FINANCIAL RISK MANAGEMENT 2

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Chapter 3. Implementing risk forecasts

In the last course FRM 1, risk measures are calculated based on the assumption that the distribution of profit and loss was known. However, in practice, the profit and loss distribution is estimated by using historical observations of the asset's returns. There are two main methods for forecasting VaR and ES which are non-parametric and parametric.

- The non-parametric risk forecasting risk refers to historical simulation. which uses the empirical distribution of data to compute risk forecasts. There is no assumption about the distribution or parameters.
- The parametric method are based on estimating the underlying distribution of returns and then obtaining risk forecasts from the estimated distribution. For most applications, the first step in the process is forecasting the covariance matrix.

3.1. Some concepts

- We define a sample to have the size T
- We separate the data into two parts, the first part is used to estimate parameters, called estimation window or W_E
- K: Number assets
- ω: $K \times 1$ vector of portfolio weights
- $y_k = \{y_{t,k}\}_{t=1}^{T}$: $T \times 1$ vector of returns on asset k
- $y = T \times K$: $T \times K$ matrix of historical returns
- Σ: $K \times K$ covariance matrix of assets
- v: the portfolio value

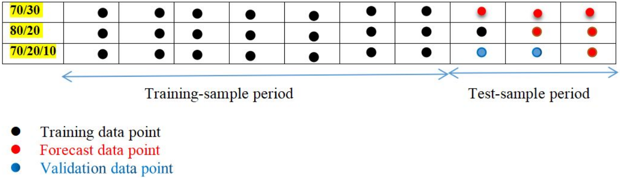
3.2. Forecasting

We consider the concepts In sample, out-of-sample, and forecasting

- If we use information from time $t = 1$ to time $t = T$ to produce about time T, it means that we are doing sample analysis
- However we want to forecast the risk
- That means using we use information $t = 1$ to $t = T$ to produce a forecast of what might happen later, perhaps $t = T + 1$

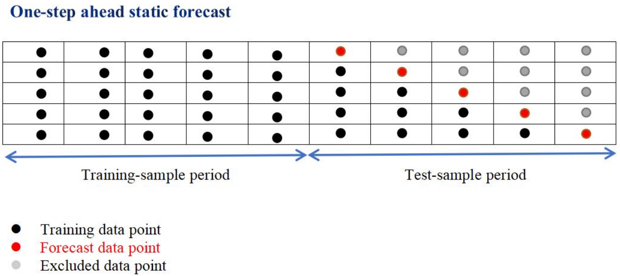
Training and testing samples

When working with forecasting we usually use the following rules

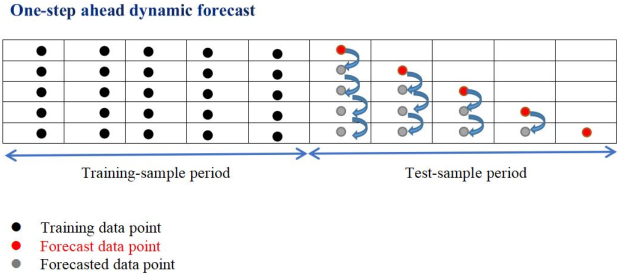


Static (Expanding) and Dynamic Forecast

Assume we have five samples and we want to forecast five values in the future we can use static forecast described as follows



Now we have five samples, if we want to forecast five values in the future we can use dynamic forecast described as follows



In general we have $y_t = f(y_{t-1}, y_{t-2}, \ldots y_{t-m})$. To forecasting h steps ahead we have

$$\hat{y}_{t+1} = f(y_t, y_{t-1}, \ldots y_{t-m+1})$$

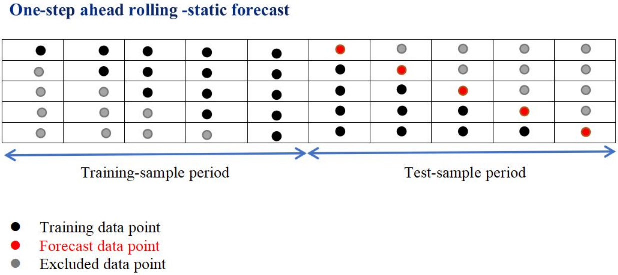
$$\hat{y}_{t+2} = f(\hat{y}_{t+1}, y_t, \ldots y_{t-m+2})$$

$$\hat{y}_{t+3} = f(\hat{y}_{t+2}, \hat{y}_{t+1}, y_t, \ldots y_{t-m+3})$$

and so on

Rolling forecast

Similarly static forecast, we use rolling window which is described as follows



Modeling process

(i) Initial steps

- Before working with data: think about the context
  - What do we expect to find in a model
  - What do you need to get from a model? for example model ARIMA=short-term forecast
  - Set a baseline: What results have been obtained by other models?
- Plot time series

