# Notes on Real Analysis for Math 171

Dieterich Lawson

April 27, 2014

## Contents



## Todo list

# 1 Foundations of the Reals

## 1.1 The Field Axioms

The field axioms are a set of axioms that we accept as the foundation of the reals.

$\forall a, b, c \in \mathbf{R}$:

**F1 Commutativity** $a + b = b + a$
**F2 Associativity** $a + (b + c) = (a + b) + c$
**F3 Distributive** $a(b + c) = ab + ac$
**F4 Identity** $\exists\, 0, 1$ such that $0 + a = a$, $1 \cdot a = a$
**F5 Additive Inverse** $\exists\, -a$ such that $a + (-a) = 0$
**F5 Multiplicative Inverse** $\exists\, 1/a$ such that $a(1/a) = 1$

## 1.2 The Order Axioms

**O1 Positive Numbers** $\exists$ a set $P \subset \mathbf{R}$ such that for all $a \in \mathbf{R}$ either $a \in P$, $-a \in P$, or $a = 0$.
**O2** $a, b \in P$ implies that $a \cdot b$ and $a + b$ are in $P$

Thus we define $a > b$ as $a - b \in P$ and similarly $a < b$ is defined as $b - a \in P$.

$F$ and $O$ axioms hold for the rationals, $\mathbf{Q}$, but $O$ does not hold for the complex numbers.

## 1.3 The Completeness Axiom

Completeness distinguishes the Reals from the Rationals. Intuitively, there are 'holes' in the rationals at irrational numbers like $\sqrt{2}$. To discuss completeness, we need to introduce some definitions.

Consider a set $S$, such that $S \subset \mathbf{R}$.
$S$ is *bounded above* if $\exists\, a \in \mathbf{R}$ such that $x \leq a\ \forall x \in S$.
$S$ is *bounded below* if $\exists\, a \in \mathbf{R}$ such that $x \geq a\ \forall x \in S$.
$S$ is *bounded* if it is bounded above and below.

**The Completeness Axiom** If $S \subset \mathbf{R}$ is nonempty and bounded above then $\exists\, a \in \mathbf{R}$ that is a least upper bound or supremum. Specifically, (i) $x \leq a\ \forall x \in S$ and (ii) $a \leq \beta\ \forall$ upper bounds, $\beta$, of $S$.

Supremums are unique by (ii) because if $a_1, a_2$ are upper bounds and $a_1 \leq a_2$ and $a_2 \leq a_1$ then $a_1 = a_2$. Thus it makes sense to talk about "the" supremum.

There is also an "infimum" or greatest lower bound that follows from repeating these arguments with $-S$.

**Note:** It is important to note that the maximum and supremum of a set are not necessarily the same.

The maximum is defined as $a \in S$ such that $x \leq a$ for all $x \in S$
The supremum is defined as $a$ such that $x \leq a$ for all $x \in S$ and (ii)

The supremum does not have to be in $S$. In fact the max of $S$ exists if and only if the supremum of $S$ is a member of $S$, in which case the max of $S$ is equal to the supremum of $S$. Conversely, if $\sup S \notin S$ then the max of $S$ does not exist.

For example take $S = (0, 1)$. Then $\sup S = 1$ but $\sup S \notin S$, so $\max S$ does not exist. The sequence $1 - 1/n$ for $n \in \mathbf{N}$ comes arbitrarily close to the max of $S$ but never reaches it.

However, any *finite*, nonempty set has a maximum.

## 1.4 Consequences of Completeness

### 1.4.1 Completeness does not hold for Q

Consider the set $\{x \mid x^2 < 2\}$. The number $\sqrt{2}$ is not a member of $\mathbf{Q}$ so the supremum of this set cannot be a member of the set. Thus $\mathbf{Q}$ is not complete, i.e. there are 'holes' at the irrational numbers.

### 1.4.2 The Archimidean Property of the Reals

Prove the Archimidean property of the Reals

### 1.4.3 Q and I are dense in R

Prove that $\mathbf{Q}$ and $\mathbf{I}$ are dense in $\mathbf{R}$

### 1.4.4 Q is countable, R and I are uncountable

Prove that $\mathbf{Q}$ is countable, $\mathbf{R}$ and $\mathbf{I}$ are uncountable

# 2 Sequences

## 2.1 Definitions

A sequence is a mapping from $\mathbf{N} \to \mathbf{R}$ and is generally written as $\{a_n\}$.

A sequence is *increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$
A sequence is *decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$
A sequences is *strictly* increasing or decreasing if equality never holds.
A sequence is *monotone* if it is increasing or decreasing.

