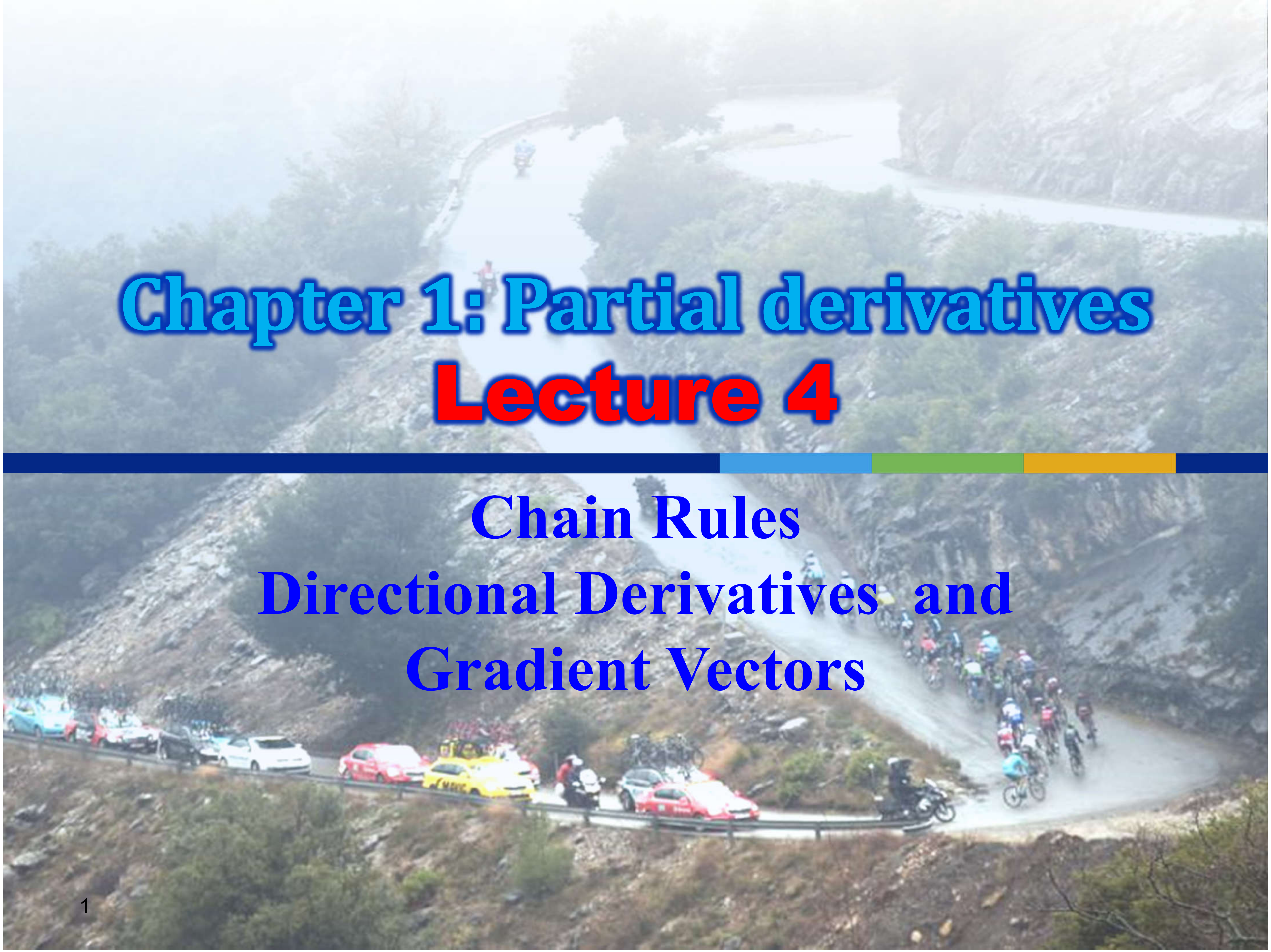


Chapter 1: Partial derivatives

Lecture 4

Chain Rules

Directional Derivatives and Gradient Vectors

1. The Chain Rule

- We recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If y=f(x) and x=g(t), where f and g are differentiable functions, then y is indirectly a differentiable function of and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Functions of several variables

