Notes on Real Analysis for Math 171

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Todo list

1 Foundations of the Reals

1.1 The Field Axioms

The field axioms are a set of axioms that we accept as the foundation of the reals.

$\forall a, b, c \in \mathbf{R}$:

F1 Commutativity $a + b = b + a$
F2 Associativity $a + (b + c) = (a + b) + c$
F3 Distributive $a(b + c) = ab + ac$
F4 Identity $\exists\ 0, 1$ such that $0 + a = a$, $1 \cdot a = a$
F5 Additive Inverse $\exists\ -a$ such that $a + (-a) = 0$
F5 Multiplicative Inverse $\exists\ 1/a$ such that $a(1/a) = 1$

1.2 The Order Axioms

O1 Positive Numbers $\exists$ a set $P \subset \mathbf{R}$ such that for all $a \in \mathbf{R}$ either $a \in P$, $-a \in P$, or $a = 0$.
O2 $a, b \in P$ implies that $a \cdot b$ and $a + b$ are in P

Thus we define $a > b$ as $a - b \in P$ and similarly $a < b$ is defined as $b - a \in P$.

F and O axioms hold for the rationals, $\mathbf{Q}$, but O does not hold for the complex numbers.

1.3 The Completeness Axiom

Completeness distinguishes the Reals from the Rationals. Intuitively, there are 'holes' in the rationals at irrational numbers like $\sqrt{2}$. To discuss completeness, we need to introduce some definitions.

Consider a set S, such that $S \subset \mathbf{R}$.
S is *bounded above* if $\exists\ a \in \mathbf{R}$ such that $x \leq a\ \forall x \in S$.
S is *bounded below* if $\exists\ a \in \mathbf{R}$ such that $x \geq a\ \forall x \in S$.
S is *bounded* if it is bounded above and below.

The Completeness Axiom If $S \subset \mathbf{R}$ is nonempty and bounded above then $\exists\ a \in \mathbf{R}$ that is a least upper bound or supremum. Specifically, (i) $x \leq a\ \forall x \in S$ and (ii) $a \leq \beta\ \forall$ upper bounds, β, of S.

Supremums are unique by (ii) because if a_1, a_2 are upper bounds and $a_1 \leq a_2$ and $a_2 \leq a_1$ then $a_1 = a_2$. Thus it makes sense to talk about "the" supremum.

There is also an "infimum" or greatest lower bound that follows from repeating these arguments with $-S$.

Note: It is important to note that the maximum and supremum of a set are not necessarily the same.

The maximum is defined as $a \in S$ such that $x \leq a$ for all $x \in S$
The supremum is defined as a such that $x \leq a$ for all $x \in S$ and (ii)

The supremum does not have to be in S. In fact the max of S exists if and only if the supremum of S is a member of S, in which case the max of S is equal to the supremum of S. Conversely, if $\sup S \notin S$ then the max of S does not exist.

For example take $S = (0, 1)$. Then $\sup S = 1$ but $\sup S \notin S$, so $\max S$ does not exist. The sequence $1 - 1/n$ for $n \in \mathbf{N}$ comes arbitrarily close to the max of S but never reaches it.

However, any *finite*, nonempty set has a maximum.

1.4 Consequences of Completeness

1.4.1 Completeness does not hold for Q

Consider the set $\{x \mid x^2 < 2\}$. The number $\sqrt{2}$ is not a member of $\mathbf{Q}$ so the supremum of this set cannot be a member of the set. Thus $\mathbf{Q}$ is not complete, i.e. there are 'holes' at the irrational numbers.

1.4.2 The Archimidean Property of the Reals

Prove the Archimidean property of the Reals

1.4.3 Q and I are dense in R

Prove that $\mathbf{Q}$ and $\mathbf{I}$ are dense in $\mathbf{R}$

1.4.4 Q is countable, R and I are uncountable

Prove that $\mathbf{Q}$ is countable, $\mathbf{R}$ and $\mathbf{I}$ are uncountable

2 Real Sequences

2.1 Definitions

A sequence is a mapping from $\mathbf{N} \to \mathbf{R}$ and is generally written as $\{a_n\}$.

A sequence is *increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$
A sequence is *decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$
A sequences is *strictly* increasing or decreasing if equality never holds.
A sequence is *monotone* if it is increasing or decreasing.

