Homework 1, Probability

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February 9, 2022

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1. Chapter 1

1.1. Problems

Problem 2. How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is $3, 4, 3, 1$ if the first roll landed on 3, the second on 4, the third on 3, and the fourth on 1?

Solution. The possible number of outcomes on a die is 6. Therefore, if a die is rolled four times, the possible outcome sequences will be $6^4 = 1296$. □

Problem 5. For years, telephone area codes in the U.S. and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third digit was any integer from 1 to 9. How many area codes were possible? How many area codes starting with a 4 were possible?

Solution. Let $\overline{abc}$ be any telephone area code in the U.S. and Canada. There are 8 integers between and including 2 and 9, there are 2 integers between and including 0 and 1, and there are 9 integers between and including 1 and 9. By the principle of counting, there are $8 \times 2 \times 9 = 144$ possible area codes. If the first digit is chosen to be 4, then the possible number of area codes will be $1 \times 2 \times 9 = 18$. □

Problem 11. In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if:

(a) The books can be arranged in any order?

(b) The mathematics books must be together and the novels must be together?

(c) The novels must be together, but the other books can be arranged in any order?

Solution.

(a) There are 6 books on the shelf, so the number of arrangements with no restriction is $6! = 720$.

(b) Since the mathematics books are all together, we can view them as a single bundle of mathematics books. Two mathematics books can be put in any order in the bundle, so the number of ways to arrange such a bundle is $2!$. Similarly there are $3!$ ways to arrange a bundle of novels. Now there are $3!$ ways to put these two bundles together with the chemistry book onto the shelf in any order, so by the multiplication principle, the number of ways is $3! \times 2! \times 3! = 72$.

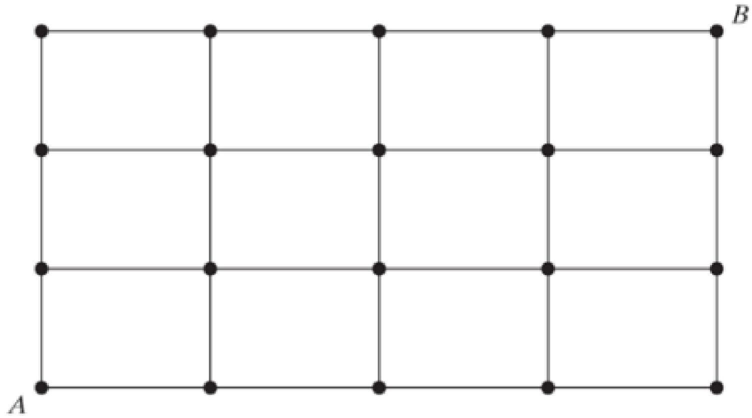
(c) There are $3!$ ways to arrange a bundle of novels and $4!$ ways to put this bundle onto the shelf with other books, so the total number of ways is $3! \times 4! = 144$.

□

Problem 15. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

Solution. There are 20 people in total, so each person shakes hands with 19 other people because they don't shake hands with themselves. It means that, there are $20 \times 19 = 380$ handshakes. But two people are involved in every handshake. Hence, there are $380/2 = 190$ handshakes taken place. $\square$

Problem 23. Consider the grid of points shown below. Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible?



Solution. Each path from A to B is a string of 3 Us and 4 Rs, where U means "up" and R means "right"; for example, $URRRRUU$. Thus there are $\binom{7}{3} = 35$ such paths from A to B. $\square$

Problem 28. The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many bridge deals are possible?

Solution. We can view this as 13 cards drawn from 52 for the first hand. 13 cards will be drawn from the remaining 39 cards for the second hand, 13 cards will be drawn from the remaining 26 cards for the third hand and 13 cards will be drawn from the remaining 13 cards for the fourth hand. Using the formula of combinations, it gives

$$\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13} \approx 5.4 \times 10^{28}$$

possible bridge deals. $\square$

Problem 29. Expand $(x_1 + 2x_2 + 3x_3)^4$.

