Real Analysis, Chapter 0
Appendix A: Set Cardinality

Quan Nguyen

International University, VNU-HCM

quannguyenuw@gmail.com

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Definition 0.1.1 (Equivalent Sets)

Two sets A and B are said to be equivalent, denoted $A \sim B$ if there exists a bijection $f : A \to B$.

Theorem 0.1.1

For any nonempty sets A, B, C :

(a) $A \sim A$;

(b) If $A \sim B$ then $B \sim A$;

(c) If $A \sim B$ and $B \sim C$ then $A \sim C$.

Guidelines:

(a) Let $f : A \to A$ be the identity mapping. Then f is bijective;

(b) If $f : A \to B$ is bijective then $f^{-1} : B \to A$ is bijective;

(c) If $f : A \to B$ and $g : B \to C$ are bijections then $g \circ f : A \to C$ is bijective.

The following set cardinal system was proposed by Georg Cantor (1870).

Definition 0.1.2 (Set Cardinality)

A set E is called:

(a) finite if $E \sim \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$, and infinite otherwise;
(b) countably infinite if $E \sim \mathbb{N}$;
(c) countable if E is either finite or countably infinite;
(d) uncountable if E is not countable.

We sometimes call a countable set **enumerable**.

Remark 0.1.1

Given two equivalent sets A and B, if A is:

(a) finite then B is also finite;
(b) countably infinite then B is also countably infinite;
(c) countable then B is also countable.

In words, two equivalent sets must have the same **cardinality**.

For any set A, we denote the cardinality of A by $|A|$.

For the sake of convenience, denote $|\mathbb{N}| = \aleph_0$. Then A is called:

- finite if $|A| < \aleph_0$, and infinite otherwise;
- countably infinite if $|A| = \aleph_0$;
- countable if $|A| \leq \aleph_0$;
- uncountable if $|A| > \aleph_0$.

Two sets A and B are equivalent if and only if $|A| = |B|$.

When A is finite, we usually define $|A|$ as the number of elements of A.

Theorem 0.1.2

A subset of a countable set is countable.

Guidelines: Pick a countable set A and any set $B \subset A$.

- If A is finite, then so is B. Hence B is countable;
- Else, A is countably infinite and has the representation $A = \{x_n\}_1^\infty$.
    - If B is finite, then it is countable;
    - Else, selecting elements of B from A is equivalent to considering a subsequence of $\{x_n\}$. Since a subsequence is countably infinite, so is B. Thus B is countable.
- In all cases, B is countable. The result follows.

Remark 0.1.2

A superset of an uncountable set is uncountable.

In general, if $B \subset A$ then $|B| \leq |A|$.

Theorem 0.1.3

For any set $A \neq \emptyset$, the following statements are equivalent:

(a) A is countable;

(b) There is a countable set B and a surjection $f : B \to A$;

(c) There is a countable set F and an injection $f : A \to F$.

Guidelines:

- ($a \Rightarrow b$) Let $f : A \to A$ be the identity mapping;
- ($b \Rightarrow a$) Since f is surjective, each element in A corresponds to at least one element in $f^{-1}(A)$. Hence $|A| \leq |f^{-1}(A)| \leq |B|$, as $f^{-1}(A) \subset B$;
- ($a \Rightarrow c$) Let $f : A \to A$ be the identity mapping;
- ($c \Rightarrow a$) Since f is injective, distinct elements in A correspond to distinct elements in $f(A)$. Hence $|A| = |f(A)| \leq |F|$, as $f(A) \subset F$.

Theorem 0.1.4

If A and B are countable, then so is $A \times B$.

Guidelines:

- Without loss of generality, assume that $|A| = |B| = \aleph_0$;
- Then we have the representation $A = \{x_n\}_1^\infty$ and $B = \{y_n\}_1^\infty$;
- Plot elements of $A \times B$ in a table, as follows:

	x_1	x_2	...
y_1	$z_{1,1}$	$z_{2,1}$	...
y_2	$z_{1,2}$	$z_{2,2}$	...
...	...	...	...

