Problem set # 1

Exercise 1. *Consider the following homeomorphism of the circle*

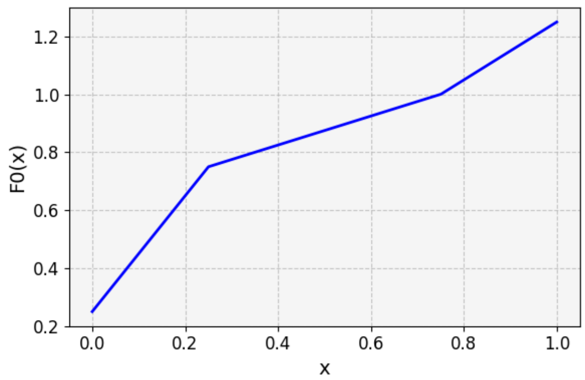
$$f(x) = \begin{cases} \frac{1}{4} + 2x \pmod 1 & \text{if } x \in [0, \frac{1}{4}] \\ \frac{5}{8} + \frac{x}{2} \pmod 1 & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ x + \frac{1}{4} \pmod 1 & \text{if } x \in [\frac{3}{4}, 1] \end{cases}$$

Draw a lift of f and compute its rotation number.

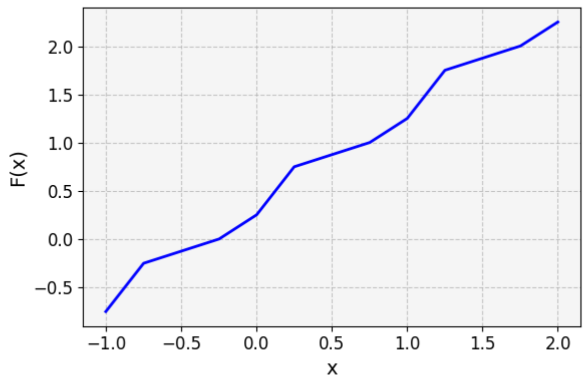
Proof. Let us define the following function:

$$F_0(x) = \begin{cases} \frac{1}{4} + 2x & \text{if } x \in [0, \frac{1}{4}] \\ \frac{5}{8} + \frac{x}{2} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ x + \frac{1}{4} & \text{if } x \in [\frac{3}{4}, 1] \end{cases}$$

This function clearly satisfies that $\pi \circ F_0 = f \circ \pi$ in the interval $[0, 1]$, where $\pi : \mathbb{R} \mapsto S^1$ is the covering map. We just need to extend this function to $\mathbb{R}$ continuously so that we end up in a lift of f. To extend it, just take $F(x) = F_0(x - n) + n$ for every interval $[n, n + 1]$[1]. Figure 1, shows the function F_0 and F.



(a) $F_0(x)$



(b) The function $F(x)$ in [-1,1]

Figure 1: At the left, the function F_0 and at the right, the lift in the interval $[-1, 1]$.

For the rotation number, notice that $f^3(0) = 0$, hence, the rotation number is either $\frac{1}{3}$ or $\frac{2}{3}$. Since $x = 0$ covers the circle just once, the rotation number is $\frac{1}{3}$. ☐

Exercise 2. *Consider $F_1(x) := x + \frac{1}{2} sin(2\pi x)$ and $F_2(x) := x + \frac{1}{4\pi} sin(2\pi x)$. Decide whether F_1 and F_2 are lifts of circle homeomorphisms. If so, decide whether that homeomorphism is orientation preserving. If it is, determine the rotation number.*

Proof. We claim that F_1 is not a lift of a circle homeomorphisms, whereas F_2 is indeed such a lift. To see that F_1 is not a lift, observe that there exist a value x^* such that $F_1'(x^*) = 0$ and $F_1''(x^*) \neq 0$. This implies that x^* is a local extrema of the function and therefore, it can not be invertible (one-to-one fails). Let us find that value:

$$F_1'(x) = 1 + \pi sin(2\pi x) = 0 \iff sin(2\pi x) = \frac{-1}{\pi}.$$

[1]Essentially, this is copying the function through the intervals

Hence, there are many values for which $F_1'(x) = 0$, we just need one, $x^* := \frac{arcsin(\frac{-1}{\pi})}{2\pi}$.

