EXERCISES AND PROBLEMS FOR CHAPTER 3: INTEGRATION

A. Exercises and Problems for everyone:

All exercises and Problems in parts B and C.

B. Non-assessed Exercises and Problems (corrected in class):

0.1.4; 0.1.8; 0.1.12; 0.1.16; 0.1.17; 0.1.18; 0.2.3; 0.3.2; 0.3.6; 0.3.8; 0.3.15; 0.3.20; 0.4.4; 0.4.7; 0.4.8; 0.4.12; 0.4.13; 0.4.14; 0.5.1; 0.5.3; 0.6.2; 0.6.3; 0.6.4; 0.7.2; 0.7.6; 0.8.2; 0.8.3; 0.8.9.

C. Assessed Assignments (to be submitted):

0.1.2; 0.1.5; 0.1.6; 0.1.10; 0.3.1; 0.3.3; 0.3.4; 0.3.9; 0.3.11; 0.3.19; 0.3.23; 0.4.3; 0.4.5; 0.4.15; 0.5.2; 0.5.4; 0.6.5; 0.7.1; 0.8.1; 0.8.4; 0.8.5, 0.8.6.

D. Bonus Exercises and Problems: Remaining exercises and problems.

0.1 MEASURABLE FUNCTIONS

In the following problems $(X, \mathcal{M})$ is a reference measurable space and measurable means with respect to $\mathcal{M}$. However, for functions defined on $\mathbb{R}^n$, otherwise stated, measurable means Lebesgue measurable.

Exercise 0.1.1. Let A and B be subsets of a set X. Prove the following relations.

(a) $\chi_\emptyset = 0$ and $\chi_X = 1$.

(b) $A \subset B$ if and if $\chi_A \leq \chi_B$.

(c) $\chi_{A\cap B} = \chi_A \cdot \chi_B = \min\{\chi_A, \chi_B\}$.

(d) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B} = \max\{\chi_A, \chi_B\}$.

(e) $\chi_{A\setminus B} = \chi_A - \chi_{A\cap B}$.

(f) If $\{A_n\}$ is a disjoint sequence of subsets of X and $A = \bigcup_{n=1}^{\infty} A_n$, then

$$\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}.$$

(g) If E is a subset of X and F a subset of Y, then $\chi_{E\times F} = \chi_E \times \chi_F$.

Exercise 0.1.2. (a) Constant functions are always measurable.

(b) If f is measurable, then the inverse image of any interval is measurable.

Exercise 0.1.3. Let D be a dense subset of $\mathbb{R}$. Show that a function $f : X \to \overline{\mathbb{R}}$ is measurable if and only if $\{x \in X : f(x) \geq \alpha\}$ is measurable for each $\alpha \in D$.

Exercise 0.1.4. For a measurable subset D of E, f is measurable on E if and only if the restrictions of f to D and $E \setminus D$ are measurable functions.

Exercise 0.1.5. If $X = A \cup B$ where $A, B \in \mathcal{M}$, a function f on X is measurable if and only if f is measurable on A and on B.

Exercise 0.1.6. Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable if and only if $f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \in \mathcal{M}$, and f is measurable on Y.

Exercise 0.1.7. Suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable. Fix $a \in \overline{\mathbb{R}}$ and define

$$h(x) = a \text{ if } f(x) = -g(x) = \pm\infty \text{ and } h(x) = f(x) + g(x) \text{ otherwise.}$$

Show that h is measurable.

Exercise 0.1.8. (a) Suppose that $\mathcal{M}$ and $\mathcal{N}$ are σ-algebras on X such that $\mathcal{M} \subset \mathcal{N}$. Show that if $f : X \to \overline{\mathbb{R}}$ is $\mathcal{M}$-measurable, then f is $\mathcal{N}$-measurable.

(b) Suppose that $A \subset X$ and $\emptyset \neq A \neq X$. Let

$$\mathcal{M} = \{\emptyset, X\} \quad \text{and} \quad \mathcal{N} = \{\emptyset, A, A^c, X\}.$$

Show that χ_A is $\mathcal{N}$-measurable but not $\mathcal{M}$-measurable.

(c) Show that every Borel measurable function on $\mathbb{R}^n$ is Lebesgue measurable.

Exercise 0.1.9. Let $f, g : X \to \overline{\mathbb{R}}$ be measurable functions and define

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in A^c \end{cases}$$

where A is a measurable subset of X. Show that h is measurable.

Exercise 0.1.10. Let X be a metric space and $\mathcal{B}(X)$ be the Borel σ-algebra on X. Show that any continuous real-valued function on X is measurable with respect to the Borel measurable space $(X, \mathcal{B}(X))$.