Define subsequences

2.2 Convergence

A sequence, $\{a_n\}$ is convergent if $\exists\ \ell \in \mathbf{R}$ such that $\forall \epsilon > 0\ \exists N$ such that $|a_n - \ell| < \epsilon$ $\forall n \geq N$. ℓ is called the limit of $\{a_n\}$.

In words, this means that a sequence is convergent if for any positive number epsilon we can pick a point in the sequence sufficiently far out such that all elements of the sequence after that point are within ϵ of ℓ. ϵ could be any positive number, but the idea is that as ϵ becomes arbitrarily small, we can find points of the sequence that are arbitrarily close to ℓ.

In general, proofs of convergence will follow a challenge-response format where given an ϵ you construct an N such that the criterion holds.

Define $\limsup$ and $\liminf$

2.2.1 Any convergent sequence is bounded

Theorem: Any convergent sequence is bounded.

Proof: Let $\{a_n\}$ be a convergent sequence with limit L. Then there exists an N such that for $n \geq N$, $|a_n - L| < \epsilon$, which implies $a_n < L + \epsilon$ for $n \geq N$. Because N is finite, we then know that $a_n \leq \max(a_1, a_2, \ldots, a_{N-1}, L + \epsilon)$ for some N and ϵ. Thus $\{a_n\}$ is bounded. $\blacksquare$

2.2.2 A bounded, monotone sequence converges

Theorem: A bounded, monotone sequence converges

Proof: Assume a sequence, $\{a_n\}$, is increasing (WLOG) and bounded, and let $\ell = \sup S$ where $S = \{a_n \mid n \in \mathbf{N}\}$ (i.e. the set of elements of the sequence). We claim that

$\lim_{n\to\infty} a_n = \ell$, or equivalently that $\{a_n\}$ converges to the limit ℓ.

Let $\epsilon > 0$. Then it must be that $\ell - \epsilon$ is *not* and upper bound because ℓ is the supremum of S. Thus there exists $a_N \in S$ such that $a_N > \ell - \epsilon$. This implies that for all $n > N$, $a_n \geq a_N > \ell - \epsilon$.

On the other hand, because all a_n are members of S and ℓ is the supremum, $a_n < \ell + \epsilon$ for all $n \geq N$. Thus for $n \geq N$, $a_n > \ell - \epsilon$ and $a_n < \ell + \epsilon$, which implies that $|a_n - \ell| < \epsilon$. Therefore by the definition of convergence, the $\{a_n\}$ converges. ■

2.2.3 The Squeeze Theorem

Prove the Squeeze Theorem

2.3 The Bolzano–Weierstrass Theorem

The Bolzano-Weierstrass Theorem is of great importance to analysis and states that any bounded sequence has a convergent subsequence.

This is not immediately obvious because the sequence $a_n = (-1)^n$ is bounded but does not converge. However, if we take the subsequences of $n_i = 2n$ or $n_i = 2n - 1$ for $n \in \mathbf{N}$ then we have only the odd or even terms of $\{a_n\}$. Those subsequences consist of only 1 and -1, respectively, and are thus convergent. We will now make this intuition more formal.

Theorem: Any bounded sequence has a convergent subsequence.

Proof: Let $\{a_n\}$ be a bounded subsequence. Then there exists a $l, u \in \mathbf{R}$ such that $l \leq a_n \leq u$ for all $n \in \mathbf{N}$. Then we know that $a_n \in [l, u]$ for all positive n. Now consider the bisection of this interval into two, giving the intervals:

$$\left[l, \frac{l+u}{2}\right], \left[\frac{l+u}{2}, u\right]$$

Because there are infinitely many terms in $\{a_n\}$, one or both of these intervals must contain infinitely many terms of $\{a_n\}$. Pick one such interval and label it I_1, with its endpoints labeled l_1 and u_1.

Now repeat this process for I_1, bisecting it into two closed intervals, picking one subinterval which contains infinitely many members of $\{a_n\}$, and labelling its endpoints l_2 and u_2. Because there are infintely many elements in $\{a_n\}$ it is possible to pick a sequence of closed intervals, I_n such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ where the width of I_n is $\frac{u-l}{2^n}$. Additionally, each of these intervals contains infinitely many elements of $\{a_n\}$.

