## Exercise set # 2.2

***Exercise 1***. *Consider the equation*

$$\begin{cases} x' = x - y - x(x^2+y^2) \\ y' = x + y - y(x^2+y^2) \end{cases}$$

*Compute the Poincaré map of it with respect to the section* $\Sigma = \{(x,y) \in \mathbb{R}^2 : x > 0, y = 0\}$ *in explicit form. Hint: Use polar coordinates.*

*Proof.* Let us follow the hint and use polar coordinates to simplify the system of differential equations. Let $(r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^1$, and consider the following change of variables

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

Since $x^2+y^2 = r^2$ and $\tan\theta = \frac{y}{x}$, taking derivatives, one can obtain the following formula for time derivative of polar coordinates

$$\begin{cases} r' = \frac{xx'+yy'}{r}, \\ \theta' = \frac{xy'-yx'}{r^2}. \end{cases}$$

Therefore,

$$\begin{aligned} rr' &= x(x-y-x(x^2+y^2)) + y(x+y-y(x^2+y^2)) = x^2+y^2-(x^2+y^2)^2 = r^2 - r^4, \\ r^2\theta' &= x(x+y-y(x^2+y^2)) - y(x-y-x(x^2+y^2)) = x^2+y^2 = r^2, \end{aligned}$$

and the initial system of differential equations becomes

$$\begin{cases} r' = (1-r^2)r, \\ \theta' = 1. \end{cases}$$

Since $(r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^1$, the mentioned section $\Sigma = \{(x,y) \in \mathbb{R}^2 : x > 0, y = 0\}$ becomes $\Sigma = \{(r,\theta) : r > 0, \theta = 0\}$. Notice that every point in $\Sigma$ returns to the section at $t = 2\pi$, hence, we can take as Poincaré map the restriction of the flow to the section $\Sigma$ computed at the time $2\pi$. Therefore, if we are able to compute the flow of the system, we will be done. Integrating, for the component $\theta$ we simply have $\theta(t) = \theta_0 + t$ and for the $r$ component we need to separate the variables and integrate:

$$\int \frac{1}{(1-r^2)r} dr = \int dt \implies \log\left(\frac{r}{\sqrt{1-r^2}}\right) = t + c.$$

Applying the exponential function to both sides of the last equation, we obtain

$$r(t) = \sqrt{\frac{e^{2(t+c)}}{1+e^{2(t+c)}}}.$$

Since $r_0 = r(0) = \sqrt{\frac{e^{2t}}{1+e^{2t}}}$, we can write $r(t)$ as

$$r(t) = \sqrt{\frac{e^{2t}r_0^2}{1+r_0^2(e^{2t}-1)}} = \sqrt{\frac{1}{1+e^{-2t}\left(\frac{1}{r_0^2}-1\right)}}.$$

Therefore, the flow of the system is

$$\Phi(r,\theta) = \left(\theta + t, \sqrt{\frac{1}{1+e^{-2t}\left(\frac{1}{r_0^2}-1\right)}}\right).$$

As previously mentioned, the Poincaré map is given by $\Phi_{2\pi}|_{\Sigma}$

$$P(r) = \sqrt{\frac{1}{1+e^{-4\pi}\left(\frac{1}{r^2}-1\right)}}$$

□

***Exercise 2.*** *Let $f(x) = \lambda x + bx^2$ be a map from $\mathbb{R}$ to $\mathbb{R}$ with $|\lambda| \neq 0, 1$. Compute the Taylor expansion of a conjugation $h$ between $f$ and $Ax = \lambda x$, such that $h(0) = 0$ and $h'(0) = 1$, up to order 3. Do you think it is possible to find the Taylor expansion to all orders? If so, are the coefficients uniquely determined?*

*Proof.* Let us start by computing the Taylor expansion of the conjugation up to order 3. Let $h$ be a conjugation between $f$ and $g(x) = \lambda x$ such that $h(0) = 0$ and $h'(0) = 1$. Then, the following diagram must be commutative:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\ h\downarrow & & \downarrow h \\ \mathbb{R} & \xrightarrow{g} & \mathbb{R} \end{array}$$

Consider the Taylor expansion of $h$ with $h(0) = 0$ and $h'(0) = 1$ up to order 3, i.e.,

$$h(x) = h(0) + h'(0)x + \frac{h''(0)}{2}x^2 + \frac{h'''(0)}{6}x^3 = x + \frac{h''(0)}{2}x^2 + \frac{h'''(0)}{6}x^3.$$

Let us define $\alpha := h''(0)$ and $\beta := h'''(0)$. Due to the commutative diagram, $h \circ f = g \circ h$, i.e.,

$$(\lambda x + bx^2) + \frac{\alpha}{2}(\lambda x + bx^2)^2 + \frac{\beta}{6}(\lambda x + bx^2)^3 = \lambda(x + \frac{\alpha}{2}x^2 + \frac{\beta}{6}x^3). \tag{0.1}$$

