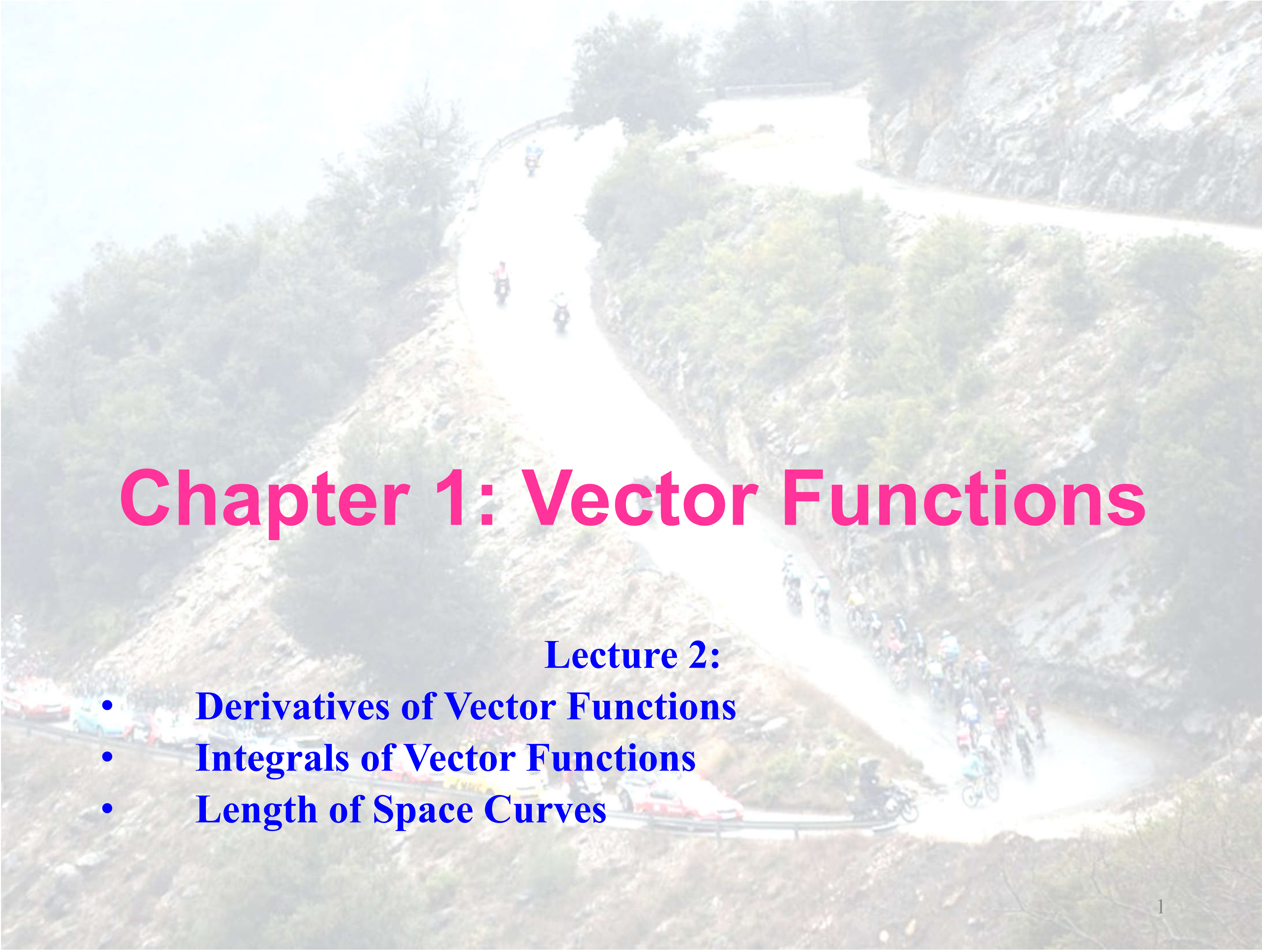


Chapter 1: Vector Functions

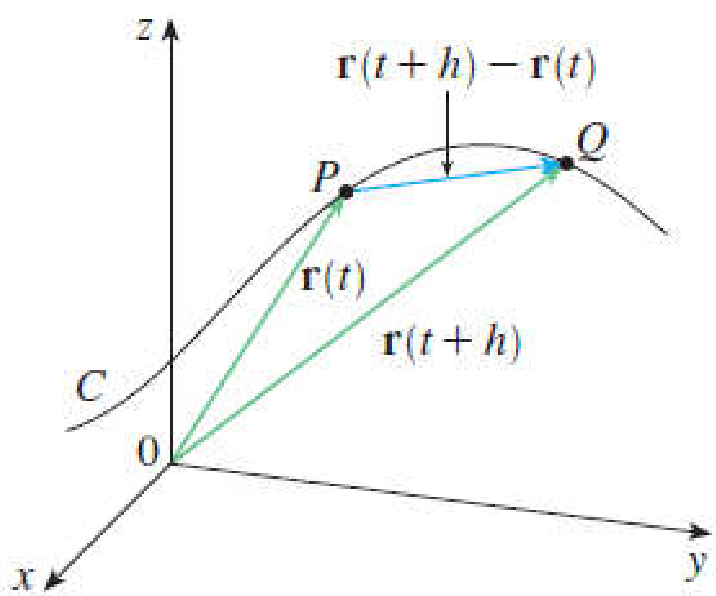
Lecture 2:

- **Derivatives of Vector Functions**
- **Integrals of Vector Functions**
- **Length of Space Curves**