Solution. Recall that

$$(a+b+c)^4 = a^4 + 4a^3(b+c) + 6a^2(b+c)^2 + 4a(b+c)^3 + (b+c)^4$$

$$= a^4 + 4a^3b + 4a^3c + 6a^2(b^2+2bc+c^2) + 4a(b^3+3b^2c+3bc^2+c^3) + (b^4+4b^3c+6b^2c^2+4bc^3+c^4)$$

$$= (a^4+b^4+c^4) + 4(a^3b+ab^3+b^3c+bc^3+c^3a+ca^3) + 6(a^2b^2+b^2c^2+c^2a^2) + 12(a^2bc+b^2ca+c^2ab)$$

thus the expansion of $(x_1 + 2x_2 + 3x_3)^4$ is

$$\sum_{i=1}^{3} \left(i^4x_i^4 + 4i^3f(i+1)x_i^3x_{f(i+1)} + 4i \cdot f(i+1)^3x_ix_{f(i+1)}^3 + 6i^2f(i+1)^2x_i^2x_{f(i+1)}^2 + 12i^2x_i^2x_{f(i+1)}x_{f(i+2)} \right)$$

where $f(1) = 1, f(2) = 2, f(3) = 3, f(4) = 1, f(5) = 2$. $\square$

1.2. Theoretical Exercises

Exercise 8. Prove that

$$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}.$$

Hint. Consider a group of n men and m women. How many groups of size r are possible?

Solution. We attempt to count the number of ways to form a group of size r from a group of n men and m women. On the one hand, the number of ways to form a group of size r from a group of $n+m$ people is $\binom{n+m}{r}$. On the other hand, we consider the following cases:

- If the group is of r women then there are $\binom{m}{r}$ ways;
- If the group is of $r-1$ women and 1 men then there are $\binom{n}{1}\binom{m}{r-1}$ ways;
- In general, if the group is of k men and $r-k$ women then there are $\binom{n}{k}\binom{m}{r-k}$ ways.

Notice that the two counting ways should yield the same result, adding up all the above cases implies the desired equality. □

Exercise 9. Use **Exercise 8** to prove that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Solution. Applying **Exercise 8** with $n = m = r$ yields

$$\binom{2n}{n} = \binom{n+n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

□

2. Chapter 2

2.1. Problems

Problem 1. A box contains 3 marbles: 1 red, 1 green, and 1 blue. Consider an experiment that consists of taking 1 marble from the box and then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat when the second marble is drawn without replacing the first marble.

Solution. Let R, B, G denote the red, blue and green marble respectively.

(a) In the first experiment, the total number of outcomes is $3^2 = 9$ and the sample space is given by

$$\Omega = \{(R,R),(R,G),(R,B),(G,R),(G,G),(G,B),(B,R),(B,G),(B,B)\}.$$

(b) In the second experiment, the second draw's result has to be different from the first one. Thus the total outcomes is $3 \times 2 = 6$ and the sample space is given by

$$\Omega = \{(R,G),(R,B),(G,R),(G,B),(B,R),(B,G)\}.$$

□

Problem 3. Two dice are thrown. Consider the following events:

E : The sum of the dice is odd;
F : At least one of the dice lands on 1;
G : The sum of the dice is 5.

Describe the events $EF, E \cup F, FG, EF^c$ and EFG.

Solution.

$$\begin{aligned} EF:&\quad \text{The sum of the dice is odd and at least one of the dice lands on 1.;}\\ E\cup F:&\quad \text{Either the sum of the dice is odd, or at least one of the dice lands on 1, or both;}\\ FG:&\quad \text{At least one of the dice lands on 1 and the sum of the dice is 5;}\\ EF^c:&\quad \text{The sum of the dice is odd and there are no dice landing on 1;}\\ EFG:&\quad \text{The sum of the dice is 5 and at least one of the dice lands on 1.}\end{aligned}$$

□

Problem 8. Suppose that A and B are mutually exclusive events for which $\mathbb{P}(A) = 0.3$ and $\mathbb{P}(B) = 0.5$. What is the probability that

(a) either A or B occurs?

(b) A occurs but B does not?

(c) both A and B occur?

Solution. Since A and B are mutually exclusive events, $AB = \emptyset$. Thus,

(a) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) = 0.3 + 0.5 = 0.8$.

(c) $\mathbb{P}(AB) = \mathbb{P}(\emptyset) = 0$.

(b) $\mathbb{P}(AB^c) = \mathbb{P}(A) - \mathbb{P}(AB) = 0.3 - 0 = 0.3$.

