VIETNAM NATIONAL UNIVERSITY-HCMC
International University

Chapter 1. Introduction to linear systems and matrices

MAFE104IU – Linear Algebra

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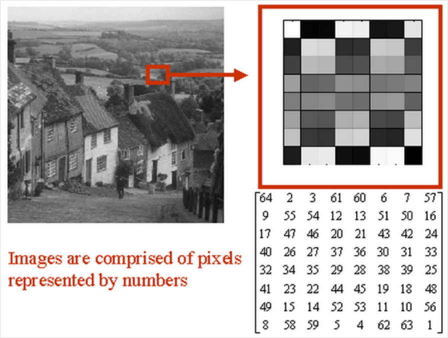
Section 1

INTRODUCTION

Introduction

Linear Algebra is the branch of mathematics. It has many application in Economics, Linear programming, Coding theory, Chemistry, Genetics, Games, Image compression,...



Section 2

LINEAR SYSTEMS, MATRIX

System of linear equations

A central problem of linear algebra is solving linear equations.

Example

The director of a trust fund has \$100.000 to invest. The rules of the trust state that both a certificate of deposit (CD) and a long-term bond must be used. The director's goal is to have the trust yield \$7800 on its investments for the year. The CD chosen returns 5% per annum, and the bond 9%. Find the amounts that the director invest in the CD and in the bond.

Let x and y be the amount to invest in the CD and in the bond, respectively. We have

$$\begin{cases} x + y = 100,000 \\ 0.05x + 0.09y = 7800 \end{cases}$$

To eliminate x, we add (-0.05) times the first equation to the second, obtaining $0.04y = 2800$, this implies $y = 70,000$. Thus, $x = 30,000$.

System of linear equations (n=2)

We consider another example

$$1x + 2y = 3 \qquad (1)$$

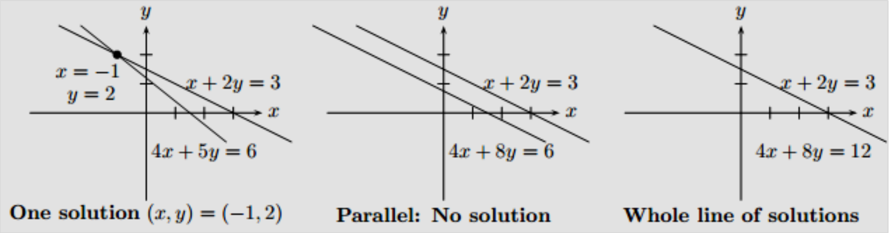
$$4x + 5y = 6 \qquad (2)$$

The two **unknowns** are **x and y**.

How to solve? $\rightarrow$ **Elimination**
$Eq.$ (2) $-$ 4 $Eq.$(1): $-3y = -6 \ \rightarrow y = 2.$
Back-substitution $x = -1$.

This method is called Gaussian Elimination. It is good to solve large systems of equations. We will consider in detail this method in this section.

The Geometry of Linear Equations (n=2)



Examples of system of linear equations (n=3)

Consider a system of $n = 3$ equations:

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - \ z = -2$$

To eliminate x, we add (-2) times the first equation to the second one and (-3) times the first equation to the third one, obtaining a system of two equations in the unknowns y and z

$$-7y - 4z = 2$$

$$-5y - 10z = -20$$

We multiply the second equation by $-1/5$, yielding

$$-7y - 4z = 2$$

$$y + 2z = 4$$

Examples of system of linear equations (n=3)

Interchanging equations

$$y + 2z = 4$$

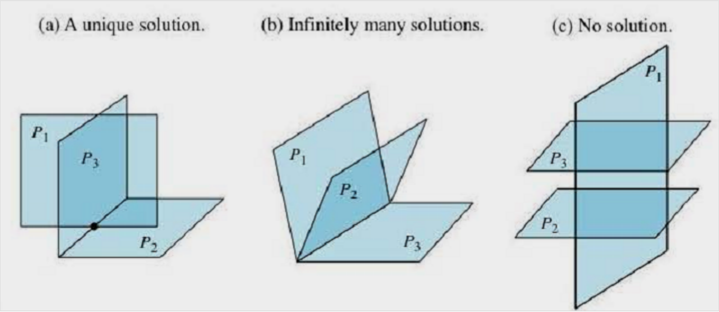
$$-7y - 4z = 2$$

We now eliminate y by adding 7 times the first equation to the second one, to obtain $10z = 30$. Thus, $z = 3$. Substitute $z = 3$ into $y + 2z = 4$ to get $y = -2$. Finally, we find $x = 1$.
We observe further that our elimination procedure has actually produced the linear system which is equivalent to the original one.

