

DIFFERENTIAL EQUATIONS

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References

William E. Boyce, Richard C. DiPrima, **Elementary Differential Equations and Boundary Problems**, 8th Ed., John Wiley & Sons, 2005.

Chapter 1 INTRODUCTION

Contents

1. Some Basic Mathematical Models
2. Classification of Differential Equations

1.1 SOME BASIC MATHEMATICAL MODELS

Why are differential equations important?

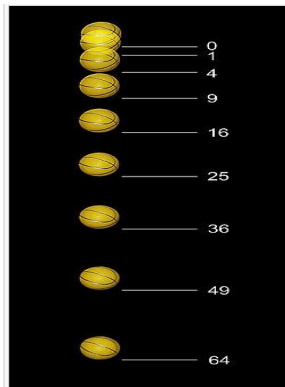
- Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen.
- When expressed in mathematical terms, the relations are equations and the rates are derivatives.
- So these mathematical models often involve an equation in which a function and its derivatives play important roles. Such equations are called **differential equations**.

A Falling Object

Suppose an object is dropped from a height at time $t = 0$.

Let $h(t)$ be the **height** of the object at time t ;

Let $a(t)$ be the **acceleration** and let $v(t)$ be the **velocity**.



A Falling Object

The relationships between a , v and h are as follows:

$$a(t) = \frac{dv}{dt}, \quad v(t) = \frac{dh}{dt}.$$

For a falling object, $a(t)$ is constant and is equal to $g = -9.8m/s^2$ (the acceleration due to gravity).

Combining the above equations, we have the differential equation

$$\frac{dh^2}{dt^2} = g.$$

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Combining the above equations, we have the differential equation

$$\frac{dh^2}{dt^2} = g.$$

Integrate both sides of the above equation to obtain $\frac{dh}{dt} = gt + v_0$.

Integrate one more time to obtain

$$h(t) = (1/2)gt^2 + v_0t + h_0$$

The above equation describes the height of a falling object, from an initial height h_0 at an initial velocity v_0 , as a function of time.

Business

A large corporation starts at time $t = 0$ to invest part of its receipts at a rate of P dollars per year in a fund for future corporate expansion.

Assume that the fund earns r percent interest per year compounded continuously. So, the rate of growth of the amount A in the fund is given by

$$\frac{dA}{dt} = rA + P,$$

where $A = 0$ when $t = 0$.

Find the amount A provided $P = \$100000$, $r = 12\%$, and $t = 5$ years.

Modeling Hybrid Selection

You are studying a population of beetles to determine how quickly characteristic D will pass from one generation to the next. This hybrid selection model is described by

$$\frac{dy}{dt} = ky(1 - y)(2 - y);$$

where y represents the percent of the population that has characteristic D and t represents the time (measured in generations) and k is a constant.

At the beginning of your study ($t = 0$), you find that half the population has characteristic D.

After four generations ($t = 4$) you find that 80% of the population has characteristic D.

Find the percent of the population that will have characteristic D after 10 generations.

Radioactivity

In the case of **radioactive decay**, we assume that **the rate of decay is proportional to the amount of radioactive substance present**. This leads to the equation

$$\frac{dA}{dt} = -kA, \quad k > 0,$$

where A is the unknown amount of radioactive substance present at time t and k is the proportionality constant.

The solution of this equation is

$$A = Ce^{-kt}.$$

The constants C and k can be determined if the initial amount of radioactive substance and the half-life of the substance are given.

Modeling a Chemical Reaction

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When $t = 0$, 60 grams of A are present, and after 1 hour ($t = 1$) only 10 grams of A remain unconverted. How much of A is present after 2 hours?

Modeling a Chemical Reaction

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Solution Let y be the unconverted amount of substance A at any time t . From the given assumption about the conversion rate, you can write the differential equation as follows.

$$\frac{dy}{dt} = ky^2$$

Diagram illustrating the components of the differential equation $\frac{dy}{dt} = ky^2$:

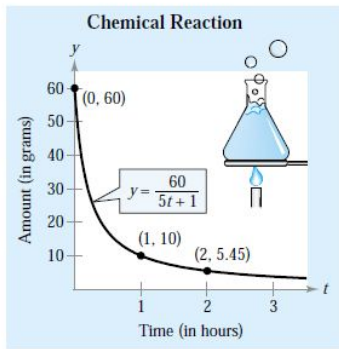
- $\frac{dy}{dt}$ is labeled "Rate of change of y ".
- k is labeled "is proportional to".
- y^2 is labeled "the square of y ".

