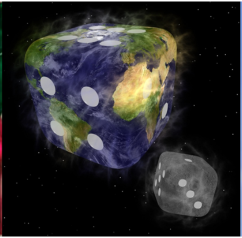
CHAPTERS 1-2: PRINCIPLE OF COUNTING. AXIOMS OF PROBABILITY

Lecturer: Nguyen Minh Quan, PhD
quannm@hcmiu.edu.vn


CONTENTS

Introduction

Q: Why do we need to study probability?

Answers:

- Any realistic model of a real-world phenomenon must take into account the possibility of randomness.
- Some biological processes seem to be directly affected by chance outcomes, e.g., occurance of genetic mutations.
- Formal statistical analysis of biological data models unexplained variation as caused by chance.
- Applications in many fields of sciences: Mathematics, Computer Science, Electrical Engineering, Optical Communications, Economics, Biology,...

The Basic Principle of Counting

Principle of Counting

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Example

How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Answer: $26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10$.

Permutations

Permutations

How many different ordered arrangements of the letters a, b, and c are possible?
By direct enumeration we see that there are 6, namely, abc, acb, bac, bca, cab, and cba. Each arrangement is known as a permutation.

Example

A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.
(a) How many different rankings are possible?
(b) If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?

Answer: (a) $10!$, (b) $6!4!$.

Permutations

Permutations

How many different letter arrangements can be formed from the letters PEPPER?

Answer: $\dfrac{6!}{3!2!}$.

Example

A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

Answer: $\dfrac{10!}{4!3!1!}$.

Combinations

Notation and terminology

We define $\begin{pmatrix} n \\ r \end{pmatrix}$, for $r \leq n$, by

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{(n-r)!r!}$$

and say that $\begin{pmatrix} n \\ r \end{pmatrix}$ represents the number of possible combinations of n objects taken r at a time.

Example

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Answer: $\begin{pmatrix} 20 \\ 3 \end{pmatrix}$

Review of Combinatorial Analysis

Example

A dance class consists of 22 students, of which 10 are women and 12 are men. (a) If 5 men and 5 women are to be chosen, how many results are possible?
(b) If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

Answer
(a) $\binom{10}{5}\binom{12}{5}$
(b) ?

Review of Combinatorial Analysis

Remark:

$$\begin{pmatrix} n \\ n_1, n_2, .., n_k \end{pmatrix} = \frac{n!}{n_1!n_2!..n_k!}$$

is the number of divisions of n items into k distinct nonoverlapping subgroups of sizes $n_1, n_2, ..., n_k$.

Example

The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many bridge deals are possible?

Sample spaces and events

- The sample space is the set of all possible outcomes, denoted by S or Ω. We will use **both notations** S or Ω during the course.
- An *event* is a single outcome or *a set of outcomes*, denoted by E.
- Mutually exclusive events are events that can **NOT** both happen at the same time.

Example

Rolling a fair 6-sided die. Outcomes: 1, 2, 3, 4, 5, 6. E_i=event of rolling an i. $E_{odd} = \{1, 3, 5\}$, $E_{even} = \{2, 4, 6\}$. E_1 and E_2 are mutually exclusive events.

Sample spaces

Example

If the experiment consists of the running of a race among the seven horses having post positions $1, 2, 3, 4, 5, 6, 7$ then

$$\Omega = \{\text{all orderings of } (1,2,3,4,5,6,7)\}.$$

Example

If the experiment consists of measuring the lifetime of a PC, then the sample space consists of all nonnegative real numbers. That is, $S = [0, \infty)$

Example

If the experiment consists of rolling two dice, then the sample space is

$$S = \left\{\begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{matrix}\right\}$$

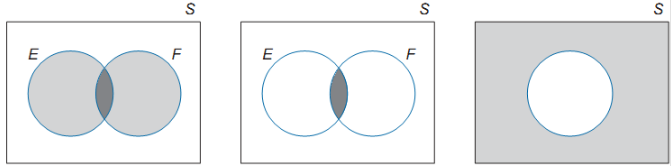
Events

Suppose that E and F are two events of a sample space S.

- We define the new event $E \cup F$, called the union of the events E and F, to consist of all outcomes that are either in E or in F or in both E and F.
- We define the new event EF, called the intersection of E and F, to consist of all outcomes that are in both E and F. Note that $EF \equiv E \cap F$.
- We define the event E^c , referred to as the complement of E , to consist of all outcomes in the sample space S that are not in E.