Matching coefficients of degree 2, we have that

$$b + \frac{\alpha\lambda^2}{2} = \frac{\alpha\lambda}{2}, \tag{0.2}$$

and matching coefficients of degree 3, we have that

$$\alpha\lambda b + \frac{\beta\lambda^3}{6} = \frac{\beta\lambda}{6}. \tag{0.3}$$

Hence, isolating $\alpha$ and $\beta$ from (0.2) and (0.3) respectively, we have that

$$\alpha = \frac{2b}{\lambda(1-\lambda)},$$

$$\beta = \frac{6\alpha b}{1-\lambda^2} = \frac{12b^2}{\lambda(1-\lambda)(1-\lambda^2)}.$$

Notice that it is possible to derive the Taylor expansion to any degree, and the coefficients are uniquely defined. This fact is evident through the expression (0.1). Notice that each $h^{(n)}(0)$ can be computed by utilizing $h^{(n-1)}(0)$ and considering the condition $0 \neq \lambda \neq 1$. □

***Exercise 3.*** *Consider the map*

$$f(x,y) = (\lambda x, \lambda^2 y + x^2), \qquad 0 < \lambda < 1.$$

*Prove that $f$ cannot be linearized with a $\mathcal{C}^2$ conjugation.*

*Proof.* Let us proof that $f$ can not be locally conjugated to its linear part by a $\mathcal{C}^2$ conjugation.

First, notice that $f(0,0) = (0,0)$ and that $(0,0)$ is an hyperbolic fixed point, since $\lambda \in (0,1)$ :

$$A = Df(0,0) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{pmatrix}.$$

We claim that $f^k(x,y) = (\lambda^k x, \lambda^{2k} y + k\lambda^{2(k-1)} x^2)$. This can be easily prove it by induction. For $k = 1$ is clear. Now, assume the result true for $k$ and let us see it for $k+1$.

$$\begin{aligned} f^{k+1}(x,y) &= f(\lambda^k x, \lambda^{2k} y + k\lambda^{2(k-1)} x^2) \\ &= (\lambda^{k+1} x, \lambda^2(\lambda^{2k} y + k\lambda^{2(k-1)} x^2) + \lambda^{2k} x^2) \\ &= (\lambda^{k+1} x, \lambda^{2(k+1)} y + (k+1)\lambda^{2k} x^2). \end{aligned}$$

Let $g(x) = Ax$ and assume $h$ be a $\mathcal{C}^2$ conjugation such that $h \circ f = g \circ h$ (we will try to reach a contradiction). We have that $h(0,0) = (0,0)$ since $h(f(0,0)) = h(0,0) = g(h(0,0))$, i.e., $A(h(0,0)) = h(0,0)$ and $h(0,0)$ is an eigenvector of eigenvalue 1. But since $\lambda \in (0,1)$, 1 is not an eigenvalue of $A$ and then $h(0,0) = (0,0)$. Let $H := h^{-1}$, $H(0,0) = (0,0)$ and $f \circ H = H \circ g$. Therefore,

$$f^k \circ H = H \circ g^k.$$

Writing it by components,

$$\begin{aligned} &(1) \qquad \lambda^k H_1(x,y) = H_1(\lambda^k x, \lambda^{2k} y), \\ &(2) \qquad \lambda^{2k} H_2(x,y) + k\lambda^{2(k-1)}(H_1(x,y))^2 = H_2(\lambda^k x, \lambda^{2k} y). \end{aligned}$$

Fix $y = 0$ and assume $x \neq 0$. Since $H_1(0,0) = 0$, equation (1) gives us

$$H_1(x,0) = \frac{H_1(\lambda^k x, 0)}{\lambda^k} = x\left[\frac{H_1(\lambda^k x, 0) - H_1(0,0)}{\lambda^k x}\right] \xrightarrow{k\to\infty} x\frac{\partial H_1}{\partial x}(0,0).$$

Moreover, if $x \neq 0$, since $H \in \mathcal{C}^2$, the equation is still true. Hence $H_1(x,0) = x\frac{\partial H_1}{\partial x}(0,0)\ \forall x$. Now, fixing $y = 0$, equation (2) becomes

$$k\lambda^{2(k-1)}(H_1(x,0))^2 = H_2(\lambda^k x, 0) - \lambda^{2k} H_2(x,0).$$

Taking $x = \lambda^k t$, since $H_2(\lambda^k x, 0) \xrightarrow{k\to\infty} 0$, we have that

$$\lim_{k\to\infty} k(H_1(\lambda^k t, 0))^2 = \lim_{k\to\infty}\left[\lambda^2 t\frac{H_2(\lambda^{2k} t, 0)}{\lambda^{2k} t} - \lambda^2 H_2(\lambda^k t, 0)\right] = \lim_{k\to\infty} \lambda^2 t\frac{H_2(\lambda^{2k} t, 0)}{\lambda^{2k} t}.$$

