NEWTON'S METHOD AS A DYNAMICAL SYSTEM

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Motivation of the project. Iterative algorithms.

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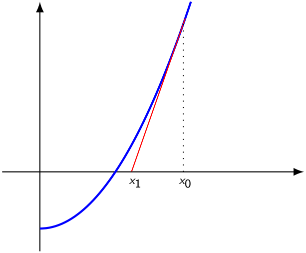
$$\boxed{\text{Newton's method in } \mathbb{R}}$$



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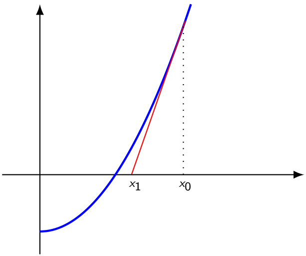
Newton's method in $\mathbb{R}$



$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

We have to choose a initial condition!!

Historical context

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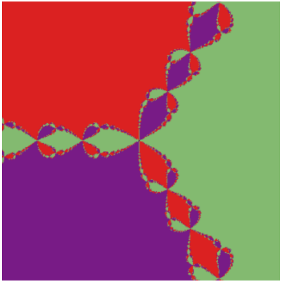
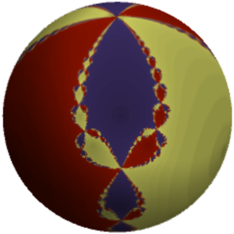
- Multiple contributions of mathematicians as Joseph Raphson or Joseph Fourier.
- In 1870 Ernst Schröder i Arthur Cayley generalise Newton's method for a functions of complex variable, with a new goal: Global study.





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Rational functions and local theory

We denote $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere.

Definition

A rational function is a function of the form $R(z) = P(z)/Q(z)$ where P and Q are polynomials.

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- Study of the fixed points $(f(z_0) = z_0) \longrightarrow$ easiest orbits to study.

$$
\begin{tikzcd}
 & \text{Attracting } (\Leftarrow |f'(z_0)| < 1) & \text{Parabolic } (f'(z_0) \text{ root of unity}) \\
\text{Fixed points} \arrow[ur] \arrow[r] \arrow[dr] & \text{Neutral } (|f'(z_0)| = 1) \arrow[ur] \arrow[dr] & \\
 & \text{Repelling } (\Leftarrow |f'(z_0)| > 1) & \text{Irrationally neutral (otherwise)}
\end{tikzcd}
$$

Local theory

Theorem (linearization)

Let $z_0 \in \mathbb{C}$, U a neighborhood of z_0 i f an holomorphic function in U such that z_0 is a fixed point of f with multiplier λ, $|\lambda| \neq 0, 1$. Then, there exist a conformal map $\phi : U \to W$, W neighborhood of the origin, such that conjugate f in U with the lineal function $g(w) = \lambda w$ in W.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"'] & U \arrow[d, "\phi"] \\ W \arrow[r, "g"'] & W \end{tikzcd}$$

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Theorem (Boettcher)

If now z_0 is a super-attracting fixed point of f $(\lambda = 0)$,

$$f(z) = z_0 + a_p(z - z_0)^p + \cdots, \qquad a_p \neq 0, p \geqslant 2.$$

Then, the same result is true with $g(w) = w^p$.

Fatou and Julia sets

Definition

Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the Fatou set is:

$$\mathcal{F}(R) = \left\{ z \in \hat{\mathbb{C}} \mid \{R^n\}_n \text{ is normal in some neighborhood of } z \right\}.$$

The Julia set is, $\mathcal{J}(R) = \hat{\mathbb{C}} \backslash \mathcal{F}(R)$.

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There are a lot of properties of these sets and lot of theory in that way!

- $\mathcal{J}(R) \neq \varnothing$.

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Properties

Let p a polynomial, consider the Newton's method, $N_p = Id - \frac{p}{p'}$.

