1 Discrete Probability

Random Variables

A random variable is a function from the sample space Ω to the real numbers $\mathbb{R}$.
A random variable X is discrete if it takes on a finite or countable number of values.

Probability Mass Function

The probability mass function (PMF) or distribution of a discrete random variable X gives the probabilities of its possible values.

$$P(X = x) \geq 0$$

(all probabilities are non-negative)

$$\sum_i P(X = x_i) = 1$$

(the probabilities sum to 1)

Cumulative Distribution Function

The cumulative distribution function (CDF) of a discrete random variable X gives the probability that X is less than or equal to x.

$$F : \mathbb{R} \to [0, 1]$$

$$F(x) = P(X \leq x)$$

Similar to PDF graph except it adds them up (cumulative)

$$\begin{aligned} &P(a < x \leq b) \\ &= P(X \leq b) - P(X \leq a) \\ &= F(b) - F(a) \end{aligned}$$

A cumulative distribution function F:

- is non-decreasing: $F(x) \leq F(y)$ for all $x \leq y$
- has limit 0: $F(-\infty) = 0$ on the left
- has limit 1: $F(\infty) = 1$ on the right

Expected Value

The expected value of a discrete random variable X is the average value of X.

$$E(X) = \sum_i x_i P(X = x_i)$$

(provided the sum exists)
Note: Expected value need not be a possible value of X.
For $g : \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned} &E(g(X)) \\ &= \sum_i g(x_i) P(X = x_i) \end{aligned}$$

(provided the sum exists)
Expectation is linear, so $E(aX + b) = aE(X) + b$ for any constants a and b.

Variance

The variance tells us how surprised we should be if we observe a value of X.

$$\mathrm{Var}(X) = E((X - E(X))^2)$$

Standard Deviation

The standard deviation is the square root of the variance.

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$$

Bernoulli and Binomial Distributions

$$X \sim \mathrm{Binom}(n, p)$$

X has the Binomial distribution with parameters n and p if, for n independent trials, each succeeding with probability p, the random variable X counts the number of successes within the n trials.
Special case $n = 1$ is called the Bernoulli distribution with parameter p. In this case, X is 1 if the trial succeeds and 0 if it fails (indication variable).

PMF

Let $X \sim \mathrm{Binom}(n, p)$ and $X = 0, 1, \ldots, n$. Then:

$$\begin{aligned} &P(X = k) \\ &= \binom{n}{k} p^k (1 - p)^{n-k} \end{aligned}$$

The Bernouilli(p) distribution can take on values 0 or 1 with properties:

$$P(X = 0) = 1 - p$$

$$P(X = 1) = p$$

Newton's Binomial Theorem:

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \\ &= (a + b)^n \end{aligned}$$

Expected Value

$$E(X) = np$$

Variance

$$\mathrm{Var}(X) = np(1 - p)$$

Poisson Distribution

$$X \sim \mathrm{Poisson}(\lambda)$$

The random variable X is Poisson distributed with parameter λ if λ is non-negative integer valued and its mass function is:

$$P(X = k) = e^{-\lambda} \times \frac{\lambda^i}{i!}$$

1.0.1 Poisson Approximation to Binomial

Take $Y \sim \mathrm{Binom}(n, p)$ with large n and small p, such that $np \approx \lambda$. Then Y is approximately Poisson(λ) distributed.

Expected Value and Variance

$$E(X) = \mathrm{Var}(X) = \lambda$$

... since Binomial expectation and variance are np and $np(1 - p)$ which both converge to λ for large n.

Geometric Distribution

When is the first success?

$$X \sim \mathrm{Geom}(p)$$

Suppose that independent trials, each succeeding with probability p, are repeated until the first success. The total number X of trials made has the Goemetric(p) distribution.
X can take on positive integers, with probabilities:

$$P(X = i) = (1 - p)^{i-1} p$$

The Goemetric random variable is (discrete) memoryless:

$$\begin{aligned} &P(X > n + k | X > n) \\ &= P(X > k) \end{aligned}$$

... for every $k \geq 1$, $n \geq 0$.

Expectation and Variance

$$E(X) = \frac{1}{p}$$

$$\mathrm{Var}(X) = \frac{1 - p}{p^2}$$