Homework 3, Probability

Tran Viet Hang
Instructor: Dr. Nguyen Minh Quan

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1 Problems

Problem 1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?

(b) What is F, the cumulative distribution function of X?

Solution.

(a) Note that

$$1 = \mathbb{P}(\mathbb{R}) = \int_{\mathbb{R}} f(x)dx = \int_{-1}^{1} c(1-x^2)dx = \frac{4c}{3} \Rightarrow c = \frac{3}{4}.$$

(b) The cumulative distribution function of X is then given by

$$\begin{aligned} F(x) &= \begin{cases} \int_{-1}^{x} 3(1-x^2)/4dx & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (3x - x^3)/4 + 1/2 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

□

Problem 2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

what is the probability that the system functions for at least 5 months?

Solution. Note that

$$1 = \mathbb{P}(\mathbb{R}) = \int_{\mathbb{R}} f(x)dx = \int_{0}^{\infty} Cxe^{-x/2}dx = 4C \Rightarrow C = 0.25.$$

Thus the probability that the system functions for at least 5 months is

$$\mathbb{P}(X \ge 5) = \int_{5}^{\infty} Cxe^{-x/2}dx = \int_{5}^{\infty} 0.25xe^{-x/2}dx = \frac{7}{2}e^{-5/2}.$$

□

Problem 4. The probability density function of the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} 10/x^2 & x > 10 \\ 0 & x \leq 0 \end{cases}$$

(a) Find $\mathbb{P}(X > 20)$.

(b) What is F, the cumulative distribution function of X?

(c) What is the probability that of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

Solution.

(a) We have

$$\mathbb{P}(X > 20) = \int_{20}^{\infty} f(x)dx = \int_{20}^{\infty} \frac{10}{x^2}dx = \frac{1}{2}.$$

(b) The cumulative distribution function of X is

$$\begin{aligned} F(x) &= \begin{cases} \int_{10}^{x} 10/x^2 & x \geq 10 \\ 0 & x < 10 \end{cases} \\ &= \begin{cases} 1 - \frac{10}{x} & x \geq 10 \\ 0 & x < 10 \end{cases}. \end{aligned}$$

(c) Let $p = \mathbb{P}(X \geq 15) = 1 - F(15) = 2/3$ and Y be the random variable representing the number of devices that function for at least 15 hours, then $Y \sim \text{Bino}(6, p) = \text{Bino}(6, 2/3)$ and the desired probability is

$$\mathbb{P}(Y \geq 3) = \sum_{i=3}^{6} \binom{6}{i} \left(\frac{2}{3}\right)^i \cdot \left(\frac{1}{3}\right)^{6-i} \approx 0.9.$$

For the solution above, we assume that lifetimes of devices are pairwise independent.

□

Problem 6. Compute $\mathbb{E}[X]$ if X has the density function given by:

(a)

$$f(x) = \begin{cases} xe^{-x/2}/4 & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f(x) = \begin{cases} \frac{5}{x^2} & x > 5 \\ 0 & x \leq 5 \end{cases}$$

Solution.

(a)

$$\mathbb{E}[X] = \int_{\mathbb{R}} xf(x)dx = \int_{0}^{\infty} \frac{1}{4}x^2e^{-x/2}dx = 4.$$

(b) From Question 1, $c = 3/4$ and hence

$$\mathbb{E}[X] = \int_{\mathbb{R}} xf(x)dx = \int_{-1}^{1} \frac{3}{4}x(1-x^2)dx = 0.$$

(c)

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x) dx = \int_5^\infty \frac{5}{x} dx = \infty.$$

$\square$

Problem 7. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $\mathbb{E}[X] = 3/5$, find a and b.

