StatLect

Algebraic and geometric multiplicity of eigenvalues

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The algebraic multiplicity of an eigenvalue is the number of times it appears as a root of the characteristic polynomial (i.e., the polynomial whose roots are the eigenvalues of a matrix).

The geometric multiplicity of an eigenvalue is the dimension of the linear space of its associated eigenvectors (i.e., its eigenspace).

In this lecture we provide rigorous definitions of the two concepts of algebraic and geometric multiplicity and we prove some useful facts about them.

Looking for a **geometric multiplicity calculator** or a **step-by-step tutorial** on how to calculate the geometric multiplicity? Follow this link.

Algebraic multiplicity

Let us start with a definition.

> **Definition** Let A be a $K\times K$ matrix. Denote by $\lambda_1,\ldots,\lambda_K$ the K possibly repeated eigenvalues of A, which solve the characteristic equation
>
> $$\det(\lambda I-A)=(\lambda-\lambda_1)\cdot\ldots\cdot(\lambda-\lambda_K)=0$$
>
> We say that an eigenvalue λ_k has algebraic multiplicity $\mu(\lambda_k)\in\mathbb{N}$ if and only if there are no more and no less than $\mu(\lambda_k)$ solutions of the characteristic equation equal to λ_k.

Let us see some examples.

Example Consider the 2×2 matrix

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}$$

The characteristic polynomial is

$$\begin{aligned} f(\lambda) &= \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} \lambda - 4 & -2 \\ -1 & \lambda - 2 \end{bmatrix}\right) \\ &= (\lambda - 4) \cdot (\lambda - 2) - (-2) \cdot (-1) \\ &= \lambda^2 - 2\lambda - 4\lambda + 8 - 2 \\ &= \lambda^2 - 6\lambda + 6 \end{aligned}$$

The roots of the polynomial, that is, the solutions of $f(\lambda) = 0$ are

$$\begin{aligned} \lambda_1 &= 3 + \sqrt{3} \\ \lambda_2 &= 3 - \sqrt{3} \end{aligned}$$

Thus, A has two distinct eigenvalues. Their algebraic multiplicities are

$$\begin{aligned} \mu(\lambda_1) &= 1 \\ \mu(\lambda_2) &= 1 \end{aligned}$$

because they are not repeated.

Example Define the 2×2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Its characteristic polynomial is

$$
\begin{aligned}
f(\lambda) &= \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \lambda - 1 & 0 \\ -2 & \lambda - 1 \end{bmatrix}\right) \\
&= (\lambda - 1) \cdot (\lambda - 1) - 0 \cdot (-2) \\
&= (\lambda - 1) \cdot (\lambda - 1)
\end{aligned}
$$

The roots of the polynomial, that is, the solutions of $f(\lambda) = 0$ are

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= 1
\end{aligned}
$$

Thus, A has one repeated eigenvalue whose algebraic multiplicity is

$$\mu(\lambda_1) = \mu(\lambda_2) = 2$$

Geometric multiplicity

Recall that each eigenvalue is associated to a linear space of eigenvectors, called eigenspace.

Definition Let A be a $K \times K$ matrix. Let λ_k be one of the eigenvalues of A and denote its associated eigenspace by E_k. The dimension of E_k is called the geometric multiplicity of the eigenvalue λ_k.

Let's now make some examples.

Definition Consider the 2×2 matrix

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

The characteristic polynomial is

$$
\begin{aligned}
f(\lambda) &= \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \lambda - 2 & 0 \\ -1 & \lambda - 1 \end{bmatrix}\right) \\
&= (\lambda - 2) \cdot (\lambda - 1) - 0 \cdot (-1) \\
&= (\lambda - 2) \cdot (\lambda - 1)
\end{aligned}
$$

The roots of the polynomial are

$$
\lambda_1 = 2 \\
\lambda_2 = 1
$$

The eigenvectors associated to $\lambda_1 = 2$ are the vectors

$$
x_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}
$$

that solve the equation

$$
\begin{bmatrix} \lambda_1 - 2 & 0 \\ -1 & \lambda_1 - 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

or

$$
\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The last equation implies that

$$
x_{11} = x_{21}
$$

Therefore, the eigenspace of λ_1 is the linear space that contains all vectors x_1 of the form

$$
x_1 = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

where α can be any scalar. Thus, the eigenspace of λ_1 is generated by a single vector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, it has dimension 1. As a consequence, the geometric multiplicity of λ_1 is 1.