Exercise 0.1.11. Show that a monotone function that is defined on an interval is m-measurable.

Exercise 0.1.12. Suppose $(X, \mathcal{M}, \mu)$ is not complete. Let E be a subset of a set of measure zero that does not belong to $\mathcal{M}$. Let $f = 0$ on X and $g = \chi_E$. Show that $f = g$ a.e. on X while f is measurable and g is not.

Exercise 0.1.13. If $|f|$ is measurable, does it follow that f is measurable?

Exercise 0.1.14. Let $f : D \subset \mathbb{R} \to \overline{\mathbb{R}}$ be a function with measurable domain D. Show that f is measurable if and only if the function g defined on $\mathbb{R}$ by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

Exercise 0.1.15. Let the function f be defined on a measurable set E. Show that f is measurable if and only if for each Borel set A, $f^{-1}(A)$ is measurable.

(*Hint:* The collection of sets A that have the property that $f^{-1}(A)$ is measurable is a σ-algebra.)

Exercise 0.1.16. Let$(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $T : X \to Y$ be a **$(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}})$-mapping**, that is $T^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. Let μ be a measure on $\mathcal{A}$. Show that the mapping $\nu : \mathcal{B} \to \overline{\mathbb{R}}$ defined by

$$\nu(B) = \mu\big(T^{-1}(B)\big), \quad B \in \mathcal{B},$$

is a measure on $\mathcal{B}$.

The measure ν is called the **image measure** of μ under T.

Application. If $(\Omega, \mathcal{F}, P)$ is a probability space, a measurable function ξ from Ω into $\mathbb{R}$ is called a **random variable**. For any random variable ξ we define

$$P_\xi(A) = P(\{\omega \in \Omega : \xi(\omega) \in A\}) = P\big(\xi^{-1}(A)\big), \quad A \in \mathcal{B}(\mathbb{R}).$$

Show that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_\xi)$ is a probability space.

P_ξ is called the **law** or the **(probability) distribution** of the random variable ξ.

Exercise 0.1.17. A random variable (see Exercise 0.1.16) that can take on at most a countable number of possible values is said to be **discrete**. For a discrete random variable ξ on the probability space $(\Omega, \mathcal{F}, P)$, we define the **probability mass function** $p(a)$ of ξ by

$$p(x) = P(\{\xi = x\}), \quad x \in \mathbb{R}.$$

If ξ must assume one of the values $x_1, x_2, \ldots$, then

$$\begin{aligned} p(x_i) &\geq 0 \quad \text{for } i = 1, 2, \ldots \\ p(x) &= 0 \quad \text{for all other values of } x. \end{aligned}$$

Show that

(a) $\sum_i p(x_i) = 1$;

(b) if $B \in \mathcal{B}(\mathbb{R})$, then

$$P_\xi(B) = P(\{\omega : \xi(\omega) \in B\}) = \sum_{x \in B} p(x).$$

Exercise 0.1.18. Let X be a nonempty set.

(i) Let x_0 belong to X and δ_{x_0} be the Dirac measure at x_0 on $\mathcal{P}(X)$. Show that two functions on X are equal δ_{x_0}-a.e. if and only if they take the same value at x_0.

(ii) Let ν be the counting measure on $\mathcal{P}(X)$. Show that two functions on X are equal ν-a.e. if and only if they take the same value at every point in X.

Exercise 0.1.19. Suppose f and g are continuous functions on $[a, b]$. Show that if $f = g$ m-a.e. on $[a, b]$, then, in fact, $f = g$ on $[a, b]$. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set E?

Exercise 0.1.20. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous a.e., then f is a Lebesgue measurable function.

Exercise 0.1.21. If a real-valued function on $\mathbb{R}$ is measurable with respect to the σ-algebra of Lebesgue measurable sets, is it necessarily measurable with respect to the Borel measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$?

Exercise 0.1.22. Suppose f and g are real-valued functions defined on all of $\mathbb{R}$, f is measurable, and g is continuous. Is the composition $f \circ g$ necessarily measurable?

Exercise 0.1.23. Assume that $f : X \to \mathbb{R}$ is a measurable function and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Show that $g \circ f$ is a measurable function.

Exercise 0.1.24. Show that the composition of two Lebesgue measurable functions need not be Lebesgue measurable.

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Exercise 0.1.25. Let f be Lebesgue measurable and finite on $[0, 1]$, and define $\varphi(t) = m\big(f^{-1}((-\infty, t))\big)$. Is φ continuous from the right or left? Is it monotone? Is it measurable? Is it invertible? What are $\lim_{t\to-\infty} \varphi(t)$, $\lim_{t\to\infty} \varphi(t)$?

