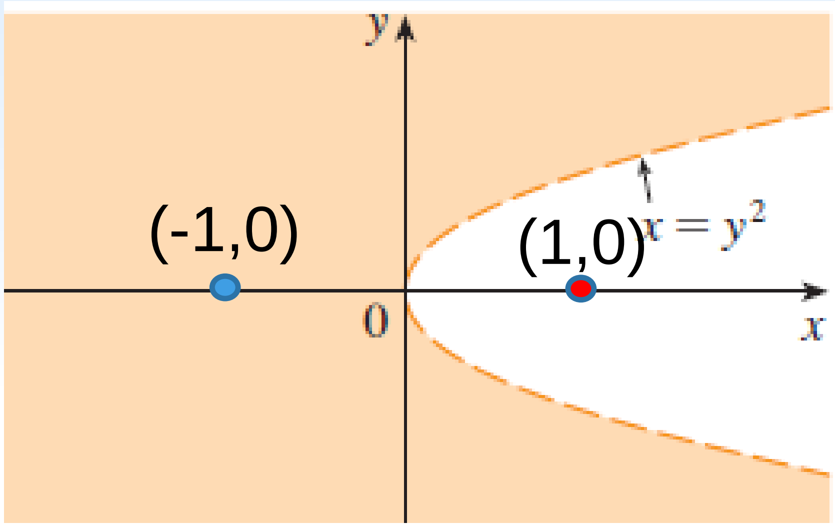
Find and sketch the domain of the function

$$f(x,y)=x\ \ln(y^2-x)$$

Solution

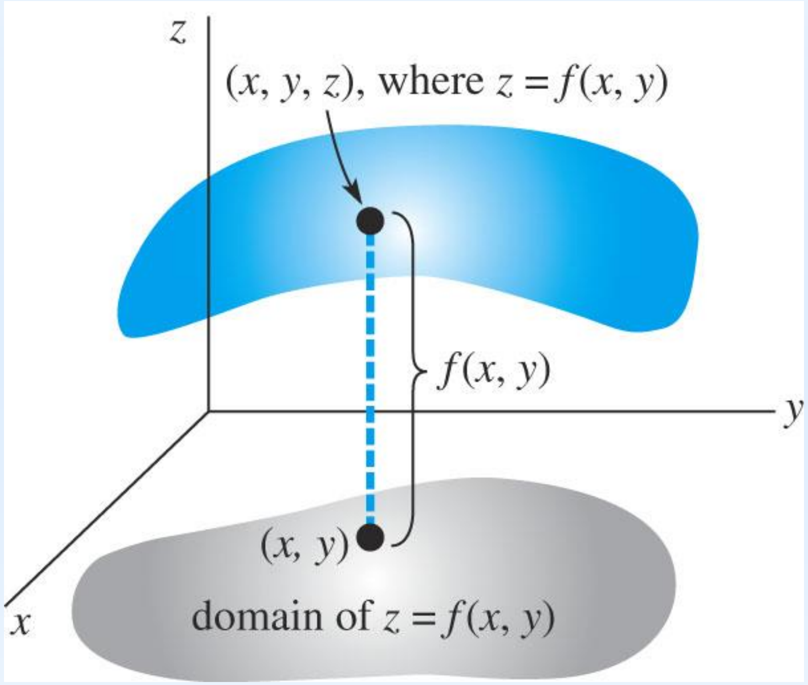
$\ln(y^2-x)$ is defined only when $(y^2-x)>0$ or $x<y^2$

So, domain of f is $D=\{(x,y)\mid x<y^2\}$



The curve x=y^2 *divides (x,y)-plan into two parts, one is D*

Graph of Function of two variables



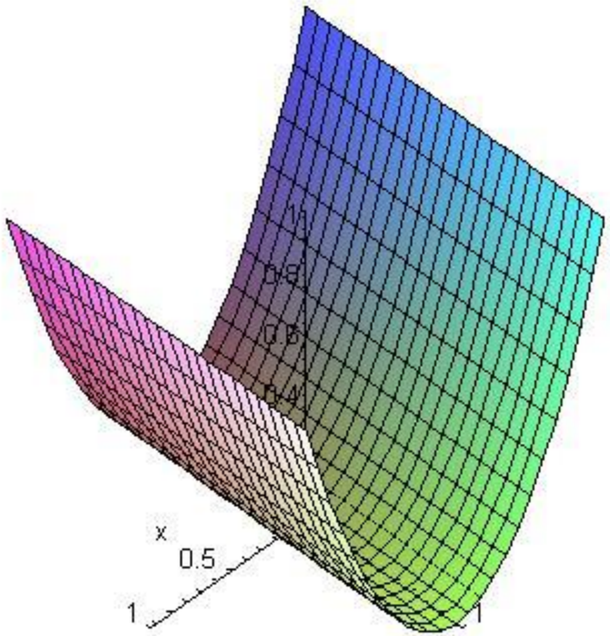
Definition. If f is a function of two variables with domain D**,** then the ***graph S*** of f is the set

