Exercise 0.1. *Let us consider the convex set(polyhedron),*

$$C = \{(x, y) \in \mathbb{R}^n \text{ such that } 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}.$$

Write this set in the form $\{z \in \mathbb{R}^n \text{ such that } Az = b, z \geq 0\}$, *compute the basic feasible solutions and, from them, the vertices.*

Proof. Given $(x, y) \in C$, let us introduce $z \geq 0$ and $t \geq 0$ two slack variables that satisfy the following:

$$\begin{cases} x + z = 1 \\ y + t = 1 \end{cases}$$

Then, we can write C as

$$\Omega := \{w = (x, y, z, t) \in \mathbb{R}^4 \text{ such that } \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ w \geq 0\}.$$

Let us now compute the basic feasible solutions. Notice that the system has four basic solutions that are $(1, 1, 0, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 0, 1, 1)$. Moreover, this are the four feasible basic solutions we were looking for. Since the set of vertices of a polytope corresponds to the set of basic feasible solutions, we are done. □

Exercise 0.2. *Assume that* $a_1, \ldots, a_m$ *are given vectors in* $\mathbb{R}^3$ *(all different from 0). Let* $b_1, \ldots, b_m$ *strictly positive numbers and let us define the set*

$$M = \{x \in \mathbb{R}^3 \text{ such that } a_i^t x \leqslant b_i \text{ for } i = 1, \ldots, m\}.$$

(a) Show that the interior of this set is not empty.

(b) We want to determine the center and the radius of the biggest sphere contained in M. Write this problem as a linear programming one.

Proof. Let us start solving *(a)*. To show that int$(M) \neq \emptyset$, notice that the origin $(0, 0, 0)$ belongs to M. Since $b_1, \ldots, b_m$ are strictly positive numbers, there exist $r_1, \ldots, r_m$ such that $0 < r_i < b_i$. Let us define a family of functions $f_i : \mathbb{R}^3 \to \mathbb{R}$ such that $f_i(x) := a_i^t x$ for every $i = 1, \ldots, m$. Since these functions are continuous in $x_0 := (0, 0, 0)$, for every $\epsilon_i > 0$, there exists a $\delta_i > 0$ such that for all $x \in \mathbb{R}^3$:

$$||x - x_0|| < \delta_i \implies |f_i(x) - f_i(x_0)| < \epsilon_i \quad (\iff ||x|| < \delta_i \implies |f_i(x)| < \epsilon_i).$$

Choosing $\epsilon_i = r_i$, and $\delta = \min(\delta_1, \ldots, \delta_m)$ we have just found a ball with center in $x_0 = (0, 0, 0)$ and radius δ contained in M. That proves that int$(M) \neq \emptyset$.

Let us prove *(b)*. We want to determine the center and the radius of the biggest sphere contained in M, so we want to maximize the following function, $v(r) := \frac{4}{3}\pi r^3$, which give us the volume of a sphere of radius r. Notice that for each point in a circle with center c and radius r to satisfy $a_i^t x \leqslant b_i$, you need that c satisfies $a_i^t c \leqslant b_i$ and we also need that the distance between the center and the line $a_i^t x = b_i$ be at least r, i.e., $d(c, a_i^t x = b_i) \geq r$. This second condition can be written as

$$\frac{|a_i^t c - b_i|}{||a_i||} \geq r.$$

Since a monotone transformation of the objective function does not change the location of the optimum, the function we will optimize is $v(r) = r$ and the linear programming problem will be the following:

$$\begin{cases} \text{Max } r \\ \text{subject to} \\ a_i^t c \leqslant b_i, i = 1, \ldots, m \\ \frac{a_i^t c - b_i}{||a_i||} \geq r, i = 1, \ldots, m \end{cases}$$

