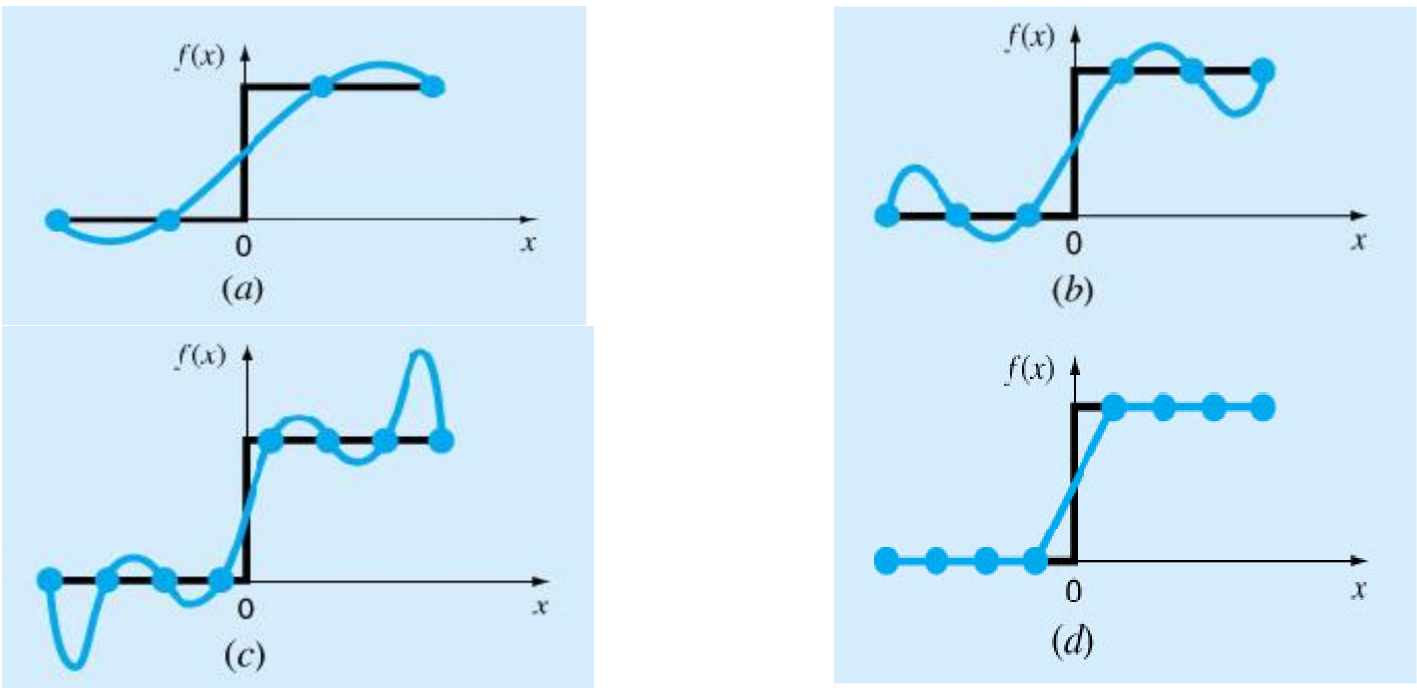
Chapter 3:
Curve Fitting & Interpolation

Lecture 3:
Spline Interpolation

Why Spline Interpolation?



Apply lower-order polynomials to subsets of data points. Spline provides a superior approximation of the behavior of functions that have local, abrupt changes.

Why Splines ?