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ z &= 3 \end{aligned}$$

The importance of this procedure is that the later system is easier to solve!

The Geometry of Linear Equations



What is a Matrix?

A matrix is a set of elements, organized into rows and columns

$$A = \begin{array}{cl} \begin{array}{cccc} \phantom{a_{11}} & \phantom{a_{12}} & & \text{Column} \end{array} & \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{l} \text{Row} \\ \\ \\ \\ \end{array} \end{array}$$

The number $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, are called the entries (or elements) of A.

Example of matrix

The following matrix gives the airline distance between the indicated cities (in statute miles):

$$
\begin{array}{l}
\\
\text{London} \\
\text{Madrid} \\
\text{New York} \\
\text{Tokyo}
\end{array}
\begin{array}{c}
\begin{array}{cccc} \text{London} & \text{Madrid} & \text{New York} & \text{Tokyo} \end{array} \\
\begin{bmatrix}
0 & 785 & 3469 & 5959 \\
785 & 0 & 3593 & 6706 \\
3469 & 3593 & 0 & 6757 \\
5959 & 6706 & 6757 & 0
\end{bmatrix}
\end{array}
$$

Example of matrix

Suppose that a manufacturer has four plants, each of which makes three products. If we let a_{ij} denote the number of units of product i made by plant j in one week, then the 3×4 matrix

$$\begin{array}{l} \\ \text{Product 1} \\ \text{Product 2} \\ \text{Product 3} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} & \text{Plant 4} \end{array} \\ \begin{bmatrix} 560 & 360 & 380 & 0 \\ 340 & 450 & 420 & 80 \\ 280 & 270 & 210 & 380 \end{bmatrix} \end{array}$$

gives the manufacturer's production for the week. For example, plant 2 makes 270 units of product 3 in one week.

Matrices

A matrix with m rows and n columns is called an $m \times n$ matrix (m by n matrix) or a matrix of order $m \times n$.

A matrix consisting of a single column is called a column vector and a matrix consisting of a single row is called a row vector.

If $m = n$ the matrix is called a square matrix. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ is a 2 by 3 matrix}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 5 & 1 \\ 7 & 0 & 9 \end{bmatrix} \text{ is a (square) 3 by 3 matrix}$$

Equality: Two matrices are said to be equal if they have the same order and all the corresponding entries are equal.

Linear systems of equations

We often deal with several linear equations at the same time:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

A system with at least one solution is called consistent. Otherwise, it is called inconsistent.

Theorem: Every system of linear equations has zero, one or infinitely many solutions; there are no other possibilities.

Coefficient and augmented matrices

Matrix of coefficients is a matrix of m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Augmented matrix is

$$\overline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Example

Consider the linear system of equations

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_2 - 8x_3 = 8$$

$$-4x_1 + 5x_2 + 9x_3 = -9$$

Matrix of Coefficients is

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -5 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Section 3

GAUSSIAN ELIMINATION

Gaussian Elimination

Motivation: Original augmented matrix $\rightarrow$ much simpler augmented matrix of an equivalent system!

For any augmented matrix of a system of equations, the following operations, which are called the elementary operations, produce the augmented matrix of an equivalent system:

1. Interchanging any two rows, denoted by: I_{ik}.

2. Multiply a row by a nonzero constant: αR_i.

3. Add a multiple of one row to another: $R_i + \alpha R_k$.

Pivot-Matrix in Echelon Form

A pivot (leading entry) of a matrix is the first nonzero entry in a row. A pivot column is a column of A that contains a pivot position.

Definition

A matrix is in (row) echelon form if
a. All rows that contain only zeros are grouped at the bottom of the matrix

b. For each row that does not contain only zeros, the pivot appears strictly to the right of the pivot of each row that appears above it

Intuitively, a matrix is in row echelon form if it has the appearance of a staircase pattern like

$$\begin{bmatrix} 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix in Row Echelon Form

Example

The following matrices are in row echelon form

$$a)\ \begin{bmatrix} 1 & -3 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$b)\ \begin{bmatrix} 1 & -3 & 2 & 7 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 2 & 9 \end{bmatrix}$$

$$c)\ \begin{bmatrix} 4 & -2 & 1 & 7 & 9 & 1 \\ 0 & 0 & -5 & 9 & 1 & 2 \\ 0 & 0 & 0 & 8 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix in Row Echelon Form

