Exercises 2

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① Prove that the automorphism group of the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is

$$G = \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\}$$

We want to prove that $\operatorname{Aut}(\mathbb{C}^+) = G$. To do this, we are going to use the following result:

Lemma: Let Γ a group acting on a set X and let $G \subset \Gamma$ a subgroup. Assume that,

i) G acts transitevely in X

ii) $\exists$ at least $a \in X$ such that $\Gamma_a = \{\gamma \in \Gamma : \gamma a = a\} \subset G$

Then, $G = \Gamma$.

Hence, if we prove that G is transitevely in $\mathbb{C}^+$ and that the set of automorphism of $\mathbb{C}^+$ leaving i fixed is included in G, we are done.

1) <u>G acts transitevely in $\mathbb{C}^+$</u>

Given $z_1, z_2 \in \mathbb{C}^+$, we need to find $g \in G$ such that $g(z_1) = z_2$.
Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $y_1, y_2 > 0$, and let us define

$$a = \frac{y_2}{y_1}\sqrt{\frac{y_1}{y_2}} \qquad c = 0$$

$$b = \left(\frac{x_2 y_1 - y_2 x_1}{y_1}\right)\sqrt{\frac{y_1}{y_2}} \qquad d = \sqrt{\frac{y_1}{y_2}}$$

Them,

$$g(z_1) = \frac{az+b}{cz+d} = \frac{\frac{y_2}{y_1}\sqrt{\frac{y_1}{y_2}}(x_1+iy_1) + \left(\frac{x_2y_1 - y_2x_1}{y_1}\right)\sqrt{\frac{y_1}{y_2}}}{\sqrt{\frac{y_1}{y_2}}}$$

$$= \frac{\sqrt{\frac{y_1}{y_2}}\, x_2 + i\sqrt{\frac{y_1}{y_2}}\, y_2}{\sqrt{\frac{y_1}{y_2}}} = x_2 + iy_2 = z_2.$$

Moreover, $a,b,c,d \in \mathbb{R}$ and $ad - bc \overset{c=0}{=} ad = 1$.

Therefore, G is transitive in $\mathbb{C}^+$.

2) <u>The set of automorphism of $\mathbb{C}^+$ leaving i fixed is included in G</u>

Let $h \in Aut(\mathbb{C}^+)$, $h(i) = i$.

Let us define $H := \varphi \circ h \circ \varphi^{-1}$ where φ is the conformal map given by

$$\varphi : \mathbb{C}^+ \to \mathbb{D} \qquad\qquad \varphi^{-1} : \mathbb{D} \to \mathbb{C}^+$$
$$z \mapsto \frac{i-z}{i+z} \qquad\qquad w \mapsto i\,\frac{1-w}{1+w}$$

Notice that $H \in Aut(\mathbb{D})$ and $H(0) = (\varphi \circ h \circ \varphi^{-1})(0) = \varphi \circ h(i)$
$= \varphi(i) = 0$.

Also, H^{-1} is a holomorphic map from $\mathbb{D}$ to $\mathbb{D}$ such that $H^{-1}(0) = 0$

By Schwarz lemma, $|H'(0)| \leq 1$ and $|(H^{-1})'(0)| \leq 1$. Thus, again, by Schwarz lemma, $\exists\, \theta \in \mathbb{R}$ such that

$$H = \varphi \circ h \circ \varphi^{-1} = e^{i\theta} z.$$

Now, we can define

$$a = \cos(\theta/2) \qquad c = -\sin(\theta/2)$$

$$b = \sin(\theta/2) \qquad d = \cos(\theta/2)$$

Then,

$$g(i) = \frac{ai+b}{ci+d} = \frac{\cos(\theta/2)\, i + \sin(\theta/2)}{-\sin(\theta/2)\, i + \cos(\theta/2)} (-i)(i) =$$

$$= \frac{-\sin(\theta/2)\, i + \cos(\theta/2)}{-\sin(\theta/2)\, i + \cos(\theta/2)}\, i = i.$$

and

$$g'(i) = \frac{a(ci+d) - (ai+b)c}{(ci+d)^2} = \frac{ad-bc}{(cz+d)^2} \overset{*}{=} \frac{1}{(cz+d)^2}$$

$$= \frac{1}{[-\sin(\theta/2)\, i + \cos(\theta/2)]^2} = e^{i\theta}.$$

Moreover, $a, b, c, d \in \mathbb{R}$ and $ad - bc = \cos^2(\theta/2) + \sin^2(\theta/2) \overset{*}{=} 1$

Hence, the map $\varphi \circ g \circ \varphi^{-1} \in Aut(\mathbb{D})$ and maps $z=0$ to $z=0$. Moreover, its derivative at $z=0$ is $e^{i\theta}$. Using Schwarz lemma, $\varphi \circ g \circ \varphi^{-1}$ is the rotation $z \mapsto e^{i\theta} z$. That is, $\varphi \circ h \circ \varphi^{-1} = \varphi \circ g \circ \varphi^{-1}$, and since φ is one-to-one, we conclude that $h = g$.