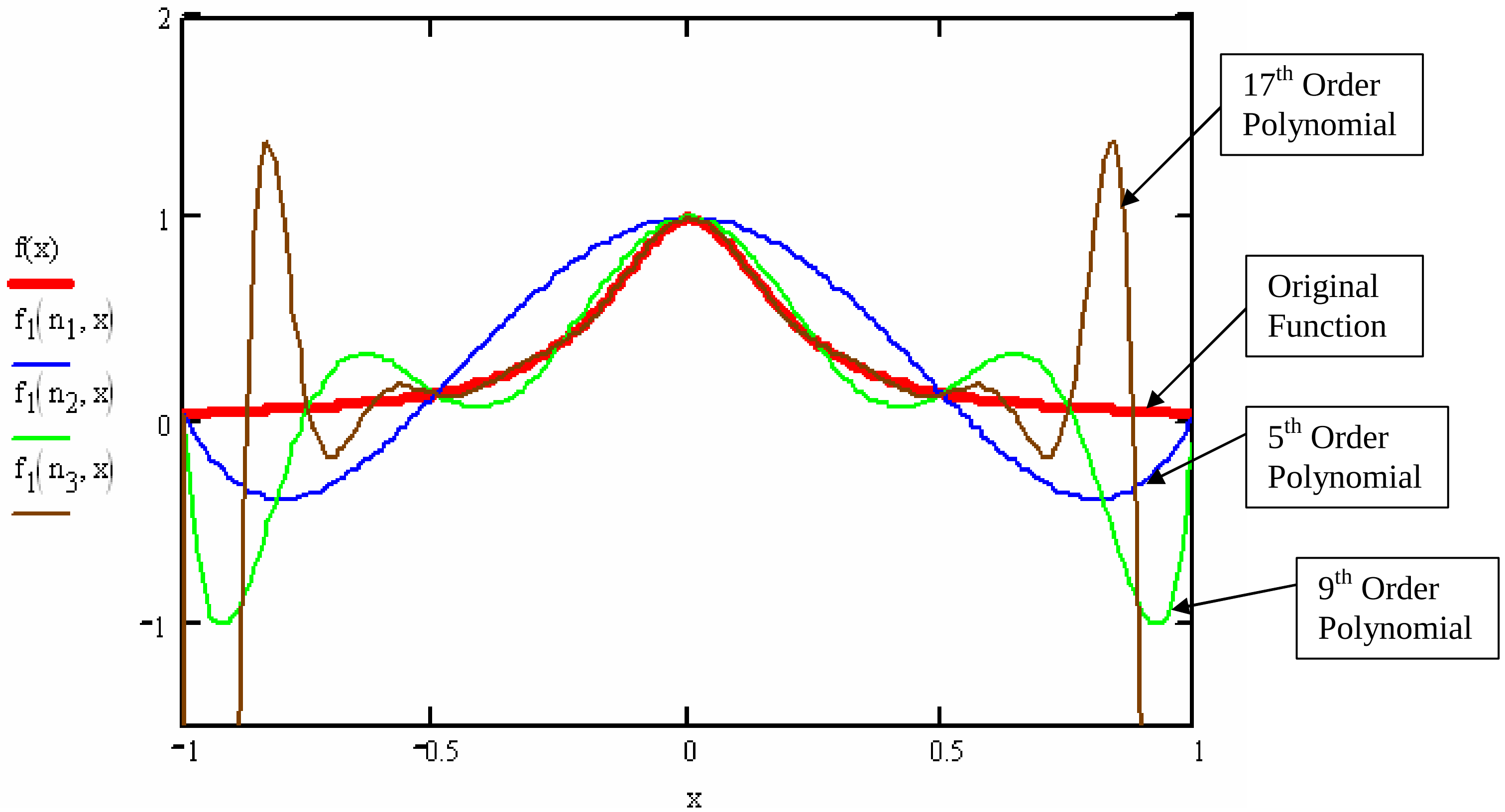
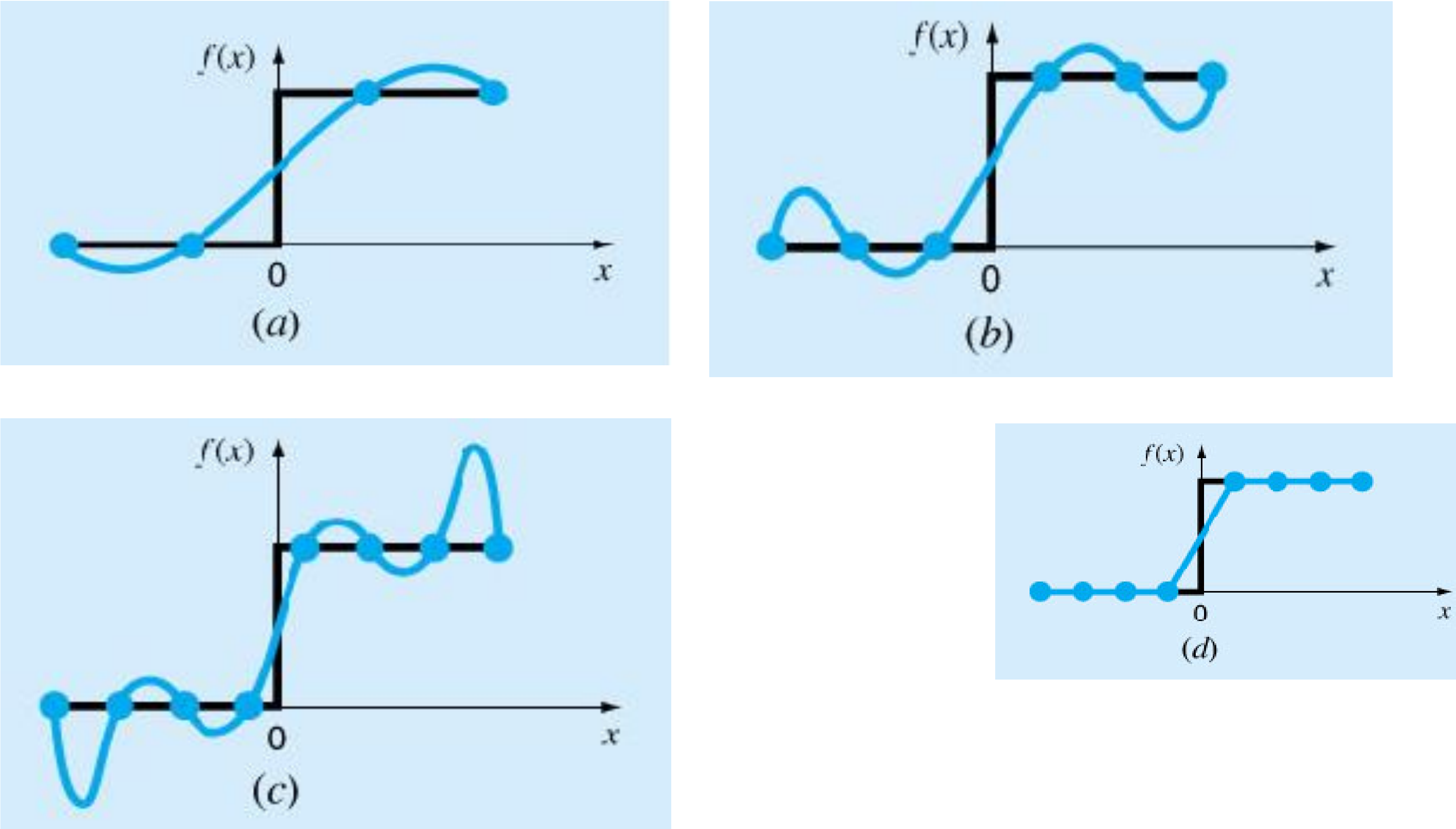