Example

The following matrices are NOT in row echelon form

$$a)\ \begin{bmatrix} 1 & -3 & 2 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$b)\ \begin{bmatrix} 1 & -3 & 2 & 7 \\ 0 & 0 & 1 & 7 \\ 0 & 2 & 2 & 9 \end{bmatrix}$$

$$c)\ \begin{bmatrix} 4 & -2 & 1 & 7 & 9 & 1 \\ 0 & 0 & -5 & 9 & 1 & 2 \\ 0 & 0 & 0 & 8 & 3 & 5 \\ 0 & 0 & 0 & 7 & 1 & 0 \end{bmatrix}$$

Matrix in Row Echelon Form

Example

Suppose that the augmented matrix for a linear system has been reduced to the given matrix in row echelon form and the variables are also given. Solve the system.

$$\begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & -3 & 2 & 17 \\ 0 & 0 & 5 & -10 \end{bmatrix}$$

Variables: x, y, z

Solution The first step is to find the associated linear system

$$2x + y - 3z = 5$$

$$-3y + 2z = 17$$

$$5z = -10$$

The system is in triangular form and we solve it by backward substitution: $5z = -10$ leads to $z = -2$. Thus, $y = -7, x = 3$.

Matrix in Row Echelon Form

Example

Solve the system

$$
\begin{aligned}
x + 4y - 4z &= 3 \\
3y + 2z &= 7 \\
-9y + 20z &= -8
\end{aligned}
$$

Solution The associated augmented matrix is

$$
\begin{bmatrix}
1 & 4 & -4 & 3 \\
0 & 3 & 2 & 7 \\
0 & -9 & 20 & -8
\end{bmatrix}
$$

Add 3 times the second row to the third, we obtain the following reduced matrix in echelon form

$$
\begin{bmatrix}
1 & 4 & -4 & 3 \\
0 & 3 & 2 & 7 \\
0 & 0 & 26 & 13
\end{bmatrix}
$$

Matrix in Row Echelon Form

Example

Solution (Cont.) The associated linear system is

$$\begin{aligned} x + 4y - 4z &= 3 \\ 3y + 2z &= 7 \\ 26z &= 13 \end{aligned}$$

This leads to the solution $(x, y, z) = (-3, 2, 1/2)$.

Matrix in Row Echelon Form

Example

Suppose that the augmented matrix for a linear system has been reduced to the given matrix in row echelon form and the variables are also given. Solve the system.

$$\begin{bmatrix} 2 & 3 & -1 & 5 & 2 \\ 0 & 3 & 2 & -1 & 2 \\ 0 & 0 & -2 & -8 & 4 \end{bmatrix}$$

Variables: x, y, z, w

Solution The associated linear system is

$$2x + 3y - z + 5w = 2$$

$$3y + 2z - w = 2$$

$$-2z - 8w = 4$$

Matrix in Row Echelon Form

Solution (Cont.)

The variables x, y, and z corresponding to the pivots of the augmented matrix are called leading variables (or dependent variables). The remaining variables are called free variables (or independent variables).
The second step is to move the free variables to the right-hand side of the equations

$$2x + 3y - z = 2 - 5w$$

$$3y + 2z = 2 + w$$

$$-2z = 4 + 8w$$

We let $w = t$, the system becomes in triangular form

$$2x + 3y - z = 2 - 5t$$

$$3y + 2z = 2 + t$$

$$-2z = 4 + 8t$$

Matrix in Row Echelon Form

Solution (Cont.)

It can be solved by backward substitution.

$$-2z = 4 + 8t, \text{ thus, } z = -2 - 4t$$

$$3y + 2(-2 - 4t) = 2 + t$$

$$3y = 6 + 9t, \; y = 2 + 3t$$

$$2x + 3(2 + 3t) - (-2 - 4t) = 2 - 5t \Rightarrow x = -3 - 9t$$

The solution is

$(x, y, z, w) = (-3 - 9t, 2 + 3t, -2 - 4t, t)$, where t is any real number

Matrix in Row Echelon Form

Exercises

Reduce the matrix A below to echelon form.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Hint

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\text{Echelon Form} \quad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian elimination

Example

Solve the following system

$$2x - y - z = 3$$

$$-6x + 6y + 5z = -3$$

$$4x + 4y + 7z = 3.$$

First, find the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 3 \\ -6 & 6 & 5 & -3 \\ 4 & 4 & 7 & 3 \end{array}\right] \xrightarrow[-2R_1+R_3\to R_3]{3R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 3 \\ 0 & 3 & 2 & 6 \\ 0 & 6 & 9 & -3 \end{array}\right]$$

Gaussian elimination

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 3 \\ 0 & 3 & 2 & 6 \\ 0 & 6 & 9 & -3 \end{array}\right] \xrightarrow{-2R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 3 \\ 0 & 3 & 2 & 6 \\ 0 & 0 & 5 & -15 \end{array}\right]$$

The matrix is now in row echelon form, so we find the associated system

$$2x - y - z = 3$$

$$3y + 2z = 6$$

$$5z = -15.$$

Thus, $z = -3$, then $y = 4$ and $x = 2$.
The solution is $(x, y, z) = (2, 4, -3)$.

