EXERCISES FOR CHAPTER 3: UNCONSTRAINED PROBLEMS

Exercises for everyone: All exercises in parts A and B.

A. Non-assessed Exercises (corrected in class):

0.1.4; 0.1.9(a); 0.1.12; 0.1.15; 0.2.4; 0.2.10; 0.3.15; 0.3.20; 0.3.24; 0.3.25; 0.4.2.

B. Assessed Assignments (to be submitted):

0.1.1; 0.1.8; 0.1.9(b), (c); 0.1.10; 0.1.11; 0.1.14; 0.2.1; 0.2.3; 0.2.6; 0.2.7; 0.2.8; 0.2.11; 0.2.13; 0.3.1; 0.3.2; 0.3.4; 0.3.5; 0.3.6; 0.3.7; 0.3.8; 0.3.10; 0.3.11; 0.3.13; 0.3.14; 0.3.17; 0.3.23; 0.3.26; 0.4.1; 0.4.3.

C. Bonus Exercises: Remaining exercises.

0.1 PRELIMINARIES

Exercise 0.1.1. Let

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{bmatrix}.$$

Show that $\mathbf{A}$ is positive definite, $\mathbf{B}$ is indefinite, and $\mathbf{C}$ is negative definite.

Exercise 0.1.2. Show that a matric $\mathbf{A}$ is positive definite if and only if its inverse is positive definite.

Exercise 0.1.3. Show that if $\mathbf{A}$ is a symmetric matrix and if there exist positive and negative elements in the diagonal of $\mathbf{A}$, then $\mathbf{A}$ is indefinite.

Exercise 0.1.4. Let $\mathbf{x} \in \mathbb{R}^n$ and let $\mathbf{A}$ be defined as

$$a_{ij} = x_i x_j, \quad i, j = 1, 2, \ldots, n.$$

Show that $\mathbf{A}$ is positive semidefinite and that it is *not* a positive definite matrix when $n > 1$.

Exercise 0.1.5. Show that the leading principal minors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix}$$

are positive, but that there are $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^2$ such that $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$.

Exercise 0.1.6. Let $\mathbf{A}$ be a square $n \times n$-matrix.

(a) Show that $\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ is symmetric.

(b) Show that $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{B}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$

(c) Conclude that $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$ in $\mathbb{R}^2$ iff $\mathbf{B}$ is positive semidefinite; $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$ iff $\mathbf{B}$ is positive definite.

(d) Show that if $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix}$, then $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^2$.

Exercise 0.1.7. Let $\mathbf{B}$ be an $n \times k$ matrix and let $\mathbf{A} = \mathbf{B}\mathbf{B}^T$.

(i) Prove $\mathbf{A}$ is positive semidefinite.

(ii) Prove that $\mathbf{A}$ is positive definite if and only if $\mathbf{B}$ has a full row rank.

Exercise 0.1.8. Let $\mathbf{A}$ be an $n \times n$ symmetric matrix. Show that $\mathbf{A}$ is positive semidefinite if and only if there exists an $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}\mathbf{B}^T$.

Exercise 0.1.9. Write each of the quadratic forms in the form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is an appropriate symmetric matrix:

(a) $3x_1^2 - x_1x_2 + 2x_2^2$.

(b) $x_1^2 + 2x_2^2 - 3x_3^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3$.

(c) $2x_1^2 - 4x_3^2 + x_1x_2 - x_2x_3$.

Exercise 0.1.10. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$f(\mathbf{x}) = c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 + c_4 x_1 x_2 + c_5 x_1 x_3 + c_6 x_2 x_3.$$

Show that $f(\mathbf{x})$ is the quadratic form and find $\frac{1}{2}\mathbf{F}(\mathbf{x})$. Discuss generalizations to higher dimensions.

Exercise 0.1.11. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined over $\mathbb{R}^n$. The function f is called **coercive** if

$$\lim_{|\mathbf{x}| \to \infty} f(\mathbf{x}) = \infty.$$

This means that for any constant M there must be a positive number r such that $f(\mathbf{x}) > M$ whenever $|\mathbf{x}| > r$.

Let $\mathbf{A}$ be a positive definite $n \times n$ matrix. Determine whether the function

$$f(\mathbf{x}) = \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{|\mathbf{x}| + 1}$$

is coercive or not.