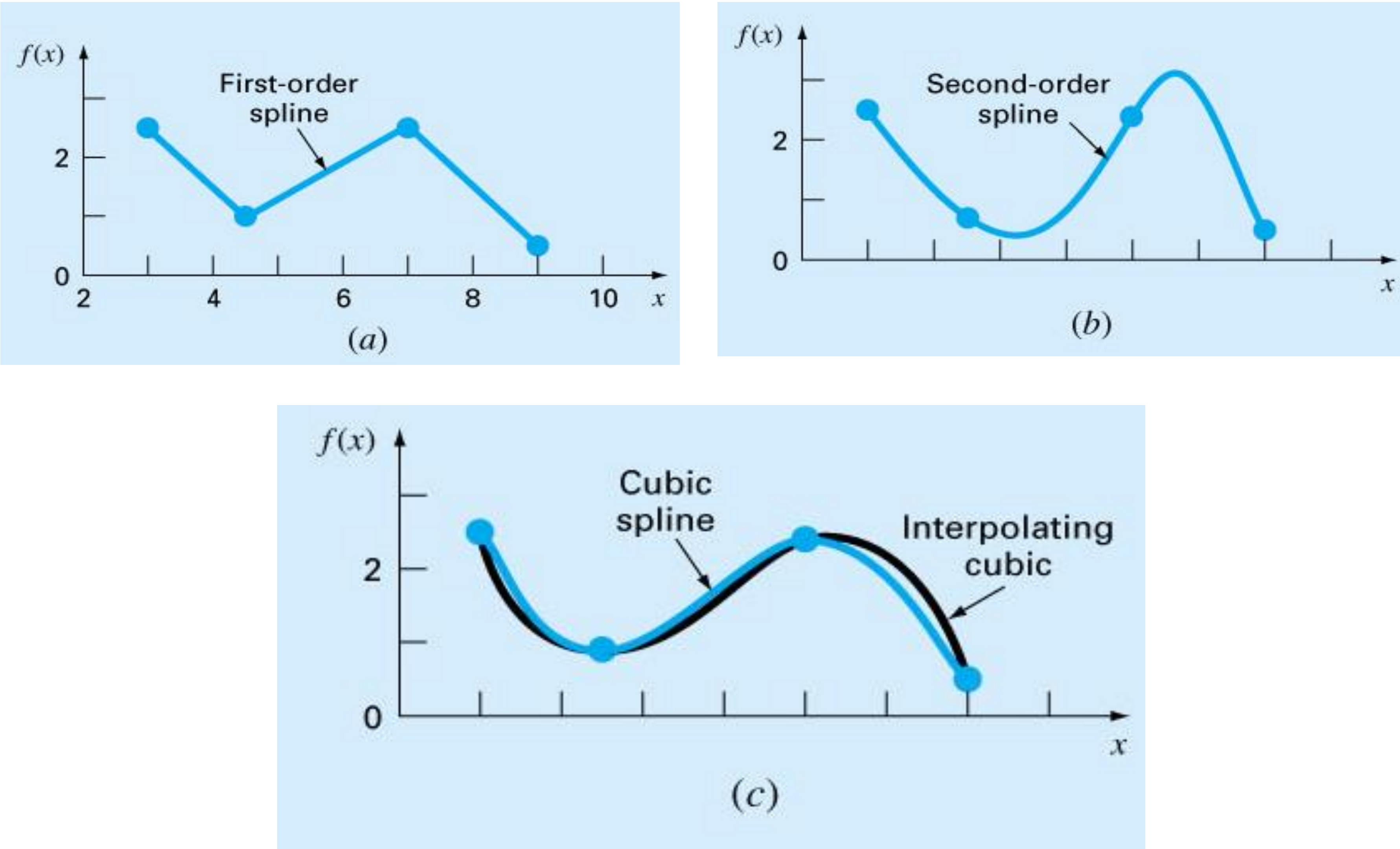
Figure : Higher order polynomial interpolation is a bad idea

Spline Interpolation

- Polynomials are the most common choice of interpolants.
- There are cases where polynomials can lead to erroneous results because of round off error and overshoot.
- Alternative approach is to apply **lower-order polynomials** to subsets of data points. Such connecting polynomials are called **spline functions.**



Spline provides a superior approximation of the behavior of functions that have local, abrupt changes (d).


f(x)
First-order
spline
2
0
2
4
6
8
10
x
(a)
f(x)
Second-order
spline
2
0
x
(b)
f(x)
Cubic
spline
Interpolating
cubic
2
0
x
(c)

Linear Spline

The first order splines for a group of ordered data points can be defined as a set of linear functions:

$$f(x) = f(x_0) + m_0(x - x_0) \qquad x_0 \le x \le x_1$$

$$f(x) = f(x_1) + m_1(x - x_1) \qquad x_1 \le x \le x_2$$

$$\vdots$$

$$f(x) = f(x_{n-1}) + m_{n-1}(x - x_{n-1}) \qquad x_{n-1} \le x \le x_n$$

$$\boxed{m_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}}$$

Linear spline - Example

Evaluate the function at x = 5 using **first order splines**:

x	$f(x)$
3.0	2.5
4.5	**1.0**
7.0	**2.5**
9.0	0.5

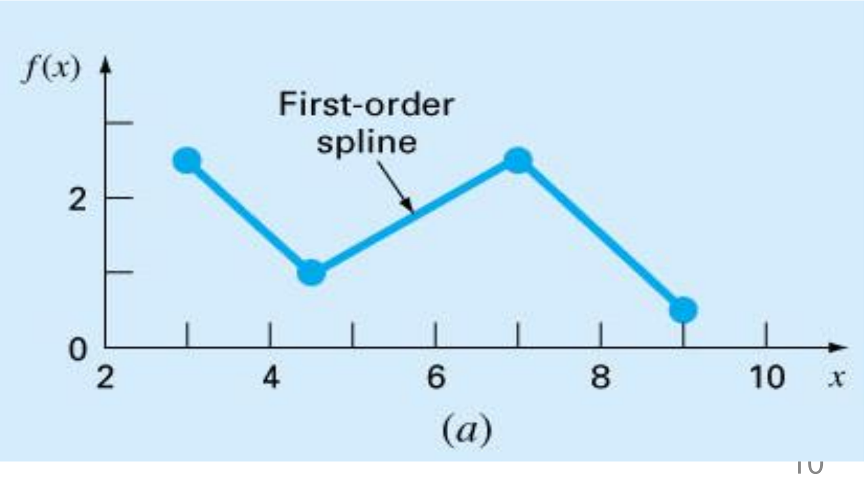
$$m_1 = \frac{2.5-1}{7-4.5} = 0.6$$

$$f(x) = f(x_1) + m_1(x - x_1)$$

$$f(5) = f(4.5) + 0.6(5-4.5)$$

$$= 1.0 + 0.6 \times 0.5$$

$$= 1.3$$



Linear Spline

- The main **disadvantage** of **linear spline** is that they are not smooth. The data points where 2 splines meets called (a knot), the changes abruptly.

