VIETNAM NATIONAL UNIVERSITY-HO CHI MINH CITY
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Chapter 3. Line Integrals and Surface Integrals

Analysis 3

Lecturer: Nguyen Minh Quan, PhD
quannm@hcmiu.edu.vn

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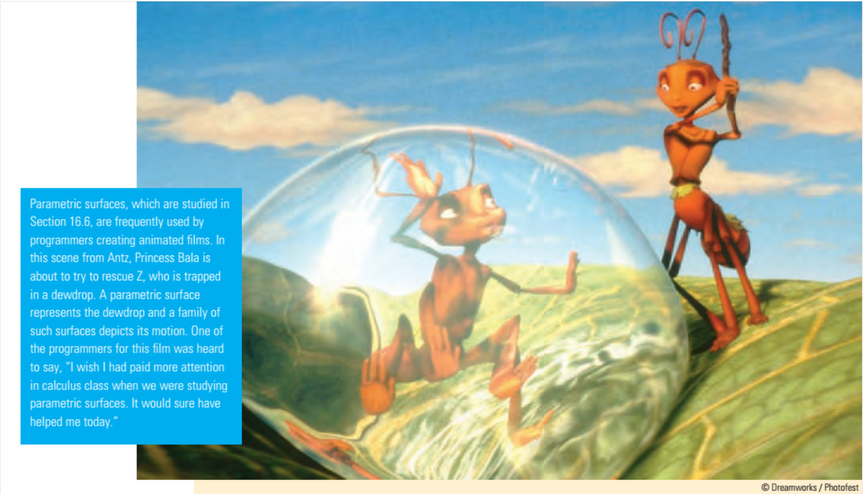
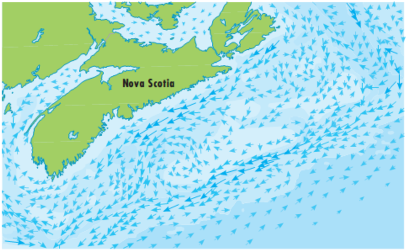


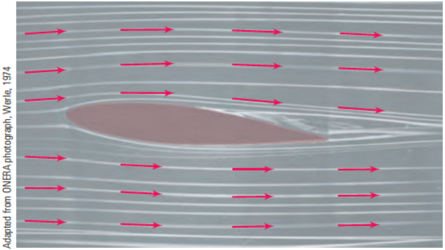
In this chapter, we study integration over curves and surfaces, and we will integrate not just functions but also vector fields.

Introduction to calculus of vector fields

How can we describe a physical object such as the wind, that consists of a large number of molecules moving in a region of space?



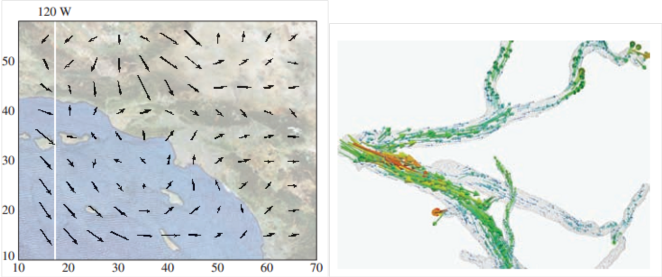
(a) Ocean currents off the coast of Nova Scotia



(b) Airflow past an inclined airfoil

Integrals of vector fields are used in the study of phenomena such as electromagnetism, fluid dynamics, wind speed, and heat transfer.

Introduction to calculus of vector fields



(a) Velocity vector field of wind velocity off the coast at Los Angeles. (b) Blood flow in an artery represented by a vector field.

The connections between these new types of integrals and the single, double, and triple integrals that we have already met are given by the higher-dimensional versions of the Fundamental Theorem of Calculus: Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

Vector functions

Definition

A vector-valued function in $\mathbb{R}^3$, or vector function, is simply a function whose domain is a set of real numbers and whose range is a set of vectors:

$$r(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

$f(t)$, $g(t)$, and $h(t)$ are real-valued functions called the component functions of $r(t)$.

Example

$$r(t) = \left\langle t^2, \ln(3-t), \sqrt{t} \right\rangle = t^2\mathbf{i} + \ln(3-t)\,\mathbf{j} + \sqrt{t}\mathbf{k}$$

is a vector function with the domain $D = [0, 3)$.

Vector functions

The limit of a vector function is defined by taking the limits of its component functions, that is,

Definition

If $r(t) = \langle f(t), g(t), h(t) \rangle$ then

$$\lim_{t \to a} r(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

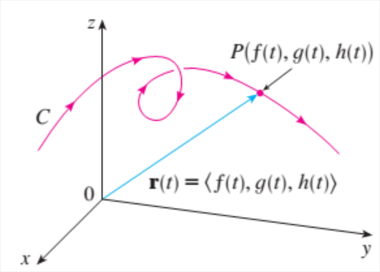
Definition

A vector function is continuous at a if

$$\lim_{t \to a} r(t) = r(a).$$

Vector functions

There is a close connection between continuous vector functions and space curves. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set of all points (x, y, z) in space, where $x = f(t)$, $y = g(t)$, and $z = h(t)$.



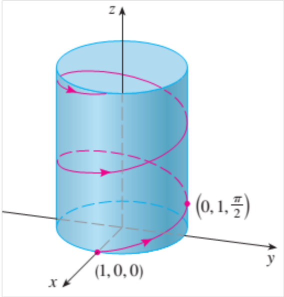
Vector functions

Example

The curve, whose vector equation is

$$r(t) = \langle \cos(t), sin(t), t \rangle,$$

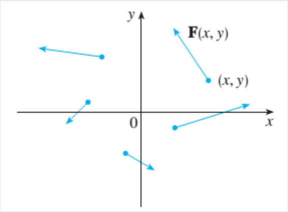
called a helix.



Vector fields in $\mathbb{R}^2$

Definition

Let D be a set in $\mathbb{R}^2$ (a plane region). A vector field on $\mathbb{R}^2$ is a function F that assigns to each point (x, y) in D a two-dimensional vector $F(x, y)$.



