Decision Making

(for Financial Engineering & Risk Management program)

Prof. DrSc. Nguyen Dinh

Department of mathematics
INTERNATIONAL UNIVERSITY, VNU-HCM

March 31, 2020

Contents

Three categories of decision making process:

• Decision making under certainty (in which the data are well defined),

• Decision making under risk (in which the payoffs associated with each decision alternative are described by probability distributions),

• Decision making under uncertainty (in which the data are ambiguous).

This chapter concerns all the three categories.

Under the condition of risk, the payoffs associated with each decision alternative are described by probability distributions.

3.1. Decision making without experimentation

Example 1. Jack wants to invest $ 10,000 in the stock market by buying shares of one of the two companies: A and B. The decision making problem can be summarized as:

Table A

Decision alternative	1-year return on $10,000 investment: "Bull" market ($)	"Bear" market ($)
Company A stock	5000	-2000
Company B stock	1500	500
Probability of occurrence	0.6	0.4

- Maximim payoff criterion
- Maximum likelihood criterion
- **Expected value criterion**

The expected value criterion seeks the maximization of expected (average) profit or the minimization of expected cost. The data of the problem assumes that the payoff (or cost) associated with each decision alternative is probabilistic.

Applying the Expected value criterion to Example 1, the expected values (payoffs for 1-year return) for the two alternatives are:

$$\begin{aligned} \text{For stock A}: \quad & 5000 \times 0.6 + (-2000) \times 0.4 = 2200(\$) \\ \text{For stock B}: \quad & 1500 \times 0.60 + 500 \times 0.40 = 1100(\$) \end{aligned}$$

Based on these results, Jack's decision is to invest in stock A.

General situation

In general situation, a decision problem may include n states of nature and m alternatives.

• Let $p_j > 0$ be the probability of occurrence for the state of nature j. Note that $p_1 + p_2 + \cdots + p_n = 1$.

• Let a_{ij} be the payoff of alternative i given state of nature j $(i = 1, 2, \cdots, m,\ j = 1, 2, 3, \cdots, n)$.

Then the expected payoff for the alternative i (denoted by EV_i) is:

$$EV_i = a_{i1}p_1 + a_{i2}p_2 + \cdots + a_{in}p_n, \quad i = 1, 2, \cdots m.$$

The best alternative V_{i_0} is the one with

$$EV_{i_0} = \max\{EV_i \mid i = 1, 2, \cdots, m\}.$$

3.1. Decision making without experimentation

Figure 3

State of nature / Alternative	SN1	SN2	SN3	...	SNn	Expected payoff
A1	a_{11}	a_{12}	a_{13}	... a_{1j}	a_{1n}	EV_1
A2	a_{21}	a_{22}	a_{23}	... a_{2j}	a_{2n}	EV_2
A3	a_{31}	a_{32}	a_{33}	...	a_{3n}	EV_3
⋮	⋮					⋮
Am	a_{m1}	a_{m2}	a_{m3}	... a_{mj}	a_{mn}	EV_m
probability of occurence	P_1	P_2	P_3		P_n	

$$\sum_{i=1}^{n} P_i = 1$$

$$EV_1 = a_{11}P_1 + a_{12}P_2 + a_{13}P_3 + \dots + a_{1j}P_j + \dots + a_{1n}P_n$$

$$EV_i = a_{i1}P_1 + a_{i2}P_2 + a_{i3}P_3 + \dots + a_{ij}P_j + \dots + a_{in}P_n$$

$i = 1, m$

The best alternative V_{i_0} is the one with

$$EV_{i_0} = \max\{EV_i \mid i = 1, 2, \dots m\}$$

In the previous Subsection 3.1, the expected value criterion (also, other criterions) relies solely on the probabilities available from historical data (called: prior probabilities).

It is natural to think of doing some more experimentation to **improve** the preliminary estimates of these prior probabilities.

The probabilities improved after experimentation are called posterior probabilities. We will **learn the way how to find posterior probabilities via an example**.

Example 2 [Example 1 revisited] Suppose that Jack does not rely solely on the probabilities of "bull" (0.6) or "bear" (0.4) markets available on the financial publications. He wants to conduct a personal investigation by consulting an expert in the stock market.