Exercise 0.1.12. Define $f(x, y)$ on $\mathbb{R}^2$ by $f(x, y) = x^4 + y^4 - 32y^2$.

(a) Find a point in $\mathbb{R}^2$ at which $\nabla^2 f$ is indefinite.

(b) Show that $f(x, y)$ is coercive.

(c) Minimize $f(x, y)$ on $\mathbb{R}^2$.

Exercise 0.1.13. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not coercive and satisfies that for any $\alpha \in \mathbb{R}$,

$$\lim_{|x| \to \infty} f(x, \alpha y) = \lim_{|y| \to \infty} f(\alpha x, y) = \infty.$$

Exercise 0.1.14. Find the first three terms of the Taylor series for

$$f(\mathbf{x}) = 3x_1^4 - 2x_1^3 x_2 - 4x_1^2 x_2^2 + 5x_1 x_2^3 + 2x_2^4.$$

at the point $\mathbf{x}_0 = (1, -1)$. Evaluate the series for $\mathbf{d} = (0.1, 0.01)$ and compare with the value of $f(\mathbf{x} + \mathbf{d})$.

Exercise 0.1.15. Consider the quadratic function given by

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c.$$

Show that $\mathbf{Q}$ is the Hessian of f and that if $\mathbf{x}_0$ is any point, then

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)\mathbf{h} + \frac{1}{2}\mathbf{h}^T\mathbf{Q}\mathbf{h},$$

where $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$. Hence f coincides with its Taylor expansion of second order about any point.

Exercise 0.1.16. Let V be a vector space. A function $p : V \to \mathbb{R}$ is called a seminorm if it satisfies the following conditions:

(a) $p(x) \geq 0$ for all $x \in V$.

(b) If $x \in V$ and α is a number, then $p(\alpha x) = |\alpha| p(x)$.

(c) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.

Show that every seminorm is convex. In particular, every norm is convex.

Exercise 0.1.17. Let $\mathbf{Q}$ be an $n \times n$ positive semidefinite matrix.

(a) Use the fact that for fixed $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the quadratic functions $\varphi(t) = (t\mathbf{x} + \mathbf{y})^T\mathbf{Q}(t\mathbf{x} + \mathbf{y})$ and $\psi(t) = (\mathbf{x} + t\mathbf{y})^T\mathbf{Q}(\mathbf{x} + t\mathbf{y})$, $t \in \mathbb{R}$, are non-negative to show that

$$\left|\mathbf{x}^T\mathbf{Q}\mathbf{y}\right| \leq \sqrt{\mathbf{x}^T\mathbf{Q}\mathbf{x} \cdot \mathbf{y}^T\mathbf{Q}\mathbf{y}} \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{0.1.1}$$

(b) Show that the function $f(\mathbf{x}) = \sqrt{\mathbf{x}^T\mathbf{Q}\mathbf{x}}$ is convex.

(c) Show that $f(\mathbf{x})$ is a seminorm on $\mathbb{R}^n$ (see Exercise 0.1.16).

(d) Show that if $\mathbf{Q}$ is positive definite, then $f(\mathbf{x})$ is a norm on $\mathbb{R}^n$.

0.2 FIRST AND SECOND-ORDER CONDITIONS

Exercise 0.2.1. Let $\Omega = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0,\ i = 1, 2, \ldots, m\}$ and assume that the functions g_i are all continuous. Prove that if $g_i(\bar{\mathbf{x}}) < 0$ for all i, then $\bar{\mathbf{x}}$ is an interior point of Ω, that is, $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \bar{\mathbf{x}}| < r\} \subset \Omega$ for some $r > 0$.

Exercise 0.2.2. Let $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$. Derive the conditions that must be satisfied by a feasible direction at a point $\mathbf{x}_0 \in S$.

Exercise 0.2.3. Consider a linear program in standard form

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Suppose that $\mathbf{x}^*$ is an optimal basic feasible solution. Show that if $\mathbf{d}$ is a feasible direction at $\mathbf{x}^*$, then

$$\begin{aligned} \mathbf{c}^T\mathbf{d} &\geq 0 \\ \mathbf{A}\mathbf{d} &= \mathbf{0} \\ d_i &\geq 0 \text{ if } x_i^* = 0. \end{aligned}$$

These conditions can be use to derive the simplex method.

