COMPLEX ANALYSIS DAVID ROSADO RODRÍGUEZ

Exercises #3

(6) For $z_1, \dots, z_m \in \mathbb{D}$ and $\gamma \in \mathbb{T}$, let

$$B(z) = \gamma \prod_{k=1}^{m} \frac{z - z_k}{1 - \overline{z_k} z} \qquad (1)$$

be a finite Blaschke product. Show that for $\zeta \in \mathbb{T}$

$$\zeta \frac{B'(\zeta)}{B(\zeta)} = \sum_{k=1}^{m} \frac{1 - |z_k|^2}{|\zeta - z_k|^2},$$

and therefore $B'(\zeta) \neq 0 \ \forall \zeta \in \mathbb{T}$.

Let us define $f_k(z) := \dfrac{z - z_k}{1 - \overline{z_k} z}$.

Taking logarithms of expression (1), we have that

$$\log(B(z)) = \log(\gamma) + \sum_{k=1}^{m} \log(f_k(z)).$$

Now, taking derivatives of the last expression, we obtain:

$$\frac{B'(z)}{B(z)} = \sum_{k=1}^{m} \frac{f_k'(z)}{f_k(z)}.$$

Notice that,

$$f_k'(z) = \frac{1 - \overline{z_k} z + (z - z_k)\overline{z_k}}{(1 - \overline{z_k} z)^2} = \frac{1 - z_k \overline{z_k}}{(1 - \overline{z_k} z)^2} = \frac{1 - |z_k|^2}{(1 - \overline{z_k} z)^2}.$$

Then,

$$\frac{f_k'(z)}{f_k(z)} = \frac{1-|z_k|^2}{(1-\bar{z}_k z)(z-z_k)} \Rightarrow \frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \frac{1-|z_k|^2}{(1-\bar{z}_k z)(z-z_k)}.$$

Now, for $\zeta \in \mathbb{T}$, $\zeta = e^{i\theta}$, $\theta \in [0, 2\pi)$, we have that

$$(1-\bar{z}_k e^{i\theta})(e^{i\theta} - z_k) = e^{i\theta}(1 - z_k e^{-i\theta} - \bar{z}_k e^{i\theta} + z_k \bar{z}_k)$$
$$= e^{i\theta}|e^{i\theta} - z_k|^2,$$

Since $|e^{i\theta} - z_k|^2 = (e^{i\theta} - z_k)(e^{-i\theta} - \bar{z}_k) = 1 - z_k e^{-i\theta} - \bar{z}_k e^{i\theta} + z_k \bar{z}_k.$

Then, for $\zeta \in \mathbb{T}$,

$$\frac{B'(\zeta)}{B(\zeta)} = \sum_{k=1}^{n} \frac{1-|z_k|^2}{\zeta|\zeta - z_k|^2} \Rightarrow \zeta \frac{B'(\zeta)}{B(\zeta)} = \sum_{k=1}^{n} \frac{1-|z_k|^2}{|\zeta - z_k|^2}. \quad (*)$$

To finish the exercise, let us prove a lemma:

Lemma If $\zeta \in \mathbb{T} \Rightarrow |B(\zeta)| = 1$.

Proof/

observe that, for $\zeta = e^{i\theta}$, $\theta \in [0, 2\pi)$,

equal

$$|e^{i\theta} - z_k|^2 = (e^{i\theta} - z_k)(e^{-i\theta} - \bar{z}_k) = 1 - \bar{z}_k e^{i\theta} - z_k e^{-i\theta} + z_k \bar{z}_k$$
$$|1 - \bar{z}_k e^{i\theta}|^2 = (1 - \bar{z}_k e^{i\theta})(1 - z_k e^{-i\theta}) = 1 - z_k e^{-i\theta} - \bar{z}_k e^{i\theta} + z_k \bar{z}_k$$

Then,

$$|B(\zeta)| = |\gamma| \prod_{k=1}^{n} \frac{|\zeta - z_k|}{|1 - \bar{z}_k \zeta|} \underset{\gamma \in \mathbb{T}}{=} 1. \quad \square$$

Therefore, taking modulus in $(*)$

$$|\zeta| \frac{|B'(\zeta)|}{|B(\zeta)|} \underset{(\text{lemma})}{=} |B'(\zeta)| = \left| \sum_{k=1}^{n} \frac{1-|z_k|^2}{|\zeta - z_k|^2} \right| \underset{(z_k \in \mathbb{D})}{>} 0 \quad \left(\Rightarrow B'(\zeta) \neq 0 \ \forall \zeta \in \mathbb{T} \right)$$