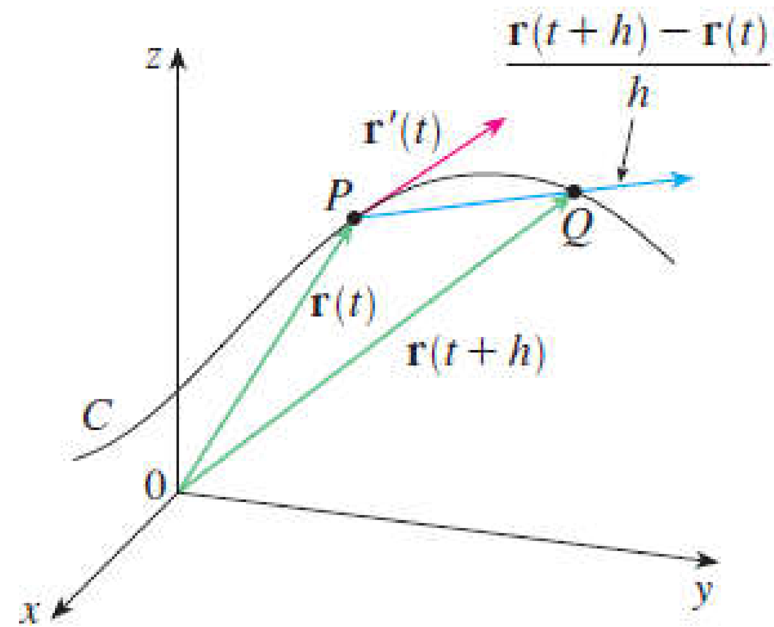
1. Derivative of Vector Function

The **derivative** $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined

$$\boxed{\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}}$$

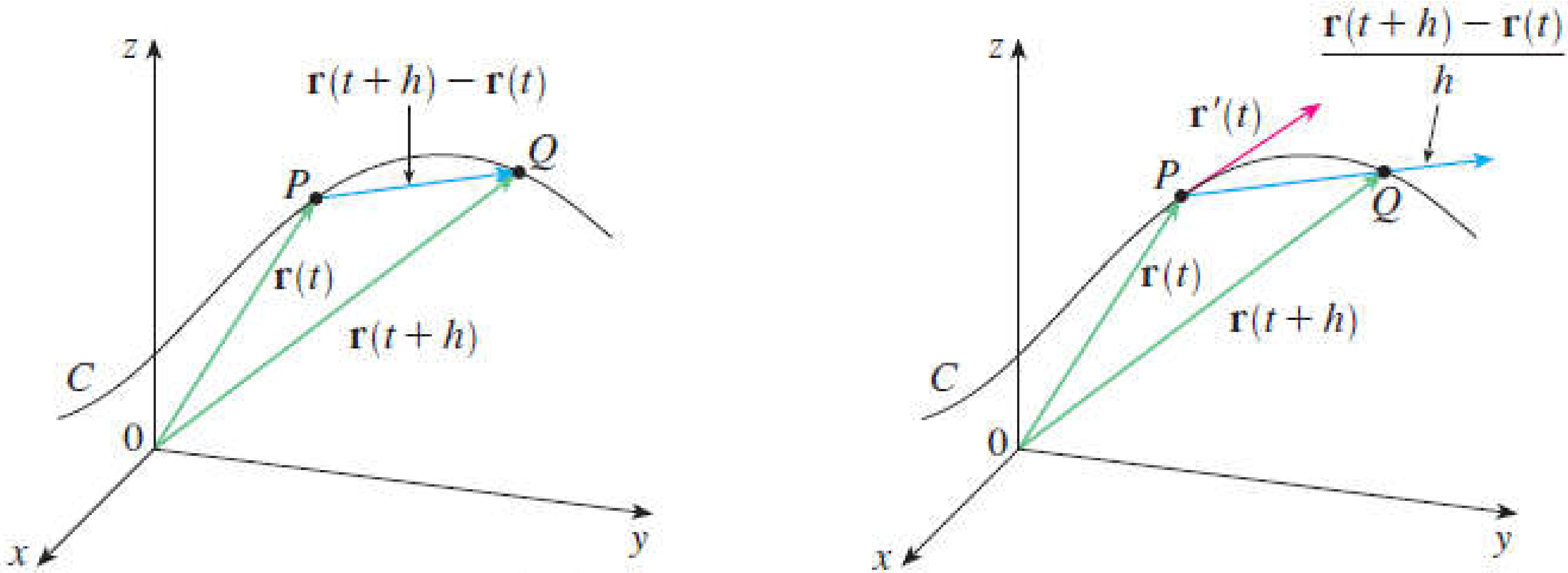


(a) The secant vector $\overrightarrow{PQ}$



(b) The tangent vector $\mathbf{r}'(t)$

Geometric Interpretation of **r**'(t)



(a) The secant vector $\overrightarrow{PQ}$

(b) The tangent vector $\mathbf{r}'(t)$

r'(t)≠0: Tangent vector to C defined by **r**(t) at P

THEOREM 1

Differentiation of a Vector Function

If $\mathbf{r}(t) = < f(t), g(t), h(t) >$, where f, g, and h are differentiable, then

$$\mathbf{r}'(t) = < f'(t), g'(t), h'(t) >$$

Proof

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[< f(t+\Delta t), g(t+\Delta t), h(t+\Delta t) > - < f(t), g(t), h(t) >]$$

$$= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$\mathbf{r}'(t) = < f'(t), g'(t), h'(t) >$$

Example 1

a) Graph the curve by $\mathbf{r}(t) = \cos 2t\ \mathbf{i} + \sin t\ \mathbf{j},\ 0 \le t \le 2\pi$.

b) Graph $\mathbf{r}'(0)$ and $\mathbf{r}'(\pi/6)$.