Modeling a Chemical Reaction

Solve the equation, to get

$$y(t) = \frac{60}{5t + 1}.$$

So $y(2) = 5.45$



Field Mice and Owls

Consider a population of field mice who inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. Denote time by t and the mouse population by $p(t)$, then

$$\frac{dp}{dt} = rp,$$

where r is called the **rate constant** or **growth rate**.

The solution is $p(t) = Ce^{rt}$.

Other Examples

Example 1.1 In the study of an electric circuit consisting of resistors, inductors, and capacitors, an application of *Kirchhoff's law* leads to the equation

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = E(t),$$

where L is the inductance, R is the resistance, C is the capacitance, $E(t)$ is the electromotive force, $q(t)$ is the charge, and t is the time.

Other Examples

Example 1.2 In the study of the gravitational equilibrium of a star, an application of *Newton's law of gravity* and of the *Stefan-Boltzmann law* for gases leads to the equilibrium equation

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi\rho G,$$

where P is the sum of the gas kinetic pressure and the radiation pressure, r is the distance from the center of the star, ρ is the density of matter, and G is the gravitational constant.

Other Examples

Example 1.3 In psychology, one model of the learning of a task involves the equation

$$\frac{dy/dt}{y^{3/2}(1-y)^{3/2}} = \frac{2p}{\sqrt{n}}.$$

Here y represents the state of the learner or the learner's skill level as a function of time t . The constants p and n depend on the individual learner and the nature of the task.

Example 1.4 In the study of vibrating strings and the propagation of waves, we find the differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,$$

where t represents time, x the location along the string, and u the displacement of the string, which is a function of time and location, $u = u(t, x)$.

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Independent and dependent variables

When an equation involves one or more derivatives with respect to a particular variable, that variable is called an **independent** variable. A variable is called **dependent** if a derivative of that variable occurs.

In the equation

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = E \omega \cos \omega t \quad (0.1)$$

i is the **dependent variable**, t is the **independent variable**, and L , R , C , E , and ω are called parameters.

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The equation

$$\frac{\partial u}{\partial t} = h^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (0.2)$$

has one dependent variable u and three independent variables.

Ordinary and Partial Differential Equations

Definition 2.1 A differential equation involving **ordinary derivatives** with respect to a **single** independent variable is called an **ordinary differential equation**.

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For instance, Equation (0.1) is an ordinary differential equation, whereas Equation (0.2) is a partial differential equation.

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Order

Definition 2.2 The **order** of a differential equation is the order of the highest-order derivatives that appear in the equation.

For instance,

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right) + y^5 = 0$$

is an equation of “order two” because $\frac{d^2y}{dx^2}$ is the highest order derivative present. It is also referred to as “**second-order differential equation**”.

The differential equation

$$y^{(4)} + 2xy''' + y'' + 2y' + x^2y = x^2 + 1,$$

is a **fourth order differential equation**.

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

More generally, the equation

$$F(x, y, y', \dots, y^{(n)}) = 0 \quad (0.3)$$

is called an **ordinary differential equation of the n th order**.

Under suitable restrictions on the function F , Equation (0.3) **can be solved explicitly** for $y^{(n)}$ in terms of the other variables $x, y, y', \dots, y^{(n-1)}$, to obtain

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}). \quad (0.4)$$

A single equation of the form (0.3) may actually represent more than one equation of the form (0.4).

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

For example, the equation

$$x(y')^2 + 4y' - 6x^2 = 0$$

actually represents two different equations,

$$y' = \frac{-2 + \sqrt{4 + 6x^3}}{x} \quad \text{or} \quad y' = \frac{-2 - \sqrt{4 + 6x^3}}{x}.$$

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Similarly, the equation

$$(y')^2 - (x + y)y' + xy = 0$$

leads to the two equations

$$y' = x \quad \text{or} \quad y' = y.$$

Linear and Nonlinear Equations

Definition 2.3 A **linear** differential equation is any equation that can be written in the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x),$$

where $a_n(x)$, $a_{n-1}(x)$, ..., $a_0(x)$, and $g(x)$ depend **only on the independent variable x , NOT on y .**

If an equation is not linear, then we call it **nonlinear**.

