

# A Continuous-time Multicast Network Model

Bomin Kim\*

Department of Statistics, Pennsylvania State University,

Aaron Schein

College of Information and Computer Sciences, UMass Amherst,

Bruce Desmarais

Department of Political Science, Pennsylvania State University,

and

Hanna Wallach

Microsoft Research NYC

May 9, 2018

## Abstract

We introduce the continuous-time multicast network model—a generative model for directed edges with multiple recipients or multiple senders. To define the model, we integrate a dynamic version of the exponential random graph model (ERGM) and generalized linear model (GLM) approach to jointly understand who communicates with whom, and when. We use the model to analyze emails sent between department managers in Montgomery county government in North Carolina. Our application demonstrates that the model is effective at predicting and explaining continuous-time networks involving edges with multiple recipients.

*Keywords:* 3 to 6 keywords, that do not appear in the title

---

\*The authors gratefully acknowledge *please remember to list all relevant funding sources in the unblinded version*

# 1 Introduction

In recent decades, real-time digitized textual communication has developed into a ubiquitous form of social and professional interaction (????). From the perspective of the computational social scientist, this has led to a growing need for methods of modeling interactions that manifest as text exchanged in continuous time. A number of models that build upon topic modeling through Latent Dirichlet Allocation (?) to incorporate link data as well as textual content have been developed recently (???). These models are innovative in their extensions that incorporate network tie information. However, none of the models that are currently available in the literature integrate the rich random-graph structure offered by state of the art models for network structure—such as the exponential random graph model (ERGM) (???). The ERGM is the canonical model for modeling the structure of a static network. It is flexible enough to specify a generative model that accounts for nearly any pattern of tie formation such as reciprocity, clustering, popularity effects (?).

Several network models have been developed that handle time-stamped ties in which tie formation is governed by structural dynamics similar to those used in the ERGM (???). Those models are very useful to understand which traits and behaviours are predictive of interactions, however, none of the models explicitly allows multicast—edge with a single sender multiple recipients—which is a common property of digitalized textual interactions such as emails and online messages. For instance, (?) develops the co-evolution model which assumes that the network structure (“network”) and the individual attributes (“behaviour”) evolve simultaneously in a dynamic process, however, the model does not allow simultaneous changes within “networks” or “behaviour”—all within changes are ordered. Relatedly, (?) treat multicast interactions via duplication (i.e., obtain pairwise interactions from the original multicast), but their model parameter estimation relies on likelihood approximation which leads to bias. Concentrating on better handling of multicast, we develop the continuous-time multicast network model which simultaneously models the two components that govern time-stamped tie formation: 1) the recipient selection process that allows multiple senders or receivers, and 2) the time-to-next interactions with flexible distributional choices.

In what follows, we introduce the continuous-time multicast network model by describ-

ing how we assume the generative process of a time-stamped network data (Section 2), and deriving the sampling equations for Bayesian inference (Section 3). Then, we apply the model to the Montgomery county government email data and perform two model validation tasks (Section 4). Finally, the paper finishes in Section 5 with conclusion and discussion.

## 2 A Continuous-time Multicast Network Model

Data generated under the model consists of  $D$  unique edges. A single edge, indexed by  $d \in [D]$ , is represented by the three components: the sender  $a_d \in [A]$ , an indicator vector of recipients  $\mathbf{r}_d = \{u_{dr}\}_{r=1}^A$ , and the timestamp  $t_d \in (0, \infty)$ . For simplicity, we assume that edges are ordered by time such that  $t_d \leq t_{d+1}$ . While the model can be applied for networks with multiple senders and a single receiver, specifically by modifying  $a_d$  to be an indicator vector of senders  $\mathbf{a}_d = \{u_{da}\}_{a=1}^A$  and  $r_d$  be the single recipient, here we only present the generative process for those involving a single sender but multiple receivers.

### 2.1 Tie Generating Process

For every possible author–recipient pair  $(a, r)_{a \neq r}$ , we define the “recipient intensity”, which is the likelihood of edge  $d$  being sent from  $a$  to  $r$ :

$$\lambda_{adr} = \mathbf{b}^\top \mathbf{x}_{adr}, \quad (1)$$

where  $\mathbf{b}$  is  $P$ –dimensional vector of coefficients and  $\mathbf{x}_{adr}$  is a set of network features which vary depending on the hypotheses regarding canonical processes relevant to network theory such as popularity, reciprocity, and transitivity. In addition, we include intercept term to account for the average (or baseline) number of recipients. We place a Normal prior  $\mathbf{b} \sim N(\boldsymbol{\mu}_b, \Sigma_b)$ .

Next, we hypothesize “If  $a$  were the sender of edge  $d$ , who would be the recipient(s)?” To do this, we draw each sender’s set of recipients from the multivariate Bernoulli (MB) distribution (?)—a model to estimate the structure of graphs with binary nodes—with probability of 1 being  $\text{logit}(\lambda_{adr})$ . In order to avoid the model degeneracy from having an empty recipient set, we define a probability measure “MB<sub>Gibbs</sub>” motivated by the non-empty

Gibbs measure (?) which excludes the all-zero vector from the support of MB distribution. As a result, we 1) allow multiple recipients or “multicast”, 2) prevent from obtaining zero recipient, and 3) ensure tractable normalizing constant. To be specific, we draw a binary vector  $\mathbf{u}_{ad} = (u_{ad1}, \dots, u_{adA})$

$$\mathbf{u}_{ad} \sim \text{MB}_{\text{Gibbs}}(\boldsymbol{\lambda}_{ad}), \quad (2)$$

where  $\boldsymbol{\lambda}_{ad} = \{\lambda_{adr}\}_{r=1}^A$ . In particular, we define  $\text{MB}_{\text{Gibbs}}(\boldsymbol{\lambda}_{ad})$  as

$$p(\mathbf{u}_{ad}|\delta, \boldsymbol{\lambda}_{ad}) = \frac{\exp\left\{\log(\mathbb{I}(\|\mathbf{u}_{ad}\|_1 > 0)) + \sum_{r \neq a} \lambda_{adr} u_{adr}\right\}}{Z(\boldsymbol{\lambda}_{ad})}, \quad (3)$$

where  $Z(\boldsymbol{\lambda}_{ad}) = \prod_{r \neq a} (\exp(\lambda_{adr}) + 1) - 1$  is the normalizing constant and  $\|\cdot\|_1$  is the  $l_1$ -norm. Again, this is equivalent to assume independent Bernoulli trial on each  $u_{adr}$  with probability of 1 being  $\logit(\lambda_{adr})$ , excluding the case when all  $u_{adr} = 0$ . We provide the derivation of the normalizing constant as a tractable form in the supplementary material.

