Exercise 0.1. *The goal of this exercise is to see the limitations of Newton's method, with an example in which Newton's method is divergent while a descent method is convergent to the minimum. Let us consider the function*

$$g(x) = -e^{-x^2}$$

that has a unique minimum at $x = 0$. Note that $g'(x) < 0$ if $x < 0$ and $g'(x) > 0$ if $x > 0$, which implies that any reasonable descent method should be able to find the minimum, no matter the starting point. Instead, let us use a Newton's method on the function g' (i.e, to solve $g'(x) = 0$).

(a) Let $\{x_n\}_n$ be the sequence of points produced by the Newton's method starting at the seed $x_0 = 1$. Prove that $\lim_{n\to\infty} x_n = \infty$

(b) Find a value α such that, if $x_0 \in [0, \alpha)$ the Newton's method converges to 0, and if $x_0 > \alpha$ the Newton's method diverges.

Proof. Let us prove *(a)*. Notice that $g'(x) = 2xe^{-x^2}$ and $g''(x) = (2 - 4x^2)e^{-x^2}$. The Newton's method on the function g' has the following iterative system:

$$x_{n+1} = x_n - \frac{g'(x_n)}{g''(x_n)} = x_n - \frac{2x_n}{2 - 4x_n^2} = \frac{4x_n^3}{4x_n^2 - 2}.$$

Let us study the following function:

$$f(x) := \frac{4x^3}{4x^2 - 2}.$$

If we prove that for $x \geq 1$, $x < f(x)$ we are done. This will prove that the sequence $\{x_n\}_n$ is strictly increasing if $x_0 \geq 1$ and since the sequence is not bounded, $\lim_{n\to\infty} x_n = \infty$. Notice that if $x > \frac{\sqrt{2}}{2}$, $4x^2 - 2 > 0$, so

$$x < \frac{4x^3}{4x^2 - 2} \iff 4x^3 - 2x < 4x^3 \iff x > 0.$$

Actually, assuming $x > \frac{\sqrt{2}}{2}$, we have $x < f(x)$ and in consequence $\lim_{n\to\infty} x_n = \infty$ if $x_0 > \frac{\sqrt{2}}{2}$.

Let us show *(b)*. We have proof that if $x > \frac{\sqrt{2}}{2}$, $x < f(x)$. In the same way, we can prove that if $x < -\frac{\sqrt{2}}{2}$, then $x > f(x)$. Therefore, if $|x_0| > \frac{\sqrt{2}}{2}$, $\lim_{n\to\infty} x_n = \infty$. Observe that if $0 < x < \frac{1}{2}$, then $-x < f(x) < x$ and if $-\frac{1}{2} < x < 0$ then $x < f(x) < -x$. This implies that the sequence $\{x_n\}_n$ converges to 0 if $x_0 \in [0, \frac{1}{2})$. Let us prove one of the inequalities, for example the first one, i.e, if $0 < x < \frac{1}{2}$, then $-x < f(x) < x$. Suppose $0 < x < \frac{1}{2}$, then $4x^2 - 2 < 0$ and

$$\frac{4x^3}{4x^2 - 2} < x \iff 4x^3 > 4x^3 - 2x \iff 2x > 0 \iff x > 0$$

and

$$-x < \frac{4x^3}{4x^2 - 2} \iff -4x^3 + 2x > 4x^3 \iff -8x^3 + 2x > 0.$$

The last statement is true if $0 < x < \frac{1}{2}$, therefore, in that case, we have $-x < f(x) < x$. In a similar way, we can prove the other inequality. Now, notice that if $\frac{1}{2} < x < \frac{\sqrt{2}}{2}$, $x < |f(x)|$. This implies that, if we choose $x_0 \in (\frac{1}{2}, \frac{\sqrt{2}}{2})$, after some iterations, the sequence will be in the case $|x_0| > \frac{\sqrt{2}}{2}$, which we know that diverges. Hence, if we choose $\alpha = \frac{1}{2}$, Newton's method converges to 0 if $x_0 \in [0, \alpha)$, and if $x_0 > \alpha$ the Newton's method diverges. □

Exercise 0.2. *Discuss if the following functions are unimodal:*

(a) $g(x) = x^3 - x$ *on* $x \in [-2, 0]$, *and on* $x \in [0, 2]$.