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

For example, the equation

$$x^2 y''' + xy' + (x^2 - 1)y = 4x^3$$

is a **linear** third order ordinary differential equation,

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

For example, the equation

$$x^2 y''' + xy' + (x^2 - 1)y = 4x^3$$

is a **linear** third order ordinary differential equation, whereas

$$y' + \sin y = 0$$

and

$$y'' - yy' = \cos x$$

are **nonlinear** because of the terms $\sin y$ and yy' .

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Solutions

Definition 2.4 An **explicit solution** of the ordinary differential equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \quad (0.5)$$

on the interval $a < x < b$ is a function ϕ such that $\phi', \phi'', \dots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(x) = f(x, \phi(x), \phi'(x), \dots, \phi^{(n-1)}(x))$$

for every x in $a < x < b$.

Example 2.1 Show that for any choice of the constant C , the function

$$\phi(x) = Ce^{2014x}$$

is a solution of the equation

$$y' = 2014y.$$

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Example: Consider the differential equation

$$y'' - 2y' + y = 0.$$

Then $\phi_1(x) = e^x, x \in \mathbb{R}$ and $\phi_2(x) = xe^x, x \in \mathbb{R}$ are solutions of the given differential equation.

Furthermore, for any $c_1, c_2 \in \mathbb{R}$, the function

$$y(x) = c_1 e^x + c_2 x e^x, \quad x \in \mathbb{R}$$

is also a solution of the given differential equation.

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Definition 2.5 A relation $G(x, y) = 0$ is said to be an **implicit solution** of Equation (0.5) on the interval (a, b) if it defines one or more explicit solutions on (a, b) .

Example 2.2 Show that

$$y^2 - x^3 + 8 = 0$$

is an implicit solution of

$$\frac{dy}{dx} = \frac{3x^2}{2y} \tag{0.6}$$

on the interval $(2, \infty)$.

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Example 2.3 Verify that

$$x + y + e^{xy} = C,$$

where C is an arbitrary constant, gives a family of implicit solutions of

$$(1 + xe^{xy}) \frac{dy}{dx} + 1 + ye^{xy} = 0 \quad (0.7)$$

and graph several of these solution curves.

For brevity, we hereafter use the term *solution* to mean either an explicit or an implicit solution.

Initial Value Problems

Definition 2.6 By an **initial value problem** for an n th order differential equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$\begin{aligned} y(x_0) &= y_0, \\ y'(x_0) &= y_1 \\ &\dots \\ y^{(n-1)}(x_0) &= y_{n-1}, \end{aligned}$$

where $x_0 \in I$ and y_0, y_1, \dots, y_{n-1} are given constants.

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Example 2.4 Show that $\phi(x) = \sin x - \cos x$ is a solution to the initial value problem

$$y'' + y = 0; \quad y(0) = -1, \quad y'(0) = 1.$$

Example 2.5 Consider again the differential equation

$$(x^2 + 1)y' + xy = 0 \tag{0.8}$$

and the initial condition

$$y(0) = y_0. \tag{0.9}$$

If $y \neq 0$, then we can rewrite the differential equation as

$$\frac{y' dx}{y} = -\frac{x dx}{x^2 + 1}.$$

By integrating both sides, we find that

$$\ln |y| = \int -\frac{x dx}{x^2 + 1} = \int -\frac{1}{2} \cdot \frac{d(x^2 + 1)}{x^2 + 1} = -\frac{1}{2} \ln(x^2 + 1) + C_1$$

where C_1 is arbitrary.

1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Then taking the exponential of both sides of this equation and solving for y , we obtain

$$y = \frac{C}{\sqrt{x^2 + 1}}, \quad (0.10)$$

where $C = \pm e^{C_1}$ is also arbitrary. Observe that $C = 0$ corresponds to the solution $y = 0$.

Finally the initial condition requires that $C = y_0$, so the solution of the initial value problem (0.8)–(0.9) is

$$y = \frac{y_0}{\sqrt{x^2 + 1}}.$$

The expression (0.10) contains **all** possible solutions of the differential equation (0.8). So it is called the **general solution** of the differential equation (0.8).

