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Exercise 13. (Compact operators)

a) Every bounded linear operator with finite rank is compact.

Assume that we have a bounded sequence $\{x_j\}$ in E. We want to see that $\{Tx_j\}$ has a convergent subsequence.
Notice that, since T has finite rank, $\operatorname{Im} T$ is a finite dimensional normed space. Moreover, since T is a bounded linear operator ($\exists M>0$ such that $\forall x \in E$ $\|Tx\| \leq M\|x\|$), the sequence $\{Tx_j\}$ is bounded in $\operatorname{Im} T$. By applying Bolzano-Weierstrass theorem, we can affirm the existence of a convergent subsequence for the sequence $\{Tx_j\}$.

□

b) If a compact operator T is of infinite rank, then T is not bounded below.

Let us prove that if T is of infinite rank and is bounded below, then T is not compact.

By the restriction of T, we get a bijective bounded linear operator

$$S : (\operatorname{Ker} T)^{\perp} \rightarrow \operatorname{Im} T$$

Now, using the open mapping theorem, S maps open sets to open sets. Let U be the open unit ball in $(\operatorname{Ker} T)^{\perp}$.
Notice that $S(U) \neq \{0\}$ and is an open set. Then $\overline{S(U)}$ can not be a compact subset of $\operatorname{Im} T$ because $\dim \operatorname{Im} T = \infty$.
Hence, T is not compact.

□

c) If T is a compact operator $\Rightarrow R(T)$ is separable

First, let us see an auxilary lemma.

Lemma If T is a compact operator and $A \subset E$ a bounded set, then the set $\overline{T(A)}$ is compact.

Proof lemma/

Assume that T is compact. Let $A \subset E$ bounded and suppose that exist a sequence $\{x_m\}$ in $\overline{T(A)}$. Then, for each $m \in \mathbb{N}$, there exist $y_m \in A$ such that $\| x_m - Ty_m \| < m^{-1}$ and the sequence $\{y_m\} \subset A$ is bounded. Since T is compact, the sequence $\{Ty_m\}$ contains a convergent subsequent, hence $\{x_m\}$ contains a convergent subsequent with limit in $\overline{T(A)}$. Then, $\overline{T(A)}$ is compact.

□

Now, let us prove the statement.

Let $B_r(0)$ be the ball of radius $r \in \mathbb{N}$ centered at the origin. Since $B_r(0)$ is bounded and T is compact, we have that $\overline{T(B_r(0))}$ is compact $\circledast$. Since $\overline{T(B_r(0))}$ is compact, is separable and, since $\overline{T(B_r(0))}$ is separable and $T(B_r(0)) \subseteq \overline{T(B_r(0))}$, we have that $T(B_r(0))$ is separable.

Finally, $\operatorname{Im} T$ is equal to the countable union $\bigcup_{r=1}^{\infty} T(B_r(0))$ of separable sets. Then, $R(T)$ is separable

$\circledast$ We have used the lemma here!!

□

d) If T is a bounded linear operator and there is a sequence $\{T_j\}$ of operators of finite rank such that $\|T_j - T\| \to 0$, then T is compact.

Since T is bounded and $\|T_j - T\| \to 0$, $\{T_j\}$ is a sequence of bounded operators of finite rank. Using 13 a), we can affirm that $\{T_j\}$ is a sequence of compact operators.

Let us prove that T is compact. Suppose $\{x_n\}$ a bounded sequence in E. We want to see that the sequence $\{Tx_n\}$ has a convergent subsequent.

By compactness of $\{T_n\}$, there exist a subsequent of $\{x_n\}$, which we label $x^1_{n_k}$, such that the sequence $\{T_1 x^1_{n_k}\}$ converges. Similarly, there exist a subsequence $\{x^2_{n_k}\}$ of $\{x^1_{n_k}\}$ such that $\{T_2 x^2_{n_k}\}$ and $\{T_1 x^2_{n_k}\}$ converges. Repeating the process, there exist a subsequence $\{x^l_{n_k}\}$ such that $\forall r \leq l$, the sequence $\{T_r x^l_{n_k}\}$ converges.