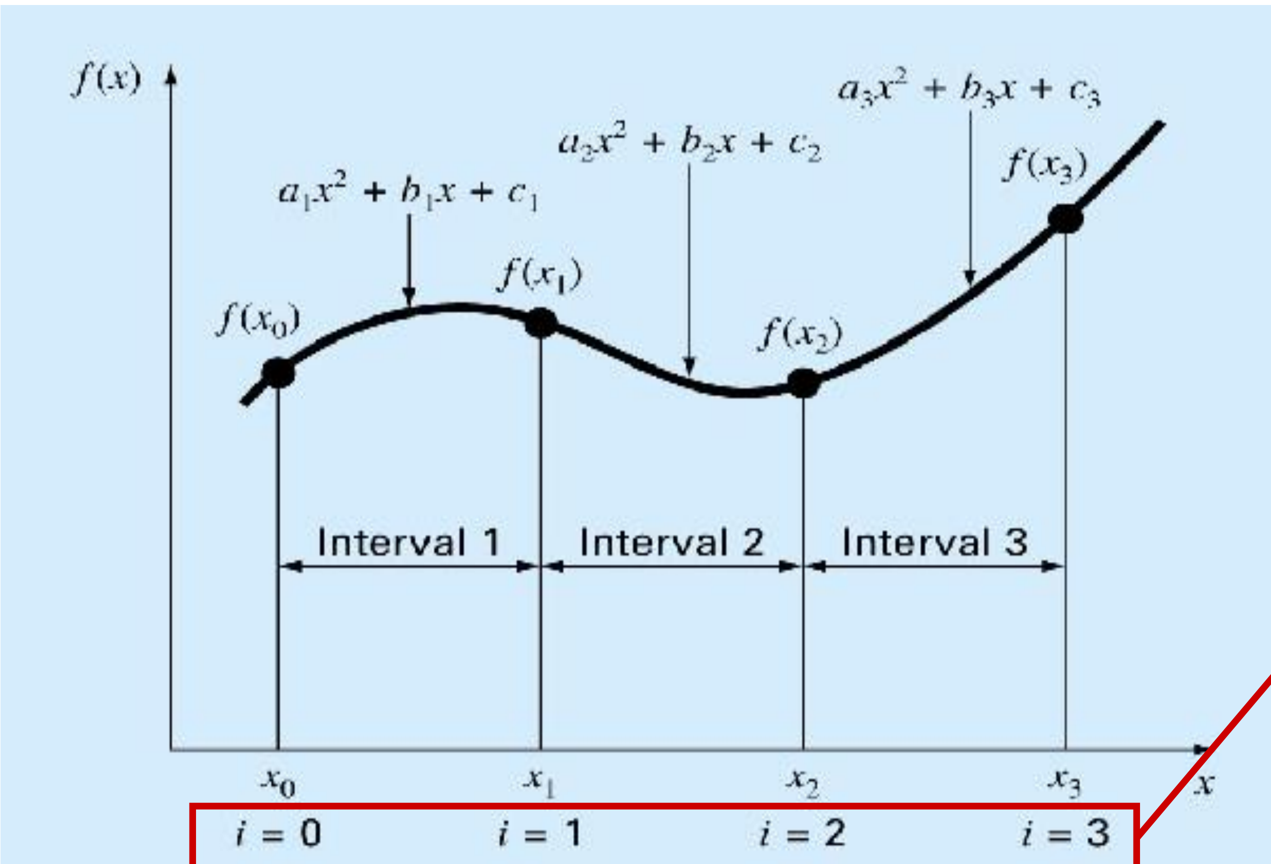
- The first derivative of the function is discontinuous at these points.

- Using **higher order polynomial splines** ensure smoothness at the knots by equating derivatives at these points.

Quadric Splines

- **Objective:** to derive a second order polynomial for each interval between data points

$$\boxed{f_i(x) = a_i x^2 + b_i x + c_i, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n}$$



For *n*+1 data points:

- *i = (0, 1, 2, …n),*
- ***n** intervals,*
- ***3n** unknown constants (a's, b's and c's)*

Quadric Splines

- The function values of adjacent polynomials must be equal at the interior knots **2(n-1)**.

$$f_i(x_i) = a_i x_i^{\,2} + b_i x_i + c_i = f(x_i)$$
$$f_{i+1}(x_i) = a_{i+1} x_i^{\,2} + b_{i+1} x_i + c_{i+1} = f(x_i) \qquad i = 1, 2, \ldots, n-1$$

- The first and last functions must pass through the end points (**2**).

$$f_1(x_0) = a_1 x_0^{\,2} + b_1 x_0 + c_1 = f(x_0)$$
$$f_n(x_n) = a_n x_n^{\,2} + b_n x_n + c_n = f(x_n)$$

Quadric Splines

- The first derivatives at the interior knots must be equal (**n-1**).

$$
\boxed{\begin{aligned}
&f_i^{'}(x) = 2a_i x + b_i \\
&f_i{}'(x_i) = f_{i+1}{}'(x_i) \Leftrightarrow 2a_i x_i + b_i = 2a_{i+1} x_i + b_{i+1}, \qquad i = 1, 2, \ldots, n-1
\end{aligned}}
$$

- Assume that the second derivate is zero at the first point (**1**)

$$a_1 = 0$$

(The first two points will be connected by a straight line)

Quadric Splines - Example

Fit the following data with **quadratic splines.** Estimate the value at x = 5.

x	3.0	4.5	7.0	9.0
$f(x)$	2.5	1.0	2.5	0.5

Solutions:

There are **3** intervals (n=3), **9** unknowns.

Example

x	3.0	4.5	7.0	9.0
$f(x)$	2.5	1.0	2.5	0.5

1. **Equal interior points:**

- **For first interior point (4.5, 1.0)**

The 1st equation:

$$f_1(x_1) = x_1^2 a_1 + x_1 b_1 + c_1 = f(x_1)$$

$$(4.5)^2 a_1 + 4.5 b_1 + c_1 = f(4.5) \longrightarrow \boxed{20.25 a_1 + 4.5 b_1 + c_1 = 1.0}$$

The 2nd equation:

$$f_2(x_1) = x_1^2 a_2 + x_1 b_2 + c_2 = f(x_1)$$

$$(4.5)^2 a_2 + 4.5 b_2 + c_2 = f(4.5) \longrightarrow \boxed{20.25 a_2 + 4.5 b_2 + c_2 = 1.0}$$

Example

x	3.0	4.5	7.0	9.0
$f(x)$	2.5	1.0	2.5	0.5

- **For second interior point (7.0, 2.5)**

The 3rd equation:

$$x_2^2 a_2 + x_2 b_2 + c_2 = f(x_2)$$

$$(7)^2 a_2 + 7b_2 + c_2 = f(7) \quad \longrightarrow \quad \boxed{49a_2 + 7b_2 + c_2 = 2.5}$$

The 4th equation:

$$x_2^2 a_3 + x_2 b_3 + c_3 = f(x_2)$$

$$(7)^2 a_3 + 7b_3 + c_3 = f(7) \quad \longrightarrow \quad \boxed{49a_3 + 7b_3 + c_3 = 2.5}$$

Example

x	3.0	4.5	7.0	9.0
$f(x)$	2.5	1.0	2.5	0.5

- First and last functions pass the end points

For the start point **(3.0, 2.5)**

$$x_0^2 a_1 + x_0 b_1 + c_1 = f(x_0) \rightarrow \boxed{9a_1 + 3b_1 + c_1 = 2.5}$$

For the end point **(9, 0.5)**

$$x_3^2 a_3 + x_3 b_3 + c_3 = f(x_3) \rightarrow \boxed{81a_3 + 9b_3 + c_3 = 0.5}$$

Example

x	3.0	4.5	7.0	9.0
$f(x)$	2.5	1.0	2.5	0.5

- Equal derivatives at the interior knots.