Direction Fields

A first order equation

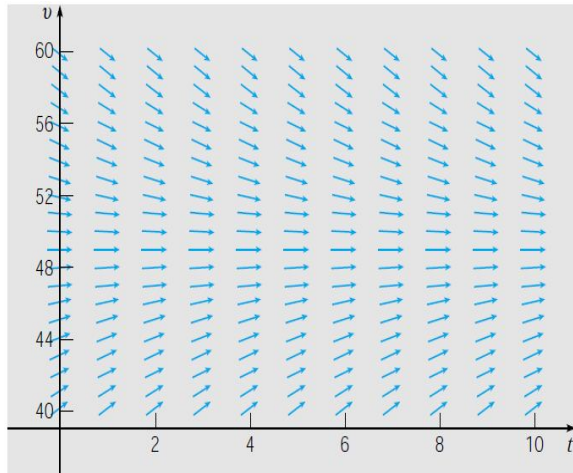
$$\frac{dy}{dx} = f(x, y)$$

specifies a **slope** at each point in the plane where f is defined. In other word, **it gives the direction that a solution to the equation must have at each point.**

A plot of the directions associated with various points in the plane is called the **direction field** for the differential equation.

Example 2.6 The direction field for the equation

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$



Chapter 2 FIRST ORDER DIFFERENTIAL EQUATIONS

Contents

1. Linear Equations
2. Separable Equations
3. Modeling with First Order Equations
4. Exact Equations

2.1 LINEAR EQUATIONS

Definition 1.1 A **linear first order** differential equation is an equation of the form

$$a_0(x) \frac{dy}{dx} + a_1(x)y = b(x), \quad (0.11)$$

where $a_0(x)$, $a_1(x)$, and $b(x)$ **depend only on the independent variable x , NOT on y .**

We assume that the functions $a_0(x)$, $a_1(x)$ and $b(x)$ are continuous on an interval and that $a_0(x) \neq 0$ on that interval. Then we can rewrite Equation (0.11) in the **standard form**

$$\frac{dy}{dx} + P(x)y = Q(x), \quad (0.12)$$

where $P(x)$ and $Q(x)$ are **continuous functions** on the interval.

2.1 LINEAR EQUATIONS

For instance, the equation

$$(\sin x) \frac{dy}{dx} = x^2 \sin x - y \cos x$$

is **linear** since it can be written in the form

$$(\sin x) \frac{dy}{dx} + (\cos x)y = x^2 \sin x.$$

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is **linear** since it can be written in the form

$$(\sin x) \frac{dy}{dx} + (\cos x)y = x^2 \sin x.$$

However, the equation

$$yy' + x^2 y^3 = e^x$$

is **not linear**, because it cannot be put in the form of Equation (0.12).

2.1 SOLVING A LINEAR EQUATION:

Step 1: Rewrite the given equation as $\frac{dy}{dx} + P(x)y = Q(x)$.

Step 2: Finding an integrating factor

We multiply Equation in Step 1 by an yet undetermined function $\mu(x)$, obtaining

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x). \quad (0.13)$$

We like to have $\mu'(x) = \mu(x)P(x)$. Thus

$$\frac{1}{\mu(x)} \frac{d\mu(x)}{dx} = P(x) \quad \text{or} \quad \frac{d}{dx} \ln |\mu(x)| = P(x).$$

It follows that

$$\mu(x) = Ce^{\int P(x)dx},$$

where C is an arbitrary constant. Choose $C = 1$ to get:

$$\mu(x) = \exp \left(\int P(x)dx \right) \text{ (integrating factor)}$$

2.1 LINEAR EQUATIONS

Step 3: Rewrite (0.13) as

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x).$$

Thus

$$\mu(x)y = \int \mu(x)Q(x) dx + C,$$

where C is an arbitrary constant. Therefore

$$y = \frac{1}{\mu(x)} \left[\int \mu(x)Q(x) dx + C \right]$$

This is the **general solution** of Equation (0.13).

2.1 LINEAR EQUATIONS

Theorem: Suppose $P(x)$ and $Q(x)$ are continuous on (a, b) that contains the point x_0 . Then for any choice of initial value y_0 , *there exists a unique solution* $y(x)$ *on the whole interval* (a, b) to the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0. \quad (0.14)$$

In fact, the solution is given by

$$y = \frac{1}{\mu(x)} \left[\int \mu(x)Q(x) dx + C \right],$$

where $\mu(x)$ is an integrating factor and C is a suitable constant.

