# Math 171 Notes

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# 1 Foundations of the Reals

## 1.1 The Field Axioms

The field axioms are a set of axioms that we accept as the foundation of the reals.

$\forall a, b, c \in \mathbf{R}$:

**F1 Commutativity** $a + b = b + a$
**F2 Associativity** $a + (b + c) = (a + b) + c$
**F3 Distributive** $a(b + c) = ab + ac$
**F4 Identity** $\exists\, 0, 1$ such that $0 + a = a$, $1 \cdot a = a$
**F5 Additive Inverse** $\exists\, -a$ such that $a + (-a) = 0$
**F5 Multiplicative Inverse** $\exists\, 1/a$ such that $a(1/a) = 1$

## 1.2 The Order Axioms

**O1 Positive Numbers** $\exists$ a set $P \subset \mathbf{R}$ such that for all $a \in \mathbf{R}$ either $a \in P$, $-a \in P$, or $a = 0$.
**O2** $a, b \in P$ implies that $a \cdot b$ and $a + b$ are in $P$

Thus we define $a > b$ as $a - b \in P$ and similarly $a < b$ is defined as $b - a \in P$.

$F$ and $O$ axioms hold for the rationals, $\mathbf{Q}$, but $O$ does not hold for the complex numbers.

## 1.3 The Completeness Axiom

Completeness distinguishes the Reals from the Rationals. Intuitively, there are 'holes' in the rationals at irrational numbers like $\sqrt{2}$. To discuss completeness, we need to introduce the "supremum".

Consider a set $S$, such that $S \subset \mathbf{R}$.
$S$ is *bounded above* if $\exists\, a \in \mathbf{R}$ such that $x \leq a\ \forall x \in S$.
$S$ is *bounded below* if $\exists\, a \in \mathbf{R}$ such that $x \geq a\ \forall x \in S$.
$S$ is *bounded* if it is bounded above and below.

**The Completeness Axiom** If $S \subset \mathbf{R}$ is nonempty and bounded above then $\exists\, a \in \mathbf{R}$ that is a least upper bound or "sup". Specifically, (i) $x \leq a\ \forall x \in S$ and (ii) $a \leq \beta\ \forall$ upper bounds, $\beta$, of $S$.

Supremums are unique by (ii) because if $a_1, a_2$ are upper bounds and $a_1 \leq a_2$ and $a_2 \leq a_1$ then $a_1 = a_2$. Thus it makes sense to talk about "the" supremum.

There is also an "infimum" or greatest lower bound that follows from repeating these arguments with $-S$.

#### 1.3.1 Maximum and Supremum are not necessarily the same

It is important to note that the maximum is not necessarily the same thing as the supremum:

The maximum is defined as $a \in S$ such that $x \leq a$ for all $x \in S$
The supremum is defined as $a$ such that $x \leq a$ for all $x \in S$ and (ii)

Note that the supremum does not have to be in $S$. In fact the max of $S$ exists if and only if the supremum of $S$ is a member of $S$, in which case the max of $S$ is equal to the supremum of $S$. Conversely, if $\sup S \notin S$ then the max of $S$ does not exist.

For example take $S = (0, 1)$. Then $\sup S = 1$ but $\sup S \notin S$, so $\max S$ does not exist. The sequence $1 - 1/n$ for $n \in \mathbf{N}$ comes arbitrarily close to the max of $S$ but never reaches it.

However, any *finite*, nonempty set has a maximum.

#### 1.3.2 Completeness does not hold for Q

Consider the set $\{x \mid x^2 < 2\}$. The number $\sqrt{2}$ is not a member of $\mathbf{Q}$ so the supremum of this set cannot be a member of the set. Thus $\mathbf{Q}$ is not complete, i.e. there are 'holes' at the irrational numbers.

# 2 Sequences

## 2.1 Definitions

A sequence is a mapping from $\mathbf{N} \to \mathbf{R}$ and is generally written as $\{a_n\}$.

A sequence is *increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$
A sequence is *decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$
A sequences is *strictly* increasing or decreasing if equality never holds.
A sequence is *monotone* if it is increasing or decreasing.

## 2.2 Convergence