Exercise set # 2.1

a) *Let $U \subset \mathbb{R}^n$ be an open set and $x_0 \in U$. Let $f : U \to \mathbb{R}^n$ be continuous with $f(x_0) = x_0$. If there exists a Liapunov function associated to f and x_0 then x_0 is stable.*

Proof. In order to show that x_0 is stable, we need to prove that:

$$\forall \epsilon, \exists \delta > 0 \text{ such that if } \ ||x - x_0|| < \delta,\ ||f^n(x) - x_0|| < \epsilon \ \forall n \geq 0.$$

Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that $\overline{B(x_0, r)} \subset U$, where $B(x_0, r)$ is the ball of radius r centered in x_0, i.e., $B(x_0, r) := \{x \in \mathbb{R}^n : ||x - x_0|| < r\}$. Let $V : U \to \mathbb{R}^n$ the Liapunov function associated to f and x_0. Let us define the following value,

$$m := \min_{x \in \partial B(x_0, r)} V(x).$$

Observe that, using point 2) of the definition 0.1, we have that $m > 0$. Now, let us define the set $\Omega_m := \{x \in B(x_0, r) : V(x) < m\}$. Observe that:

- $\Omega_m \neq \emptyset$ since $x_0 \in \Omega_m$.
- $\Omega_m \subset B(x_0, r)$.

Now, let C_{x_0} be the connected component of x_0 in Ω_m. Let us prove the following lemma

Lemma 0.1. *With the introduced notation, $f(C_{x_0}) \subseteq C_{x_0}$. In particular, $f^n(C_{x_0}) \subseteq C_{x_0}$ for every $n \geq 1$.*

Proof. To prove this lemma, let us first see that $f(C_{x_0}) \subseteq B(x_0, r)$. We will proceed by contradiction, assume that $f(C_{x_0}) \not\subseteq B(x_0, r)$. In that case, there would be a value $y \in C_{x_0}$ such that $f(y) \in \partial B(x_0, r)$, otherwise, the sets

$$\begin{aligned} S &:= \{x \in C_{x_0} : f(x) \in B(x_0, r)\} = C_{x_0} \cap f^{-1}(B(x_0, r)) \\ R &:= \{x \in C_{x_0} : f(x) \notin \overline{B(x_0, r)}\} = C_{x_0} \cap f^{-1}(U \backslash \overline{B(x_0, r)}), \end{aligned}$$

will be non-empty, since $x_0 \in S$ and we are assuming that $f(C_{x_0}) \not\subseteq B(x_0, r)$. Moreover, since f is continuous, R and S are open sets and $f(C_{x_0})$ is connected. Hence, we have two non-empty disjoint sets such that $R \cup S = f(C_{x_0})$, in contradiction with the connectivity.

Therefore, $f(y) \in \partial B(x_0, r)$ and, since $\overline{B(x_0, r)} \subset U$, we can use point 3) of the definition 0.1. to see that

$$m < V(f(y)) \leq V(y) < m. \tag{0.1}$$

The first inequality is due to the definition of m, and the last one, is due to the fact that $y \in C_{x_0} \subseteq \Omega_m$. We have reached a contradiction! Therefore, we have that $f(C_{x_0}) \subseteq B(x_0, r)$.

Now, since $f(C_{x_0}) \subseteq B(x_0, r)$, we can apply again the inequality used in (0.1) to see that $f(C_{x_0}) \subseteq \Omega_m$. Hence, $f(C_{x_0})$ is a connected set contained in Ω_m, such that $x_0 \in f(C_{x_0})$ (since x_0 is a fixed point), this implies that $f(C_{x_0}) \subseteq C_{x_0}$. □

Let us finish the proof. Since V is continuous and $V(x_0) = 0$, we have that

$$\exists 0 < \delta < r \text{ such that if } ||x - x_0|| < \delta, |V(x) - V(x_0)| = |V(x)| < m.$$

Now, since $\delta < r$, we have that $B(x_0, \delta) \subset \Omega_m$. Moreover $B(x_0, \delta)$ is a connected set such that $x_0 \in B(x_0, \delta)$, hence, using the same connectivity argument, $B(x_0, \delta) \subseteq C_{x_0}$. Therefore,

$$x \in B(x_0, \delta) \implies x \in C_{x_0} \implies f^n(x) \in C_{x_0} \implies f^n(x) \in \Omega_m \implies f^n(x) \in B(x_0, r),$$

i.e.,

$$\text{if } ||x - x_0|| < \delta, ||f^n(x) - x_0|| < r \leq \epsilon.$$

□

b) *If there exists a Liapunov function associated to f and x_0 and if no positive semiorbit $\mathcal{O}_+(x) = \{f^k(x) : k \geq 0\}$ is contained in Z, except $\mathcal{O}_+(x_0) = \{x_0\}$, then x_0 is asymptotically stable.*

Proof. Let $V : U \to \mathbb{R}^n$ be the Liapunov function associated to f and x_0. Using the previous exercise, we have that x_0 is stable. It is enough to see that,

$$\exists \eta > 0 \text{ such that if } \ ||x - x_0|| < \eta, \lim_{n\to\infty} f^n(x) = x_0.$$

Let us define the ω-limit set of a point as

$$\omega(x) := \{y \in U : \exists n_k \to \infty \text{ such that } \lim_{k\to\infty} f^{n_k}(x) = y\}.$$

Let $\epsilon > 0$ be such that $\overline{B(x_0, \epsilon)} \subset U$ and $\delta > 0$ be given by the definition of stability, depending on ϵ. Let $x \in B(x_0, \delta)$. Notice that $V(f^n(x))$ is non-increasing since, using point 3) of the definition 0.1., we have that

$$V(f^n(x)) \leq V(f^{n-1}(x)) \leq \cdots \leq V(x).$$

Also, notice that $V(f^n(x))$ is well defined due to the stability of x_0. Now, using point 2) of the definition 0.1., we can see that $V(f^n(x))$ is bounded below, hence,

$$\exists c \in \mathbb{R} \text{ such that } V(f^n(x)) \xrightarrow[n\to\infty]{} c. \tag{0.2}$$

Consider now $y \in \omega(x)$. By definition of ω-limit,

$$\exists n_k \to \infty \text{ such that } f^{n_k}(x) \xrightarrow[k\to\infty]{} y. \tag{0.3}$$

Notice that, from (0.2), since $V(f^n)$ converges to c, the subsequence also converges, i.e.,

$$V(f^{n_k}(x)) \xrightarrow[k\to\infty]{} c.$$

and, from (0.3), using that V is continuous, we have that

$$V(f^{n_k}(x)) \xrightarrow[k\to\infty]{} V(y).$$

By uniqueness of limits, $V(y) = c$. Since the ω-limit is invariant, $f(y) \in \omega(x)$, and, using the same argument, $V(f(y)) = c$.

Therefore, if $y \in \omega(x)$, $\Delta V(y) = V(f(y)) - V(y) = c - c = 0$, i.e., $\omega(x) \subset Z$. Then, as the only positive semiorbit contained in Z is $\mathcal{O}_+(x_0) = \{x_0\}$,

$$\lim_{n\to\infty} f^n(x) = x_0.$$

□