Define subsequences

## 2.2 Convergence

A sequence, $\{a_n\}$ is convergent if $\exists\ \ell \in \mathbf{R}$ such that $\forall \epsilon > 0\ \exists N$ such that $|a_n - \ell| < \epsilon$ $\forall n \geq N$. $\ell$ is called the limit of $\{a_n\}$.

In words, this means that a sequence is convergent if for any positive number epsilon we can pick a point in the sequence sufficiently far out such that all elements of the sequence after that point are within $\epsilon$ of $\ell$. $\epsilon$ could be any positive number, but the idea is that as $\epsilon$ becomes arbitrarily small, we can find points of the sequence that are arbitrarily close to $\ell$.

In general, proofs of convergence will follow a challenge-response format where given an $\epsilon$ you construct an $N$ such that the criterion holds.

Define $\limsup$ and $\liminf$

### 2.2.1 Any convergent sequence is bounded

*Theorem:* Any convergent sequence is bounded.

*Proof:* Let $\{a_n\}$ be a convergent sequence with limit $L$. Then there exists an $N$ such that for $n \geq N$, $|a_n - L| < \epsilon$, which implies $a_n < L + \epsilon$ for $n \geq N$. Because $N$ is finite, we then know that $a_n \leq \max(a_1, a_2, \ldots, a_{N-1}, L + \epsilon)$ for some $N$ and $\epsilon$. Thus $\{a_n\}$ is bounded. $\blacksquare$

### 2.2.2 A bounded, monotone sequence converges

*Theorem:* A bounded, monotone sequence converges

*Proof:* Assume a sequence, $\{a_n\}$, is increasing (WLOG) and bounded, and let $\ell = \sup S$ where $S = \{a_n \mid n \in \mathbf{N}\}$ (i.e. the set of elements of the sequence). We claim that

$\lim_{n\to\infty} a_n = \ell$, or equivalently that $\{a_n\}$ converges to the limit $\ell$.

Let $\epsilon > 0$. Then it must be that $\ell - \epsilon$ is *not* and upper bound because $\ell$ is the supremum of $S$. Thus there exists $a_N \in S$ such that $a_N > \ell - \epsilon$. This implies that for all $n > N$, $a_n \geq a_N > \ell - \epsilon$.

On the other hand, because all $a_n$ are members of $S$ and $\ell$ is the supremum, $a_n < \ell + \epsilon$ for all $n \geq N$. Thus for $n \geq N$, $a_n > \ell - \epsilon$ and $a_n < \ell + \epsilon$, which implies that $|a_n - \ell| < \epsilon$. Therefore by the definition of convergence, the $\{a_n\}$ converges. ■

### 2.2.3 The Squeeze Theorem

Prove the Squeeze Theorem

## 2.3 The Bolzano–Weierstrass Theorem

The Bolzano-Weierstrass Theorem is of great importance to analysis and states that any bounded sequence has a convergent subsequence.

This is not immediately obvious because the sequence $a_n = (-1)^n$ is bounded but does not converge. However, if we take the subsequences of $n_i = 2n$ or $n_i = 2n - 1$ for $n \in \mathbf{N}$ then we have only the odd or even terms of $\{a_n\}$. Those subsequences consist of only 1 and $-1$, respectively, and are thus convergent. We will now make this intuition more formal.

*Theorem:* Any bounded sequence has a convergent subsequence.

*Proof:* Let $\{a_n\}$ be a bounded subsequence. Then there exists a $l, u \in \mathbf{R}$ such that $l \leq a_n \leq u$ for all $n \in \mathbf{N}$. Then we know that $a_n \in [l, u]$ for all positive $n$. Now consider the bisection of this interval into two, giving the intervals:

$$\left[l, \frac{l+u}{2}\right], \left[\frac{l+u}{2}, u\right]$$

Because there are infinitely many terms in $\{a_n\}$, one or both of these intervals must contain infinitely many terms of $\{a_n\}$. Pick one such interval and label it $I_1$, with its endpoints labeled $l_1$ and $u_1$.

Now repeat this process for $I_1$, bisecting it into two closed intervals, picking one subinterval which contains infinitely many members of $\{a_n\}$, and labelling its endpoints $l_2$ and $u_2$. Because there are infintely many elements in $\{a_n\}$ it is possible to pick a sequence of closed intervals, $I_n$ such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ where the width of $I_n$ is $\frac{u-l}{2^n}$. Additionally, each of these intervals contains infinitely many elements of $\{a_n\}$.

