Homework 4, Probability

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1 Chapter 6

Problem 2. Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the i^{th} ball selected is white, and 0 otherwise. Give the joint probability mass function of (a) X_1, X_2 and (b) X_1, X_2, X_3.

Solution.

(a) Values of $p(i,j) = \mathbb{P}(X_1 = i, X_2 = j)$ are given below.

i \ j	0	1
0	14/39	10/39
1	10/39	5/39

(b) Values of $p(i,j,k) = \mathbb{P}(X_1 = i, X_2 = j, X_3 = k)$ are given below.

i \ (j,k)	(0,0)	(0,1)	(1,0)	(1,1)
0	28/143	70/429	70/429	40/429
1	70/429	40/429	40/429	5/143

□

Problem 4. Repeat Problem 2 above if the ball selected is replaced in the urn before the next selection.

Solution.

(a) Values of $p(i,j) = \mathbb{P}(X_1 = i, X_2 = j)$ are given below.

i \ j	0	1
0	64/169	40/169
1	40/169	25/169

(b) Values of $p(i,j,k) = \mathbb{P}(X_1 = i, X_2 = j, X_3 = k)$ are given below.

i \ (j,k)	(0,0)	(0,1)	(1,0)	(1,1)
0	512/2197	320/2197	320/2197	200/2197
1	320/2197	200/2197	200/2197	125/2197

□

Problem 6. The severity of a certain cancer is designated by one of the grades $1, 2, 3, 4$ with 1 being the least severe and 4 the most severe. If X is the score of an initially diagnosed patient and Y the score of that patient after 3 months of treatment, hospital data indicates that $p(i, j) = \mathbb{P}(X = i, Y = j)$ is given by

$i \backslash j$	1	2	3	4
1	0.08	0.06	0.04	0.02
2	0.06	0.12	0.08	0.04
3	0.03	0.09	0.12	0.06
4	0.01	0.03	0.07	0.09

(a) Find the probability mass functions of X and Y.

(b) Find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(c) Find $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.

Solution. The probability mass functions of X and Y are given by

x	1	2	3	4
$\mathbb{P}(X = x)$	0.2	0.3	0.3	0.2

and

y	1	2	3	4
$\mathbb{P}(Y = y)$	0.18	0.3	0.31	0.21

so

$$\begin{cases} \mathbb{E}[X] = 0.2 + 2 \cdot 0.3 + 3 \cdot 0.3 + 4 \cdot 0.2 = 2.5 \\ \mathbb{E}[Y] = 0.18 + 2 \cdot 0.3 + 3 \cdot 0.31 + 4 \cdot 0.21 = 2.55 \end{cases}$$

and

$$\begin{cases} \mathrm{Var}(X) = 0.2(1 - 2.5)^2 + 0.3(2 - 2.5)^2 + 0.3(3 - 2.5)^2 + 0.2(4 - 2.5)^2 = 1.05 \\ \mathrm{Var}(Y) = 0.18(1 - 2.55)^2 + 0.3(2 - 2.55)^2 + 0.31(3 - 2.55)^2 + 0.21(4 - 2.55)^2 = 1.0275 \end{cases}$$

$\square$

Problem 8. The joint probability density function of X and Y is given by

$$f(x, y) = c(y^2 - x^2)e^{-y}, \qquad -y \le x \le y, \quad 0 < y < \infty$$

(a) Find c.

(b) Find the marginal densities of X and Y.

(c) Find $\mathbb{E}[X]$.

Solution.

(a) Note that

$$f_Y(y) = c \int_{-y}^{y} (y^2 - x^2)e^{-y} dx = \frac{4}{3} c y^3 e^{-y}, \qquad 0 < y < \infty$$

and

$$\int_0^\infty f_Y(y) dy = 8c = 1, \text{ so } c = \frac{1}{8}.$$

(b) By part (a),

$$f_Y(y) = \frac{y^3 e^{-y}}{6}, \quad 0 < y < \infty$$

and

$$f_X(x) = \frac{1}{8} \int_{|x|}^{\infty} (y^2 - x^2)e^{-y} dy = \frac{1}{4} e^{-|x|}(1 + |x|).$$

(c) Since f_X is an odd function,

$$\mathbb{E}[X] = \int_{-y}^{y} \frac{1}{4} x e^{-|x|}(1+|x|)dx = 0.$$

$\square$

Problem 9. The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), \qquad 0 < x < 1, \quad 0 < y < 2.$$

(a) Verify that f is a joint density function.

(b) Compute the density function of X.

(c) Find $\mathbb{P}(X \geq Y)$.

