COMPLEX ANALYSIS

DAVID ROSADO RODRÍGUEZ

Exercises 4

(4) Let f be entire (i.e. $f \in \mathcal{H}(\mathbb{C}^n)$). Prove:

a) If f is bounded then it is constant (Liouville's theorem)

By Liouville's theorem in one variable, the restriction of a bounded holomorphic function f to each complex line through the origin is constant (since this is a function in $\mathbb{C}$ and Liouville's theorem in one variable apply).

Therefore, $f(z) \equiv f(0)$ on $\mathbb{C}^n$

$\square$

b) If $|f(z)| \leq A + B|z^\alpha|$ $\forall z \in \mathbb{C}^n$, then f is a polynomial of the form $f(z) = \sum_{\beta \leq \alpha} c_\beta z^\beta$

Since f is entire, we have that $f(z) = \sum_{\beta \in \mathbb{N}^n} \frac{\partial^\beta f(0)}{\beta!} z^\beta$.

Since Cauchy formula holds in several variables, Cauchy estimates also hold and therefore,

$$|\partial^\beta f(0)| \leq \frac{\beta!}{r^\beta} \sup_{z \in D^n(0,r)} |f(z)| \overset{|f(z)| \leq A + B|z^\alpha|}{\leq} \frac{\beta!}{r^\beta}(A + Br^\alpha).$$

Then, for every $\beta > \alpha$, $|\partial^\beta f(0)| \leq \frac{\beta!}{r^\beta} A + \frac{\beta!}{r^{\beta-\alpha}} B \xrightarrow{r \to \infty} 0.$

Hence,

$$f(z) = \sum_{\beta \leq \alpha} \frac{\partial^\beta f(0)}{\beta!} z^\beta = \sum_{\beta \leq \alpha} c_\beta z^\beta.$$

$\square$

⑤ If $D \subset \mathbb{C}^m$ is the domain of convergence of the power series $\sum_{\alpha \in \mathbb{N}^m} c_\alpha z^\alpha$ then

$$\Lambda := \{ (\log|z_1|, \dots, \log|z_m|) : z \in D \}$$

is convex in $\mathbb{R}^m$.

Since D is the domain of convergence of the power series, D is a complete Reinhardt domain.

Now, let $z, w \in D$, since D is open, we can take $\lambda > 1$ such that $\lambda z, \lambda w \in D$.

Since $\lambda z, \lambda w \in D$, $\sup_{\alpha \in \mathbb{N}^m} \{ |c_\alpha| \lambda^{|\alpha|} |z^\alpha|, |c_\alpha| \lambda^{|\alpha|} |w^\alpha| \} \le C$ for some $C > 0$. Then, $\sup_{\alpha \in \mathbb{N}^m} \{ |c_\alpha| \lambda^{|\alpha|} |z^\alpha|^t |w^\alpha|^{(1-t)} \} \le C$ $\forall t \in [0,1]$. Indeed,

$$|c_\alpha| \lambda^{|\alpha|} |z^\alpha|^t |w^\alpha|^{(1-t)} = \overbrace{|c_\alpha| \lambda^{|\alpha|} |w^\alpha|}^{\le C} \overbrace{|c_\alpha|^t \lambda^{|\alpha| t} |z^\alpha|^t}^{\le C^t} \overbrace{|c_\alpha|^{-t} \lambda^{-|\alpha| t} |w^\alpha|^{-t}}^{\le C^{-t}}$$

$$\le C \, C^t \, C^{-t} = C.$$

By Abel's lemma, the power series $\sum_\alpha c_\alpha z^\alpha$ converge ucs in $\{ x \in \mathbb{C}^m : |x_i| < |\lambda_i z_i w_i|, i = 1, \dots, m \}$. In particular, since $\lambda > 1$, the series converges at the point

$$\mathfrak{z}_t := (|z_1|^t |w_1|^{1-t}, \dots, |z_m| |w_m|^{1-t}).$$

Therefore, $\mathfrak{z}_t \in D$ and

$$t \log|z| + (1-t) \log|w| \in \{ (\log|z_1|, \dots, \log|z_m|) : z \in D \}$$

and the set Λ is convex

□