Problem set # 2.2

Exercise 1. *Let $P(z)=(z-\alpha)(z-\beta)$ where $\alpha,\beta\in\mathbb{C}$ and $\alpha\neq\beta$. Let $N_P(z)=z-\frac{P(z)}{P'(z)}$ be the Newton's method of P. Describe precisely (with proofs) the basins of attraction of α and β, the Fatou set and the Julia set. What can you say about the dynamics on the Julia set? (Hint: Conjugate N_P (on the whole Riemann sphere) by the Möbius transformation $M(z)=\frac{z-\alpha}{z-\beta}$ and see what the resulting map is.)*

Proof. Let us define the map $f(z):=z^2$. We are going to prove that the Newton method of P is conjugate to f via the Möbius transformation M, i.e., $M\circ N_P=f\circ M$. First, observe that

$$N_P(z)=z-\frac{P(z)}{P'(z)}=z-\frac{(z-\alpha)(z-\beta)}{2z-(\alpha+\beta)}=\frac{z^2-\alpha\beta}{2z-(\alpha+\beta)}.$$

Hence,

$$\begin{aligned}M(N_P(z))=M\left(\frac{z^2-\alpha\beta}{2z-(\alpha+\beta)}\right)&=\frac{z^2-\alpha\beta-\alpha(2z-(\alpha+\beta))}{z^2-\alpha\beta-\beta(2z-(\alpha+\beta))}=\\&=\frac{z^2-2z\alpha+\alpha^2}{z^2-2z\beta+\beta^2}=\left(\frac{z-\alpha}{z-\beta}\right)^2=f(M(z)).\end{aligned}$$

Notice that the Möbius transformation sends α to zero and β to ∞. Hence, to study of the basins of attractions of α and β for the polynomial P, it is equivalent to study the basins of attractions of 0 and ∞ of the function f. Recall that

$$\mathcal{A}_f(0)=\{z\in\hat{\mathbb{C}}:f^n(z)\xrightarrow[n\to\infty]{}0\},$$
$$\mathcal{A}_f(\infty)=\{z\in\hat{\mathbb{C}}:f^n(z)\xrightarrow[n\to\infty]{}\infty\}.$$

Since $f^n(z)=z^{2^n}$, it is clear that $\mathcal{A}_f(0)=D(0,1)$ and $\mathcal{A}_f(\infty)=\hat{\mathbb{C}}\setminus\overline{D(0,1)}$. Now, observe that,

$$\mathcal{A}_{N_P}(\alpha)=M^{-1}(D(0,1))=\{z\in\hat{\mathbb{C}}:|M(z)|<1\}=\{z\in\hat{\mathbb{C}}:|z-\alpha|<|z-\beta|\},$$
$$\mathcal{A}_{N_P}(\beta)=M^{-1}(\hat{\mathbb{C}}\setminus\overline{D(0,1)})=\{z\in\hat{\mathbb{C}}:|M(z)|>1\}=\{z\in\hat{\mathbb{C}}:|z-\alpha|>|z-\beta|\}.$$

Now, let us compute the Fatou and Julia sets for the map f. Clearly, the family $(f^n)_n$ is normal in $\hat{\mathbb{C}}\setminus\overline{D(0,1)}$ and in $D(0,1)$. In the first case, $f^n\rightrightarrows g\equiv\infty$ and in the second one, $f^n\rightrightarrows g\equiv 0$. Now, let us prove that the family is not normal on any neighborhood of $\mathbb{S}^1$. Take U neighborhood of any $z_0\in\mathbb{S}^1$. Then,

$$\begin{cases}\exists w_1\in U\cap D(0,1), f^n(w_1)\xrightarrow[n\to\infty]{}0 \quad\text{and,}\\ \exists w_2\in U\cap\hat{\mathbb{C}}\setminus\overline{D(0,1)}, f^n(w_2)\xrightarrow[n\to\infty]{}\infty\end{cases}$$

If $f^{n_k}\rightrightarrows g$, since the family $(f^n)_n$ is holomorphic, g would be holomorphic and in particular, continuous. But $g(w_1)=0, g(w_2)=\infty$ and $|g(z_0)|=1$, which is a contradiction. Hence, $\mathcal{F}(f)=\hat{\mathbb{C}}\setminus\mathbb{S}^1$ and $\mathcal{J}(f)=\mathbb{S}^1$. Using the same reasoning as before, we obtain that

$$\mathcal{J}(N_P)=\{z\in\hat{\mathbb{C}}:|z-\alpha|=|z-\beta|\},$$
$$\mathcal{F}(N_P)=\{z\in\hat{\mathbb{C}}:|z-\alpha|\neq|z-\beta|\}.$$

To complete the exercise, a final comment regarding the dynamics on the Julia set: When using the Newton method to find the roots of a function, a crucial consideration is the selection of an initial guess to start the root-finding algorithm. As demonstrated in this straightforward example, initiating the process from a point that isn't equidistant to both roots leads to convergence towards some roots. On the contrary, any other point that is equidistant to both roots, residing in the Julia set, will fail to converge owing to the chaotic dynamics involved.

