Exercise 0.1. ***(optional)*** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $-f$ is also a convex function. Prove that there exist $a \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $f(x) = a^T x + c$*

Proof. Assume that f is a convex function and $-f$ is also a convex function. By the definition, we have that for all $\lambda \in [0,1]$ and for all $x, y \in \mathbb{R}^n$,

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leqslant \lambda f(x) + (1-\lambda) f(y), \\ -f(\lambda x + (1-\lambda)y) &\leqslant -\lambda f(x) - (1-\lambda) f(y). \end{aligned}$$

This implies that

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y),$$

Let $g(x) = f(x) - f(0)$. If we are able to see that g is linear, we will obtain that $f(x) - f(0)$ is linear, hence f will be an affine function, i.e, $f(x) = a^T x + c$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Let us see that g is a linear function. We have to proof that $g(\lambda x) = \lambda g(x)$ and $g(x+y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}^n$ and for all $\lambda \in \mathbb{R}$. For the first one, if we set $y = 0$, we obtain that

$$g(\lambda x) = f(\lambda x) - f(0) = \lambda f(x) + (1-\lambda) f(0) - f(0) = \lambda(f(x) - f(0)) = \lambda g(x).$$

For the second one, if we set $\lambda = \frac{1}{2}$, we obtain that

$$\begin{aligned} g\left(\frac{x}{2} + \frac{y}{2}\right) = f\left(\frac{x}{2} + \frac{y}{2}\right) - f(0) &= \frac{1}{2}(f(x) - f(0)) + \frac{1}{2}(f(y) - f(0)) = \frac{1}{2} g(x) + \frac{1}{2} g(y) \\ &= g\left(\frac{x}{2}\right) + g\left(\frac{y}{2}\right). \end{aligned}$$

□

Exercise 0.2. *Use the Kuhn-Tucker conditions to solve the following problems*

$$(a)\begin{cases} \text{Min } f(x) = x_1 x_2 \\ \text{subject to} \\ x_1 + x_2 \geq 2 \\ x_2 \geq x_1 \end{cases} \quad (b)\begin{cases} \text{Min } f(x) = (x_1 - 1)^2 + x_2 - 2 \\ \text{subject to} \\ x_2 - x_1 = 1 \\ x_1 + x_2 \leqslant 2 \end{cases} \quad (c)\begin{cases} \text{Min } f(x) = x_1^2 + 2x_2^2 + 3x_3^2 \\ \text{subject to} \\ x_1 - x_2 - 2x_3 \leqslant 12 \\ x_1 + 2x_2 - 3x_3 \leqslant 8 \end{cases}$$

Proof. Let us recall the Kuhn-Tucker conditions. Consider the following problem:

$$\begin{cases} \text{Min } f(x) \\ \text{subject to} \\ g_i(x) \leqslant 0 \quad i \in I = \{1, \ldots, m\} \\ h_l(x) = 0 \quad l \in L = \{1, \ldots, p\} \\ x \in \mathbb{R}^n \end{cases}$$

Theorem 0.3. *A necessary condition for x^0 to be a local minimum of the previous problem is that there exist numbers $\lambda_i \geq 0$ $(i \in I)$ and μ_l $(l \in L)$ (μ_l not sign-restricted) such that*

$$\begin{cases} \nabla f(x^0) + \sum_{i \in I} \lambda_i \nabla g_i(x^0) + \sum_{l \in I} \mu_i \nabla h_l(x^0) = 0 \\ \lambda_i g_i(x^0) = 0 \ \ (\forall i \in I) \end{cases}$$

Let us start solving (a). Maximize a function f is equivalent to minimize $-f$,thus we can rewrite the problem as

$$\begin{cases} \text{Min } \; f(x) = x_1 x_2 \\ \text{subject to} \\ x_1 + x_2 \geq 2 \\ x_2 \geq x_1 \end{cases}$$

Calling $g_1(x) = -x_1 - x_2 + 2$ and $g_2(x) = x_1 - x_2$, the Kuhn-Tucker conditions becomes to solve the following system:

$$\begin{cases} x_2 - \lambda_1 + \lambda_2 = 0 \\ x_1 - \lambda_1 - \lambda_2 = 0 \\ \lambda_1(-x_1 - x_2 + 2) = 0 \\ \lambda_2(x_1 - x_2) = 0 \end{cases}$$

Notice that from the las two equations, we have that $\lambda_1 = 0$ or $-x_1 - x_2 + 2 = 0$ and $\lambda_2 = 0$ or $x_1 - x_2 = 0$. Therefore, we can distinct four cases.