Gaussian elimination

Example

Solve the following system

$$\begin{aligned} 3y + 2z &= 7 \\ x + 4y - 4z &= 3 \\ 3x + 3y + 8z &= 1 \end{aligned}$$

Hint

$$\left[\begin{array}{ccc|c} 0 & 3 & 2 & 7 \\ 1 & 4 & -4 & 3 \\ 3 & 3 & 8 & 1 \end{array}\right] \xrightarrow[3R_2+R_3\to R_3]{I_{12};-3R_1+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 4 & -4 & 3 \\ 0 & 3 & 2 & 7 \\ 0 & 0 & 26 & 13 \end{array}\right]$$

Gaussian elimination

Example

Solve the following system with three equations and three unknowns:

$$2x + y + 5z = 1$$

$$x - 3y + 6z = 2$$

$$3x + 5y + 4z = 0.$$

First, write down the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & 5 & 1 \\ 1 & -3 & 6 & 2 \\ 3 & 5 & 4 & 0 \end{array}\right] \xrightarrow[-3/2R_1+R_3 \to R_3]{-1/2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 2 & 1 & 5 & 1 \\ 0 & -7/2 & 7/2 & 3/2 \\ 0 & 7/2 & -7/2 & -3/2 \end{array}\right]$$

Gaussian elimination

$$\xrightarrow{R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 2 & 1 & 5 & 1 \\ 0 & -7/2 & 7/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row gives us no information whatsoever. If we let $z = t$ arbitrarily, then $y = t - 3/7$.

Substituting this into the first row, we get $x = (1 - y - 5z)/2 = 5/7 - 3t$.

Thus, this system has an infinite number of solutions parameterized by the parameter t, i.e., for every value of t, there corresponds a solution to the system.

Gaussian elimination

Example

Solve the following system with three equations and four unknowns:

$$x - 2y - z - w = -4$$

$$3x + y + z - 2w = 11$$

$$x + 12y + 7z + w = 31.$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -4 \\ 3 & 1 & 1 & -2 & 11 \\ 1 & 12 & 7 & 1 & 31 \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-3R_1+R_2\to R_2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -4 \\ 0 & 7 & 4 & 1 & 23 \\ 0 & 14 & 8 & 2 & 35 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -4 \\ 0 & 7 & 4 & 1 & 23 \\ 0 & 0 & 0 & 0 & -11 \end{array}\right]$$

The last equation can not be solved. So the system has no solutions.

Exercises

Exercise 1

Solve the following system

$$2x + 5y - z = 15$$

$$x - y + 3z = 4$$

$$3x + 3y - 5z = 2.$$

Answer: $(x, y, z) = (1, 3, 2)$.

Exercise 2

Solve the following system

$$x - 2y + z = 0$$

$$2y - 8z = 8$$

$$-4x + 5y + 9z = -9$$

Answer: $(x, y, z) = (29, 16, 3)$.

Exercise 3

Solve the following system

$$2x + 8y - z + w = 0$$

$$4x + 16y - 3z - w = -10$$

$$-2x + 4y - z + 3w = -6$$

$$-6x + 2y + 5z + w = 3$$

Answer: $(x, y, z, w) = (3, -1/2, 4, 2)$.

Exercise 4

A certain brand of razor blades comes in packages of 6, 12, and 24 blades, costing \$2, \$3, and \$4 per package, respectively. A store sold 12 packages, containing a total of 162 razor blades and took in \$35. How many packages of each type were sold?

Answer: $(x, y, z) = (5, 3, 4)$.

Exercise 5

It is known that three brands of fertilizer (Fertifun, Big Grow and Soakem) are available that provide Nitrogen, Phosphoric Acid and Soluble Potash to the soil. One bag of each brand provides the following units of each nutrient:

$$\begin{array}{cc} & \begin{array}{ccc} Fertifun & BigGrow & Soakem \end{array} \\ \begin{array}{c} Nitrogen \\ Phosphoric\ Acid \\ Potash \end{array} & \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \end{array}$$

The soil of Dong Nai farm needs 18 units of nitrogen, 23 units of phosphoric acid and 13 units of potash per acre. The corresponding units for An Giang are 31, 24, and 11. How many bags of each brand of fertilizer should be used per acre for each farm?

