TOPOLOGICAL DATA ANALYSIS

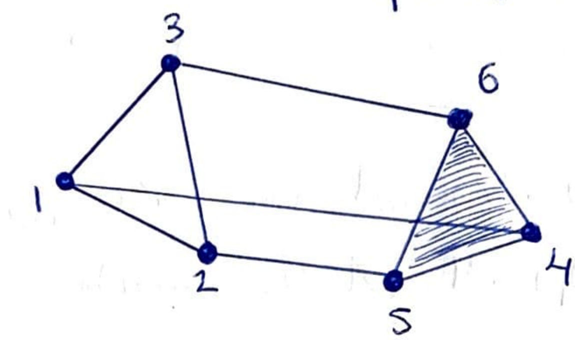
Exercise

Find the homology groups with coefficients in $\mathbb{Z}$ of the abstracts simplicial complex whose maximal faces are

$$(12)\,(13)\,(14)\,(23)\,(25)\,(36)\,(456)$$

Let us visualize the abstract simplicial complex (let us call it K)

K



Let us describe the n-chains of the abstract simplicial complex

$$C_0(K) = \mathbb{Z}(1) \oplus \mathbb{Z}(1) \oplus \mathbb{Z}(2) \oplus \mathbb{Z}(3) \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(5) \oplus \mathbb{Z}(6)$$

$$C_1(K) = \mathbb{Z}(12) \oplus \mathbb{Z}(13) \oplus \mathbb{Z}(14) \oplus \mathbb{Z}(23) \oplus \mathbb{Z}(25) \oplus \mathbb{Z}(36) \oplus \mathbb{Z}(45) \oplus \mathbb{Z}(46) \oplus \mathbb{Z}(56)$$

$$C_2(K) = \mathbb{Z}(456)$$

$$C_i(K) = 0, \; i \geq 3$$

The boundary operators are:

$$0 \xrightarrow{\partial_3} C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$$

Notice that the 0-homology group is given by

$$H_0(K) = \frac{\operatorname{Ker}\partial_0}{\operatorname{Im}\partial_1} = \frac{C_0(K)}{\operatorname{Im}\partial_1}$$

Let us compute $\operatorname{Im}\partial_1$

Matrix of ∂_1:

$$\begin{array}{c|ccccccccc}
 & (12) & (13) & (14) & (23) & (25) & (36) & (45) & (46) & (56) \\
\hline
(1) & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2) & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
(3) & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
(4) & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\
(5) & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\
(6) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}$$

Notice that $\operatorname{rk}(\operatorname{Im}\partial_1) = 5$ and since $\operatorname{rk}(C_1(K)) = 9$, we obtain that $\operatorname{rk}(\operatorname{Ker}\partial_1) = 4$. Let us describe a basis of $\operatorname{Im}\partial_1$.

$$\operatorname{Im}\partial_1 = \langle (2)-(1), (3)-(1), (4)-(1), (6)-(4), (6)-(5) \rangle$$

Then,

$$H_0(K) = \frac{\langle (1), (2), (3), (4), (5), (6) \rangle}{\langle (2)-(1), (3)-(1), (4)-(1), (6)-(4), (6)-(5) \rangle} \cong \mathbb{Z}$$

$$[1] = [2] = [3] = [4] = [6] = [5]$$

generated by the class $[1]$ of (1).

Notice that the first homology group is given by

$$H_1(K) = \operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2$$

Let us describe a basis of $\operatorname{Ker}\partial_1$. Recall that $rk(\operatorname{Ker}\partial_1)=4$

$$\operatorname{Ker}\partial_1 = \langle (45)-(46)+(56),\ (12)-(13)+(23),\ (23)-(25)+(36)-(56),$$
$$-(12)+(14)-(25)+(45)\rangle$$

Let us continue computing the matrix of ∂_2, in order to calculate $\operatorname{Im}\partial_2$

$$\begin{array}{c|c}
 & (456) \\
\hline
(12) & 0 \\
(13) & 0 \\
(14) & 0 \\
(23) & 0 \\
(25) & 0 \\
(36) & 0 \\
(45) & 1 \\
(46) & -1 \\
(56) & 1
\end{array}$$

Notice that $rk(\operatorname{Im}\partial_2)=1$ and since $rk(C_2(K))=1$, we obtain that $rk(\operatorname{Ker}\partial_2)=0$. Let us describe a basis of $\operatorname{Im}\partial_2$

$$\operatorname{Im}\partial_2 = \langle (45)-(46)+(56)\rangle$$

Then,

$$H_1(K) = \operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2 \cong \mathbb{Z}^3 \quad \text{generated by}$$

$$\langle (12)-(13)+(23),\ (23)-(25)+(36)-(56),\ -(-12)+(14)-(25)+(45)\rangle$$

Finally, let us compute the second homology group.

$$H_2(K) = \operatorname{Ker}\partial_2 / \operatorname{Im}\partial_3 = \operatorname{Ker}\partial_2$$

Since $\pi K(\ker \partial_2) = 0$, we can conclude that $H_2(K) = 0$.

In summary, we have obtain the following:

$$H_i(K) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^3 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$