• The expert offers the general opinion of "for" or "against" the investment.

• The expert's opinion is further quantified in the following manner:

☐ If it is a "bull" market, there is a 90% chance the vote will be "for".

☐ If it is a "bear" market, the chance of a "for" vote is only 50%.

Questions:

1. How to make use of this additional information?

2. If the expert's recommendation is "for", would Jack invest in stock A or B?

3. If the expert's recommendation is "against", would Jack invest in stock A or B?

To process the extra information, we introduce the followings:

- Findings after experimentation:

$$\nu_1 = \text{“for” vote}, \quad \nu_2 = \text{“against” vote}.$$

- States of nature:

$$m_1 = \text{“bull” market}, \quad m_2 = \text{“bear” market}.$$

Then the expert statements can be rewritten as conditional probabilities (This is GIVEN):

□ If it is a “bull” market, there is a 90% chance the vote will be “for”.

$$P(\nu_1 \mid m_1) = 0.9, \quad P(\nu_2 \mid m_1) = 1 - 0.9 = 0.1,$$

□ If it is a “bear” market, the chance of a “for” vote is only 50%.

$$P(\nu_1 \mid m_2) = 0.5, \quad P(\nu_2 \mid m_2) = 1 - 0.5 = 0.5.$$

To process the extra information, we introduce the followings:

- Findings after experimentation:

$$\nu_1 = \text{“for” vote}, \quad \nu_2 = \text{“against” vote}.$$

- States of nature:

$$m_1 = \text{“bull” market}, \quad m_2 = \text{“bear” market}.$$

□ If - “bull” market, there is a 90% chance the vote will be “for”.

□ If - “bear” market, the chance of a “for” vote is only 50%.

These can be summarized as (here $P(\nu_i \mid m_j)$):

	ν_1	ν_2
m_1	0.9	0.1
m_2	0.5	0.5

Table 1.

Problem: How to calculate the posterior (conditional) probabilities $P\{m_i \mid \nu_j\}$? (for what?). Recall

	1-year return on $10,000 investment		
Decision alternative	"Bull" market ($)	"Bear" market ($)	"for" / "against"
Company A	5000	-2000	
Company B	1500	500	
Prob. occur.	?(α)	?(β)	"for"
Prob. occur.	?	?	"against"

- $\alpha = P\{m_1 \mid \nu_1\} = P\{\text{"bull"} \mid \text{"for"}\}$ (known: $P(\nu_i \mid m_j)$)
- If the opinion of the expert is "for" then What are the expectation of A and of B?
- If "for", then $E(A) = 5000\alpha - 2000\beta$
  $E(B) = 1500\alpha + 500\beta$

Problem: How to calculate the posterior probabilities $P\{m_i \mid \nu_j\}$?

Algorithm (for finding $P\{m_i \mid \nu_j\}$)

Step 1. Compute the joint probabilities as [Q1: Where comes (1)?]

$$P\{m_i, \nu_j\} = P\{\nu_j \mid m_i\}P\{m_i\}, \text{ for all } i, j. \tag{1}$$

Note that $P\{m_i, \nu_j\}$ is the probability for the two events m_i and ν_j occurring at the same time.

Taking the prior probabilities $P\{m_1\} = 0.6$, $P\{m_2\} = 0.4$ and the conditional probabilities given in Table 1 into account, the joint probabilities can be computed by (1) and are given in the Table 2 (multiply rows 1 and 2 in Table 1 by 0.6 and 0.4, resp.).

$P\{m_i, \nu_j\}$:

	ν_1	ν_2
m_1	$0.54 = P(m_1, \nu_1)$	$0.06 = P(m_1, \nu_2)$
m_2	0.20	0.20

Table 2.

Step 2. Compute the absolute probabilities $P\{\nu_j\}$ as

$$P\{\nu_j\} = \sum_i P\{m_i, \nu_j\}, \text{ for all } j. \qquad (2)$$

- Ex. $P(\nu_1) = P(m_1, \nu_1) + P(m_2, \nu_1)$

(computed from Table 2 by summing respective columns).