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Properties

- *N_p is a rational function.*
- *$\alpha \in \mathbb{C}$ is a fixed point of N_p if and only if $p(\alpha) = 0$.*
- *If $p(\alpha) = 0$, then α is an attracting fixed point of N_p. If the root is simple, then is super-attracting.*
- *The infinite point is a repelling fixed point of N_p.*

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Corollary (Linearization and Boettcher theorem)

Let z_0 a simple root of the polynomial p. Then N_p is locally conjugate to $z \mapsto z^k$ with $k \geqslant 2$. If the root is not simple, calling m the order of the root, the algorithm is locally conjugate to $z \mapsto \frac{m-1}{m} z$.

Basin of attraction

Theorem

Let p a polynomial of degree $d \geqslant 2$, α a root of p. Then, $\mathcal{A}^*_{N_p}(\alpha)$, is not bounded.

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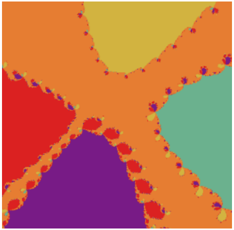
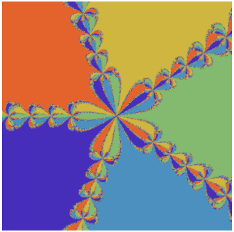
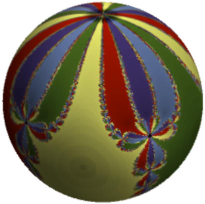






Numerical applications

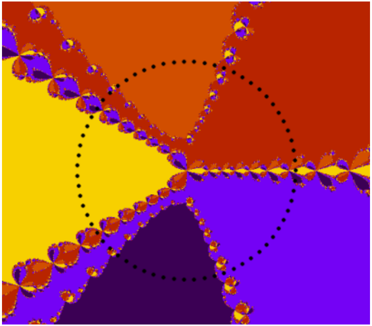
Theorem (Hubbard, Schleicher and Sutherland)

For each $d \geqslant 2$, exist a set $\mathcal{S}_d$ of $1.11 d \log^2 d$ points of $\mathbb{C}$ as maximum, such that, for each polynomial p of degree d and for each of its roots, exist a point $s \in \mathcal{S}_d$ in the basin of attraction of the chosen root.

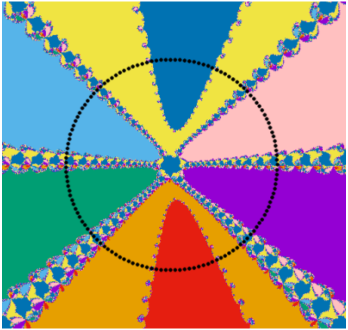
$$\mathcal{S}_d = \{(1+\sqrt{2}) \left(\frac{d-1}{d}\right)^{\frac{2v-1}{4s}} exp(i\frac{2\pi j}{N}) \,|\, 1 \leqslant v \leqslant s\,,\, 0 \leqslant j \leqslant N-1\}.$$

on $s = \lceil 0.26632 \log d \rceil$ i $N = \lceil 8.32547 d \log d \rceil$.

Examples



(a) The algorithm applied to a polynomial of degree 5. We need $s = 1$ circles and $N = 67$ points distributed in the circle of radius $r = 2.28322$. Actually, we only need 46.



(b) The algorithm applied to a polynomial of degree 8. We need $s = 1$ circles and $N = 139$ points distributed in the circle of radius $r = 2.33495$. Actually, we only need 107.

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Newton's method and the exponential function

Now, we study the method applied to the function $F(z) = exp(z)p(z)$.

