TOPOLOGICAL DATA ANALYSIS

① Prove that a morphism f of persistence modules is an isomorphism if and only if f_t is an isomorphism of vector spaces for all t

Assume that $f:(V,\pi) \longrightarrow (V',\pi')$ is an isomorphism of persistence modules. Then, there exist a morphism $g:(V',\pi') \longrightarrow (V,\pi)$ such that $g\circ f = id_V$ and $f\circ g = id_{V'}$. This implies the following:

$$\begin{cases} (f\circ g)_t = f_t \circ g_t = id_t & \text{for all } t \\ (g\circ f)_t = g_t \circ f_t = id_t & \text{for all } t \end{cases}$$

This implies that f_t is invertible for all t and $g_t = f^{-1}_t$. Thus, f_t is a bijection and in consequence f_t is an isomorphism of vector spaces for all t.

Conversily, if f_t is an isomorphism of vector spaces for all t, there exist f_t^{-1} and it is well-defined. Let us define a morphism $g:(V',\pi') \longrightarrow (V,\pi)$ of persistence modules, such that $g_t: V'_t \longrightarrow V_t$ is defined as $g_t := f^{-1}_t$ for all t. Let us check that g is actually a morphism of persistence modules:

Since $f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$ for $s\leq t$ and f_t is an isomorphism of vector spaces for all t, we obtain that $\pi_{s,t}\circ f_s^{-1} = f_t^{-1} \circ \pi'_{s,t}$.

Hence, g is a morphism of persistence modules. Let us review this argument with diagrams:

$$\begin{tikzcd} V_s \arrow[r, "f_s"] \arrow[d, "\pi_{s,t}"'] \arrow[dr, phantom, "///"] & V'_s \arrow[d, "\pi'_{s,t}"] \\ V_t \arrow[r, "f_t"'] & V'_t \end{tikzcd} \rightsquigarrow \begin{tikzcd} V'_s \arrow[r, "f_s^{-1}"] \arrow[d, "\pi'_{s,t}"'] \arrow[dr, phantom, "///"] & V_s \arrow[d, "\pi_{s,t}"] \\ V'_t \arrow[r, "f_t^{-1}"'] & V_t \end{tikzcd}$$

Moreover, to confirm that f is an isomorphism of persistence modules and conclude the proof, we have to see that $g \circ f = id_V$ and $f \circ g = id_{V'}$. This is immediate since:

$$\begin{cases} (g \circ f)_t = g_t \circ f_t = f_t^{-1} \circ f_t = id_t \\ (f \circ g)_t = f_t \circ g_t = f_t \circ f_t^{-1} = id_t \end{cases}$$

$\square$

② Prove that two isomorphic persistence modules of finite type have the same spectrum

Assume that $f:(V,\pi) \longrightarrow (V',\pi')$ is an isomorphism of persistence modules of finite type and suppose that they have different spectrums. Let us call $A = \{a_0, \dots, a_m\}$ and $A' = \{a'_0, \dots, a'_m\}$ the spectrums of (V,π) and (V',π') respectively. Notice that if $A \neq A'$, there must exist $a_i \neq a'_j$ for som i and j. We have the following conditions:

- If $a_i \neq a'_j$, there exist $\eta > 0$ such that $a'_j \notin [a_i - \eta, a_i + \eta]$.
- If $a_i \in A$, there exist $\varepsilon > 0$ such that if $a_i \leq t < a_i + \varepsilon$, then $\pi_{a_i,t}$ is an isomorphism while if $a_i - \varepsilon < s < a$, then π_{s,a_i} is not an isomorphism
- If $a_i \notin A'$, there exist $\delta > 0$ such that $\pi_{s,t}$ is an isomorphism for $a_i - \delta < s \leq t < a_i + \delta$.