Solution. Note that

$$\frac{3}{5} = \mathbb{E}[X] = \int_{\mathbb{R}} x f(x) dx = \int_0^1 (ax + bx^3) dx = \frac{a}{2} + \frac{b}{4} \tag{1}$$

and

$$1 = \mathbb{P}(\mathbb{R}) = \int_{\mathbb{R}} f(x) dx = \int_0^1 (a + bx^2) dx = a + \frac{b}{3} \tag{2}$$

thus (1) and (2) implies $a = 3/5$ and $b = 6/5$. $\square$

Problem 13. You arrive at a bus stop at 10:00, knowing that the bus will arrive at some time uniformly distributed between 10:00 and 10:30.

(a) What is the probability that you will have to wait longer than 10 minutes?

(b) If at 10:15 the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

Solution.

(a) Let X be the bus arrival time, then X is a uniform random variable of density

$$f(x) = \begin{cases} 1/30 & 0 \le x \le 30 \\ 0 & \text{otherwise} \end{cases}$$

and the probability that you will have to wait longer than 10 minutes is

$$\mathbb{P}(X > 10) = \int_{10}^{30} \frac{1}{30} dx = \frac{2}{3}.$$

(b) Similarly to part (a),

$$\mathbb{P}(X > 25) = \int_{25}^{30} \frac{1}{30} dx = \frac{1}{6} \quad \text{and} \quad \mathbb{P}(X > 15) = \int_{15}^{30} \frac{1}{30} dx = \frac{1}{2}$$

hence the conditional probability that you will have to wait at least 10 minutes more is

$$\mathbb{P}(X > 25 | X > 15) = \frac{\mathbb{P}(X > 25, X > 15)}{\mathbb{P}(X > 15)} = \frac{\mathbb{P}(X > 25)}{\mathbb{P}(X > 15)} = \frac{1/6}{1/2} = \frac{1}{3}.$$

$\square$

Problem 14. Let X be a uniform $(0, 1)$ random variable. Compute $\mathbb{E}[X^n]$ by using Proposition 2.1, and then check the result by using the definition of expectation.

Solution. By Proposition 2.1,

$$\mathbb{E}[X^n] = \int_0^1 x^n \cdot f_X(x)dx = \int_0^1 x^n dx = \frac{1}{n+1}.$$

By definition,

$$F_{X^n}(x) = \mathbb{P}(X^n \leq x) = \mathbb{P}(X \leq x^{1/n}) = x^{1/n}$$

and thus

$$\mathbb{E}[X^n] = \int_0^1 x f_{X^n}(x)dx = \int_0^1 x \cdot \frac{d}{dx} F_{X^n}(x)dx = \int_0^1 x \cdot \frac{x^{1/n-1}}{n} dx = \frac{1}{n}\int_0^1 x^{1/n} dx = \frac{1}{n+1}.$$

□

Problem 15. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute (a) $\mathbb{P}(X > 5)$, (b) $\mathbb{P}(4 < X < 16)$, (c) $\mathbb{P}(X < 8)$, (d) $\mathbb{P}(X < 20)$ and (e) $\mathbb{P}(X > 16)$.

Solution. Let $Z \sim \mathcal{N}(0,1)$ and Φ be its cumulative distribution function, then:

(a)

$$\mathbb{P}(X > 5) = \mathbb{P}\left(Z > \frac{5-10}{6}\right) = 1 - \mathbb{P}(Z < -5/6) = 1 - \Phi(-5/6) \approx 0.798.$$

(b)

$$\begin{aligned}\mathbb{P}(4 < X < 16) &= \mathbb{P}\left(\frac{4-10}{6} < Z < \frac{16-10}{6}\right) = \mathbb{P}(Z < 1) - \mathbb{P}(Z \leq 1) \\ &= \Phi(1) - \Phi(-1) \approx 0.8413 - 0.1586 = 0.683.\end{aligned}$$

(c)

$$\mathbb{P}(X < 8) = \mathbb{P}\left(Z < \frac{8-10}{6}\right) = \mathbb{P}(Z < -1/3) = \Phi(-1/3) \approx 0.37.$$

(d)

$$\mathbb{P}(X < 20) = \mathbb{P}\left(Z < \frac{20-10}{6}\right) = \mathbb{P}(Z < 5/3) = \Phi(5/3) \approx 0.953.$$

(e)

$$\mathbb{P}(X > 16) = \mathbb{P}\left(Z > \frac{16-10}{6}\right) = 1 - \mathbb{P}(Z < 1) = 1 - \Phi(1) \approx 0.159.$$

□

Problem 16. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that starting with this year, it will take more than 10 years before a year occurs having a rainfall of more than 50 inches? What assumptions are you making?

