ANALYSIS 2

2. Applications of Integrals
(Chapter 6)

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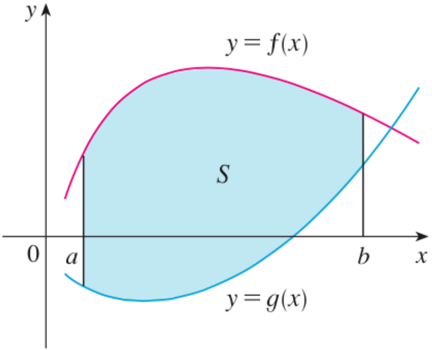
Contents

Section 1

Areas between curves

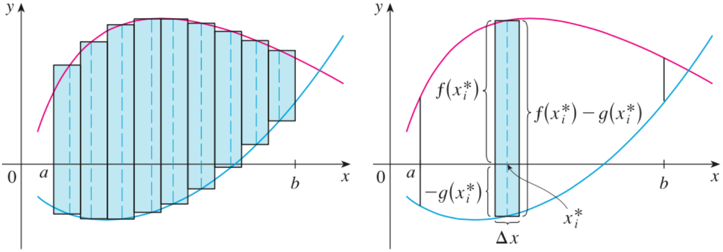
Area between curves

Suppose f and g are continuous and let S be the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$.



$$S = \{(x, y) \mid a \leqslant x \leqslant b,\, g(x) \leqslant y \leqslant f(x)\}$$

Area between curves



We approximate the area by

$$\sum_i [f(x_i^*) - g(x_i^*)]\Delta x.$$

This is a Riemann sum of the function $f(x) - g(x)$.
Thus, as $n \to \infty$, the sum converges to

$$\int_a^b [f(x) - g(x)]dx.$$

Formula

- If $f(x) \geq g(x)$ on $[a, b]$ then the area of S is

$$A = \int_a^b [f(x) - g(x)]dx.$$

- In general, the area is

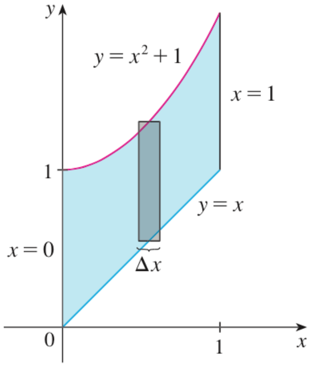
$$A = \int_a^b |f(x) - g(x)|dx.$$

Notes: In general, we may need to find the zeros of $f(x) - g(x)$ and determine its signs in between these points.

Example

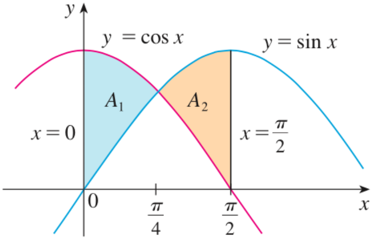
- Find the area of the region bounded by $y = x^2 + 1, y = x, x = 0, x = 1$.
- Solution:

$$A = \int_0^1 [(x^2 + 1) - x]dx$$
$$= \frac{x^3}{3} - \frac{x^2}{2} + x\Big|_0^1$$
$$= \frac{5}{6}$$



Example

Find the area of the region bounded by the curves $y = \sin x, y = \cos x, x = 0, x = \pi/2$.

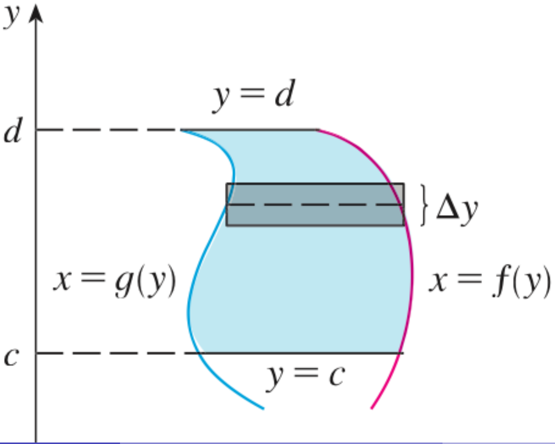


$$
\begin{aligned}
A &= \int_0^{\pi/2} |\cos x - \sin x|\, dx = A_1 + A_2 \\
&= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx \\
&= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\
&= 2\sqrt{2} - 2
\end{aligned}
$$