(d) Find $\mathbb{P}(Y > 1/2 | X < 1/2)$.

(e) Find $\mathbb{E}[X]$.

(f) Find $\mathbb{E}[Y]$.

Solution.

(a) Note that

$$\int\int_{\mathbb{R}^2} f(x,y)dxdy = \int_0^1 \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx = 1$$

and $f(x,y) \geq 0, \forall x, y > 0$, so f is a joint probability density function.

(b) The density function of X is

$$f_X(x) = \frac{6}{7}\int_0^2 (x^2 + \frac{xy}{2})dy = \frac{6}{7}(2x^2 + x).$$

(c)

$$\mathbb{P}(X \geq Y) = \frac{6}{7}\int_0^1 \int_0^x \left(x^2 + \frac{xy}{2}\right) dydx = \frac{15}{56}.$$

(d)

$$\begin{aligned}\mathbb{P}(Y > 1/2 | X < 1/2) &= \mathbb{P}(Y > 1/2, X < 1/2) : \mathbb{P}(X < 1/2) \\ &= \left(\int_{1/2}^2 \int_0^{1/2} \left(x^2 + \frac{xy}{2}\right) dxdy\right) : \left(\int_0^{1/2} (2x^2 + x)dx\right) \\ &= \frac{23}{128} : \frac{5}{24} = 0.8625.\end{aligned}$$

(e)

$$\mathbb{E}[X] = \int_0^1 \frac{6x}{7} \cdot (2x^2 + x)dx = 1.$$

(f)

$$\mathbb{E}[Y] = \int_0^1 \int_0^2 \frac{6y}{7}\left(x^2 + \frac{xy}{2}\right) dxdy = \frac{8}{7}.$$

$\square$

Problem 10. The joint probability density function of X and Y is given by

$$f(x,y) = e^{-(x+y)}, \quad 0 \le x < \infty, \quad 0 \le y < \infty.$$

Find (a) $\mathbb{P}(X < Y)$ and (b) $\mathbb{P}(X < a)$.

Solution. The marginal densities of X and Y are

$$f_X(x) = \int_0^\infty e^{-(x+y)} dy = e^{-x}$$

and

$$f_Y(y) = \int_0^\infty e^{-(x+y)} dx = e^{-y}$$

so $\mathbb{P}(X < Y) = 1/2$ and $\mathbb{P}(X < a) = 1 - e^{-a}$. □

Problem 19. Show that

$$f(x,y) = \frac{1}{x}, \quad 0 < y < x < 1$$

is a joint density function. If f is the joint density function of X, Y, find (a) the marginal densities and (b) the expectation of X, Y.

Solution. Note that

$$\int\int_{\mathbb{R}^2} f(x,y)dxdy = \int_0^1 \int_0^x \frac{1}{x} dydx = 1$$

and $f(x,y) \ge 0, \forall x, y > 0$, so f is a joint probability density function. The marginal densities of X and Y are

$$f_Y(y) = \int_y^1 f(x,y)dx = \int_y^1 \frac{1}{x}dx = -\ln y, \quad 0 < y < 1$$

and

$$f_X(x) = \int_0^x f(x,y)dy = \int_0^x \frac{1}{x}dy = 1, \quad 0 < y < 1$$

which gives the expectation as

$$\mathbb{E}[X] = \int_0^1 x\ dx = \frac{1}{2} \quad \text{and} \quad \mathbb{E}[Y] = \int_0^1 -y \ln y\ dy = \frac{1}{4}.$$

□

Problem 21. Let

$$f(x,y) = \begin{cases} 24xy & 0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that (a) f is a joint probability density function and find (b) $\mathbb{E}[X]$ and (c) $\mathbb{E}[Y]$.

Solution.

(a) Note that

$$\int\int_{\mathbb{R}^2} f(x,y)dxdy = \int_0^1 \int_0^{1-y} 24xy\ dxdy = 1$$

and $f(x,y) \ge 0, \forall x, y \ge 0$, so f is a joint probability density function.

(b)

$$\mathbb{E}[X] = \int_0^1 x f_X(x)dx = \int_0^1 x \int_0^{1-x} 24xy\ dydx = \int_0^1 12x^2(1-x)^2 dx = \frac{2}{5}.$$

(c)

$$\mathbb{E}[Y] = \int_0^1 x f_Y(y)dy = \int_0^1 y \int_0^{1-x} 24xy\ dxdy = \int_0^1 8x(-x+1)^3 dx = \frac{2}{5}.$$

□

Problem 22. The joint density function of X and Y is

$$f(x,y) = \begin{cases} x+y & 0 < x < 1, \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?