- The Chain Rule has two versions:

1) $z = f(x, y), \quad x = x(t), y = y(t), \quad t \in \mathbb{R}$

2) $z = f(x, y), \quad x = x(s, t), y = y(s, t), \quad s, t \in \mathbb{R}$

The Chain Rule (Case I)

- Theorem: z=f(x,y), where x=x(t) and y=y(t)
- Then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$or$$

$$z'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$$

Example

- If $z=x^2y+3xy^4$, where $x=\sin 2t$ and $y=\cos t$, find dz/dt when $t=0$.
- Solution. The Chain Rule gives

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$=(2xy+3y^4)(2\cos 2t)+(x^2+12xy^3)(-\sin t)$$

- It's not necessary to substitute the expressions for x and y in terms of t. We simply observe that when t=0 we have x=sin 0 =0 and y=cos 0 = 1. Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2\cos 0) + (0+0)(-\sin 0) = 6$$

The Chain Rule (Case 2)

- Theorem: z=f(x,y), where x=g(s,t) and y=h(s,t).
- Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

or

$$z_s = z_x x_s + z_y y_s, \qquad z_t = z_x x_t + z_y y_t$$

Example: If $z=e^x \sin y$, where $x=st^2$ and $y=s^2t$, find $\partial z/\partial t$ and $\partial z/\partial s$

Solution: Apply the Chain Rule (case 2), we get

$$z_s = z_x x_s + z_y y_s$$

$$= (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$= t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t)$$

$$z_t = z_x x_t + z_y y_t$$

$$= (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

$$= 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t)$$

Chain rule: General Case

- $u=u(x_1, \ldots, x_n)$,
- $x_k=x_k(t_1,\ldots,t_m)$.
- Then for i=1,2,…,m:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

$$= \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i}$$

Implicit Differentiation

$$F(x, y) = 0 \quad \text{defines} \quad y = f(x)$$

$$0 = \frac{d}{dx} F(x, y) = F_x(x, y) + F_y(x, y) \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

Implicit Differentiation

$$F(x, y, z) = 0 \ \text{ defines } \ z = f(x, y)$$

$$0 = \frac{\partial}{\partial x} F(x, y, z) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