0.2 CONVERGENCE A.E.

Exercise 0.2.1. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable?

(*Hint:* $\lim f_n(x)$ exists if and only if $\limsup f_n(x) = \liminf f_n(x)$.)

Exercise 0.2.2. Let $\{f_n\}$ be a sequence of real-valued measurable functions on X. Then show that the following sets

$$A = \{x \in X : \lim_{n\to\infty} f_n(x) \to \infty\}$$
$$B = \{x \in X : \lim_{n\to\infty} f_n(x) \to -\infty\}$$
$$C = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\}$$

are all measurable.

Exercise 0.2.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiate function. Show that f' is Lebesgue measurable.

(*Hint:* Note that $f'(x) = \lim_{n\to\infty} n\big[f(x+\frac{1}{n}) - f(x)\big]$.)

0.3 INTEGRATION

In the following problems, $(X, \mathcal{M}, \mu)$ is a measure space, measurable means with respect to $\mathcal{M}$, and integrable means with respect to μ.

Exercise 0.3.1. (a) If $A, B \subset X$, show that $|\chi_A - \chi_B| = \chi_{A\Delta B}$. (Recall that $A\Delta B = (A \setminus B) \cup (B \setminus A)$.)

(b) If A and B are measurable sets, find $\int_X |\chi_A - \chi_B| d\mu$.

Exercise 0.3.2. Let f be a measurable function on X. Suppose that f is bounded on X and vanishes outside a set of finite measure, that is, there exists a subset E of X such that $\mu(E) < \infty$ and $f = 0$ on $X \setminus E$. Show that f is integrable over X.

Exercise 0.3.3. If $f = g$ almost everywhere, does it follow that $f^+ = g^+$ and $f^- = g^-$ almost everywhere? What can be said of the converse?

Exercise 0.3.4. Let X be a compact metric space and $\mathcal{A}$ a σ-algebra of subsets of X that contains all open sets in X. Suppose f is a continuous real-valued function on X and $(X, \mathcal{A}, \mu)$ is a finite measure space.

(a) Show that f is $\mathcal{A}$-measurable.

(b) Show that f is integrable over X with respect to μ.

Exercise 0.3.5. Let f be nonnegative and measurable. Prove that $\int_X f d\mu = \int_A d\mu$, where $A = \{x : f(x) > 0\}$.

Exercise 0.3.6. Let f be integrable on X. Show that $\{|f| > 0\}$ is σ-finite.

Exercise 0.3.7. If f is measurable and g integrable and α, β are real numbers such that $\alpha \leq f \leq \beta$ a.e., then there exists $\gamma \in [\alpha, \beta]$ such that $\int f g d\mu = \gamma \int |g| d\mu$.

Exercise 0.3.8. Show that if f is a measurable function and there exist two integrable functions h and g such that $h \leq f \leq g$ a.e., then f is also an integrable function.

Exercise 0.3.9. Let f be a measurable function on X and A a measurable subset of X. Show that f is integrable over X if and only if it is integrable over both A and $A^c = X \setminus A$.

Exercise 0.3.10. Let X be the disjoint union of the measurable sets $\{X_n\}_{n=1}^{\infty}$. For a measurable function f on X, characterize the integrability of f on X in terms of the integrability and the integral of f over the X_n's.

Exercise 0.3.11. If f and g are integrable functions on E, show that the functions $\max\{f, g\}$ and $\min\{f, g\}$ are integrable.

Exercise 0.3.12. Let $E_1, \ldots, E_k$ be measurable sets in X and let F_j $(j = 1, ..., k)$ be the sets of points belonging to precisely j of the E_i. Show that

$$\sum_{i=1}^{k} \mu(E_i) \geq \sum_{j=1}^{k} j\mu(F_j).$$

(*Hint:* $F_j = \{f = j\}$ where $f = \sum_{i=1}^{n} \chi_{E_i}$.)

Exercise 0.3.13. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space. The **(cumulative) distribution function** of a random variable $\xi : \Omega \to \mathbb{R}$ is defined by

$$F_\xi(y) = \mathbb{P}\big(\{\omega : \xi(\omega) \leq y\}\big).$$

Find F_ξ and $\int_\Omega \xi d\mathbb{P}$ for

(a) a constant random variable ξ, $\xi(\omega) = a$ for all ω.

(b) $\xi : [0,1] \to \mathbb{R}$ given by $\xi(\omega) = \min\{\omega, 1-\omega\}$ (the distance to the nearest endpoint of the interval $[0,1]$.)

(c) $\xi : [0,1]^2 \to \mathbb{R}$, the distance to the nearest edge of the square $[0,1]^2$.