Hint: a. Dong Nai: $(x, y, z) = (5, 2, 3)$.
b. An Giang: $(x, y, z) = (2, 4, 7)$.

The Cost of Elimination

Q: How many separate arithmetical operations does elimination require, for n equations in n unknowns?

Since all the steps are known, we should be able to predict the number of operations.

Theorem

If n is at all large, a good estimate for the number of operations is $\dfrac{n^3}{3}$.

Proof:

(1) Suppose we call each division, and each *multiplication-subtraction*, one operation.
In column 1, *takes n operations for every zero we achieve*-one to find the multiple, and the other to find the new entries along the row.

(2) There are $n-1$ rows underneath the first one, so the first stage of elimination needs $n(n-1) = n^2 - n$ operations.

The Cost of Elimination

Proof (cont.):

(3) When the elimination is down to k equations, only $k^2 - k$ operations are needed to clear out the column below the pivot.

(4) Altogether, the total number of operations is the sum of $k^2 - k$ over all values of k from 1 to n.

$$\sum_{k=1}^{n} \left(k^2 - k\right) = \frac{n\left(n+1\right)\left(2n+1\right)}{6} - \frac{n\left(n+1\right)}{2}$$

$$= \frac{n^3 - n}{3}$$

Therefore, a good estimate for the number of operations is $\frac{n^3}{3}$.

Section 4

MATRIX OPERATIONS

Product of two vectors

Product of a row matrix and a column matrix of the same dimension: Let $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix}^T$. The product AB is the 1×1 matrix whose entry is

$$a_1 b_1 + a_2 b_2 + ... + a_n b_n = \sum_1^n a_i b_i$$

i.e., it is the dot product of the row vector A and the column vector B.

Example

If

$$u = (3, 6, 2)\,, v = (4, 2, 4)\,,$$

then

$$u \cdot v = 3 \cdot 4 + 6 \cdot 2 + 2 \cdot 4 = 32$$

$$u \cdot u = 3 \cdot 3 + 6 \cdot 6 + 2 \cdot 2 = 49 = \|u\|^2$$

Basic Matrix Operations

Addition, Subtraction, Multiplication: creating new matrices.

a. Addition: Just add elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

b. Subtraction: Just subtract elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Generally, if both matrices A and B are of the same size, then

$$A \pm B = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$$

Basic Matrix Operations

c. Scalar Multiplication

$$5 \cdot \begin{bmatrix} 2 & -3 \\ 4 & -1 \\ 1/5 & 6 \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 20 & -5 \\ 1 & 30 \end{bmatrix}$$

$$\alpha A = A\alpha = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$

d. Multiplication: Multiply each row by each column

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Is $AB = BA$? Maybe, but maybe not! In general, multiplication is NOT commutative!

Matrix Multiplication

The product AB of two matrices A and B is defined if and only if the number of columns of A is equal to the number of rows of B.

$$(A)_{m\times n}(B)_{n\times p} = (AB)_{m\times p}$$

Furthermore, the entry in the i-th row and j-th column of AB is given by the dot product of the i-th row of A and the j-th column of B. That is,

If $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$, then $AB = C = (c_{ij})_{m\times p}$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$, i.e.,

$$AB = \left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)_{m\times p}$$

Scalar Multiplication

Example

Let

$$p = \begin{bmatrix} 18.95 \\ 14.75 \\ 8.60 \end{bmatrix}$$

be a 3-vector that represents the current prices of three items at a store. Suppose that the store announces a sale so that the price of each item is reduced by 20%.

(a) Determine a 3-vector that gives the price changes for the three items.

(b) Determine a 3-vector that gives the new prices of the items.

Scalar Multiplication

Solution

(a) Since each item is reduced by 20%, the 3-vector

$$(-0.20)\,p = \begin{bmatrix} (-0.20)\,(18.95) \\ (-0.20)\,(14.75) \\ (-0.20)\,(8.60) \end{bmatrix} = -\begin{bmatrix} 3.79 \\ 2.95 \\ 1.72 \end{bmatrix}$$

gives the price changes for the three items.