Now choose a positive integer n_1 such that $a_{n_1} \in I_1$. Because I_2 contains infinitely many elements of $\{a_n\}$, there exists a positive integer n_2 such that $n_2 > n_1$ and $a_{n_2} \in I_2$. Continue picking elements of $\{a_n\}$ in this way to construct a subsequence, $\{a_{n_i}\}$, such that $a_{n_i} \in I_i$ for all n_i. We will show that $\{a_{n_i}\}$ converges.

Ending 1 We know that there must be one element, x in all I_n. Let $\epsilon > 0$ and pick an interval, I_N such that the width of I_N, $\frac{u-l}{2^N}$ is less than epsilon, and pick an element, a_{n_K} such that $a_{n_K} \in I_N$. x must be in this interval, and by construction $a_{n_i} \in I_N$ for all $n_i > n_K$. Thus $|a_{n_i} - x| < \epsilon$ for $n_i > n_K$ and $\{a_{n_i}\}$ converges to x. ■

Ending 2 Consider the sequence of upper bounds on these intervals $u_1, u_2, \ldots$, and note that they are bounded and decreasing and therefore converge to some limit, U. Similarly, the lower bounds of the intervals, $l_1, l_2, \ldots$ are bounded and increasing and therefore converge to some limit L. Because the width of interval n is $\frac{u-l}{2^n}$, $\lim_{n\to\infty} u_n - l_n = 0$. Finally we know that $u_i \geq a_{n_i} \geq l_i$ for all i, so by the squeeze theorem $\{a_{n_i}\}$ converges. ■

2.4 Cauchy Sequences

A sequence $\{a_n\}$ is called Cauchy if for all $\epsilon > 0$ there exists a positive integer N such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$. Intuitively, this says a sequence is Cauchy if it has a tail where the elements are arbitrarily close together. Note that this is not a statement about consecutive elements in $\{a_n\}$, it is a statement about all elements past N.

2.4.1 A sequence converges if and only if it is Cauchy

Theorem: A sequence converges if and only if it is Cauchy.

Proof: ($\Longrightarrow$) Assume that $\{a_n\}$ is a convergent series with limit L. Let $\epsilon > 0$ and choose N such that $|a_n - L| < \epsilon/2$ for all $n \geq N$.

Now choose $m, n \geq N$. By the triangle inequality we know that $|a_n - a_m| \leq |a_n - L| + |L - a_m| \leq \epsilon/2 + \epsilon/2 = \epsilon$. Thus for all $n, m \geq N$, $|a_n - a_m| < \epsilon$, and $\{a_n\}$ must be Cauchy.

($\Longleftarrow$) Assume that $\{a_n\}$ is a Cauchy sequence. We will show that $\{a_n\}$ is convergent in three steps: (1) Show that any Cauchy sequence is bounded. (2) Use Bolzano-Weierstrass to obtain a convergent subsequence. (3) Show that (2) implies that the whole sequence converges.

Take $\epsilon = 1$. Because $\{a_n\}$ is Cauchy we know that there exists an N such that $|a_n - a_m| < 1$

for all $n, m \geq N$. This implies for all $n \geq N$:

$$\begin{aligned}
|a_n - a_N| &< 1 \\
||a_n| - |a_N|| &< 1 \\
|a_n| - |a_N| &< 1 \\
|a_n| &< |a_N| + 1
\end{aligned}$$

Because N is finite, there are finitely many elements of $\{a_n\}$ where $n < N$, so we know that for all n (not just $n \geq N$), $|a_n| \leq \max(|a_1|, |a_2|, \ldots, |a_N| + 1)$. Thus $\{a_n\}$ is bounded.

Because $\{a_n\}$ is bounded we know by the Bolzano-Weierstrass theorem that there must exist a convergent subsequence of $\{a_n\}$, $\{a_{n_i}\}$ with limit L.

Take $\epsilon > 0$. Then we know that there exists an N such that $|a_{n_i} - L| < \epsilon/2$ for all $n_i \geq N$. Additionally, because $\{a_n\}$ is Cauchy we know that there exists an N' such that $|a_n - a_m| < \epsilon/2$ for all $n, m \geq N'$. Then, by the triangle inequality we know that $|a_n - L| \leq |a_n - a_{n_i}| + |a_{n_i} - L| < \epsilon/2 + \epsilon/2 = \epsilon$. Thus $\{a_n\}$ must converge, and its limit must be L. $\blacksquare$

3 Real Series

A *series* is the sum of the terms of a sequence, and is often written as $\sum_{k=1}^{\infty} a_k$ where $\{a_n\}$ is the related sequence. The sum of the first n terms of a sequence, i.e. $s_n = \sum_{k=1}^{n} a_k$ is called the nth partial sum of a sequence and is generally denoted as $\{s_n\}$.

