Exercises 1

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(3) Let Ω be a domain (take $\Omega = \mathbb{D}$) in $\mathbb{C}$ with $0 \in \Omega$ and let $\mathcal{F} \subset H(\Omega)$ be such that $\mathcal{F}'$ is a normal family. Prove that if $\sup\{|f(0)| ; f \in \mathcal{F}\} < \infty \Rightarrow \mathcal{F}$ normal in $H(\Omega)$

Let (f_n) be a sequence in $\mathcal{F}$. We want to see that there exist a subsequence that converge ucs.

Since $\mathcal{F}'$ is normal, there exist a subsequence f_{n_k} such that f'_{n_k} is convergent uniformly in compact sets of $\mathbb{D}$. Moreover, $\{f_{n_k}(0) : f \in \mathcal{F}\}$ is bounded and by Bolzano-Weierstrass has a convergent subsequence. Hence, by taking another subsequence, we have that

$$\left\{ \begin{array}{l} f'_{n_{k_j}} \xrightarrow[j\to\infty]{} g \quad \text{UCS in } \mathbb{D} \\ f_{n_{k_j}}(0) \xrightarrow[j\to\infty]{} w \end{array} \right. \quad (*)$$

Now, let us define, for $z \in \mathbb{D}$

$$G(z) = w + \int_{[0,z]} g(t)\,dt,$$

where $[0,z]$ is the line from 0 to z.

Notice that G is holomorphic and $G' = g$. Observe also, that the following equality is true:

$$f_{n_{k_j}}(z) - f_{n_{k_j}}(0) = \int_{[0,z]} f'_{n_{k_j}}(t)\,dt.$$

Therefore, the following equality is also true:

$$f_{mK_j}(z) - G(z) = f_{mK_j}(0) - w + \int_{[0,z]} (f_{mK_j}'(t) - g(t))\,dt$$

Then, if $K \subset \mathbb{D}$ compact, choose a closed disc $\overline{B}$ such that $0 \in \overline{B}$ and $K \subset \overline{B} \subset \mathbb{D}$. Then $[0,z] \subset \overline{B}$ for all $z \in K$ and

$$|f_{mK_j}(z) - G(z)| \leq |f_{mK_j}(0) - w| + \operatorname{diam}(\mathbb{D}) \sup_{t\in\overline{B}} |f_{mK_j}'(t) - g(t)|$$

$$= |f_{mK_j}(0) - w| + 2 \sup_{t\in\overline{B}} |f_{mK_j}'(t) - g(t)|.$$

Therefore, using ✱, we have that f_{mK_j} converge uniformly on compact sets.

□

7. A compact set $K \subset \mathbb{C}$ is polynomially convex if the polynomial convex hull

$$\hat{K} = \{z \in \mathbb{C} : |p(z)| \leq \sup_{w \in K} |p(w)|, \text{ for all } p \text{ polynomial}\}$$

satisfies $\hat{K} = K$. Prove that:

$$K \text{ polynomially convex} \iff \mathbb{C} \setminus K \text{ connected}$$

$\boxed{\Rightarrow}$

Let us state two general topology results.

I) If $U \subset \mathbb{C}$ is bounded $\Rightarrow$ $\mathbb{C} \setminus U$ has exactly one unbounded component

II) If $K \subset \mathbb{C}$ compact, every component C of $\mathbb{C} \setminus K$ satisfies that $\partial C \subset K$

Since a compact set is bounded, we can use I) to see that all the others connected components of $\mathbb{C}\setminus K$ are contained in a certain bounded set and therefore, they are bounded.

Let us prove the following lemma to obtain the desire result.

Lemma If $K\subset\mathbb{C}$ compact, then any bounded component of $\mathbb{C}\setminus K$ is contained in $\hat{K}$.

Proof of the lemma/

Assume that $\mathbb{C}\setminus K$ has a bounded component C. Any polynomial p is holomorphic in C and continous in $\bar{C}$. It follows from the maximumm modulus principle that, $\forall z\in C$

$$|p(z)| \leq \max_{w\in\partial C}|p(w)| \leq \max_{w\in K}|p(w)| .$$

The second inequality holds because, using II), $\partial C\subset K$. Therefore $z\in\hat{K}$, i.e, $C\subset\hat{K}$.

$\square$

Hence, if K is polynomially convex ($\hat{K}=K$), then $\mathbb{C}\setminus K$ can not have a bounded component (otherwise $\mathbb{C}\setminus K\subset K$!!). Therefore $\mathbb{C}\setminus K$ is connected.

$\boxed{\Leftarrow}$

$\square$

Reciprocally, let us see that K is polynomially convex, i.e., $K=\hat{K}$. To see this, we are going to show that $\mathbb{C}\setminus K=\mathbb{C}\setminus\hat{K}$.