For first interior point **(4.5, 1.0)**

$$f_1'(x_1) = f_2'(x_1)$$

$$2x_1 a_1 + b_1 = 2x_1 a_2 + b_2 \longrightarrow \boxed{9a_1 + b_1 = 9a_2 + b_2}$$

For second interior point **(7.0, 2.5)**

$$f_2'(x_2) = f_3'(x_2)$$

$$2x_2 a_2 + b_2 = 2x_2 a_3 + b_3 \longrightarrow \boxed{14a_2 + b_2 = 14a_3 + b_3}$$

- Second derivative at the first point is 0:

$$\boxed{f''(x_0) = a_1 = 0}$$

Example

System of linear equations $AX=B$

$$\begin{bmatrix} 4.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20.25 & 4.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 49 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 49 & 7 & 1 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 81 & 9 & 1 \\ 1 & 0 & -9 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & 1 & 0 & -14 & -1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2.5 \\ 2.5 \\ 2.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$$

Example

x	3.0	4.5	7.0	9.0
$f(x)$	2.5	1.0	2.5	0.5

Solving these 8 equations with 8 unknowns

$$a_1 = 0, \quad b_1 = -1, \quad c_1 = 5.5$$

$$a_2 = 0.64, \; b_2 = -6.76, \; c_2 = 18.46$$

$$a_3 = -1.6, \; b_3 = 24.6, \quad c_3 = -91.3$$

$$f_1(x) = -x + 5.5, \qquad 3.0 \le x \le 4.5$$

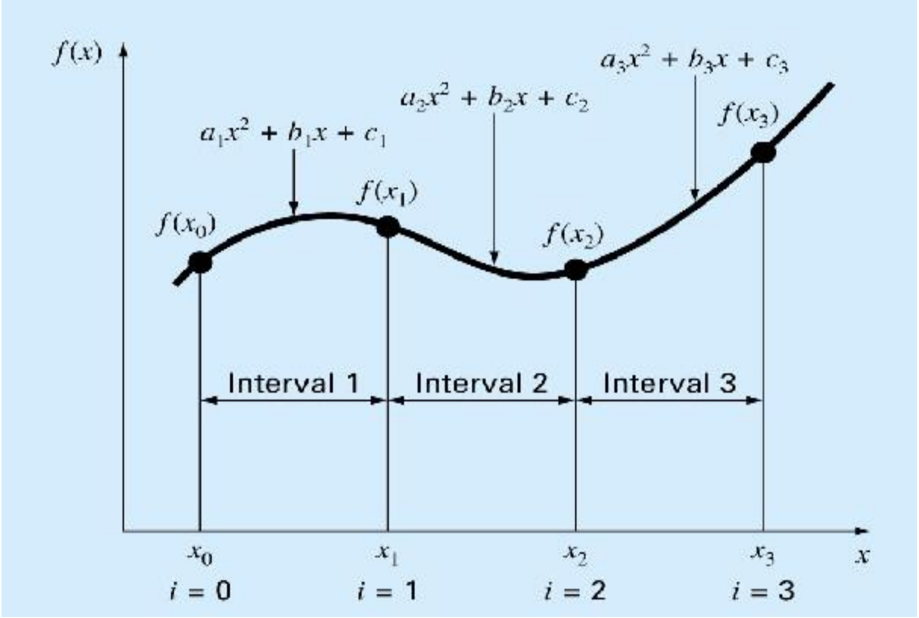
$$f_2(x) = 0.64x^2 - 6.76x + 18.46, \qquad 4.5 \le x \le 7.0$$

$$f_3(x) = -1.6x^2 + 24.6x - 91.3, \qquad 7.0 \le x \le 9.0$$

$$f(5) = f_2(5) = 0.64 \times 5^2 - 6.76 \times 5 + 18.46 = 0.66$$

Quadratic Splines: A Practical way

Consider each polynomial starting from f_1
Calculations are simpler



f_1 passes through x_0, x_1:

$$\boxed{a_1 = 0, \begin{cases} b_1 x_0 + c_1 = f(x_0) \\ b_1 x_1 + c_1 = f(x_1) \end{cases} \Rightarrow b_1, c_1}$$

f_i passes through x_{i-1}, x_i, and $f_i'(x_{i-1}) = f_{i-1}'(x_{i-1})$:

$$\boxed{\begin{cases} a_i x_{i-1}{}^2 + b_i x_{i-1} + c_i = f(x_{i-1}) \\ a_i x_i{}^2 + b_i x_i + c_i = f(x_i) \\ 2a_i x_{i-1} + b_i = 2a_{i-1} x_{i-1} + b_{i-1} \end{cases} \Rightarrow a_i, b_i, c_i, \quad i = 2, 3, \ldots, n}$$

Exercise

Fit the following data with **quadratic splines.**
Estimate the value at x = 3 and x=6.

