INTERNATIONAL UNIVERSITY (IU) - VIETNAM NATIONAL UNIVERSITY - HCMC

FINAL EXAMINATION
August 2018
Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Head of Dept. of Mathematics:	Lecturer:
Assoc. Prof. Pham Huu Anh Ngoc	Assoc. Prof. Nguyen Ngoc Hai

INSTRUCTIONS: *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1 *(25 marks)* Let μ be a measure on $\mathcal{B}(\mathbb{R})$ such that $\mu(I) < \infty$ for each bounded interval I. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} -\mu\big((x,0]\big) & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ \mu\big((0,x]\big) & \text{if } x > 0. \end{cases}$$

Show that F is increasing.

Question 2 (a) *(10 marks)* Let $(X, \mathcal{M})$ be a measurable space and E a nonmeasurable subset of X. Show that $f = \chi_E - \chi_{E^c}$ is not measurable on X but $|f|$ is measurable.

(b) *(15 marks)* Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \overline{\mathbb{R}}$ a measurable function. For $E, F \in \mathcal{M}$ define $E\Delta F = (E \setminus F) \cup (F \setminus E)$. Show that if $\mu(E\Delta F) = 0$, then $\int_E f d\mu = \int_F f d\mu$ provided one of the integrals exists.

------continued on next page------

Question 3 Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = \frac{nx^{n-1}}{1+x}$ for $x \in [0,1]$.

(a) *(20 marks)* Show that $\lim \int_0^1 f_n(x)dx = \frac{1}{2}$.
(*Hint:* Apply integration by parts to the integral $\int_0^1 f_n(x)dx$.)

(b) *(5 marks)* Show that $f_n \to 0$ m-a.e.

Question 4 *(25 marks)* Let μ, ν be positive measures on the measurable space $(X, \mathcal{M})$. Assume that $\nu = \nu_1 + \nu_2$ where $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$, $\mu(A) = \nu_2(A^c) = 0$. Show that ν_2 has the form

$$\nu_2(E) = \nu(E \cap A) \quad \text{for every } E \in \mathcal{M}.$$

----------END OF QUESTION PAPER----------

SOLUTIONS

Question 1 For $x < y$, we have

$$F(y) - F(x) = \begin{cases} -\mu\big((y,0]\big) + \mu\big((x,0]\big) = \mu\big((x,y]\big) & \text{if } x < y < 0, \\ 0 + \mu\big((x,0]\big) = \mu\big((x,0]\big) & \text{if } x < y = 0, \\ \quad \mu\big((0,y]\big) - 0 = \mu\big((0,y]\big) & \text{if } x = 0 < y, \\ \quad \mu\big((0,y]\big) - \mu\big((0,x]\big) = \mu\big((x,y]\big) & \text{if } 0 < x < y. \end{cases}$$

Thus $F(y) - F(x) \geq 0$ whenever $x < y$, so F is increasing.

Question 2

(a) Since $X = E \cup E^c$ and $E \cap E^c = \emptyset$, we have

$$f(x) = \begin{cases} 1 & \text{if } x \in E \\ -1 & \text{if } x \notin E. \end{cases}$$

Since the set $f^{-1}(\{1\}) = E$ is not measurable, we conclude that f is not measurable. However, $|f| = 1$, which is a constant function. Hence $|f|$ is measurable on X.

(b) We write $E \cup F$ as disjoint unions:

$$E \cup F = F \cup (E \setminus F) = E \cup (F \setminus E).$$

$E \setminus F$ and $F \setminus E$ are measurable subsets of $E \Delta F$ and by assumption, $\mu(E\Delta F) = 0$, so $\mu(E \setminus F) = \mu(F \setminus E) = 0$. It follows that $\int_{E\setminus F} f d\mu = \int_{F\setminus E} f d\mu = 0$ and that

$$\int_E f d\mu = \int_E f d\mu + \int_{F\setminus E} f d\mu = \int_{E\cup F} f d\mu = \int_F f d\mu + \int_{E\setminus F} f d\mu = \int_F f d\mu$$

whenever either $\int_E f d\mu$ or $\int_F f d\mu$ exists.

Question 3 (a) Since the functions f_n are continuous on $[0, 1]$, their Riemann integrals exist. Applying integration by parts we get

$$\int_0^1 f_n(x)dx = \frac{x^n}{1+x}\Big|_0^1 + \int_0^1 \frac{x^n}{(1+x)^2}dx = \frac{1}{2} - \int_0^1 \frac{x^n}{(1+x)^2}dx. \tag{1}$$

Since $\lim_{n\to\infty} \frac{x^n}{(1+x)^2} = 0$ for $0 \leq x < 1$ and $0 \leq \frac{x^n}{(1+x)^2} \leq \frac{1}{(1+0)^2} = 1$ for all n, we can apply the Dominated Convergence Theorem to obtain $\lim_{n\to\infty} \int_0^1 \frac{x^n}{(1+x)^2}dx = 0$. Therefore (1) gives $\lim \int_0^1 f_n(x)dx = \frac{1}{2}$.

(b) For each $0 < x < 1$ consider the function $\varphi_x(t) = \frac{tx^{t-1}}{1+x}$ on $[1, \infty)$. Applying L'Hospital's Rule we get

$$\lim_{t\to\infty} \varphi_x(t) = \lim_{t\to\infty} \frac{t}{(1+x)x^{1-t}} = \lim_{t\to\infty} \frac{1}{(1+x)x^{1-t}\ln x} = 0.$$

It follows that $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \varphi_x(n) = 0$ for all $x \in (0,1)$, that is, $f_n \to 0$ m-a.e.

Alternative solution Consider the series $\sum_{n=1}^{\infty} f_n(x)$ on $(0,1)$. Applying the Ratio Test we get

$$\lim_{n\to\infty} \frac{f_{n+1}(x)}{f_n(x)} = \lim_{n\to\infty} \frac{\frac{(n+1)x^n}{1+x}}{\frac{nx^{n-1}}{1+x}} = \lim_{n\to\infty} \frac{n+1}{n}x = x < 1.$$

Thus the series $\sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in (0,1)$. It follows that $f_n(x) \to 0$ for all $x \in (0,1)$, that is, $f_n \to 0$ m-a.e.

Question 4 Let $E \in \mathcal{M}$. Since $\nu_2(A^c) = 0$ and ν_2 is a measure, $\nu_2(E \cap A^c) = 0$. Thus

$$\nu_2(E) = \nu_2(E \cap A) + \nu_2(E \cap A^c) = \nu_2(E \cap A). \tag{2}$$

Likewise, as $\mu(A) = 0$, $\mu(E \cap A) = 0$. Since $\nu_1 \ll \mu$, we have $\nu_1(E \cap A) = 0$, so

$$\nu(E \cap A) = \nu_1(E \cap A) + \nu_2(E \cap A) = \nu_2(E \cap A). \tag{3}$$

Equalities (2) and (3) imply $\nu_2(E) = \nu(E \cap A)$ for all $E \in \mathcal{M}$.