(b) The new prices of the items are given by the expression

$$p - 0.20p = \begin{bmatrix} 18.95 \\ 14.75 \\ 8.60 \end{bmatrix} - \begin{bmatrix} 3.79 \\ 2.95 \\ 1.72 \end{bmatrix} = \begin{bmatrix} 15.16 \\ 11.80 \\ 6.88 \end{bmatrix}$$

Examples: Matrix Multiplication

Example

$$\begin{pmatrix} 5 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} = 10 - 2 - 12 = -4.$$

Example

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 4 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 25 & 7 \\ 15 & 1 \end{pmatrix}$$

Examples: Matrix Multiplication

Example

$$\text{If } A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 0 & -1 \end{pmatrix},\ B = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & 3 \\ 2 & 3 & -1 \end{pmatrix}$$

then

$$AB = \begin{pmatrix} 1 & 8 & 17 \\ 0 & -5 & 5 \end{pmatrix}$$

Example

Let

$$A = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix}$$

If $AB = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$, find x and y.

Answer: $x = -2$, $y = 6$.

Examples: Matrix Multiplication

Last month, Nguyen bought 3 pencils, 3 notebooks, and 1 eraser. This month, he bought 3 pencils, 2 notebooks, and 5 erasers. The prices at IU are \$2, \$5 and \$1 respectively. Hence

$$QUANTITY \cdot PRICE = COST$$

$$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 21 \end{bmatrix}$$

Exercise

Multiply

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{pmatrix}, B = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$$

Exercises

A store brand X and brand Y dishwashers. The following matrices give the sales figures and costs of these items for three months. Use matrix multiplication to determine the total dollar sales and total costs of these items for the three months.

$$\begin{array}{lc} & \begin{array}{ccc} Dec. & Apr. & Aug. \end{array} \\ \begin{array}{l} Brand\ X \\ Brand\ Y \end{array} & \begin{bmatrix} 18 & 10 & 12 \\ 19 & 12 & 14 \end{bmatrix} \end{array}, \qquad \begin{array}{lc} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{l} Retail\ Price \\ Dealer\ Cost \end{array} & \begin{bmatrix} 350 & 260 \\ 240 & 190 \end{bmatrix} \end{array}$$

Properties of Matrix Addition

Theorem

Let A, B, and C be $m \times n$ matrices.
(a) $A + B = B + A$.
(b) $A + (B + C) = (A + B) + C$.
(c) There is a unique $m \times n$ matrix O such that $A + O = A$, for any $m \times n$ matrix A. The matrix O is called the $m \times n$ zero matrix
(d) For each $m \times n$ matrix A, there is a unique $m \times n$ matrix D such that $A + D = O$, where O is the $m \times n$ zero matrix.
We will write D as $-A$.

Proof:
(a) Let $A = [a_{ij}]$, $B = [b_{ij}]$, $A + B = C = [c_{ij}]$, and $B + A = D = [d_{ij}]$.
By the definition of matrix addition, we have $c_{ij} = a_{ij} + b_{ij}$ and $d_{ij} = b_{ij} + a_{ij}$. Since a_{ij} and b_{ij} are real numbers, we get $a_{ij} + b_{ij} = b_{ij} + a_{ij}$, which implies $c_{ij} = d_{ij}$ for all i, j.
Thus, $C = D$, or $A + B = B + A$.
(b)-(d): Similarly (exercises!).

Properties of Matrix Multiplication

Theorem

If A, B, and C are matrices of the appropriate sizes then
(a) $A(BC) = (AB)C$.
(b) $(A+B)C = AC + BC$.
(c) $C(A+B) = CA + CB$.

Proof:

Properties of Scalar Multiplication

Theorem

If r and s are real numbers and A, B, and C are matrices of the appropriate sizes then
(a) $r(sA) = (rs)A$
(b) $(r+s)A = rA + sA$
(c) $r(A+B) = rA + rB$
(d) $A(rB) = r(AB) = (rA)B$.

Proof:

Remark

Remark: If a, b, and c are real numbers for which $ab = ac$ and $a \neq 0$ it follows that $b = c$. That is, we can cancel out the nonzero factor a. However, the cancellation law does NOT hold for matrices, as the following example shows.

Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$$

$$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix}$$

but $B \neq C$.

Remark

Remark: We also note two other peculiarities of matrix multiplication. If a and b are real numbers, then $ab = 0$ can hold only if a or b is zero. However, this is not true for matrices.

Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$$

but

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Special Types of Matrices

- An $n \times n$ matrix $A = [a_{ij}]$ is called a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 \cdots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

- An $n \times n$ diagonal matrix whose entries on the diagonal are all 1 is called an identity matrix (or a unit matrix). It is denoted by I_n or simply I.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  It is easy to show that $I_n A = A I_n$ for any square $n \times n$ matrix A.
- A scalar matrix is a diagonal matrix whose diagonal elements are equal.