## 2.2 Time Generating Process

Similarly, we hypothesize “If  $a$  were the sender of edge  $d$ , when would it be sent?” and define the “timing rate” for sender  $a$

$$\mu_{ad} = g^{-1}(\boldsymbol{\eta}^\top \mathbf{y}_{ad}), \quad (4)$$

where  $\boldsymbol{\eta}$  is  $Q$ -dimensional vector of coefficients with a Normal prior  $\boldsymbol{\eta} \sim N(\boldsymbol{\mu}_\eta, \Sigma_\eta)$ ,  $\mathbf{y}_{ad}$  is a set of time-related covariates, e.g., any feature that could affect timestamps of the edge, and  $g(\cdot)$  is the appropriate link function such as identity, log, or inverse.

In modeling “when”, we do not directly model the timestamp  $t_d$ . Instead, we assume that each sender’s the time-increment or “time to next interaction” (i.e.,  $\tau_d = t_d - t_{d-1}$ ) is drawn from a specific distribution in the exponential family. We follow the generalized linear model (GLM) framework (?):

$$\begin{aligned} E(\tau_{ad}) &= \mu_{ad}, \\ V(\tau_{ad}) &= V(\mu_{ad}), \end{aligned} \quad (5)$$

where  $\tau_{ad}$  here is a positive real number. Possible choices of distribution include Exponential, Weibull, Gamma, and lognormal<sup>1</sup> distributions, which are commonly used in time-to-

---

<sup>1</sup>lognormal distribution is not exponential family but can be used via modeling of  $\log(\tau_d)$ .

event modeling (??). Based on the choice of distribution, we may introduce any additional parameter (e.g.,  $\sigma_\tau^2$ ) to account for the variance. We use  $f_\tau(\cdot; \mu, \sigma_\tau^2)$  and  $F_\tau(\cdot; \mu, \sigma_\tau^2)$  to denote the probability density function (p.d.f) and cumulative density function (c.d.f), respectively, with mean  $\mu$  and variance  $\sigma^2$ .

## 2.3 Observed Data

Finally, we choose the sender, recipients, and timestamp—which will be observed—by selecting the sender–recipient-set pair with the smallest time-increment (?):

$$\begin{aligned} a_d &= \operatorname{argmin}_a(\tau_{ad}), \\ \mathbf{r}_d &= \mathbf{u}_{a_d d}, \\ t_d &= t_{d-1} + \tau_{a_d d}. \end{aligned} \tag{6}$$

Therefore, it is a sender-driven process in that the recipients and timestamp of an edge is determined by the sender’s urgency to send the edge to chosen recipients. Note that our generative process accounts for tied events such that in case of tied events (i.e., multiple senders generated exactly same time increments  $\tau_d$ ), we observe all of the tied events without assigning the orders of tied events. Algorithm ?? summarizes the generative process in Section ??.

## 3 Posterior Inference

Our inference goal is to invert the generative process to obtain the posterior distribution over the unknown parameters, conditioned on the observed data and hyperparameters  $(\boldsymbol{\mu}_b, \Sigma_b, \boldsymbol{\mu}_\eta, \Sigma_\eta)$ . We draw the samples using Markov chain Monte Carlo (MCMC) methods, repeatedly resampling the value of each parameter from its conditional posterior given the observed data, hyperparameters, and the current values of the other parameters. We express each parameters conditional posterior in a closed form using the data augmentation schemes in **u** ?). In this section, we provide each latent variable’s conditional posterior.

First, since  $u_{adr}$  is a binary random variable, new values may be sampled directly using

$$\begin{aligned} P(u_{adr} = 1 | \mathbf{u}_{ad \setminus r}, \mathbf{b}, \mathbf{x}) &\propto \exp(\lambda_{adr}); \\ P(u_{adr} = 0 | \mathbf{u}_{ad \setminus r}, \mathbf{b}, \mathbf{x}) &\propto \mathbb{I}(\|\mathbf{u}_{ad \setminus r}\|_1 > 0), \end{aligned} \tag{7}$$

---

**Algorithm 1** Generating Process
 

---

```

for  $d=1$  to  $D$  do
  for  $a=1$  to  $A$  do
    for  $r=1$  to  $A$  ( $r \neq a$ ) do
      form  $\mathbf{x}_{adr}$ 
      set  $\lambda_{adr} = \mathbf{b}^\top \mathbf{x}_{adr}$ 
    end
    draw  $\mathbf{u}_{ad} \sim \text{Gibbs}(\boldsymbol{\lambda}_{ad})$ 
    form  $\mathbf{y}_{ad}$ 
    set  $\mu_{ad} = g^{-1}(\boldsymbol{\eta}^\top \mathbf{y}_{ad})$ 
    draw  $\tau_{ad} \sim f_\tau(\mu_{ad}, \sigma_\tau^2)$ 
  end
  For no tied events,
  set  $a_d = \text{argmin}_a(\tau_{ad})$ 
  set  $\mathbf{r}_d = \mathbf{u}_{a_d d}$ 
  set  $t_d = t_{d-1} + \min_a \tau_{ad}$ 

  For  $n$  tied events, (i.e.,  $n$  senders generated same minimum time increments)
  set  $a_d, \dots, a_{d+n} = \text{argmin}_a(\tau_{ad})$ 
  set  $\mathbf{r}_d = \mathbf{u}_{a_d d}, \dots, \mathbf{r}_{d+n} =, \mathbf{u}_{a_{d+n} d}$ 
  set  $t_d, \dots, t_{d+n} = t_{d-1} + \min_a \tau_{ad}$ 
  jump to  $d = d + n + 1$ 
end

```

---

where  $I(\cdot)$  is the indicator function that is used to prevent from the instances where a sender chooses zero number of recipients.