2.1 LINEAR EQUATIONS

Example 1.1 Solve the equation

$$\frac{dy}{dx} + \frac{2}{x}y = 8x.$$

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$$\frac{dy}{dx} + \frac{2}{x}y = 8x.$$

Solution: The integrating factor is given by

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \int \frac{1}{x} dx} = e^{2 \ln |x|} = e^{\ln |x|^2} = x^2.$$

Multiplying both sides of the given equation by x^2 , we have

$$x^2 \frac{dy}{dx} + 2xy = 8x^3,$$

or equivalently,

$$\frac{d}{dx}(x^2 y) = 8x^3.$$

Thus

$$x^2 y = \int 8x^3 dx = 2x^4 + C, \quad \text{or equivalently, } y = 2x^2 + \frac{C}{x^2}.$$

Example 1.2 Find the general solution to

$$\frac{dy}{dx} + y \tan x = \sin 2x.$$

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$$\frac{dy}{dx} + y \tan x = \sin 2x.$$

Solution: The integrating factor is defined by

$$\mu(x) = e^{\int \tan x dx} = e^{\int \frac{\sin x}{\cos x} dx} = e^{-\ln |\cos x|} = \frac{1}{|\cos x|}.$$

Note that $\mu(x) = \frac{1}{\cos x}$ is also an integrating factor to the given equation. Multiplying both sides of the given equation by $\frac{1}{\cos x}$, we have

$$\frac{1}{\cos x} \frac{dy}{dx} + y \frac{\sin x}{(\cos x)^2} = 2 \sin x,$$

or equivalently,

$$\frac{d}{dx} \left(\frac{1}{\cos x} y \right) = 2 \sin x.$$

Thus

$$\frac{1}{\cos x} y = \int 2 \sin x dx = -2 \cos x + C, \text{ or equivalently, } y = -2(\cos x)^2 + C \cos x.$$

Example 1.3 Solve the initial value problem

$$(x^2 + 1)y' + xy = -x, \quad y(0) = 1.$$

2.1 LINEAR EQUATIONS

Example 1.4 A rock contains two radioactive isotopes RA_1 and RA_2 that belong to the same radioactive series; that is, RA_1 decays into RA_2 which then decays into stable atoms. Assume that the rate that RA_1 decays into RA_2 is $50e^{-10t}$ kg/sec. Since the rate of decay of RA_2 is proportional to the mass $y(t)$ of RA_2 present, then the rate of change in RA_2 is

$$\begin{aligned}\frac{dy}{dt} &= \text{rate of creation} - \text{rate of decay}, \\ \frac{dy}{dt} &= 50e^{-10t} - ky,\end{aligned}$$

where $k > 0$ is the decay constant. If $k = 2/\text{sec}$ and initially $y(0) = 40$ kg, find the mass of RA_2 for $t \geq 0$.

2.2 SEPARABLE EQUATIONS

Definition 2.1 A differential equation of the form

$$N(y)dy = M(x)dx$$

is said to be **separable**.

For instance, the equation

$$\frac{dy}{dx} = \frac{2x + xy}{y^2 + 1}$$

is separable. However, the equation

$$\frac{dy}{dx} = 1 + xy$$

admits no factorization of the right-hand side and so is **NOT** separable.

2.2 SEPARABLE EQUATIONS

In general, *an equation that can be written in the form*

$$\frac{dy}{dx} = g(x)h(y)$$

is a separable.

Example 2.1 Solve the equation

$$x \frac{dy}{dx} = 2y.$$

Example 2.2 Solve the initial value problem

$$\frac{dy}{dx} = 6y^2x, \quad y(0) = \frac{1}{25}.$$

Example 2.3 Solve the initial value problem

$$2x(y+1)dx - ydy = 0, \quad y(0) = -2.$$

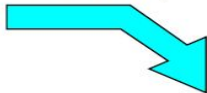
2.3 MODELING WITH FIRST ORDER EQUATIONS

Mixing Problem:

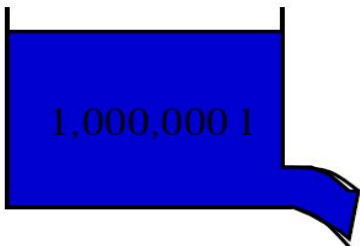
Initially, a water tower contains 1 million litres of pure water. Two valves are then opened. One valve allows a solution of water and fluoride, with a concentration of 0.1 kg of fluoride per litre of water, to flow into the tower at a rate of 80 litres per minute. The other valve allows the solution in the tank to be drained at 80 litres per minute. Assume that the solution is mixed constantly, so that we have a homogeneous fluid in the tank, i.e., at any point in time, the concentration of fluoride in the water is uniform throughout the tank.

- (a) Find an expression for the amount (in kg) of fluoride in the water tower t minutes after the valves are opened.
- (b) Determine how long it will take for the concentration of fluoride in the water to reach .05 kg/l.