Solution

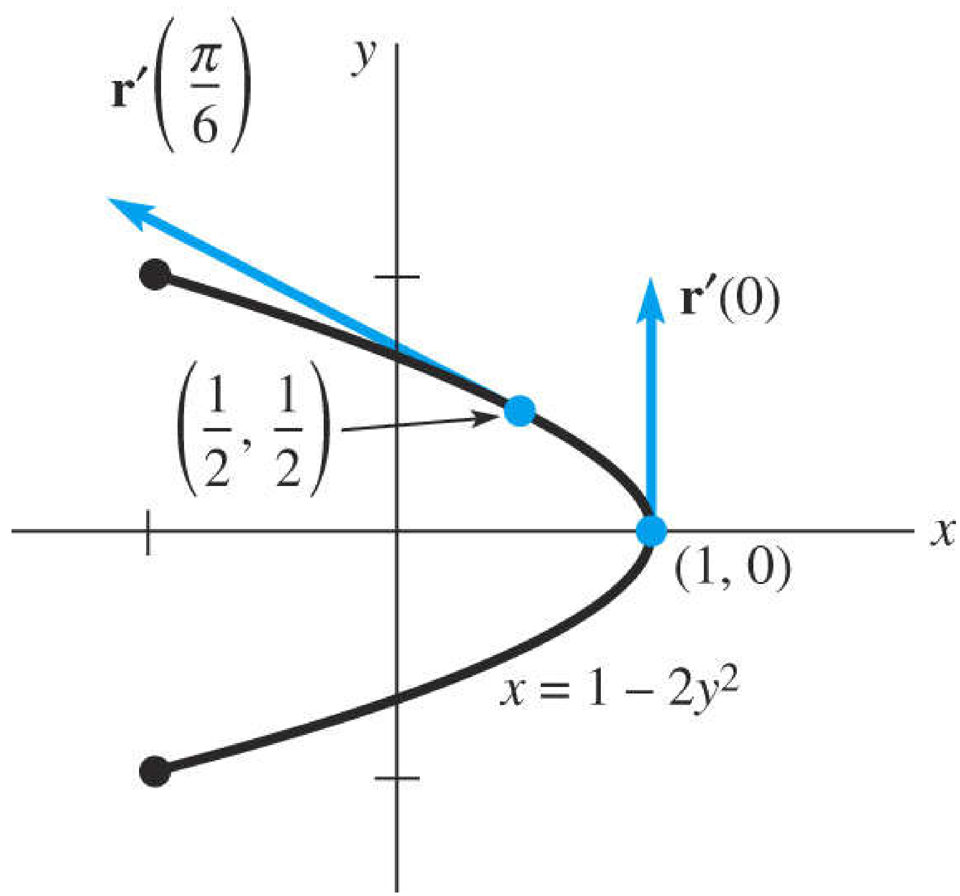
a) $x = \cos 2t,\ y = \sin t$, then
$x = 1 - 2y^2,\ -1 \le x \le 1$

b) $\mathbf{r}'(t) = -2\sin 2t\mathbf{i} + \cos t\mathbf{j}$,

$\mathbf{r}'(0) = < 0,1 > = \overrightarrow{PQ}$
P(1,0), Q(1,1)

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = \left\langle -\sqrt{3}, \frac{\sqrt{3}}{2} \right\rangle = \overrightarrow{AB}$$

$$A\left(\frac{1}{2}, \frac{1}{2}\right), B\left(-\sqrt{3} + \frac{1}{2}, \frac{\sqrt{3}}{2} + \frac{1}{2}\right)$$



Example 2

Find the tangent line to

$$x = t^2,\ y = t^2 - t,\ \ z = -7t \text{ at } t = 3$$

Solution

$$x' = 2t,\ y' = 2t - 1,\ \ z' = -7$$

$\mathbf{r}(3) = 9\mathbf{i} + 6\mathbf{j} - 21\mathbf{k}$**.** So, tangent point: $P(9, 6, -21)$

$\mathbf{r}'(3) = 6\mathbf{i} + 5\mathbf{j} - 7\mathbf{k} = \langle 6, 5, -7 \rangle$

Then

$$x = 9 + 6t,\ y = 6 + 5t,\ z = -21 - 7t$$

Higher Derivatives

- Second derivative: $\mathbf{r}''(t) = (\mathbf{r}'(t))'$
- nth derivative: $\mathbf{r}^{(n)}(t) = (\mathbf{r}^{(n-1)}(t))'$

- Example 6: If $\mathbf{r}(t) = (t^3 - 2t^2)\mathbf{i} + 4t\mathbf{j} + e^{-t}\mathbf{k}$, then

$$\mathbf{r}'(t) = (3t^2 - 4t)\mathbf{i} + 4\mathbf{j} - e^{-t}\mathbf{k}, \text{ and}$$

$$\mathbf{r}''(t) = (6t - 4)\mathbf{i} + e^{-t}\mathbf{k}.$$

THEOREM 2

Chain Rule

If $\mathbf{r}(s)$ is a differentiable vector function and $s = u(t)$ is a differentiable scalar function, then the derivative of $\mathbf{r}(s)$ with respect to t is