(ii) Estimation

- Fit initial model, explore simpler & more complex models
- Check residuals for problems
  -Ljung-Box test of residual autocorrelations
  - Residual plots show outliers, other anomalies

(iii) Forecasting

- Check for normality
- Extrapolate pattern implied by dependence
- Compare to baseline estimates

Example 1

Assume that the time series y_t with mean μ is fitted by following MA(2) model

$$y_t = \mu + \omega_t + \theta_1 \omega_{t-1} + \theta_2 \omega_{t-2}$$

- Forecasting 1-step:

$$y_{n+1} = \mu + \epsilon_{n+1} + \theta_1 \epsilon_n + \theta_2 \epsilon_{n-1}$$
$$\hat{y}_{n+1} = \mu + \theta_1 \epsilon_n + \theta_2 \epsilon_{n-1}$$

hence

$$y_{n+1} - \hat{y}_{n+1} = \epsilon_{n+1}$$
$$var(y_{n+1} - \hat{y}_{n+1}) = \sigma^2$$

2-steps

$$y_{n+2} = \mu + \epsilon_{n+2} + \theta_1\epsilon_{n+1} + \theta_2\epsilon_n$$
$$\hat{y}_{n+2} = \mu \qquad\qquad + \theta_2\epsilon_n$$

Hence,

$$y_{n+2} - \hat{y}_{n+2} = \epsilon_{n+2} + \theta_1\epsilon_{n+1}$$
$$var(y_{n+2} - \hat{y}_{n+2}) = \sigma^2(1 + \theta_1^2)$$

3-steps

$$y_{n+3} - \hat{y}_{n+3} = \epsilon_{n+2} + \theta_1\epsilon_{n+2} + \theta_2\epsilon_{n+1}$$

$$var(y_{n+3} - \hat{y}_{n+3}) = \sigma^2(1 + \theta_1^2 + \theta_2^2)$$

Example 2

Consider ARMA(2,1) model, we have

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t + \theta_1 \epsilon_{t-1}$$

- For one-step ahead

$$\hat{y}_{n+1} = \delta + \phi_1 y_n + \phi_2 y_{n-1} + (\hat{\epsilon}_{n+1} = 0) + \theta_1 \hat{\epsilon}_n$$

- 2-steps

$$\hat{y}_{n+2} = \delta + \phi_1 \hat{y}_{n+1} + \phi_2 y_n + (\hat{\epsilon}_{n+2} = 0) + \theta_1 (\hat{\epsilon}_{n+1} = 0)$$

  3-steps

$$\hat{y}_{n+3} = \delta + \phi_1 \hat{y}_{n+2} + \phi_2 \hat{y}_{n+1} + (\hat{\epsilon}_{n+3} = 0) + \theta_1 (\hat{\epsilon}_{n+2} = 0)$$

3.3. Historical simulation for one asset

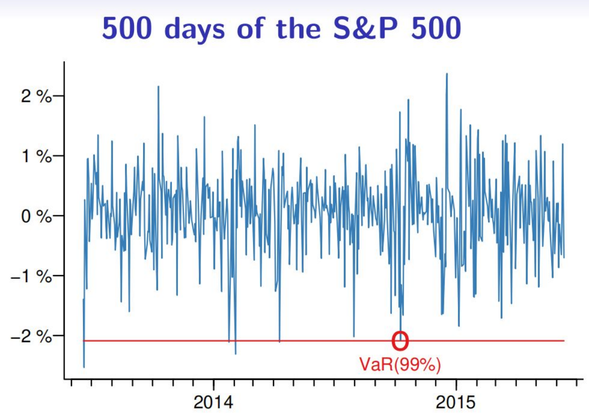
- Consider a portfolio of one stock with the observations in the estimation window will be next day return.
- VaR is one of the observations in the estimation window, multiplied by the monetary of asset holding, the portfolio value V_t is