x	**2.0**	**4**	**7.0**	**8.0**
f(x)	**0.5**	**2.0**	**4**	**2.5**

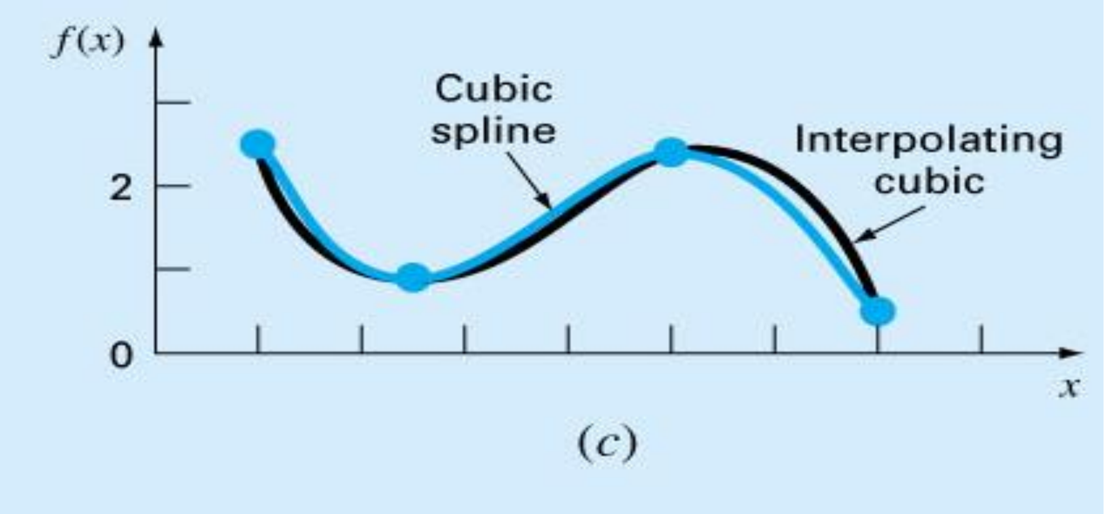
Cubic Splines

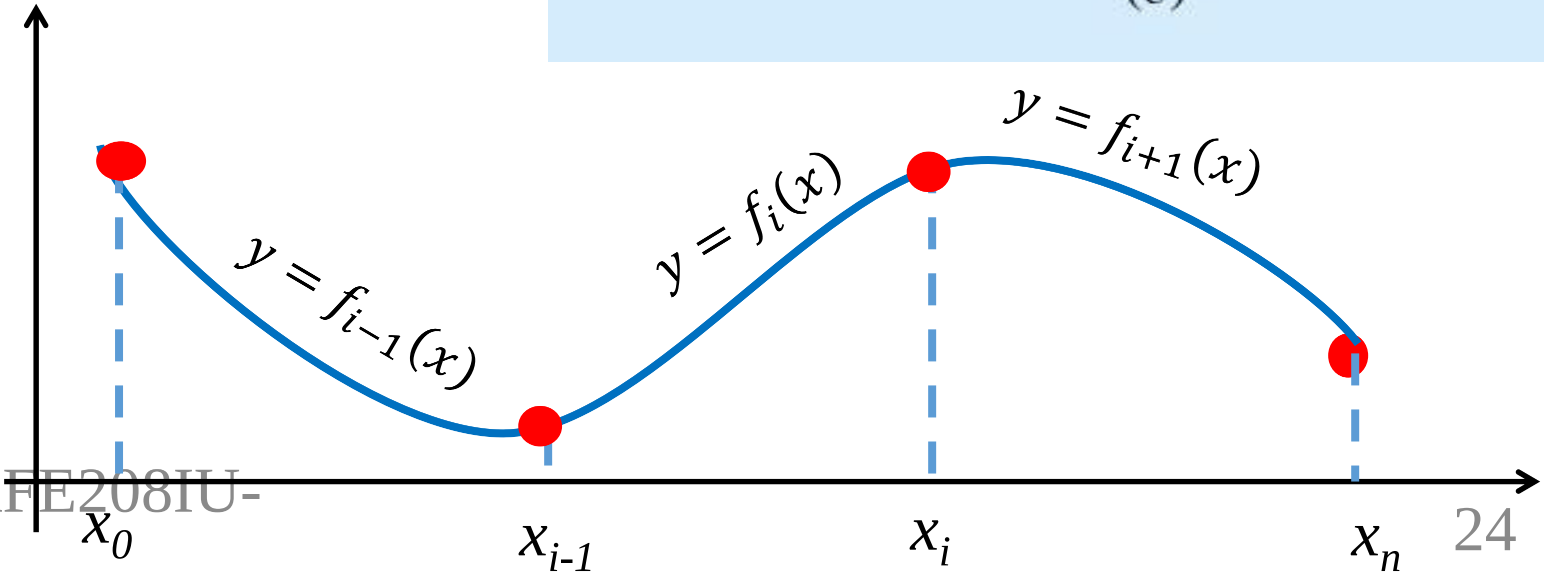
Objective: Derive a third order polynomial for each interval between data points

$$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n$$

For *n*+1 data points:

- *i = (0, 1, 2, …n),*
- ***n** intervals,*
- ***4n** unknown constants*
- *(a's, b's ,c's and d's)*





Function values coincide at the interior knots **2(n-1):**

$$f_{i+1}(x_i) = f(x_i),\ f_i(x_i) = f(x_i), \quad i = 1,2,\dots,n-1$$

First and last functions pass through the endpoints **(2):**

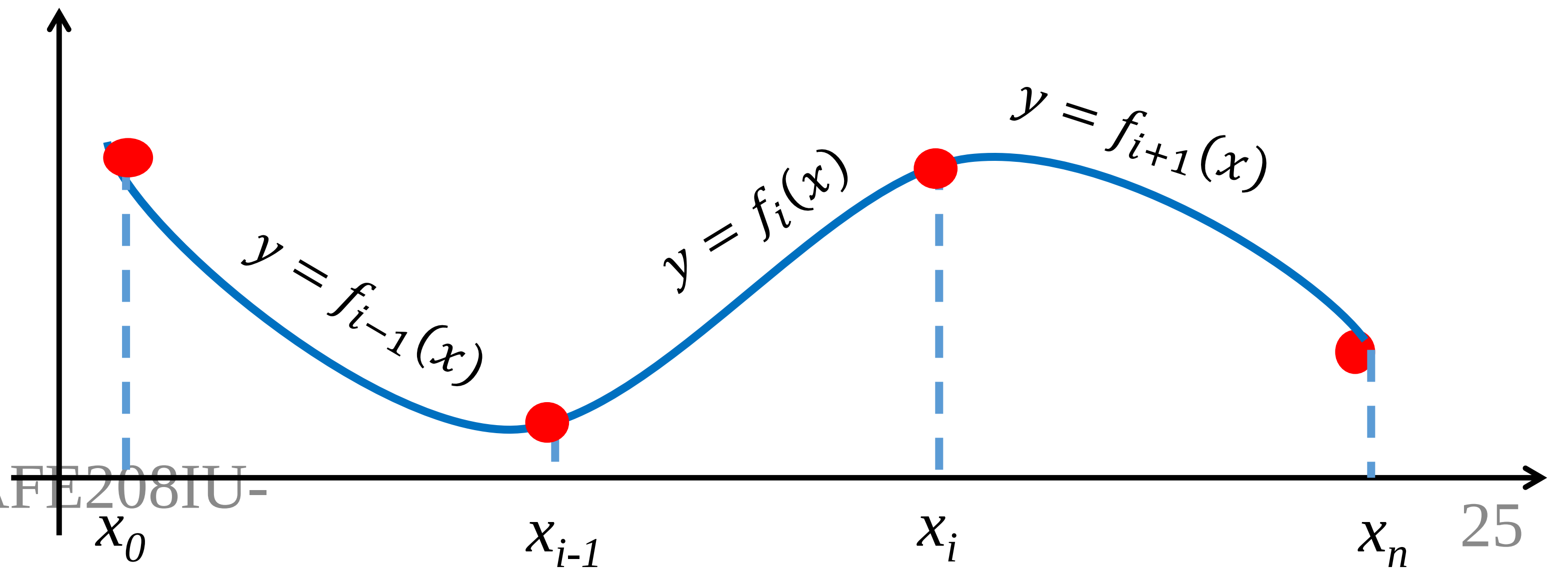
$$f_1(x_0) = f(x_0),\ f_n(x_n) = f(x_n)$$