□

⑥ Show that the hyperbolic distance in $\mathbb{D}$ is given by

$$d(z,w) = \log\left(\frac{1+ph(z,w)}{1-ph(z,w)}\right)$$

where

$$ph(z,w) = \left|\frac{z-w}{1-z\bar{w}}\right|, \quad z,w \in \mathbb{D}$$

is the pseudo-hyperbolic distance.

Let us prove it first for $x,y \in (-1,1)$. Assume $-1<x<y<1$, we want to prove that

$$d(x,y) = \log\left(\frac{1+\frac{y-x}{1-xy}}{1-\frac{y-x}{1-xy}}\right). \quad (*)$$

Consider a smooth curve γ joining x to y in $\mathbb{D}$ and write $\gamma(t) = u(t)+iv(t)$, where $0 \le t \le 1$. Then

$$\ell(\gamma) \underset{\substack{\text{(using the} \\ \text{hyperbolic metric)}}}{=} \int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}\,dt \geq \int_0^1 \frac{2u'(t)}{1-u(t)^2}\,dt$$

because $|\gamma'(t)| \geq |u'(t)| \geq u'(t)$ and $|\gamma(t)|^2 \geq |u(t)|^2 = u(t)^2$.

Since $\int \frac{2dx}{1-x^2} = \log\left(\frac{x+1}{-x+1}\right) + C$, the last integral can be evaluated and gives

$$\ell(\gamma) \geq \log\left(\frac{1+y}{1-y}\cdot\frac{1-x}{1+x}\right) = \log\left(\frac{1+\frac{y-x}{1-xy}}{1-\frac{y-x}{1-xy}}\right)$$

Taking $\gamma(t) = x + t(y-x)$, $0 \le t \le 1$, then $\gamma(t) = u(t)$ and the previous inequalities becomes equalities. Since the distance is defined as $d(x,y) = \inf_{\gamma} \rho(\gamma)$, we have that for $0<x<y<1$,

$$d(x,y) = \log\left(\frac{1 + \frac{y-x}{1-xy}}{1 - \frac{y-x}{1-xy}} \right).$$

Now, we have to extend the result to $z, w \in \mathbb{D}$. Let us prove a couple of auxilary results.

Lemma 1: Take $z, w \in \mathbb{D}$ and let $f(z) := \dfrac{z-w}{1-z\bar{w}}$. Then f is an isometry for d.

Proof/

Since $\mathrm{Aut}(\mathbb{D}) = \left\{ e^{i\theta} \dfrac{\alpha - z}{1-\bar{\alpha} z}, \ \theta \in \mathbb{R}, \alpha \in \mathbb{D} \right\}$, it is clear that $f \in \mathrm{Aut}(\mathbb{D})$ and in consequence, f is an isometry for d.

Lemma 2: For every $z \in \mathbb{D}$, $d(0,z) = d(0,|z|)$ □

Proof/

We just need to find $\varphi \in \mathrm{Aut}(\mathbb{D})$ such that $\varphi(0) = 0$ and $\varphi(z) = |z| \ \forall z \in \mathbb{D}$.

Take $\varphi(z) = -e^{i\theta} z$ where $\theta = \pi - \arg(z)$. Clearly $\varphi \in \mathrm{Aut}(\mathbb{D})$ (just take $\alpha = 0$ and $\theta = \pi - \arg(z) \in \mathbb{R}$).

Moreover $\varphi(0) = 0$ and

(modulus and argument of complex exponential formula)

$$\varphi(z) = -e^{i\pi} e^{-i\arg(z)} z = e^{-i\arg(z)} z = |z|$$

□

Therefore,

$$d(z,w) = d(w,z) \overset{\text{lemma 1}}{=} d(f(w), f(z)) = d(0, f(z))$$

$$\underset{\text{lemma 2}}{=} d(0, |f(z)|) = d(0, ph(z,w))$$

Hence, using the formula ⊛ with $x=0$, $y = ph(z,w)$, we obtain that:

$$d(z,w) = \log\left(\frac{1+ph(z,w)}{1-ph(z,w)}\right).$$