Choose $\nu = \min\{\eta, \varepsilon, \delta\}$ and notice that there exist s such that $a_i - \nu < s < a_i$ and π_{s,a_i} is not an isomorphism and π'_{s,a_i} is an isomorphism. Notice that we have the following commutative diagram:

$$\begin{tikzcd}
V_s \arrow[r, "f_s"] \arrow[d, "\pi_{s,a_i}"'] \arrow[dr, phantom, "\circlearrowleft"] & V_s' \arrow[d, "\pi'_{s,a_i}"] \\
V_{a_i} \arrow[r, "f_{a_i}"'] & V'_{a_i}
\end{tikzcd}
\qquad f_{a_i} \circ \pi_{s,a_i} = \pi'_{s,a_i} \circ f_s$$

Moreover, since f is an isomorphism, we have that f_s and f_{a_i} are isomorphism. Then, $\pi_{s,a_i} = f_{a_i}^{-1} \circ \pi'_{s,a_i} \circ f_s$ is an isomorphism, since the composition of isomorphism is an isomorphism. Thus, we reach a contradiction which comes from assuming $A \neq A'$. Therefore, $A = A'$ and we conclude the proof.

$\square$

③ Prove that there is a nonzero morphism $\mathbb{F}[a,b) \longrightarrow \mathbb{F}[c,d)$ if and only if $c \leq a$ and $a < d \leq b$.

Let us prove first that if $c \leq a$ and $a < d \leq b$, then, there exist a nonzero morphism $\mathbb{F}[a,b) \longrightarrow \mathbb{F}[c,d)$. Let us draw the values to keep them in mind.

Notice that if $a = c$ and $b = d$, the problem becomes immediate since the identity morphism satisfy what we wanted. Assume then $c < a$ and $d < b$.

Let us define $f_t : \mathbb{F}[a,b)_t \longrightarrow \mathbb{F}[c,d)_t$ such that

$$f_t = \begin{cases} 0 & \text{if } t \in [c,a) \\ \mathrm{Id} & \text{if } t \in [a,d) \\ 0 & \text{if } t \in [d,b) \\ 0 & \text{otherwise} \end{cases}$$

We claim that f is a non-zero morphism $\mathbb{F}[a,b) \longrightarrow \mathbb{F}[c,d)$. To prove this, we have to see that for $s \leq t$, is satisfied the following:

$$f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$$

Notice that if $t < c$ or $s \geq b$ this is immediate. Let us check the interesting cases.

① Assume $t \in [c,a)$. Then, either $s < c$ or $s \in [c,a)$.

⊙ If $s < c$, $\mathbb{F}[a,b)_s = \mathbb{F}[c,d)_s = 0$ and $\mathbb{F}[a,b)_t = 0$ but $\mathbb{F}[c,d)_t = \mathbb{F}$.

The diagram becomes

$$\begin{tikzcd}
0 \arrow[r, "f_s = 0"] \arrow[d, "\pi_{s,t} = 0"'] \arrow[dr, phantom, "="] & 0 \arrow[d, "\pi'_{s,t} = 0"] \\
0 \arrow[r, "f_t = 0"'] & \mathbb{F}
\end{tikzcd}$$

$f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$ ✓

⊙ If $s \in [c,a)$, $\mathbb{F}[a,b)_s = 0$, $\mathbb{F}[c,d)_s = \mathbb{F}$ and $\mathbb{F}[a,b)_t = 0$, $\mathbb{F}[c,d)_t = \mathbb{F}$

The diagram becomes

$$\begin{tikzcd}
0 \arrow[r, "f_s = 0"] \arrow[d, "\pi_{s,t} = 0"'] \arrow[dr, phantom, "="] & \mathbb{F} \arrow[d, "\pi'_{s,t} = id"] \\
0 \arrow[r, "f_t = 0"'] & \mathbb{F}
\end{tikzcd}$$

$f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$ ✓

② Assume $t \in [a,d)$. Then, either $s < c$, $s \in [c,a)$ or $s \in [a,d)$.