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$$N_F(z) = z - \frac{exp(z)p(z)}{exp(z)p(z) + exp(z)p'(z)} = z - \frac{p(z)}{p(z) + p'(z)}.$$

Properties

- *N_F is a rational function.*
- *$\alpha \in \mathbb{C}$ is a fixed point of N_F if and only if $F(\alpha) = 0$ ($\iff$ $p(\alpha) = 0$).*
- *If $F(\alpha) = 0$, then α is an attracting fixed point (super-attracting) of N_F.*
- **The infinite point is a parabolic fixed point of N_F.**

Asymptotic behaviour

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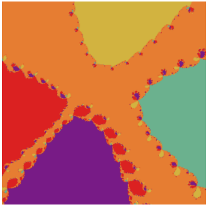

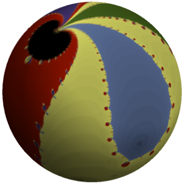

Numerical applications with N_F

Question

Can we build an algorithm to find all of the roots of a polyonmial with N_F?

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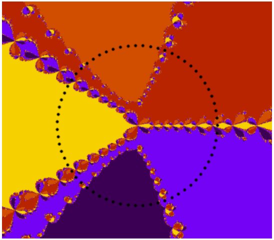
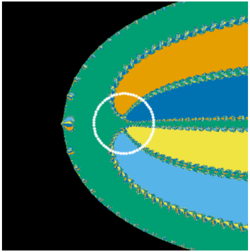
Applying the same algorithm at N_F (Hubbard, Schleicher and Sutherland)...



(a) The algorithm applied at a polynomial of degree 5 by N_p. Of the 67 initial conditions, only 46 were needed.



(b) The algorithm applied at a polynomial of degree 5 by N_p. Of the 67 initial conditions, only 29 were needed.

New set of initial condtions

New approach

We propose the next set to find all of the roots of a polynomial with N_F*:*

$$\mathcal{T}_d = \{a_d + b_d i \mid -L_d \leqslant b_d \leqslant L_d\},$$

where a_d *is a fixed value, not "too big" and* L_d *follows the next condition. If* $|b_d| \geqslant L_d$*, the orbit of the initial condition* $a_d + b_d i$ *tends to infinite.*

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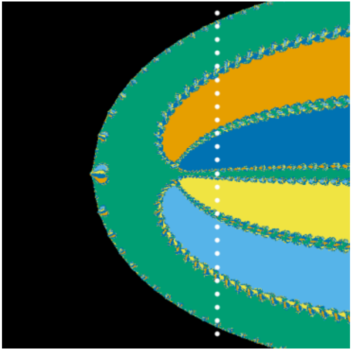
where a_d is a fixed value, not "too big" and L_d follows the next condition. If $|b_d| \geqslant L_d$, the orbit of the initial condition $a_d + b_d i$ tends to infinite.

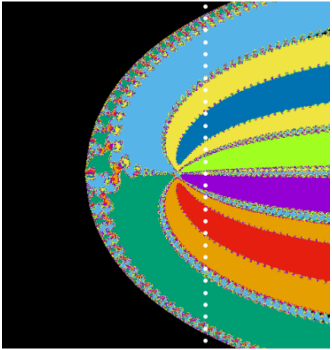
Observation

For a given polynomial, we do not have the analytic support of the existence of this line.

Examples



(a) New algorithm applied to a polynomial of degree 5 by N_F. Of the 25 initial conditions, only 17 were needed.



(b) New algorithm applied to a polynomial of degree 8. Of the 29 initial conditions, only 18 were needed.

Numerical evidences. Conclusions

$$p_a(z) = z(z-1)(z-a), \; a = \omega e^{2\pi i\theta}, 0 < w < 1, 0 \leqslant \theta < 2\pi$$

Results

For 4000 different values of a → success in 3997 of the cases, putting 28 initial conditions in the line.

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Results

For 4000 *different values of a* $\rightarrow$ *success in 3997 of the cases, putting* 28 *initial conditions in the line.*

Observations

- *It is not necessary the* 28 *initial conditions in most cases. For* $a = i$, *only* 12 *are needed.*
- *Problem when we have two "close" roots.*
- *Reduction of the number of initial conditions!!*

Final

Thank you!