Example Consider the 2×2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

The characteristic polynomial is

$$\begin{aligned} f(\lambda) &= \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} \lambda - 1 & 0 \\ 1 & \lambda - 1 \end{bmatrix} \right) \\ &= (\lambda - 1) \cdot (\lambda - 1) - 0 \cdot 1 \\ &= (\lambda - 1) \cdot (\lambda - 1) \end{aligned}$$

and its roots are

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 1 \end{aligned}$$

Thus, there is a repeated eigenvalue ($\lambda_1 = \lambda_2 = 1$) with algebraic multiplicity equal to 2. Its associated eigenvectors

$$x_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$$

solve the equation

$$\begin{bmatrix} \lambda_1 - 1 & 0 \\ 1 & \lambda_1 - 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equation is satisfied for $x_{11} = 0$ and any value of x_{21}. As a consequence, the eigenspace of λ_1 is the linear space that contains all vectors x_1 of the form

$$x_1 = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where α can be any scalar. Since the eigenspace of λ_1 is generated by a single vector

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

it has dimension 1. As a consequence, the geometric multiplicity of λ_1 is 1, less than its algebraic multiplicity, which is equal to 2.

Example Define the 2×2 matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The characteristic polynomial is

$$\begin{aligned} f(\lambda) &= \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 2 \end{bmatrix} \right) \\ &= (\lambda - 2) \cdot (\lambda - 2) - 0 \cdot 0 \\ &= (\lambda - 2) \cdot (\lambda - 2) \end{aligned}$$

and its roots are

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 2 \end{aligned}$$

Thus, there is a repeated eigenvalue ($\lambda_1 = \lambda_2 = 2$) with algebraic multiplicity equal to 2. Its associated eigenvectors

$$x_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$$

solve the equation

$$\begin{bmatrix} \lambda_1 - 2 & 0 \\ 0 & \lambda_1 - 2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equation is satisfied for any value of x_{11} and x_{21}. As a consequence, the eigenspace of λ_1 is the linear space that contains all vectors x_1 of the form

$$x_1 = x_{11} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_{21} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where x_{11} and x_{21} are scalars that can be arbitrarily chosen. Thus, the eigenspace of λ_1 is generated by the two linearly independent vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence, it has dimension 2. As a consequence, the geometric multiplicity of λ_1 is 2, equal to its algebraic multiplicity.

A takeaway message from the previous examples is that the algebraic and geometric multiplicity of an eigenvalue do not necessarily coincide.

Relationship between algebraic and geometric multiplicity

The following proposition states an important property of multiplicities.

Proposition Let A be a $K \times K$ matrix. Let λ_k be one of the eigenvalues of A. Then, the geometric multiplicity of λ_k is less than or equal to its algebraic multiplicity.

Proof

Defective eigenvalues

When the geometric multiplicity of a repeated eigenvalue is strictly less than its algebraic multiplicity, then that eigenvalue is said to be **defective**.

An eigenvalue that is not repeated has an associated eigenvector which is different from zero. Therefore, the dimension of its eigenspace is equal to 1, its geometric multiplicity is equal to 1 and equals its algebraic multiplicity. Thus, an eigenvalue that is not repeated is also non-defective.

Solved exercises

Below you can find some exercises with explained solutions.

Exercise 1

Find whether the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

has any defective eigenvalues.

Solution

Exercise 2

Define

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Determine whether A possesses any defective eigenvalues.

Solution

How to cite

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