This document corresponds to the explanation of the long exercise proposed on November 10 and its main goal is to show and explain the software used to solve the exercise, as well as, interpret the results and extract some conclusions. The delivery consists in three files, a sage code, a book of algebraic topology used as a reference and this *pdf* file.

The longer exercise requires the computation of homology groups with coefficients in $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{F}_2$ of two abstract simplicial complexes, K and L, whose maximal faces are described in the Virtual Campus. In previous exercises, we have seen that the geometric realizations of these abstract simplicial complex are homeomorphic to a real projective plane and a torus respectively. This information is relevant because there is an isomorphism between the homology group of the abstract simplicial complex and their geometric realization. More precisely,

Theorem 0.1. *Given an ordered simplicial complex K and its geometric realization $|K|$, if $|K| \cong Y$, the morphism*

$$H_*(v) : H_*(K) \to H_*(Y)$$

is an isomorphism.[1]

In our case, $|K| \cong \mathbb{RP}^2$ and $|L| \cong \mathbb{T}$, hence, by applying the previous theorem, we obtain that $H_*(K) \cong H_*(\mathbb{RP}^2)$ and $H_*(L) \cong H_*(\mathbb{T})$. Therefore, notice that we only have to compute the homology groups of the real projective plane and the torus. Nevertheless, in our code, we compute the homology groups of all of them in order to corroborate the theory explained.

The software used to compute homology groups is *SageMath*.[2] We have chosen this software, essentially because is the one which is recommended and also because it is easy and intuitive to use. We have used the online source provided by this software, called *CoCalc*. As you can see in the deliver code, the implementation is really simple. We have defined the simplicial complexes K and L with the function *SimplicialComplex* provided by *CoCalc*, in which we have to add the maximal faces of the abstract simplicial complex you are working with, and then, we have computed the homology groups with the function *homology*, specifying the ring, ZZ for $\mathbb{Z}$, QQ for $\mathbb{Q}$ and $GF(2)$ for $\mathbb{F}_2$. The first part of the code corresponds to the computation of the homology groups by considering the abstract simplicial complexes K and L and the second part, corresponds to the corroboration of the computation, in where we determine the homology groups of the real projective plane and the torus.

Notice that this sage document is a notebook in which we have cells and we can execute them to obtain the results. For example, notice that

$$H_p(K;\mathbb{Z}) = \begin{cases} \mathbb{Z} \ \text{ if } p = 0 \\ \mathbb{Z}/2\mathbb{Z} \ \text{ if } p = 1 \\ 0 \ \text{ otherwise} \end{cases}$$

Be careful with the notation! In the sage program, the group of two elements $\mathbb{Z}/2\mathbb{Z}$ is given by C_2. As we can expect, in the homology group of K with coefficients in $\mathbb{Q}$, the torsion group is not appearing and the first homology group is a vector space of dimension 0 over $\mathbb{Q}$. Nevertheless, for the $\mathbb{F}_2$ case, the first homology group corresponds to a vector space of dimension 1 over $\mathbb{F}_2$, so we can see reflected the torsion group that appears in $H_1(K;\mathbb{Z})$. As

[1] To see more details and the proof of the theorem, you can review the book attached in the delivery. More precisely, Theorem 1.6 and Theorem 9.2.

[2] https://www.sagemath.org/

we have said, this homology groups coincide with the homology groups of the real projective plane, as expected. In the case of L notice that,

$$H_p(L;\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } p = 0 \\ \mathbb{Z} \times \mathbb{Z} \quad \text{if } p = 1 \\ \mathbb{Z} \text{ if } p = 2 \\ 0 \text{ otherwise} \end{cases}$$

The results of the homology group with coefficients in $\mathbb{Q}$ and $\mathbb{F}_2$ are what we expected, a vector space of dimension 1 for H_0, a vector space of dimension 2 for H_1 and a vector space of dimension 1 for H_2 in both cases. As we have said, this homology groups coincide with the homology groups of the torus, as was foreseeable.