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{r}'(s)u'(t)$$

Example 3

If $\mathbf{r}(s) = \cos 2s\mathbf{i} + \sin 2s\mathbf{j} + e^{-3s}\mathbf{k}$, $s = t^4$, then

$$\frac{d\mathbf{r}}{dt} = [-2\sin 2s\mathbf{i} + 2\cos 2s\mathbf{j} - 3e^{-3s}\mathbf{k}]4t^3$$

$$= -8t^3\sin(2t^4)\mathbf{i} + 8t^3\cos(2t^4)\mathbf{j} - 12t^3 e^{-3t^4}\mathbf{k}$$

THEOREM 3

Differentiation Rule

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be differentiable vector functions and $u(t)$ be a differentiable scalar function.

(i) $$\frac{d}{dt}[\mathbf{r}_1(t)+\mathbf{r}_2(t)]=\mathbf{r}_1'(t)+\mathbf{r}_2'(t)$$

(ii) $$\frac{d}{dt}[u(t)\mathbf{r}_1(t)]=u(t)\mathbf{r}_1'(t)+u'(t)\mathbf{r}_1(t)$$

(iii) $$\frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)]=\mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)+\mathbf{r}_1'(t)\cdot\mathbf{r}_2(t)$$

(iv) $$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]=\mathbf{r}_1(t)\times\mathbf{r}_2'(t)+\mathbf{r}_1'(t)\times\mathbf{r}_2(t)$$

2. Integrals of Vector Functions

Vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

Define integral of vector function $\mathbf{r}(t)$ by

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k}$$

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}$$

Example 4

If $\mathbf{r}(t) = 6t^2\mathbf{i} + 4e^{-2t}\,\mathbf{j} + 8\cos 4t\ \mathbf{k}$, then

$$\int \mathbf{r}(t)\,dt = \left[\int 6t^2\,dt\right]\mathbf{i} + \left[\int 4e^{-2t}\,dt\right]\mathbf{j} + \left[\int 8\cos 4t\,dt\right]\mathbf{k}$$

$$= [2t^3 + c_1]\,\mathbf{i} + [-2e^{-2t} + c_2]\,\mathbf{j} + [2\sin 4t + c_3]\,\mathbf{k}$$

$$= 2t^3\mathbf{i} - 2e^{-2t}\mathbf{j} + 2\sin 4t\mathbf{k} + \mathbf{c}$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