Notice that, as $l \to \infty$, we can find a single sequence $\{x^l_{n_l}\}$ such that for each $r \in \mathbb{N}$, $\{T_r x^l_{n_l}\}$ converges.

Now, we will show that the sequence $\{T x^l_{n_l}\} := \{T x_{n_l}\}$ converge. Since E is a Banach space, we will see that $\{T x_{n_l}\}$ is a Cauchy sequence. Fix $K \in \mathbb{N}$.

$$\|Tx_{n_l} - Tx_{n_s}\| \leq \|Tx_{n_l} - T_K x_{n_l}\| + \|T_K x_{n_l} - T_K x_{n_s}\| + \|T_K x_{n_s} - Tx_{n_s}\|$$

$$\leq \|T - T_K\| \|x_{n_l}\| + \|T_K x_{n_l} - T_K x_{n_s}\| + \|T - T_K\| \|x_{n_s}\|$$

Then, If K is big enough, we have:

- $\|T - T_K\| \|x_{m_\ell}\| \to 0$. Since $\{x_{m_\ell}\}$ bounded and $\|T_K - T\| \to 0$ by hipothesis

- $\|T_K x_{m_\ell} - T_K x_{m_s}\| \to 0$ Since $\{T_K x_{m_\ell}\}$ converge and E is a Banach space ($\Rightarrow$ Cauchy)

- $\|T - T_K\| \|x_{m_s}\| \to 0$ Same reason of the first one

Then, $\|Tx_{m_\ell} - Tx_{m_s}\| \to 0$ and $\{Tx_{m_\ell}\}$ is a Cauchy sequence. Hence, T is compact.

$\square$

e) Conversely, if T is a compact operator on a Hilbert space E, then T is the norm limit of operators of finite rank.

Again, we need an auxilary lemma

Lemma If H is a Hilbert space and $Y \subset H$ is a linear subspace, then Y is a Hilbert space iff Y is closed $\square$

Let us prove the statement.

If T had finite rank it would be obvious, let us assume that T has infinite rank. Using the lemma and the ideas of 13 c), $\overline{\operatorname{Im} T}$ is of infinite dimension and separable Hilbert space, so it has an orthonormal basis $\{e_n\}$.

Let P_K be the orthogonal projection from $\overline{\operatorname{Im} T}$ onto $\operatorname{Span}\{e_1, \dots, e_K\}$ and let $T_K = P_K T$. Since $\operatorname{Im} T_K \subset \operatorname{Span}\{e_1, \dots, e_K\}$, T_K is of finite rank. We will show that $\|T_K - T\| \to 0$

Suppose that $\|T_K - T\| \not\to 0$. Then, $\exists \varepsilon > 0$ such that $\|T_K - T\| \geq \varepsilon \ \forall K$. Thus, there exist a sequence of unit vectors $\{x_K\}$ such that $\|(T_K - T)x_K\| \geq \frac{\varepsilon}{2} \ \forall K$. Since T is compact, we may suppose $Tx_K \to y$ for some $y \in \bar{E}$ (after taking a subsequence if necessary). Now, the following equalities are true:

$$(T_K - T)x_K = (P_K - I)Tx_K = (P_K - I)y + (P_K - I)(Tx_K - y)$$
$$= -\sum_{n=K+1}^{\infty} \langle y, e_n \rangle e_n + (P_K - I)(Tx_K - y)$$

Taking norms and using that $\|P_K\| = 1$, we obtain that

$$\frac{\varepsilon}{2} \leq \|(T_K - T)x_K\| \underset{\|P_K - I\| \leq 2}{\leq} \left(\sum_{n=K+1}^{\infty} \langle y, e_n \rangle^2 \right)^{1/2} + 2\|Tx_K - y\|$$

The right-hand side of the inequality tends to 0 when $K \to \infty$, which is a contradiction and proves the theorem.

$\square$