□

Problem 11. A total of 28% of American males smoke cigarettes, 7% smoke cigars, and 5% smoke both cigars and cigarettes.

(a) What percentage of males smokes neither cigars nor cigarettes?

(b) What percentage smokes cigars but not cigarettes?

Solution. Let E, F be the event that an American male smokes cigarettes and smokes cigars, respectively. Then we are given $\mathbb{P}(E) = 0.28, \mathbb{P}(F) = 0.07, \mathbb{P}(EF) = 0.05$ and thus

(a) $\mathbb{P}(E^cF^c) = 1 - \mathbb{P}(E \cup F) = 1 - (\mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(EF)) = 1 - (0.28 + 0.07 - 0.05) = 0.7$, i.e. 70% of males smokes neither cigars nor cigarettes.

(b) $\mathbb{P}(FE^c) = \mathbb{P}(F) - \mathbb{P}(EF) = 0.07 - 0.05 = 0.02$, i.e. 2% of males smokes cigars but not cigarettes.

□

Problem 18. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

Solution. In an ordinary 52-card playing deck, there are 4 aces, 4 tens, 4 jacks, 4 queens and 4 kings. Therefore, the probability that 2 random cards form a blackjack is

$$\frac{\binom{4}{1}\binom{16}{1}}{\binom{52}{2}} = \frac{32}{663} \approx 0.048.$$

□

Problem 28. An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be

(a) of the same color?

(b) of different colors?

Repeat under the assumption that whenever a ball is selected, its color is noted and it is then replaced in the urn before the next selection. This is known as *sampling with replacement.*

Solution. Let A, B be the events mentioned in parts (a) and (b), respectively. First, we consider the case without the assumption.

(a) If the 3 balls are of the same color, then either they are all red, or all blue, or all green (three mutually exclusive events). Thus,

$$\mathbb{P}(A) = \frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}} = \frac{86}{969} \approx 0.089.$$

(b) To choose three balls of different colors, we must choose 1 of 5 red balls, and 1 of 6 blue balls, and 1 of 8 green balls. Thus,

$$\mathbb{P}(B) = \frac{\binom{5}{1}\binom{6}{1}\binom{8}{1}}{\binom{19}{3}} = \frac{80}{323} \approx 0.248.$$

Next, we consider the case of *sampling with replacement.*

(a) There are now 5^3 ways to choose 3 red balls and similarly for other color. Thus,

$$\mathbb{P}(A) = \frac{5^3 + 6^3 + 8^3}{19^3} = \frac{853}{6859} \approx 0.124.$$

(b) There are $5 \times 6 \times 8$ combinations of 3 balls, one of each color and each combination can be ordered in 3! ways. Therefore,

$$\mathbb{P}(B) = \frac{(5 \times 6 \times 8)(3!)}{19^3} = \frac{1440}{6859} \approx 0.21.$$

□

Problem 41. If a die is rolled 4 times, what is the probability that 6 comes up at least once?

Solution. Denote A_i the event that 6 comes up on the i^{th} roll, $i = \overline{1,4}$. Then $\{A_i^c\}$ are independent and

$$\mathbb{P}(A_i^c) = 1 - \mathbb{P}(A_i) = 1 - \frac{1}{6} = \frac{5}{6}, \forall i = \overline{1,4},$$

implying

$$\mathbb{P}(A_1^c A_2^c A_3^c A_4^c) = \mathbb{P}(A_1^c) \cdot \mathbb{P}(A_2^c) \cdot \mathbb{P}(A_3^c) \cdot \mathbb{P}(A_4^c) = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$$

and therefore

$$\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \mathbb{P}(A_1^c A_2^c A_3^c A_4^c) = \frac{671}{1296} \approx 0.517,$$

i.e. the probability that 6 comes up at least once is 0.517. □

Problem 42. Two dice are thrown n times in succession. Compute the probability that double 6 appears at least once. How large need n be to make this probability at least $\frac{1}{2}$?