Integrals in y

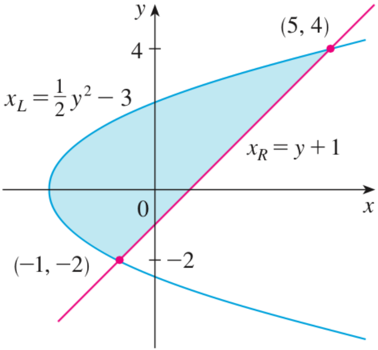
We can also integrate in y:

$$A = \int_c^d (x_R - x_L)dy$$



Example

- Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.
- Intersections: $(-1, -2)$ and $(5, 4)$
- Write equations in y



$$x_L = \tfrac{1}{2}y^2 - 3 \quad \text{and} \quad x_R = y + 1$$

$$A = \int_{-2}^{4} \left[(y+1) - (\tfrac{1}{2}y^2 - 3)\right] dy$$

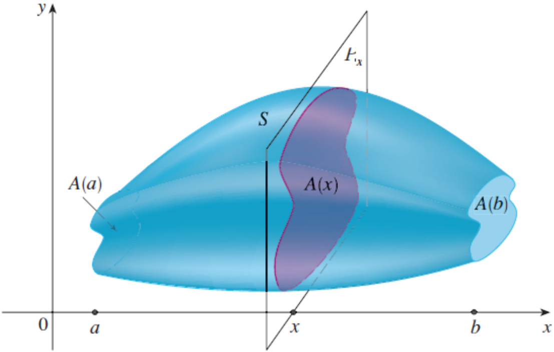
$$= -\frac{y^3}{6} + \frac{y^2}{2} + 4y\Big|_{-2}^{4} = 18$$

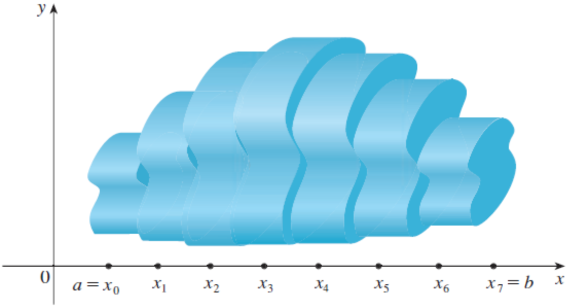
Section 2

Volumes

Volumes

Suppose a solid S lies between the planes $x = a$ to $x = b$. Assume that we know be the area $A(x)$ of the cross section of the solid perpendicular to the $x-$axis at x.





- Cut the solid into n parts by dividing $[a, b]$ into n equal intervals $[x_{i-1}, x_i]$ of length $\Delta x = \frac{b-a}{n}$.
- Then V is approximately $\sum_{i=1}^{n} A(x_i^*)\Delta x$, where $x_i^* \in [x_{i-1}, x_i]$.
- This is a Riemann sum for the function $A(x)$!
- Hence, passing to the limit $n \to \infty$, we arrive at

$$V = \int_a^b A(x)dx.$$

Section 3

Disk Method

Solids of Revolution - Disk Method

- When S is a solid obtained by rotating a two dimensional figure about the $x-$axis, the cross-sections are annuli. Thus,

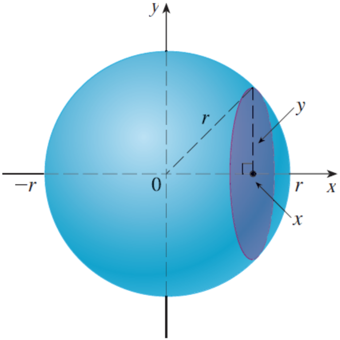
$$A(x) = \pi[(\text{outer radius})^2 - (\text{inner radius})^2].$$

- When S is obtained by rotating about the $y-$axis, use a similar formula for $A(y)$ and integrate in y.