New values for continuous variables  $\mathbf{b}$ , and  $\boldsymbol{\eta}$  and  $\sigma_\tau^2$  (if applicable) cannot be sampled directly from their conditional posteriors, but may instead be obtained using the Metropolis–Hastings algorithm. With uninformative priors (i.e.,  $N(0, \infty)$ ), the conditional posterior over  $\mathbf{b}$  is

$$P(\mathbf{b}|\mathbf{u}, \mathbf{x}) \propto \prod_{d=1}^D \prod_{a=1}^A \frac{\exp \left\{ \log(I(\|\mathbf{u}_{ad}\|_1 > 0)) + \sum_{r \neq a} \lambda_{adr} u_{adr} \right\}}{Z(\boldsymbol{\lambda}_{ad})}, \quad (8)$$

where the two variables share the conditional posterior and thus can be jointly sampled. Likewise, assuming uninformative priors on  $\boldsymbol{\eta}$  (i.e.,  $N(0, \infty)$ ) and  $\sigma_\tau^2$  (i.e., half-Cauchy( $\infty$ )), the conditional posterior for no-tied event case is

$$P(\boldsymbol{\eta}, \sigma_\tau^2|\mathbf{u}, \mathbf{y}) \propto \prod_{d=1}^D \left( f_\tau(\tau_d; \mu_{a_d d}, \sigma_\tau^2) \times \prod_{a \neq a_d} (1 - F_\tau(\tau_d; \mu_{ad}, \sigma_\tau^2)) \right), \quad (9)$$

where  $f_\tau(\tau_d; \mu_{ad}, \sigma_\tau^2)$  is the probability that the  $d^{th}$  observed time-increment comes from the specified distribution  $f_\tau(\cdot)$  with the observed sender's mean parameter, and  $\prod_{a \neq a_d} (1 - F_\tau(\tau_d; \mu_{ad}, \sigma_\tau^2))$  is the probability that the latent (or unobserved) senders for event  $d$  all generated time-increments greater than  $\tau_d$ . Moreover, under the existence of tied-event, the conditional posterior of  $\boldsymbol{\eta}$  and  $\sigma_\tau^2$  should be written as

$$P(\boldsymbol{\eta}, \sigma_\tau^2 | \mathbf{u}, \mathbf{y}) \propto \prod_{m=1}^M \left( \prod_{d:t_d=t_m^*} f_\tau(t_m^* - t_{m-1}^*; \mu_{ad}, \sigma_\tau^2) \times \prod_{a \notin \{a_d\}_{d:t_d=t_m^*}} (1 - F_\tau(t_m^* - t_{m-1}^*; \mu_{ad}, \sigma_\tau^2)) \right), \quad (10)$$

where  $t_1^*, \dots, t_M^*$  are the unique timepoints across  $D$  events ( $M \leq D$ ). If  $M = D$  (i.e., no tied events), Equation (10) reduces to Equation (9). Algorithm ?? provides the pseudocode.

---

**Algorithm 2** MCMC Algorithm

---

set initial values of  $\mathbf{b}, \boldsymbol{\eta}, \sigma_\tau^2$

**for**  $o=1$  to *outer* **do**

**for**  $d=1$  to  $D$  **do**

**for**  $a = 1$  to  $A$  **do**

**for**  $r = 1$  to  $A$  ( $r \neq a$ ) **do**

                | update  $u_{adr}$  using Gibbs update — Equation (7)

**end**

**end**

**end**

**for**  $n=1$  to *inner1* **do**

        | update  $\mathbf{b}$  using Metropolis-Hastings — Equation (8)

**end**

**for**  $n=1$  to *inner2* **do**

        | update  $\boldsymbol{\eta}$  using Metropolis-Hastings — Equation (10)

**end**

**for**  $n=1$  to *inner3* **do**

        | update  $\sigma_\tau^2$  using Metropolis-Hastings — Equation (10)

**end**

**end**

Summarize the results with:

last chain of  $\mathbf{b}$ , and last chain of  $\boldsymbol{\eta}$  and  $\sigma_\tau^2$

---

## 4 Case Studies

We now present two case studies applying our method to 1) county government emails (single sender multiple recipients) and 2) international sanctions (multiple senders single

recipient). For each case study, we formulate the network statistics  $\mathbf{x}$  and timestamp statistics  $\mathbf{y}$  and ground them in illustrative examples. We then report a suite of experiments that test our methods ability to form the posterior distribution over latent variables for different types of data.

## 4.1 Case Study 1: County Government Emails

### 4.1.1 Montgomery County Data

Our data come from the North Carolina county government email dataset collected by ?) that includes internal email corpora covering the inboxes and outboxes of managerial-level employees of North Carolina county governments. Out of over twenty counties, we chose Montgomery County to 1) test our model using data with a lot of multicast edges (16.76%), and 2) limit the scope of this initial application. The Montgomery County email network contains 680 emails, sent and received by 18 department managers over a period of 3 months (March–May) in 2012.

### 4.1.2 Covariates

In the example of email networks, we form the covariate vector  $\mathbf{x}_{adr}$  using time-varying network statistics based on the time interval prior to and including  $t_{d-1}$ . Along with intercept term, we compute the network statistics (?), where the time interval tracks 7 days prior to the last email was sent  $l_d = (t_{d-1} - 7\text{days}, t_{d-1}]$ . To be specific, we use

1. intercept  $x_{adr1} = 1$ ;
2. outdegree  $x_{adr2} = \sum_{d': t_{d'} \in l_d} I(a_{d'} = a)$ ;
3. indegree  $x_{adr3} = \sum_{d': t_{d'} \in l_d} I(u_{d'r} = 1)$ ;
4. send  $x_{adr4} = \sum_{d': t_{d'} \in l_d} I(a_{d'} = a)I(u_{d'r} = 1)$ ;
5. receive  $x_{adr5} = \text{send}(r, a)$ ;
6. 2-send  $x_{adr6} = \sum_{h \neq a, r} \text{send}(a, h)\text{send}(h, r)$ ;
7. 2-receive  $x_{adr7} = \sum_{h \neq a, r} \text{send}(h, a)\text{send}(r, h)$ ;



8. sibling  $x_{adr8} = \sum_{h \neq a, r} \text{send}(h, a) \text{send}(h, r);$
9. cosibling  $x_{adr9} = \sum_{h \neq a, r} \text{send}(a, h) \text{send}(r, h);$
10. Nreceive  $x_{adr10} = \sum_{d': t_{d'} \in l_d} \sum_{r=1}^A I(a_{d'} = a) I(u_{d'r} = 1);$
11. outdegree\*Nreceive  $x_{adr11} = x_{adr2} \times x_{adr10};$

where  $I(\cdot)$  is an indicator function, and the last covariate is the interaction term between outdegree and Nreceive. As with the nodal (2, 3, 10, 11) and dyadic (4, 5) effects, the triadic (6–9) effects are designed so that their coefficient have a straightforward interpretation, which are illustrated in Figure ??.

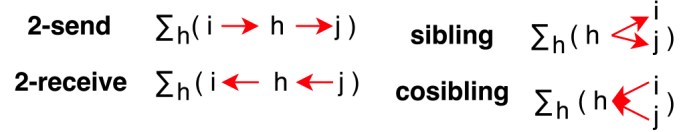


Figure 1: Visualization of triadic statistics.