- Assume $\lambda_1 = \lambda_2 = 0$. Then, from the first two equations, $x_1 = x_2 = 0$, but the point $(0, 0)$ is not satisfying the constrains.
- Assume $\lambda_2 = 0$ and $-x_1 - x_2 + 2 = 0$. In this case, $x_1 = -x_2 + 2$ and the first two equations becomes to the following system

$$\begin{cases} x_2 - \lambda_1 = 0 \\ -x_2 + 2 - \lambda_1 = 0 \end{cases}$$

  which implies that $x_2 = -x_2 + 2$ and in consequence $x_2 = 1, x_1 = 1$ and $\lambda_1 = 1$. Since $\lambda \geq 0$, and $(1, 1)$ is satisfying the constrains, $(1, 1)$ is a candidate to be a local minimum of the problem.
- Assume $\lambda_1 = 0$ and $x_1 = x_2$. In this case, the first two equations becomes to the following system

$$\begin{cases} x_2 + \lambda_2 = 0 \\ x_2 - \lambda_2 = 0 \end{cases}$$

  which implies that $x_2 = -x_2$ and in consequence $x_2 = 0$ and $\lambda_2 = 0$. Therefore $x_1 = 0$, but the point $(0, 0)$ is not satisfying the constrains.
- Assume $-x_1 - x_2 + 2 = 0$ and $x_1 = x_2$. In this case, notice that this first assumption becomes $-x_1 - x_1 + 2 = 0$, which implies that $x_1 = 1$ and in consequence $x_2 = 1$. From the first two equations, we have

$$\begin{cases} 1 - \lambda_1 + \lambda_2 = 0 \\ 1 - \lambda_1 - \lambda_2 = 0 \end{cases}$$

  which implies $\lambda_2 = 0$ and $\lambda_1 = 1$. Since $\lambda \geq 0$, and $(1, 1)$ is satisfying the constrains, $(1, 1)$ is a candidate to be a local minimum of the problem.

Nevertheless, $(1,1)$ is not a minimum of the problem, notice that $f(1,1) > f(-1,5)$ and $(-1,5)$ is satisfying the constrains. Therefore, the problem does not have solutions.

Let us continue showing (b). Calling $g(x) = x_1+x_2-2$ and $h(x) = x_2-x_1-1$, the Kuhn-Tucker conditions becomes to solve the following system:

$$\begin{cases} 2x_1 - 2 + \lambda - \mu = 0 \\ 1 + \lambda + \mu = 0 \\ \lambda(x_1 + x_2 - 2) = 0 \end{cases}$$

From the last equation, we can distinct two cases, either $\lambda = 0$ or $x_1 + x_2 - 2 = 0$.

- Assume $\lambda = 0$. In this case, the first two equations becomes to the following system

$$\begin{cases} 2x_1 - 2 - \mu = 0 \\ 1 + \mu = 0 \end{cases}$$

  Then $\mu = -1$ and $x_1 = \frac{1}{2}$. Since $h(x) = 0$, we have that $x_2 = x_1 + 1$, so $x_2 = \frac{3}{2}$. Since $\lambda \geq 0$ and $(\frac{1}{2}, \frac{3}{2})$ is satisfying the constrains, $(\frac{1}{2}, \frac{3}{2})$ is a candidate to be a local minimum of the problem.

- Assume now $x_1 + x_2 - 2 = 0$, then, since $h(x) = 0$, we have that $x_2 - x_1 - 1 = 0$. Solving this system, one can find that $(x_1, x_2) = (\frac{1}{2}, \frac{3}{2})$. Then, the first two equations becomes to the following system

$$\begin{cases} -1 + \lambda - \mu = 0 \\ 1 + \lambda + \mu = 0 \end{cases}$$

  which implies that $\lambda = 0$ and $\mu = -1$. Since $\lambda \geq 0$ and $(\frac{1}{2}, \frac{3}{2})$ is satisfying the constrains, this point is also a candidate to be a local minimum of the problem.

Let us see if $(x_1, x_2) = (\frac{1}{2}, \frac{3}{2})$ is the solution of the problem by computing the Hessian matrix and observing that is positive semi-definite. Notice that the Hessian matrix is given by

$$H_f = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

Since all the eigenvalues of this matrix are positive, the matrix is clearly semipositve definite, then, the point $(x_1, x_2) = (\frac{1}{2}, \frac{3}{2})$ is a minimum.

Let us continue with (c). Calling $g_1(x) = x_1 - x_2 - 2x_3 - 12$ and $g_2(x) = x_1 + 2x_2 - 3x_3 - 8$, the Kuhn-Tucker conditions becomes to solve the following system:

$$\begin{cases} 2x_1 + \lambda_1 + \lambda_2 = 0 \\ 4x_2 - \lambda_1 + 2\lambda_2 = 0 \\ 6x_3 - 2\lambda_1 - 3\lambda_2 = 0 \\ \lambda_1(x_1 - x_2 - 2x_3 - 12) = 0 \\ \lambda_2(x_1 + 2x_2 - 3x_3 - 8) = 0 \end{cases}$$

From the last two equations we have that $\lambda_1 = 0$ or $x_1 - x_2 - 2x_3 - 12 = 0$ and $\lambda_2 = 0$ or $x_1 + 2x_2 - 3x_3 - 8 = 0$. Therefore, we can distinct four cases.