Venn diagrams and the algebra of events

A graphical representation of events that is very useful for illustrating logical relations among them is the Venn diagram.

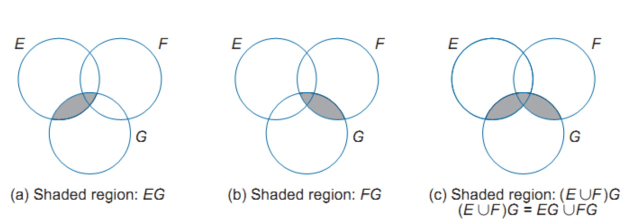


(a) Shaded region: $E \cup F$ (b) Shaded region: EF (c) Shaded region: E^c

Commutative law	$E \cup F = F \cup E$	$EF = FE$
Associative law	$(E \cup F) \cup G = E \cup (F \cup G)$	$(EF)G = E(FG)$
Distributive law	$(E \cup F)G = EG \cup FG$	$EF \cup G = (E \cup G)(F \cup G)$

Venn diagrams and the algebra of events

The distributive law may be verified by the following sequence of diagrams:



DeMorgan's laws:

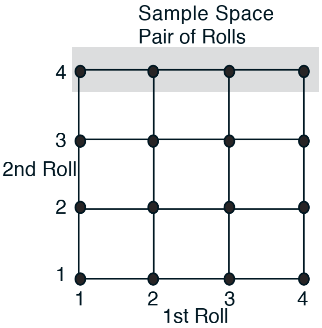
$$(E \cup F)^c = E^c F^c, \quad (EF)^c = E^c \cup F^c$$

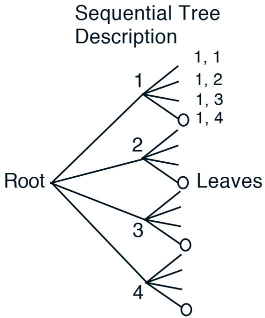
In general,

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c, \quad \left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$$

Probability model: Sequential model

- Many experiments have an inherently sequential character, such as observing the value of a stock on five successive days. It is useful to describe the experiment by means of a tree-based sequential description. For example, the following two equivalent tree diagrams for tossing a 4-sided die twice:





$P(\{\text{the second roll is } 4\}) = 4/16.$

Axioms of probability

- For every event E, we denote $P(E)$ by the probability of the event E. It measures the relative likelihood that a random event will occur.

- Axioms of probability.
  Axiom 1 (Nonnegativity). $0 \leqslant P(E) \leqslant 1$.

  Axiom 2 (Normalization). $P(S) = 1$ where S is the sample space.

  Axiom 3 (Additivity). If $E_1, E_2, ..$ is a sequence of mutually exclusive events (that is, $E_i E_j = \emptyset$ for all i, j) then

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i), \quad n \text{ can be a positive interger or } \infty$$

  Note: Axiom 3 states that for any set of mutually exclusive events the probability that at least one of these events occurs is equal to the sum of their respective probabilities.

Probability measure

Definition: σ-algebra [σ-field]

Let Ω be a sample space. A σ-algebra on Ω is a non-empty set $\mathcal{F}$ of subsets of Ω such that
(1) if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$,
(2) if $A_1, A_2, \dots \in \mathcal{F}$ then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

Example of σ-algebra

(1) The trivial σ-algebra: $\mathcal{F} = \{\emptyset, \Omega\}$.

(2) $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$, where A is a fixed subset of Ω.

Probability measure and Kolmogorov's axioms

Definition: Probability measure and Kolmogorov axioms

Let Ω be a set and let $\mathcal{F}$ be a σ-algebra on Ω. A probability measure on $\mathcal{F}$ is a function $P : \mathcal{F} \to [0, 1]$ such that

(1) if $A_1, A_2, \dots \in F$ are disjoint then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

[Countable additivity.]

(2) $P(\Omega) = 1$ (Normalization).

Remark: For any $A \in \mathcal{F}$, $P(A)$ is called the probability of the event A.

Probability space

Definition: Probability space

Let Ω be a set, $\mathcal{F}$ a σ-algebra on Ω, and P a probability measure on $\mathcal{F}$. The triple (Ω, F, P) is called a probability space.

- The set Ω is called the sample space, and an element ω of Ω is called an outcome.
- A subset of Ω which is an element of $\mathcal{F}$ is called an event
- Let $A \in \mathcal{F}$ be an event. If $P(A) = 1$ then A is called an almost sure event, and if $P(A) = 0$ then A is called a null event.