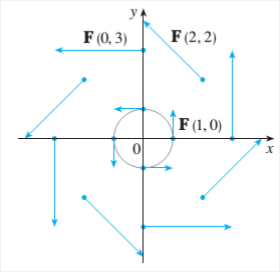
Since $F(x, y)$ is a two-dimensional vector, we can write it in terms of its component functions P and Q as follows:

$$F(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y), Q(x, y) \rangle$$

Vector fields in $\mathbb{R}^2$

Example

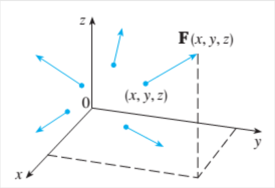
A vector field on $\mathbb{R}^2$ is defined by $F(x, y) = -y\mathbf{i} + x\mathbf{j}$. Describe by sketching some of the vectors $F(x, y)$.



Vector fields in $\mathbb{R}^3$

Definition

Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function F that assigns to each point (x, y, z) in E a three-dimensional vector $F(x, y, z)$.



$F(x, y, z)$ can be written as follows:

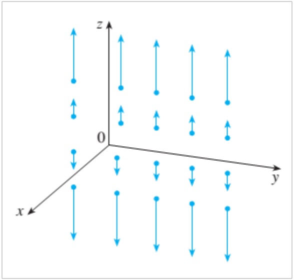
$$F(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

$$F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

Vector fields in $\mathbb{R}^3$

Example

Sketch the vector field on $\mathbb{R}^3$ given by $F(x, y, z) = z\mathbf{k}$.



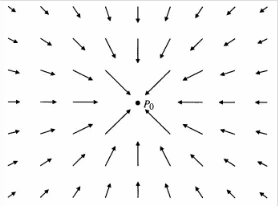
Example: Gravitational field

The gravitational field of a point mass

The gravitational force field due to a point mass m located at point P_0 having position vector r_0 is

$$F(x, y, z) = F(r) = \frac{-km(r - r_0)}{|r - r_0|^3}$$

where $k > 0$ is a constant. F points toward the point r_0 and has magnitude $|F| = \frac{km}{|r-r_0|^2}$.

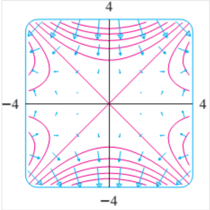


Gradient fields

Gradient fields

If is a scalar function of two variables then the gradient $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$ is really a vector field on $\mathbb{R}^2$ and is called a gradient vector field.

The figures below shows the gradient vector field of $f(x, y) = x^2 y - y^3$.



Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$.

Conservative vector field

Conservative vector field

A vector field F is called a conservative vector field if it is the gradient of some scalar function, that is, if there exists a function V such that $F = \nabla V$. In this situation V is called a potential function for F.

Example

$V(x, y, z) = xy + yz^2$ is a potential function for the vector field $F = \left\langle y, x + z^2, 2yz \right\rangle$ since $F = \nabla V$.

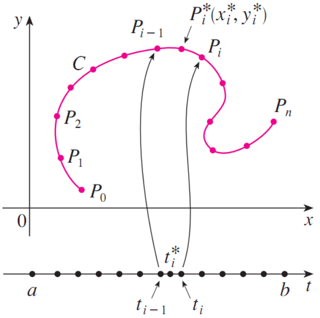
Example

If $F(x) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$, $\mathbf{x} = \langle x, y, z \rangle$, then $F = \nabla f$, where $f = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$.
So, F is conservative.

Line integrals

We start with a plane curve given by the parametric equations:

$$x = x(t), y = y(t), a \leq x \leq b.$$



Riemann sum: $\sum_{i=1}^{n} f\left(x_i^*, y_i^*\right) \Delta s_i$ We take the limit of Riemann sum and make the definition by analogy with a single integral.

Line integrals

Definition

If f is defined on a smooth curve C given by $x = x(t)$, $y = y(t)$, $a \leq x \leq b$, then the line integral of f along C is

$$\int\limits_C f(x, y)\, dS = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

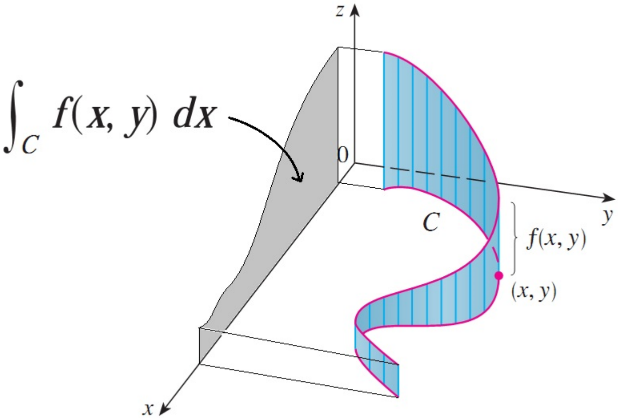
if this limit exists.

Theorem

If f is defined on a smooth curve C given by $x = x(t)$ and $y = y(t)$, then the line integral of f along C is:

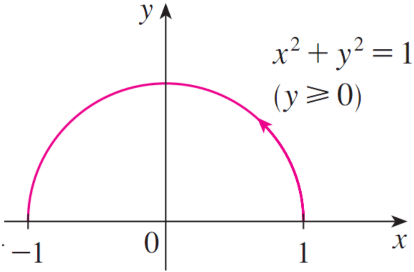
$$\int_C f(x,y)\mathrm{d}s = \int_a^b f\big(x(t), y(t)\big)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t$$

Line integrals: Geometric meaning



$\int_C f(x,y)\mathrm{d}s$ is the area of the blue fence (the blue strip) and $\int_C f(x,y)\mathrm{d}s$ is the area of its shadow (projection) on Oxy-plane.

Example: Evaluate $\int_C (2 + x^2 y)\mathrm{d}s$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.



Solution

The the upper half of the unit circle can be parametrized by $x = \cos t, y = \sin t, 0 \le t \le \pi$.