⊙ If $s < c$, $\mathbb{F}[a,b)_s = \mathbb{F}[c,d)_s = 0$ and $\mathbb{F}[a,b)_t = \mathbb{F}[c,d)_t = \mathbb{F}$

The diagram becomes

$$\begin{tikzcd}
0 \arrow[r, "f_s = 0"] \arrow[d, "\pi_{s,t} = 0"'] \arrow[dr, phantom, "="] & 0 \arrow[d, "\pi'_{s,t} = 0"] \\
\mathbb{F} \arrow[r, "f_t = id"'] & \mathbb{F}
\end{tikzcd}$$

$f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$ ✓

⊙ If $s \in [c,a)$, $\mathbb{F}[a,b)_s = 0$, $\mathbb{F}[c,d)_s = \mathbb{F}$ and $\mathbb{F}[a,b)_t = \mathbb{F}[c,d)_t = \mathbb{F}$
The diagram becomes

$$
\begin{tikzcd}
0 \arrow[r, "f_s=0"] \arrow[d, "\pi_{s,t}=0"'] \arrow[dr, phantom, "="] & \mathbb{F} \arrow[d, "\pi'_{s,t}=\mathrm{id}"] \\
\mathbb{F} \arrow[r, "f_t=\mathrm{id}"'] & \mathbb{F}
\end{tikzcd}
\qquad f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s
$$

⊙ If $s \in [a,d)$, $\mathbb{F}[a,b)_s = \mathbb{F} = \mathbb{F}[c,d)_s$ and $\mathbb{F}[a,b)_t = \mathbb{F}[c,d)_t = \mathbb{F}$

The diagram becomes

$$
\begin{tikzcd}
\mathbb{F} \arrow[r, "f_s=\mathrm{id}"] \arrow[d, "\pi_{s,t}=\mathrm{id}"'] \arrow[dr, phantom, "="] & \mathbb{F} \arrow[d, "\pi'_{s,t}=\mathrm{id}"] \\
\mathbb{F} \arrow[r, "f_t=\mathrm{id}"'] & \mathbb{F}
\end{tikzcd}
\qquad f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s
$$

③ Assume $t \in [d,b)$. Then, either $s<c$, $s \in [c,a)$, $s \in [a,d)$ or $s \in [d,b)$

⊙ If $s<c$, $\mathbb{F}[a,b)_s = \mathbb{F}[c,d)_s = 0$ and $\mathbb{F}[a,b)_t = \mathbb{F}$ but $\mathbb{F}[c,d)_t = 0$

The diagram becomes

$$
\begin{tikzcd}
0 \arrow[r, "f_s=0"] \arrow[d, "\pi_{s,t}=0"'] \arrow[dr, phantom, "="] & 0 \arrow[d, "\pi'_{s,t}=0"] \\
\mathbb{F} \arrow[r, "f_t=0"'] & 0
\end{tikzcd}
\qquad f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s
$$

⊙ If $s \in [c,a)$, $\mathbb{F}[a,b)_s = 0$, $\mathbb{F}[c,d)_s = \mathbb{F}$ and $\mathbb{F}[a,b)_t = \mathbb{F}$ but $\mathbb{F}[c,d)_t = 0$
The diagram becomes

$$
\begin{tikzcd}
0 \arrow[r, "f_s=0"] \arrow[d, "\pi_{s,t}=0"'] \arrow[dr, phantom, "="] & \mathbb{F} \arrow[d, "\pi_{s,t}=0"] \\
\mathbb{F} \arrow[r, "f_t=0"'] & 0
\end{tikzcd}
\qquad f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s
$$

• If $s \in [a,d)$, $\mathbb{F}[a,b)_s = \mathbb{F}[c,d)_s = \mathbb{F}$ and $\mathbb{F}[a,b)_t = \mathbb{F}$ but $\mathbb{F}[c,d)_t = 0$

$$\begin{tikzcd} \mathbb{F} \arrow[r, "f_s = id"] \arrow[d, "\pi_{s,t} = id"'] \arrow[dr, phantom, "\equiv"] & \mathbb{F} \arrow[d, "\pi'_{s,t} = 0"] \\ \mathbb{F} \arrow[r, "f_t = 0"'] & 0 \end{tikzcd}$$

