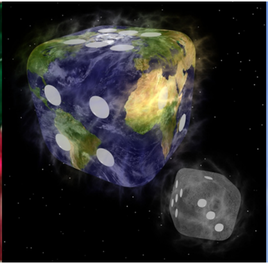
CHAPTER 7: Properties of Expectation

Lecturer: Nguyen Minh Quan, PhD
Department of Mathematics
HCMC International University



CONTENTS

Expectation of Sums of Random Variables

Proposition

If X and Y have a joint probability mass function $p(x, y)$, then

$$E\left[g\left(X, Y\right)\right] = \sum_{y} \sum_{x} g\left(x, y\right) p\left(x, y\right)$$

If X and Y have a joint probability density function $f(x, y)$, then

$$E\left[g\left(X, Y\right)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g\left(x, y\right) f\left(x, y\right) dxdy$$

For an important application of this proposition, suppose that $E(X)$ and $E(Y)$ are both finite and let $g(X, Y) = X + Y$:

$$E\left(X + Y\right) = E\left(X\right) + E\left(Y\right)$$

Expectation of Sums of Random Variables

By induction, we have

$$E\left[X_1 + X_2 + ... + X_n\right] = E\left(X_1\right) + ... + E\left(X_n\right)$$

Example: The sample mean

Let $X_1, \cdots, X_n$ be independent and identically distributed (i.i.d.) random variables having distribution function F and expected value μ. Such a sequence of random variables is said to constitute a sample from the distribution F. The quantity

$$\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$$

is called the sample mean. Compute $E(\overline{X})$

$E\left(\overline{X}\right) = \frac{1}{n} \sum_{i=1}^{n} E\left(X_i\right) = \mu.$

Expectation of Sums of Random Variables

Example: Expectation of a binomial random variable

Let X be a binomial random variable with parameters n and p. Recalling that such a random variable represents the number of successes in n independent trials when each trial has probability p of being a success, we have that

$$X = X_1 + X_2 + ... + X_n$$

where

$$X_i = \begin{cases} 1, \text{ if the ith trial is a sucess} \\ 0 \text{ if the ith trial is a failure} \end{cases}$$

Hence, X_i is a Bernoulli random variable having expectation $E(X_i) = 1(p) + 0(1-p) = p$.
Thus,

$$E(X) = E(X_1) + ... + E(X_n) = np$$

Covariance

Proposition

If X and Y are independent, then, for any functions h and g,

$$E\left[g\left(X\right)h\left(Y\right)\right] = E\left[g\left(X\right)\right]E\left[h\left(Y\right)\right]$$

Definition

The covariance between X and Y, denoted by $Cov(X, Y)$, is defined by

$$Cov\left(X, Y\right) = E\left[\left(X - E\left(X\right)\right)\left(Y - E\left(Y\right)\right)\right]$$

That is,

$$Cov\left(X, Y\right) = E\left[XY\right] - E\left(X\right)E\left(Y\right)$$

Covariance

Note that if X and Y are independent then $Cov(X, Y) = 0$.

Q: Is the converse true?

A: No!

Example

A simple example of two dependent random variables X and Y having zero covariance is obtained by letting X be a random variable such that $P(X = 0) = P(X = 1) = P(X = -1) = 1/3$ and defining

$$Y = \begin{cases} 0, & \text{if } X \neq 0 \\ 1, & \text{if } X = 0 \end{cases}$$

We have $XY = 0$, so $E(XY) = 0$, also it is easy to check that $E(X) = 0$. Therefore,

$$Cov(X, Y) = E(XY) - E(X)E(Y) = 0$$

Covariance

Properties of Covariance

(i) $Cov\,(X,Y) = Cov\,(Y,X)$
(ii) $Cov\,(X,X) = Var\,(X)$
(iii) $Cov\,(aX,Y) = aCov\,(X,Y)$
(iv) $Cov\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{m} Y_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov\,(X_i, Y_j)$

It follows from parts (ii) and (iv) that

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var\,(X_i) + 2\sum_{i<j}\sum Cov\,(X_i, X_j)$$

Covariance

Example

Let $X_1, \cdots, X_n$ be independent and identically distributed random variables having expected value μ and variance σ^2, and let $\bar{X} = \sum_{i=1}^{n} X_i$ be the sample mean. The quantities $X_i - \bar{X}$, $i = 1, \cdots, n$, are called deviations, as they equal the differences between the individual data and the sample mean. The random variable

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

is called the sample variance. Find (a) $Var(X)$ and (b) $E[S^2]$.