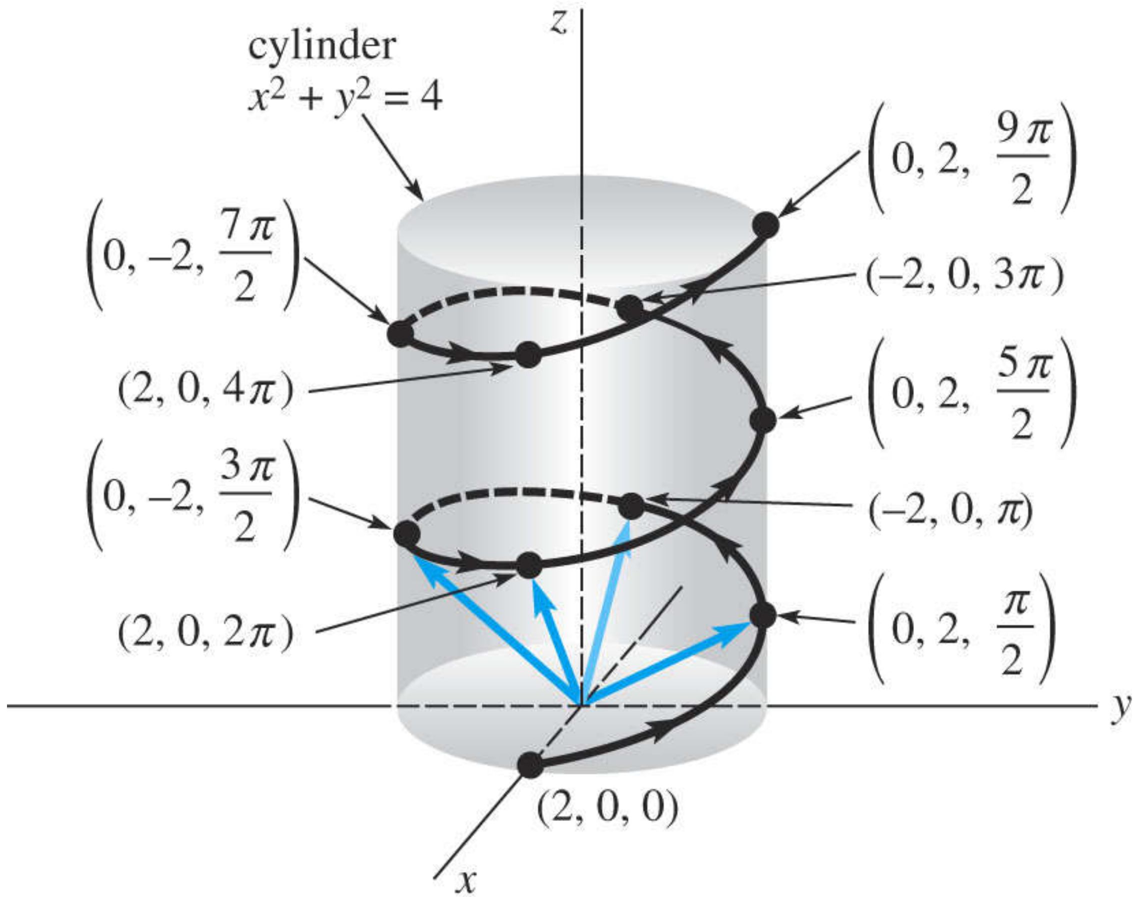
3. Length of a Space Curve

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, $a \le t \le b$, then the length of this smooth curve is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt$$

Example 5

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$$



Example 6

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$

Since $|\mathbf{r}'(t)| = \sqrt{5}$, from (3) the length from $\mathbf{r}(0)$ to $\mathbf{r}(t)$ is

$$s = \int_0^t \sqrt{5}\, du = \sqrt{5}t$$

Using $t = s/\sqrt{5}$ then

$$\mathbf{r}(s) = 2\cos\frac{s}{\sqrt{5}}\mathbf{i} + 2\sin\frac{s}{\sqrt{5}}\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k} \tag{4}$$

Thus

$$f(s) = 2\cos\frac{s}{\sqrt{5}}, \quad g(s) = 2\sin\frac{s}{\sqrt{5}}, \quad h(s) = \frac{s}{\sqrt{5}}$$

Fundamental Theorem of Calculus for vector functions

We can extend the Fundamental Theorem of Calculus to continuous vector functions:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a) \quad \text{where } \mathbf{R}'(t) = \mathbf{r}(t)$$