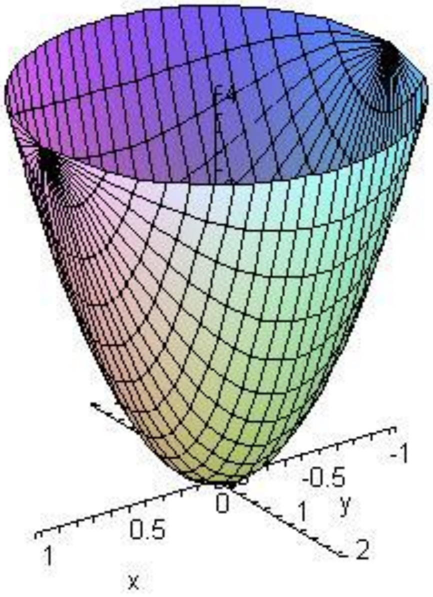
$$S = \{(x, y, z) | z = f(x, y), (x, y) \in D\}$$

Graph S of a function of two variables is a surface with equation $z=f(x,y)$

- Graph $f(x,y)=x^2$

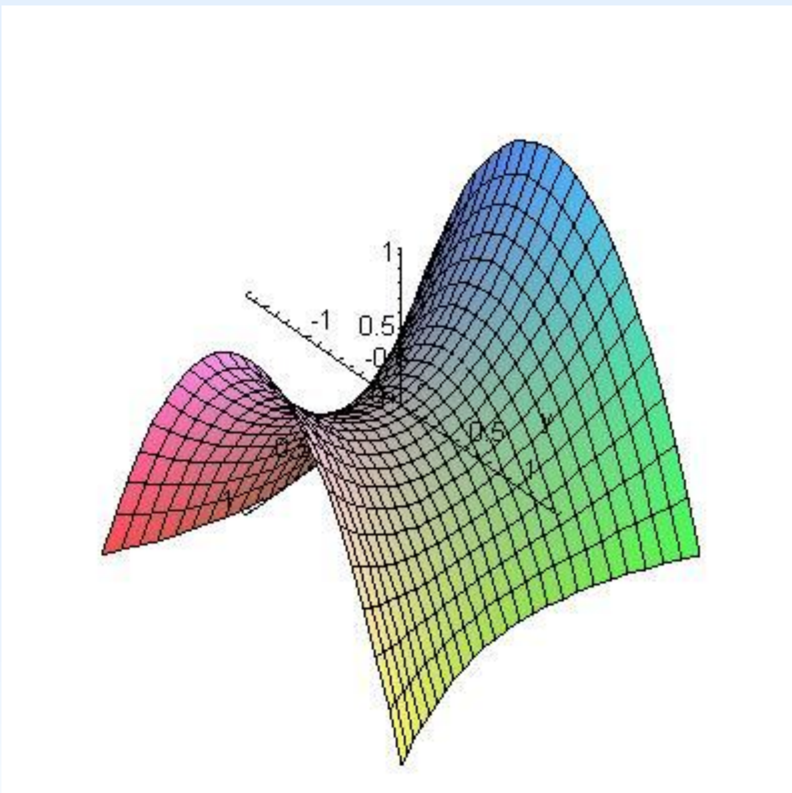
- Graph $f(x,y)=4x^2+y^2$

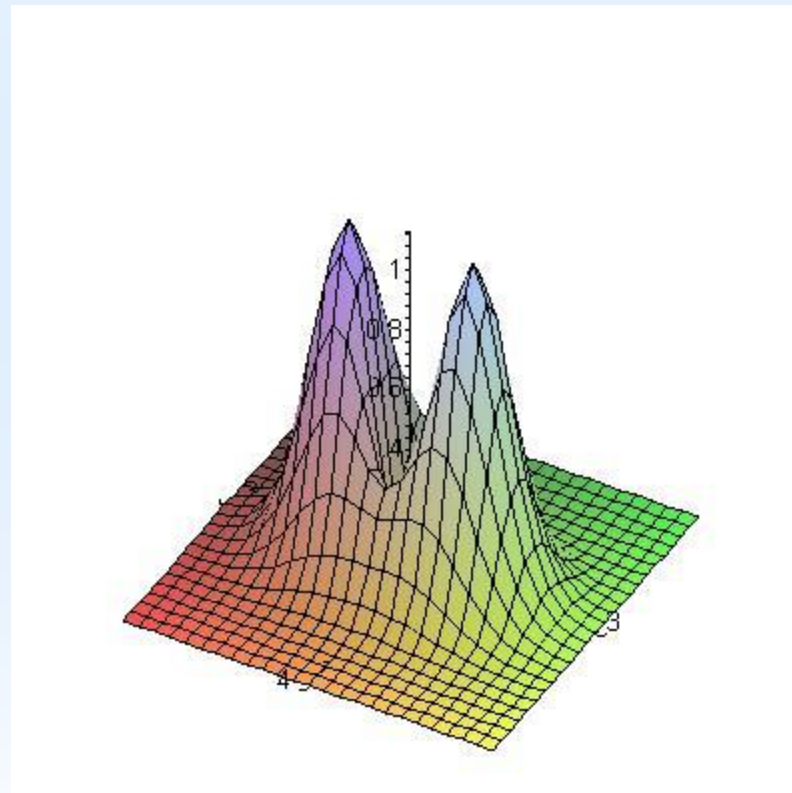




- Graph $z=x^2 -y^2$

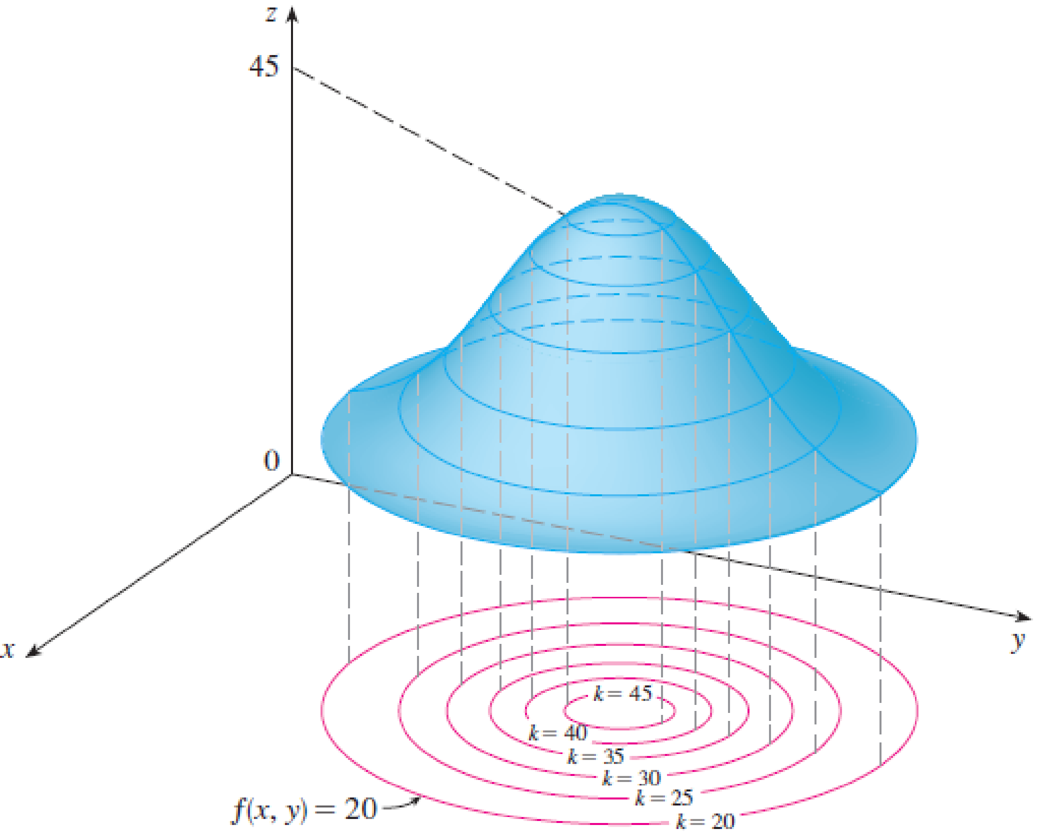
$z=(x^2 +3y^2)\exp(-x^2 -y^2)$