Exercise 0.2.4. Let f be a function defined on $\Omega \subset \mathbb{R}^n$ and $\mathbf{x} \in \Omega$. We say that the vector $\mathbf{d} \in \mathbb{R}^n$ is a **feasible descent direction** at a point $\mathbf{x} \in \Omega$ if there exists a $\delta > 0$ such that

$$\mathbf{x} + \alpha\mathbf{d} \in \Omega \quad \text{and} \quad f(\mathbf{x} + \alpha\mathbf{d}) < f(\mathbf{x}) \quad \text{for all } 0 < \alpha \leq \delta.$$

Assume that f is continuously differentiable on Ω and $\mathbf{x} \in \Omega$. Show that if $\nabla f(\mathbf{x})\mathbf{p} < 0$, then $\mathbf{p}$ is a descent direction with respect to f at $\mathbf{x}$.

Exercise 0.2.5. Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x_1, x_2) = -x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(i) Determine the feasible directions at $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(0, 2)$.

(ii) Determine whether there exist feasible descent directions at these points, and hence determine which (if any) of the points can be local minimizers.

Exercise 0.2.6. Show that the function $f(x_1, x_2) = x_1^3 x_2^3$ satisfies two conditions

(i) $\nabla f(\mathbf{x}^*) = \mathbf{0}$

(ii) $\mathbf{d}^T \nabla^2 f(\mathbf{x}^*)\mathbf{d} \geq 0$ for all $\mathbf{d}$

at the point $\mathbf{x}^* = (0, 0)$ but $\mathbf{x}^*$ is not a local minimum point.

Exercise 0.2.7. Investigate the stationary points of the function

$$f(x, y) = \frac{x + y}{x^2 + y^2 + 1}.$$

Exercise 0.2.8. Find optimum point(s) of

$$f(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1 + (x_1^2 + x_2^2 + x_3^2)^2.$$

Exercise 0.2.9. Find all the values of the parameter a such that $(1, 0)$ is the minimizer or maximizer of the function

$$f(x_1, x_2) = a^3 x_1 e^{x_2} + 2a^2 \log(x_1 + x_2) - (a + 2)x_1 + 8ax_2 + 16x_1x_2.$$

Exercise 0.2.10. Consider the problem

$$\text{minimize} \quad f(x_1, x_2) = (x_2 - x_1^2)(x_2 - 2x_1^2).$$

(i) Show that the first- and second-order necessary conditions for optimality are satisfied at $(0, 0)$.

(ii) Show that the origin is a local minimizer of f along any line passing through the origin (that is, $x_2 = mx_1$).

(iii) Show that the origin is not a local minimizer of f (consider, for example, curves of the form $x_2 = kx_1^2$). What conclusions can you draw from this?

Exercise 0.2.11. Let

$$f(x_1, x_2) = cx_1^2 + x_2^2 - 2x_1x_2 - 2x_2,$$

where c is some constant.

(i) Determine the stationary points of f for each value of c.

(ii) For what values of c can f have a minimizer? For what values of c can f have a maximizer? Determine the minimizers/maximizers corresponding to such values of c and indicate what kind of minima or maxima (local, global, strict, etc.) they are.

Exercise 0.2.12. Consider the following unconstrained problem:

$$\text{minimize} \quad f(x_1, x_2) = x_1^2 - x_1x_2 + 2x_2^2 - 2x_1 + e^{x_1+x_2}.$$

(i) Write down the first-order necessary conditions for optimality.

(ii) Is $\mathbf{x} = (0, 0)$ a local optimum? If not, find a direction $\mathbf{d}$ along which the function decreases.

(iii) Attempt to minimize the function starting from $\mathbf{x} = (0, 0)$ along the direction $\mathbf{d}$ that you have chosen in part (ii).

[*Hint:* Consider $\varphi(t) = f(\mathbf{x} + t\mathbf{d})$.]

Exercise 0.2.13. Consider the quadratic function

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{c}^T\mathbf{x}.$$

(i) Write the first-order necessary condition. When does a stationary point exist?

(ii) Under what conditions on $\mathbf{Q}$ does a local minimizer exist?

(iii) Under what conditions on $\mathbf{Q}$ does f have a stationary point, but no local minima nor maxima?

