HARMONIC ANALYSIS David Rosado Rodríguez

1. Let $T \in S'$. Show that

a) $\hat{T}^{(K)} = \left[(-2\pi i t)^K T\right]^\wedge$ b) $\widehat{T^{(K)}} = (2\pi i \xi)^K \hat{T}$

Let us recall two definitions.

<u>Definition 1</u>: The Fourier transform of a distribution $T \in S'$ is defined by the action

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle, \quad \varphi \in S$$

<u>Definition 2</u>: The derivative $T' \in S'$ is the distribution defined by

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle, \quad \varphi \in S$$

Let us start with a)

$$\langle \hat{T}^{(K)}, \varphi \rangle \overset{\text{def 2}}{=} -\langle \hat{T}, \varphi^{(K)} \rangle \overset{\text{def 1}}{=} -\langle T, \widehat{\varphi^{(K)}} \rangle \overset{\text{rules for Fourier and differentiation for functions}}{=} -\langle T, (2\pi i \xi)^K \hat{\varphi} \rangle$$

$$= -\langle (2\pi i \xi)^K T, \hat{\varphi} \rangle = \langle (-2\pi i \xi)^K T, \hat{\varphi} \rangle$$

Product of a C^∞ function, $(2\pi i \xi)^K$, with a distribution.

Therefore, we have just proved that $\hat{T}^{(K)} = \left[(-2\pi i t)^K T\right]^\wedge$

Let us continue with b)

$$\langle \widehat{T^{(K)}}, \varphi \rangle \overset{\text{def 1}}{=} \langle T^{(K)}, \hat{\varphi} \rangle \overset{\text{def 2}}{=} -\langle T, (\hat{\varphi})^{(K)} \rangle \overset{\text{rules for Fourier and differentiation for functions}}{=} -\langle T, \left((-2\pi i \xi)^K \varphi\right)^\wedge \rangle$$

$$\underset{\text{def 1}}{=} -\langle \hat{T}, (-2\pi i \xi)^K \varphi \rangle = \langle (2\pi i \xi)^K \hat{T}, \varphi \rangle$$

Product of a C^∞ function, $(2\pi i \xi)^K$, with a distribution.

Them, we have proved that

$$\langle \widehat{T^{(k)}} - (2\pi i \xi)^k \hat{T}, \varphi \rangle = 0 \quad \forall \varphi \in S$$

This implies that $\widehat{T^{(k)}} = (2\pi i \xi)^k \hat{T}$

③ Let $H(x) = \chi_{(0,\infty)}(x)$. Prove that $H' = \delta_0$

Following the second definition of the previous exercise, notice that H' is the distribution defined by

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle, \quad \varphi \in S$$

Notice that,

Fundamental theorem of calculus and $\varphi \in S$

$$\langle H, \varphi' \rangle = \int_{\mathbb{R}} H(x)\varphi'(x)\,dx = \int_0^\infty \varphi'(x)\,dx = -\varphi(0) =$$

$$= -\int_{\mathbb{R}} \delta_0(x)\varphi(x)\,dx = -\langle \delta_0, \varphi \rangle$$

Thus, $H' = \delta_0$