HARMONIC ANALYSIS

Fourier series exercises

⑦ Let $f \in \mathcal{C}^K[0,2\pi]$ with $f^{(j)}(0) = f^{(j)}(2\pi) \quad \forall j \leq K$.

Prove that $\lim\limits_{|n|\to\infty} |n|^K |\hat{f}(n)| = 0$

Let us prove it for $K=1$. Assume $K=1$ ($f \in \mathcal{C}^1[0,2\pi]$, $f(0)=f(2\pi)$).

Integrating by parts:

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{f(t)}_{u}\underbrace{e^{-int}\,dt}_{dv} = \frac{1}{2\pi}\left[f(t)\frac{e^{-int}}{-in}\right]_0^{2\pi} + \frac{1}{2\pi}\int_0^{2\pi} f'(t)\frac{e^{-int}}{in}\,dt$$

$$\underset{f(0)=f(2\pi)}{=} \frac{1}{2\pi}\int_0^{2\pi} f'(t)\frac{e^{-int}}{in}\,dt = \frac{\hat{f'}(n)}{in}$$

Therefore, $\hat{f'}(n) = in\,\hat{f}(n)$. Then,

$$\lim_{|n|\to\infty} |n||\hat{f}(n)| = \lim_{|n|\to\infty} |n|\frac{|\hat{f'}(n)|}{|in|} = \lim_{|n|\to\infty} |\hat{f'}(n)| = 0$$

(Riemann-Lebesgue lemma for continous functions)

To prove it for each K, notice that, by induction, we obtain that

$$\hat{f^{(K)}}(n) = (in)^K \hat{f}(n)$$

Therefore,

$$\lim_{|n|\to\infty} |n|^K |\hat{f}(n)| = \lim_{|n|\to\infty} |n|^K \frac{|\hat{f^{(K)}}(n)|}{|in|^K} = \lim_{|n|\to\infty} |\hat{f^{(K)}}(n)| = 0$$

$f \in \mathcal{C}^K[0,2\pi]$

Finally, let us prove the Riemann-Lebesgue lemma for continous functions.

Lemma (Riemann-Lebesgue) Given $f \in C[0,2\pi]$ $(f(0)=f(2\pi))$, $\lim\limits_{|n|\to\infty} |\hat{f}(n)| = 0$.

Proof:

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt = -\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}e^{in\pi/n}\,dt =$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-in(t-\pi/n)}\,dt \overset{t-\pi/n=s}{=} -\frac{1}{2\pi}\int_{-\pi/n}^{2\pi-\pi/n} f(s+\pi/n)e^{-ins}\,ds$$

$$\underset{2\pi\text{-periodic}}{=} -\frac{1}{2\pi}\int_0^{2\pi} f(s+\pi/n)e^{-ins}\,ds$$

Notice that,

$$\hat{f}(n) = \frac{\hat{f}(n)+\hat{f}(n)}{2} = \frac{1}{4\pi}\int_0^{2\pi}\big(f(t)-f(t+\pi/n)\big)e^{-int}\,dt$$

$$\Rightarrow |\hat{f}(n)| \leq \frac{1}{4\pi}\int_0^{2\pi}|f(t)-f(t+\pi/n)|\,dt$$

$$\underset{\substack{\text{continous function in}\\ \text{a compact set}}}{\leq} \frac{1}{4\pi}\max_{t\in[0,2\pi]}|f(t)-f(t+\pi/n)|\,2\pi \xrightarrow[n\to\infty]{} 0$$