Finally, I provide both the code and the corresponding image (Figure 1) displaying the basins of attraction generated by the Newton method. Notice that a color intensity palette has been incorporated, wherein points converging more rapidly exhibit higher color intensity compared to others.

```
import numpy as np
import matplotlib.pyplot as plt
# Functions definition
def f(z):
  return z*(z-1)*(z-1j)
def der_f(z):
  return 3*z**2 - 2*z*(1j) - 2*z + (1j)

# Define a function compute the Newton-Raphson method
def newton(f,der_f, z,tol):

  # Maximum 50 iteration
  for i in range(50):

    if abs(der_f(z)) > tol:
      z_new = z-f(z)/der_f(z)
    else:
      print('The derivative is 0 at z=',z)

    # Stop the method when two iterates are close or f(z) = 0
    if abs(f(z))< tol or abs(z_new-z) < tol:
      return z, i
    else:
      # Update iterations
      z = z_new

  # If no convergence, return None
  return None


# Define a function to plot the roots of a complex function using the
    Newton-Raphson method
def plot(f, der_f, tol):
    # List to store unique roots
    roots = []

    # Define the ranges for z_x and z_y
    z_x_range = np.linspace(-2, 2.5, 5000)
    z_y_range = np.linspace(-2, 2.5, 5000)

    # Create a meshgrid from the ranges
```

```
z_x, z_y = np.meshgrid(z_x_range, z_y_range)

# Create an array to store the colors for each point
colors = np.zeros_like(z_x, dtype=float)

# Iterate over every point in the grid
for i in range(len(z_x_range)):
    for j in range(len(z_y_range)):
        # Create a complex number from the grid point
        point = complex(z_x[i, j], z_y[i, j])

        # Apply the Newton-Raphson method to find a root and the number of
            iterations
        root, iterations = newton(f, der_f, point, tol)

        # Check if a root is found
        if root:
            flag = False
            # Check if the root is already in the list with a certain tolerance
            for test_root in roots:
                if abs(test_root - root) < 10e-3:
                    root = test_root
                    flag = True
                    break

            # If the root is not in the list, append it
            if not flag:
                roots.append(root)

            # Assign color intensity based on the number of iterations
            color_intensity = (iterations / 50)**.4
            # Assign a unique color index to each root
            color_index = roots.index(root) + 1
            colors[i, j] = color_intensity * color_index

# Plot the colored picture
plt.figure(figsize=(10, 8))
plt.scatter(z_x, z_y, c=colors, cmap='viridis', marker='.')

# Plot the root points in black
root_markers = np.array(roots)
plt.scatter(root_markers.real, root_markers.imag, color='black', marker='o',
    s=20)

# Remove the axis
plt.axis('off')

# Display the plot
plt.show()
```