For time-related covariates  $\mathbf{y}_{ad}$  in Section ??, we form the covariate vector  $\mathbf{y}_{ad}$  using the seven statistics which may possibly effect “time to send the next document”. Note that these statistics could depend on  $a$  only (nodal effect),  $d$  (or  $t_d$ ) only (temporal effect), or both  $a$  and  $d$  (nodal & temporal effect). Specifically, the statistics are defined as

1. intercept  $y_{ad1} = 1;$
2. outdegree  $y_{ad2} = \sum_{d': t_{d'} \in l_d} I(a_{d'} = a);$
3. indegree  $y_{ad3} = \sum_{d': t_{d'} \in l_d} I(u_{d'a} = 1);$
4. gender  $y_{ad4} = I(a = \text{female});$
5. manager  $y_{ad5} = I(a = \text{County Manager});$
6. weekend  $y_{ad6} = I(t_{d-1} = \text{weekend});$
7. am/pm  $y_{ad7} = I(t_{d-1} = \text{pm}).$

### 4.1.3 Parameter Estimation and Interpretation

Based on some preliminary experiments, we choose lognormal distribution to model time-to-next-email in Montgomery county email data, thus we estimate the variance parameter  $\sigma_\tau^2$  with the prior specified as inverse-Gamma(2, 1). In this section, we present the results based on 55,000 MCMC outer iterations with a burn-in of 15,000, where we thin by keeping every 40th sample. While the inner iterations for  $\boldsymbol{\eta}$  and  $\sigma_\tau^2$  are fixed as 1, we use 10 inner iterations for  $\mathbf{b}$  to adjust for its slower convergence rate.

Figure ?? shows the boxplots summarizing posterior samples of the network effect coefficients  $\mathbf{b}$ . Since we use logit probability of  $\lambda_{adr}$  for tie generating process (Section ??), we have

$$\text{logit}(\lambda_{adr}) = \log\left(\frac{\lambda_{adr}}{1 - \lambda_{adr}}\right) = b_1 + b_2 x_{adr2} \dots + b_{11} x_{adr11},$$

and can interpret the parameter estimates in terms of odds ratio  $\frac{\lambda_{adr}}{1 - \lambda_{adr}} = \exp(b_1 + b_2 x_{adr2} \dots + b_{11} x_{adr11})$ . Firstly, we can see that the effect of send (i.e., “number of times the actor  $a$  sent emails to the actor  $r$  over the last week”) is positive, implying that if the actor  $a$  sent  $n$  number of emails to  $r$  last week, then the actor  $a$  is approximately

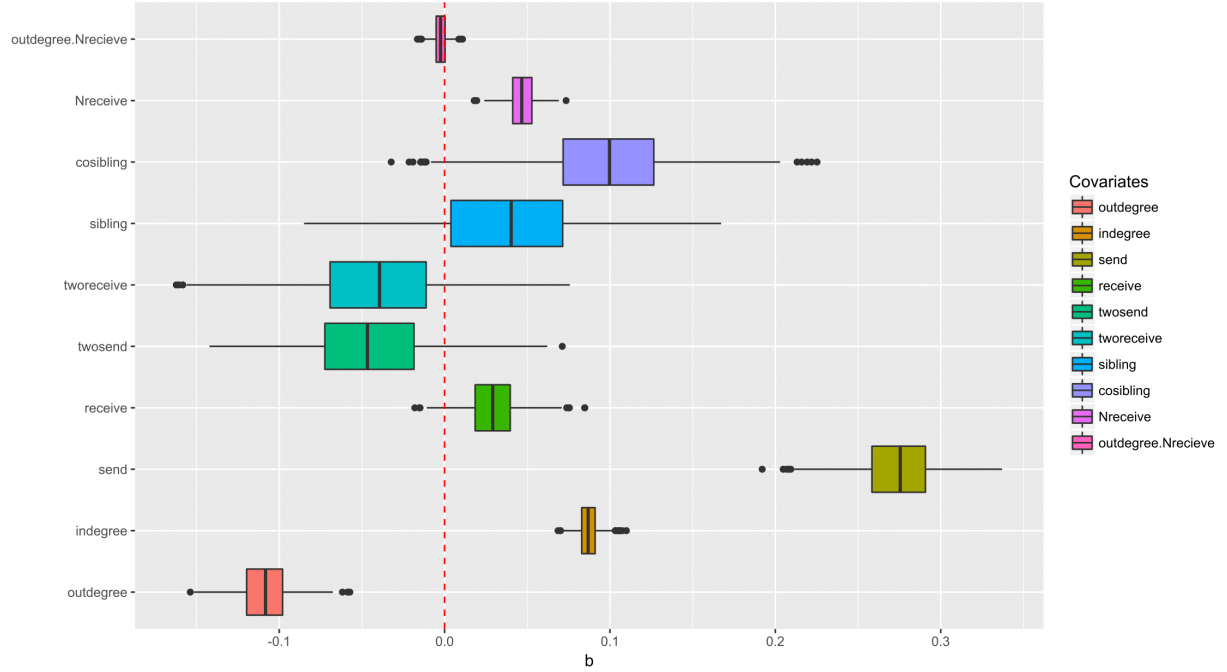


Figure 2: Posterior distribution of  $\mathbf{b}$  estimates.

$\exp(0.274 \times n) \approx (1.315)^n$  times more likely to send an email to  $r$ . Similar interpretation can be applied for the statistics indegree and Nreceive. For example, as the actor  $r$  received  $n$  emails and the actor  $a$  sent emails to  $n$  receivers over the last week, the actor  $a$  is  $\exp(0.086 \times n) \approx (1.091)^n$  times and  $\exp(0.047 \times n) \approx (1.048)^n$  times, respectively, more likely to send an email to  $r$ . On the other hand, outdegree statistic has negative effect—i.e., if the actor  $a$  sent  $n$  number of emails to anyone last week, then the actor  $a$  is approximately  $\exp(-0.109 \times n) \approx (0.897)^n$  times likely to send an email to  $r$ —possibly due to its multicollinearity with the statistics send and Nreceive. This can be interpreted as “if the actor  $a$  sent large number of emails over the last week but 1) not to  $r$  (excluding the effect of send) or 2) no broadcast or multicast emails (excluding the effect of Nreceive)” then the actor  $a$  is less likely to send an email to  $r$ . Rest of statistics are not statistically significant because their 95% credible interval do not include 0. Surprisingly, none of the triadic effects seem to have significant effect on the tie generating process.

