FINAL EXAMINATION
January 2020
Duration: 120 minutes

SUBJECT: REAL ANALYSIS	
Head of Dept. of Mathematics:	Lecturer:
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INSTRUCTIONS: *All documents and electronic devices are forbidden.*

Question 1

(a) *(10 marks)* State (without proof) the dominated convergence theorem.

(b) *(15 marks)* Suppose $\mathcal{F}$ is a family of subsets of X with $\emptyset \in \mathcal{F}$ and $\mu : \mathcal{F} \to [0, \infty]$ satisfies $\mu(\emptyset) = 0$. For any $A \subset X$, define

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : A_n \in \mathcal{F} \text{ and } A \subset \bigcup_{n=1}^{\infty} A_n \right\}.$$

If there is no sequence $\{A_n\}$ of $\mathcal{F}$ such that $A \subset \bigcup_{n=1}^{\infty} A_n$, then we let $\mu^*(A) = \infty$. Prove that

(i) μ^* is an outer measure, and

(ii) $\mu^*(A) \leq \mu(A)$ for all $A \in \mathcal{F}$.

Question 2

(a) *(10 marks)* Prove that if μ^* is an outer measure on X and if $B \subset X$, $\mu^*(B) = 0$, then $\mu^*(A \cup B) = \mu^*(A \setminus B) = \mu^*(A)$ for all $A \subset X$.

(b) *(15 marks)* Let $(X, \mathcal{M})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$ a measurable function. Let c be a positive real number. Show that the set $A = \{x \in X : |f(x)| \leq c\}$ is measurable and the function $g : X \to \overline{\mathbb{R}}$ defined by $g(x) = f(x)$ if $|f(x)| \leq c$ and $g(x) = 0$ if $|f(x)| > c$, is measurable.

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Question 3 Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that there exists an integrable function f on X which satisfies the condition: $f(x) > 0$ for all $x \in X$.

(a) *(15 marks)* Show that for each $n \in \mathbb{N}$, the set

$$A_n = \left\{ x \in X : f(x) > \frac{1}{n} \right\}$$

has finite measure.

(b) *(10 marks)* Show that the measure μ is σ-finite.

Question 4

(a) *(15 marks)* Given a signed measure ν on a measurable space $(X, \mathcal{M})$. Show that a measurable set A is null for ν if and only if $|\nu|(A) = 0$.

(b) *(10 marks)* Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. If $B \in \mathcal{F}$ with $\mathbf{P}(B) > 0$ is given, then the set function $\mathbf{Q} : \mathcal{F} \to \mathbb{R}$ defined by

$$\mathbf{Q}(A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}, \qquad A \in \mathcal{F},$$

is a probability measure on $\mathcal{F}$. Show that $\mathbf{Q} \ll \mathbf{P}$ and find the Radon-Nikodym derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}$.

----------END OF QUESTION PAPER----------

SOLUTIONS

Question 1 (b) (i) It is clear that $\mu^*(A) \geq 0$ for all subset A of X and $\mu^*(\emptyset) = 0$ (take $E_n = \emptyset$ for all n). Suppose $A, A_n \in \mathcal{P}(X)$ with $A \subset \bigcup_n A_n$. We can assume that $\sum_{n=1}^{\infty} \mu^*(A_n) < \infty$ since otherwise, $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n) = \infty$. For every $\epsilon > 0$ and every $n \in \mathbb{N}$, since $\mu^*(A_n) < \infty$, there is a sequence $\{E_{n,k}\}_{k \in \mathbb{N}}$ in $\mathcal{E}$ with

$$A_n \subset \bigcup_{k=1}^{\infty} E_{n,k} \quad \text{and} \quad \sum_{k=1}^{\infty} \mu(E_{n,k}) \leq \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then $A \subset \bigcup_{n=1}^{\infty} (\bigcup_{k=1}^{\infty} E_{n,k})$ and we get

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu^*\big(E_{n,k}\big) \leq \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) = \epsilon + \sum_{n=1}^{\infty} \mu^*(A_n).$$

Because $\epsilon > 0$ is arbitrary, $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ and σ-subadditivity follows. Thus μ^* is an outer measure.

(ii) If $A \in \mathcal{E}$, choose $A_1 = A$ and $A_n = \emptyset$ for $n \geq 2$. Then $\{A_n\} \subset \mathcal{E}$ and $A = \bigcup_{n=1}^{\infty} A_n$. Thus $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu(A_n) = \mu(A)$.

Question 2 (a) We express $A \cup B$ as union of disjoint sets $A \cup B = (A \setminus B) \cup B$ and apply monotonicity and subadditivity to obtain

$$\mu^*(A) \leq \mu^*(A \cup B) \leq \mu^*(A \setminus B) + \mu^*(B) = \mu^*(A \setminus B) \leq \mu^*(A).$$

Thus all inequalities are equalities, that is, $\mu^*(A) = \mu^*(A \setminus B) = \mu^*(A \cup B)$.

(b) Since f is measurable, so is $|f|$. Thus $A = \{x \in X : |f(x)| \leq c\}$ is measurable and so is χ_A. Consider the function $f\chi_A$. If $x \in A$, $f\chi_A(x) = f(x) = g(x)$. If $x \notin A$, $f\chi_A(x) = 0 = g(x)$. Thus $g = f\chi_A$, a measurable function.

Question 3 (a) Since $A \subset \big\{f \geq 1/n\big\}$, we apply monotonicity of μ and Chebychev's inequality to obtain

$$\mu(A_n) \leq \mu(\{f \geq 1/n\}) \leq n \int_X f d\mu < \infty \quad \text{for all } n.$$

(b) For each $x \in X$, $f(x) > 0$, so there is $n \in \mathbb{N}$ such that $f(x) > 1/n$. It follows that $X = \bigcup_{n=1}^{\infty} A_n$. By part (a), each set A_n has finite measure. Thus μ is σ-finite.

Question 4 (a) Suppose that A is null for ν. Let $\{P, N\}$ be a Hahn decomposition for ν. As $A \cap P$ and $A \cap N$ are measurable subsets of A and A is null for ν, we have $\nu(A \cap P) = \nu(A \cap N) = 0$. Hence,

$$|\nu|(A) = \nu^+(A) + \nu^-(A) = \nu(A \cap P) - \nu(A \cap N) = 0.$$

Conversely, assume $|\nu|(A) = 0$. Then for any measurable subset B of A,

$$|\nu(B)| = \left|\nu^+(B) - \nu^-(B)\right| \leq \nu^+(B) + \nu^-(B) = |\nu|(B) \leq |\nu|(A) = 0.$$

The last inequality holds since $|\nu|$ is a measure. Thus $\nu(B) = 0$, and hence, A is null for ν.

(b) If $\mathbf{P}(E) = 0$, then $\mathbf{P}(E \cap B) = 0$, so $\mathbf{Q}(E) = 0$. This shows that $\mathbf{Q} \ll \mathbf{P}$. Furthermore,

$$\mathbf{Q}(A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{1}{\mathbf{P}(B)} \int_A \chi_B d\mathbf{P} = \int_A \frac{1}{\mathbf{P}(B)} \cdot \chi_B d\mathbf{P} \quad \text{for all } A \in \mathcal{F}.$$

Therefore $d\mathbf{Q}/d\mathbf{P} = \frac{1}{\mathbf{P}(B)} \cdot \chi_B$.