- Assume $\lambda_1 = \lambda_2 = 0$. In this case, the first three equations becomes to solve $(2x_1, 4x_2, 6x_3) = (0, 0, 0)$ which implies that $(x_1, x_2, x_3) = (0, 0, 0)$. Since $\lambda_1 = \lambda_2 \geq 0$ and the point $(0, 0, 0)$ is satisfying the constrains, this point is a candidate to be a local minimum of the problem.

- Assume $\lambda_1 = 0$ and $x_1 + 2x_2 - 3x_3 - 8 = 0$. Then $x_1 = -2x_2 + 3x_3 + 8$ and the first three equations becomes to the following system

$$\begin{cases} -4x_2 + 6x_3 + 16 + \lambda_2 = 0 \\ 2x_2 + \lambda_2 = 0 \\ 2x_3 - \lambda_2 = 0 \end{cases}$$

  Then, from the last two equations, we have that $x_2 = -x_3$ and the first equation becomes $-10x_2 + \lambda_2 + 16 = 0$. Since $\lambda_2 = -2x_2$, we have that $x_2 = \frac{4}{3}$ and in consequence $x_3 = -\frac{4}{3}$, $x_1 = \frac{4}{3}$ and $\lambda_2 = -\frac{8}{3}$. But notice that $\lambda_2 < 0$, so the system has no solutions in the Kuhn-Tucker sense, as all λ_i must be positive.

- Assume $\lambda_2 = 0$ and $x_1 - x_2 - 2x_3 - 12 = 0$. Then $x_1 = x_2 + 2x_3 + 12$ and the first three equations becomes to the following system

$$\begin{cases} 2x_2 + 4x_3 + 24 + \lambda_1 = 0 \\ 4x_2 - \lambda_1 = 0 \\ 3x_3 - \lambda_1 = 0 \end{cases}$$

  Then, from the last two equations, we have that $4x_2 = 3x_3$ and the first equation becomes $\frac{11}{2}x_3 + 24 + \lambda_1 = 0$. Since $\lambda_1 = 3x_3$, we have that $x_3 = -\frac{48}{17}$ and in consequence $x_2 = -\frac{36}{17}$, $x_1 = \frac{72}{17}$ and $\lambda_1 = \frac{-144}{17}$. But notice that $\lambda_1 < 0$, so the system has no solutions in the Kuhn-Tucker sense, as all λ_i must be positive.

- Assume $x_1 - x_2 - 2x_3 - 12 = 0$ and $x_1 + 2x_2 - 3x_3 - 8 = 0$. Then $x_2 + 2x_3 + 12 = -2x_2 + 3x_3 + 8$, which implies that $3x_2 - x_3 + 4 = 0$, i.e., $x_3 = 3x_2 + 4$. The first three equations becomes to the following system

$$\begin{cases} 14x_2 + 40 + \lambda_1 + \lambda_2 = 0 \\ 4x_2 - \lambda_1 + 2\lambda_2 = 0 \\ 18x_2 + 24 - 2\lambda_1 - 3\lambda_2 = 0 \end{cases}$$

  The solution of this system[1], is $(\lambda_1, \lambda_2, x_2) = (-\frac{112}{13}, \frac{8}{39}, -\frac{88}{39})$. Notice that $\lambda_1 < 0$, so the system has no solutions in the Kuhn-Tucker sense, as all λ_i must be positive.

Let us see if $(x_1, x_2, x_3) = (0, 0, 0)$ is the solution of the problem by computing the Hessian matrix and observing that is positive semi-definite. Notice that the Hessian matrix is given by

$$H_f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Since all the eigenvalues of this matrix are positive, the matrix is clearly semipostive definite, actually, it is positive definite and in consequence, the point $(x_1, x_2, x_3) = (0, 0, 0)$ is a minimum. $\square$

[1]This system is of the form $Az = b$, with A non singular matrix

Exercise 0.4. ***(optional)*** *Consider the problem*

$$\begin{cases} \text{Min } f(x) \\ \text{subject to} \\ g(x) \leqslant 0 \\ x \in S \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are two convex functions and $S \in \mathbb{R}^n$ is a convex set. If x^* is an optimal solution of this problem such that $g(x^*) < 0$, show that x^* is also an optimal solution of the problem

$$\begin{cases} \text{Min } f(x) \\ \text{subject to} \\ x \in S \end{cases}$$

Proof. Notice that, if x^* is an optimal solution of the first problem and $g(x^*) < 0$, then, there exist a neighborhood of x^*, $B(x^*, \epsilon)$, for which $g(B(x^*, \epsilon)) \leqslant 0$, due to the convexity of g(actually we only need continuity). Therefore, x^* is a local minimum of f and since f is a convex function and we are in a convex set, this local minimum, must be a global minimum. Hence, x^* is also the solution of the second problem. □