Figure ?? shows the boxplots summarizing posterior samples of the time-related coefficients  $\boldsymbol{\eta}$ . For this specific dataset, we assume lognormal distribution (Section ??) with the unit of time being an hour, so that the interpretation of  $\hat{\boldsymbol{\eta}}$  should be based on

$$\log(\tau_{ad}) \sim N(\mu_{ad}, \sigma_{\tau}^2), \text{ with}$$

$$\mu_{ad} = \eta_1 + \eta_2 y_{ad2} \dots + \eta_7 y_{ad7}.$$

To begin with, the posterior estimates of two temporal effects—weekend and pm—indicate that if the  $(d-1)^{th}$  email was sent during the weekend or after 12pm, then the time to  $d^{th}$  email is expected to take  $\exp(1.552) \approx (4.722)$  hours and  $\exp(0.980) \approx (2.665)$  hours longer, respectively, compared to their counterparts (i.e., weekdays and am). On the contrary, the statistics manager, outdegree, and indegree turn out to shorten the time to next email. For example, being a county manager (i.e., the manager of the managers) lowers the expected  $\log(\tau_{ad})$  by -1.070, since he or she needs to respond fastly to emails with important decisions. In addition, the manager in general sends or receives a lot more emails which may shorten the response time. This argument is supported by the posterior estimates for outdegree and indegree, where the estimated coefficients are approximately -0.206 and -0.060, respectively. Gender of the department manager is shown to have no significant effect on time-to-next-email. The posterior mean estimate of the variance parameter  $\sigma_{\tau}^2$  is 14.093 with its 95%

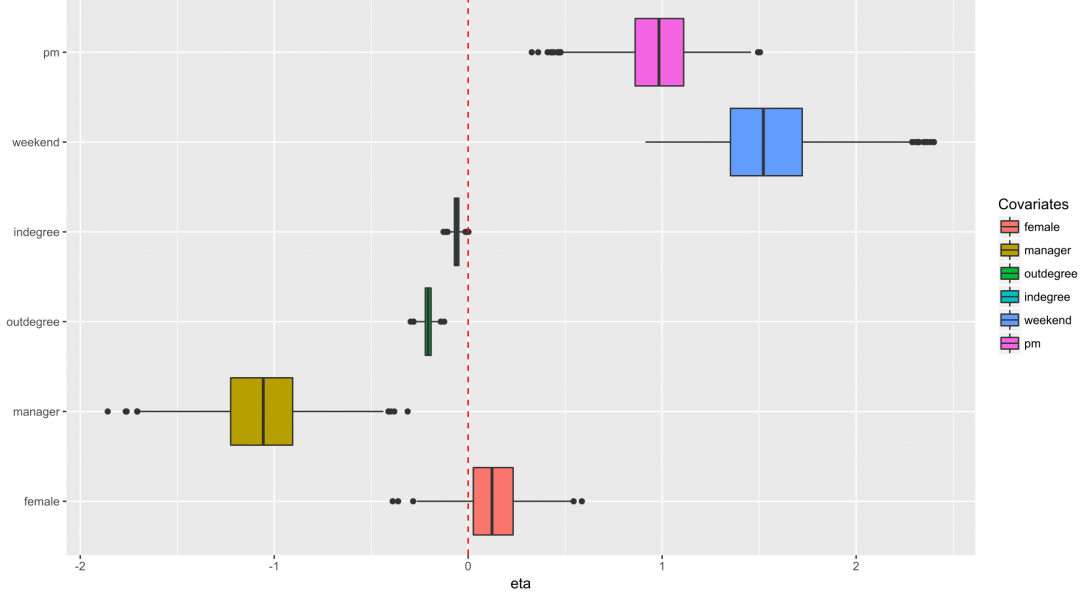


Figure 3: Posterior distribution of  $\eta$  estimates.

credible interval (12.709, 15.555), indicating that there exists huge variability in the time-increments in this data.

#### 4.1.4 Posterior Predictive Checks

We perform posterior predictive checks (?) to evaluate the appropriateness of our model specification for the Montgomery County email data. We formally generated entirely new data, by simulating ties and timestamps  $\{(a_d, \mathbf{r}_d, t_d)\}_{d=1}^D$  from the generative process in Section ??, conditional upon a set of inferred parameter values from the inference in Section ?. For the test of goodness-of-fit in terms of network dynamics, we use multiple network statistics that summarize meaningful aspects of the Montgomery County email data: outdegree distribution, indegree distribution, recipient size distribution, and time-increments probability-probability (PP) plot. We then generated 500 synthetic networks from the posterior predictive distribution, according to Algorithm ?.

Figure ?? illustrates the results of posterior predictive checks. Upper two plots show node-specific degree distributions, where the left one is the boxplots of 500 posterior predictive outdegree statistic and the right plot is the same one for indegree statistic. For both plots, the x-axis represents the node id's from 1 to 18, and the y-axis represents the number

---

**Algorithm 3** Generate new data for PPC

---

**Input:** number of new data to generate  $R$ , covariates  $\mathbf{x}$  and  $\mathbf{y}$   
estimated latent variables  $(\mathbf{u}, \mathbf{b}, \boldsymbol{\eta}, \sigma_\tau^2)$

```
for  $r = 1$  to  $R$  do  
  for  $d = 1$  to  $D$  do  
    Draw  $(a_d, \mathbf{r}_d, t_d)$  following Section ??  
  end for  
  Store every  $r^{th}$  new data  $\{(a_d, \mathbf{r}_d, t_d)\}_{d=1}^D$   
end for
```

---

of emails sent or received by the node. When compared with the observed outdegree and indegree statistics (red lines), our model recovers the overall distribution of ties across the nodes. For example, node 1 and 10 show significantly higher level of both sending and receiving activities relative to the rest, and the model-simulated data captures those big jumps, without using node-specific indicators. Outdegree distribution of some low-activity nodes are not precisely recovered, however, indegree distribution looks much better. Since we use more information in the recipient selection process (i.e., network effects) while we only rely on minimum time-increments when choosing the observed sender, these results are expected. Lower left plot is the distribution of recipient sizes, where the x-axis spans over the size of recipients 1 to 14 (which is the maximum size of observed recipients) and the y-axis denotes the number of documents with x-number of recipients. The result shows that our model is underestimating single-receiver documents while overestimating documents with two, three, and four recipients. One explanation behind what we observe is that the model is trying to recover so-called “broadcast” emails, which are the documents with  $\geq 10$  number of recipients, so that the intercept estimate  $b_1$  is slightly moved toward right. To improve the goodness of fit for recipient sizes, we can add more covariates (e.g., sender indicators) or assign strong prior structure on  $b_1$ . In the end, the plot on the lower right is the PP plot for time-increments, which is a graphical measure commonly used for assessing how closely two data sets agree, by plotting the two cumulative distribution functions against each other. The resulting goodness of fit of the diagonal line gives a measure of the difference between the distribution of simulated time-increments and the observed ones, and our PP plot shows that we have great performance in reproducing the observed time distribution.