First derivatives at the interior knots coincide **(n-1):** $f'_{i+1}(x_i) = f_i'(x_i)$

Second derivatives at the interior knots coincide **(n-1):** $f''_{i+1}(x_i) = f_i''(x_i)$

Natural splines: Second derivatives at the endpoints are zero **(2):**

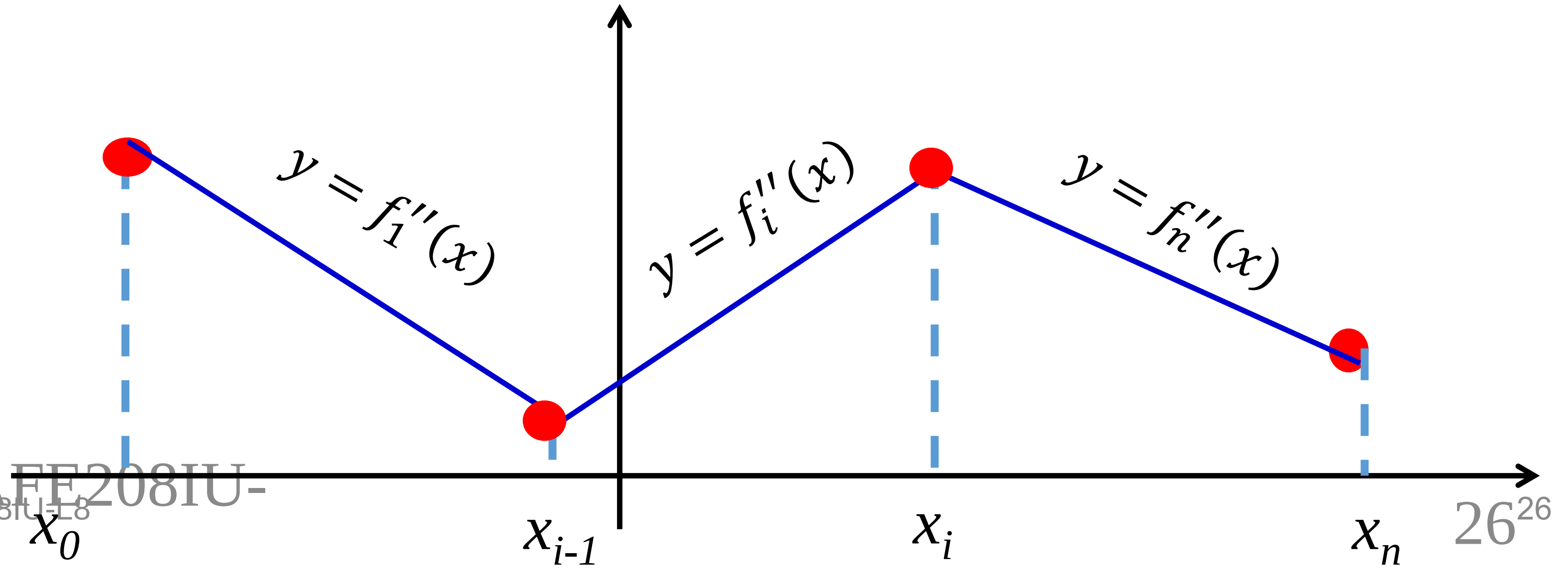
$$f_1''(x_0) = f_n''(x_n) = 0$$



Alternative technique to get Cubic Splines

The second derivative within each interval [x_{i-1}, x_i] is a **straight line,** represented by first order Lagrange interpolating polynomials:

$$\boxed{f_i''(x) = f''(x_{i-1})\frac{x - x_i}{x_{i-1} - x_i} + f''(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}}}$$



Cubic Splines

- The last equation can be **integrated twice**

  2 unknown constants of integration can be evaluated by applying the boundary conditions:

  1. $f_i(x) = f(x_{i-1})$ at x_{i-1}
  2. $f_i(x) = f(x_i)$ at x_i

$$f_i(x) = \frac{f''(x_{i-1})}{6(x_i - x_{i-1})}(x_i - x)^3 + \frac{f''(x_i)}{6(x_i - x_{i-1})}(x - x_{i-1})^3$$

$$+ \left[\frac{f(x_{i-1})}{x_i - x_{i-1}} - \frac{f''(x_{i-1})(x_i - x_{i-1})}{6}\right](x_i - x)$$

$$+ \left[\frac{f(x_i)}{x_i - x_{i-1}} - \frac{f''(x_i)(x_i - x_{i-1})}{6}\right](x - x_{i-1})$$

Unknowns:

$f''(x_i)$

$i = 0, 1, \ldots, n$

For each interior point x_i : $\quad f'_{i-1}(x_i) = f'_i(x_i)$

This equation result with **n-1** unknown second derivatives

$$(x_i - x_{i-1}) f''(x_{i-1}) + 2(x_{i+1} - x_{i-1}) f''(x_i) + (x_{i+1} - x_i) f''(x_{i+1})$$
$$= \frac{6}{x_{i+1} - x_i}\left[f(x_{i+1}) - f(x_i) \right] + \frac{6}{x_i - x_{i-1}}\left[f(x_{i-1}) - f(x_i) \right]$$