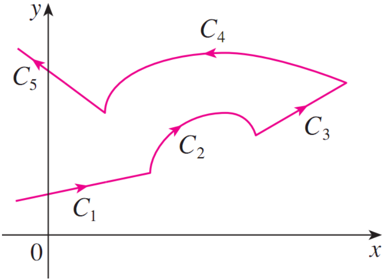
$$\int\limits_C \left(2 + x^2 y\right) dS = \int\limits_0^\pi \left(2 + \cos^2 t \sin t\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int\limits_0^\pi \left(2 + \cos^2 t \sin t\right) \sqrt{\sin^2 t + \cos^2 t} dt = 2t - \frac{\cos^3 t}{3}\bigg|_0^\pi = 2\pi + \frac{2}{3}.$$

Remark on piecewise-smooth curves

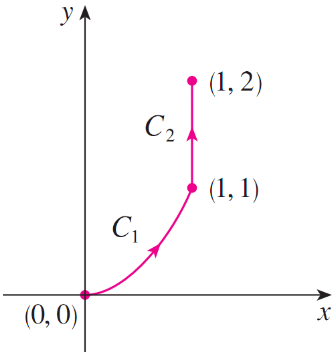
If C is a piecewise-smooth curve, that is, C is a union of a finite number of smooth curves $C_1, C_2, .., C_n$: $C = C_1 \cup \cdots \cup C_n$ then

$$f(x,y)\mathrm{d}s = \int_{C_1} f(x,y)\mathrm{d}s + \cdots + \int_{C_n} f(x,y)\mathrm{d}s$$



Example

Evaluate $\int_C 2x\mathrm{d}s$, where C consists of the arc C_1 of the parabola from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$



Solution

The parametric equations for C_1:

$$x = t, y = t^2, 0 \leqslant t \leqslant 1$$

Therefore

$$\int\limits_{C_1} 2xds = \int\limits_0^1 2t\sqrt{1+4t^2}dt = \frac{5\sqrt{5}-1}{6}$$

The parametric equations of C_2 are $x = 1, y = t, 1 \leqslant t \leqslant 2$

$$\int\limits_{C_2} 2xds = \int\limits_1^2 2\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \int\limits_1^2 2dt = 2$$

$$\int\limits_C 2xds = \int\limits_{C1} 2xds + \int\limits_{C_2} 2xds = \frac{5\sqrt{5}-1}{6} + 2$$

Solution 2

Remark:

We can also use x or y as an parameter as follows.

The parametric equations for C_1:

$$x = x, y = x^2, 0 \leqslant x \leqslant 1$$

Therefore

$$\int\limits_{C_1} 2xds = \int\limits_0^1 2x\sqrt{1+4x^2}dx = \frac{5\sqrt{5}-1}{6}$$

The parametric equations of C_2 are $x = 1, y = y, 1 \leqslant y \leqslant 2$

$$\int\limits_{C_2} 2xds = \int\limits_1^2 2\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}dy = \int\limits_1^2 2dy = 2$$

$$\int\limits_C 2xds = \int\limits_{C1} 2xds + \int\limits_{C_2} 2xds = \frac{5\sqrt{5}-1}{6} + 2$$

Line integral with respect to arc length

In the Definition of line integral, two other line integrals are obtained by replacing Δs_i by either Δx_i or Δy_i. They are called the line integrals of f along with respect to x and y.

If C is a smooth curve given by $x = x(t)$, $y = y(t)$, $t \in [a, b]$ and $f(x, y)$ is continuous, then:

$$\int_C f(x,y)\mathrm{d}x = \int_a^b f\big(x(t), y(t)\big)x'(t)\mathrm{d}t$$

$$\int_C f(x,y)\mathrm{d}y = \int_a^b f\big(x(t), y(t)\big)y'(t)\mathrm{d}t$$

Line integral with respect to arc length

It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing:

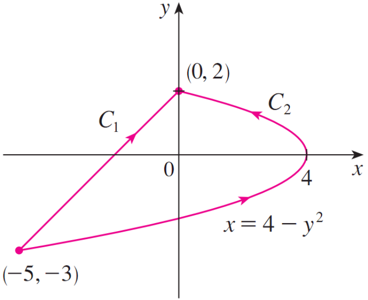
$$\int_C P(x,y)\mathrm{d}x + \int_C Q(x,y)\mathrm{d}y = \int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$$

Example

Evaluate $\int_C y^2\mathrm{d}x + x\mathrm{d}y$, where:

a. $C = C_1$, is the line segment from $(-5,-3)$ to $(0,2)$
b. $C = C_2$, is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$
3. $C = -C_1$ is the line segment from $(0,2)$ to $(-5,-3)$

Line integral with respect to arc length

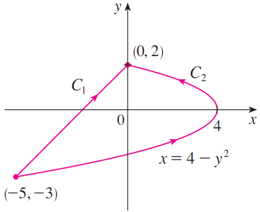


Solution

(a) A parametric representation for the line segment is $x = 5t - 5$, $y = 5t - 3$, $0 \leqslant t \leqslant 1$. Thus,

$$\int_{C_1} y^2 dx + x dy = \int_0^1 (5t-3)^2 (5dt) + (5t-5)(5dt) = -\frac{5}{6}$$

Solution (Cont.)



(b) Let's take y as the parameter and write C_2 as

$$x = 4 - y^2, y = y, -3 \leqslant y \leqslant 2$$

Therefore,

$$\int_{C_2} y^2 dx + x dy = \int_{-3}^{2} y^2 \left(-2y dy\right) + \left(4 - y^2\right) dy = 40\frac{5}{6}$$

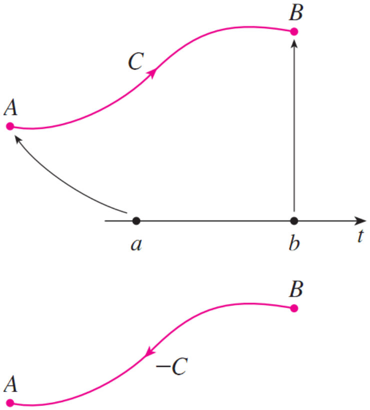
(c) Parametrization: $x = -5t, y = 2 - 5t, 0 \leqslant t \leqslant 1$.
Therefore, $\int_{-C_1} y^2 dx + x dy = \frac{5}{6}$.

Remark 1

A vector representation of the line segment that starts at r_0 and ends at r_1 is given by

$$r(t) = (1 - t)\, r_0 + t r_1, 0 \leqslant t \leqslant 1$$

Remark 2



If $-C$ denotes the curve consisting of the same points as C but with the *opposite orientation*. Then:

$$\int_{-C} f(x,y)\mathrm{d}x = -\int_{C} f(x,y)\mathrm{d}x$$

$$\int_{-C} f(x,y)\mathrm{d}y = -\int_{C} f(x,y)\mathrm{d}y$$

But if we integrate with respect to arc length, the value of the line integral does not change:

$$\int_{-C} f(x,y)\mathrm{d}s = \int_{C} f(x,y)\mathrm{d}s$$

Line Integrals in Space

Suppose that C is a smooth space curve given by the parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad a \leq t \leq b$$

or by a vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If f is a function of

three variables that is continuous on some region containing C, then we define the line integral of along C:

$$\int_C f(x, y, z)\mathrm{d}s = \int_a^b f\left(x(t), y(t), z(t)\right) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\mathrm{d}t$$