Example

Volume of a sphere of radius r.

- Lies between $x = -r$ to $x = r$
- Cross-section at x is a circle of radius $\sqrt{r^2 - x^2}$, so $A(x) = \pi(r^2 - x^2)$.
- Thus,

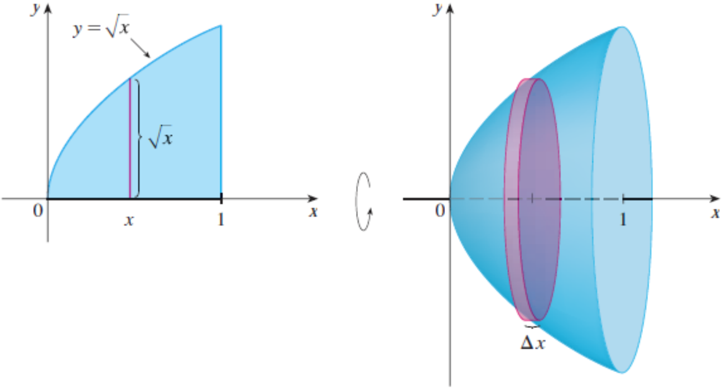


$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = 2\pi \int_0^r (r^2 - x^2)\,dx$$

$$= 2\pi\left[r^2 x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \tfrac{4}{3}\pi r^3$$

Example

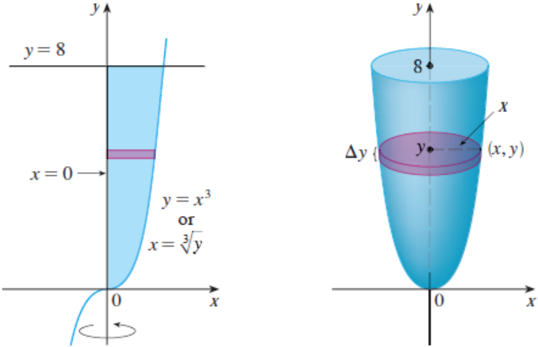
Find volume of a solid obtained by rotating about the $x-$ axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



Answer: $V = \int_0^1 A(x)dx = \int_0^1 \pi x dx = \frac{\pi}{2}$.

Example

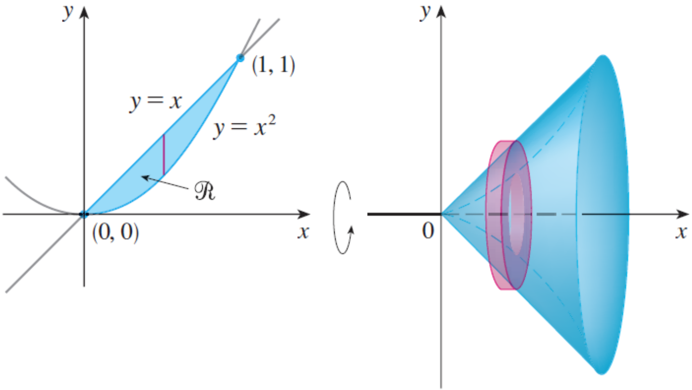
Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the $y-$axis.



Answer: $V = \int_0^8 A(y)dy = \int_0^8 \pi y^{2/3} dy = \frac{96\pi}{5}$.

Example

The region enclosed by the curves $y = x$ and $y = x^2$ is rotated about the $x-$axis. Find the volume of the resulting solid.