On the other hand, F_2 is indeed a lift. To see this, we will prove that F_2 is a lift of

$$f(\theta) = \theta + \frac{1}{4\pi} sin(2\pi\theta) \text{ (mod 1)}$$

Clearly, $\pi \circ F_2 = f \circ \pi$, where $\pi : \mathbb{R} \mapsto S^1$ is the covering map. It remains to see that f is indeed an homeomorphism. Notice that

$$f'(\theta) = 1 + \frac{1}{2} cos(2\pi\theta) \geq 1 - \frac{1}{2} > 0,$$

so f is strictly increasing and therefore, one-to-one. The function is clearly surjective and since S^1 is Hausdorff and compact, the continuous bijective function f becomes an homeomorphism.

Observe that the degree of the circle homeomorphisms is equal to $F_2(x+1) - F_2(x) = 1$, so it is orientation preserving. Finally, since $f(0) = 0$, f has a fixed point and the rotation number is zero. □

Exercise 3.*Let $f(\theta) = \theta + \frac{\epsilon}{2\pi} sin(2\pi n\theta)$ (mod 1) for $0 < \epsilon < 1/n, n \in \mathbb{N}$. Find an expression for the lifts F. Calculate the periodic points of f and determine their character. Draw the phase portrait of f and calculate its rotation number.*

Proof. The family of lifts are

$$F_k(x) = x + \frac{\epsilon}{2\pi} \sin(2\pi nx) + k, \;\; k \in \mathbb{Z}$$

since $\pi \circ F_k = \exp(2\pi F_k(x)) = f(\exp(2\pi x)) = f \circ \pi$. Now, to find the periodic points of f, let us first find the fixed points, i.e., the values θ such that $f(\theta) = \theta$. We need to solve the following equation:

$$\frac{\epsilon}{2\pi} \sin(2\pi n\theta) = 0 \;\; (\text{mod } 1) \iff \sin(2\pi n\theta) = \frac{2\pi m}{\epsilon}, \;\; m \in \mathbb{Z}.$$

Since $\epsilon < \frac{1}{n}$, we have that $\frac{2\pi m}{\epsilon} > 2\pi mn$, but $\sin(2\pi n\theta) \in [-1, 1]$. This necessarily implies that $m = 0$ and therefore, the equation to solve is

$$\sin(2\pi n\theta) = 0.$$

The solutions of the equation are $\theta = \frac{k}{2n}$, with $k \in \{1, \ldots, 2n\}$ (since $\theta \in [0, 1)$). Therefore, the fixed points of f are $p_k = \frac{k}{2n}$, with $k \in \{1, \ldots, 2n\}$.

Observe that there are not periodic points of period greater or equal than 2. To see this, observe first that, since $\epsilon < \frac{1}{n}$, f is strictly increasing:

$$f'(\theta) = 1 + n\epsilon \cos(2\pi nx) \geq 1 - n\epsilon > 0. \tag{0.1}$$

Now, assume we have a periodic point of f of period $r > 1$, i.e., x is fixed by f^r, but not for any f^i, $0 \leq i < r$. In that case, either $x < f(x) < f^2(x) < \cdots < f^r(x) = x$ or $x > f(x) > f^2(x) > \cdots > f^r(x) = x$, which gives a contradiction.

Let us continue determining the character of the fixed points. Taking the derivative of f, see (0.1), we obtain that

$$f'(p_k) = 1 + (-1)^k n\epsilon.$$

Hence, if k is even, $f'(p_k) > 1$ and if it is odd, $f'(p_k) \in (0,1)$. Thus, p_k is attractor when k is odd and repelling when k is even. See Figure 2 to visualize the phase portrait of f.

Finally, observe that since $f(0) = 0$, f has a fixed point and the rotation number is zero.