$P\{\nu_1\}$	$P\{\nu_2\}$
0.74	0.26

Table 3.

Recall:

$P\{m_i, \nu_j\}$:

	ν_1	ν_2
m_1	$0.54 = P(m_1, \nu_1)$	$0.06 = P(m_1, \nu_2)$
m_2	$0.20 = P(m_2, \nu_1)$	0.20

Table 2.

Step 3. Determine the desired posterior probabilities $P\{m_i \mid \nu_j\}$ as

$$P\{m_i \mid \nu_j\} = \frac{P\{m_i, \nu_j\}}{P\{\nu_j\}}. \qquad (3)$$

These posterior probabilities for Example 2 are shown in the Table 4 (by dividing each column of Table 2 by the element of the corresponding column of Table 3).

$P\{m_i \mid \nu_j\}$:

	ν_1	ν_2
m_1	0.730	0.231
m_2	0.270	0.769

Table 4.

Recall

$sP\{m_i, \nu_j\}$:

	ν_1	ν_2
m_1	$0.54 = P(m_1, \nu_1)$	$0.06 = P(m_1, \nu_2)$
m_2	$0.20 = P(m_2, \nu_1)$	0.20

Table 2.

$P\{\nu_1\}$	$P\{\nu_2\}$
0.74	0.26

Table 3.

$P\{m_i \mid \nu_j\}$:

	ν_1 "for"	ν_2 "against"
m_1	0.730 (bull, α)	0.231
m_2	0.270 (bear, β)	0.769

Table 4.

Note: These are different from the prior probabilities $P\{m_1\} = 0.6$, $P\{m_2\} = 0.4$.

$P\{m_i \mid \nu_j\}$:

	ν_1 "for"	ν_2 "against"
m_1	0.730 (bull, α)	0.231
m_2	0.270 (bear, β)	0.769

Table 4.

Coming back to the problem and supply with posterior Probability:

Decision alternative	1-year return on $10,000 investment "Bull" market ($)	 "Bear" market ($)	"for" /"against"
Company A	5000	-2000	
Company B	1500	500	
Prob. occur.	$\alpha = 0.730$	$\beta = 0.270$	"for"
Prob. occur.	0.231	0.769	"against"

- $\alpha = P\{m_1 \mid \nu_1\} = P\{\text{"bull"} \mid \text{"for"}\}$
- If "for", $E(A) = 5000 \times 0.730 - 2000 \times 0.270 = 3110(\$)$
  $E(B) = 1500 \times 0.730 + 500 \times 0.270 = 1230\$$
- If "against", $E(A) = 5000 \times 0.231 - 2000 \times 0.769 = -383(\$)$
  $E(B) = 1500 \times 0.231 + 500 \times 0.769 = 731(\$)$

Some Questions:

Q1: One question remains unanswered: Is it worth conducting an experimentation? (e.g., for the problem in Example 2).
For an answer of this question, see [1], Chapter 16, page 694, Subsection "The value of experimentation".

Q2: What would be changed in our previous calculation if Jack is asked to pay \$ 200 for the opinion of the expert?

Assignment 1. [15 marks] I. Apply the method of Section 3 (without and with experimentation) to the Prototype Example in Section 16.1 of [1], pages 683-692.
II. Consider a generalization to case where there are m alternatives and n State of Sciences and consider only the case with experimentation.

Instruction

You work on A4 papers with the following remarks:
1. In the first page, write your full name and your ID.
2. Scan or make photos by your phone... and then change to PDF file and combine all page into ONE file.
3. Name your file by the rule: (files that do not follow this rule are excluded - means NOT accepted)
yourfullname-DM-(5 last digits of your ID)-ASS1.pdf
Example: nguyenvanvu-DM-19027-ASS1.pdf
(to be continued in next page)

4. Submission: Send the PDF file by e-mail to me:
ndinh02@gmail.com
with the SUBJECT: "ASS1, Student of Decision Making class"

5. Very important: Students can discussing/asking together BUT NOT COPY from each other! This is strictly. ANY 2 assignments of any 2 or 3 students (which contain some part(s)) which are the same, all of these students get zero marks.