Exercise 0.2.14. Consider the problem

$$\text{minimize} \quad f(\mathbf{x}) = |\mathbf{A}\mathbf{x} - \mathbf{b}|^2,$$

where $\mathbf{A}$ is an $m \times n$ matrix with $m \geq n$, and $\mathbf{b}$ is a vector of length m. Assume that the rank of $\mathbf{A}$ is equal to n.

(i) Write down the first-order necessary condition for optimality. Is this also a sufficient condition?

(ii) Write down the optimal solution in closed form.

0.3 CONVEX FUNCTIONS AND OPTIMIZATION OF CONVEX FUNCTIONS

0.3.1 Convex Functions

Exercise 0.3.1. Let $\mathbf{x}_0 \in \mathbb{R}^n$ and $r > 0$. Let $\|\cdot\|$ be an arbitrary norm defined on $\mathbb{R}^n$.

(a) Show that the function $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_0\|$ is convex.

(b) Show that the open ball $B(\mathbf{x}_0, r) = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < r\}$ and the closed ball $\overline{B}(\mathbf{x}_0, r) = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ are convex.

Application: Let $x_k, y_k \in \mathbb{R}$, $1 \leq k \leq n$, be such that $x_k^2 + y_k^2 = 1$ for all k. Let $c_1, \ldots, c_n \in [0, 1]$ with $\sum_{k=1}^n = 1$. Show that

$$\left(\sum_{k=1}^n c_k x_k\right)^2 + \left(\sum_{k=1}^n c_k y_k\right)^2 \leq 1.$$

Exercise 0.3.2. An **ellipsoid** is a set of the form

$$E = \{\mathbf{x} : \mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{b}^T \mathbf{x} + c \leq 0\},$$

where $\mathbf{Q}$ is a positive semidefinite symmetric $n \times n$ matrix, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Show that every ellipsoid is convex.

Show that if $\mathbf{Q}$ is positive definite, then the ellipsoid E is compact.

Exercise 0.3.3. Show that the set

$$K = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T \mathbf{Q} \mathbf{x} \leq (\mathbf{a}^T \mathbf{x})^2,\ \mathbf{a}^T \mathbf{x} \geq \mathbf{0}\},$$

where $\mathbf{Q}$ is an $n \times n$ positive definite matrix and $\mathbf{a} \in \mathbb{R}^n$ is a convex cone.

Exercise 0.3.4. Consider the function $f(x, y) = a x^p y^q$, defined on $C = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ For what values of α, p, and q is the function convex? Strictly convex? For what values is it concave? Strictly concave?

Exercise 0.3.5. Show that any norm on $\mathbb{R}^n$ is convex.

Exercise 0.3.6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and let $g : \mathbb{R} \to \mathbb{R}$ be a convex increasing function. Prove that the composite function $h : \mathbb{R}^n \to \mathbb{R}$ defined by $h(\mathbf{x}) = g\big(f(\mathbf{x})\big)$ is convex.

Exercise 0.3.7. Show that the function $f(x) = x^4$ is strictly convex on $\mathbb{R}$ and that $g(x) = x^p$ for $p > 1$ is strictly convex over $[0, \infty)$.

Exercise 0.3.8. Show that the following functions are convex over the given specified domain:

(i) $f(x_1, x_2, x_3) = -\sqrt{x_1 x_2} + 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$ over $\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} > \mathbf{0}\}$.

(ii) $f(\mathbf{x}) = |\mathbf{x}|^p$, $p > 1$, over $\mathbb{R}^n$.

(iii) $f(\mathbf{x}) = \sum_{i=1}^n x_i \log(x_i) - \left(\sum_{i=1}^n x_i\right) \log\left(\sum_{i=1}^n x_i\right)$ over $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} > \mathbf{0}\}$.

(iv) $f(\mathbf{x}) = \sqrt{\mathbf{x}^T \mathbf{Q} \mathbf{x} + 1}$ over $\mathbb{R}^n$, where $\mathbf{Q}$ is a semidefinite $n \times n$ matrix.

(v) $f(x_1, x_2) = (2x_1^2 + 3x_2^2)\left(\frac{1}{2}x_1^2 + \frac{1}{3}x_2^2\right)$.