Example 1

$$\sin(x-y) = xe^y \qquad (\Rightarrow y = f(x) \text{ implicitely})$$

$$\Rightarrow (\sin(x-y))' = (xe^y)'$$

$$\Rightarrow (1-y')\cos(x-y) = e^y + xe^y y'$$

$$\Rightarrow (xe^y + \cos(x-y))y' = -e^y + \cos(x-y)$$

$$\Rightarrow y' = \frac{-e^y + \cos(x-y)}{xe^y + \cos(x-y)}$$

Example 2

$$\ln(x+yz)=1+xy^2z^3 \qquad (F(x,y,z)=0 \Rightarrow z=z(x,y))$$

$$\Rightarrow (\ln(x+yz))_x=(1+xy^2z^3)_x$$

$$\frac{1+yz_x}{x+yz}=y^2(z^3+3xz^2z_x)$$

$$\Rightarrow [3xy^2z^2(x+yz)-y]z_x=1-y^2z^3(x+yz)$$

$$\Rightarrow z_x=\frac{1-y^2z^3(x+yz)}{3xy^2z^2(x+yz)-y}$$

Exercises

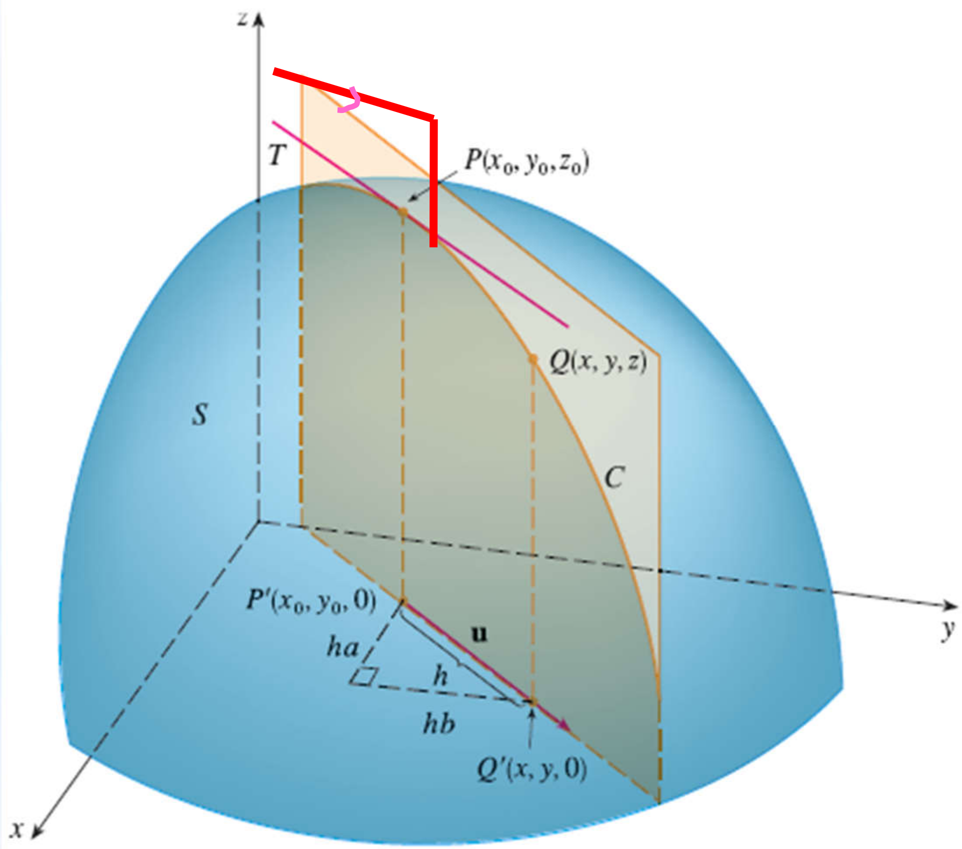
1. Find all the partial derivatives $\partial z/\partial t$ and $\partial z/\partial s$

$$z = e^x \cos y, \quad x = st, \quad y = \sqrt{s^2 + t^2}$$

2. Find all partial derivatives $\partial z/\partial r$, $\partial z/\partial t$ and $\partial z/\partial s$ when $r=1, s=-1, t=0$

$$z = x / y, \quad x = re^{st}, \quad y = rse^t$$

2. Directional Derivative and Gradient



$$D_u f(P_0) = \lim_{h\to 0}\frac{f(P_0+hu)-f(P_0)}{h}$$

$$= \tan\alpha$$

Definition. **The directional derivative** of z=*f(x,y)* at (x_0 , y_0) in the direction of a unit vector u=<a,b> (i.e., |u|=1) is defined by

$$D_u f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha,y_0+hb)-f(x_0,y_0)}{h}$$

Theorem. If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector u=<a,b> and

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$$

If the unit vector makes an angle with the positive x-axis, then we can write u=<cos θ, sin θ> and the above formula becomes

$$D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

Gradient Vector

Definition. If f is a function of two variables x and y, then the gradient of f is the vector function defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = f_x(x,y)i + f_y(x,y)j$$

Gradient is also denoted by **grad *f***

We can rewrite Directional Derivative:

$$\boxed{D_u f(x,y) = f_x(x,y)a + f_y(x,y)b = \nabla f(x,y).u}$$

Example

$$f(x,y)=\sin x+e^{xy}$$

$$\nabla f(x,y)=<f_x(x,y),f_y(x,y)>=<\cos x+ye^{xy},xe^{xy}>$$

Directional Derivative of Functions of three variables

Definition. **The directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector u=<a,b,c> is defined by

$$D_u f(x_0,y_0,z_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb, z_0+hc)-f(x_0,y_0,z_0)}{h}$$

$$=\lim_{h\to 0}\frac{f(P_0+hu)-f(P_0)}{h}, \qquad P_0=(x_0,y_0,z_0)$$

if this limit exists

If *f* is a function of three variables, the **gradient** of *f* is the vector function defined by

$$\nabla f(x, y, z) = \left\langle f_x(x, y, z), f_y(x, y, z), f_z(z, y, z) \right\rangle$$