Covariance

Solution

(a)

$$Var\left(\overline{X}\right) = \left(\frac{1}{n}\right)^2 Var\left(\sum_{i=1}^{n} X_i\right) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} Var\left(X_i\right) = \frac{\sigma^2}{n}$$

(b)

$$(n-1)\, S^2 = \sum_{i=1}^{n} \left(X_i - \mu + \mu - \overline{X}\right)^2 = \sum_{i=1}^{n} \left(X_i - \mu\right)^2 - n\left(\overline{X} - \mu\right)^2$$

$$(n-1)\, E\left(S^2\right) = \sum_{i=1}^{n} E\left[\left(X_i - \overline{X}\right)^2\right] - nE\left[\left(\overline{X} - \mu\right)^2\right]$$

$$= n\sigma^2 - nVar\left(\overline{X}\right) = (n-1)\, \sigma^2$$

Covariance

Example

Compute the variance of a binomial random variable X with parameters n and p.

Solution

Note that

$$X = X_1 + X_2 + ... + X_n$$

where

$$X_i = \begin{cases} 1, & \text{if the ith trial is a sucess} \\ 0 & \text{if the ith trial is a failure} \end{cases}$$

$$Var\left(X_i\right) = p - p^2 \rightarrow Var\left(X\right) = np\left(1 - p\right)$$

Conditional Expectation

Definition

Let X and Y be jointly discrete random variables, we define the conditional expectation of X given that $Y = y$, for all values of y such that $p_Y(y) > 0$

$$E[X|\,Y = y] = \sum_x xP(X = x|\,Y = y) = \sum_x xp_{X|Y}(x|\,y)$$

where

$$p_{X|Y}(x|\,y) = \frac{p(x,y)}{p_Y(y)}$$

Similarly, for continuous case,

$$E[X|\,Y = y] = \int_{-\infty}^{\infty} xf_{X|Y}(x|\,y)\,dx$$

provided that $f_Y(y) > 0$.

Conditional Expectation

Example

Suppose that the joint density of X and Y is given by

$$f\left(x,y\right)=\frac{e^{-x/y}e^{-y}}{y},0<x<\infty,0<y<\infty$$

Compute $E[X|Y=y]$.

Solution

$$f_{X|Y}\left(x|\,y\right)=\frac{f\left(x,y\right)}{f_{Y}\left(y\right)}=\frac{\frac{e^{-x/y}e^{-y}}{y}}{\int_{-\infty}^{\infty}f\left(x,y\right)dx}$$

$$f_{X|Y}\left(x|\,y\right)=\frac{1}{y}e^{-x/y}$$

$$E\left[X|\,Y=y\right]=\int_{-\infty}^{\infty}\frac{x}{y}e^{-x/y}dx=y$$

Moment Generating Functions

Definition

The moment generating function $M(t)$ of the random variable X is defined for all real values of t by

$$M(t) = E[e^{tX}]$$

That is, if X is discrete with mass function $p(x)$ then

$$M(t) = E\left[e^{tX}\right] = \sum_x e^{tx} p(x)$$

If X is continuous with density function $f(x)$ then

$$M(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Moment Generating Functions

We call $M(t)$ the moment generating function because all of the moments of X can be obtained by successively differentiating $M(t)$ and then evaluating the result at $t = 0$.

For example,

$$M'(t) = \frac{d}{dt} E\left[e^{tX}\right] = E\left[\frac{d}{dt}\left(e^{tX}\right)\right] = E\left[Xe^{tX}\right]$$

Therefore,

$$M'(0) = E[X]$$

Similarly, $M''(t) = \frac{d}{dt} M'(t) = E\left(X^2 e^{tX}\right)$. Thus, $M''(0) = E\left[X^2\right]$.

In general, $M^{(n)}(0) = E[X^n]$.

Moment Generating Functions

Example: Poisson distribution with mean λ

If X is a Poisson random variable with parameter λ, then

$$M(t) = E\left(e^{tX}\right) = \sum_{n=0}^{\infty} e^{tn} \frac{e^{-\lambda}\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\left(\lambda e^t\right)^n}{n!}$$

$$M(t) = e^{-\lambda} e^{\lambda e^t} = \exp\left[\lambda\left(e^t - 1\right)\right]$$

$$E(X) = M'(t) = \lambda$$

$$E\left(X^2\right) = \lambda^2 + \lambda$$

$$Var(X) = E\left(X^2\right) - \left(E(X)^2\right) = \lambda$$

Moment Generating Functions

Example: Moment Generating Function of Normal distribution

We first compute the moment generating function of a unit normal random variable Z with parameters 0 and 1.

$$M_Z(t) = E\left[e^{tZ}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2} dx$$

$$M_Z(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-t)^2}{2} + \frac{t^2}{2}\right] dx = e^{t^2/2}$$

Moment Generating Functions

For $X = \mu + \sigma Z$, we have

$$M_X(t) = E\left[e^{tX}\right] = E\left[e^{t(\mu+\sigma Z)}\right] = e^{t\mu} M_Z(t\sigma)$$

$$M_X(t) = \exp\left\{\frac{\sigma^2 t^2}{2} + \mu t\right\}.$$

Therefore,

$$E(X) = M'(0) = \mu$$

$$E\left(X^2\right) = M''(0) = \mu^2 + \sigma^2.$$

-END OF CHAPTER 7-