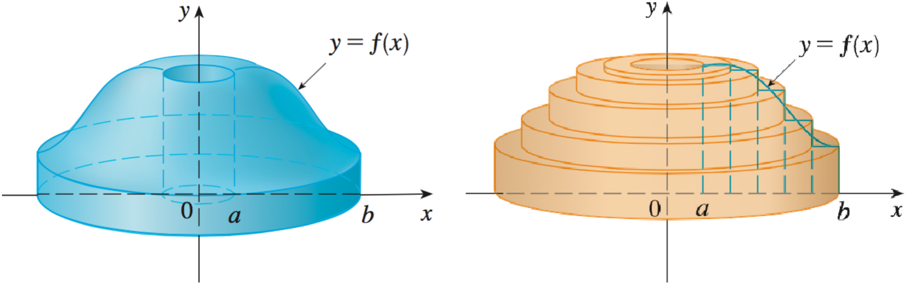
Answer: $V = \int_0^1 A(x)dx = \int_0^1 \pi(x^2 - x^4)dx = \frac{2\pi}{15}$.

Section 4

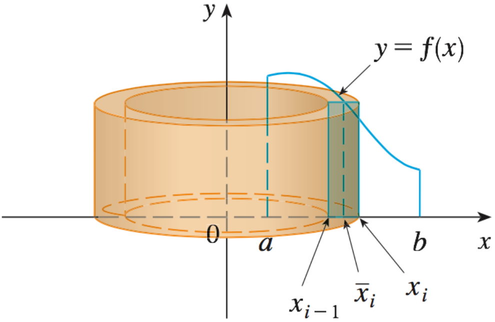
Cylindrical Shell Method

Cylindrical shell method

- It's easier if we cut the solid into cylindrical shells.
- Divide the interval of x into n parts $[x_{i-1}, x_i]$ with length Δx
- Approximate the solid in the region $x_{i-1} \leq r \leq x_i$ by a cylindrical shell with height $f(\bar{x}_i)$ where $\bar{x}_i \in [x_{i-1}, x_i]$.



Volume of one shell



- Think of a shell as a thin rectangle box folded into circular shape.
- Volume = Circumference × height × thickness, i.e.

$$V_i = (2\pi\bar{x}_i)f(\bar{x}_i)\Delta x.$$

Adding up

- Sum of shell volumes

$$S_n = \sum_{i=1}^{n} V_i$$
$$= \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i) \Delta x$$

- S_n is a Riemann sum
- Let $n \to \infty$ then S_n converges to the volume of the solid

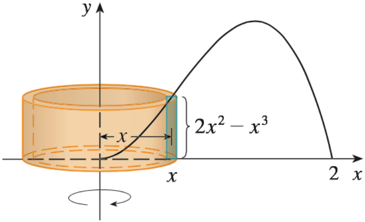
$$V = \lim_{n\to\infty} S_n = \int_a^b 2\pi x f(x) dx.$$

- To memorize: $V = \int_a^b 2\pi [\text{radius}][\text{height}] dx$

Example

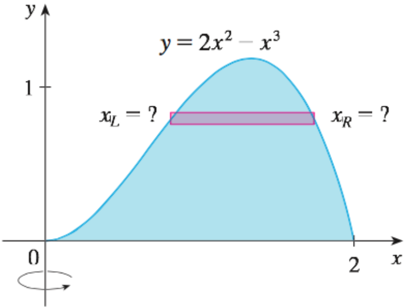
Find the volume of the solid created by rotating about $y-$axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

- Shell has radius x, interval $[0, 2]$
- the height $f(x) = 2x^2 - x^3$
- Volume



$$V = \int_0^2 2\pi x(2x^2 - x^3)dx = \frac{16}{5}\pi$$

- If we use the disk method, i.e. integrating in y, then we have to solve for x_L and x_R.



- This means solving the cubic equation

$$2x^2 - x^3 = y.$$

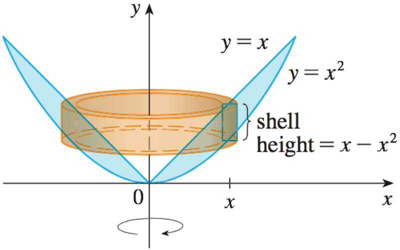
- Much more complicated than using cylindrical shell method!