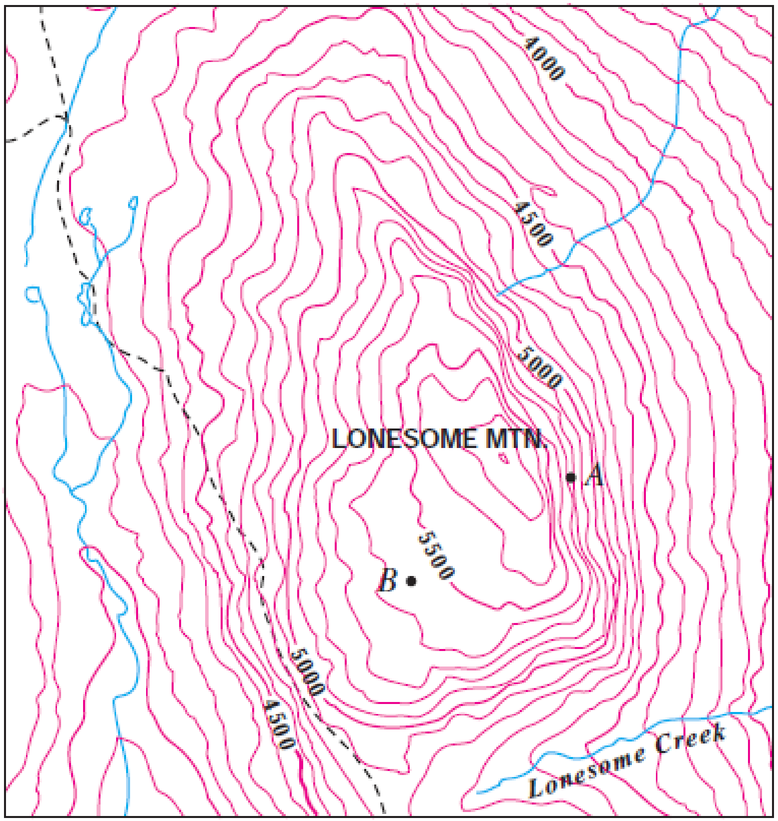




Level Curves

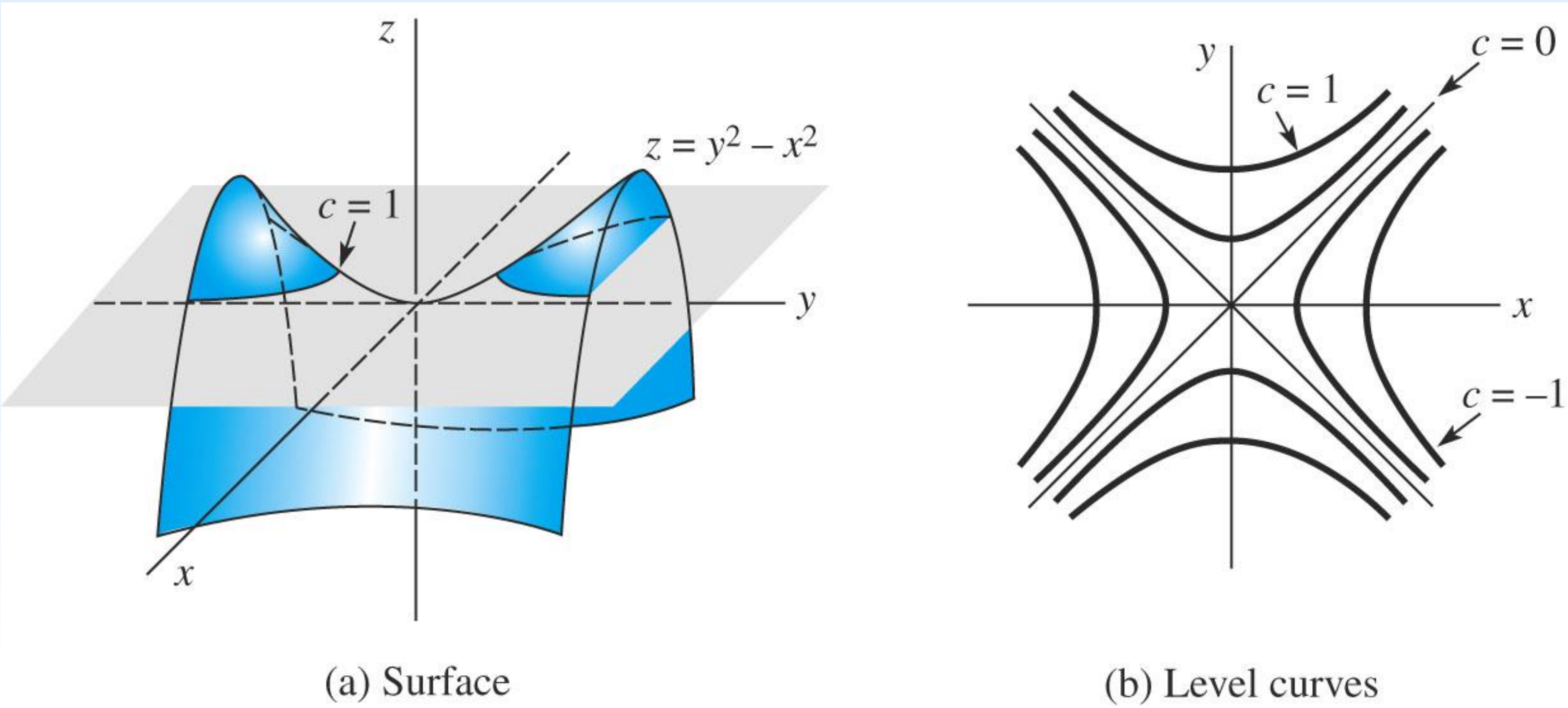
- The curves in domain of f defined by $f(x, y) = k$ are called the ***level curves*** of f.