Line Integrals in Space

Line integrals along C with respect to x, y, and z can also be defined:

$$\int_C f(x,y,z)\,\mathrm{d}x = \int_a^b f\big(x(t),y(t),z(t)\big)x'(t)\,\mathrm{d}t$$
$$\int_C f(x,y,z)\,\mathrm{d}y = \int_a^b f\big(x(t),y(t),z(t)\big)y'(t)\,\mathrm{d}t$$
$$\int_C f(x,y,z)\,\mathrm{d}z = \int_a^b f\big(x(t),y(t),z(t)\big)z'(t)\,\mathrm{d}t$$

Line integrals in the plane:

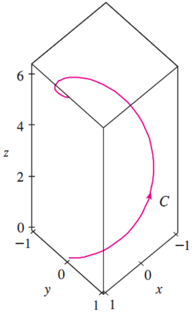
$$\int_C P(x,y,z)\,\mathrm{d}x + \int_C Q(x,y,z)\,\mathrm{d}y + \int_C R(x,y,z)\,\mathrm{d}z$$
$$= \int_C P(x,y,z)\,\mathrm{d}x + Q(x,y,z)\,\mathrm{d}y + R(x,y,z)\,\mathrm{d}z$$

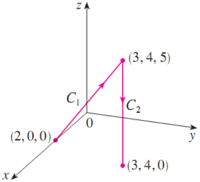
Example

1. Evaluate $\int_C y \sin z \mathrm{d}s$, where C is the circular helix given by the equations $x = \cos t, y = \sin t, z = t, 0 \leq t \leq 2\pi$
2. Evaluate $\int_C y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z$, where C consists of the line segments $(2,0,0)$, $(3,4,5)$, $(3,4,0)$

Answers

1. $\sqrt{2}\pi$
2. $\dfrac{49}{2}$





Line Integrals of Vector Fields

How to compute the work done by a force field along a curve?

Definition

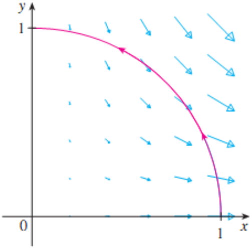
Let F be a continuous vector field defined on a smooth curve C given by a vector function $r(t)$, $a \leq t \leq b$. Then the line integral of F along C is

$$\int_C F \cdot dr = \int_a^b F\left(r\left(t\right)\right) \cdot r'\left(t\right) dt = \int_C F \cdot T ds$$

Example

Find the work done by the force field $F(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter-circle $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \leq t \leq \pi/2$.

Solution



Since $x = \cos t$ and $y = \sin t$, we have

$$F(r(t)) = \cos^2 t\mathbf{i} - \cos t \sin t\mathbf{j}$$

$$r'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\int_C F \cdot dr = \int_0^{\pi/2} F(r(t)) \cdot r'(t)\, dt = \int_0^{\pi/2} -2\cos^2 t \sin t dt = -\frac{2}{3}$$

Line Integrals of Vector Fields

Remarks: If $F = \langle P, Q, R \rangle$ then

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt = \int_C Pdx + Qdy + Rdz$$

Exercise

Evaluate $\int_C F \cdot dr$, where $F(x, y, z) = xyi + yzj + zxk$ and C is the twisted cubic given by $x = t, y = t^2, z = t^3, 0 \le t \le 1$.

Hint:

$$r(t) = \left\langle t, t^2, t^3 \right\rangle$$

$$\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t)\, dt = \int_0^1 \left(t^3 + 5t^6\right) dt = \frac{27}{28}$$

The Fundamental Theorem for Line Integrals

Recall that Part 2 of the Fundamental Theorem of Calculus can be written as

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

If we think of the gradient vector ∇f of a function of two or three variables as a sort of derivative of f, then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

Theorem

Let C be a smooth curve given by the vector function $r(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

The Fundamental Theorem for Line Integrals

Proof:

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \qquad \text{(by the Chain Rule)}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

The Fundamental Theorem for Line Integrals

Example

Find the work done by the gravitational field

$$F(x) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x},$$

$\mathbf{x} = \langle x, y, z \rangle$, in moving a particle with mass from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C.

Solution

We have $F = \nabla f$, where $f = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$. That is, F is a conservative vector field.

Therefore, the work done is

$$W = \int_C F \cdot dr = \int_C \nabla f \cdot dr = f(2,2,0) - f(3,4,12)$$

$$W = \frac{mMG}{\sqrt{2^2+2^2}} - \frac{mMG}{\sqrt{3^2+4^2+12^2}} = mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)$$

Independence of Path

Definition

If F is a continuous vector field with domain D, we say that the line integral $\int_C F \cdot dr$ is independent of path if $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$ for any two paths C_1 and C_2 in that have the same initial and terminal points.

For example, line integrals of conservative vector fields are independent of path.

Independence of Path

Definition

A curve is called closed if its terminal point coincides with its initial point, that is, $r(b) = r(a)$.

Theorem

$\int\limits_C F \cdot dr$ is independent of path in D if and only if $\int\limits_C F \cdot dr = 0$ for every closed path C in D.

Conservative vector field

Theorem

Suppose F is a vector field that is continuous on an open connected region D. If $\int_C F \cdot dr$ is independent of path in D, then F is a conservative vector field on D; that is, there exists a function f such that $\nabla f = F$.

The question remains: How is it possible to determine whether or not a vector field is conservative?

Theorem

If F is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

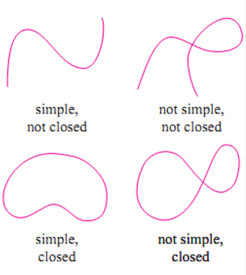
Q: Is the converse is true?

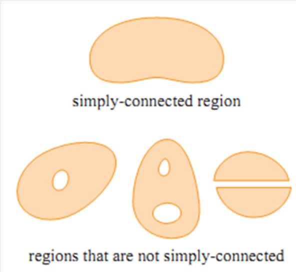
Simply-connected region

Definition

1. A simple curve is a curve that doesn't intersect itself any-where between its endpoints.

2. A simply-connected region in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D.





Intuitively speaking, a simply-connected region contains no hole and can't consist of two separate pieces

Conservative vector fields

Theorem

Let $F = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then F is conservative.

Example

Determine whether or not the vector field $F(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$ is conservative.