Since $H_2(0,0) = 0$, we can rewrite the previous equation to obtain

$$\lim_{k\to\infty} k(H_1(\lambda^k t, 0))^2 = \lim_{k\to\infty} \lambda^2 t\left[\frac{H_2(\lambda^{2k} t, 0) - H_2(0,0)}{\lambda^{2k} t}\right] = \lambda^2 t\frac{\partial H_2}{\partial x}(0,0).$$

Since $H_1(\lambda^k x, 0) = \lambda^k x \frac{\partial H_1}{\partial x}(0,0)$, the limit becomes

$$\lim_{k\to\infty} k\lambda^{2k}x^2 \left(\frac{\partial H_1}{\partial x}(0,0)\right)^2 = x\lambda^2 \frac{\partial H_2}{\partial x}(0,0),$$

and since $k\lambda^{2k} \xrightarrow{k\to\infty} 0$, we have that $\frac{\partial H_2}{\partial x}(0,0) = 0$.

Now, we can take partial derivatives of (1) with respect to $y$ to obtain

$$\lambda^k \frac{\partial H_1}{\partial y}(x,y) = \lambda^{2k}\frac{\partial H_1}{\partial y}(\lambda^k x, \lambda^{2k} y).$$

Hence,

$$\frac{\partial H_1}{\partial y}(x,y) = \lambda^k \frac{\partial H_1}{\partial y}(\lambda^k x, \lambda^{2k} y) \xrightarrow{k\to\infty} 0,$$

and $\frac{\partial H_1}{\partial y}(x,y) = 0 \;\; \forall x, y$. This implies that $H_1$ does not depend on $y$, i.e., $H_1(x,y) = H_1(x)$. Now, taking partial derivatives of (2) with respect to $x$, we have that

$$\lambda^{2k}\frac{\partial H_2}{\partial x}(x,y) + 2k\lambda^{2(k-1)}H_1(x)\frac{\partial H_1}{\partial x}(x) = \lambda^k \frac{\partial H_2}{\partial x}(\lambda^k x, \lambda^{2k} y).$$

Then,

$$2k\lambda^{-2}H_1(x)\frac{\partial H_1}{\partial x}(x) = \lambda^{-k}\frac{\partial H_2}{\partial x}(\lambda^k x, \lambda^{2k} y) - \frac{\partial H_2}{\partial x}(x,y).$$

Fixing $y = 0$, since we have seen that $\frac{\partial H_2}{\partial x}(0,0) = 0$, the previous equation becomes

$$\lim_{k\to\infty} 2k\lambda^{-2}H_1(x)\frac{\partial H_1}{\partial x}(x) = x\left[\frac{\frac{\partial H_2}{\partial x}(\lambda^k x, 0) - \frac{\partial H_2}{\partial x}(0,0)}{\lambda^k x}\right] - \frac{\partial H_2}{\partial x}(x,0) = x\frac{\partial^2 H_2}{\partial x^2}(0,0) - \frac{\partial H_2}{\partial x}(x,0).$$

Notice that the right-hand side of the equation does not depends on $k$ and, the limit of the left-hand side goes to infinity unless $H_1(x) = 0$ or $\frac{\partial H_1}{\partial x}(x) = 0$. Then, we face two possibilities:

$$\begin{cases} i) \;\; H_1(x) = 0 \;\; \forall x \;\text{ and }\; x\frac{\partial^2 H_2}{\partial x^2}(0,0) = \frac{\partial H_2}{\partial x}(x,0), \\ ii) \;\; \frac{\partial H_1}{\partial x}(x) = 0 \;\; \forall x \;\text{ and }\; x\frac{\partial^2 H_2}{\partial x^2}(0,0) = \frac{\partial H_2}{\partial x}(x,0). \end{cases}$$

Hence, $H_1(x)\frac{\partial H_1}{\partial x}(x) = 0 \;\forall x$. This implies that $\frac{\partial H_1^2}{\partial x}(x) = 2H_1(x)\frac{\partial H_1}{\partial x}(x) = 0 \;\forall x$. Therefore, $H_1^2$ does not depend on $x$ and in consequence neither does $H_1$. Since we have seen that $H_1$ was neither depending on $y$, $H_1$ is constant. Then, there exist a fixed value $k$ such that $H = (H_1, H_2) = (k, H_2)$, leading to a contradiction with $H$ being injective. □