Example 1

The level curves of $f(x, y) = y^2 - x^2$ are defined by $y^2 - x^2 = c$. For $c = 0$, we obtain the lines $y = x$, $y = -x$.



Functions of three variables

- A **function of three variables**, *f* **, is a rule that assigns to each ordered triple** (x,y,z) in a domain D of R^3 a unique real number denoted by $w=f(x,y,z)$.
- For instance, the temperature at a point on the surface of the Earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T=f(x,y,t)$.

Example

- Find the domain of f if

$$f(x,y,z)=\ln(z-y)+xy\cos(x+2y)$$

- The expression for $f(x,y,z)$ is defined as long as

$$z-y>0$$

so the domain of f is

$$D=\{(x,y,z) \mid z>y\}$$

Functions of n variables

Functions of any number of variables can be considered. A ***function of n variables*** is a rule that assigns a unique number

$$y=f(x_1, x_2, \ldots, x_n)$$

to each n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by R^n the set of all such *n*-tuples.

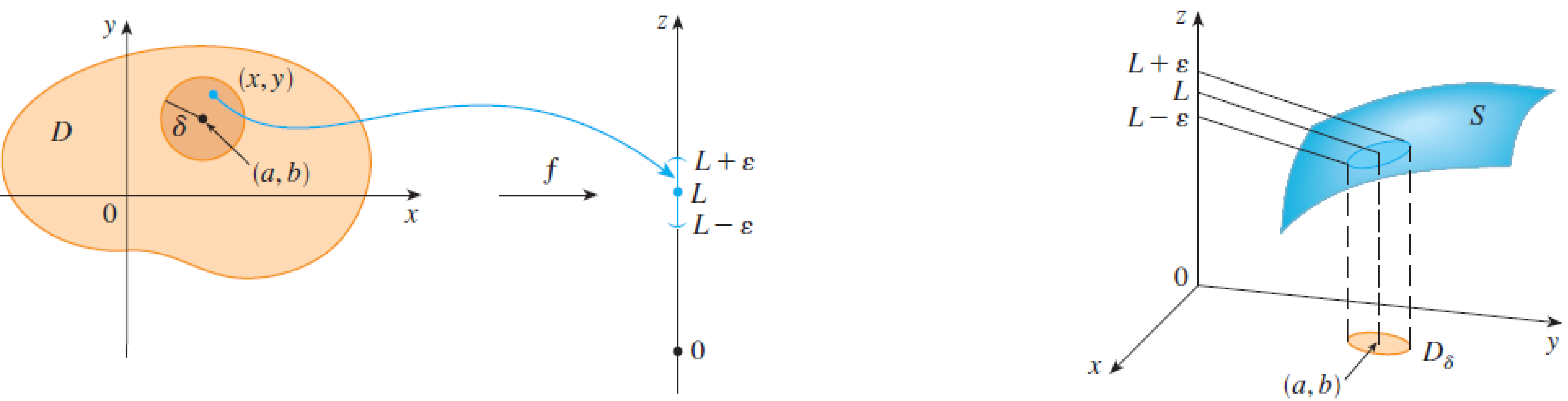
2. LIMITS AND CONTINUITY

We study the concepts of Limits and Continuity for Functions of several variables

Definition

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \text{ if for every } \varepsilon > 0, \text{ there exists } \delta > 0$$

$$\text{such that if } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \text{ then } |f(x,y) - L| < \varepsilon$$



That is, we can make *f(x,y)* as close to *L* as we like by taking *(x,y)* sufficiently close to, but not equal to, *(a,b)*.

Example 1

Evaluate $\lim\limits_{(x,y)\to(2,2)} \dfrac{x+y-4}{(x+y)^2-16}$

$$\lim_{(x,y)\to(2,2)} \frac{x+y-4}{(x+y)^2-16} = \lim_{(x,y)\to(2,2)} \frac{x+y-4}{(x+y-4)(x+y+4)}$$