Solution. Let X_i denote the annual rainfall in year i from now, $i \in \mathbb{N}$. Then $X_i \sim \mathcal{N}(\mu, \sigma^2)$ and the desired probability is

$$\begin{aligned}\mathbb{P}(X_1 \leq 50, ..., X_{10} \leq 50) &= \prod_{i=1}^{10} \mathbb{P}(X_i \leq 50) = \prod_{i=1}^{10} \mathbb{P}\left(Z \leq \frac{50-40}{4}\right) \\ &= (\Phi(2.5))^{10} \approx 0.94.\end{aligned}$$

For the solution above, we assume that annual rainfall in different years are pairwise independent. □

Problem 17. The salaries of physicians in a certain speciality are approximately normally distributed. If 25% of these physicians earn less than \$180,000 and 25% earn more than \$320,000, approximately what fraction earn

(a) less than \$200,000?

(b) between \$280,000 and \$320,000?

Solution. Let X be the salary of a physician, then $X \sim \mathcal{N}(\mu, \sigma^2)$ for some μ and σ. Note that

$$0.25 = \mathbb{P}(X < 180000) = \mathbb{P}\left(Z < \frac{180000-\mu}{\sigma}\right) = \Phi\left(\frac{180000-\mu}{\sigma}\right)$$

and

$$0.75 = \mathbb{P}(X \le 320000) = \mathbb{P}\left(Z \le \frac{320000-\mu}{\sigma}\right) = \Phi\left(\frac{320000-\mu}{\sigma}\right),$$

it implies that

$$\begin{cases}(180000-\mu)/\sigma = \Phi^{-1}(0.25) \approx -0.674 \\ (320000-\mu)/\sigma = \Phi^{-1}(0.75) \approx 0.674\end{cases} \Rightarrow \begin{cases}\mu = 250000 \\ \sigma \approx 103857.57\end{cases}.$$

(a) The desired probability is

$$\mathbb{P}(X < 200000) = \mathbb{P}\left(Z < \frac{200000-\mu}{\sigma}\right) \approx \mathbb{P}(Z < -0.481) = \Phi(-0.481) \approx 0.315.$$

(b) The desired probability is

$$\begin{aligned}\mathbb{P}(280000 \le X \le 320000) &= \mathbb{P}\left(\frac{280000-\mu}{\sigma} \le Z \le \frac{320000-\mu}{\sigma}\right) \\ &= \mathbb{P}\left(\frac{280000-250000}{103857.57} \le Z \le \frac{320000-250000}{103857.57}\right) \\ &= \mathbb{P}(0.289 \le Z \le 0.674) \\ &\approx \mathbb{P}(Z \le 0.674) - \mathbb{P}(Z < 0.289) \\ &= \Phi(0.674) - \Phi(0.289) \approx 0.136.\end{aligned}$$

□

Problem 18. If X is a normal random variable with mean 5 and $\mathbb{P}(X > 9) = 0.2$, determine $\mathrm{Var}(X)$.

Solution. Let $\mathrm{Var}(X) = \sigma^2$, then

$$0.2 = \mathbb{P}(X > 9) = \mathbb{P}\left(Z > \frac{9-5}{\sigma}\right) = 1 - \mathbb{P}(Z < 4/\sigma) = 1 - \Phi(4/\sigma)$$

implying $4/\sigma = \Phi^{-1}(0.8) \approx 0.84$ and $\mathrm{Var}(X) \approx 22.676$. □

Problem 19. Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $\mathbb{P}(X > c) = 0.1$.