$$V_t = \text{number stocks owned} \times P_t$$

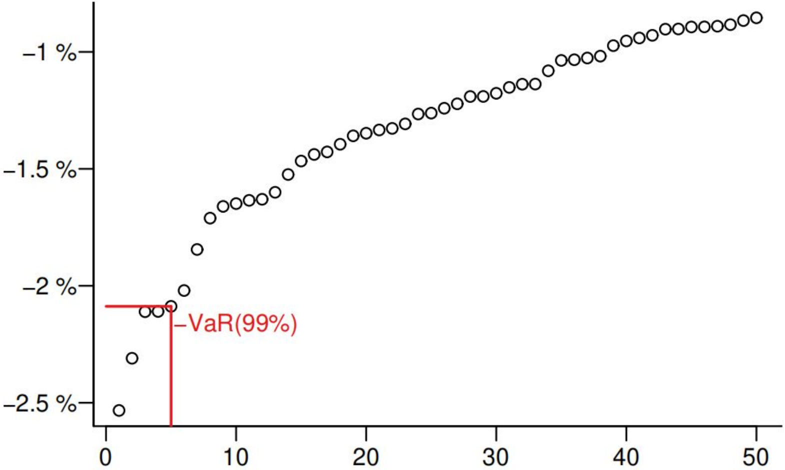
- Then we have VaR is calculated by

$$VaR_t = -(T \times \alpha)^{th}$$

the smallest return, times V_t The following figures describe VaR of S&P 500 return with confidence level 99% (or 1%)



After sorting historical returns we obtain



Procedure historical simulation

To calculate VaR using historical simulation we do the following steps.

- Decide on a probability α
- Have the sample of returns y with length T, e.g., 1000
- sort y from the smallest to the largest, call that ys
- Take $(T \times \alpha)^{th}$ the smallest value of y_s called $ys_{T\times\alpha}$
- We have

$$VaR_t = -ys_{T\times\alpha}P_{t-1} \times \text{number of stocks}$$

- The expected shortfall is

$$ES = \frac{1}{(T \times \alpha)^{th}} \sum_{i=1}^{(T\times\alpha)^{th}} ys_i$$

For Multiple assets

In case we have a portfolio of multiple assets then we do the following steps to find VaR

- Take a matrix of historical portfolio return
- Denote y the $T \times K$ matrix returns
- w is the $K \times 1$ vector of portfolio weights
- Then we get the time series vector of portfolio return

$$y_p = y \times w$$

- Then we can simply treat the portfolio as if we were a single asset and apply Historical Simulation

Example We have stock prices of Microsoft and IBM download from internet. Then we calculate VaR of Microsoft and the portfolio of Microsoft and IBM

```
library("tseries")
library("zoo")
p1 = get.hist.quote(instrument = "msft",start = "2015-01-01",
                    end = "2020-12-31",quote = "AdjClose")
p2 = get.hist.quote(instrument = "ibm", start = "2015-01-01",
                    end = "2020-12-31",quote = "AdjClose")
y1=coredata(diff(log(p1)))
y2=coredata(diff(log(p2)))
portfolio=1000
w=matrix(c(0.4, 0.6))
alpha=0.01
T=length(y1)
op=ceiling(alpha*T)# alpha percent smallest, rounded up
## VaR of microsoft
VaRm=-sort(x1)[op]
## VaR of portfolio
y=cbind(y1, y2)%*%w
VaR_p=-sort(y)[op]*portfolio
## Expected shortfall of portfolio
ES=-mean(sort(y)[1:op])*portfolio
```

Importance of sample size and Issue

- The most extreme observations fluctuate o lot more than observations that are less extreme. Therefore, the bigger sample size the more precise the estimation of HS.
- If there is a structural break in data (like in crisis 2008) the VaR forecast take longer to adjust to structural change in risk
- No model assumptions needed
- In the absence of structural break HS tends to perform well.
- It captures nonlinear dependence directly. But performs badly when data has structural breaks

3.4. Parametric methods

We need to calculate VaR and ES on day t conditional on

- a return density on day t, it is $f(.)$ with the distribution $F(.)$
- and parameter θ of perhaps, Normal distribution with (μ, σ) or student distribution with (μ, σ, ν)

Recall that

$$\mathbb{P}(V \leq -VaR^{\alpha}(V)) = \int_{-\infty}^{-VaR^{\alpha}(V)} f(x)dx = \alpha$$

The profit and loss when we own one stock

$$V_t = P_t - P_{t-1}$$

From the definition of return

$$R_t = \frac{P_t - P_{t-1}}{P_r}$$

So we have

$$\begin{aligned}\alpha &= \mathbb{P}(V_t \le -VaR^{\alpha}(V_t)) \\ &= \mathbb{P}(R_t P_{t-1} \le -VaR^{\alpha}(V_t)) \\ &= \mathbb{P}(\frac{R_t}{\sigma} \le -\frac{VaR^{\alpha}(V_t)}{P_{t-1}})\end{aligned}$$