$$= \lim_{(x,y)\to(2,2)} \frac{1}{(x+y+4)} = \frac{1}{8}$$

Example 2

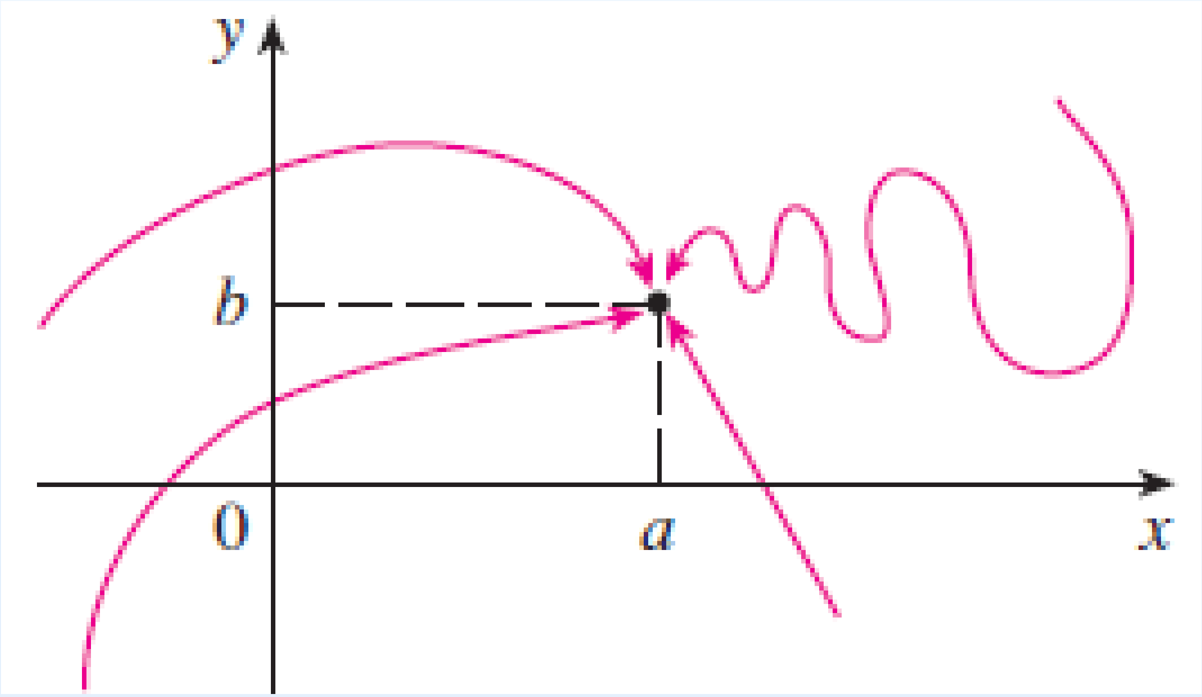
Evaluate $\lim\limits_{(x,y)\to(0,1)} \dfrac{xy-x}{\sqrt{y}-1}$

$$\lim_{(x,y)\to(0,1)} \frac{xy-x}{\sqrt{y}-1} = \lim_{(x,y)\to(0,1)} \frac{x(y-1)(\sqrt{y}+1)}{(\sqrt{y}-1)(\sqrt{y}+1)}$$

$$= \lim_{(x,y)\to(0,1)} \frac{x(y-1)(\sqrt{y}+1)}{y-1} = \lim_{(x,y)\to(0,1)} x(\sqrt{y}+1) = 0$$

How to prove limit does not exist?

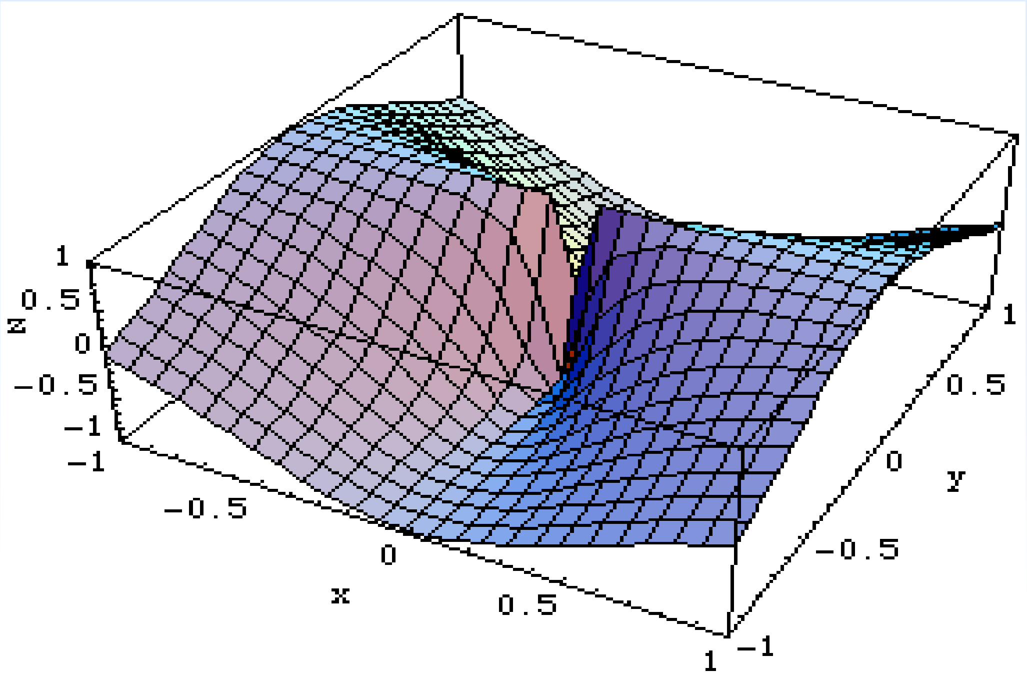
- Limit must be the same along every path that "leads" to (a,b)
- Otherwise, limit does not exist



If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along path C_1,
$f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along path C_2,
$L_1 \neq L_2$, then the limit of $f(x,y)$ as $(x,y) \to (a,b)$ does not exist.