Let $P = x - y$, $Q = x - 2$. Since $\frac{\partial P}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = 1$, F is not conservative.

Conservative vector fields

Example

Determine whether or not the vector field
$F(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is conservative.

Solution

Let $P = 3 + 2xy$, $Q = x^2 - 3y^2$. Since $\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$.
Also, the domain of F is the entire plane ($D = \mathbb{R}^2$), which is open and simply-connected.
Thus, F is conservative.

Conservative vector fields

Exercise

(a) If $F(x,y) = (3+2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, find a function f such that $F = \nabla f$.
(b) Evaluate the line integral $\int_C F \cdot dr$, where C is the curve given by $r(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j}$, where $0 \leq t \leq \pi$.

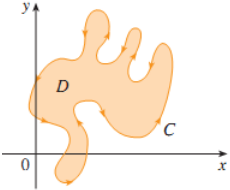
Hint
(a) $f(x,y) = 3x + x^2y - y^3 + C$
(b)

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(0, -e^{\pi}) - f(0,1) = e^{3\pi} + 1$$

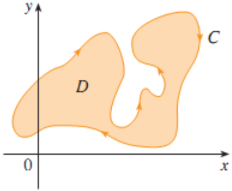
Green Theorem

Definition: Positive Orientation

The positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C. That is, if C is given by the vector function $r(t)$, $a \leq t \leq b$, then the region D is always on the left as the point traverses C.

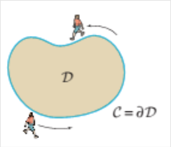


(a) Positive orientation



(b) Negative orientation

Green Theorem



Green Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then:

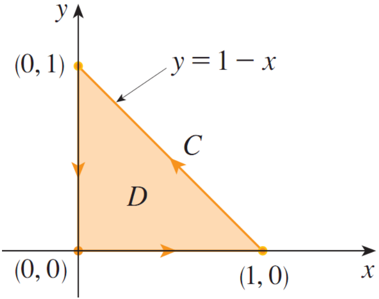
$$\oint_C P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y$$

The equation in Green's Theorem can be written as

$$\oint_{\partial D} P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y$$

Example

1. Evaluate $I_1 = \oint_C x^4 \mathrm{d}x + xy\mathrm{d}y$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.
2. Evaluate $I_2 = \oint_C \left(3y - e^{\sin x}\right) \mathrm{d}x + \left(7x + \sqrt{y^4+1}\right) \mathrm{d}y$, where C is the circle $x^2 + y^2 = 9$

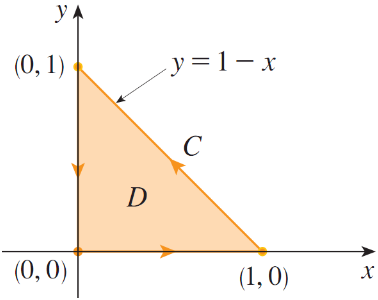


Solutions

1. Using Green's Theorem

$$I_1 = \oint_C x^4 dx + xy dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_0^{1-x} (y - 0)\, dy dx$$

Therefore, $I_1 = \frac{1}{2} \int_0^1 (1-x)^2 = \frac{1}{6}$.



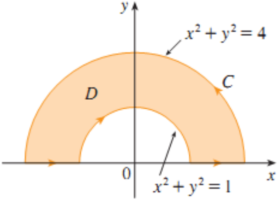
2. Hint: $I_2 = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 4 \int_0^{2\pi} d\theta \int_0^3 r dr = 36\pi$.

Example

3. Evaluate

$$I_3 = \oint_C y^2 dx + 3xydy$$

where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



Hint: $D = \{(r, \theta) : 1 \le r \le 2, 0 \le \theta \le \pi\}$

$$I_3 = \iint_D \left(\frac{\partial (3xy)}{\partial x} - \frac{\partial (y^2)}{\partial y} \right) dA = \int_0^\pi \int_1^2 (r \sin\theta) r dr d\theta = \frac{14}{3}$$

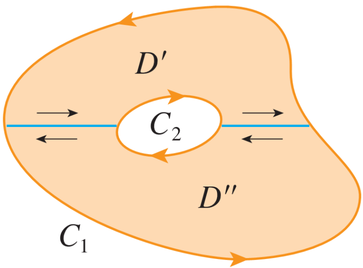
Remarks

- The Green's Theorem gives the following formulas for the area of D:
$$A = \oint_C x\mathrm{d}y = -\oint_C y\mathrm{d}x = \frac{1}{2}[\oint_C x\mathrm{d}y - y\mathrm{d}x]$$

- Extended Versions of Green's Theorem for bounded domain

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y =$$

$$\oint_{C_1} P\mathrm{d}x + Q\mathrm{d}y + \oint_{C_2} P\mathrm{d}x + Q\mathrm{d}y$$



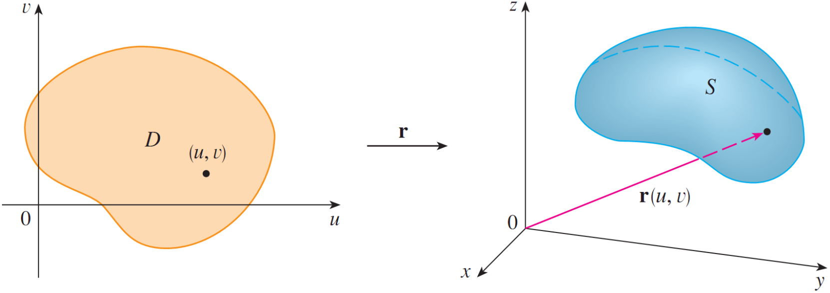
Parametric Surfaces

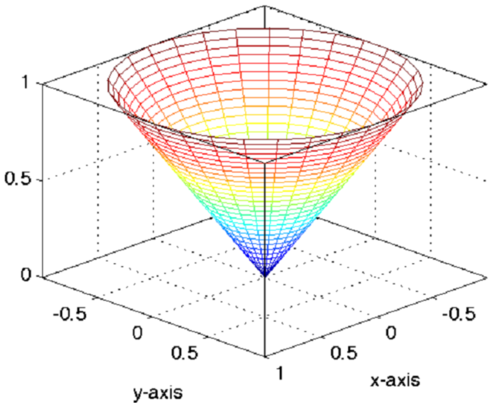
The set of all points $(x, y, z) \in \mathbb{R}^3$ such that:

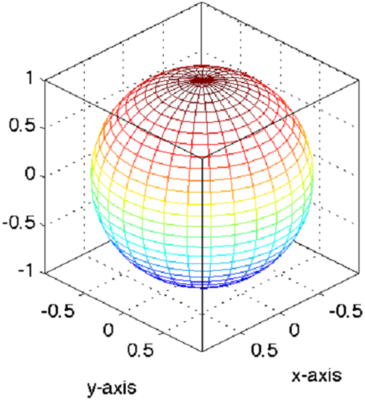
$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

where $(u, v) \in D$ is called a *parametric surface* S and the equations above are called *parametric equations* of S.