Exercise 0.3.14. (The Continuity of Integration) Let f be integrable over X.

(a) If $\{X_n\}$ is a sequence of measurable subsets of X, $X_n \subset X_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} X_n = X$, then

$$\int\limits_X f d\mu = \lim_{n\to\infty} \int\limits_{X_n} f d\mu$$

(b) If $\{X_n\}$ is a sequence of measurable subsets of X, $X_n \supset X_{n+1}$ for all n, then

$$\int\limits_{\cap_{n=1}^{\infty} X_n} f d\mu = \lim_{n\to\infty} \int\limits_{X_n} f d\mu.$$

Exercise 0.3.15. (a) Let X be a nonempty set, and let δ be the Dirac measure on $\mathcal{P}(X)$ with respect to the point a. Show that every function $f : X \to \mathbb{R}$ is integrable and that $\int_X f d\delta = f(a)$.

(b) Let μ be the counting measure on $\mathbf{N}$. Show that a function $f : \mathbf{N} \to \mathbb{R}$ is integrable if and only if $\sum_{n=1}^{\infty} |f(n)| < \infty$. Also, show that in this case $\int_{\mathbf{N}} f d\mu = \sum_{n=1}^{\infty} f(n)$.

Exercise 0.3.16. (a) Show that $|x|^{k-1} \leq \max\{|x|^k, 1\} \leq |x|^k + 1$ for any $x \in \mathbb{R}$.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let ξ be a random variable. The **moments** and **absolute moments** of ξ are

$$E(\xi^k) = \int\limits_{\Omega} \xi^k d\mathbb{P} \quad \text{and} \quad E(|\xi|^k) = \int\limits_{\Omega} |\xi^k| d\mathbb{P}, \quad k \in \mathbf{N},$$

respectively, provided that there quantities exist. The first moment, $E(\xi)$, is usually called the **expectation** or **mean**. The number

$$\mathrm{Var}(\xi) = E\big[(\xi - E(\xi))^2\big]$$

is called **variance**.

(i) Use part (a) to show that if $E\big(|\xi|^k\big)$ is finite for some $k > 1$, then so are $E\big(|\xi|^n\big)$, $E\big(\xi^n\big)$ for $0 \le n \le k-1$.

(ii) Show that $\operatorname{Var}(\xi) = E(\xi^2) - \big[E(\xi)\big]^2 = \inf_{a\in\mathbb{R}} E\big[(X-a)^2\big]$.

(iii) Show that $E(\xi^2) < \infty$ if and only if ξ is integrable and its variance is finite.

Exercise 0.3.17. Let ξ be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see Exercise 0.1.16). The **expectation** (or **expected value**) of ξ, denoted by $E(\xi)$, is defined as

$$E(\xi) = \int_\Omega \xi d\mathbb{P}$$

provided the integral is well defined. The **variance** of X is defined as $\operatorname{Var}(\xi) = E\big[(\xi - E(\xi))^2\big]$, provided $E(\xi^2) < \infty$.

Let ξ be a random variable with $E(\xi^2) < \infty$ and finite expectation $E(\xi) = m$.

(a) Show that for all real number $\alpha > 0$,

$$\mathbb{P}(|\xi - m| \ge \alpha) \le \frac{\operatorname{Var}(\xi)}{\alpha^2}.$$

In words, the probability that ξ differs from its expectation by more than α is bounded above by its variance divided by α^2.

(*Hint:* Use Chebychev's inequality.)

(b) Let $\sigma := \sqrt{\operatorname{Var}(\xi)}$. Show that for any $0 < k < \infty$,

$$\mathbb{P}\big(|X - m| \ge k\sigma\big) \le \frac{1}{k^2}.$$

This is a quantitative result to the effect that a random variable with small variance is likely to be close to its mean.

Exercise 0.3.18. Let ξ be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The real number m is called a **median** of the random variable ξ if

$$\mathbb{P}(\xi < m) \le \frac{1}{2} \le \mathbb{P}(\xi \le m).$$

Show that every random variable has at least one median. Show that if ξ is non-negative, then

$$\frac{1}{2} \leq \mathbb{P}(\xi \geq m) \leq \frac{E(\xi)}{m},$$

where $E(\xi)$ is the expectation of ξ (see Exercise 0.3.17).

Exercise 0.3.19. Let ν be another measure on $\mathcal{M}$. For an extended real-valued function f on X that is measurable with respect to the measurable space $(X, \mathcal{M})$, under what conditions is it true that

$$\int\limits_X f d[\mu + \nu] = \int\limits_X f d\mu + \int\limits_X f d\nu?$$

(*Hint:* Consider the cases: f is simple and nonnegative, f is nonnegative, f is arbitrary.)