Exercises

1. Let $\mathbf{r}(t) = (1+2\cos t)\mathbf{i} + (2-\sin t)\,\mathbf{j}$

a) Find parametric equations for tangent line to the curve $\mathbf{r}(t)$ at $t = \pi/6$

b) Graph the curve $\mathbf{r}(t)$ for t in $[0, \pi]$, the position vector $\mathbf{r}(\pi/6)$ and tangent vector $\mathbf{r}'(\pi/6)$

2. Let $\mathbf{r}(t) = < -\cos 2t, \sin 2t, 6t^{3/2} >$

Find the length $s = \int_0^{\pi/4} |\mathbf{r}'(t)|\, dt$

Exercises

1. Let $\mathbf{r}(t) = (1+2\cos t)\mathbf{i} + (1+\sin t)\,\mathbf{j}$

a) Find parametric equations for tangent line to the curve $\mathbf{r}(t)$ at $t = \pi/4$

b) Graph the curve $\mathbf{r}(t)$ for t in $[0, \pi/2]$, the position vector $\mathbf{r}(\pi/6)$ and tangent vector $\mathbf{r}'(\pi/6)$

2. Let $\mathbf{r}(t) = < t, \cos 2t, \sin 2t >$

a) Find the unit tangent vector to this curve at $t = \pi / 6$

b) Find $\int_0^{\pi/4} \mathbf{r}(t)\, dt$ and the length $s = \int_0^{\pi/4} |\mathbf{r}'(t)|\, dt$