□

Exercise 0.3. *Use a software package to solve*

$$(a) \begin{cases} \text{Min } \; -8x_1 - 9x_2 - 5x_3 \\ \text{subject to} \\ x_1 + x_2 + 2x_3 \leqslant 2 \\ 2x_1 + 3x_2 + 4x_3 \leqslant 3 \\ 6x_1 + 6x_2 + 2x_3 \leqslant 8 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{cases} \qquad (b) \begin{cases} \text{Min } \; 5x_1 - 3x_2 \\ \text{subject to} \\ x_1 - x_2 \geq 2 \\ 2x_1 + 3x_2 \leqslant 4 \\ -x_1 + 6x_2 = 10 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

$$(c) \begin{cases} \text{Max } \; 3x_1 + 2x_2 - 5x_3 \\ \text{subject to} \\ 4x_1 - 2x_2 + 2x_3 \leqslant 4 \\ -2x_1 + x_2 - x_3 \leqslant -1 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{cases} \qquad (d) \begin{cases} \text{Max } \; 4x_1 + 6x_2 + 3x_3 + x_4 \\ \text{subject to} \\ 1.5x_1 + 2x_2 + 4x_3 + 3x_4 \leqslant 550 \\ 4x_1 + x_2 + 2x_3 + x_4 \leqslant 700 \\ 2x_1 + 3x_2 + x_3 + 2x_4 \leqslant 200 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \end{cases}$$

Proof. We will work in matlab environment. The code will go as follows:

- Define the variables we want to optimize with the command *optimvar(variables)*
- Define the function we want to optimize with the command *optimproblem.Objective(function)*
- Define all the constraints with the command *optimproblem.Constraints(constraints)*
- Solve the problem with the command *solve(optimproblem)*

Let us see the code to solve the first problem.

```
x1=optimvar('x1');
x2=optimvar('x2');
x3=optimvar('x3');
prob=optimproblem;
prob.Objective=-8*x1-9*x2-5*x3;
prob.Constraints.cons1=x1+x2+2*x3<=2;
prob.Constraints.cons2=2*x1+3*x2+4*x3<=3;
prob.Constraints.cons3=6*x1+6*x2+2*x3<=8;
prob.Constraints.cons4=x1>=0;
prob.Constraints.cons5=x2>=0;
prob.Constraints.cons6=x3>=0;
sol=solve(prob)
```

The solution is $(x_1, x_2, x_3) = (1, \frac{1}{3}, 0)$. Let us give the solution of the others problem. Notice that in *(c)* and *(d)* we want to maximize instead of minimize. In order to deal with that, we just change the sign of the objective function. The results are the following: No solutions for *(b)*, unbounded for *(c)* and $(x_1, x_2, x_3, x_4) = (0, 25, 125, 0)$ for *(d)*. □

Exercise 0.4. *Consider a linear programme (P) in standard form and its dual programme (D),*

$$(P) \begin{cases} \text{Min } \ z = cx \\ Ax = b \\ x \geq 0 \end{cases} \qquad (D) \begin{cases} \text{Max } \ w = ub \\ uA \leqslant c \end{cases}$$

Let us denote by A_j the jth column of A. Prove that two solutions $(\bar{x}, \bar{u})$ of, respectively, (P) and (D) are optimal if and only if

$$(\bar{u}A_j - c_j)\bar{x}_j = 0, \quad \forall j = 1, \dots, n.$$

Proof. Let us first do an observation. If $(\bar{x}, \bar{u})$ are solutions of, respectively, (P) and (D), we have that

$$A\bar{x} = b \implies \bar{u}A\bar{x} - c\bar{x} = \bar{u}b - c\bar{x}.$$

From left to right, if $(\bar{x}, \bar{u})$ is a pair of optimal solutions, we have $c\bar{x} = \bar{u}b$, hence $(\bar{u}A - c)\bar{x} = 0$. Since $\bar{x} \geq 0$ and $\bar{u}A - c \leqslant 0$, the zero scalar product implies that

$$(\bar{u}A_j - c_j)\bar{x}_j = 0, \quad \forall j = 1, \dots, n.$$

Conversely, if $(\bar{u}A - c)\bar{x} = 0$ then $c\bar{x} = \bar{u}b$ and by a Corollary given in class, the pair $(\bar{x}, \bar{u})$ are optimal solutions. The mentioned corollary is the following:

Corollary 0.5. *If x^* and u^* are, respectively, solutions of the primal and of the dual for which $cx^* = u^*b$, then x^* is an optimal solution of the primal, and u^* is an optimal solution of the dual.*

□