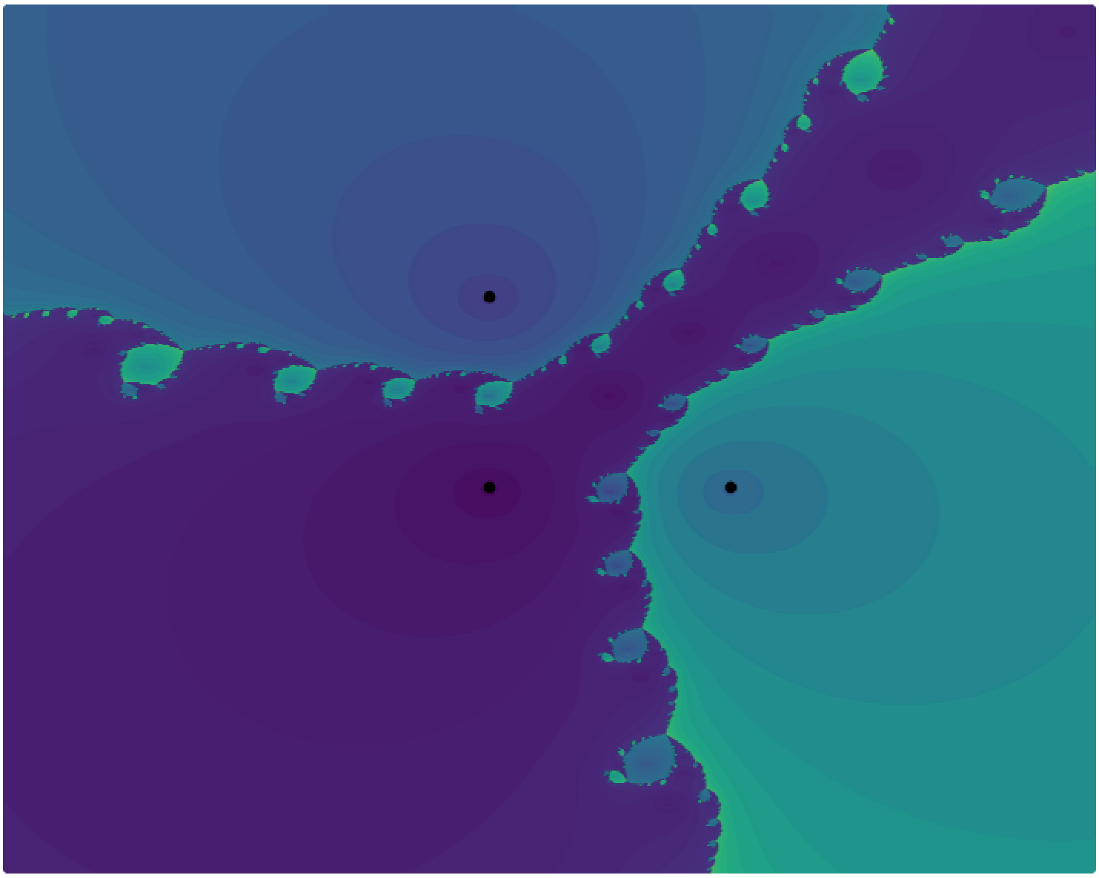

Figure 1: Basins of attraction for the Newton method of $p(z) = z(z-1)(z-i)$.

$\square$

Exercise 2. ***(The quadratic family).*** *Let $Q_c(z) = z^2 + c$. Let $\mathcal{A}_c(\infty)$ denote the basin of attraction of ∞ for Q_c, and $K_c := \mathbb{C} \setminus \mathcal{A}_c(\infty)$ denote the filled Julia set.*

a) *Prove that $K_c \subset \overline{D(0,R)}$ where $R = \max\{|c|, 2\}$.*

b) *Deduce that if $|c| > 2$ then the orbit of the critical point $z = 0$ escapes to infinity.*

c) *Show that for every value of $c \in \mathbb{C}$, Q_c has at most one attracting cycle.*

d) *Calculate and draw the sets:*

$$\Omega_1 := \{c \in \mathbb{C} : Q_c \text{ has an attracting fixed point}\},$$
$$\Omega_2 := \{c \in \mathbb{C} : Q_c \text{ has an attracting 2-cycle}\}.$$

a)

Proof. Let us see that if $|z| > c$ and $|z| > 2$, then $|Q_c^n(z)| \xrightarrow[n\to\infty]{} \infty$. This would mean that $\mathbb{C} \setminus \overline{D(0,R)} \subset \mathcal{A}_c(\infty)$, and therefore, $K_c \subset \overline{D(0,R)}$. Notice that, since $|z| > c$,

$$|Q_c(z)| = |z^2 + c| \geq |z^2| - |c| > |z|^2 - |z| = |z|(|z| - 1).$$

Now, since $|z| > 2$, $|z| - 1 = 1 + \epsilon$, with $\epsilon > 0$. Hence,

$$|Q_c(z)| > (1+\epsilon)|z| \implies |Q_c^n(z)| > (1+\epsilon)^n|z| \xrightarrow[n\to\infty]{} \infty.$$

□

b)

Proof. Using the previous exercise, since $|c| > 2$, we have that $R = \max\{|c|, 2\} = |c|$ and, $\mathbb{C} \setminus \overline{D(0,|c|)} \subset \mathcal{A}_c(\infty)$. Notice that $|Q_c(0)| = |c|$ and $|Q_c^2(0)| = |c^2 + c| > |c|$. Then, $Q_c^2(0) \in \mathbb{C} \setminus \overline{D(0,|c|)} \subset \mathcal{A}_c(\infty)$ and therefore, the orbit of $z = 0$ escapes to infinity. □

c)

Proof. We saw in class the following theorem:

Theorem. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a rational map. If $< z_0 >$ is an attracting cycle of period k, then, there exist a critical point of f^k in every component of $\mathcal{A}^*(< z_0 >)$. In particular, there exist a critical point of f in at least one component of $\mathcal{A}^*(< z_0 >)$.