Exercise 0.3.9. Let $C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} > 0\}$ and $\phi : C \to \mathbb{R}$ be a convex function. Then the function $f : C \to \mathbb{R}$ defined by

$$f(x, y) = y\phi\left(\frac{x}{y}\right), \quad x, y > 0$$

is convex over C.

Application: Show that the function $f(x, y) = -x^p y^{1-p}$, $0 < p < 1$, is convex on $\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

Exercise 0.3.10. Let f be a convex function defined on a convex set C. Suppose that f is not strictly convex on C. Prove that there exist $\mathbf{x}, \mathbf{y} \in C$, $\mathbf{x} \neq \mathbf{y}$, such that f is affine over the segment $[\mathbf{x}, \mathbf{y}]$, that is,

$$f\big((1 - \lambda)\mathbf{x} + \lambda\mathbf{y}\big) = (1 - \lambda)f(\mathbf{x}) + \lambda f(\mathbf{y}), \quad \text{for all } \lambda \in [0, 1].$$

Exercise 0.3.11. Let $C \subset \mathbb{R}^n$ be convex and $f : C \to \mathbb{R}$. Prove that f is convex if and only if for any $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{d} \neq \mathbf{0}$, the one-dimensional function

$$\varphi_{\mathbf{x},\mathbf{d}}(t) := f(\mathbf{x} + t\mathbf{d})$$

is convex on $I_{\mathbf{x}} := \{t \in \mathbb{R} : \mathbf{x} + t\mathbf{d} \in C\}$.

Exercise 0.3.12. Let $C \subset \mathbb{R}^n$ be a convex set. Let f be a convex function over C, and let g be a strictly convex function over C. Show that the sum function $f + g$ is strictly convex over C.

Exercise 0.3.13. Show that any affine function $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} + \alpha$, where $\mathbf{a} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, is convex.

Exercise 0.3.14. Show that $f : \mathbb{R}^n \to \mathbb{R}$ is both convex and concave if and only if f is affine, that is, there exists $\mathbf{a} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} + \alpha$ for all $\mathbf{x} \in \mathbb{R}$.

Exercise 0.3.15. (Jensen's Inequality) Let f be a function on a convex set $C \subset \mathbb{R}^n$. Prove that f is convex if and only if

$$f\left(\sum_{i=1}^{k} \alpha_i \mathbf{x}_i\right) \leq \sum_{i=1}^{k} \alpha_i f(\mathbf{x}_i).$$

for all $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in C$ and $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^{k} \alpha_i = 1$.

Exercise 0.3.16. Prove that if f and g are convex, twice differentiable, nondecreasing, and positive on $\mathbb{R}$, then the product fg is convex over $\mathbb{R}$. Show by an example that the positivity assumption is necessary to establish the convexity.

Exercise 0.3.17. Let $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}\mathbf{x} + c$ be a quadratic function over $\mathbb{R}^n$. Suppose that $\mathbf{A}$ is positive definite. Determine the global minimizer and the minimal value of f.

Exercise 0.3.18. Let $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}+2\mathbf{b}\mathbf{x}+c$, where $\mathbf{A}$ is an $n\times n$ symmetric matrix, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Show that f is coercive if and only if $\mathbf{A}$ is positive definite.

Exercise 0.3.19. Show that the functions $f(\mathbf{x}) = |\mathbf{x}|^4$ and $g(\mathbf{x}) = (|\mathbf{x}|^2+1)^2$ are strictly convex on $\mathbb{R}^n$.

Exercise 0.3.20. Let $f : C \to \mathbb{R}$ be a convex function which is not constant over the convex set $C \subset \mathbb{R}^n$. Show that f does not attain a maximum at a point in $\text{int}(C)$.

Exercise 0.3.21. Let $C \subset \mathbb{R}^n$ be a nonempty closed convex set and $\mathbf{y} \in \mathbb{R}^n$. Consider the problem

$$\begin{aligned} \text{minimize} \qquad & f(\mathbf{x}) = |\mathbf{x} - \mathbf{y}|^2 \\ \text{subject to} \qquad & \mathbf{x} \in C. \end{aligned}$$

(a) Show that the problem has a unique solution.