Notice that, since $K \subseteq \hat{K}$ (obvious by the definition), we have that $\mathbb{C} \setminus \hat{K} \subseteq \mathbb{C} \setminus K$. Now, let us see that $\mathbb{C} \setminus K \subseteq \mathbb{C} \setminus \hat{K}$ and we will be done.

Let $z_0 \in \mathbb{C} \setminus K$. By hipothesis, $\mathbb{C} \setminus K$ is connected, hence, $\mathbb{C} \setminus (K \cup \{z_0\})$ is also connected. Let us define the function

$$f(z) := \begin{cases} 0 & \text{if } z \in K \\ 1 & \text{if } z = z_0 \end{cases}$$

which can be extended to a holomorphic function in a neighborhood of $K \cup \{z_0\}$.

Now, since $K \cup \{z_0\}$ is compact (the union of two compact sets is compact as well) and $\mathbb{C} \setminus (K \cup \{z_0\})$ is connected, by Runge's theorem there exist a polynomial p such that $|p - f| < \frac{1}{2}$ on $K \cup \{z_0\}$. The polynomial satisfies

$$|p(z_0)| > \frac{1}{2} > \max_{z \in K} |p(z)|.$$

Then $z_0 \in \mathbb{C} \setminus \hat{K}$

$\square$

1) Let $\{f_n\}_n$ be a normal sequence of holomorphic functions in $\mathbb{D}$, and for any $n \geq 0$, let $f_n(z) = \sum_k c_{n,k} z^k$ be its Taylor expansion around the origin. Prove that

$$\lim_n f_n = f_0 \text{ in } H(\mathbb{D}) \iff \forall k \geq 0, \ \lim_n c_{n,k} = c_{0k}$$

$\boxed{\Rightarrow}$

From left to right, is an immediate consequence of Weierstrass theorem, let us see it.

Recall first that the coefficients of the Taylor expansion are essentially the derivatives of the function, more precisely,

$$c_{n,k} = \frac{f_n^{(k)}(0)}{k!}.$$

Now, since $\lim_n f_n = f_0$ in $H(\mathbb{D})$, by Weierstrass theorem,

$$\forall k > 0, \ \lim_n f_n^{(k)} = f_0^{(k)} \text{ in } H(\mathbb{D}).$$

Since the derivatives converge uniformly on compact sets of $\mathbb{D}$, when we evaluate at the origin, we obtain pointwise convergence at that point, i.e.,

$$\forall k > 0, \ \lim_n f_n^{(k)}(0) = f_0^{(k)}(0).$$

Therefore,

$$\lim_n c_{n,k} = c_{0,k} \quad \forall k > 0.$$

$\square$

$\boxed{\Leftarrow}$

Reciprocally, assume that $\lim_m c_{m,k} \overset{*}{=} c_{0,k}$.

Since $\{f_m\}$ is normal, for every subsequence m_j, $\{f_{m_j}\}$ is also normal. By definition of normality, it exists a further subsequence m_{j_π} such that $\lim_\pi f_{m_{j_\pi}} = f$ in $H(\mathbb{D})$, where f is an holomorphic function.

Consider the Taylor expansion of f around the origin,

$$f(z) = \sum_k c_k z^k.$$

Using the fact that we have already proved, since $\lim_\pi f_{m_{j_\pi}} = f$ in $H(\mathbb{D})$, $\lim_\pi c_{m_{j_\pi},k} \overset{*}{=} c_k \quad \forall k \geq 0$.

Moreover, since $\lim_m c_{m,k} = c_{0,k}$, $\{c_{m,k}\}$ is Cauchy, i.e.,

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall m, m_{j_\pi} > N$, $|c_{m_{j_\pi},k} - c_{m,k}| \overset{*}{<} \varepsilon$.

Then,

$$|c_{0,k} - c_k| \leq \overset{*}{|c_{0,k} - c_{m,k}|} + \overset{*}{|c_{m,k} - c_{m_{j_\pi},k}|} + \overset{*}{|c_{m_{j_\pi},k} - c_k|} \lesssim \varepsilon.$$

Therefore, $c_k = c_{0,k}$ and $f(z) = \sum_k c_k z^k = \sum_k c_{0,k} z^k = f_0(z)$.

We have proved that, for every subsequence of $\{f_m\}$, exists a further subsequence such that converge to f_0 in $H(\mathbb{D})$. Then, $\{f_m\}$ converge to f_0 in $H(\mathbb{D})$, i.e.,

$$\lim_m f_m = f_0 \quad \text{in } H(\mathbb{D}). \qquad \square$$