Special Types of Matrices

- Suppose that A is a square matrix. The powers of a matrix, A^p, for p a positive integer, is defined by

$$A^p = A.A...A \qquad \text{(p factors)}$$

  We also define A^0 by the $n \times n$ identity matrix, I_n.

- An $n \times n$ matrix $A = [a_{ij}]$ is called upper triangular if $a_{ij} = 0$ for $i > j$. Example

$$A = \begin{bmatrix} 1 & 7 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

- An $n \times n$ matrix $A = [a_{ij}]$ is called lower triangular if $a_{ij} = 0$ for $i < j$. Example:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 4 & 2 \end{bmatrix}$$

Transpose, Symmetric

- The transpose of an $m \times n$ matrix $A = [a_{ij}]$, denoted by A^T is obtained by interchanging the rows and columns of A, i.e.

$$[a_{ij}]^T = [a_{ji}]$$

Example: Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 8 & 7 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & -4 \\ 2 & 8 \\ 3 & 7 \end{pmatrix}$$

- A matrix A is symmetric if $A^T = A$. Example:

$$A = \begin{pmatrix} 1 & -1 & 4 \\ -1 & 0 & 2 \\ 4 & 2 & 6 \end{pmatrix}$$

Properties of Transpose

Theorem

If r and s are real numbers and A, B, and C are matrices of the appropriate sizes then
(a) $(A^T)^T = A$
(b) $(A + B)^T = A^T + B^T$
(c) $(AB)^T = B^T A^T$
(d) $(rA)^T = r(A^T)$.

Proof:

Example: $(AB)^T = B^T A^T$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 12 & 5 \\ 7 & -3 \end{bmatrix}, (AB)^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}$$

On the other hand,

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix}, B^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix} = (AB)^T$$

Skew Symmetric

- A matrix A with real entries is called skew symmetric if $A^T = -A$.
  Example:
  $$A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

  Remark:

- If A is symmetric or skew symmetric, then A is a square matrix.
- If A is a symmetric matrix, then the entries of A are symmetric with respect to the main diagonal of A.
- A is symmetric if and only if $a_{ij} = a_{ji}$, and A is skew symmetric if and only if $a_{ij} = -a_{ji}$.
- If A is an $n \times n$ matrix, then we can show that $A = S + K$, where S is symmetric and K is skew symmetric (Exercise).

Diagonal

- Let A be a square, $n \times n$, matrix. Then its (principal) diagonal consists of the entries $a_{11}, a_{22}, ..., a_{nn}$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Remark on representing linear systems via matrices

Consider the linear system of m equations in n unknowns,

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

Then the linear system can be re-written as $AX = b$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, X = (x_1, x_2, ..., x_n)^T \text{and } b = (b_1, b_2, ..., b_m)^T$$

Q: Can we find X as "$X = A^{-1}b$"? And what is "A^{-1}"?

Section 5

INVERSE MATRIX

Inverse of a Matrix

Definition

Let A be an $n \times n$ matrix. A matrix B is said to be the inverse of A if

$$AB = BA = I_n,$$

where I_n is the $n \times n$ identity matrix. In this case, we denote B by A^{-1}. And A is said to be invertible or nonsinglar.

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Inverse of a Matrix

- If $AB = I_n$, then $BA = I_n$ (why?).

  Thus, to verify that B is an inverse of A, we need verify only that $AB = I_n$.

Theorem

The inverse of a matrix, if it exists, is unique.

Proof: Let B and C be inverses of A. Then $AB = BA = I_n$ and $AC = CA = I_n$. We then have

$$B = BI_n = B(AC) = (BA)C = I_nC = C,$$

which proves that the inverse of a matrix, if it exists, is unique.

Inverse of a Matrix

Theorem

If A and B are both nonsingular $n \times n$ matrices, then AB is nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof:...

Hint: Show that $(AB)(B^{-1}A^{-1}) = I_n$ and $(B^{-1}A^{-1})(AB) = I_n$.

Corollary

If A_1, A_2,..., and A_k are nonsingular $n \times n$ matrices, then $A_1A_2...A_k$ is nonsingular and

$$(A_1A_2...A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}...A_1^{-1}$$

Inverse of a Matrix

Theorem

If A is a nonsingular matrix, then A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

Hint: Show that $(A^{-1})A = A(A^{-1}) = I_n$!

Theorem

If A is a nonsingular matrix, then A^T is nonsingular and

$$(A^{-1})^T = (A^T)^{-1}$$

Proof: Taking transposes of the equation $AA^{-1} = I_n$ both sides, we get

$$\left(A^{-1}\right)^T A^T = I_n^T = I_n$$

Similarly, $A^T \left(A^{-1}\right)^T = (A^{-1}A)^T = I_n$. These equation implies that the inverse of A^T is $\left(A^{-1}\right)^T$.