Now choose a positive integer $n_1$ such that $a_{n_1} \in I_1$. Because $I_2$ contains infinitely many elements of $\{a_n\}$, there exists a positive integer $n_2$ such that $n_2 > n_1$ and $a_{n_2} \in I_2$. Continue picking elements of $\{a_n\}$ in this way to construct a subsequence, $\{a_{n_i}\}$, such that $a_{n_i} \in I_i$ for all $n_i$. We will show that $\{a_{n_i}\}$ converges.

**Ending 1** We know that there must be one element, $x$ in all $I_n$. Let $\epsilon > 0$ and pick an interval, $I_N$ such that the width of $I_N$, $\frac{u-l}{2^N}$ is less than epsilon, and pick an element, $a_{n_K}$ such that $a_{n_K} \in I_N$. $x$ must be in this interval, and by construction $a_{n_i} \in I_N$ for all $n_i > n_K$. Thus $|a_{n_i} - x| < \epsilon$ for $n_i > n_K$ and $\{a_{n_i}\}$ converges to $x$. ■

**Ending 2** Consider the sequence of upper bounds on these intervals $u_1, u_2, ...$, and note that they are bounded and decreasing and therefore converge to some limit, $U$. Similarly, the lower bounds converge to some limit $L$. Because the width of interval $n$ is $\frac{u-l}{2^n}$, $\lim_{n\to\infty} u_n - l_n = 0$. Finally we know that $u_i \geq a_{n_i} \geq l_i$ for all $i$, so by the squeeze theorem $\{a_{n_i}\}$ converges.

Make the alternate ending of BW more rigorous

■

## 2.4 Cauchy Sequences

A sequence $\{a_n\}$ is called Cauchy if for all $\epsilon > 0$ there exists a positive integer $N$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$. Intuitively, this says a sequence is Cauchy if it has a tail where the elements are arbitrarily close together. Note that this is not a statement about consecutive elements in $\{a_n\}$, it is a statement about all elements past $N$.

There are various convergence results related to Cauchy sequences.

### 2.4.1 A sequence converges if and only if it is Cauchy

*Theorem:* A sequence converges if and only if it is Cauchy.

*Proof:* ($\implies$) Assume that $\{a_n\}$ is a convergent series with limit $L$. Let $\epsilon > 0$ and choose $N$ such that $|a_n - L| < \epsilon/2$ for all $n \geq N$.

Now choose $m, n \geq N$. By the triangle inequality we know that $|a_n - a_m| \leq |a_n - L| + |L - a_m| \leq \epsilon/2 + \epsilon/2 = \epsilon$. Thus for all $n, m \geq N$, $|a_n - a_m| < \epsilon$, and $\{a_n\}$ must be Cauchy.

($\impliedby$) Assume that $\{a_n\}$ is a Cauchy sequence. We will show that $\{a_n\}$ is convergent in three steps: (1) Show that any Cauchy sequence is bounded. (2) Use Bolzano-Weierstrass to obtain a convergent subsequence. (3) Show that (2) implies that the whole sequence converges.

Take $\epsilon = 1$. Because $\{a_n\}$ is Cauchy we know that there exists an $N$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$. This implies for all $n \geq N$:

$$\begin{aligned} |a_n - a_N| &< 1 \\ ||a_n| - |a_N|| &< 1 \\ |a_n| - |a_N| &< 1 \\ |a_n| &< |a_N| + 1 \end{aligned}$$

Because $N$ is finite, there are finitely many elements of $\{a_n\}$ where $n < N$, so we know that for all $n$ (not just $n \geq N$), $|a_n| \leq \max(|a_1|, |a_2|, \ldots, |a_N| + 1)$. Thus $\{a_n\}$ is bounded.

Because $\{a_n\}$ is bounded we know by the Bolzano-Weierstrass theorem that there must exist a convergent subsequence of $\{a_n\}$, $\{a_{n_i}\}$ with limit $L$.

Take $\epsilon > 0$. Then we know that there exists an $N$ such that $|a_{n_i} - L| < \epsilon/2$ for all $n_i \geq N$. Additionally, because $\{a_n\}$ is Cauchy we know that there exists an $N'$ such that $|a_n - a_m| < \epsilon/2$ for all $n, m \geq N'$. Then, by the triangle inequality we know that $|a_n - L| \leq |a_n - a_{n_i}| + |a_{n_i} - L| < \epsilon/2 + \epsilon/2 = \epsilon$. Thus $\{a_n\}$ must converge, and its limit must be $L$. ■