(b) Show that $\mathbf{x}^* \in C$ is the solution of the problem if and only if

$$(\mathbf{y} - \mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \leq 0 \quad \text{for any } \mathbf{x} \in C.$$

Exercise 0.3.22. Suppose that $\Omega \subset \mathbb{R}^n$ is convex and that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and in C^1 on Ω. Then, the value of $\nabla f(x)$ is constant on the optimal solution set Γ. Further, suppose that $\mathbf{x}^* \in \Gamma$. Then

$$\Gamma = \big\{\mathbf{x} \in \Omega : \nabla f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) = 0 \quad \text{and} \quad \nabla f(\mathbf{x}) = \nabla f(\mathbf{x}^*)\big\}.$$

0.3.2 Use Convex Functions to Prove Inequalities

Exercise 0.3.23. (a) Prove that the function $f(x) = \frac{1}{1+e^x}$ is strictly convex over $[0, \infty)$.

(b) Prove that for any $a_1, a_2, \ldots, a_n \geq 1$ the inequality

$$\sum_{i=1}^{n} \frac{1}{1+a_i} \geq \frac{n}{1 + \sqrt[n]{a_1 a_2 \cdots a_n}}$$

holds.

0.3.3 Convex Optimization

Exercise 0.3.24. Given a nonempty closed convex set $C \subset \mathbb{R}^n$, the **orthogonal projection operator** $P_C : \mathbb{R}^n \to C$ is defined by

$$P_C(\mathbf{x}) = \{\mathbf{z} \in C : |\mathbf{x} - \mathbf{z}|^2 \leq |\mathbf{x} - \mathbf{y}|^2 \text{ for all } \mathbf{y} \in C\}.$$

(a) Show that $P_C(\mathbf{x})$ is singleton for all $\mathbf{x} \in \mathbb{R}^n$.

(b) Show that for $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{z} = P_C(\mathbf{x})$ if and only if

$$(\mathbf{x} - \mathbf{z})^T(\mathbf{y} - \mathbf{z}) \leq 0 \quad \text{for any } \mathbf{y} \in C. \tag{0.3.1}$$

Exercise 0.3.25. Find the global minimizers of

(a) $f(x, y) = e^{x-y} + e^{y-x}$ (b) $g(x, y, z) = e^{x-y} + e^{x+y}$.

Exercise 0.3.26. Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g(\mathbf{x}) \leq 0, \\ & \mathbf{x} \in \Omega, \end{aligned}$$

where f and g are convex functions over $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is a convex set. Suppose that $\mathbf{x}^*$ is an optimal solution of the above problem that satisfies $g(\mathbf{x}^*) < 0$. Show that $\mathbf{x}^*$ is also an optimal solution of the problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{x} \in \Omega. \end{aligned}$$

Exercise 0.3.27. Let f be a strictly convex function over $\mathbb{R}^m$ and let g be a convex function over $\mathbb{R}^n$. Define the function

$$h(\mathbf{x}) = f(\mathbf{A}\mathbf{x}) + g(\mathbf{x}),$$

where $\mathbf{A}$ is an $m \times n$ matrix. Assume that $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal solutions of the unconstrained problem of minimizing h. Show that $\mathbf{A}\mathbf{x}^* = \mathbf{A}\mathbf{y}^*$.

0.4 NEWTON METHODS

Exercise 0.4.1. Use Newton's method to solve

$$\text{minimize} \quad f(x_1, x_2) = 5x_1^4 + 6x_2^4 - 6x_1^2 + 2x_1x_2 + 5x_2^2 + 15x_1 - 7x_2 + 13.$$

Use the initial guess $(1, 1)$. Make sure that you have found a minimum and not a maximum.

Exercise 0.4.2. Construct the Newton's Method sequence $\{\mathbf{x}_k\}$ for minimizing the function

$$f(x_1, x_2) = x_1^4 + 2x_1^2x_2^2 + x_2^4$$

with initial point $\mathbf{x}_0 = (a, a)$, where $a \in \mathbb{R}$.

ANS. $\mathbf{x}_k = \left(\left(\frac{2}{3}\right)^k a, \left(\frac{2}{3}\right)^k a \right).$

Exercise 0.4.3. Consider the problem

$$\text{minimize} \quad f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{c}^T\mathbf{x},$$

where $\mathbf{Q}$ is a positive-definite matrix. Prove that Newton's method will determine the minimizer of f in one iteration, regardless of the starting point.