INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

**FINAL EXAMINATION**
**June 2017**
**Duration: 120 minutes**

| **SUBJECT: REAL ANALYSIS** | |
|---|---|
| Head of Dept. of Mathematics: | Lecturer: |
| Assoc. Prof. Pham Huu Anh Ngoc | Assoc. Prof. Nguyen Ngoc Hai |

**INSTRUCTIONS:** *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

**Question 1** Let $f$ be integrable on $E$.

(a) *(15 marks)* Show that $\int_E f d\mu = \int_A f d\mu$ where $A = \{x \in E : f(x) \neq 0\}$.

(b) *(15 marks)* Show that $\int_E f d\mu \leq \int_P f d\mu$ where $P = \{x \in E : f(x) \geq 0\}$.

**Question 2** *(15 marks)* Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{2^n} \text{ if } n-1 \leq |x| < n, \quad n = 1, 2, \ldots .$$

Evaluate $\int_{\mathbb{R}} f dm$ where $m$ is the Lebesgue measure on $\mathbb{R}$.

**Question 3** *(15 marks)* Suppose that $f, g, f_1, f_2, \ldots$ are measurable functions on $X$, $|f_n| \leq g$ for all $n$, $g^2$ is integrable on $X$, and $f_n \to f$ a.e. Show that $f^2$ is integrable and $\int_X |f_n - f|^2 d\mu \to 0$ as $n \to \infty$.

**Question 4** *(20 marks)* Suppose that $f$ is of bounded variation on $[a, b]$ and there is a real constant $c > 0$ such that $f \geq c$ on $[a, b]$. Show that $1/f$ is of bounded variation on $[a, b]$.

**Question 5**

(a) *(10 marks)* Let $A$ be a negative set for a signed measure $\nu$ and let $B$ be a nonempty measurable subset of $A$. Show that $B$ is also a negative set for $\nu$.

(b) *(10 marks)* Let $X = P \cup N$ be a Hahn decomposition for $\nu$ where $\nu(P) \geq 0$ and $\nu(N) \leq 0$. For each measurable subset $E$ of $X$, set $\varphi(E) = -\nu(E \cap N)$. Show that

$$\varphi(E) = \sup \big\{ -\nu(F) : F \subset E,\ F \text{ is measurable} \big\}.$$

(*Hint:* If $F \subset E$, then $E \cap N = (F \cap N) \cup \big[(E \setminus F) \cap N\big]$.)

$* * *$ END OF QUESTION PAPER $* * *$

## REAL ANALYSIS-JUNE, 2017
## SOLUTIONS

**Question 1** (a) Let $B = \{x \in E : f(x) = 0\}$. Then $A$ and $B$ are measurable, $A \cap B = \emptyset$, and $E = A \cup B$. Thus

$$\int_E f d\mu = \int_A f d\mu + \int_B f d\mu = \int_A f d\mu + \int_B 0 d\mu = \int_A f d\mu.$$

(b) Let $N = \{x \in E : f(x) < 0\}$. Since $N$ is measurable and $f < 0$ on $N$, $\int_N f d\mu \le 0$. Noting that $P \cap N = \emptyset$ and $E = P \cup N$, we have

$$\int_E f d\mu = \int_P f d\mu + \int_N f d\mu \le \int_P f d\mu.$$

*Solution 2.* Since $f\chi_P \ge 0$ and $f \le f\chi_P$, we get $\int_E f d\mu \le \int_E f\chi_P d\mu = \int_P f d\mu$.

*Solution 3.* If $f(x) < 0$, then $(f\chi_P)(x) = 0 = f^+(x)$; if $f(x) \ge 0$, then $(f\chi_P)(x) = f(x) = f^+(x)$. Thus $f\chi_P = f^+$, implying $\int_P f d\mu = \int_X f\chi_P d\mu = \int_X f^+ d\mu \ge \int_X f^+ d\mu - \int_X f^- d\mu = \int_X f d\mu$.

**Question 2** Let $A_n = \{x \in \mathbb{R} : n - 1 \le |x| < n\}$, $n = 1, 2, \ldots$. Then $A_n$ is Lebesgue measurable and $f = \sum_{n=1}^{\infty} \frac{1}{2^n}\chi_{A_n}$. Thus,

$$\int_{\mathbb{R}} f dm = \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{1}{2^n}\chi_{A_n} dm = \sum_{n=1}^{\infty} \frac{1}{2^n} m(A_n) = \sum_{n=1}^{\infty} \frac{2}{2^n} = 2.$$

**Question 3** As $f_n^2 \le g^2$ for all $n$, $0 \le f^2 = \lim_{n\to\infty} f_n^2 \le g^2$. By assumption, $g^2$ is integrable. It follows that $f_n^2$ and $f^2$ are integrable, too. Thus $f_n$ and $f$ are finite a.e. and so $f_n - f$ is defined a.e. Also $|f_n - f|^2 \le (|f_n| + |f|)^2 \le 4g^2$, which is integrable. Applying the Dominated Convergence Theorem we get

$$\lim_{n\to\infty} \int_X |f_n - f|^2 d\mu = \int_X \left(\lim_{n\to\infty} |f_n - f|^2\right) d\mu = \int_X 0 d\mu = 0.$$

**Question 4** Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a, b]$. We have

$$\begin{aligned} V\left(\frac{1}{f}; P\right) &= \sum_{i=1}^{n} \left|\frac{1}{f(x_{i-1})} - \frac{1}{f(x_i)}\right| = \sum_{i=1}^{n} \frac{|f(x_i) - f(x_{i-1})|}{f(x_i)f(x_{i-1})} \\ &\le \frac{1}{c^2} \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)| = \frac{1}{c^2} V(f; P) \le \frac{1}{c^2} V_a^b(f). \end{aligned}$$

As $V_a^b(f) < \infty$, $1/f$ is of bounded variation on $[a, b]$.

**Question 5** (a) Suppose $C$ is a measurable subset of $B$. As $B \subset A$, $C$ is a measurable subset of $A$. Since $A$ is a negative set for $\nu$, we have $\nu(C) \le 0$. By definition, $B$ is a negative set for $\nu$.

(b) As $F \subset E$, we express $E \cap N$ as a disjoint union $E \cap N = (F \cap N) \cup \big[(E \setminus F) \cap N\big]$ and obtain

$$\nu(E \cap N) = \nu(F \cap N) + \nu\big[(E \setminus F) \cap N\big] \leq \nu(F \cap N) \leq \nu(F \cap N) + \nu(F \cap P) = \nu(F)$$

(since $N$ is negative and $P$ is positive for $\nu$). Thus $\varphi(E) = -\nu(E \cap N) \geq -\nu(F)$ so that

$$\varphi(E) \geq \sup\big\{-\nu(F) : F \subset E,\ F \text{ is measurable}\big\}.$$

Conversely, let $F_0 = E \cap N$. Clearly $F_0$ is a measurable subset of $E$ and $-\nu(F_0) = -\nu(E \cap N) = \varphi(E)$. Therefore, $\varphi(E) = \sup\big\{-\nu(F) : F \subset E,\ F \text{ is measurable}\big\}$.