(b) $g(x) = e^{-x}$ *on* $x \in [0, 1]$.

(c) $g(x) = |x| + |x - 1|$ *on* $x \in [-2, 2]$.

Proof. First of all, we recall the definition of unimodal function.
Definition We say that a function g is unimodal on the real interval $[A, B]$ if it has a minimum $\bar{\alpha} \in [A, B]$ and if $\forall \alpha_1 \in [A, B]$, $\forall \alpha_2 \in [A, B]$ with $\alpha_1 < \alpha_2$ we have

$$\begin{aligned} \alpha_2 \leqslant \bar{\alpha} &\implies g(\alpha_1) > g(\alpha_2) \\ \alpha_1 \geq \bar{\alpha} &\implies g(\alpha_1) < g(\alpha_2). \end{aligned}$$

Let us start discussing *(a)*. Consider $g(x) = x^3 - x$ with $x \in [-2, 0]$. If we consider $x \in [-2, 0]$, this function has a minimum in $\bar{\alpha} = -2$ with the value $g(-2) = -6$. Notice that if we choose $\alpha_1 = -\frac{1}{2}$ and $\alpha_2 = 0$, we have $\alpha_1 < \alpha_2$ and $\alpha_1 \geq \bar{\alpha}$ but, $g(\alpha_1) = \frac{3}{8} \not< g(\alpha_2) = 0$. Hence, this function is not unimodal in $[-2, 0]$.

Nevertheless, this function is unimodal on $[0, 2]$, let us prove it. First notice that in $[0, 2]$ this function has a minimum in $\bar{\alpha} = \sqrt{\frac{1}{3}}$. Then, observe that for $x \in \left[0, \sqrt{\frac{1}{3}}\right)$, $g'(x) < 0$, thus g is strictly decreasing. On the other hand, if $x \in \left(\sqrt{\frac{1}{3}}, 2\right]$, then $g'(x) > 0$, thus g is strictly increasing. Then, this function satisfies the conditions of a unimodal function.

Let us continue discussing *(b)*. The function $g(x) = e^{-x}$ in $x \in [0, 1]$ is unimodal. Notice that in this interval, the function has a minimum in $\bar{\alpha} = 1$. Observe also that $g'(x) < 0$ for all $x \in [0, 1]$. This implies that the function is strictly decreasing and both conditions of unimodals functions definition are satisfied, since the second one has not to be check because the minimum is at one extreme.

To finish the exercise, let us discuss *(c)*. Notice that definition of unimodal function implies the uniqueness of a local minimum. This function, $g(x) = |x| + |x - 1|$, has multiple minimums in $[-2, 2]$, for example, $\bar{\alpha} = \frac{1}{2}$, $\bar{\mu} = 0$ and $\bar{\eta} = 1$ are local minimums of the function. Actually this function has a line of local minimum, the line $0 \leqslant x \leqslant 1$. Thus, this function can not be an unimodal function. □

Exercise 0.3. *(**Optional**) Look for the Golden section search method. Explain it.*

The Golden section search method is a technique for finding a minimum of an unimodal function defined on the interval $[a, b]$. The method operates by successively narrowing the range of values on the specified interval, which makes it relatively slow, but very robust. The method is the following: Given an unimodal function f in $[a, b]$, compute $c = b + \frac{a-b}{\phi}$ and $d = a + \frac{b-a}{\phi}$ where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Then, if $f(c) < f(d)$, we rename $b \leftarrow d$, $d \leftarrow c$, $f(d) \leftarrow f(c)$ and $c \leftarrow b + \frac{a-b}{\phi}$. If not, i.e., if $f(c) \geq f(d)$, we rename $a \leftarrow c$, $c \leftarrow d$, $f(c) \leftarrow f(d)$ and $d \leftarrow a + \frac{b-a}{\phi}$. Then, if $b - a < \epsilon$, we stop and the minimum is $\frac{a+b}{2}$. If not, we return to ask if $f(c) < f(d)$ or $f(c) \geq f(d)$ and do the corresponding computations.