Solution. Note that

$$0.1 = \mathbb{P}(X > c) = \mathbb{P}\left(Z > \frac{c-12}{2}\right) = 1 - \mathbb{P}(Z \le c/2 - 6) = 1 - \Phi(c/2 - 6),$$

implying $c/2 - 6 = \Phi^{-1}(0.9) \approx 1.28$ and $c = 14.56$. □

Problem 20. If 65% of the population of a large community is in favor of a proposed rise in school taxes, approximate the probability that a random sample of 100 people will contain (a) at least 50 who are in favor of the proposition, (b) between 60 and 70 inclusive who are in favor, and (c) fewer than 75 in favor.

Solution. Denote X the number of people who are in favor of a proposed rise in school taxes, then $X \sim \mathrm{Bino}(100, 0.65)$.

(a) The normal approximation of the desired probability is

$$\mathbb{P}(X \geq 50) \approx \mathbb{P}\left(Z \geq \frac{49.5 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right) \approx 1 - \mathbb{P}(Z < -3.25) = 1 - \Phi(-3.25) \approx 1.$$

(b) The normal approximation of the desired probability is

$$\begin{aligned}\mathbb{P}(60 \leq X \leq 70) &\approx \mathbb{P}\left(\frac{59.5 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}} \leq Z \leq \frac{70.5 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right) \\ &\approx \mathbb{P}(Z \leq 1.15) - \mathbb{P}(Z < -1.15) = \Phi(1.15) - \Phi(-1.15) \approx 0.75.\end{aligned}$$

(c) The normal approximation of the desired probability is

$$\mathbb{P}(X < 75) \approx \mathbb{P}\left(Z \leq \frac{74.5 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right) \approx \mathbb{P}(Z \leq 1.99) = \Phi(1.99) \approx 0.977.$$

□

Problem 23. 1000 independent rolls of a fair die will be made. Compute an approximation to the probability that the number 6 will appear between 150 and 200 times inclusively. If the number 6 appears exactly 200 times, find the probability that the number 5 will appear less than 150 times.

Solution. Let X be the number of times the number 6 appears, then $X \sim \text{Bino}(1000, 1/6)$. The probability that the number 6 will appear between 150 and 200 times is normally approximated by

$$\begin{aligned}\mathbb{P}(150 \leq X \leq 200) &\approx \mathbb{P}\left(\frac{149.5 - 1000 \cdot 1/6}{\sqrt{1000 \cdot 1/6 \cdot 5/6}} \leq Z \leq \frac{200.5 - 1000 \cdot 1/6}{\sqrt{1000 \cdot 1/6 \cdot 5/6}}\right) \\ &\approx \mathbb{P}(Z \leq 2.87) - \mathbb{P}(Z < -1.46) = \Phi(2.87) - \Phi(-1.46) \approx 0.926.\end{aligned}$$

Given that the number 6 appears exactly 200 times, the number 5 appears less than 150 times if it does so in the remaining 800 trials. Let Y be the number of times the number 5 appears in the remaining 800 trials, then $Y \sim \text{Bino}(800, 1/5)$. The probability that the number 5 will appear less than 150 times is normally approximated by

$$\mathbb{P}(Y < 150) \approx \mathbb{P}\left(Z \leq \frac{149.5 - 800 \cdot 1/5}{\sqrt{800 \cdot 1/5 \cdot 4/5}}\right) \approx \mathbb{P}(Z \leq -0.93) = \Phi(-0.93) \approx 0.176.$$

□

Problem 24. The lifetimes of interactive computer chips produced by a certain semiconductor manufacturer are normally distributed with parameters $\mu = 1.4 \cdot 10^6$ hours and $\sigma = 3 \cdot 10^5$ hours. What is the approximate probability that a batch of 100 chips will contain at least 20 whose lifetimes are less than $1.8 \cdot 10^6$?