We write $(S) : \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$

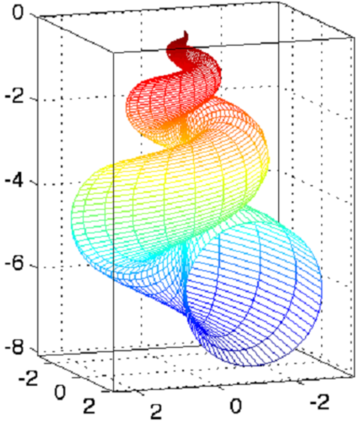


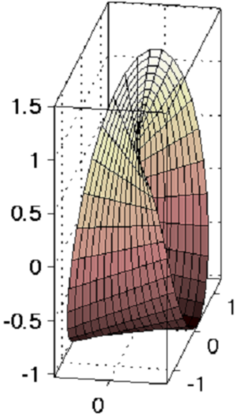




$$x = r\cos\theta \qquad 0 \le r \le 1$$
$$y = r\sin\theta \qquad 0 \le \theta \le 2\pi$$
$$z = r,$$

$$x = \sin\phi\cos\theta \qquad 0 \le \phi \le \pi$$
$$y = \sin\phi\sin\theta \qquad 0 \le \theta \le 2\pi$$
$$z = \cos\phi,$$





$$x = 2\left[1 - e^{u/(6\pi)}\right]\cos u \cos^2\left(\frac{v}{2}\right)$$
$$y = 2\left[-1 + e^{u/(6\pi)}\right]\sin u \cos^2\left(\frac{v}{2}\right)$$
$$z = 1 - e^{u/(3\pi)} - \sin v + e^{u/(6\pi)}\sin v,$$

$$0 \le u \le 6\pi \qquad 0 \le v \le 2\pi$$

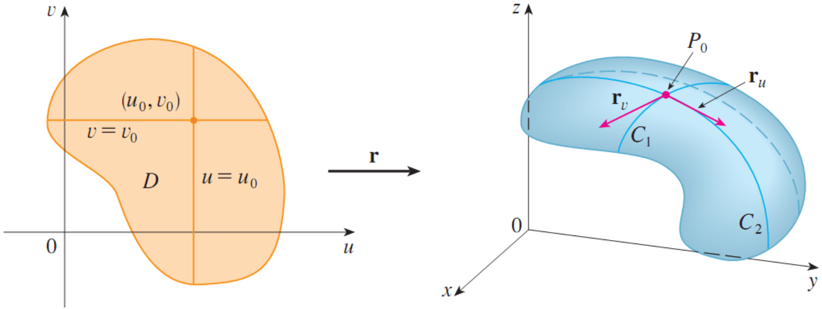
$\left(\textit{Mobius Strip}\right)$

$$x = \frac{v}{2}\sin\frac{u}{2}$$
$$y = \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u$$
$$z = \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u,$$

$$0 \le u \le 2\pi \qquad -1 \le v \le 1$$

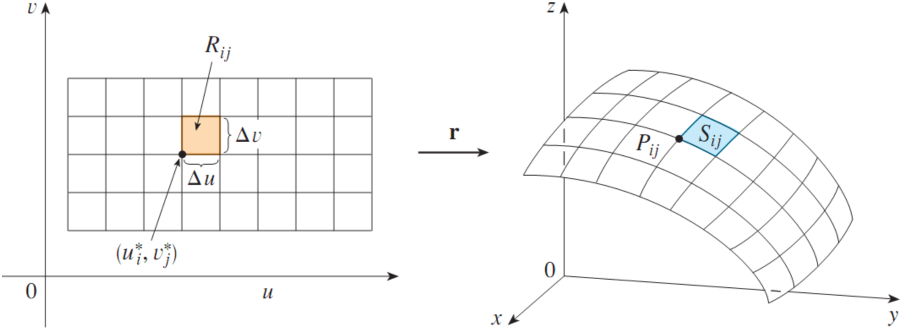
Normal vector to the tangent plane

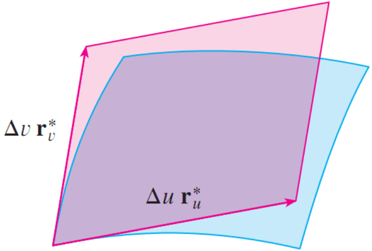
$$
\begin{aligned}
\mathbf{r}(u,v) &= x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D \\
\mathbf{r}_u(x_0,y_0) &= \frac{\partial x}{\partial u}(x_0,y_0)\mathbf{i} + \frac{\partial y}{\partial u}(x_0,y_0)\mathbf{j} + \frac{\partial z}{\partial u}(x_0,y_0)\mathbf{k} \\
\mathbf{r}_v(x_0,y_0) &= \frac{\partial x}{\partial v}(x_0,y_0)\mathbf{i} + \frac{\partial y}{\partial v}(x_0,y_0)\mathbf{j} + \frac{\partial z}{\partial v}(x_0,y_0)\mathbf{k}
\end{aligned}
$$



The vector $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ is the normal vector to the tangent plane.