Since Q_c has only one critical point, $z = 0$, Q_c has at most one attracting cycle. □

d)

Proof. Let us start with Ω_1. We want to find the values of $c \in \mathbb{C}$ such that Q_c has an attracting fixed point, i.e., the values of $c \in \mathbb{C}$ such that,

$$\begin{cases} z^2 + c = z, \\ |Q_c'(z)| = |2z| < 1. \end{cases}$$

Let us find the boundary of the region, i.e.,

$$\begin{cases} z^2 + c = z, \\ |Q_c'(z)| = |2z| = 1 \implies |z| = \frac{1}{2}. \end{cases}$$

Hence, $z = \frac{1}{2}e^{i\theta}$ with $\theta \in [0, 2\pi)$, and from the first equation, we obtain that $c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$[1].

Now, to obtain Ω_2, we need to find the values of $c \in \mathbb{C}$ such that $(z^2+c)^2+c = z$ and its multiplier is less than one. From the equation, factorizing the fixed points, we obtain that $z^2+z+c+1 = 0$. Hence, $z_\pm = \frac{-1\pm\sqrt{-3-4c}}{2}$. Then, the multiplier is given by $2z_+2z_- = 1-(-3-4c) = 4(c+1)$. Hence, the cycle is attracting if $4|c+1| < 1$, i.e., $|c-(-1)| < \frac{1}{4}$[2]. Figure 2 shows the cardioid (in green) and the disk (in red).

[1] A cardioid.
[2] Disk centered in -1 and radius $\frac{1}{4}$

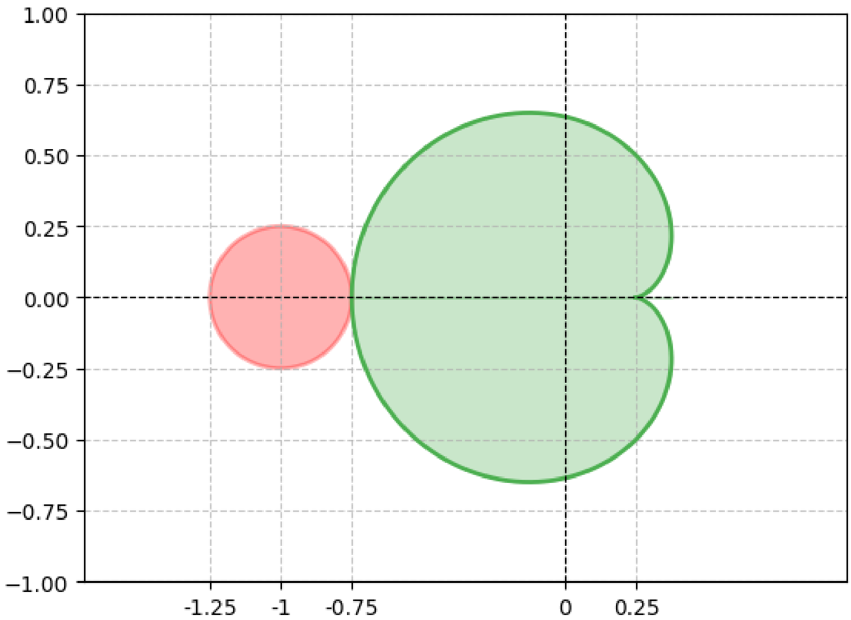

Figure 2: In green, the set Ω_1 and, in red, the set Ω_2. The graph has been generated with python.

The code to generate the figure:

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.patches import Circle

# Define the cardioid
def cardioid(x):
    z = (1/2) * np.exp(1j*x) - (1/4) * np.exp(1j*2*x)
    return z

# Generate values for x in the specified range
x_values = np.linspace(0, 2*np.pi, 1000)

# Compute the values for the cardioid
z_values = cardioid(x_values)

# Plot in the complex plane
plt.plot(np.real(z_values), np.imag(z_values), color='#4CAF50', linewidth=2)

# Fill the area between the curve and the x-axis
plt.fill_between(np.real(z_values), np.imag(z_values), color='#4CAF50', alpha=0.3)

# Add axes
plt.axhline(0, color='black', linestyle='--', linewidth=0.8)
plt.axvline(0, color='black', linestyle='--', linewidth=0.8)

x_ticks = [-5/4, -1, -3/4, 0, 1/4]
plt.xticks(x_ticks, [f'{val}' for val in x_ticks])