- Now we can list the elements of $A \times B$ in a 'zig-zag' order: $(1,1) \to (2,1) \to (1,2) \to (1,3) \to (2,2) \to (3,1) \to \ldots$
- Thus $|A \times B| = \aleph_0$.

Inductively, if $A_1, A_2, \ldots A_n$ are countable then so is $\prod_{i=1}^{n} A_i = A_1 \times \ldots \times A_n$.

Theorem 0.1.5

If I is countable and A_i is countable for each $i \in I$, then so is $A = \bigcup_{i \in I} A_i$.

Guidelines:

- Without loss of generality, assume that $I = \mathbb{N}$ and $|A_i| = \aleph_0, \forall i \in \mathbb{N}$;
- Then we have the representation $A_i = \{x_{n,i}\}_{n=1}^{\infty}, \forall i \in \mathbb{N}$;
- Plot elements of A in a table, as follows:

A_1	$x_{1,1}$	$x_{2,1}$	...
A_2	$x_{1,2}$	$x_{2,2}$	...
...	...	...	...

- Now we can list the elements of A in a 'zig-zag' order;
- Thus $|A| = \aleph_0$.

Theorem 0.1.6

If $a, b \in \mathbb{R}$ and $a < b$, then the closed interval $[a, b]$ is uncountable.

Guidelines: Let $A = [a, b]$.

- Define a sequence $\{c_n\} : c_n = a + \frac{b-a}{2^n}, \forall n \in \mathbb{N}$;
- Then $\{c_n\} \subset A$, thus A is infinite. Assume conversely that $|A| = \aleph_0$;
- Then we have the representation $A = \{x_n\}_1^\infty$;
- Construct two new sequence $\{a_n\}$ and $\{b_n\}$ as follows:
  - $a_1 = a$ and $b_1 = b$;
  - If $x_n \in (a_n, b_n)^c$ then $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \frac{a_{n+1}+b_n}{2}$;
  - Else, $a_{n+1} = \frac{x_n+b_n}{2}$ and $b_{n+1} = \frac{a_{n+1}+b_n}{2}$;
- Then $\{a_n\}$ is strictly increasing and bounded above, while $\{b_n\}$ is strictly decreasing and bounded below. Thus $\exists a_0 = \lim_{n\to\infty} a_n$ and $\exists b_0 = \lim_{n\to\infty} b_n$. Moreover, $a_0 \leq b_0$ and $x_n \notin [a_{n+1}, b_{n+1}], \forall n \in \mathbb{N}$;
- Thus if $c_0 = \frac{a_0+b_0}{2}$ then $c_0 \in A$ and $c_0 \neq x_n, \forall n \in \mathbb{N}$, a contradiction;
- Hence $|A| > \aleph_0$.

Example 0.1.1

$\mathbb{Z}$ is countable.

Hint. $|\mathbb{Z}| = |\mathbb{N} \cup \{0, -1, -2, ...\}| = |\mathbb{N} \cup \mathbb{N}| = \aleph_0$.

Example 0.1.2

$\mathbb{Q}$ is countable.

Hint. $|\mathbb{Q}| = \left|\left\{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}\right| = |\{m : m \in \mathbb{Z}\} \times \{n : n \in \mathbb{N}\}| = \aleph_0$.

Example 0.1.3

$\mathbb{R}$ is uncountable. In set theory, we usually denote $|\mathbb{R}| = \aleph_1 = 2^{\aleph_0}$.

Hint. If $\mathbb{R}$ is countable then $[a, b] \subset \mathbb{R}$ is countable, a contradiction.

Example 0.1.4

The set of all irrational numbers, denoted $\mathbb{Q}^c$, is uncountable.

Hint. If $\mathbb{Q}^c$ is countable then $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ is countable, a contradiction.