Surface Area





$$\Delta S_{ij} \approx |(\Delta u \mathbf{r}_u^*) \times (\Delta v \mathbf{r}_v^*)|$$

$$= |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \Delta v$$

Surface Area

$$S \approx \sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \Delta v$$

Surface Area

If a smooth parametric surface S is given by the equation $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ and is covered just once as (u, v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \mathrm{d}u\mathrm{d}v$$

where

$$r_u = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k, r_v = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j + \frac{\partial z}{\partial v} k$$

Surface Area

Example

1. Find the surface area of a sphere of radius a
2. Surface Area of the Graph of a Function: Show that the surface area of $S : z = f(x, y)$, where $(x, y) \in D$ is

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \mathrm{d}x\mathrm{d}y$$

Surface Area

Solution

1. We have

$$x = a\cos\theta\sin\phi, y = a\sin\theta\sin\phi, z = a\cos\phi$$

where

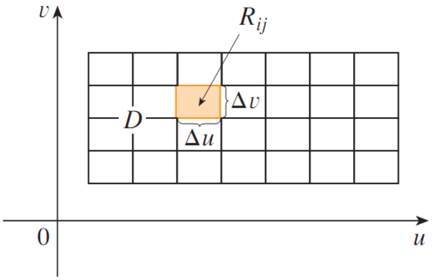
$$0 \le \theta \le 2\pi, 0 \le \phi \le \pi.$$

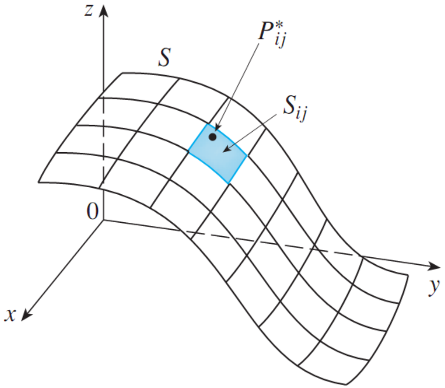
$$|r_\phi \times r_\theta| = \left| \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial x}{\partial \theta} \end{vmatrix} \right| = a^2 \sin\phi$$

Therefore, the surface area of a sphere of radius a is

$$A = \iint\limits_D |r_\phi \times r_\theta|\, dA = \int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi d\phi d\theta = 4\pi a^2$$

Surface Integral





Riemann sum:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij} \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)|\mathbf{r}_u \times \mathbf{r}_v|\Delta u \Delta v$$

Surface Integral

Surface integral of f over the surface S:

$$\iint_S f(x,y,z)\mathrm{d}\sigma = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\mathrm{d}u\mathrm{d}v$$

Example:

1. Evaluate $\displaystyle\iint_S x^2\mathrm{d}\sigma$ where S is the unit sphere.
2. Let $S : z = g(x,y)$, where $(x,y) \in D$. Show that:

$$\iint_S f(x,y,z)\mathrm{d}\sigma = \iint_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\mathrm{d}x\mathrm{d}y$$

Surface Area

Solution

1. We have

$$x = \cos\theta\sin\phi, y = \sin\theta\sin\phi, z = \cos\phi$$

where $0 \le \theta \le 2\pi, 0 \le \phi \le \pi$.

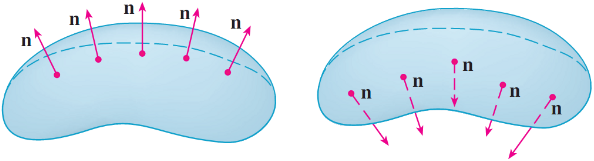
$$|r_\phi \times r_\theta| = |\begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial x}{\partial \theta} \end{vmatrix}| = \sin\phi$$

Therefore,

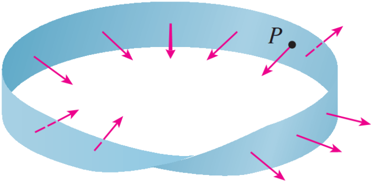
$$\iint\limits_S x^2 dS = \iint\limits_D (\sin\phi\cos\theta)^2 |r_\phi \times r_\theta| \, dA = \int_0^{2\pi}\int_0^{\pi} cos^2\theta\sin^3\phi d\phi d\theta = \frac{4\pi}{3}$$

Oriented Surfaces

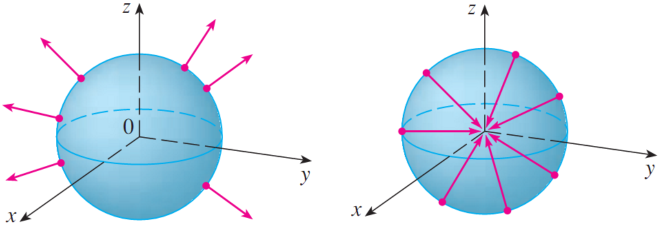
If it is possible to choose a *unit normal vector* **n** at every such point (x, y, z) so that $\mathbf{n}(x, y, z)$ varies continuously over S, then S is called an *oriented surface* and the given choice of **n** provides with an orientation.



Not all surfaces can be oriented. For example, Möbius surface.



For a closed surface, the convention is that the *positive orientation* is the one for which the normal vectors point outward from, and inward-pointing normals give the negative orientation.



If S is oriented and defined by $\mathbf{r}(u, v)$ then the unit normal vector is

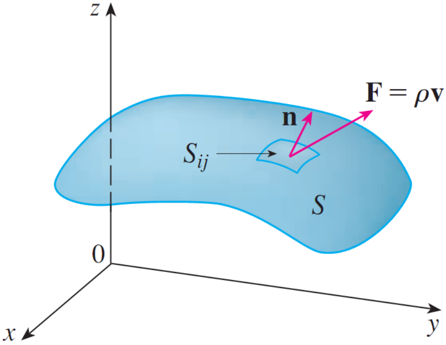
$$\mathbf{n} = \pm \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

The unit normal vector of $z = g(x, y)$:

$$\mathbf{n} = \frac{-g_x \mathbf{i} - g_y \mathbf{j} + \mathbf{k}}{\sqrt{1 + (g_x)^2 + (g_y)^2}}$$

Surface Integrals of Vector Fields

Consider a fluid with density $\rho(x, y, z)$ flowing S with velocity field $\mathbf{v}(x, y, z) = \big(v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)\big)$
Then the rate of flow (mass per unit time) per unit area is: $\mathbf{F} = \rho\mathbf{v}$



We can approximate the mass of fluid per unit time crossing S_{ij} *in the direction of the normal* $\mathbf{n}$:

$$(\rho\mathbf{v} \cdot \mathbf{n})A(S_{ij})$$

Surface Integrals of Vector Fields

The total mass of fluid per unit time crossing S (per unit time)

$$\iint_S \rho(x,y,z)\mathbf{v}(x,y,z)\cdot\mathbf{n}(x,y,z)\mathrm{d}\sigma = \iint_S \mathbf{F}\cdot\mathbf{n}\mathrm{d}\sigma$$

Definition

If $\mathbf{F}$ *is a continuous vector field defined on an oriented surface* S *with unit normal vector* $\mathbf{n}$*, then the surface integral of over* S *is*

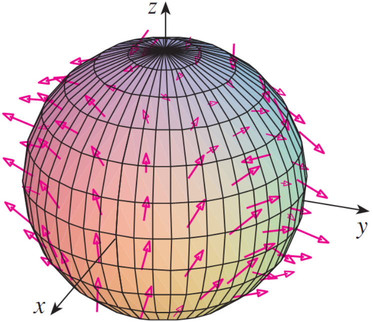
$$\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\mathrm{d}\sigma$$

This integral is also called the flux $\mathbf{F}$ of across S.

