TOPOLOGICAL DATA ANALYSIS

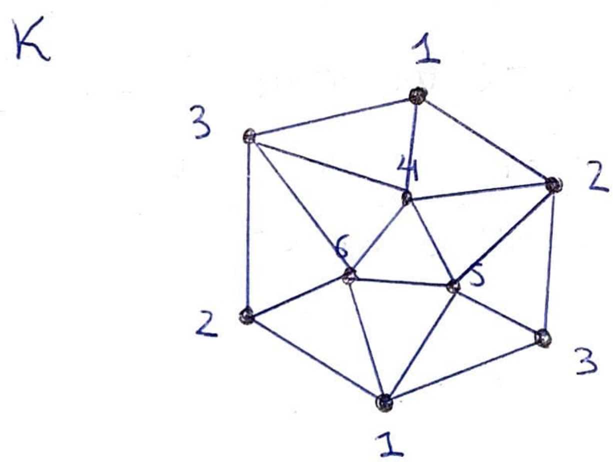
① Let $K$ and $L$ be the abstract simplicial complexes whose maximal faces are, respectively,

a) $K$: (124) (125) (135) (136) (146) (234) (236) (256) (345) (456)

b) $L$: (014) (015) (023) (027) (035) (047) (126) (128) (148) (156) (236) (278) (346) (348) (358) (467) (567) (578)

Prove that the geometric realizations $|K|$ and $|L|$ are compact surfaces, and find out which surfaces they are.

Let us start by drawing $K$.



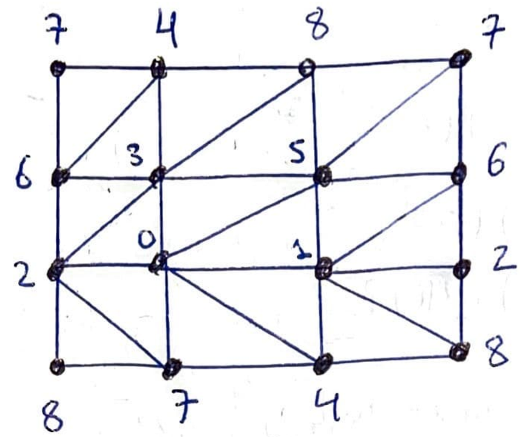

Notice that this is a triangulation of the real projective plane. Since this surface is a quotient of two compact surfaces is compact. Moreover, notice that the Euler characteristic is 1, hence, applying the classification theorem for compact surfaces, we obtain that this surface is, in fact, as we said, a projective plane.

Remark: Remember that the Euler characteristic is given by

$$\chi = \text{vertices} - \text{edges} + \text{faces}$$

observation: In this case, $\chi = 6 - 15 + 10 = 1$

Let us continue with L. If we draw L, we obtain the following:

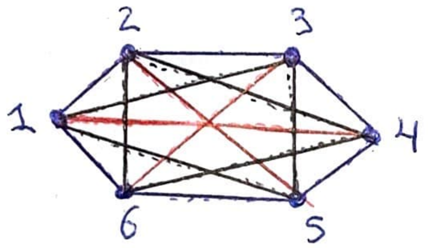



This is a triangulation of the Klein bottle. Again, since the quotient of two compacts is compact, this surface is compact. Moreover, if we compute the Euler characteristic, we obtain $\chi = 0$. Taking into account the orientability of this surface, applying the classification theorem for compact surfaces, we obtain that this is a Klein bottle.

TOPOLOGICAL DATA ANALYSIS

② List the maximal faces of the Čech complex $C_\varepsilon(X)$ and the Vietoris-Rips complex $R_\varepsilon(X)$, depending on $\varepsilon$, if $X$ is the set of vertices of a regular hexagon of radius 1.

Assume $X=\{x_1, x_2, x_3, x_4, x_5, x_6\}$ the set of vertices of a regular hexagon of radius 1. To simplify notation, we will denote $X=\{1,2,3,4,5,6\}$.




Using basic geometry and Pythagoras theorem, one can see that the distance between two vertices connected with a blue line is 1, the distance between two vertices connected with a black line is $\sqrt{3}$ and the distance between two vertices connected with a red line is 2.

With this information, we can easily describe Vietoris-Rips complex:

- For $0 \leq \varepsilon < 1$, $R_\varepsilon(X)$: (1) (2) (3) (4) (5) (6)
- For $1 \leq \varepsilon < \sqrt{3}$, $R_\varepsilon(X)$: (12) (16) (23) (34) (45) (56)
- For $\sqrt{3} \leq \varepsilon < 2$, $R_\varepsilon(X)$: (123) (234) (345) (456)
  (126) (246) (351)
  (156)
- For $2 \leq \varepsilon$, $R_\varepsilon(X)$: (123456)

Let us describe now the Čech complex:

- For $0 \leq \varepsilon < 1$, $C_\varepsilon(X)$: (1) (2) (3) (4) (5) (6)
- For $1 \leq \varepsilon < \sqrt{3}$, $C_\varepsilon(X)$: (12) (16) (23) (34) (45) (56)

- For $\sqrt{3} \leq \varepsilon < 2$, $C_\varepsilon(X)$: $(123)\,(126)\,(156)\,(234)\,(345)\,(456)$

- For $2 \leq \varepsilon$, $C_\varepsilon(X)$: $(123456)$

Remark: The only difference between Čech and Vietoris-Rips complexs is in $\sqrt{3} \leq \varepsilon < 2$. The Čech complex is a cylinder while the Vietoris Rips complex is an octahedron. This is due to the barycenter of the triangles $(246)$ and $(135)$.