If $\{s_n\}$ has a limit S, i.e. if the sequence of partial sums converges, then we say that the series $\sum_{k=1}^{\infty} a_k$ converges and has sum S. If $\{s_n\}$ does not converge, i.e. does not have a limit, then we say that the series diverges.

3.1 Convergence of Series

Theorem: If $\sum_{i=1}^{\infty} a_n$ converges, then $\lim a_n = 0$

Proof:

Prove that if $\sum_{i=1}^{\infty} a_n$ converges, then $\lim a_n = 0$

Note: The converse of the above statement is not true. For example $\sum_{i=1}^{\infty} 1/n$ does not converge, but $\lim a_n = 0$.

Theorem: If $\sum a_n, \sum b_n$ converge to S and T respectively, and $\alpha, \beta \in \mathbf{R}$ then $\lim_{n\to\infty} \sum_{k=1}^{n} \alpha a_n + \sum_{k=1}^{n} \beta b_n = \alpha S + \beta T$.

Proof:

Prove linearity of convergence for series

Theorem: If all terms of a series $\sum a_n$ are greater than 0, then $\sum a_n$ converges if and only if the sequence of partial sums $\{s_n\}$ is bounded.

Proof:

Prove that if all terms of a series $\sum a_n$ are greater than 0, then $\sum a_n$ converges if and only if the sequence of partial sums $\{s_n\}$ is bounded

3.2 Absolute Convergence

A series, $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. If a series converges but does not converge absolutely then it is called conditionally convergent. This can happen when terms in the series cancel in the normal case but when wrapped in absolute value do not cancel.

You can think of absolute value of a number as a combination of the positive and negative parts. More specifically, $|a_n| = (a_n)^+ + (a_n)^-$. Then you can rewrite $\sum_{n=1}^{\infty} a_n$ as $\sum_{n=1}^{\infty}((a_n)^+ - (a_n)^-)$. Thus the series is absolutely convergent if and only of both $\sum_{n=1}^{\infty}(a_n)^+$

and $\sum_{n=1}^{\infty}(a_n)^-$ converge.

Theorem: Absolute convergence implies converges.

Proof:

Prove that absolute convergence implies convergence

3.3 Rearrangements

A rearrangement of a sequence $\{a_n\}$ is a 1-1 onto mapping $f : \mathbf{N} \to \mathbf{N}$. Intuitively, it's exactly like what it sounds like – it just changes the ordering of a sequence.

If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, it is possible to reorder the terms and get different limits. Put another way, not all rearrangements converge or converge to the same limit.

Find example of series that has rearrangements with different limits

Theorem: If $\sum a_n$ is absolutely convergent then any rearrangement is also absolutely convergent and has the same sum as $\sum a_n$.

Proof:

Prove that if If $\sum a_n$ is absolutely convergent then any rearrangement is absolutely convergent, and has the same sum as $\sum a_n$

3.4 Convergence Tests

3.4.1 Ratio Test

Write up ratio test

3.4.2 Root Test

Write up root test

3.4.3 Alternating Series Test

Write up alternating series test

3.4.4 Comparison Test

Write up comparison test

3.5 Power Series

A power series is an infinite series of the form

$$P(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

This power series is centered at 0, and is the special case of a power series with an arbitrary center, c:

$$P(x,c) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

For all power series one of three possibilities holds

(i) $P(x)$ converges only for $x = 0$. This occurs if a_n grows extremely rapidly. For example

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots + n!x^n + \cdots$$

By the ratio test we have:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!x^{n+1}}{n!x^n} = \lim_{n\to\infty} (n+1)x = \infty$$

Thus for all non-zero x, this power series diverges.

(ii) $P(x)$ converges for all x. This occurs if the a_n go to 0 quickly, for example

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

To see that this power series is convergent we will use the ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{x}{n+1} = 0$$

Thus the power series for e^x converges for all finite x.