Solution. Denote A_i the event that double 6 comes up on the i^{th} roll, $i = \overline{1,n}$. Then $\{A_i^c\}$ are independent and

$$\mathbb{P}(A_i^c) = 1 - \mathbb{P}(A_i) = 1 - \frac{1}{36} = \frac{35}{36}, \forall i = \overline{1,n},$$

implying

$$\mathbb{P}\left(\prod_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} \mathbb{P}(A_i^c) = \left(\frac{35}{36}\right)^n$$

and therefore

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \mathbb{P}\left(\prod_{i=1}^{n} A_i^c\right) = 1 - \left(\frac{35}{36}\right)^n,$$

i.e. the probability that double 6 appears at least once is $1 - \left(\frac{35}{36}\right)^n$. Also,

$$1 - \left(\frac{35}{36}\right)^n \geq \frac{1}{2} \Leftrightarrow n \geq \log_{35/36}\left(\frac{1}{2}\right) \approx 24.605,$$

i.e. n should be at least 25 to make this probability at least $\frac{1}{2}$. $\square$

2.2. Theoretical Exercises

Exercise 5. For any sequence of events $E_1, E_2, \ldots$ define a new sequence $F_1, F_2, \ldots$ of disjoint events (that is, events such that $F_i F_j = \emptyset$ whenever $i \neq j$) such that for all $n \geq 1$,

$$\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i.$$

Solution. Define

$$F_j = \left(\bigcup_{i=1}^{j} E_i\right) \setminus \left(\bigcup_{i=1}^{j-1} E_i\right), \forall j = \overline{1, n}.$$

We verify that the defined sequence $\{F_n\}$ satisfies the given conditions.

- Assume conversely that there is $u < v$ such that $\exists x_0 \in F_u F_v$. Then

$$x \in F_u = \left(\bigcup_{i=1}^{u} E_i\right) \setminus \left(\bigcup_{i=1}^{u-1} E_i\right) \subset \bigcup_{i=1}^{u} E_i$$

  while

$$x \in F_v = \left(\bigcup_{i=1}^{v} E_i\right) \setminus \left(\bigcup_{i=1}^{v-1} E_i\right) \subset \left(\bigcup_{i=1}^{v-1} E_i\right)^c \subset \left(\bigcup_{i=1}^{u} E_i\right)^c,$$

  a contradiction. Thus the sequence $\{F_n\}$ is pairwise disjoint.

- Clearly

$$F_u \subset \bigcup_{i=1}^{u} E_i \subset \bigcup_{i=1}^{v} E_i, \forall v \geq u$$

  and therefore

$$\bigcup_{i=1}^{n} F_i \subset \bigcup_{i=1}^{n} E_i, \forall n \in \mathbb{N}.$$

  Now take any $x \in \bigcup_{i=1}^{n} E_i$ and define $k = \inf\{i : x \in E_i\}$, then $x \in E_k$ and $x \notin E_i, \forall i < k$ which implies

$$x \in E_k \setminus \left(\bigcup_{i=1}^{k-1} E_i\right) \subset \left(\bigcup_{i=1}^{k} E_i\right) \setminus \left(\bigcup_{i=1}^{k-1} E_i\right) = F_k \subset \bigcup_{i=1}^{n} F_i.$$

  Hence

$$\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i.$$

$\square$

Exercise 9. Suppose that an experiment is performed n times. For any event E of the sample space, let $n(E)$ denote the number of times that event E occurs and define $f(E) = \frac{n(E)}{n}$. Show that $f(\cdot)$ satisfies Axioms 1, 2, and 3.

Solution. Denote S the sample space.

- Since each event E occurs at least 0 times and at most n times,

$$0 \leq \frac{n(E)}{n} \leq \frac{n}{n} = 1.$$

- Since the experiment is performed n times, the number of times that "something" happen (i.e. there is an outcome) is n, so

$$f(S) = \frac{n(S)}{n} = \frac{n}{n} = 1.$$

- If $\{E_i\}_{i=1}^{\infty}$ is a sequence of mutually exclusive events, the number of times the event $\bigcup_{i=1}^{\infty} E_i$ occurs is equal to the sum of the number of times that each event E_i occurs, implying

$$f\left(\bigcup_{i=1}^{\infty} E_i\right) = \frac{n\left(\bigcup_{i=1}^{\infty} E_i\right)}{n} = \frac{\sum_{i=1}^{\infty} n(E_i)}{n} = \sum_{i=1}^{\infty} \frac{n(E_i)}{n} = \sum_{i=1}^{\infty} f(E_i).$$

□

3. Chapter 3

3.1. Problems

Problem 1. Two fair dice are rolled. What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?