80 l/min



1,000,000 l



80 l/min

Solution: (a) Let $y = y(t)$ be the number of kilograms of fluoride in the tank at time t . We need to **find both the input and output rates**. Since water (with fluoride) is flowing in and out at 80 l/min, the **volume in the tank remains constant at 1,000,000 l**.

Solution: (a) Let $y = y(t)$ be the number of kilograms of fluoride in the tank at time t . We need to **find both the input and output rates**. Since water (with fluoride) is flowing in and out at 80 l/min, the **volume in the tank remains constant at 1,000,000 l**.

Input Rate: We have 80 litres per minute entering the tank, with each litre containing 0.1 kg of fluoride, so each minute we have $(80)(0.1) = 8$ kg of fluoride **entering the tank**. Thus, we have: **Input Rate = 8 kg/min**.

Solution: (a) Let $y = y(t)$ be the number of kilograms of fluoride in the tank at time t . We need to **find both the input and output rates**. Since water (with fluoride) is flowing in and out at 80 l/min, the **volume in the tank remains constant at 1,000,000 l**.

Input Rate: We have 80 litres per minute entering the tank, with each litre containing 0.1 kg of fluoride, so each minute we have $(80)(0.1) = 8$ kg of fluoride **entering the tank**. Thus, we have: **Input Rate = 8 kg/min**.

Output Rate: The entire system contains y kg of fluoride at any given time, distributed throughout the one million litres in the tank. This means that each litre of water in the tank contains $\frac{y}{1000000}$ kg of fluoride at any given moment in time. We have 80 litres per minute leaving the system, therefore in any one minute we have $(80)(\frac{y}{1000000})$ kg of fluoride **leaving the tank**. That is, we have: **Output Rate = $(8)(\frac{y}{100000})$ kg/min**.

We use the differential equation

$$\frac{dy}{dt} = (\text{input rate}) - (\text{output rate}).$$

Substituting the rates above into this formula, we get:

$$\frac{dy}{dt} = 8 - (8)\left(\frac{y}{100000}\right)$$

or equivalently

$$\frac{dy}{dt} + 0.00008y = 8.$$

The general solution is

$$y = 100,000 + Ce^{-.00008t}.$$

We use the differential equation

$$\frac{dy}{dt} = (\text{input rate}) - (\text{output rate}).$$

Substituting the rates above into this formula, we get:

$$\frac{dy}{dt} = 8 - (8)\left(\frac{y}{100000}\right)$$

or equivalently

$$\frac{dy}{dt} + 0.00008y = 8.$$

The general solution is

$$y = 100,000 + Ce^{-.00008t}.$$

At time $t = 0$ there was no fluoride in the tank, so we have the side condition $y(0) = 0$. It implies that $C = -100000$.

We substitute this into the general solution to get the **specific solution**:

$$y = 100000 - 100000e^{-0.00008t}.$$

(b) We need to determine how long it will take for the concentration to reach a level of $.05\text{ kg/l}$. This concentration of fluoride in the 1,000,000 l in the tank corresponds to having $(1000000)(.05) = 50000$ kg of fluoride in total in the tank.

Thus we see that we need to find the value of t that satisfies $y = 50000$. We do this using the formula we found in part (a).

$$50000 = 100000 - 100000e^{-0.00008t}.$$

This yields

$$t = \frac{\ln 2}{0.00008} \simeq 8664.34.$$

3.2 Newton's Law of Cooling

Experiment has shown that under certain conditions, a good approximation to the temperature of an object can be obtained by using **Newton's law of cooling**:

The temperature of a body changes at a rate that is proportional to the difference in temperature between the outside medium and the body itself.

This means

$$\frac{dT}{dt} = k(T - R)$$

where T is the temperature of the object at time t , R is the temperature of the surrounding environment (constant) and k is a constant of proportionality.

Example:

A pot of liquid is put on the stove to boil. The temperature of the liquid reaches $170^{\circ}F$ and then the pot is taken off the burner and placed on a counter in the kitchen. The temperature of the air in the kitchen is $76^{\circ}F$. After two minutes the temperature of the liquid in the pot is $123^{\circ}F$. How long before the temperature of the liquid in the pot will be $84^{\circ}F$?

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Solution: Let T be the temperature of the pot of liquid at time t . By the Newton's law of cooling, we have

$$\frac{dT}{dt} = k(T - 76), \quad T(0) = 170.$$

Solve it, to get $T(t) = 76 + 94e^{kt}$, $t \geq 0$.