$f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$ ✓

⊙ If $s \in [d,b)$, $\mathbb{F}[a,b)_s = \mathbb{F}$, $\mathbb{F}[c,d)_s = 0$ and $\mathbb{F}[a,b)_t = \mathbb{F}$ but $\mathbb{F}[c,d)_t = 0$

$$\begin{tikzcd} \mathbb{F} \arrow[r, "f_s = 0"] \arrow[d, "\pi_{s,t} = id"'] \arrow[dr, phantom, "\equiv"] & 0 \arrow[d, "\pi_{s,t} = 0"] \\ \mathbb{F} \arrow[r, "f_t = 0"'] & 0 \end{tikzcd}$$

$f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$ ✓

This concludes that f is a non-zero morphism and finish the first part of the proof.

Conversely, let us assume $c > a$ or $a \geq d$ or $d > b$ and proof that the only possible morphism $\mathbb{F}[a,b) \to \mathbb{F}[c,d)$ is the 0-morphism. Suppose first that $c > a$. Then, either $b \geq d$ or $b < d$.

Assume $c > a$ and $b \geq d$. Then, we have that

$$\mathbb{F}[a,b)_t = \begin{cases} \mathbb{F} & \text{if } t \in [a,c) \\ \mathbb{F} & \text{if } t \in [c,d) \\ \mathbb{F} & \text{if } t \in [d,b) \\ 0 & \text{otherwise} \end{cases} \qquad \mathbb{F}[c,d)_t = \begin{cases} 0 & \text{if } t \in [a,c) \\ \mathbb{F} & \text{if } t \in [c,d) \\ 0 & \text{if } t \in [d,b) \\ 0 & \text{otherwise} \end{cases}$$

The only possible non-zero morphism is the following:

$$f_t = \begin{cases} id & \text{if } t \in [c,d) \\ 0 & \text{otherwise} \end{cases}$$

Let us check that actually, this is not a morphism. Notice that if $t \in [c, d)$ and $s \in [a, c)$, $s \leq t$ and the following diagram should commute

$$\begin{tikzcd} \mathbb{F} \arrow[r, "f_s = 0"] \arrow[d, "\pi_{s,t} = id"'] & 0 \arrow[d, "\pi'_{s,t} = 0"] \\ \mathbb{F} \arrow[r, "f_t = id"'] & \mathbb{F} \end{tikzcd} \qquad f_t \circ \pi_{s,t} \neq \pi'_{s,t} \circ f_s$$

We reach a contradiction!! The only possible morphism is the zero.

Assume now $c > a$ and $b < d$. Then, we have that

$$\mathbb{F}[a,b)_t = \begin{cases} \mathbb{F} & \text{if } t \in [a,c) \\ \mathbb{F} & \text{if } t \in [c,b) \\ 0 & \text{if } t \in [b,d) \\ 0 & \text{otherwise} \end{cases} \qquad \mathbb{F}[c,d)_t = \begin{cases} 0 & \text{if } t \in [a,c) \\ \mathbb{F} & \text{if } t \in [c,b) \\ \mathbb{F} & \text{if } t \in [b,d) \\ 0 & \text{otherwise} \end{cases}$$

The only possible non-zero morphism is the following

$$f_t = \begin{cases} id & \text{if } t \in [c,b) \\ 0 & \text{otherwise} \end{cases}$$

Let us check that actually, this is not a morphism. Notice that if $t \in [c, b)$ and $s \in [a, c)$, $s \leq t$ and the following diagram should be commutative

$$\begin{tikzcd} \mathbb{F} \arrow[r, "f_s = 0"] \arrow[d, "\pi_{s,t} = id"'] & 0 \arrow[d, "\pi'_{s,t} = 0"] \\ \mathbb{F} \arrow[r, "f_t = id"'] & \mathbb{F} \end{tikzcd} \qquad f_t \circ \pi_{s,t} \neq \pi'_{s,t} \circ f_s$$

Contradiction ∇

Doing exactly the same argument with the remaining cases, $a \geq d$ and $d > b$, we will reach a contradiction and we will conclude that the only possible morphism is the zero, which conclude the proof.

$\square$