Solution. Consider the following events:

A : No die lands on 6;
B : The dice land on the same number.

Then $\Omega = \left\{(i,j) : i,j = \overline{1,6}\right\}$ and

$$A = \left\{(i,j) : i,j = \overline{1,5}\right\}, B = \left\{(i,i) : i = \overline{1,6}\right\}, AB = \left\{(i,i) : i = \overline{1,5}\right\},$$

implying

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{25}{36}, \mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}, \mathbb{P}(AB) = \frac{|AB|}{|\Omega|} = \frac{5}{36}$$

and therefore

$$\mathbb{P}(A^c|B^c) = \frac{\mathbb{P}(A^cB^c)}{\mathbb{P}(B^c)} = \frac{1-\mathbb{P}(A\cup B)}{1-\mathbb{P}(B)} = \frac{1-(\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(AB))}{1-\mathbb{P}(B)} = \frac{1}{3},$$

i.e. the conditional probability that at least one die lands on 6 given that the dice land on different numbers is $\frac{1}{3}$. □

Problem 5. An urn contains 6 white and 9 black balls. If 4 balls are to be randomly selected without replacement, what is the probability that the first 2 selected are white and the last 2 black?

Solution. By the multiplicative rule, we get the probability that the first 2 selected are white and the last 2 black is

$$\frac{(6\times 5)\times(9\times 8)}{15\times 14\times 13\times 12} = \frac{6}{91} \approx 0.066.$$

□

Problem 10. Three cards are randomly selected, without replacement, from an ordinary deck of 52 playing cards. Compute the conditional probability that the first card selected is a spade given that the second and third cards are spades.

Solution. Consider the following events:

A : The first card selected is a spade;
B : The second and third cards selected are spades;
AB : All selected cards are spades.

Then

$$\mathbb{P}(AB) = \frac{\binom{13}{3}}{\binom{52}{3}} = \frac{11}{850}, \mathbb{P}(B) = \frac{\binom{13}{2}}{\binom{52}{2}} = \frac{1}{17}$$

implying

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{11}{50},$$

i.e. the probability that the first card selected is a spade given that the second and third cards are spades is $\frac{11}{50}$. □

Problem 16. An ectopic pregnancy is twice as likely to develop when the pregnant woman is a smoker as it is when she is a nonsmoker. If 32% of women of childbearing age are smokers, what percentage of women having ectopic pregnancies are smokers?

Solution. Consider the following events:

E : A randomly chosen woman of childbearing age has ectopic pregnancy;
S : A randomly chosen woman of childbearing age is a smoker.

Then

$$\mathbb{P}(E|S) = 2 \cdot \mathbb{P}(E|S^c), \mathbb{P}(S) = 0.32$$

and thus by applying Bayes' Theorem,

$$\begin{aligned}\mathbb{P}(S|E) &= \frac{\mathbb{P}(S) \cdot \mathbb{P}(E|S)}{\mathbb{P}(S) \cdot \mathbb{P}(E|S) + \mathbb{P}(S^c) \cdot \mathbb{P}(E|S^c)} = \frac{\mathbb{P}(S) \cdot 2 \cdot \mathbb{P}(E|S^c)}{\mathbb{P}(S) \cdot 2 \cdot \mathbb{P}(E|S^c) + \mathbb{P}(S^c) \cdot \mathbb{P}(E|S^c)} \\ &= \frac{2 \cdot \mathbb{P}(S)}{2 \cdot \mathbb{P}(S) + \mathbb{P}(S^c)} = \frac{2 \cdot \mathbb{P}(S)}{2 \cdot \mathbb{P}(S) + 1 - \mathbb{P}(S)} = \frac{16}{33} \approx 0.4848,\end{aligned}$$

i.e. 48.48% of women having ectopic pregnancies are smokers. □

Problem 18. In a certain community, 36% of the families own a dog and 22% of the families that own a dog also own a cat. In addition, 30% of the families own a cat. What is

(a) the probability that a randomly selected family owns both a dog and a cat?

(b) the conditional probability that a randomly selected family owns a dog given that it owns a cat?

Solution. Consider the following events:

D : A randomly selected family owns a dog;
C : A randomly selected family owns a cat.