(iii) There exists $\rho > 0$ such that $P(x)$ converges absolutely for $|x| < \rho$, diverges for $|x| > \rho$. When $|x| = \rho$ there is no statement. In this case ρ is known as "the radius of convergence". For example

$$\sum_{n=0}^{\infty} 2^n x^n = 1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots$$

Using the ratio test we have that:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}x^{n+1}}{2^n x^n} \right| = \lim_{n\to\infty} 2|x|$$

The series converges when $\rho < 1$, which occurs when $|x| < 1/2$. Technically, we should check what happens when $|x| = 1/2$, but spoiler alert: it diverges...

We can actually prove that one of these three possibilities always holds.

Proof:

Prove 3 possibilities for power sequences

4 Continuity of Real Functions

Let f be a function $f : \mathbf{R} \to \mathbf{R}$. Then f is continuous at $c \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

A function is continuous on an interval $[a, b]$ if it is continuous at every point $c \in [a, b]$.

4.1 Algebraic Properties of Continuous Functions

Continuous functions are closed under addition, subtraction, multiplication, division when the denominator is not 0, and composition.

Prove algebraic properties of continuous functions

4.2 The Heine–Borel Theorem

The Heine-Borel theorem is a result of fundamental importance to analysis that states that if we cover a closed interval of the real line with open intervals we can extract a finite subset that still covers the closed interval. This result has several consequences for continuous functions.

This is not true for the $\mathbf{Q}$, for example, because you could take the interval to be $[1, 2]$, which includes $\sqrt{2}$. You would need an infinite number of subintervals of rationals to come close to $\sqrt{2}$ so it is impossible to extract a finite subcovering.

Theorem: Let $[a, b] \in \mathbf{R}$ and let $\mathcal{I}$ be an infinite collection of open intervals such that $[a, b] \subset \cup\mathcal{I}$. Then there exists a finite subset $I_1, \ldots, I_n \in \mathcal{I}$ which already cover $[a, b]$, i.e. $[a, b] \subset \cup_{i=1}^{n} I_i$

Proof: Let the set X be the set of points in $[a, b]$ that are coverable with a finite number of I. Formally,

$$X = \{x \in (a, b] \mid [a, x] \subset \bigcup_{i=1}^{n} I_i \text{ for some } I_1, \ldots I_n \in \mathcal{I}\}$$

To see that X is not empty note that $a \in I = (l_0, u_0)$ for some $I \in \mathcal{I}$. Then we can choose x_0 such that $a < x_0 < u_0$, which means that x_0 is also in I and therefore $x_0 \in X$. Because X is nonempty and bounded above (by b), it must have a least upper bound c, such that $a < x_0 \leq c$. The remainder of the proof will show that $c \in X$, and then that $c = b$, thus proving the theorem.

We know that $c \in I = (l_1, u_1)$ for some $I \in \mathcal{I}$. Additionally, $l_1 < c$, and thus l_1 cannot be an upper bound for X. Because l_1 is less than the upper bound for X, there exists an

$x \in X, x > a$ such that $l_1 < x \leq c$. Then, by the definition of X there exists $I_1, \ldots, I_n \in \mathcal{I}$ such that

$$[a, x] \subset \bigcup_{i=1}^{n} I_i$$

But because x and c are both contained in the chosen interval I, we know that

$$[a, c] \subset \left(\bigcup_{i=1}^{n} I_i \right) \cup I$$

And so the supremum of X, c, is a member of X. Next we will show that $c = b$.

To see that $c = b$, first note that by construction $c \in [a, b]$ and so $c \leq b$. Thus it suffices to show that c is not less than b. Assume the contradiction, namely that $c < b$. Because $c \in X$ we know that we can cover $[a, c]$ with a finite union of open intervals, i.e.

$$[a, c] \subset \bigcup_{i=1}^{n} I_i$$

for some $I_1, \ldots I_n \in \mathcal{I}$. Thus $c \in I_j = (l_2, u_2)$ for some j, $1 \leq j \leq n$. However, because I_j is open we know that we can pick a $d \in I_j$ such that $c < d < b$ and $c < d < u_2$. But we have already covered $[a, d]$ with $\cup_{i=1}^{n} I_i$, i.e. we have covered $[a, d]$ with a finite open cover and thus $d \in X$. Furthermore, we previously picked c such that it was the supremum of X but $d > c$, and so we have reached a contradiction and $c = b$. ■

4.3 Consequences of Heine-Borel

Prove that if f is continuous on an interval then it is bounded above and below and achieves its max and min

5 Set Theory

6 Metric Spaces