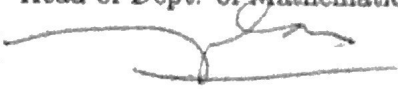



FINAL EXAMINATION

January 2021

Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Head of Dept. of Mathematics:	Lecturer:
	
Prof. Pham Huu Anh Ngoc	Assoc. Prof. Nguyen Ngoc Hai

INSTRUCTIONS: Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.

Question 1 Let (X, \mathcal{A}, μ) be a measure space and let $\mathcal{B} \subset \mathcal{A}$ be a sub- σ -algebra. Denote $\nu = \mu|_{\mathcal{B}}$, the restriction of μ to \mathcal{B} .

- (a) (15 marks) Show that ν is again a measure.
- (b) (5 marks) Assume that μ is a finite measure. Is ν still a finite measure?
- (c) (10 marks) Assume that μ is a σ -finite measure. Is ν still a σ -finite measure?

Question 2 (a) (10 marks) Let f be a measurable function such that $f \geq g$ where g is μ -integrable over X . Show that $\int_X f d\mu$ exists.

(b) (15 marks) Let $(f_n)_{n=1}^{\infty}$ be real-valued integrable functions on the measure space (X, \mathcal{M}, μ) such that $|f_n| \leq f$ for all $n = 1, 2, \dots$. Define $F : (0, \infty) \rightarrow \mathbb{R}$ by

$$F(\alpha) := \sup \left\{ \int_{\{|f_n| \geq \alpha\}} |f_n| d\mu : n \in \mathbb{N} \right\} \leq \int_{\{|f_n| \geq \alpha\}} f d\mu \leq \int_X f d\mu < \infty$$

Show that $F(\alpha) < \infty$ for all $0 < \alpha < \infty$, F is decreasing, and $\lim_{\alpha \rightarrow \infty} F(\alpha) = 0$. (Hint. Use Chebyshev's inequality.)

$$\int_X f = \int_X f^+ - \int_X f^-$$

-----continued on next page-----

$$\text{ind.} \Rightarrow \int g^+ - \int g^- \text{ finite.}$$

$$\Rightarrow \alpha \int f^- \leq \int f < \infty$$

1

$$f \geq g$$

$$\Rightarrow f^- = -\min\{f, 0\} \leq \min\{g, 0\} = g^-$$

Question 3 Let f be a measurable function on the measure space (X, \mathcal{M}, μ) .

(a) (10 marks) Show that if $\int_X f d\mu$ is defined, then $\int_X f d\mu = \int_A f d\mu$, where $A = \{f \neq 0\}$.

(b) (15 marks) Show that if f is integrable over X , then

$$\lim_{n \rightarrow \infty} \int_{A_n} f d\mu = \int_X f d\mu,$$

where $A_n = \{|f| \leq n\}$.

Question 4 (10 marks) Show that the Dirac measure δ_a on \mathbb{R} has no density with respect to the Lebesgue m on $\mathcal{L}(\mathbb{R})$.

(b) (10 marks) Let μ and ν be two measures on the measurable space (X, \mathcal{M}) . Suppose that $\nu(E) \leq \mu(E)$ for all E in \mathcal{M} . Show that ν is absolutely continuous with respect to μ .

$$\delta_a(\{a\}) = 1 \\ m(\{a\}) = 0 \Rightarrow \delta_a \not\ll m.$$

-----END OF QUESTION PAPER-----