Exercise 0.3.20. Let f be an integrable function such that $f(x) > 0$ holds for almost all x. If A is a measurable set such that $\int_A f d\mu = 0$, then $\mu(A) = 0$.

Exercise 0.3.21. If μ is σ-finite on X, f and g are measurable, $\int_X f d\mu$ and $\int_X g d\mu$ exist, and $\int_A f d\mu \leq \int_A g d\mu$ for all measurable set $A \subset X$, then $f \leq g$ a.e.

Exercise 0.3.22. Let f and g be nonnegative integrable functions on X for which $g \leq f$ a.e. on X. Show that $f = g$ a.e. on X if and only if $\int_X f d\mu = \int_X g d\mu$.

Exercise 0.3.23. Suppose f and g are nonnegative measurable functions on X for which f^2 and g^2 are integrable over X with respect to μ. Show that fg also is integrable over X with respect to μ.

0.4 CONVERGENCE THEOREMS

In the following problems, $(X, \mathcal{M}, \mu)$ is a measure space, measurable means with respect to $\mathcal{M}$, and integrable means with respect to μ.

Exercise 0.4.1. (a) Give an example in which strict inequality occurs in Fatou's Lemma.

(b) Give an example to show that the nonnegativity hypothesis cannot be dropped from Fatou's Lemma.

Exercise 0.4.2. Show that if $\{f_n\}$ is a sequence of measurable functions on E and there exists an integrable function g on E such that $f_n \geq g$ a.e. on E for all n, then

$$\int_E \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int_E f_n d\mu.$$

Exercise 0.4.3. Define $f_n(x)$ to be n if $|x| \leq 1/n$ and to be 0 otherwise. What are $\int_{\mathbb{R}} \lim f_n dx$ and $\lim \int_{\mathbb{R}} f_n dx$?

Exercise 0.4.4. (Beppo Levi's Lemma) Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on X. If the sequence of integrals $\left\{\int_X f_n d\mu\right\}$ is bounded, then $\{f_n\}$ converges pointwise on X to a measurable function f that is finite a. e. on X and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu < \infty.$$

Exercise 0.4.5. Let f_n be measurable functions on X such that $f_n \geq f_{n+1} \geq 0$ for all n and $\int_X f_n d\mu \searrow 0$. Prove that $f_n \searrow 0$ a.e.

Exercise 0.4.6. Find a sequence of simple functions f_n converging uniformly to 0, yet $\int |f_n| \not\to 0$.

Exercise 0.4.7. (Extended Monotone Convergence Theorem) Let $f_1, f_2, ..., f, g$ be measurable on X.

(a) Show that if $f_n \geq g$ for all n, $f_n \nearrow f$, and g is integrable on X, then

$$\int_X f_n d\mu \nearrow \int_X f d\mu.$$

(b) Show that if $f_n \leq g$ for all n, $f_n \searrow f$, and g is integrable on X, then

$$\int_X f_n d\mu \searrow \int_X f d\mu.$$

Exercise 0.4.8. (a) Let $\alpha, \beta \in \overline{\mathbb{R}}$ and set $a_n = \alpha$ if n is odd and $a_n = \beta$ if n is even. Determine $\liminf_n a_n$.

(b) Let g, h be continuous function on $[a, b]$. Set $f_n = g$ if n is odd and $f_n = h$ if n is even. Determine $\int_a^b (\liminf f_n) dx$ and $\liminf \int_a^b f_n dx$.

(c) Let $(X, \mathcal{A}, \mu)$ be a measure space and E be a measurable set such that $0 < \mu(E) < \mu(X)$. Define $f_n = \chi_E$ when n is even and $f_n = 1 - \chi_E = \chi_{E^c}$ when n is odd. Show that $\liminf f_n = 0$ and

$$\int_X (\liminf f_n) d\mu = 0 < \min\{\mu(E), \mu(E^c)\} = \liminf \int_X f_n d\mu.$$

Thus the inequality in Fatou's lemma can be strict.

Exercise 0.4.9. Let μ be counting measure on $\mathbf{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorems as statements about infinite series.

Exercise 0.4.10. Let g be a nonnegative function that is integrable over X. Define

$$\nu(E) = \int_E g d\mu, \quad E \in \mathcal{M}.$$

(i) Show that ν is a measure on the measurable space $(X, \mathcal{M})$.