Example 3

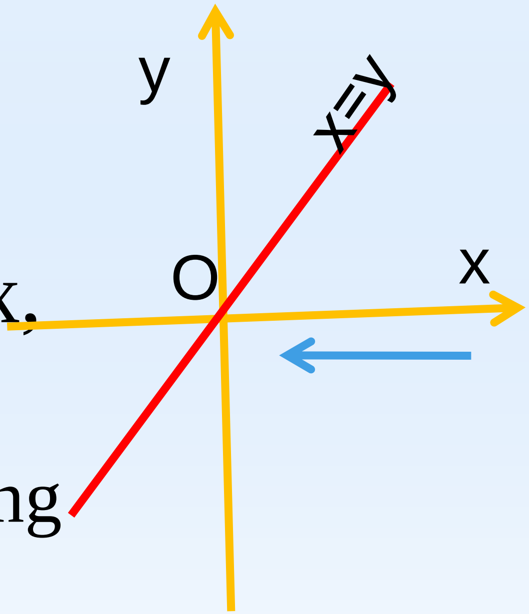
Find the limit if it exists, or show that limit does not exist: $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$



Solution

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

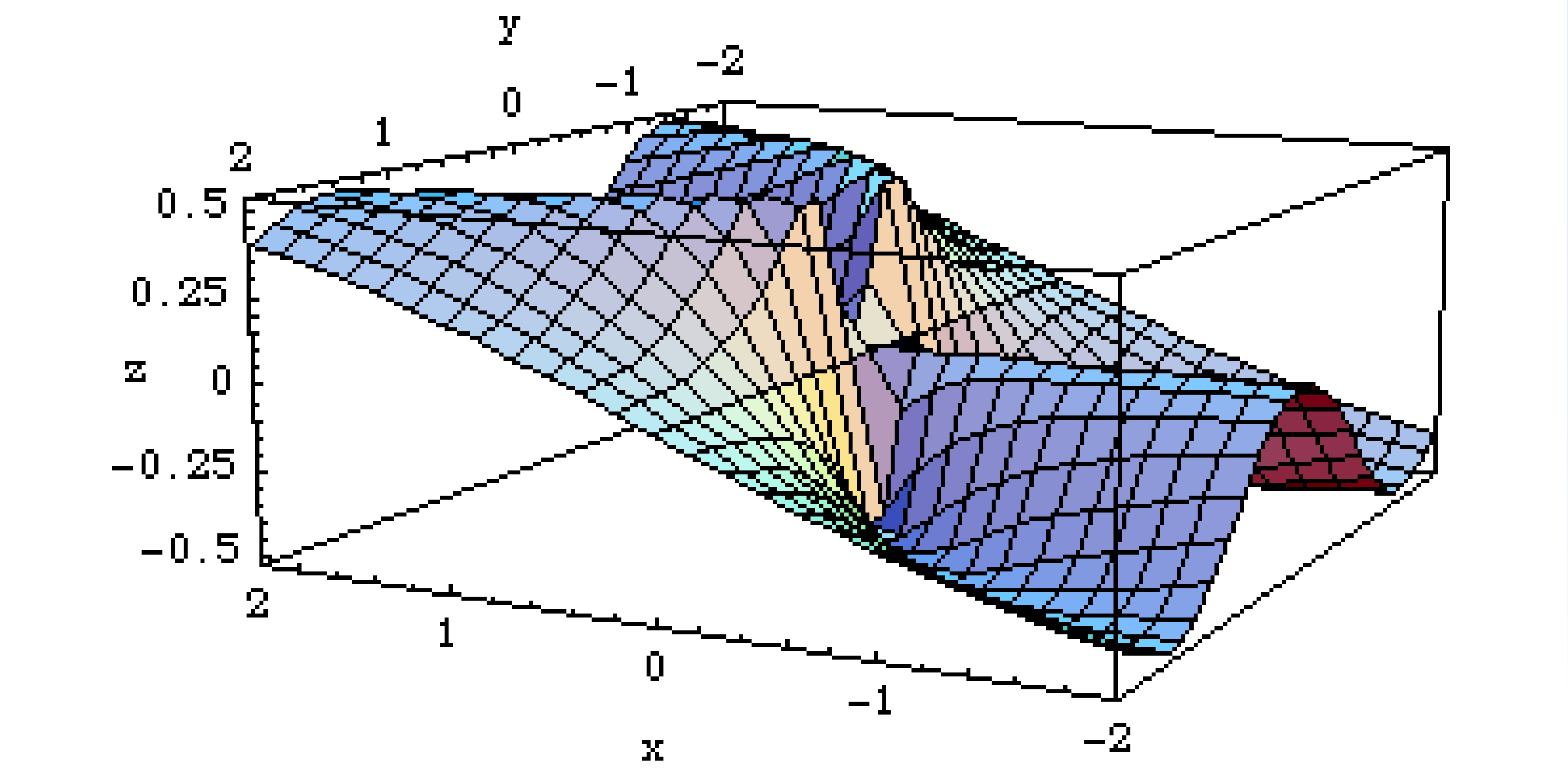
- If we let (x,y)→ (0,0) along the straight line x=y
  - Then f(x,y)=0
  - f(x,y)=0 → 0 along C: x=y
- If we let (x,y)→ (0,0) along Ox,
  - Then y=0 and therefore
  - f(x,y)=1. Thus, f(x,y)=1→ 1 along Ox