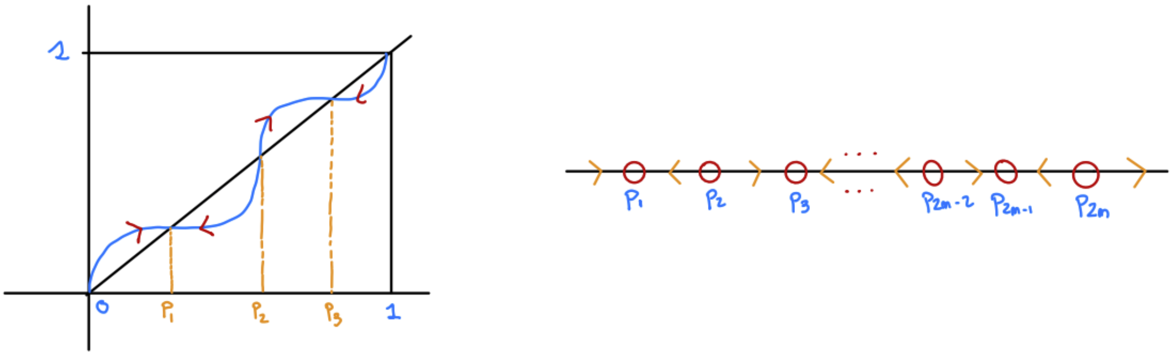


Figure 2: At the left, an example for $n = 2$ of the phase portrait of f. At the right, the general phase portrait of the function f.

□

Exercise 4. *Let f be an orientation preserving homeomorphism of the circle. Show that all periodic orbits of f must have the same period. Is this also true for orientation reversing homeomorphisms? Prove it or give a counterexample.*

Proof. Let us strart proving that all periodic orbits of f (OPH) must have the same period. Assume that there are two periodic orbits of f with different periods, m and n, $m \neq n$. As seen in class, we have that

$$\rho(f) = \frac{k}{m}, \text{ with } k \in \{1, \dots, m-1\},$$
$$\rho(f) = \frac{k'}{n}, \text{ with } k' \in \{1, \dots, n-1\}.$$

First, we will assume that $(k, m) = (k', n) = 1$. Since the rotation number is unique, $\frac{k}{m} = \frac{k'}{n}$. This implies that $nk = mk'$. On the one hand, $n|mk'$, and since $(n, k') = 1$, $n|m$. With the same reasoning, $m|nk$, and since $(m, k) = 1$, $m|n$. Therefore, since $m|n$ and $n|m$, $n = m$ which is a contradiction.

Now, for the general case, assume that $(k, m) \neq 1 \neq (k', n)$. Then, one can reduce the fraction to obtain

$$\rho(f) = \frac{\hat{k}}{\hat{m}}, \text{ with } \hat{k} \in \{1, \dots, \hat{m}-1\},$$
$$\rho(f) = \frac{\hat{k}'}{\hat{n}}, \text{ with } \hat{k}' \in \{1, \dots, \hat{n}-1\}.$$

with $(\hat{k}, \hat{m}) = (\hat{k}', \hat{n}) = 1$. Then, the map actually have periodic orbits of period $\hat{m}$ and $\hat{n}$. Using the previous case, one can see that $\hat{n} = \hat{m}$ and then, all periodic orbits must have the same period.

For orientation reversing homeomorphisms is not true. For instance, take the function $f(\theta) = -\theta \pmod 1$. Notice that f has two fixed points (0 and $\frac{1}{2}$) and all other points have period 2. $\square$

Exercise 5.(The Arnold family of circle maps) *Given $\alpha, \epsilon \in [0,1)$ and $\theta \in [0,1)$, consider the circle map*

$$f_{\alpha,\epsilon}(\theta) = \theta + \alpha + \frac{\epsilon}{2\pi}\sin(2\pi\theta) \pmod 1.$$

Let $\rho(f_{\alpha,\epsilon})$ denote the rotation number of the map $f_{\alpha,\epsilon}$. Fixed $\epsilon \in (0,1)$, and writing $f_\alpha = f_{\alpha,\epsilon}$, the graph of $\alpha \mapsto \rho(f_\alpha)$ is a devil's staircase since it increases from 0 to 1 continuously, while having a derivative equal to 0 almost everywhere.

a)*Show that the map $\alpha \mapsto \rho(f_\alpha)$ is not absolutely continuous.*

Proof. Let us call that map $G : [0,1) \mapsto \mathbb{R}$. Assume that G is absolutely continuous. Since G has a derivative G' almost everywhere, an equivalent definition of absolute continuity is that

$$G(\alpha) = G(0) + \int_0^\alpha G'(t)dt.$$

However, since G' is equal to 0 almost everywhere, the integral is zero and $G(\alpha) = G(0)$ for every $\alpha \in [0,1)$, which is a contradiction since G is a strictly increasing function. $\square$

Exercises 5b) and 5c) are in the attached python jupyter notebook!