HARMONIC ANALYSIS

(2) Consider the Hilbert space of entire functions

$$\mathcal{F} = \left\{ f \in H(\mathbb{C}) : \|f\|^2 := \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \frac{dm(z)}{\pi} < \infty \right\}$$

a) Prove that $K_\lambda(z) = \sum_{m \geq 0} \overline{e_m(\lambda)} e_m(z)$, where $\{e_m\}_{m\geq 0}$ is any orthonormal basis of $\mathcal{F}$

Given any orthonormal basis of $\mathcal{F}$, $\{e_m\}_{m\geq 0}$, since $K_\lambda \in \mathcal{F}$, there exist some coefficients $a_m(\lambda)$ such that

$$K_\lambda(z) = \sum_{m\geq 0} a_m(\lambda) e_m(z). \qquad (0)$$

Since $\{e_m\}_{m\geq 0}$ is an orthonormal basis,

$$a_m(\lambda) = \langle K_\lambda, e_m \rangle = \overline{\langle e_m, K_\lambda \rangle} = \overline{e_m(\lambda)}. \qquad (1)$$

The last equality comes from the Riesz representation theorem, more precisely, by the Riesz representation, $\exists K_\lambda \in \mathcal{F}$ such that

$$f(\lambda) = \langle f, K_\lambda \rangle, \ \forall f \in \mathcal{F}.$$

Hence, using the expression (1) and replacing it in (0), we obtain

$$K_\lambda(z) = \sum_{m\geq 0} \overline{e_m(\lambda)} e_m(z)$$

b) Prove that $\{ z^m/\sqrt{m!} \}_{m\geq 0}$ is an orthonormal basis and deduce the value of $K_\lambda(z)$

Assuming that $\{ z^m/\sqrt{m!} \}_{m\geq 0}$ is an orthonormal basis, we can deduce the value of $K_\lambda(z)$.

$$K_\lambda(z) = \sum_{m\geq 0} \overline{e_m(\lambda)}\, e_m(z) = \sum_{m\geq 0} \frac{\overline{\lambda}^m z^m}{\sqrt{m!}\sqrt{m!}} = \sum_{m\geq 0} \frac{(\overline{\lambda} z)^m}{m!}$$

$$= \exp(\overline{\lambda} z)$$

c) Let $\Lambda = \{\lambda_k\}_{k\geq 1}$ be a discrete sequence in $\mathbb{C}$. Prove that the family of normalised reproducing kernels $k_{\lambda_k} = K_{\lambda_k}/\|K_{\lambda_k}\|$, $k\geq 1$ is a frame for $\mathcal{F}$ if and only if there exist $A, B > 0$ such that

$$A\|g\|^2 \leq \sum_{k\geq 1} |g(\lambda_k)|^2 e^{-|\lambda_k|^2} \leq B\|g\|^2, \forall g \in \mathcal{F}$$

Recall that k_{λ_k} is a frame for $\mathcal{F}$ if $\exists A, B > 0$ such that

$$A\|g\|^2 \leq \sum_{k=1}^{\infty} |\langle g, k_{\lambda_k}\rangle|^2 \leq B\|g\|^2 \quad \forall g \in \mathcal{F}$$

By the Riesz representation theorem,

$$|g(\lambda_k)|^2 = |\langle g, K_{\lambda_k}\rangle|^2$$

Since $k_{\lambda_k} = K_{\lambda_k}/\|K_{\lambda_k}\|$ and $\|K_{\lambda_k}\|^2 = \langle K_{\lambda_k}, K_{\lambda_k}\rangle = K_{\lambda_k}(\lambda_k)$
$= \exp(\overline{\lambda_k}\lambda_k) = \exp(|\lambda_k|^2)$, we obtain that

$$\sum_{k=1}^{\infty} |g(\lambda_k)|^2 e^{-|\lambda_k|^2} = \sum_{k=1}^{\infty} |\langle g, k_{\lambda_k}\rangle|^2$$

Hence,

$$\sum_{k=1}^{\infty} |g(\lambda_k)|^2 e^{-|\lambda_k|^2} = \|\bar{K}_{\lambda_k}\|^2 \sum_{k=1}^{\infty} |\langle g, K_{\lambda_k}\rangle|^2$$

Therefore, defining $C := \|\bar{K}_{\lambda_k}\|^2 > 0$, we have that

$$CA\|g\|^2 \leq \sum_{k\geq 1} |g(\lambda_k)|^2 e^{-|\lambda_k|^2} \leq CB\|g\|^2 \quad \forall g \in \mathcal{F}$$