(ii) Let f be a nonnegative function on X that is measurable with respect to $\mathcal{M}$. Show that

$$\int_X f d\nu = \int_X fg d\nu.$$

(*Hint:* Consider the following cases: $f = \chi_E$, f is nonnegative simple, and f is nonnegative. For the last case, apply the Monotone Convergence Theorem.)

Exercise 0.4.11. Suppose that $(X\mathcal{A}, \mu)$ and $Y, \mathcal{B}, \nu)$ are measure spaces, and $\varphi : X \to Y$ is a function such that $\varphi^{-1}(B) \in \mathcal{A}$ and $\mu\big(\varphi^{-1}(B)\big) = \nu(B)$ for every $B \in \mathcal{B}$. If $f = g \circ \varphi$ is the composition of φ and a measurable function $g : Y \to \overline{\mathbb{R}}$, then $f : X \to \overline{\mathbb{R}}$ is a measurable function, the integral

$\int_X f d\mu$ exists if and only if $\int_Y g d\nu$ exists (including the cases when the integrals are equal to ∞ or $-\infty$), and

$$\int\limits_X f d\mu = \int\limits_Y g d\nu.$$

(*Hint:* Consider the following cases: g is nonnegative simple, and g is nonnegative, g is arbitrary. For the second case, apply the Monotone Convergence Theorem.)

Exercise 0.4.12. Let X be the union of an increasing sequence of measurable sets $\{X_n\}$ and f a nonnegative measurable function on X. Show that $\lim \int_{X_n} f d\mu = \int_X f d\mu$ and that f is integrable over X if and only if there is an $M > 0$ for which $\int_{X_n} f d\mu \leq M$ for all n.

Exercise 0.4.13. If $f_1, f_2, \ldots, f, g$ are measurable, $|f_n| \leq g$ for all n, where $|g|^p$ is integrable ($p > 0$, fixed), and $f_n \to f$ a.e., then $|f|^p$ is integrable and $\int_X |f_n - f|^p d\mu \to 0$ as $n \to \infty$.

Exercise 0.4.14. Suppose $\{f_n\}$ is a sequence of integrable functions on X that converges uniformly to f, i.e., for each $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$ and $n \geq N$.

(a) If $\mu(X) < \infty$, show that f is integrable on X and $\int_X f_n d\mu \to \int_X f d\mu$.

(b) Show that if $\mu(X) = \infty$ the conclusions of (a) can fail.

Exercise 0.4.15. Let $\{f_n\}$ be a sequence of integrable functions such that $0 \leq f_{n+1} \leq f_n$ a.e. holds for each n. Then show that $f_n \to 0$ a.e. holds if and only if $\int_X f_n d\mu \to 0$.

Exercise 0.4.16. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f, f_1, f_2, \ldots$ be non-negative integrable functions satisfying $f_n \to f$ a.e. and $\lim \int_X f_n d\mu = \int_X f d\mu$. If E is a measurable set, then show that

$$\lim_{n\to\infty} \int\limits_E f_n d\mu = \int\limits_E f d\mu.$$

Exercise 0.4.17. Suppose we are given three sequences of integrable functions $\{f_n\}$, $\{g_n\}$, and $\{h_n\}$ such that $g_n \le f_n \le h_n$ a.e and

$$\lim f_n = f, \quad \lim g_n = g, \quad \lim h_n = h.$$

If g and h be integrable and

$$\lim \int_E g_n d\mu = \int_E g d\mu, \quad \lim \int_E h_n d\mu = \int_E h d\mu,$$

then f is integrable and

$$\lim \int_E f_n d\mu = \int_E f d\mu.$$

(*Hint:* Rework the proof of the dominated convergence theorem.)

0.5 THE RIEMANN INTEGRAL AS A LEBESGUE INTEGRAL

Exercise 0.5.1. Show that if f and g are Riemann integrable on $[a, b]$, then so is fg.

Exercise 0.5.2. Show that if f is Riemann integrable on $[a, b]$ with $f([a, b]) \subset [c, d]$ and if $g : [c, d] \to \mathbb{R}$ is continuous, then the composition $g \circ f$ is Riemann integrable on $[a, b]$. In particular, if f is Riemann integrable on $[a, b]$ then so are $|f|$ and f^n, $n \in \mathbf{N}$.

Exercise 0.5.3. Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a, b]$ such that $\lim f_n(x) = f(x)$ holds for each $x \in [a, b]$ and f is Riemann integrable. Also, assume that there exists a constant M such that $|f_n(x)| \le M$ holds for all $x \in [a, b]$ and all n. Show that

$$\lim_{n \to \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx.$$

Exercise 0.5.4. Which of the following functions are Lebesgue integrable on $(0, 1)$:

(a) $f(x) = x^{-1}$;

(b) $g(x) = \frac{1}{\sqrt{x}}$;

(c) $h(x) = \exp(-\frac{1}{x})$;

(d) $k(x) = \log x$.