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Solve it, to get $T(t) = 76 + 94e^{kt}$, $t \geq 0$.

"After two minutes the temperature of the liquid in the pot is $123^{\circ}F$ " means

$$T(2) = 123.$$

This gives

$$123 = 76 + 94e^{2k}.$$

Thus

$$k = \frac{1}{2} \ln \frac{1}{2} \simeq -0.3465 \text{ and } T(t) = 76 + 94e^{-0.3465t}.$$

We have to find t such that

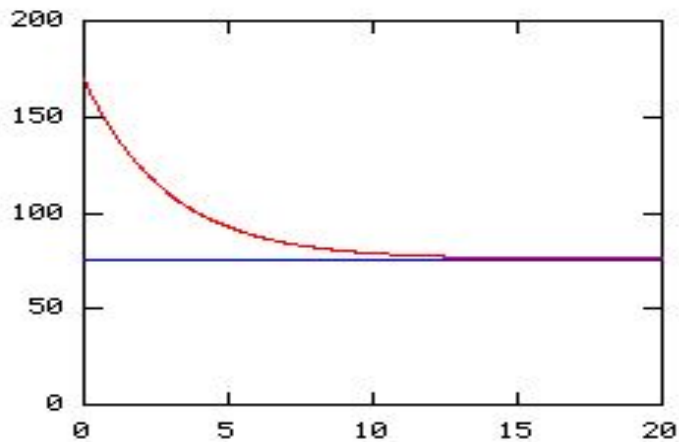
$$T(t) = 84.$$

This gives

$$84 = 76 + 94e^{-0.3465t}, \text{ or } \ln \frac{4}{47} = -0.3465t.$$

So we have $T \simeq 7.11$ minutes.

Solution Function Graph



3.3 Population Models

Let $p(t)$ be the population of a species at time t .

As long as there are sufficient space and ample food supply for the species, we can assume that the growth and dead rates are proportional to the population present.

Hence a model for population of the species is

$$\frac{dp}{dt} = k_1 p - k_2 p, \quad p(0) = p_0,$$

where k_1 , k_2 are the proportionality constants for the growth rate and the dead rate, respectively, and p_0 is the population at time $t = 0$.

2.3 MODELING WITH FIRST ORDER EQUATIONS

This gives the mathematical model

$$\frac{dp}{dt} = kp, \quad p(0) = p_0,$$

which is called the **Malthusian** or **exponential law** of population growth.

Solving this initial value problem for $p(t)$, gives

$$p(t) = p_0 e^{kt}$$

Example 3.3 In 1790 the population of the United States was 3.93 million, and in 1890 the population was 62.95 million. Using the Malthusian model, estimate the population of the United States as a function of time.

2.3 MODELING WITH FIRST ORDER EQUATIONS

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2.3 MODELING WITH FIRST ORDER EQUATIONS

- The Malthusian model is satisfactory as long as the population is not too large. It considers only death by natural causes and does not reflect the fact that individual members are competing with each other for the limited space, natural resources and so on.
- A suitable choice of a competition term is $-bp^2$, where b is a constant. We have, therefore, the modified equation

$$\frac{dp}{dt} = ap - bp^2, \quad p(0) = p_0 \quad (0.15)$$

This is called the **logistic model**.

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This is called the **logistic model**.

The equation (0.15) is separable and can be solved easily. Its solution is

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)e^{-at}}. \quad (0.16)$$

The function $p(t)$ given in (0.16) is called the **logistic function**. Its graph is called the **logistic curve**.

2.3 MODELING WITH FIRST ORDER EQUATIONS

An important property of the logistic function is that

$$\lim_{t \rightarrow \infty} p(t) = \frac{a}{b} \quad \text{if } p_0 > 0.$$

The two functions $p(t) \equiv 0$ and $p(t) \equiv a/b$ are also solutions of the logistic equation (0.15). They are referred to as **equilibrium populations**.

Example 3.4 Taking the 1790 population of 3.93 million as the initial population and given the 1840 and 1890 populations of 17.07 and 62.95 million, respectively, use the logistic model to estimate the population at time t .