For boundary points: $\quad f''(x_0) = f''(x_n) = 0$

Set $u_i = f''(x_i), \quad f_i = f(x_i), \quad h_i = x_i - x_{i-1}$

This yields n-1 linear equations

$$\boxed{h_i u_{i-1} + 2(h_i + h_{i+1}) u_i + h_{i+1} u_{i+1} = 6\left(\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i} \right), \quad i = 1, 2, \ldots, n-1}$$

Matrix form of $n-1$ linear equations: $Au = b$

$$A = \begin{bmatrix} 2(h_1+h_2) & h_2 & 0 & \cdots & 0 & 0 \\ h_2 & 2(h_2+h_3) & h_3 & \cdots & 0 & 0 \\ 0 & h_3 & 2(h_3+h_4) & \cdots & & \\ \vdots & \vdots & \vdots & \cdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & \cdots & h_{n-1} & 2(h_{n-1}+h_n) \end{bmatrix}$$

$$u = \begin{bmatrix} f_1'' \\ f_2'' \\ \vdots \\ f_{n-1}'' \end{bmatrix}, \quad b = 6\begin{bmatrix} \dfrac{f_2-f_1}{h_2} - \dfrac{f_1-f_0}{h_1} \\ \dfrac{f_3-f_2}{h_3} - \dfrac{f_2-f_1}{h_2} \\ \vdots \\ \dfrac{f_n-f_{n-1}}{h_n} - \dfrac{f_{n-1}-f_{n-2}}{h_{n-1}} \end{bmatrix}$$

$$\boxed{\begin{aligned} f_i(x) &= \frac{u_{i-1}}{6h_i}\left(x_i - x\right)^3 + \frac{u_i}{6h_i}\left(x - x_{i-1}\right)^3 \\ &+ \left[\frac{f_{i-1}}{h_i} - \frac{u_{i-1}h_i}{6}\right]\left(x_i - x\right) \\ &+ \left[\frac{f_i}{h_i} - \frac{u_i h_i}{6}\right]\left(x - x_{i-1}\right), \quad x_{i-1} \le x \le x_i \end{aligned}}$$

Special case: Equally-spaced data: $h_i = h, i = 1,2,\dots,n$

$$A = \begin{bmatrix} 4h & h & 0 & \cdots & 0 & 0 \\ h & 4h & h & \cdots & 0 & 0 \\ 0 & h & 4h & \cdots & & \\ \vdots & \vdots & \vdots & \cdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 4h & h \\ 0 & 0 & 0 & \cdots & h & 4h \end{bmatrix} \qquad b = \frac{6}{h}\begin{bmatrix} f_2 - 2f_1 + f_0 \\ f_3 - 2f_2 + f_1 \\ \vdots \\ f_n - 2f_{n-1} + f_{n-2} \end{bmatrix} \qquad u = \begin{bmatrix} f_1'' \\ f_2'' \\ \vdots \\ f_{n-1}'' \end{bmatrix}$$

$$Au = b$$

$$\boxed{\begin{aligned} f_i(x) &= \frac{u_{i-1}}{6h}(x_i - x)^3 + \frac{u_i}{6h}(x - x_{i-1})^3 \\ &+ \left[\frac{f_{i-1}}{h} - \frac{u_{i-1}h}{6}\right](x_i - x) \\ &+ \left[\frac{f_i}{h} - \frac{u_i h}{6}\right](x - x_{i-1}), \quad x_{i-1} \le x \le x_i \end{aligned}}$$

Example

Fit the following data with **cubic splines**

Use the results to estimate the value at x=5.

x	3.0	4.5	7.0	9.0
$f(x)$	2.5	1.0	2.5	0.5

Solution:

➢Natural Spline:

$$f''(x_0) = f''(3) = 0, \quad f''(x_3) = f''(9) = 0$$

Solution

x	3.0	4.5	7.0	9.0
$f(x)$	2.5	1.0	2.5	0.5

$$b = 6\begin{bmatrix} \dfrac{f_2 - f_1}{h_2} - \dfrac{f_1 - f_0}{h_1} \\ \dfrac{f_3 - f_2}{h_3} - \dfrac{f_2 - f_1}{h_2} \\ \vdots \\ \dfrac{f_n - f_{n-1}}{h_n} - \dfrac{f_{n-1} - f_{n-2}}{h_{n-1}} \end{bmatrix}$$

$$A=\begin{bmatrix} 2(1.5+2.5) & 2.5 \\ 2.5 & 2(2.5+2) \end{bmatrix}=\begin{bmatrix} 8 & 2.5 \\ 2.5 & 9 \end{bmatrix}$$

$$b=6\begin{bmatrix} \dfrac{2.5-1}{2.5} - \dfrac{1-2.5}{1.5} \\ \dfrac{0.5-2.5}{2} - \dfrac{2.5-1}{2.5} \end{bmatrix}=\begin{bmatrix} 48/5 \\ -48/5 \end{bmatrix}$$

$$Au=b \Rightarrow u=\begin{bmatrix} 1.6791 \\ -1.5331 \end{bmatrix}$$

$$f_1(x) = 0.186566(x-3)^3 + 1.6667\,(4.5-x) + 0.24689(x-3)$$

$$f_2(x) = 0.111939(7-x)^3 - 0.102205\,(x-4.5)^3 - 0.29962(7-x) + 1.638783(x-4.5)$$

$$f_3(x) = -0.127757(9-x)^3 + 1.761027\,(9-x) + 0.25(x-7)$$

$$f\,(5) = f_2(5) = 1.102886$$

Exercise

Reconstruct the function $f(x) = e^{-x^2}$ in [0, 3/2] using the values of $f(x)$ at x=0, 1/2, 1, and 3/2 by

a) linear splines, and use it to find f(3/4) and error
b) quadratic splines, and use it to find f(3/4) and error
c) cubic splines, and use it to find f(3/4) and error

Exercise

Reconstruct the function $f(x) = \ln(x^2 + 1)$ in [0,1] using the values of *f(x)* at *x*=0, ¼, 1/2, 3/4 and 1 by a cubic spline, and use it to find f(2/3) and error

Exercise

The oxygen level O (mg/L) at different temperature T of the sea is given in the following table

a) Fit the data with linear **splines.** Use the results to estimate the oxygen level value at T=20.

b) Fit the data with quadratic **splines.** Use the results to estimate the oxygen level value at T=20.

c) Fit the data with **cubic splines.** Use the results to estimate the oxygen level value at T=20.

x=T	8	16	24	32
f(x)	11.843	9.87	8.418	7.305