Example

Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

- Shell has radius x, interval $[0, 1]$
- Height: $f(x) = x - x^2$
- Volume:

$$V = \int_0^1 2\pi x(x - x^2)dx = \frac{\pi}{6}$$



Note: The Disk Method uses integration along the axis of rotation, while the Shell Method uses integration along the other axis.

Section 5

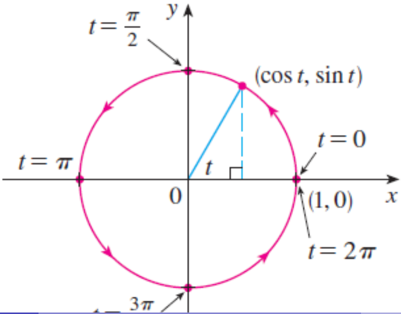
Lengths of Curves

Parametric curves

- To describe curves that fail the vertical line test.
- Write both coordinates x and y of points on the curve as function of a parameter $t \in [a, b]$:

$$x = x(t), \quad y = y(t).$$

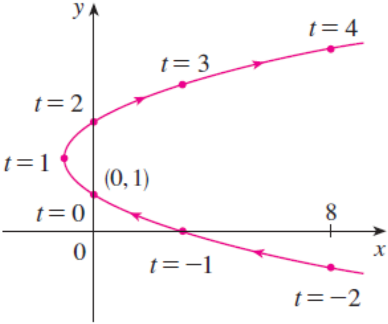
Example: The circle centered at $(0, 0)$ of radius 1 can be parameterized by $x = \cos(t), \quad y = \sin(t), \ t \in [0, 2\pi]$



Example

- The line segment connecting (a, b) and (c, d) can be paremeterized by $x(t) = a + t(c - a)$, $y(t) = b + t(d - b)$ for $t \in [0, 1]$
- The parabola

$$x = y^2 - 4y + 3$$

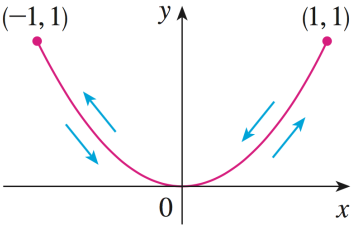


can be parameterized by

- Parametric equations describe a moving point.
- The range of this point is the curve.
- There are more than one way to parameterize a curve. E.g.
  - $x = \cos(2t), \quad y = \sin(2t),\ t \in [0, 2\pi]$ is another parameterization for the unit cirle.
    The point now goes around the circle twice, counterclockwise, starting from $(1, 0)$
  - $x = \sin(2t), \quad y = \cos(2t),\ t \in [0, 2\pi]$ is another parameterization for the unit cirle.
    The point goes around the circle twice, clockwise, starting from $(0, 1)$.

Sketching Parametric Curves

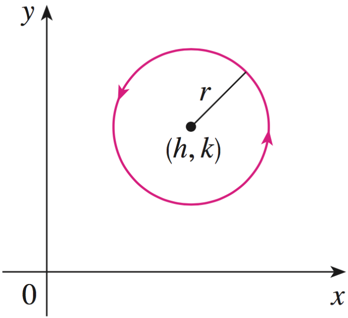
- There is no general method to sketch parametric curves.
- For certain curves, we can find relationship between x and y from the parametric equations. E.g.
  - For the curve $x = \sin t, y = \sin^2 t$, we notice that $y = x^2$, so the point moves on the parabola.
  - Since $-1 \leq x = \sin t \leq 1$, the curve is only a part of the parabola.
  - $\sin t$ is periodic, hence the point moves back and forth between $(-1, 1)$and $(1, 1)$.



Example

Sketch: $x = h + r\cos t, y = k + r\sin t$
$0 \leq t \leq 2\pi$

- Remove t to find relation between x and y:
  $(x-h)^2 + (y-k)^2 = r^2 \to$ a circle of radius r.
- The point goes around the circle once, in counterclockwise direction.