Exercise 0.5.5. Show that $\int_0^\infty x^n e^{-x} dx = n!$ by differentiating the equation $\int_0^\infty e^{-tx} dx = 1/t$. Similarly, show that $\int_{-\infty}^\infty x^{2n} e^{-x^2} dx = (2n!)\sqrt{\pi}/4^n n!$ by differentiating the equation $\int_{-\infty}^\infty e^{-tx^2} dx = \sqrt{\pi/t}$.

0.6 PRODUCT MEASURES

Exercise 0.6.1. Let μ be the counting measure on $\mathcal{P}(\mathbf{N})$. Let $(X, \mathcal{A}, \nu)$ a general measure space. Consider $\mathbf{N} \times X$ with the product measure $\mu \times \nu$.

(i) Show that a subset E of $\mathbf{N} \times X$ is measurable with respect to $\mu \times \nu$ if and only if for each natural number k, $E_k = \{x \in X : (k, x) \in E\}$ is measurable with respect to ν.

(ii) Show that a function $f : \mathbf{N} \times X \to \mathbb{R}$ is measurable with respect to $\mu \times \nu$ if and only if for each natural number k, $f(k, \cdot) : X \to \mathbb{R}$ is measurable with respect to ν.

(iii) Show that a function $f : \mathbf{N} \times X \to \mathbb{R}$ is integral over $\mathbf{N} \times X$ with respect to $\mu \times \nu$ if and only if for each natural number k, $f(k, \cdot) : X \to \mathbb{R}$ is integral over X with respect to ν and

$$\sum_{k=1}^{\infty} \int_X |f(k, x)| d\nu(x) < \infty.$$

(iv) Show that a function $f : \mathbf{N} \times X \to \mathbb{R}$ is integral over $\mathbf{N} \times X$ with respect to $\mu \times \nu$, then

$$\int_{\mathbf{N} \times X} f d(\mu \times \nu) = \sum_{k=1}^{\infty} \int_X |f(k, x)| d\nu(x) < \infty.$$

Exercise 0.6.2. Show that if $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$, with $f(0, 0) = 0$, then

$$\int_0^1 \left(\int_0^1 f(x, y) dx \right) dyn = -\frac{\pi}{4} \quad \text{and} \quad \int_0^1 \left(\int_0^1 f(x, y) dy \right) dx = \frac{\pi}{4}.$$

Exercise 0.6.3. Let $g : X \to \mathbb{R}$ be a μ-integrable function, and let $h : Y \to \mathbb{R}$ be a ν-integrable function. Define $f : X \times Y \to \mathbb{R}$ by $f(x,y) = g(x)h(y)$ for each x and y. Show that f is $\mu \times \nu$-integrable and that

$$\int f d(\mu \times \nu) = \Big(\int_X g d\mu\Big) \cdot \Big(\int_X h d\nu\Big).$$

(Note: We do not need to assume that μ and ν are σ-finite.)

Exercise 0.6.4. Let $I = \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dm =: \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy dx$.

(a) Show that $I = \lim_{n\to\infty} \int_{D_n} e^{-(x^2+y^2)} dm$, where D_n is the disk with radius $n \in \mathbf{N}$ and center the origin.

(b) Show that $I = \pi$.

(c) Show that $I = \lim_{n\to\infty} \int_{S_n} e^{-(x^2+y^2)} dm$, where $S_n = \{(x,y) \in \mathbb{R}^2 : |x| \leq n,\ |y| \leq n\}$ and $n \in \mathbf{N}$. Use this to show that

$$\Big(\int_{-\infty}^{\infty} e^{-x^2} dx\Big) \cdot \Big(\int_{-\infty}^{\infty} e^{-y^2} dy\Big) = \pi.$$

(d) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\,.$$

(e) By making the change of variable $t = \sqrt{2}\,x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}\,.$$

(This is a fundamental result for probability and statistics.)

Exercise 0.6.5. Show that if $f(x,y) = ye^{-(1+x^2)y^2}$ for each x and y, then

$$\int_0^{\infty} \Big(\int_0^{\infty} f(x,y) dx\Big) dy = \int_0^{\infty} \Big(\int_0^{\infty} f(x,y) dy\Big) dx.$$

Use the previous equality to give an alternate proof of the formula

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

0.7 SIGNED MEASURES

Exercise 0.7.1. If μ is a signed measure, does it follow that $-\mu$ is also a signed measure? Are sums and differences of signed measures signed measures?