(b) Find the density function of X.

(c) Find $\mathbb{P}(X+Y<1)$.

Solution.

(a) The marginal densities of X and Y are

$$f_X(x) = \int_0^1 (x+y)dy = x + \frac{1}{2}, \quad 0 < x < 1$$

and

$$f_Y(y) = \int_0^1 (x+y)dx = y + \frac{1}{2}, \quad 0 < y < 1$$

so $f_X(x) \cdot f_Y(y) = (x+1/2)(y+1/2) \neq f_{X,Y}(x,y)$, i.e. X and Y are not independent.

(b) Already given in part (a).

(c)

$$\mathbb{P}(X+Y<1) = \int_0^1 \int_0^{1-x} (x+y)dydx = \int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2} \right) dx = \frac{1}{3}.$$

□

Problem 23. The random variables X and Y have joint density function

$$f(x,y) = \begin{cases} 12xy(1-x) & 0 < x < 1, \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent? Find (b) $\mathbb{E}(X)$, (c) $\mathbb{E}(Y)$, (d) $\mathrm{Var}(X)$, and (e) $\mathrm{Var}(Y)$.

Solution.

(a) The marginal densities of X and Y are

$$f_X(x) = \int_0^1 12xy(1-x)dy = 6x(1-x), \quad 0 < x < 1$$

and

$$f_Y(y) = \int_0^1 12xy(1-x)dx = 2y, \quad 0 < y < 1$$

so $f_X(x) \cdot f_Y(y) = 12xy(1-x) = f_{X,Y}(x,y)$, i.e. X and Y are independent.

(b)

$$\mathbb{E}(X) = \int_0^1 6x^2(1-x)dx = \frac{1}{2}.$$

(c)

$$\mathbb{E}(Y) = \int_0^1 2y^2dy = \frac{2}{3}.$$

(d)

$$\mathrm{Var}(X) = \int_0^1 6x^3(1-x)dx - \frac{1}{4} = \frac{1}{20}.$$

(e)

$$\mathrm{Var}(Y) = \int_0^1 2y^3 dy - \frac{4}{9} = \frac{1}{18}.$$

□

Problem 42. Choose a number X at random from the set $\{1,2,3,4,5\}$. Now choose a number Y at random from the set $\{1,2,...,X\}$.

(a) Find the joint mass function of X and Y.

(b) Find the conditional mass function of X given that $Y = i$, for $i = 1,2,3,4,5$.

(c) Are X and Y independent? Why?

Solution.

(a) Values of $p(i,j) = \mathbb{P}(X = i, Y = j)$ are given below.

$i \backslash j$	1	2	3	4	5
1	0.2	0	0	0	0
2	0.1	0.1	0	0	0
3	1/15	1/15	1/15	0	0
4	0.05	0.05	0.05	0.05	0
5	0.04	0.04	0.04	0.04	0.04

(b) The conditional mass function $X|Y = i$ are given below.

x	1	2	3	4	5
$\mathbb{P}(X = x\|Y = 1)$	60/137	30/137	20/137	15/137	12/137

x	2	3	4	5
$\mathbb{P}(X = x\|Y = 2)$	30/77	20/77	15/77	12/77

x	3	4	5
$\mathbb{P}(X = x\|Y = 3)$	20/47	15/47	12/47

x	4	5
$\mathbb{P}(X = x\|Y = 4)$	5/9	4/9

x	5
$\mathbb{P}(X = x\|Y = 5)$	1

(c) X and Y are not independent, since the conditional mass of X varies as Y varies.

□

2 Chapter 7

2.1 Problems

Exercise 4. If X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} 1/y & 0 < y < 1, \quad 0 < x < y \\ 0 & \text{otherwise} \end{cases},$$

find (a) $\mathbb{E}[XY]$, (b) $\mathbb{E}[X]$ and (c) $\mathbb{E}[Y]$.

Solution. (a)

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dxdy = \int_0^1\int_0^y x\ dxdy = \frac{1}{6}.$$

(b)

$$\mathbb{E}[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)dxdy = \int_0^1\int_0^y \frac{1}{y}dxdy = \frac{1}{4}.$$

(c)

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)dxdy = \int_0^1\int_0^y 1dxdy = \frac{1}{2}.$$

□

Exercise 6. A fair die is rolled 10 times. Calculate the expected sum of the 10 rolls.

Solution. Note that each roll has an expected result of 3.5, so the linearity of expectation gives the total expected sum of 35. □

Exercise 16. Let $Z \sim \mathcal{N}(0,1)$ and, for a fixed x, set

$$X = \begin{cases} Z & Z > x \\ 0 & \text{otherwise} \end{cases}$$

Show that $\mathbb{E}[X] = 1/(\sqrt{2\pi}e^{-x^2}/2)$.