Then $\mathbb{P}(D) = 0.36, \mathbb{P}(C) = 0.3, \mathbb{P}(C|D) = 0.22$ and thus

(a) $\mathbb{P}(CD) = \mathbb{P}(D) \cdot \mathbb{P}(C|D) = 0.0792$, i.e the probability that a randomly selected family owns both a dog and a cat is 0.0792.

(b) $\mathbb{P}(D|C) = \frac{\mathbb{P}(CD)}{\mathbb{P}(C)} = 0.264$, i.e. the conditional probability that a randomly selected family owns a dog given that it owns a cat is 0.264.

□

Problem 22. A total of 500 married working couples were polled about their annual salaries, with the following information resulting:

Wife \ Husband	Less than \$125,000	More than \$125,000
Less than \$125,000	212	198
More than \$125,000	36	54

For instance, in 36 of the couples, the wife earned more and the husband earned less than \$125,000. If one of the couples is randomly chosen, what is

(a) the probability that the husband earns less than \$125,000?

(b) the conditional probability that the wife earns more than \$125,000 given that the husband earns more than this amount?

(c) the conditional probability that the wife earns more than \$125,000 given that the husband earns less than this amount?

Solution.

(a) The probability that the husband earns less than \$125,000 is $\frac{212+36}{500} = 0.496$;

(b) The probability that the wife earns more than \$125,000 given that the husband earns more than this amount is $\frac{54}{198+54} \approx 0.214$;

(c) The probability that the wife earns more than \$125,000 given that the husband earns less than this amount is $\frac{36}{212+36} \approx 0.145$.

$\square$

Problem 24. Urn I contains 2 white and 4 red balls, whereas urn II contains 1 white and 1 red ball. A ball is randomly chosen from urn I and put into urn II, and a ball is then randomly selected from urn II. What is

(a) the probability that the ball selected from urn II is white?

(b) the conditional probability that the transferred ball was white given that a white ball is selected from urn II?

Solution. Consider the following events:

A : The transferred ball is white;
B : The ball selected from urn II is white.

Then $\mathbb{P}(A) = \frac{1}{3}$ and $\mathbb{P}(A^c) = \frac{2}{3}$.

(a) If the transferred ball is white, then after the transfer urn II has 2 white balls and 1 red ball. Now, if a ball is randomly chosen from urn II, the probability that it is a white ball is $\frac{2}{3}$, i.e. $\mathbb{P}(B|A) = \frac{2}{3}$.

On the other hand, if the transferred ball is red, then after the transfer urn II has 1 white and 2 red balls. Now, if a ball is randomly chosen from urn II, the probability that it is a white ball is $\frac{1}{3}$, i.e $\mathbb{P}(B|A^c) = \frac{1}{3}$. Therefore,

$$\mathbb{P}(B) = \mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B|A^c) \cdot \mathbb{P}(A^c) = \frac{4}{9},$$

i.e. the probability that the ball selected from urn II is white is $\frac{4}{9}$.

(b) Applying Bayes' Theorem yields

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A) \cdot \mathbb{P}(B|A)}{\mathbb{P}(A) \cdot \mathbb{P}(B|A) + \mathbb{P}(A^c) \cdot \mathbb{P}(B|A^c)} = \frac{1}{2},$$

i.e. the conditional probability that the transferred ball was white given that a white ball is selected from urn II is $\frac{1}{2}$.

$\square$

Problem 28. Suppose that 5% of men and 0.25% of women are color-blind. A color-blind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females. What if the population consisted of twice as many males as females?

Solution. Consider the following events:

A : A randomly chosen person is male;
B : A randomly chosen person is color-blind.

If there are an equal number of males and females, applying Bayes' Theorem yields

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\cdot\mathbb{P}(A)}{\mathbb{P}(B|A)\cdot\mathbb{P}(A)+\mathbb{P}(B|A^c)\cdot\mathbb{P}(A^c)} = \frac{5\%\cdot 50\%}{5\%\cdot 50\% + 0.25\%\cdot 50\%} = \frac{20}{21} \approx 0.952,$$

i.e. the probability of a randomly chosen color-blind person being male is 0.952. If indeed the population consisted of twice as many males as females, then this probability is

$$\mathbb{P}(A|B) = \frac{5\%\cdot(2/3)}{5\%\cdot(2/3)+0.25\%\cdot(1/3)} = \frac{40}{41} \approx 0.976.$$

$\square$

Problem 35. On rainy days, Joe is late to work with probability .3; on nonrainy days, he is late with probability .1. With probability .7, it will rain tomorrow.