If S is defined by $\mathbf{r}(u, v)$ $((u, v) \in D)$, then:

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\mathrm{d}u\mathrm{d}v$$

Example: Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the unit sphere S: $x^2 + y^2 + z^2 = 1$

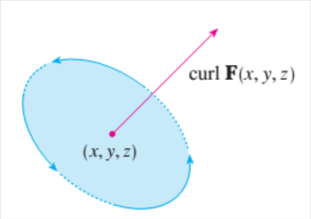


Curl

Definition

If $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of F is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$



Curl

Recall:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

We can consider the formal cross product of ∇ with the vector field F as follows:

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

So the easiest way to remember Definition is by means of the symbolic expression:

$$\text{curl } F = \nabla \times F$$

Example

If $F(x, y, z) = xzi + xyzj - y^2k$, find curl F.

Solution

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} =$$

$$\left[\frac{\partial}{\partial y}\left(-y^2\right) - \frac{\partial}{\partial z}(xyz)\right] i - \left[\frac{\partial}{\partial x}\left(-y^2\right) - \frac{\partial}{\partial z}(xz)\right] j + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right] k$$

$$\text{curl } F = (-2y - xy)\, i + xj + yzk$$

Curl

Theorem

If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl } (\nabla f) = 0$$

Proof:...

Remark: Since a conservative vector field is one for which $F = \nabla f$, thus if F is conservative, then curl $(F) = 0$.

This gives us a way of verifying that a vector field is not conservative.

Example

Show that the vector field $F = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ is not conservative.

Solution

We have

$$\text{curl } F = (-2y - xy)\, i + xj + yzk$$

Therefore, curl $F \neq 0$, so F is not conservative.

Curl

The converse of previous Theorem is not true in general, but the following theorem says the converse is true if F is defined everywhere.

Theorem

If F is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $F = 0$, then F is a conservative vector field.

Example

(a) Show that $F(x, y, z) = y^2z^3i + 2xyz^3j + 3xy^2z^2k$ is a conservative vector field.
(b) Find a function such that $F = \nabla f$.

Hint:
(a) Show that curl $F = 0$, then F is thus a conservative vector field.
(b) $f(x, y, z) = xy^2z^3 + C$.

Curl

Example

(a) Show that $F(x,y,z) = y^2z^3i + 2xyz^3j + 3xy^2z^2k$ is a conservative vector field.
(b) Find a function such that $F = \nabla f$.

Hint:
(a) Show that curl $F = 0$, then F is thus a conservative vector field.
(b) $f(x,y,z) = xy^2z^3 + C$.

Divergence

If $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the divergence of F is the function

$$\text{div } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot F$$

Example

If $F(x, y, z) = xzi + xyzj - y^2k$, find div F.

$$\text{div } F = \nabla \cdot F = z + xz$$

Divergence

Theorem

If $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } F = 0$$

Divergence Theorem

Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E \mathrm{div}\mathbf{F}\,\mathrm{d}V$$

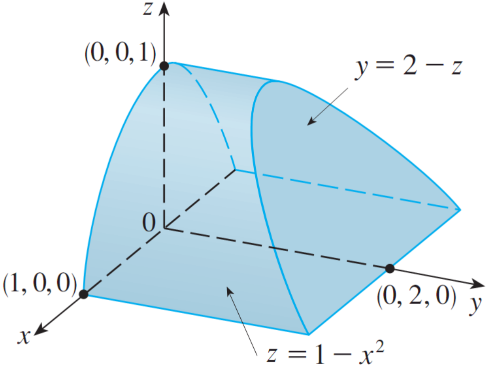
The Divergence Theorem is sometimes called Gauss's Theorem.

Example

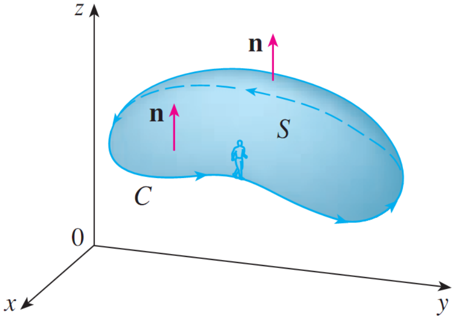
Evaluate $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\mathbf{k}$$

and S be the boundary surface of E bounded by $z = 1 - x^2$ and the planes $z = 0,\ y = 0,\ y + z = 2$



Stokes Theorem



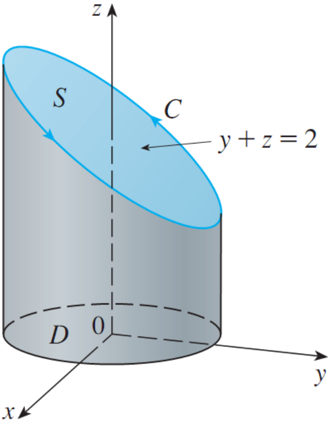
Theorem

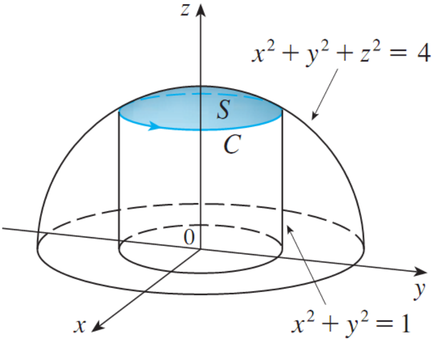
Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let F be a vector field whose components have continuous partial derivatives on an open region in that contains S. Then

$$\iint_S \mathrm{curl} F \cdot \mathrm{d}\mathbf{S} = \oint_C P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$$

Example

1. Evaluate $\int_C -y^2 \mathrm{d}x + x \mathrm{d}y + z^2 \mathrm{d}z$, where C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient to be counterclockwise when viewed from above.)
2. Use Stokes' Theorem to compute the integral $\int_S \mathrm{curl}\mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where $\mathbf{F}(x, y, z) = (xz, yz, xy)$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xt-plane.





HAPPY NEW YEAR AND GOOD LUCK!