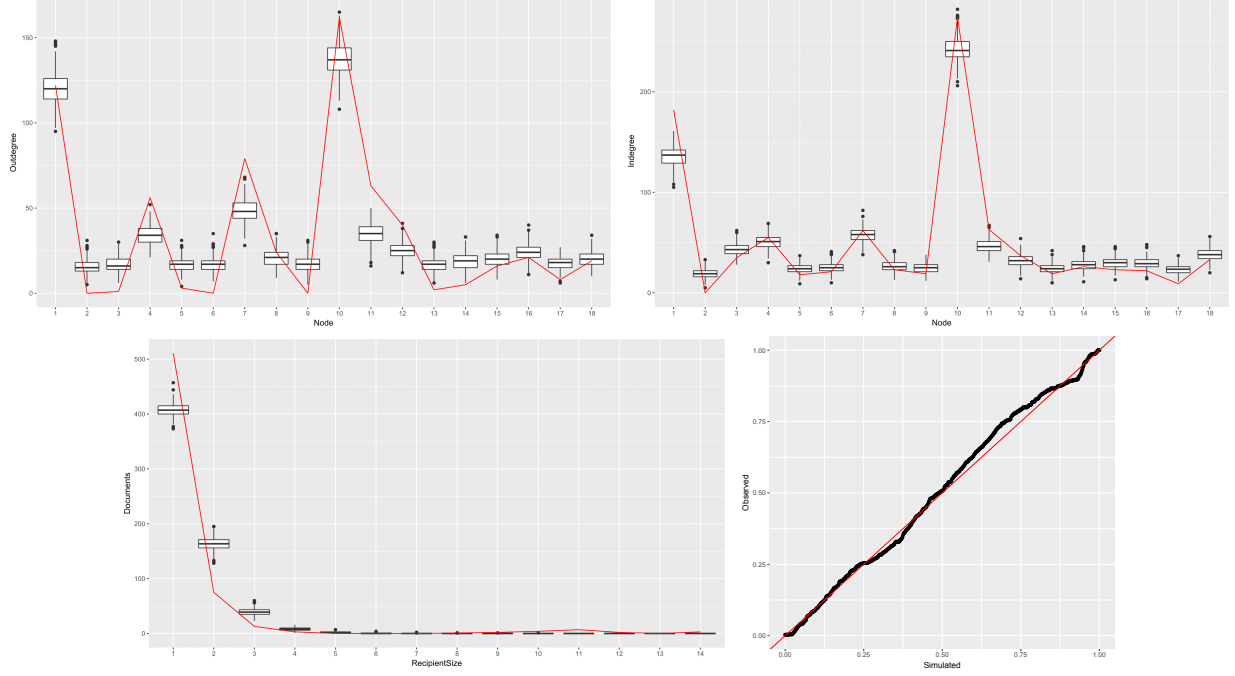


Figure 4: PPC results: outdegree distribution (*upper left*), indegree distribution (*upper right*), recipient size distribution (*lower left*), and time-increments probability-probability (PP) plot (*lower right*). Red lines in the first three plot depict the observed statistics, and the red line in the last plot is the diagonal line connecting  $(0,0)$  and  $(1,1)$ .

#### 4.1.5 Prediction Experiment

The HeEM (Hyper-edge event model) has many component parts that need to be specified by the user (i.e., the selection of the event timing features ( $y$ ), the recipient selection features ( $x$ ), and the event time distribution ( $g$ )). Many of these components will be specified based on user expertise (e.g., regarding which features would drive recipient selection), but some decisions may require a data-driven approach to model specification. For example, though theoretical considerations may inform the specification of features, subject-matter expertise is unlikely to inform the decision regarding the family of the event time distribution. Furthermore, since different distribution families (and model specifications more generally) may involve different size parameter spaces, any data-driven approach to model comparison must guard against over-fitting the

data. In this section we present a general-purpose approach to evaluating a HeEM specification using out-of-sample prediction. We illustrate this approach by comparing alternative distributional families for the event timing component of the model.

Lastly, we evaluate the model’s ability to predict ties from the Montgomery County email data, conditioned on their “training” part of the data. To perform the experiment, we separately formed a test split of each three components—sender, recipients, and timestamps—by randomly selecting “test” data with probability  $p = 0.1$ . Any missing variables were imputed by drawing samples from their conditional posterior distributions, given the observed data, parameter estimates, and current values of test data. We then run inference to update the latent variables given the imputed and observed data. We iterate the two steps—imputation and inference—multiple times to obtain enough number of estimates for “test” data. Algorithm ?? outlines this procedure in detail, including the full conditional distributions for missing sender, receiver, and timestamps.

The goal of this task is to provide an effective answer to the question “how are we filling in the missing information of the emails?” as well as to offer one standard way to determine the time-increment distribution in Section ??. To do this, we compare the predictive performance from two distributions—lognormal and Exponential. We particularly choose Exponential distribution as an alternative to what we used earlier (i.e., lognormal) since Exponential is the most commonly specified distribution for time-to-event data which is also used in the stochastic actor oriented models (?) as well as their extensions (?).

After running the experiment, we measure the predictive performance of two separate time distributions using 500 predicted samples, which are depicted in Figure ??.