6. Deadline: 23:00, Sunday, March 29, 2020.
Late submission (-5 marks): 11:00 (AM) Tuesday, next week, March 31, 2020

Coming back to the problem and supply with posterior Probability:

Decision alternative	1-year return on \$10,000 investment: "Bull" market (\$)	"Bear" market (\$)	"for" / "against"
Company A	5000	-2000	
Company B	1500	500	
Prob. occur.	$\alpha = 0.730$	$\beta = 0.270$	"for"
Prob. occur.	0.231	0.769	"against"

- $\alpha = P\{m_1 \mid \nu_1\} = P\{\text{"bull"} \mid \text{"for"}\}$
- If "for", $E(A) = 5000 \times 0.730 - 2000 \times 0.270 = 3110(\$)$
  $E(B) = 1500 \times 0.730 + 500 \times 0.270 = 1230\$$
- If "against", $E(A) = 5000 \times 0.231 - 2000 \times 0.769 = -383(\$)$
  $E(B) = 1500 \times 0.231 + 500 \times 0.769 = 731(\$)$

Extra page (On the algorithm)

What are we doing with the algorithm (for finding $P\{m_i \mid \nu_j\}$)?

Knowing $P\{\nu_i \mid m_j\}$ for all $i, j = 1, 2$, we find $P\{m_j \mid \nu_i\}$. We used:

$$
\begin{aligned}
\text{(step 1)} \quad P\{m_i, \nu_j\} &= P\{\nu_j \mid m_i\} P\{m_i\}, \text{ for all } i, j, && (4)\\
\text{(step 2)} \quad P\{\nu_j\} &= \sum_i P\{m_i, \nu_j\}, \text{ for all } j, && (5)\\
\text{(step 3)} \quad P\{m_i \mid \nu_j\} &= \frac{P\{m_i, \nu_j\}}{P\{\nu_j\}}. && (6)
\end{aligned}
$$

- With "for": We know $P\{\nu_1 \mid m_1\}$, $P\{\nu_1 \mid m_2\}$, then for $i = 1, 2$:

$$
P\{m_i \mid \nu_1\} \overset{(6)}{=} \frac{P\{m_i, \nu_1\}}{P\{\nu_1\}} \overset{(4)}{=} \frac{P\{\nu_1 \mid m_i\} P\{m_i\}}{P\{\nu_1\}}
$$

$$
P\{m_i \mid \nu_1\} \overset{(5)}{=} \frac{P\{\nu_1 \mid m_i\} P\{m_i\}}{P\{m_1, \nu_1\} + P\{m_2, \nu_1\}}.
$$

You know, what is this?

- With "for": We know $P\{\nu_1 \mid m_1\}$, $P\{\nu_1 \mid m_2\}$, then for $i = 1, 2$:

$$P\{m_i \mid \nu_1\} = \frac{P\{\nu_1 \mid m_i\}P\{m_i\}}{P\{m_1, \nu_1\} + P\{m_2, \nu_1\}}.$$

You know, what is this?

- With "against": ν_2 ....

This is the Bayes' theorem (or Bayes' formula)

Recall on Bayes' Theorem

- F and F' are disjoint events
  $F \cup F'$ = Sample space
  E is an event.
  $P(E|F)$ and $P(E|F')$ known
  Then

Bayes's formula

$$\boxed{P(F|E) = \frac{P(E|F) \cdot P(F)}{P(E|F) \cdot P(F) + P(E|F') \cdot P(F')}}$$

- General case
  $F_1, \ldots, F_n$ disjoint events
  $\bigcup_{i=1}^{n} F_i$ = sample space
  E – an event | $P(E|F_i)$ – known

Then

$$\boxed{P(F_i|E) = \frac{P(E|F_i) \cdot P(F_i)}{P(E|F_1) \cdot P(F_1) + \cdots + P(E|F_n) \cdot P(F_n)}}$$

Bayes' formula

Decision tree (method) base on expected value criterion. Consider again the stock investment in Example 1.

3.3.1. The case without experimentation

The problem can also be represented as a decision tree as shown in Figure 13.4.