Probability space

Example

If the experiment consists of a single roll of an ordinary die, the natural sample space is the set $\Omega = \{1, 2, 3, 4, 5, 6\}$, consisting of 6 elements. The outcome $\omega = 2$ indicates that the result of the roll was 2.

Example

If the experiment consists of five consecutive rolls of an ordinary die, the natural sample space is the set $\Omega = \{1, 2, 3, 4, 5, 6\}^5$. The element $\omega = (2, 1, 1, 3, 6)$ is an example of a possible outcome.

Example

If the experiment consists of measuring the velocity of a vehicle with infinite precision, a natural sample space is the set $\mathbb{R}$ of real numbers.

Probability space

Example

Consider a single toss of a coin. If we believe that heads (H) and tails (T) are equally likely, the following is an appropriate model. We set $\Omega = \{\omega_1, \omega_2\}$, where $\omega_1 = H$ and $\omega_2 = H$, and let $p_1 = p_2 = 1/2$. Here, $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$ and $P(\emptyset) = 0$, $P(H) = P(T) = 1/2$, $P(\{H, T\}) = 1$.

Example

Let $\Omega = \mathbb{N}$, and $p_k = (1/2)^k$, for k = 1,2,... More generally, given a parameter $p \in (0, 1)$, we can define $p_k = (1 - p)p^{k-1}$, for k = 1,2,... This results in a legitimate probability space because

$$\sum_{k=1}^{\infty} (1 - p)\, p^{k-1} = 1.$$

Basic Propositions

Theorem

- $P(\emptyset) = 0$.
- $P(E^c) = 1 - P(E)$.
- If $E \subset F$ then $P(E) \leqslant P(F)$.
- $P(E \cup F) = P(E) + P(F) - P(EF)$.

Proof:

Propositions

Corollary

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(FG) - P(EG)$$

$$+P(EFG).$$

Theorem

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \ldots$$

$$+(-1)^{r+1} \sum_{i_1 < i_2 < \ldots < i_r} P(E_{i_1} E_{i_2} \ldots E_{i_r}) + \ldots + (-1)^{n+1} P(E_{i_1} E_{i_2} \ldots E_{i_n})$$

The summation $\sum_{i_1 < i_2 < \ldots < i_r} P(E_{i_1} E_{i_2} \ldots E_{i_r})$ is taken over all of the $\binom{n}{r}$ possible subsets of size r of the set $\{1, 2, \ldots, n\}$.

Propositions

Example

Suppose that we toss two fair coins. Let E be the event that the first coin falls heads, and F be the event that the second coin falls heads. Find the probability that either the first or the second coin falls heads.

Solution

$$E = \{(H,H),(H,T)\}, \text{ and } \mathrm{F} = \{(H,H),(T,H)\}$$

The probability that either the first or the second coin falls heads is

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$P(E \cup F) = \frac{1}{2} + \frac{1}{2} - P(\{H,H\}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Propositions

Example

J is taking two books along on her holiday vacation. With probability 0.5, she will like the first book; with probability 0.4, she will like the second book; and with probability .3, she will like both books. What is the probability that she likes neither book?

Solution Let B_i denote the event that J likes book i, $i = 1, 2$. Then the probability that she likes at least one of the books is

$$P(B_1 \cup B_2) = P(B_1) + P(B_2) - P(B_1 B_2) = 0.5 + 0.4 - 0.3 = 0.6$$

We obtain the result

$$P(\text{J likes neither book}) = P(B_1^c B_2^c) = P\left((B_1 \cup B_2)^c\right) = 1 - 0.6 = 0.4$$

Propositions

Exercise

A total of 28 percent of American males smoke cigarettes, 7 percent smoke cigars, and 5 percent smoke both cigars and cigarettes.
(a) What percentage of males smokes neither cigars nor cigarettes?
(b) What percentage smokes cigars but not cigarettes?

Venn diagrams and the algebra of events

Exercise

An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. The classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students that are in both Spanish and French, 4 that are in both Spanish and German, and 6 that are in both French and German. In addition, there are 2 students taking all 3 classes.

(a) If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?

(b) If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?

(c) If 2 students are chosen randomly, what is the probability that at least 1 is taking a language class?

Finite sample spaces

Let S be a finite sample space, i.e., $S = \{a_1, a_2, ..., a_n\}$. A finite probability space is obtained by assigning to each point $a_i \in S$ a real number p_i, called the probability of a_i, satisfying the following properties:

- Each p_i is nonnegative, $p_i \geq 0$.
- The sum of the p_i is one, $p_1 + p_2 + .. + p_n = 1$.