Inverse of a Matrix

Note: If A is nonsingular. Then $AB = AC$ implies that $B = C$ (Exercise).

Theorem

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $ad - bc \neq 0$ then A is nonsingular and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\to A^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

Elementary Matrices

Definition

Let e be an elementary row operation. Then the $n \times n$ elementary matrix E associated with e is the matrix obtained by applying e to the $n \times n$ identity matrix. Thus

$$E = e(I)$$

Example

$$M_3 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -5 & \\ & & & 1 \end{bmatrix}$$

Multiply the third row by -5. the M is for "multiply" and the subscript 3 is for the row acted upon.

Elementary Matrices

Example

$$P_{12} = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix}$$

Permute (or interchange) the first and second rows. The P is for "permute" and the subscript indicate the rows interchanged.

Example

$$E_{12} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix}$$

Add -3 times row 1 to row 2. The E stands for "elementary" and the sub-scripts 1 and 2 represent the rows acted upon.

Elementary Matrices

Theorem

Let e be an elementary operation and let E be the corresponding $m \times m$ elementary matrix $E = e(I)$. Then for every $m \times n$ matrix A,

$$e(A) = EA$$

That is, an elementary row operation can be performed on A by multiplying A on the left by the corresponding elementary matrix.

Example

$$A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}, M_1 = \begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

M_1 was obtained from I_3 by multiplying the first row by 2.

Elementary Matrices

Example (Cont.)

$$A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} \xrightarrow{(2R_1)\to R_1} e(A) = \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

And

$$M_1 A = \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

As can be seen, $e(A) = M_1 A$

Elementary Matrices

Theorem

Each elementary matrix is invertible and its inverse is an elementary matrix of the same type.

The theorem in the previous slide state that if we can obtain a matrix $B := I_n$ from a matrix A by performing elementary row operations, then we have $E_k ... E_2 E_1 A = B = I_n$. Thus, we can get from B back to A:

$$A = E_1^{-1} E_2^{-1} ... E_k^{-1} I_n$$

Therefore, $A^{-1} = E_k ... E_1 I_n$.

This leads to the following method to find A^{-1}!

An algorithm for finding A^{-1}

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1}.

ALGORITHM FOR FINDING A^{-1} If A is row equivalent to I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, A does not have an inverse.

An algorithm for finding A^{-1}

To obtain A^{-1} for any $n \times n$ invertible matrix A, follow these steps

Algorithm for finding A^{-1}

1. Form the matrix $[A \quad I]$, where I is the identity matrix.
2. Perform row operations on $[A \quad I]$ to get a matrix of the form $[I \quad B]$.
3. Matrix B is A^{-1}.

Finding A^{-1}

Example

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 6 & 7 \end{bmatrix}$$

Solution: Step 1: We form the matrix $[A|I]$ as

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{array}\right]$$

Solution (Cont.)

Step 2: Using Row Operations on $[A|I]$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1\to R_3]{R_2-R_1\to R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-3R_2\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-3R_3\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] = [I|A^{-1}]$$

Step 3: Conclusion

$$A^{-1} = \begin{bmatrix} 10 & -3 & -3 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

An algorithm for finding A^{-1}

Example

Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix},$$

if it exists.

$$[\,A \quad I\,] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

An algorithm for finding A^{-1}

Example (Cont.)

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Since $A \sim I$, A is nonsingular and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

Check: $AA^{-1} = I_3$ (?).

An algorithm for finding A^{-1}

Example

Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

$$[A \quad I] \sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}$$

So $[A \ I]$ is row equivalent to a matrix of the form $[B \ D]$, where B has a row of zeros. Further row operations will not transform B into I. Thus, A does NOT have the inverse.

An algorithm for finding A^{-1}

Theorem

If A is an invertible (nonsingular) $n \times n$ matrix , then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

Example

Solve the system

$$\begin{cases} 3x_1 + 4x_2 = 3 \\ 5x_1 + 6x_2 = 7 \end{cases}$$

This system is equivalent to $Ax = b$, where $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, so

$$x = A^{-1}b = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Homeworks

Textbook: B. Kolman and David R. Hill, Elementary Linear Algebra with Applications, 9th edition, Prentice Hall, 2008

-Section 1.1: 3, 4, 8, 12

-Section 1.2: 6, 8

-Section 1.3: 11

-Section 2.1: 1, 2

-Section 2.2: 2, 8

-Section 2.3: 11

Deadline: March 4th, 2022