HARMONIC ANALYSIS

6

a) Prove that the Fourier transform of the function $f(t) = e^{-2\pi|t|}$ is

$$\hat{f}(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}$$

$$\left(\text{Hint: } \int_{\mathbb{R}} e^{-2\pi|t|} e^{-2\pi i \xi t}\, dt = 2\int_0^\infty e^{-2\pi t} \cos(2\pi\xi t)\, dt\right)$$

By definition of the Fourier transform,

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t}\, dt = \int_{\mathbb{R}} e^{-2\pi|t|} e^{-2\pi i \xi t}\, dt \overset{\text{Hint}}{=}$$

$$= 2\int_0^\infty e^{-2\pi t} \cos(2\pi\xi t)\, dt =$$

Integrating by parts

$$= 2\cancelto{0}{\left[e^{-2\pi t}\frac{1}{2\pi\xi}\sin(2\pi\xi t)\right]_0^\infty} + 4\pi\int_0^\infty e^{-2\pi t} \cdot \frac{1}{2\pi\xi}\sin(2\pi\xi t)\, dt$$

$$= \frac{2}{\xi}\int_0^\infty e^{-2\pi t}\sin(2\pi\xi t)\, dt$$

Integrating by parts

$$= \frac{2}{\xi}\left[e^{-2\pi t}\frac{1}{2\pi\xi}(-\cos(2\pi\xi t))\right]_0^\infty - \frac{4\pi}{\xi}\int_0^\infty e^{-2\pi t} \cdot \frac{1}{2\pi\xi}\cos(2\pi\xi t)\, dt$$

$$= \frac{2}{\xi}\cdot\frac{1}{2\pi\xi} - \frac{2}{\xi^2}\int_0^\infty e^{-2\pi t}\cos(2\pi\xi t)\, dt$$

Notice that we have obtained the following:

$$\left(2 + \frac{2}{\xi^2}\right)\int_0^\infty e^{-2\pi t}\cos(2\pi\xi t)\, dt = \frac{1}{\pi\xi^2}$$

Then,

$$\int_0^\infty e^{-2\pi t}\cos(2\pi \xi t)\,dt = \frac{1}{\pi \xi^2(2+2/\xi^2)} = \frac{1}{\pi(2+2\xi^2)}$$

Since $\hat{f}(\xi) = 2\int_0^\infty e^{-2\pi t}\cos(2\pi \xi t)\,dt$, we obtain that

$$\boxed{\hat{f}(\xi) = \frac{1}{\pi(1+\xi^2)}}$$

The proof of the hint is immediate taking into account $e^{-2\pi i \xi t} = \cos(2\pi \xi t) - i\sin(2\pi \xi t)$ and the parity of the functions.

b) Let $g \in \mathcal{C}(\mathbb{R}) \cap L^1(\mathbb{R})$. Find $u \in \mathcal{C}^2(\mathbb{R})$ such that $u, u', u'' \in L^1(\mathbb{R})$ and solving the differential equation

$$u'' - u = g$$

Prove also that $u(\infty) = 0$

Let us take Fourier transform in both sides of the equation.

$$\widehat{(u''-u)} = \hat{g} \iff \widehat{(u'')} - \hat{u} = \hat{g} \iff (2\pi i \xi)^2 \hat{u}(\xi) - \hat{u}(\xi) = \hat{g}(\xi)$$

$$\iff \hat{u}(\xi)[4\pi^2\xi^2 + 1] = -\hat{g}(\xi)$$

$$\iff \hat{u}(\xi) = -\frac{\hat{g}(\xi)}{1+4\pi^2\xi^2}$$

Notice that, $\hat{u}(\xi) = -\pi\,\hat{g}(\xi)\,\hat{f}(2\pi\xi)$

Let us define $\hat{F}(\xi) := -\pi \hat{f}(2\pi\xi)$. Observe that, since $g \in L^1(\mathbb{R})$

$\hat{u}(\xi) = \hat{g}(\xi)\hat{F}(\xi) = (g * F)^{\wedge}(\xi)$

We are going to take advantage of the convolution to obtain the function u and its properties. First of all, notice that, since $g \in L^1(\mathbb{R})$ and F, which is essentially f, also is in $L^1(\mathbb{R})$, we have that $g * F \in L^1(\mathbb{R})$. Then, by a corollary seen in class, since $g * F \in L^1(\mathbb{R})$ and $\widehat{(g*F)}(\xi) = \hat{u}(\xi)$, we have that $u(\xi) = (g * F)(\xi)$ a.e.

Now that we have our function u defined almost everywhere, let us obtain regularity properties.