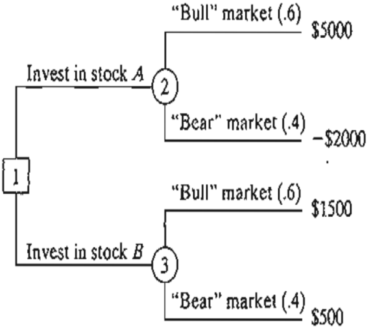


FIGURE 13.4

Decision-tree representation of the stock market problem

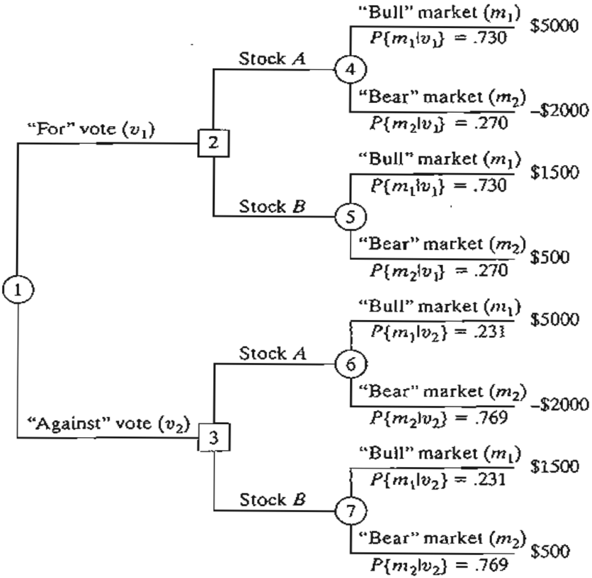
Decision tree base on expected value criterion. Consider again the stock investment in Example 1.
Two types of nodes are used in the tree:

- a square (□) represents a decision point and
- a circle (O) represents a chance event.

Thus, two branches emanate (originate) from decision point 1 to represent the two alternatives of investing in stock A or stock B.

Two branches emanating from chance events 2 and 3 represent the "bull" and the "bear" markets with probabilities and payoffs:
For stock A = \$5000 X .6 + (-2000) X .4 = \$ 2200
For stock B = \$1500 X .6 + \$ 500 X .4 = \$1100
Based on these computations, your decision is to invest in stock A.

Consider Example 2: Decision tree with posterior probabilities



Base on the tree, we can calculate the expectation value for each of the cases and make a decision.

Utility functions

In the preceding presentation, the expected value criterion has been applied to situations where the payoff is real money. There are cases where the utility rather than the real value should be used in the analysis...

Student see the text book (chapter 16).

Decision making under uncertainty, as under risk, involves alternative actions whose payoffs depend on the (random) states of nature.

$$
\begin{array}{c|cccc|}
 & s_1 & s_2 & \cdots & s_n \\
\hline
a_1 & v(a_1, s_1) & v(a_1, s_2) & \cdots & v(a_1, s_n) \\
a_2 & v(a_2, s_1) & v(a_2, s_2) & \cdots & v(a_2, s_n) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_m & v(a_m, s_1) & v(a_m, s_2) & \cdots & v(a_m, s_n) \\
\hline
\end{array}
$$

4. Making decision under uncertainty

• This case , we have: m alternative actions and n states of nature.

• a_i represents action i,
• s_j represents state of nature j,
• $v(a_i, s_j)$ represent the payoff (or outcome) associated with action a_i and state s_j,

• The difference between making a decision under risk and under uncertainty is that in the case of **uncertainty,** the probability distribution associated with the states s_j, $j = 1, 2, ..., n$, is either unknown or cannot be determined.

Criteria for analyzing the decision problem:

1. Laplace criterion
2. Minimax
3. Savage criterion [1]
4. Hurwicz criterion

These criteria differ in how conservative the decision maker is in the face of uncertainty.

[1] t/c "khac khe"

Laplace criterion.