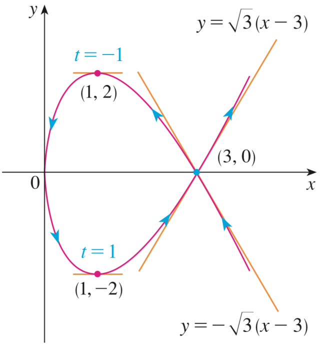


Tangent line

- Suppose $x(t)$ and $y(t)$ are differentiable. Then the tangent line to the curve at $(x(t), y(t))$ has direction $(x'(t), y'(t))$. Thus, the formula is
$$\frac{x - x(t)}{x'(t)} = \frac{y - y(t)}{y'(t)}.$$
- Example: Let C be the parametric curve
$$x = t^2, \quad y = t^3 - 3t.$$
Show that C has two tangent lines at $(3, 0)$ and find their equations.

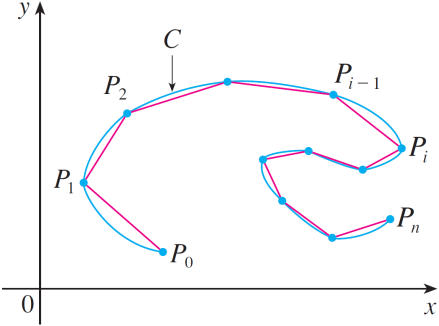
Solution

- Solve $y = t^3 - 3t = 0$ we have $t = 0$ and $t = \pm\sqrt{3}$.
- Both 2 values $t = \pm\sqrt{3}$ give the point $(3, 0)$: C crosses itself at $(0, 3)$
- At $t = -\sqrt{3}$, the tangent line is
$$-\frac{x-3}{2\sqrt{3}} = \frac{y}{6}.$$
At $t = \sqrt{3}$, the tangent line is
$$\frac{x-3}{2\sqrt{3}} = \frac{y}{6}.$$



Arc length

- To find the length of a parametric curve $(x(t), y(t))$, $t \in [a, b]$, we approximate it by simpler curves, i.e. line segments.
- Partition the interval $[a, b]$ by $a = t_0 < t_1 < \cdots < t_n = b$, then we approximate the curve by the line segment $P_{i-1}P_i$, where $P_i = (x(t_i), y(t_i))$.



- Let $P_i = (x(t_i), y(t_i))$. Then

$$|P_{i-1}P_i| \approx \sqrt{(x'(t_i))^2 + (y'(t_i))^2}\Delta t.$$

- Thus,

$$L \approx \sum_{i=1}^{n} \sqrt{(x'(t_i))^2 + (y'(t_i))^2}\Delta t.$$

- This is a Riemann sum for the function $\sqrt{(x'(t))^2 + (y'(t))^2}$. Thus, passing to the limit $n \to \infty$, we obtain

$$\boxed{L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt.}$$

Special cases

- If f' is continuous on $[a, b]$ then the length of the arc $y = f(x),\ a \leq x \leq b$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}dx = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}dx$$

- If the curve has equation $x = g(y),\ c \leq y \leq d$ then the length is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}dy = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}dy$$

Example

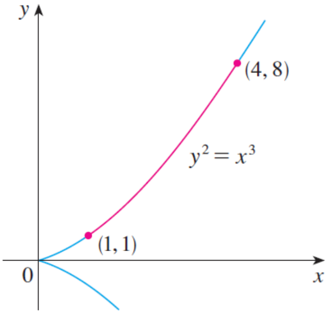
Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1,1) and (4,8).

- $y = x^{3/2}$ so $y' = \frac{3}{2}x^{1/2}$
- Arc length

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}dx$$

Subs: $u = 1 + \frac{9}{4}x$

$$L = \frac{4}{9}\int_{13/4}^{10} \sqrt{u}du$$
$$= \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$$



Exercises

- 6.1: 9, 12, 21, 27
- 6.2: 5, 9, 25, 31
- 6.3: 15, 23, 37, 38
- 6.4: 8, 10, 12, 27, 28, 29