# Add a grid f
plt.grid(True, linestyle='--', alpha=0.7)
```

```
# Add a disk centered at -1 with radius 1/4 and fill it with color
disk = Circle((-1, 0), 1/4, color='red', fill=True, alpha=0.3,
    edgecolor='black',linewidth=2)
plt.gca().add_patch(disk)

# Set equal aspect ratio for a more accurate representation of the complex plane
plt.axis('equal')

plt.xlim([-1.7, 1])
plt.ylim([-1, 1])

# Show the plot
plt.show()
```

□

Exercise 3. *Let R be a rational function and suppose that C is a round circle such that $R^{-1}(C) \subset C$ Prove that $\mathcal{J}(R) = C$ or $\mathcal{J}(R)$ is a totally disconnected subset of C. Hint: There are several ways of solving this problem. Some key words that might be related to possible solutions are: conjugacy, unit circle, Schwarz reflection, invariance, Denjoy-Wolff, normality...*

Proof. Before starting the proof, let us recall the definition of the exceptional set and two needed properties.

Definition: *The exceptional set of R is defined as*

$$\epsilon(R) = \{w \in \hat{\mathbb{C}} : R^n(z) \neq w \,\forall z \in U \text{ where } U \cap \mathcal{J}(R) \neq \emptyset\}.$$

Properties:

1. $\#\epsilon(R) \leq 2$.
2. *Given a rational map R. If $z_0 \in \hat{\mathbb{C}} \setminus \epsilon(R)$, then*

$$\mathcal{J}(R) \subset \overline{\bigcup_n R^{-n}(z_0)}.$$

Let us start the proof. First of all, let us prove that $\mathcal{J}(R) \subseteq C$. Since C is a round circle and has more than two points, there exist a point $z_0 \in C$ such that $z_0 \notin \epsilon(R)$. Then, since $R^{-1}(C) \subset C$, $R^{-n}(z_0) \in C$ for every $n \in \mathbb{N}$. Using the second mentioned property and the fact that C is closed, we have that

$$\mathcal{J}(R) \subset \overline{\bigcup_n R^{-n}(z_0)} \subset \overline{C} = C.$$

To finish the prove, let us assume that $\mathcal{J}(R) \neq C$ and let us see that $\mathcal{J}(R)$ is a totally disconnected subset of C. Since $\mathcal{J}(R) \neq C$, there exist $z_1 \in C$ such that $z_1 \notin \mathcal{J}(R)$, i.e., $z_1 \in \mathcal{F}(R)$. Since $\mathcal{F}(R)$ is an open set, there exist a neighborhood $U \ni z_1$ such that $U \subset \mathcal{F}(R)$. Since $z_1 \in C$, $z_1 \in C \cap U$ and there exist an arc of C, S ("essentially a piece of C") included in the Fatou set. Again, since $\#\epsilon(R) \leq 2$, there exist $z_2 \in S \subset \mathcal{F}(R)$ such that $z_2 \notin \epsilon(R)$. Now, observe that, since the Fatou set is invariant, $R^{-n}(z_2) \in \mathcal{F}(R)$ for every $n \in \mathbb{N}$ and, since

$z_2 \in C$, $R^{-n}(z_2) \in C$ for every $n \in \mathbb{N}$ (due to the fact that $R^{-1}(C) \subset C$). Moreover, since $z_2 \notin \epsilon(R)$, using the second property,

$$\mathcal{J}(R) \subset \overline{\bigcup_n R^{-n}(z_2)}.$$

Therefore, for every $w \in \mathcal{J}(R)$ and for every $\epsilon > 0$, there exist $\sigma \in \bigcup_n R^{-n}(z_2)$ such that $||w - \sigma|| < \epsilon$. But observe that $\sigma \in \mathcal{F}(R) \cap C$. Essentially, we have that for every point w in the Julia set, we can find points in the circle of the Fatou set as close as we want from w.

Now, take a connected component of the Julia set containing a point $w_1 \in \mathcal{J}(R)$. Since the Julia set is included in C, the connected component is an arc of C. Assume that there are at least two point in the connected component, w_1, w_2. We have just seen that, there exist $\sigma_1, \sigma_2 \in \mathcal{F}(R) \cap C$ such that $||w_1 - \sigma_1|| < \epsilon$ and $||w_2 - \sigma_2|| < \epsilon$. Then, the arc, which is entirely contained in the Julia set must be reduced to a single point, due to the fact that we can find points of the Fatou set as close as we want from w_1 and w_2. Then, the connected componet is just the point w_1 and the Julia set is a totally disconnected subset of C. $\square$