Denote F_R the distribution function of R_t/σ, we have

$$VaR^{\alpha}(V_t) = -\sigma F_R^{-1}(\alpha) P_{t-1} \qquad (*)$$

If we use log return

$$Y_t = \log(\frac{P_t}{P_{t-1}})$$

similarly, we obtain

$$VaR^{\alpha}(V_t) = -(e^{F_Y^{-1}(\alpha)\sigma} - 1)P_{t-1} \qquad (**)$$

Note that

$$e^{F_Y^{-1}(\alpha)\sigma} - 1 \approx F_Y^{-1}(\alpha)\sigma$$

and the distribution function $F_Y \approx F_R$. Therefore, the formula $(**)$ can be approximated by $(*)$. Meaning that the VaR for continuously compounded returns is approximately the same as the VaR using simple returns

3.5.VaR with time dependent mean and volatility

Let r_t denote the log-return of a asset or portfolio between periods $t-1$ and t and $\mathcal{F}_t$ denote the information filtration generated by these returns.

$$r_t = \mu_t + a_t$$

where

$$\mu_t = \mathbb{E}(r_t \mid \mathcal{F}_t)$$

and

$$\sigma_t^2 := var(r_t \mid \mathcal{F}_t)) = var(a_t \mid \mathcal{F}_t))$$

Note that μ_t and σ_t are known at time $t-1$. The mean μ_t are assumed to follow a stationary time series model such that ARMA (p, q)

$$\mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \gamma_j a_{t-j}$$

The coefficients ϕ_i and γ_j will be estimated from the observations of r_t.

Note that a_t is the random component of log-return and the variance σ_t^2 is time varying and follows $GARCH(L_1, L_2)$ model, i.e.,

$$a_t = \sigma_t \epsilon_t$$

and

$$\sigma_t^2 = \omega + \sum_{i=1}^{L_1} \alpha_i a_{t-i}^2 + \sum_{j=1}^{L_2} \beta_j \sigma_{t-j}^2$$

where $\omega > 0 \alpha_i \geq 0, \beta_j \geq 0$, ϵ_t is i.i.d random variable with zero mean and variance 1, usually $\epsilon_t \sim N(0, 1)$.

Estimating Risk Measures

Suppose that today is date t and we wish to estimate the portfolio VaR and ES over period t to $t+1$. We use ARMA and GARCH models to forecast the value of r_{t+1} as follows

$$r_{t+1} = \hat{\mu}_{t+1} + \hat{\sigma}_{t+1}\epsilon_{t+1}$$

where $\hat{\mu}_{t+1}$ and $\hat{\sigma}_{t+1}$ are the estimates of the next periods mean log-return and volatility. So we have

$$\widehat{VaR}^{\alpha}_{t+1}(r_{t+1}) = -\hat{\mu}_{t+1} + \hat{\sigma}_{t+1} VaR^{\alpha}(\epsilon_{t+1})$$

and the Expected Shortfall

$$\widehat{ES}^{\alpha}_{t+1}(r_{t+1}) = \hat{\mu}_{t+1} + \hat{\sigma}_{t+1} ES^{\alpha}(\epsilon_{t+1})$$

Special case

In case the residuals $\epsilon_t \sim N(0,1)$ and the initial value of the portfolio is P_0 we have

$$VaR^{\alpha}(\epsilon_{t+1}) = -\Phi^{-1}(\alpha)$$

and

$$ES^{\alpha}(\epsilon_{t+1}) = \frac{\phi(\Phi^{-1}(\alpha))}{\alpha}$$

where ϕ and Φ is the density and distribution functions of standard normal distribution. So we obtain

$$\widehat{VaR}^{\alpha}_{t+1}(r_{t+1}) = -(\hat{\mu}_{t+1} + \hat{\sigma}_{t+1}\Phi^{-1}(\alpha))P_0$$

and

$$\widehat{ES}^{\alpha}_{t+1}(r_{t+1}) = (-\hat{\mu}_{t+1} + \hat{\sigma}_{t+1}\frac{\phi(\Phi^{-1}(\alpha))}{\alpha})P_0$$

Time aggregation of VaR with mean

- If returns are i.i.d., then both mean and variance aggregate are at the same rate
- Mean and variance over T days is equals to T times mean and variance over one day
- The T-period VaR is therefore

$$\begin{aligned} VaR^{\alpha}(Tday) &= -\sigma(Tday)F^{-1}(\alpha) - \mu(Tday) \\ &= -\sqrt{T}\sigma(1day)F^{-1}(\alpha) - T\mu(1day) \end{aligned}$$