- Since 0 ≠1, the limit does not exist.

Example 4

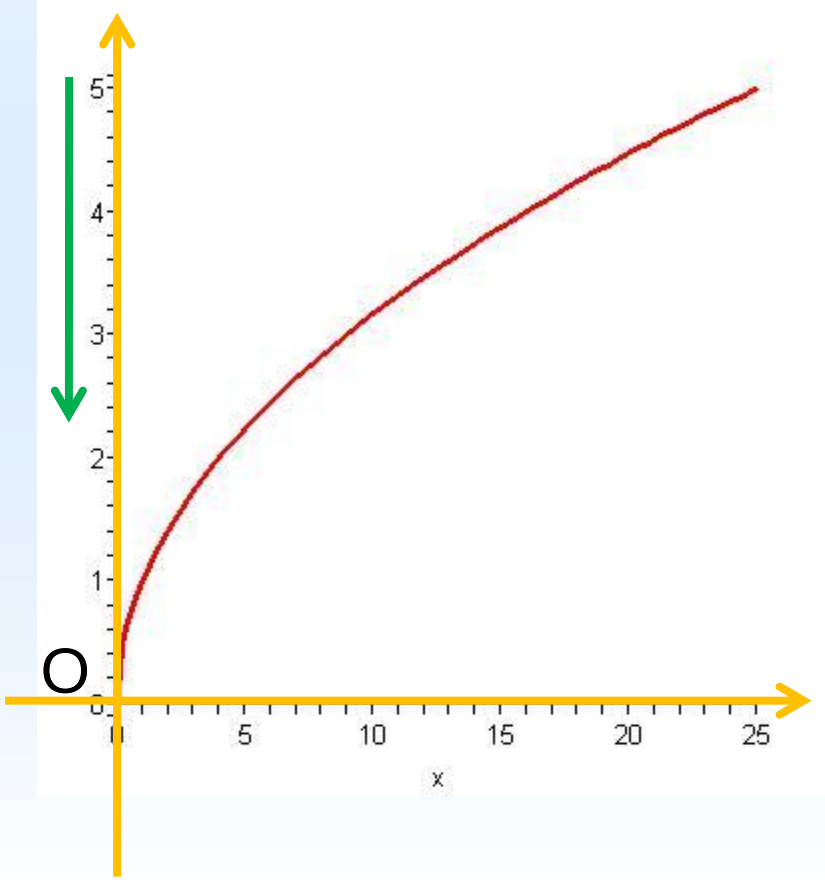
$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$



Solution

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

- If we let (x,y)→ (0,0) along Oy, then x=0. Thus, f(x,y)=0. So
  - f(x,y)→ 0 along Oy
- If we let (x,y) → (0,0) along $x=y^2$ then f(x,y)=1/2.
  - So f(x,y)→ ½ along $x=y^2$
- Thus, limit does not exist.



Continuity

<u>Definition:</u> *f(x,y)* is *continuous* at *(a,b)* if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

If f(x,y) is not continuous at (a,b), we say that it is discontinuous at (a,b). The point (a,b) is then called the point of discontinuity

Homework Chapter 2

1. Function of several variables: 13, 14, 18, 35, 43, 44
2. Limits and Continuity: 14, 15, 18, 22, 25, 36
3. Partial derivatives: 21, 32, 36, 42, 55, 66, 68
4. Tangent Plane and Linear approximations: 5, 6, 19, 20
5. The chain rule: 6, 10, 12, 21, 22, 27
6. Directional Derivatives and Gradient Vectors : 8, 12, 15, 21, 28
7. Maximum and Minimum Values: 9, 15, 18, 32, 35, 44, 45
8. Lagrange Multipliers: 8, 11, 17, 20