

Probability Review

1. Consider a binomial model for stock price. The stock goes up by a factor u with probability p and goes down by a factor d with probability q after each period.

Suppose that at initial your wealth X_0 is used to buy Δ_0 shares of stock whose initial price is S_0 and to invest the remain $X_0 - \Delta_0 S_0$ in money market with interest rate r (same interest rate for investing and borrowing). It means that one dollar invested in the money market at time zero will yield $1 + r$ dollars at time one. Conversely, one dollar borrowed from the money market at time zero will result in a debt of $1 + r$ at time one

Assume that $0 < d < 1 + r < u$ (this assumption guarantees that there is no arbitrage opportunity for trading in stock and money market, i.e you can get profit from nothing by using some trading strategy.). Your portfolio value at time 1 is given by

$$X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0)$$

- (a) If $X_0 = 0$, determine all possible values of X_1 .
 - (b) Find $P(X_1 > 0)$, $P(X_1 < 0)$ when $X_0 = 0$.
2. **Asian option** Consider a three-period binomial asset pricing model with $S_0 = 4$, $u = 2$, $d = 1/2$, $p = q = 1/2$. Define

$$Y_n = S_0 + S_1 + \dots + S_n$$

to be the sum of stock price from time zero to time n . Consider an (arithmetic) Asian call option that expires at time three and has trike price $K = 4$, whose payoff at time 3 is

$$V_3 = \left(\frac{1}{4} Y_3 - 4 \right)^+.$$

This is like a European call, except the payoff of the option is based on the average stock rather than the final stock price. Determine the pmf of V_3 and the expectation of V_3 .

3. **Look back option** Consider a binomial asset pricing with $S_0 = 4$, $u = 2$, $d = 1/2$, $p = q = 1/2$. As Asian option, look back option is a type of path-dependent options that's value depends not only on the price of the underlying asset but the path that asset took during all or part of the life of the option. Consider a look back call option that expires at time 3 and has floating strike price S_3 . The payoff of the option is given by

$$V_3 = \max_{0 \leq n \leq 3} S_n - S_3.$$

Determine the pmf of V_3 and the expectation of V_3 .

4. Consider a binomial asset pricing with $S_0 = 4$, $u = 2$, $d = 1/2$, $p = q = 1/2$.
 - (a) Determine conditional distribution of S_4 given $S_3 = 8$. Hence compute $E(S_4|S_3 = 8)$.
 - (b) Find $E(S_4|S_3)$.
5. Let X and Y be jointly normal random variables with parameters $\mu_X = 1$, $\sigma_X^2 = 1$, $\mu_Y = 0$, $\sigma_Y^2 = 4$, and $\rho = 1/2$.
 - (a) Find $P(2X + Y < 3)$.
 - (b) Find $P(Y > 1|X = 2)$.
 - (c) Find conditional expectation of Y given $X = 2$.