---

**Algorithm 4** Out-of-Sample Tie Predictions

---

**Input:** data  $\{(a_d, \mathbf{r}_d, t_d)\}_{d=1}^D$ , number of new data to generate  $R$ , hyperparameters

**Test splits:**

draw test authors with  $p = 0.1$  (out of  $D$  authors)

draw test recipients with  $p = 0.1$  (out of  $D \times (A-1)$  recipient indicators  $\{\{\mathbf{r}_{dr}\}_{r \in [A] \setminus a_d}\}_{d=1}^D$ )

draw test timestamps with  $p = 0.1$  (out of  $D$  timestamps)

set the “test” data as “missing” (NA)

**Imputation and inference:**

initialize the parameters  $(\mathbf{b}, \boldsymbol{\eta}, \mathbf{u}, \sigma_\tau^2)$

**for**  $r = 1$  **to**  $R$  **do**

**for**  $d = 1$  **to**  $D$  **do**

**if**  $a_d = \text{NA}$  **then**

**for**  $a = 1$  **to**  $A$  **do**

                compute  $\pi_a$  using  $P(a_d = a|\cdot) = f_\tau(\tau_d; \mu_{a_d}, \sigma_\tau^2) \times \prod_{a \neq a_d} (1 - F_\tau(\tau_d; \mu_{ad}, \sigma_\tau^2))$

**end for**

            draw  $a_d \sim \text{Multinomial}(\pi)$

**end if**

**for**  $r \in [A] \setminus a_d$  **do**

**if**  $r_{dr} = \text{NA}$  **then**

            draw  $r_{dr}$  using  $P(r_{dr} = 1|\cdot)$  and  $P(r_{dr} = 0|\cdot)$ —Equation (7)

**end if**

**end for**

**if**  $t_d = \text{NA}$  **then**

        draw  $\boldsymbol{\tau}_d^{\text{new}}$  from proposal distribution (e.g.,  $\boldsymbol{\tau}_d^{\text{new}} \sim N(\boldsymbol{\tau}_d^{\text{old}}, \sigma_{\text{prop}}^2)$ )

        use Metropolis-Hastings to decide accept or reject using the probability

$$\frac{f_\tau(\boldsymbol{\tau}_d^{\text{new}}; \mu_{a_d}, \sigma_\tau^2) \times \prod_{a \neq a_d} (1 - F_\tau(\tau_d; \mu_{ad}, \sigma_\tau^2))}{f_\tau(\boldsymbol{\tau}_d^{\text{old}}; \mu_{a_d}, \sigma_\tau^2) \times \prod_{a \neq a_d} (1 - F_\tau(\tau_d; \mu_{ad}, \sigma_\tau^2))},$$

        where  $\boldsymbol{\tau}_d^{\text{old}}$  is from earlier iteration.

**end if**

    run inference and update  $(\mathbf{b}, \boldsymbol{\eta}, \mathbf{u}, \sigma_\tau^2)$  given the imputed and observed data

**end for**

store the estimates for “test” data

**end for**

---

## 4.2 Case Study 2: International Sanctions

### 4.2.1 Threat and Imposition of Sanctions (TIES) Data

We processed the threat and imposition of sanctions (TIES) data (?) that includes the records of international sanctions—defined as actions that one



or more countries take to limit or end their economic relations with a target country in an effort to persuade that country to change one or more of its policies—from 1945 to 2005.

#### 4.2.2 Covariates

#### 4.2.3 Parameter Estimation and Interpretation

#### 4.2.4 Posterior Predictive Checks

#### 4.2.5 Prediction Experiment

## 5 Conclusion

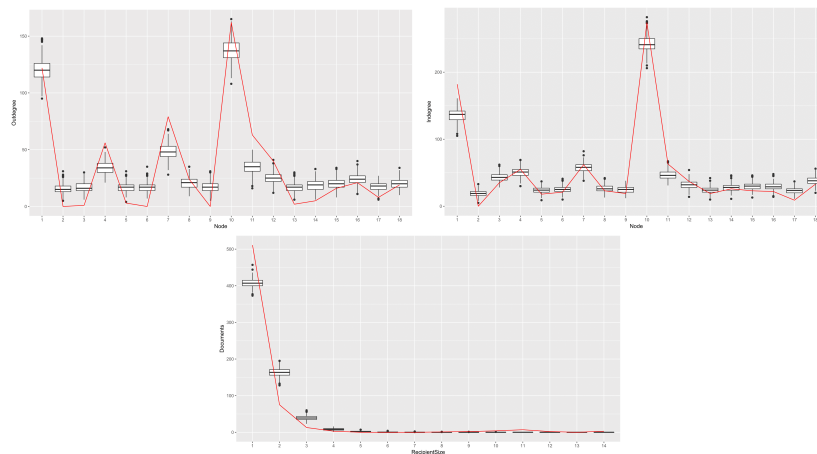


Figure 5: PPE results

## References

- ben Aaron, J., Denny, M., Desmarais, B. and Wallach, H. (2017), ‘Transparency by conformity: A field experiment evaluating openness in local governments’, *Public Administration Review* 77(1), 68–77.
- Blei, D. M., Ng, A. Y. and Jordan, M. I. (2003), ‘Latent dirichlet allocation’, *J. Mach. Learn. Res.* 3, 993–1022.
- Burgess, A., Jackson, T. and Edwards, J. (2004), Email overload: Tolerance levels of employees within the workplace, *in* ‘Innovations Through Information Technology: 2004 Information Resources Management Association International Conference, New Orleans, Louisiana, USA, May 23-26, 2004’, Vol. 1, IGI Global, p. 205.
- Butts, C. T. (2008), ‘A relational event framework for social action’, *Sociological Methodology* 38(1), 155–200.
- Chatterjee, S., Diaconis, P. et al. (2013), ‘Estimating and understanding exponential random graph models’, *The Annals of Statistics* 41(5), 2428–2461.
- Dai, B., Ding, S., Wahba, G. et al. (2013), ‘Multivariate bernoulli distribution’, *Bernoulli* 19(4), 1465–1483.
- Desmarais, B. A. and Cranmer, S. J. (2017), Statistical inference in political networks research, *in* J. N. Victor, A. H. Montgomery and M. Lubell, eds, ‘The Oxford Handbook of Political Networks’, Oxford University Press.
- Fellows, I. and Handcock, M. (2017), Removing phase transitions from gibbs measures, *in* ‘Artificial Intelligence and Statistics’, pp. 289–297.
- Hunter, D. R., Handcock, M. S., Butts, C. T., Goodreau, S. M. and Morris, M. (2008), ‘ergm: A package to fit, simulate and diagnose exponential-family models for networks’, *Journal of statistical software* 24(3), nihpa54860.