Gradient is also denoted by grad *f*

- Directional Derivatives can be expressed as the dot product:

$$D_u f(x, y, z) = \nabla f(x, y, z).u, \qquad |u| = 1$$

- If v≠0, then directional derivative of *f* in the direction v is given by

$$D_u f(x, y, z) = \nabla f(x, y, z).u, \qquad u = \frac{v}{|v|}$$

Maximizing the Directional Derivative

Theorem. The maximum value of $D_u f(x,y,z)$ is $|\nabla f(x,y,z)|$ and it occurs when u has the same direction as $\nabla f(x,y,z)$

Proof

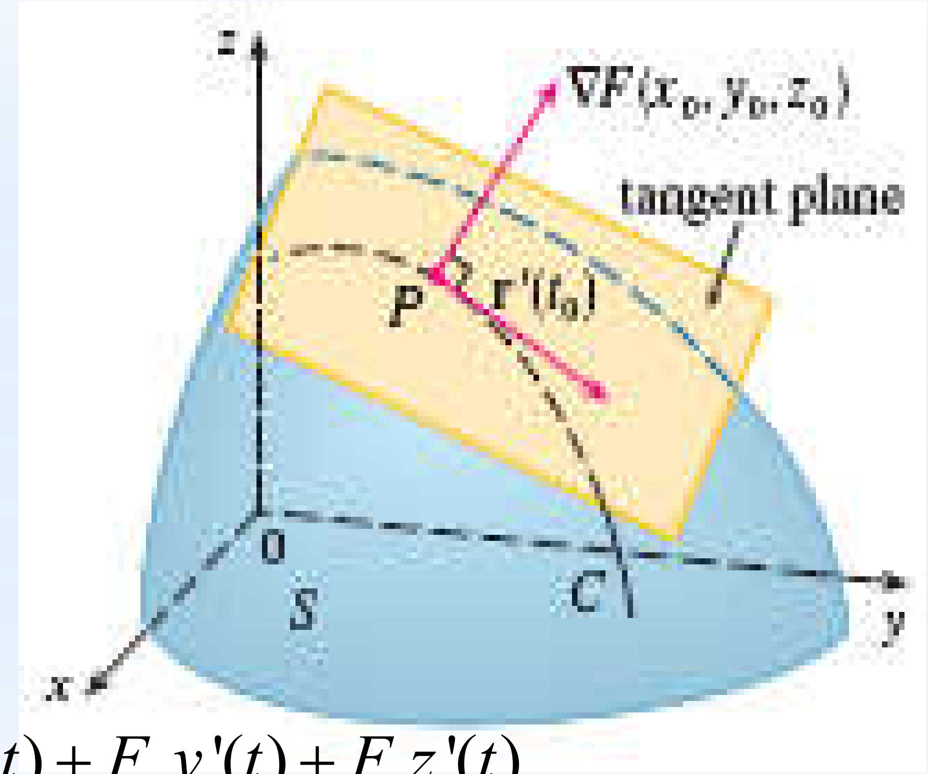
$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot u \leq |\nabla f||u| = |\nabla f|$$

"=" occurs iff $\nabla f(x,y,z)$ has the same direction as u

Tangent Planes to Level Surfaces

- Surface S: F(x,y,z)=k=constant
- P: point on S
- C: curve on S through P

  $r(t)=<x(t),y(t),z(t)>$, $r(t_0)=P$



It holds that

$$F(x(t), y(t), z(t)) = k$$

$$\Rightarrow 0 = \frac{d}{dt} F(x(t), y(t), z(t)) = F_x x'(t) + F_y y'(t) + F_z z'(t)$$