Example

A coin is weighted so that heads is twice as likely to appear as tails. Find $P(T)$ and $P(H)$.
Answer $P(T) = 1/3$ and $P(H) = 2/3$

Finite equiprobable spaces

A finite probability space S, where each sample point has the same probability (equally likely to occur), is called an equiprobable or uniform space.

$$P(E) = \frac{\text{number of elements in E}}{\text{number of elements in S}}$$

Example

Let a card be selected at random from an ordinary deck of 52 cards. Let $A = \{\text{the card is a spade}\}$, and $B = \{\text{the card is a face}\}$. Find $P(A), P(B)$ and $P(AB)$.

Solution

$$P(A) = \frac{\text{number of spades}}{\text{number of cards}} = \frac{13}{52} = \frac{1}{4}$$

$$P(B) = \frac{\text{number of face cards}}{\text{number of cards}} = \frac{12}{52} = \frac{3}{13}$$

$$P(AB) = \frac{\text{number of spade face cards}}{\text{number of cards}} = \frac{3}{52}$$

Finite equiprobable spaces

Example

Two cards are drawn at random from an ordinary deck of 52 cards. Find the probability p that (a) both are spades, (b) one is a spade and one is a heart.

Answer

(a)

$$\frac{\binom{13}{2}}{\binom{52}{2}} = \frac{78}{1326} = \frac{1}{17}$$

(b) $13/102$.

Finite equiprobable spaces

Example

If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two black?

Answer

$$\frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{4}{11}$$

Finite equiprobable spaces

Example

A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Answer

$$\frac{\begin{pmatrix} 6 \\ 3 \end{pmatrix}\begin{pmatrix} 9 \\ 2 \end{pmatrix}}{\begin{pmatrix} 15 \\ 5 \end{pmatrix}} = \frac{240}{1001}$$

Finite equiprobable spaces

Example

An urn contains n balls, one of which is special. If k of these balls are withdrawn one at a time, with each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

Answer

$$p(\text{ special ball is selected }) = \frac{\binom{1}{1}\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

Finite equiprobable spaces

Example

A total of 36 members of a club play tennis, 28 play squash, and 18 play badminton. Furthermore, 22 of the members play both tennis and squash, 12 play both tennis and badminton, 9 play both squash and badminton, and 4 play all three sports. How many members of this club play at least one of three sports?

Hint

Let N denote the number of members of the club. Let T be the set of members that plays tennis, S be the set that plays squash, and B be the set that plays badminton. We compute

$$P\left(T \cup S \cup B\right) = \frac{43}{N}$$

Hence, we can conclude that 43 members play at least one of the sports.

Probability as a Continuous Set Function

Definition

A sequence of events $\{E_n, n \geqslant 1\}$ is said to be an increasing sequence if

$$E_1 \subset E_2 \subset \dots \subset E_n \subset E_{n+1} \subset \dots$$

whereas it is said to be a decreasing sequence if

$$E_1 \supset E_2 \supset \dots \supset E_n \supset E_{n+1} \supset \dots$$

If $\{E_n, n \geqslant 1\}$ is an increasing sequence of events, then we define a new event, denoted by $\lim_{n\to\infty} E_n$, by

$$\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i$$

Similarly, if $\{E_n, n \geqslant 1\}$ is a decreasing sequence of events, we define $\lim_{n\to\infty} E_n$ by $\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_i$.

Probability as a Continuous Set Function

Example

If $\{E_n, n \geqslant 1\}$ is either an increasing or a decreasing sequence of events, then

$$\lim_{n\to\infty} P(E_n) = P\left(\lim_{n\to\infty} E_n\right)$$

Probability as a Measure of Belief

$P(A)$ can be interpreted either as a long-run relative frequency or as a measure of one's degree of belief.

Example

Suppose that, in a 7-horse race, you feel that each of the first 2 horses has a 20 percent chance of winning, horses 3 and 4 each have a 15 percent chance, and the remaining 3 horses have a 10 percent chance each. Would it be better for you to wager at even money that the winner will be one of the first three horses or to wager, again at even money, that the winner will be one of the horses 1, 5, 6, and 7?

Solution.

On the basis of your personal probabilities concerning the outcome of the race, your probability of winning the first bet is $.2 + .2 + .15 = .55$, whereas it is $.2 + .1 + .1 + .1 = .5$ for the second bet.
Hence, the first wager is more attractive.

–End of Chapters 1-2. Thank you!–