2.3 MODELING WITH FIRST ORDER EQUATIONS

A comparison of the Malthusian and logistic models with U.S. census data (Population is given in millions)

| Year | U.S. Census | Malthusian (Ex. 3.3) | Logistic (Ex. 3.4) |
|------|-------------|----------------------|--------------------|
| 1790 | 3.93 | 3.93 | 3.93 |
| 1800 | 5.31 | 5.19 | 5.30 |
| 1810 | 7.24 | 6.84 | 7.13 |
| 1820 | 9.64 | 9.03 | 9.58 |
| 1830 | 12.87 | 11.92 | 12.82 |
| 1840 | 17.07 | 15.73 | 17.07 |
| 1850 | 23.19 | 20.76 | 22.60 |
| 1860 | 31.44 | 27.39 | 29.70 |
| 1870 | 39.82 | 36.15 | 38.65 |
| 1880 | 50.16 | 47.70 | 49.69 |
| 1890 | 62.95 | 62.95 | 62.95 |

2.3 MODELING WITH FIRST ORDER EQUATIONS

A comparison of the Malthusian and logistic models with U.S. census data (Population is given in millions)

| Year | U.S. Census | Malthusian (Ex. 3.3) | Logistic (Ex. 3.4) |
|------|-------------|----------------------|--------------------|
| 1900 | 75.99 | 83.07 | 78.37 |
| 1910 | 91.97 | 109.63 | 95.64 |
| 1920 | 105.71 | 144.67 | 114.21 |
| 1930 | 122.78 | 190.91 | 133.28 |
| 1940 | 131.67 | 251.94 | 152.00 |
| 1950 | 151.33 | 332.47 | 169.56 |
| 1960 | 179.32 | 438.75 | 185.35 |
| 1970 | 203.21 | 579.00 | 199.01 |
| 1980 | 226.50 | 764.08 | 210.46 |
| 1990 | 249.63 | 1008.32 | 219.77 |
| 2000 | ? | 1330.63 | 227.19 |

2.4 EXACT EQUATIONS

Many equations in this chapter may be written

$$\frac{dy}{dx} = f(x, y).$$

It is sometimes convenient to write the equation in the form

$$M(x, y)dx + N(x, y)dy = 0 \quad (0.17)$$

To solve Equation (0.17), it is helpful to know whether the left hand side is a total differential.

- Recall that the **total differential** $dF(x, y)$ of a function $F(x, y)$ of two variables is defined by

$$dF(x, y) = \frac{\partial F}{\partial x}(x, y)dx + \frac{\partial F}{\partial y}(x, y)dy,$$

where dx and dy are arbitrary increments.

2.4 EXACT EQUATIONS

Definition 4.1 An equation

$$M(x, y)dx + N(x, y)dy = 0 \quad (0.18)$$

is called **exact** in a region R if there is a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all (x, y) in R .

In other words, Equation (0.18) is exact if we can write it in the form

$$dF(x, y) = M(x, y)dx + N(x, y)dy = 0$$

for some function $F(x, y)$.

2.4 EXACT EQUATIONS

Example 4.1 Show that the equation

$$(y - 3x^2)dx + (x - 1)dy = 0$$

is exact and find its solutions.

If the equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{0.19}$$

is exact, then by definition F exists such that

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y).$$

These two equations lead to

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

2.4 EXACT EQUATIONS

But, from calculus,

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x},$$

provided that these partial derivatives are continuous.

Therefore, if Equation (0.19) is exact, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \quad (0.20)$$

Conversely, it is showed that if condition (0.20) is satisfied, (0.19) is an exact equation.

2.4 EXACT EQUATIONS

Theorem Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a region R . Then

$$M(x, y)dx + N(x, y)dy = 0$$

is an exact equation in R if and only if

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

for all $(x, y) \in R$.

2.4 EXACT EQUATIONS

Example 4.2 Solve

$$(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0.$$

Example 4.3 Solve

$$(1 + e^x y + x e^x y)dx + (x e^x + 2)dy = 0.$$

| Pages | Exercises | Assignments |
|--------|----------------------|---|
| 24–26 | 3, 8, 12, 19, 22, 25 | 4, 6, 10, 13, 14, 18, 20, 21, 24, 26, 27, 30 |
| 39–41 | 13, 15, 30 | 16, 19, 20, 27, 28, 31, 33, 35, 38, 39 |
| 47–50 | 3, 7, 21, 26, 30 | 22, 25, 31, 36, 38 |
| 59–68 | 3, 13 | 1, 4, 6, 12, 24, 29 |
| 99–101 | 3, 5, 15 | 4, 6, 9, 11, 12, 16, 20, 22 |