Exercise 0.7.2. Let $\int_X f d\mu$ be defined and $\nu(E) = \int_E f d\mu$. Show that A is positive for ν if and only if $\mu(A \cap \{f < 0\}) = 0$, B is negative for ν if and only if $\mu(B \cap \{f > 0\}) = 0$.

Exercise 0.7.3. Let $(X, \mathcal{A})$ be a measurable space and μ a signed measure on $\mathcal{A}$. Let $E, F \in \mathcal{A}$ and $E \subset F$. Show that

(i) If $\mu(F)$ is finite then so is $\mu(E)$;

(ii) If $\mu(E) = +\infty$ then $\mu(F) = +\infty$;

(iii) If $\mu(E) = -\infty$ then $\mu(F) = -\infty$.

Exercise 0.7.4. (i) Show that every measurable subset of a positive set is positive.

(ii) Show that if the sets A_n are positive, then $A = \cup_n A_n$ is also positive.
Hint: Set $B_n = A_n \cap \big(\cap_{m=1}^{n-1} A_m^c \big)$. Then B_n is positive, disjoint, and $\cup_n B_n = \cup_n A_n$.

Exercise 0.7.5. Prove that the measures ν^+ and ν^- in the Jordan decomposition of ν have the following properties that could be taken for their definitions:

$$\nu^+(A) = \sup \big\{ \nu(B) : B \subset A,\ B \in \mathcal{A} \big\},$$
$$\nu^-(A) = \sup \big\{ -\nu(B) : B \subset A,\ B \in \mathcal{A} \big\}.$$

(*Hint:* Use the Hahn Decomposition Theorem.)

Exercise 0.7.6. Prove that a signed measure μ is monotone on a positive set, that is, if $A \subset B \subset S$, where S is a positive set, then $\mu(A) \leq \mu(B)$.

Exercise 0.7.7. Show that if a signed measure ν on the σ-algebra $\mathcal{A}$ is the difference of two measures ν_1 and ν_2, $\nu = \nu_1 - \nu_2$, show that $\nu_1 \geq \nu^+$ and $\nu_2 \geq \nu^-$.

Exercise 0.7.8. Let P be an arbitrary probability measure on $\mathcal{B}(\mathbb{R})$. Find ν^+, ν^- and $|\nu|$ of the signed measure $\nu = P - \delta_0$, where δ_0 is the Dirac measure at 0 on $\mathcal{B}(\mathbb{R})$.

0.8 THE RADON-NIKODYM THEOREM

Exercise 0.8.1. (i) Show that if $\mu \ll 0$, then $\mu = 0$.

(ii) Show that if μ, ν are (positive) measures and $\nu \leq \mu$, then $\nu \ll \mu$.

Exercise 0.8.2. Show that if $\mu_1 \ll \mu_2$ and $\mu_2 \perp \nu$, then $\mu_1 \perp \nu$.

Exercise 0.8.3. Verify that

$$\nu \ll \mu \iff (\nu^+ \ll \mu \text{ and } \nu^- \ll \mu) \iff |\nu| \ll \mu.$$

Exercise 0.8.4. Show that if $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$.

Exercise 0.8.5. If μ is σ-finite and $\nu_1, \nu_2 \ll \mu$, show that $\nu_1 + \nu_2 \ll \mu$ and

$$\frac{d}{d\mu}(\nu_1 + \nu_2) = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}.$$

Exercise 0.8.6. Show that if ν and μ are mutually singular on X and if g is ν-integrable on X, then the set function $\lambda(E) = \int_E g d\nu$ is singular with respect to μ.

Exercise 0.8.7. If $\nu \ll \mu$ and $f \geq 0$, then

$$\int_E f d\mu = \int_E f \frac{d\nu}{d\mu} d\mu.$$

Exercise 0.8.8. Let $X = [0,1]$ with Lebesgue measure and consider measures μ, ν given by densities χ_A, χ_B respectively. Find a condition on the sets A, B so that μ dominates ν (that is, $0 \leq \nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$) and find the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Exercise 0.8.9. Let $(X, \mathcal{M}, \mu)$ be a measure space. Fix $A \in \mathcal{M}$ and define $\nu(E) = \mu(E \cap A)$ for all $E \in \mathcal{M}$.

(a) Is ν absolutely continuous with respect to μ?

(b) If $\int_X f d\nu$ exists, is the equation $\int_E f d\nu = \int_{E\cap A} f d\mu$ true for all $E \in \mathcal{M}$?

(c) Suppose that $\mu(A) < \infty$. Is $\varphi := \mu - \nu$ a measure? Find a condition on the set A^c so that $\mu = \nu + \varphi$ is a Lebesgue decomposition of μ.