Since $g \in L^1(\mathbb{R})$ and F, which is essentially f, is in $L^\infty(\mathbb{R})$, we have that $g * F \in \mathcal{C}(\mathbb{R})$. Also, we have just seen that $g * F \in L^1(\mathbb{R})$, then, $u \in \mathcal{C}(\mathbb{R})$ and $u \in L^1(\mathbb{R})$.

Using properties of the differentiability of the convolution and similar reasonings, we will obtain that $u \in \mathcal{C}^2(\mathbb{R})$ and $u, u', u'' \in L^1(\mathbb{R})$.

Let us prove the fact that $u(\infty) = 0$

We want to prove that

$$\lim_{\xi \to \infty} u(\xi) = \lim_{\xi \to \infty} \int_{\mathbb{R}} g(s) F(\xi - s)\, ds = 0$$

Notice that,

- $\lim_{\xi \to \infty} g(s) F(\xi - s) = 0$, since F it is esentially $f(t) = e^{-2\pi|t|}$
- $|g(s)||F(\xi - s)| \underset{\substack{\downarrow \\ |e^{-2\pi|t|}| \leq 1\ \forall t}}{\leq} |g(s)| \in L^1(\mathbb{R})$

By the dominated convergence theorem,

$$\lim_{z \to \infty} u(z) = \int_{\mathbb{R}} \lim_{z \to \infty} g(s) F(z-s)\, ds = \int_{\mathbb{R}} 0\, ds = 0$$

$$\Rightarrow \quad u(\infty) = 0$$

(10) Use the Fourier transform to compute

$$\int_{\mathbb{R}} \frac{\sin x}{x(x^2+1)}\,dx$$

Let us define $f(t) := e^{-2\pi|t|}$ and remember that (exercise 6 a))

$$\hat{f}(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}$$

Let us also define $g(t) := \chi_{[-1/2\pi,\, 1/2\pi]}(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.
Notice that

$$\hat{g}(\xi) = \int_{-1/2\pi}^{1/2\pi} e^{-2\pi i \xi t}\,dt = \left[\frac{e^{-2\pi i \xi t}}{-2\pi i \xi}\right]_{-1/2\pi}^{1/2\pi} = \frac{\sin(\xi)}{\pi \xi}$$

Moreover, since g and $\hat{g}$ are $L^2(\mathbb{R})$ functions and $\hat{g} \in \mathcal{C}(\mathbb{R})$, we have the following:

$$\check{g}(\xi) = \int_{-1/2\pi}^{1/2\pi} e^{2\pi i \xi t}\,dt = \left[\frac{e^{2\pi i \xi t}}{2\pi i \xi}\right]_{-1/2\pi}^{1/2\pi} = \frac{\sin(\xi)}{\pi \xi}$$

Therefore, $\hat{g}(\xi) = \check{g}(\xi) \Rightarrow \hat{\hat{g}}(\xi) = \hat{\check{g}}(\xi) = g(\xi)$

Now, let us compute the desire integral,

$$\int_{\mathbb{R}} \frac{\sin x}{x}\frac{1}{(x^2+1)}\,dx = \int_{\mathbb{R}} \frac{\sin \xi}{\xi}\,\pi \hat{f}(\xi)\,d\xi \overset{\left(\frac{\sin\xi}{\xi}\right),\, f \in L^2}{=}$$

$$= \pi \int_{\mathbb{R}} \widehat{\left(\frac{\sin \xi}{\xi}\right)} f(\xi)\,d\xi = \pi^2 \int_{\mathbb{R}} \hat{\hat{g}}(\xi) f(\xi)\,d\xi = \pi^2 \int_{\mathbb{R}} g(\xi) f(\xi)\,d\xi$$

$$= \pi^2 \int_{-1/2\pi}^{1/2\pi} e^{-2\pi|t|} \, dt = \pi^2 \left[\int_{-1/2\pi}^{0} e^{2\pi t} \, dt + \int_{0}^{1/2\pi} e^{-2\pi t} \, dt \right]$$

$$= \pi^2 \left[\frac{e^{2\pi t}}{2\pi} \right]_{-1/2\pi}^{1/2\pi} + \pi^2 \left[\frac{e^{-2\pi t}}{-2\pi} \right]_{-1/2\pi}^{1/2\pi} = \pi \left(\frac{1}{2} - \frac{1}{2e} + \frac{1}{2} - \frac{1}{2e} \right)$$

$$= \pi - \pi/e$$

$$\Rightarrow \quad \boxed{\int_{\mathbb{R}} \frac{\sin x}{x(x^2+1)} \, dx = \pi - \pi/e}$$