- Kanungo, S. and Jain, V. (2008), ‘Modeling email use: a case of email system transition’, *System Dynamics Review* 24(3), 299–319.
- Krafft, P., Moore, J., Desmarais, B. and Wallach, H. M. (2012), Topic-partitioned multinet network embeddings, *in* F. Pereira, C. Burges, L. Bottou and K. Weinberger, eds, ‘Advances in Neural Information Processing Systems 25’, Curran Associates, Inc., pp. 2807–2815.
- Lim, K. W., Chen, C. and Buntine, W. (2013), Twitter-network topic model: A full bayesian treatment for social network and text modeling, *in* ‘NIPS2013 Topic Model workshop’, pp. 1–5.
- McCallum, A., Corrada-Emmanuel, A. and Wang, X. (2005), The author-recipient-topic model for topic and role discovery in social networks, with application to enron and academic email, *in* ‘Workshop on Link Analysis, Counterterrorism and Security’, p. 33.
- Morgan, T. C., Bapat, N. and Kobayashi, Y. (2014), ‘Threat and imposition of economic sanctions 1945–2005: Updating the ties dataset’, *Conflict Management and Peace Science* 31(5), 541–558.
- Nelder, J. A. and Baker, R. J. (1972), *Generalized linear models*, Wiley Online Library.
- Perry, P. O. and Wolfe, P. J. (2013), ‘Point process modelling for directed interaction networks’, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 75(5), 821–849.
- Pew, R. C. (2016), ‘Social media fact sheet’, *Accessed on 03/07/17*.
- Rao, P. (2000), ‘Applied survival analysis: regression modeling of time to event data’, *Journal of the American Statistical Association* 95(450), 681–681.
- Rizopoulos, D. (2012), *Joint models for longitudinal and time-to-event data: With applications in R*, CRC Press.

- Robins, G., Pattison, P., Kalish, Y. and Lusher, D. (2007), ‘An introduction to exponential random graph ( $p^*$ ) models for social networks’, *Social networks* 29(2), 173–191.
- Rubin, D. B. et al. (1984), ‘Bayesianly justifiable and relevant frequency calculations for the applied statistician’, *The Annals of Statistics* 12(4), 1151–1172.
- Snijders, T. A. (1996), ‘Stochastic actor-oriented models for network change’, *Journal of mathematical sociology* 21(1-2), 149–172.
- Snijders, T. A., Van de Bunt, G. G. and Steglich, C. E. (2010), ‘Introduction to stochastic actor-based models for network dynamics’, *Social networks* 32(1), 44–60.
- Snijders, T., Steglich, C. and Schweinberger, M. (2007), *Modeling the coevolution of networks and behavior*, na.
- Szóstek, A. M. (2011), ‘?dealing with my emails?: Latent user needs in email management.’, *Computers in Human Behavior* 27(2), 723–729.
- Tanner, M. A. and Wong, W. H. (1987), ‘The calculation of posterior distributions by data augmentation’, *Journal of the American statistical Association* 82(398), 528–540.
- Vu, D. Q., Hunter, D., Smyth, P. and Asuncion, A. U. (2011), Continuous-time regression models for longitudinal networks, *in* J. Shawe-Taylor, R. Zemel, P. Bartlett, F. Pereira and K. Weinberger, eds, ‘Advances in Neural Information Processing Systems 24’, Curran Associates, Inc., pp. 2492–2500.

# Appendices

## A Normalizing constant of $\text{MB}_{\text{Gibbs}}$

Our probability measure “ $\text{MB}_{\text{Gibbs}}$ ”—the multivariate Bernoulli distribution with non-empty Gibbs measure (?)—defines the probability of author  $a$  selecting the binary recipient vector  $\mathbf{u}_{ad}$  as

$$P(\mathbf{u}_{ad}|\boldsymbol{\lambda}_{ad}) = \frac{\exp \left\{ \log(\mathbf{I}(\|\mathbf{u}_{ad}\|_1 > 0)) + \sum_{r \neq a} \lambda_{adr} u_{adr} \right\}}{Z(\boldsymbol{\lambda}_{ad})}.$$

To use this distribution efficiently, we derive a closed-form expression for  $Z(\boldsymbol{\lambda}_{ad})$  that does not require brute-force summation over the support of  $\mathbf{u}_{ad}$  (*i.e.*  $\forall \mathbf{u}_{ad} \in [0, 1]^A$ ). We recognize that if  $\mathbf{u}_{ad}$  were drawn via independent Bernoulli distributions in which  $P(u_{adr} = 1|\delta, \boldsymbol{\lambda}_{ad})$  was given by  $\text{logit}(\lambda_{adr})$ , then

$$P(\mathbf{u}_{ad}|\boldsymbol{\lambda}_{ad}) \propto \exp \left\{ \sum_{r \neq a} \lambda_{adr} u_{adr} \right\}.$$

This is straightforward to verify by looking at

$$P(u_{adr} = 1|\mathbf{u}_{ad[-r]}, \boldsymbol{\lambda}_{ad}) = \frac{\exp(\lambda_{adr})}{\exp(\lambda_{adr}) + 1}.$$

We denote the logistic-Bernoulli normalizing constant as  $Z^l(\boldsymbol{\lambda}_{ad})$ , which is defined as

$$Z^l(\delta, \boldsymbol{\lambda}_{ad}) = \sum_{\mathbf{u}_{ad} \in [0, 1]^A} \exp \left\{ \sum_{r \neq a} \lambda_{adr} u_{adr} \right\}.$$

Now, since

$$\begin{aligned} & \exp \left\{ \log(\mathbf{I}(\|\mathbf{u}_{ad}\|_1 > 0)) + \sum_{r \neq a} \lambda_{adr} u_{adr} \right\} \\ &= \exp \left\{ \sum_{r \neq a} \lambda_{adr} u_{adr} \right\}, \end{aligned}$$

except when  $\|\mathbf{u}_{ad}\|_1 = 0$ , we note that

$$\begin{aligned} Z(\boldsymbol{\lambda}_{ad}) &= Z^l(\boldsymbol{\lambda}_{ad}) - \exp \left\{ \sum_{\forall u_{adr}=0} \lambda_{adr} u_{adr} \right\} \\ &= Z^l(\lambda_a^{(d)}) - 1. \end{aligned}$$

We can therefore derive a closed form expression for  $Z(\delta, \lambda_{ad})$  via a closed form expression for  $Z^l(\delta, \lambda_{ad})$ . This can be done by looking at the probability of the zero vector under the logistic-Bernoulli model:

$$\frac{\exp \left\{ \sum_{\forall u_{adr}=0} \lambda_{adr} u_{adr} \right\}}{Z^l(\lambda_{ad})} = \prod_{r \neq a} \left( 1 - \frac{\exp(\lambda_{adr})}{\exp(\lambda_{adr}) + 1} \right).$$

Then, we have

$$\frac{1}{Z^l(\lambda_{ad})} = \prod_{r \neq a} \frac{1}{\exp(\lambda_{adr}) + 1}.$$

Finally, the closed form expression for the normalizing constant under the non-empty Gibbs measure is

$$Z(\lambda_{ad}) = \prod_{r \neq a} (\exp(\lambda_{adr}) + 1) - 1.$$