$$= \nabla F(x(t), y(t), z(t)).r'(t), \qquad r(t) = (x(t), y(t), z(t))$$

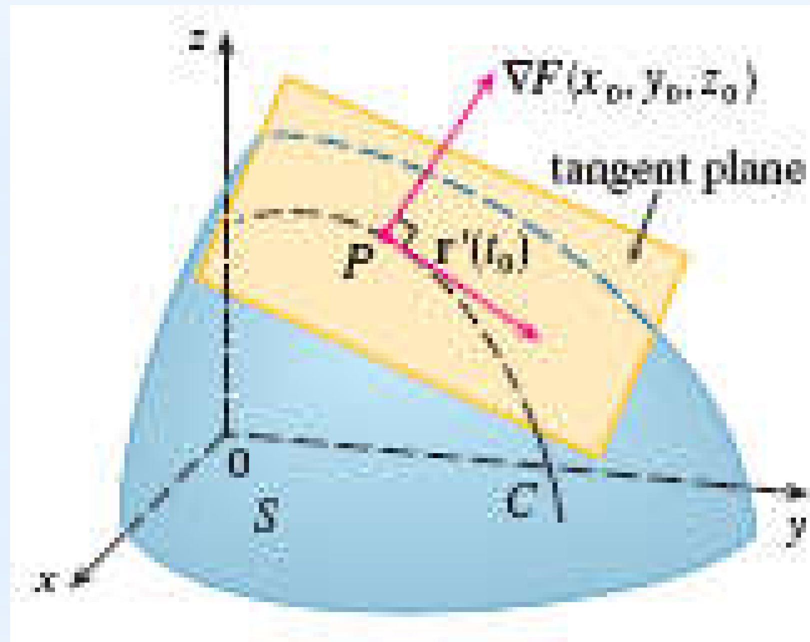
$$\Rightarrow \nabla F(x_0, y_0, z_0).r'(t_0) = 0$$

Tangent Planes to Level Surfaces

$$\nabla F(x_0, y_0, z_0) \perp r'(t_0)$$

for any tangent vector r '(t 0) to any curve C on S that passes through P

Tangent plane to surface *F(x,y,z)=k* at P is the plane passing through P and has normal vector $\nabla F(x_0, y_0, z_0)$

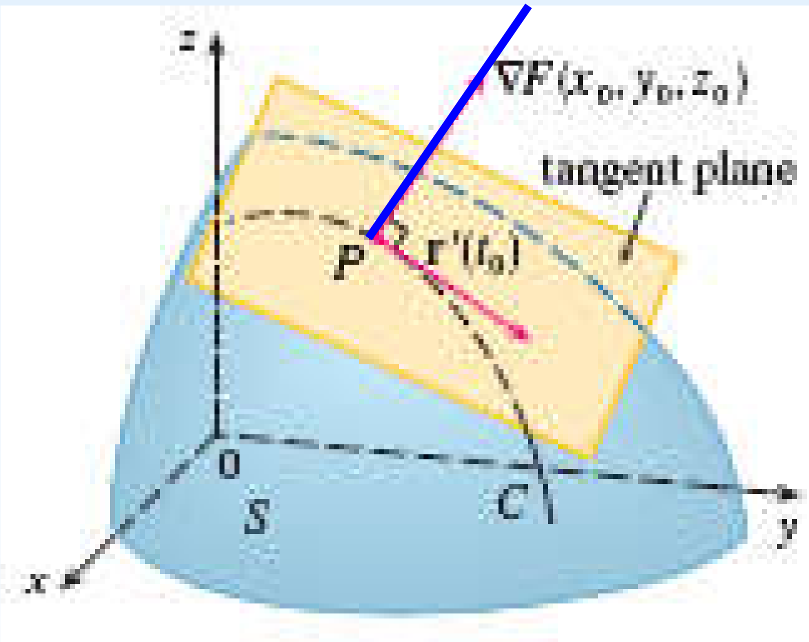


Tangent plane to S has equation:

$$\boxed{F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0}$$

Normal line to Level Surface

The ***normal line*** to surface $F(x,y,z)=k$ at P is the line passing through P and perpendicular to the tangent plane



Direction of normal line:

$$\nabla F(x_0, y_0, z_0)$$

Equation of normal line:

$$\boxed{\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}}$$