Solution. Let X be the lifetime of an interactive computer chip, then $X \sim \mathcal{N}(\mu, \sigma^2)$ and

$$\mathbb{P}(X < 1.8 \cdot 10^6) = \mathbb{P}\left(Z < \frac{1.8 \cdot 10^6 - 1.4 \cdot 10^6}{3 \cdot 10^5}\right) \approx \mathbb{P}(Z < 1.33) = \Phi(1.33) \approx 0.908.$$

Thus, if N is the number of the chips among the batch whose life is less than $1.8 \cdot 10^6$, then $N \sim \text{Bino}(100, 0.908)$. Hence the normal approximation of the desired probability is

$$\mathbb{P}(N \geq 20) \approx \mathbb{P}\left(Z \geq \frac{19.5 - 100 \cdot 0.908}{\sqrt{100 \cdot 0.908 \cdot 0.092}}\right) \approx 1 - \mathbb{P}(Z < -24.87) = 1 - \Phi(-24.87) \approx 1.$$

□

Problem 25. Each item produced by a certain manufacturer is, independently, of acceptable quality with probability 0.95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

Solution. Let X denote the number of unacceptable items among the next 150 produced, then $X \sim \text{Bino}(150, 0.05)$. Hence the normal approximation of the desired probability is

$$\mathbb{P}(X \leq 10) \approx \mathbb{P}\left(Z \leq \frac{10.5 - 150 \cdot 0.05}{\sqrt{150 \cdot 0.05 \cdot 0.95}}\right) \approx \mathbb{P}(Z \leq 1.12) = \Phi(1.12) \approx 0.869.$$

□

Problem 27. In 10,000 independent tosses of a coin, the coin landed on heads 5,800 times. Is it reasonable to assume that the coin is not fair? Explain.

Solution. Assume that the given coin is fair. Denote X the number of heads in the total of 10,000 independent tosses of a coin, then $X \sim \text{Bino}(10000, 0.5)$ and we have the normal approximation

$$\mathbb{P}(X \geq 5800) \approx \mathbb{P}\left(Z \geq \frac{5799.5 - 10000 \cdot 0.5}{\sqrt{10000 \cdot 0.5 \cdot 0.5}}\right) \approx 1 - \mathbb{P}(Z < 15.99) = 1 - \Phi(15.99) \approx 0.$$

Hence, it is reasonable to assume that the coin is not fair. □

Problem 28. 12% of the population is left-handed. Approximate the probability that there are at least 20 left-handers in a school of 200 students. State your assumptions.

Solution. Denote X the number of left-handers, then $X \sim \text{Bino}(200, 0.12)$. Hence the normal approximation of the desired probability is

$$\mathbb{P}(X \geq 20) \approx \mathbb{P}\left(Z \geq \frac{19.5 - 200 \cdot 0.12}{\sqrt{200 \cdot 0.12 \cdot 0.88}}\right) \approx 1 - \mathbb{P}(Z < -0.98) = 1 - \Phi(-0.98) \approx 0.837.$$

For the solution above, we assume that left-handed abilities of different people are pairwise independent. □

Problem 29. A model for the movement of a stock supposes that if the present price of the stock is s, then after one period, it will be either us with probability p or ds with probability $1-p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30% after the next 1000 periods if $u = 1.012, d = 0.99$ and $p = 0.52$.

Solution. Let X be the number of increments among the next 1000 periods, then $X \sim \text{Bino}(1000, 0.52)$ and the stock price afterwards is

$$S = s \cdot u^X \cdot d^{1000-X} = sd^{1000} \cdot \left(\frac{u}{d}\right)^X \approx \frac{1.022^X \cdot s}{23163.57}$$

and hence $S \geq 130\% \cdot s \Leftrightarrow X \geq \log_{1.022}(1.3 \cdot 23163.57) \approx 473.9$. Thus the normal approximation of the desired probability is

$$\mathbb{P}(X \geq 474) \approx \mathbb{P}\left(Z \geq \frac{473.5 - 1000 \cdot 0.52}{\sqrt{1000 \cdot 0.52 \cdot 0.48}}\right) \approx 1 - \mathbb{P}(Z < -2.943) = 1 - \Phi(-2.943) \approx 0.998.$$

□

Problem 32. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is:

(a) What is the probability that a repair time exceeds 2 hours?