(a) Find the probability that Joe is early tomorrow.

(b) Given that Joe was early, what is the conditional probability that it rained?

Solution. Consider the following events:

A : It is raining;
B : Joe is late.

Then $\mathbb{P}(B|A) = 0.3, \mathbb{P}(B^c|A) = 0.7, \mathbb{P}(B|A^c) = 0.1, \mathbb{P}(B^c|A^c) = 0.9, \mathbb{P}(A) = 0.7$ and thus

(a) $\mathbb{P}(B^c) = \mathbb{P}(B^c|A)\cdot\mathbb{P}(A) + \mathbb{P}(B^c|A^c)\cdot\mathbb{P}(A^c) = 0.76$, i.e. the probability that Joe is early tomorrow is 0.76.

(b) Applying Bayes' Theorem yields

$$\mathbb{P}(A|B^c) = \frac{\mathbb{P}(A)\cdot\mathbb{P}(B^c|A)}{\mathbb{P}(A)\cdot\mathbb{P}(B^c|A)+\mathbb{P}(A^c)\cdot\mathbb{P}(B^c|A^c)} = \frac{49}{76} \approx 0.645,$$

i.e. the conditional probability that it rained given that Joe was early is 0.645.

$\square$

3.2. Theoretical Exercises

Problem 1. Show that if $\mathbb{P}(A) > 0$, then

$$\mathbb{P}(AB|A) \geq \mathbb{P}(AB|A\cup B).$$

Solution. By the conditional probability formula,

$$\mathbb{P}(AB|A) = \frac{\mathbb{P}(AB\cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(AB)}{\mathbb{P}(A)}$$

and

$$\mathbb{P}(AB|A \cup B) = \frac{\mathbb{P}(AB \cap (A \cup B))}{\mathbb{P}(A \cup B)} = \frac{\mathbb{P}(AB)}{\mathbb{P}(A \cup B)}.$$

Since $A \subset A \cup B$, the monotonicity of $\mathbb{P}$ implies $\mathbb{P}(A) \leq \mathbb{P}(A \cup B)$ and thus

$$\mathbb{P}(AB|A) = \frac{\mathbb{P}(AB)}{\mathbb{P}(A)} \geq \frac{\mathbb{P}(AB)}{\mathbb{P}(A \cup B)} = \mathbb{P}(AB|A \cup B).$$

$\square$

Problem 6. Prove that if $E_1, E_2, ..., E_n$ are independent events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = 1 - \prod_{i=1}^{n}[1 - \mathbb{P}(E_i)].$$

Solution. We will prove by induction on n.

- If E_1 and E_2 are independent then by **Proposition 4.1**, E_1^c and E_2^c are independent and thus

$$\mathbb{P}(E_1 \cup E_2) = 1 - \mathbb{P}(E_1^c E_2^c) = 1 - \mathbb{P}(E_1^c) \cdot \mathbb{P}(E_2^c) = 1 - [1 - \mathbb{P}(E_1)][1 - \mathbb{P}(E_2)].$$

- Assume that the stated property holds for $k \in \mathbb{N}$. Let $\{E_i\}_{i=1}^{k+1}$ be independent, then

$$\mathbb{P}\left(\bigcup_{i=1}^{k+1} E_i\right) = 1 - \left[1 - \mathbb{P}\left(\bigcup_{i=1}^{k} E_i\right)\right][1 - \mathbb{P}(E_{k+1})],$$

  since E_{k+1} and $\displaystyle\bigcup_{i=1}^{k} E_i$ are independent. But $\{E_i\}_{i=1}^{k}$ are independent, so

$$\mathbb{P}\left(\bigcup_{i=1}^{k} E_i\right) = 1 - \prod_{i=1}^{k}[1 - \mathbb{P}(E_i)],$$

  impying

$$\mathbb{P}\left(\bigcup_{i=1}^{k+1} E_i\right) = 1 - \prod_{i=1}^{k+1}[1 - \mathbb{P}(E_i)].$$

Thus the stated property holds for all $n \in \mathbb{N}$. $\square$