The Laplace criterion is based on the principle of insufficient reason.
"Because the probability distributions are not known, there is no reason to believe that the probabilities associated with the states of nature are different. The alternatives are thus evaluated using the optimistic assumption that **all states are equally likely to occur** - that is,

$$P\{s_1\} = P\{s_2\} = ... = P\{s_n\} = 1/n.$$

The expectation of the state of nature s_1 then is:

$$E(a_i) = \sum_{j=1}^{n} \frac{1}{n} v(a_i, v_j) = \frac{1}{n} \sum_{j=1}^{n} v(a_i, v_j).$$

• payoff $v(a_i, s_j)$ represents gain, the best alternative a_{i_0} is the one that yields:

$$E(a_{i_0}) = \max_i E(a_i).$$

• payoff $v(a_i, s_j)$ represents loss, the best alternative a_{i_0} is the one that yields: ....

$$E(a_{i_0}) = \min_i E(a_i).$$

The Minimax criterion

The maximin (minimax) criterion is based on the conservative attitude of making the best of the worst possible conditions.

• If $v(a_i, s_j)$ is **loss**, then we select the action that corresponds to the minimax criterion

$$\min_i \max_j v(a_i, s_j).$$

• If $v(a_i, s_j)$ is **gain**, then we select the action that corresponds to the miximin criterion:

$$\max_i \min_j v(a_i, s_j).$$

Hurwicz Criterion

The Hurwicz criterion is designed to reflect decision-making attitudes, ranging from the most optimistic to the most pessimistic (or conservative). Define $0 \leq \alpha \leq 1$.

• If $v(a_i, s_j)$ represents the **"gain"**. Then the selected action must be associated with

$$\max_i \left\{ \alpha \max_j v(a_i, s_j) + (1-\alpha) \min_j v(a_i, s_j) \right\}$$

• If $v(a_i, s_j)$ represents the **"loss"**. Then the selected action must be associated with

$$\min_i \left\{ \alpha \min_j v(a_i, s_j) + (1-\alpha) \max_j v(a_i, s_j) \right\}$$

Hurwicz Criterion

Remark. The parameter α is called the index of optimism.

• If $\alpha = 0$, the criterion is conservative because it is the regular minimax criterion.

• If $\alpha = 1$, the criterion produces optimistic results because it seeks the best of the best conditions.

• We can adjust the degree of optimism (or pessimism) through a proper selection of the value of α in the specified $(0, 1)$ range. In the absence of strong feeling regarding optimism and pessimism, $\alpha = 0.5$ may be an appropriate choice.

The Savage regret criterion

[Student red in the file "Decision Game - Taha, page 516 (the material given at the beginning of the chapter)].

Example 13.3-1

National Outdoors School (NOS) is preparing a summer campsite in the heart of Alaska to train individuals in wilderness survival. NOS estimates that attendance can fall into one of four categories: 200, 250, 300, and 350 persons. The cost of the campsite will be the smallest when its size meets the demand exactly. Deviations above or below the ideal demand levels incur additional costs resulting from building surplus (unused) capacity or losing income opportunities when the demand is not met. Letting a_1 to a_4 represent the sizes of the campsites (200, 250, 300, and 350 persons) and s_1 to s_4 the level of attendance, the following table summarizes the cost matrix (in thousands of dollars) for the situation.

	s_1	s_2	s_3	s_4
a_1	5	10	18	25
a_2	8	7	12	23
a_3	21	18	12	21
a_4	30	22	19	15

Laplace. Given $P\{s_j\} = \frac{1}{4}$, $j = 1$ to 4, the expected values for the different actic
puted as

$$E\{a_1\} = \tfrac{1}{4}(5 + 10 + 18 + 25) = \$14{,}500$$

$$E\{a_2\} = \tfrac{1}{4}(8 + 7 + 12 + 23) = \$12{,}500 \leftarrow \text{Optimum}$$

$$E\{a_3\} = \tfrac{1}{4}(21 + 18 + 12 + 21) = \$18{,}000$$

$$E\{a_4\} = \tfrac{1}{4}(30 + 22 + 19 + 15) = \$21{,}500$$

Minimax. The minimax criterion produces the following matrix:

	s_1	s_2	s_3	s_4	Row max
a_1	5	10	18	25	25
a_2	8	7	12	23	23
a_3	21	18	12	21	21 ← Minimax
a_4	30	22	19	15	30