(b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

Solution.

(a) Let X be the time required to repair a machine, then $X \sim \text{Exp}(\lambda)$ and the desired probability is

$$\mathbb{P}(X > 2) = 1 - \mathbb{P}(X \leq 2) = 1 - (1 - e^{-1}) = e^{-1}.$$

(b) By the memoryless property, the desired conditional probability is

$$\mathbb{P}(X \geq 10 | X > 9) = \mathbb{P}(X > 10 - 9) = \mathbb{P}(X > 1) = 1 - \mathbb{P}(X \leq 1) = 1 - (1 - e^{-1/2}) = e^{-1/2}.$$

$\square$

Problem 35. If X is an exponential random variable with parameter λ and $c > 0$, find the density function of cX. What kind of random variable is cX?

Solution. Note that if $x \geq 0$, then

$$F_{cX}(x) = \mathbb{P}(cX \leq x) = \mathbb{P}\left(X \leq \frac{x}{c}\right) = 1 - e^{-\lambda x/c} = 1 - e^{-(\lambda/c)x}$$

and thus $cX \sim \text{Exp}(\lambda/c)$. The density of cX is given by

$$f(x) = \begin{cases} \frac{\lambda}{c} e^{-\lambda x/c} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

$\square$

2 Theoretical Exercises

Exercise 1. The speed of a molecule in a uniform gas at equilibrium is a random variable whose probability density function is given by

$$f(x) = \begin{cases} ax^2 e^{-bx^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $b = m/2kT$ and k, T, m denote, respectively, Boltzmann's constant, the absolute temperature of the gas, and the mass of the molecule. Evaluate a in terms of b.

Solution. Let $u = -x/2b$ and $v = e^{-bx^2}$, then $du = -1/2b$ and $dv = -2bxe^{-bx^2}$. Applying integration by parts method,

$$\int_0^\infty x^2 e^{-bx^2} dx = \frac{-xe^{-bx^2}}{2b} \int_0^\infty + \frac{1}{2b} \int_0^\infty e^{-bx^2} dx$$

$$= \frac{1}{(2b)^{3/2}} \int_0^\infty e^{-y^2/2} dy (with y = x\sqrt{2b})$$

$$= \frac{\sqrt{2\pi}}{2} \frac{1}{(2b)^{3/2}} = \frac{\sqrt{\pi}}{4b^{3/2}}$$

where the above uses that

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-y^2/2} dy = 1/2$$

Hence, $a = \frac{4b^{3/2}}{\sqrt{\pi}}$ $\square$

Exercise 2. Show that

$$\mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y > y) dy - \int_0^\infty \mathbb{P}(Y < -y) dy.$$

Solution.

$$\int_0^\infty \mathbb{P}(Y < -y)dy = \int_0^\infty \int_{-\infty}^{-y} f_Y(x)dxdy = -\int_{-\infty}^0 xf_Y(x)dx.$$

Similarly,

$$\int_0^\infty \mathbb{P}(Y > y)dy = \int_0^\infty xf_Y(x)dx.$$

Subtracting these equalities give the desired result. □

Exercise 4. Prove Corollary 2.1.

Solution.

$$\mathbb{E}[aX + b] = \int (ax + b)f(x)dx = a \int xf(x)dx + b \int f(x)dx = a \cdot \mathbb{E}[X] + b.$$

□

Exercise 7. The standard deviation of X, denoted $\mathrm{SD}(X)$, is given by

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}.$$

Find $\mathrm{SD}(aX + b)$ if $\mathrm{Var}(X) = \sigma^2$.

Solution. $\mathrm{SD}(aX + b) = \sqrt{\mathrm{Var}(aX + b)} = \sqrt{a^2\sigma^2} = |a|\sigma.$ □