Solution.

$$\mathbb{E}[X] = \int_{y>x} \frac{y}{\sqrt{2\pi}} \cdot e^{-y^2/2}dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

□

Exercise 30. If X, Y are i.i.d. with mean μ and variance σ^2, find $\mathbb{E}[(X-Y)^2]$.

Solution. By the linearity of expectation,

$$\begin{aligned}\mathbb{E}[(X-Y)^2] &= \mathbb{E}(X^2) - 2\cdot\mathbb{E}(XY) + \mathbb{E}(Y^2) = \mathrm{Var}(X) + \mathbb{E}(X)^2 - 2\cdot\mathbb{E}(XY) + \mathrm{Var}(Y) + \mathbb{E}(Y)^2 \\ &= 2\sigma^2 + 2\mu^2 - 2(\mathrm{cov}(X,Y) + \mathbb{E}(X)\cdot\mathbb{E}(Y)) = 2\sigma^2.\end{aligned}$$

□

Exercise 33. If $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 5$, find (a) $\mathbb{E}[(2+X)^2]$ and (b) $\mathrm{Var}(4+3X)$.

Solution. By the linearity of expectation and variance,

$$\mathbb{E}[(2+X)^2] = \mathbb{E}[X^2] + 4\cdot\mathbb{E}[X] + 4 = \mathrm{Var}(X) + \mathbb{E}(X)^2 + 8 = 14$$

and

$$\mathrm{Var}(4+3X) = 9\cdot\mathrm{Var}(X) = 45.$$

□

Exercise 40. The random variables X and Y have a joint density function given by

$$f(x,y) = \begin{cases} 2e^{-2x}/x & 0 \le x < \infty,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathrm{cov}(X,Y)$.

Solution. We have

$$\mathbb{E}[X] = \int_0^\infty \int_0^x 2e^{-2x} dy dx = \int_0^\infty 2xe^{-2x} dx = \frac{1}{2},$$

$$\mathbb{E}[Y] = \int_0^\infty \int_0^x \frac{2ye^{-2x}}{x} dy dx = \int_0^\infty xe^{-2x} dx = \frac{1}{4},$$

$$\mathbb{E}[XY] = \int_0^\infty \int_0^x 2ye^{-2x} dy dx = \int_0^\infty x^2 e^{-2x} dx = \frac{1}{4}$$

so $\text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 1/8$. □

Exercise 52. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y}, \qquad 0 < x < \infty, \quad 0 < y < \infty.$$

Compute $\mathbb{E}[X^2|Y = y]$.

Solution. The marginal density of Y is

$$f_Y(y) = \int_0^\infty f(x, y) dx = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y} dx = e^{-y}$$

so the conditional density $X|Y$ is

$$f_{X|Y}(x|y) = f(x, y)/f_Y(y) = \frac{e^{-x/y}e^{-y}}{y} : e^{-y} = \frac{e^{-x/y}}{y}$$

implying

$$\mathbb{E}[X^2|Y = y] = \int_0^\infty x^2 f_{X|Y}(x|y) dx = \int_0^\infty \frac{x^2 e^{-x/y}}{y} dx = 2y^2.$$

□

2.2 Theoretical Exercises

Exercise 1. Show that $\mathbb{E}[(X - a)^2]$ is minimized at $a = \mathbb{E}[X]$.

Solution. By the linearity of expectation,

$$\begin{aligned}\mathbb{E}[(X - a)^2] &= \mathbb{E}[X^2 - 2aX + a^2] = \mathbb{E}[X^2] - 2a \cdot \mathbb{E}[X] + a^2 \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 + (\mathbb{E}[X] - a)^2 \geq \mathbb{E}[X^2] - \mathbb{E}[X]^2\end{aligned}$$

so the minimum value of $\mathbb{E}[(X - a)^2]$ is $\mathbb{E}[X^2] - \mathbb{E}[X]^2$, obtained at $a = \mathbb{E}[X]$. □

Exercise 20. Show that if X, Y are identically distributed but not necessarily independent, then

$$\text{cov}(X + Y, X - Y) = 0.$$

Solution.

$$\begin{aligned}\text{cov}(X + Y, X - Y) &= \text{cov}(X, X) - \text{cov}(X, Y) + \text{cov}(Y, X) - \text{cov}(Y, Y) \\ &= \text{Var}(X) - \text{Var}(Y) = 0.\end{aligned}$$

□