Math 171 Notes

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Contents

Todo list

1 Foundations of the Reals

1.1 The Field Axioms

The field axioms are a set of axioms that we accept as the foundation of the reals.

$\forall a, b, c \in \mathbf{R}$:

F1 Commutativity $a + b = b + a$
F2 Associativity $a + (b + c) = (a + b) + c$
F3 Distributive $a(b + c) = ab + ac$
F4 Identity $\exists\ 0, 1$ such that $0 + a = a$, $1 \cdot a = a$
F5 Additive Inverse $\exists\ -a$ such that $a + (-a) = 0$
F5 Multiplicative Inverse $\exists\ 1/a$ such that $a(1/a) = 1$

1.2 The Order Axioms

O1 Positive Numbers $\exists$ a set $P \subset \mathbf{R}$ such that for all $a \in \mathbf{R}$ either $a \in P$, $-a \in P$, or $a = 0$.
O2 $a, b \in P$ implies that $a \cdot b$ and $a + b$ are in P

Thus we define $a > b$ as $a - b \in P$ and similarly $a < b$ is defined as $b - a \in P$.

F and O axioms hold for the rationals, $\mathbf{Q}$, but O does not hold for the complex numbers.

1.3 The Completeness Axiom

Completeness distinguishes the Reals from the Rationals. Intuitively, there are 'holes' in the rationals at irrational numbers like $\sqrt{2}$. To discuss completeness, we need to introduce the "supremum".

Consider a set S, such that $S \subset \mathbf{R}$.
S is *bounded above* if $\exists\ a \in \mathbf{R}$ such that $x \leq a\ \forall x \in S$.
S is *bounded below* if $\exists\ a \in \mathbf{R}$ such that $x \geq a\ \forall x \in S$.
S is *bounded* if it is bounded above and below.

The Completeness Axiom If $S \subset \mathbf{R}$ is nonempty and bounded above then $\exists\ a \in \mathbf{R}$ that is a least upper bound or supremum. Specifically, (i) $x \leq a\ \forall x \in S$ and (ii) $a \leq \beta\ \forall$ upper bounds, β, of S.

Supremums are unique by (ii) because if a_1, a_2 are upper bounds and $a_1 \leq a_2$ and $a_2 \leq a_1$ then $a_1 = a_2$. Thus it makes sense to talk about "the" supremum.

There is also an "infimum" or greatest lower bound that follows from repeating these arguments with $-S$.

1.3.1 The maximum and supremum of a set are not necessarily the same

It is important to note that the maximum is not necessarily the same thing as the supremum:

The maximum is defined as $a \in S$ such that $x \leq a$ for all $x \in S$
The supremum is defined as a such that $x \leq a$ for all $x \in S$ and (ii)

Note that the supremum does not have to be in S. In fact the max of S exists if and only if the supremum of S is a member of S, in which case the max of S is equal to the supremum of S. Conversely, if $\sup S \notin S$ then the max of S does not exist.

For example take $S = (0, 1)$. Then $\sup S = 1$ but $\sup S \notin S$, so $\max S$ does not exist. The sequence $1 - 1/n$ for $n \in \mathbf{N}$ comes arbitrarily close to the max of S but never reaches it.

However, any *finite*, nonempty set has a maximum.

1.4 Consequences of Completeness

1.4.1 Completeness does not hold for Q

Consider the set $\{x \mid x^2 < 2\}$. The number $\sqrt{2}$ is not a member of $\mathbf{Q}$ so the supremum of this set cannot be a member of the set. Thus $\mathbf{Q}$ is not complete, i.e. there are 'holes' at the irrational numbers.

1.4.2 The Archimidean Property of the Reals

Prove the Archimidean property of the Reals

1.4.3 Q and I are dense in R

Prove that $\mathbf{Q}$ and $\mathbf{I}$ are dense in $\mathbf{R}$

1.4.4 Q is countable, R and I are uncountable

Prove that $\mathbf{Q}$ is countable, $\mathbf{R}$ and $\mathbf{I}$ are uncountable

2 Sequences

2.1 Definitions

A sequence is a mapping from $\mathbf{N} \to \mathbf{R}$ and is generally written as $\{a_n\}$.

A sequence is *increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$
A sequence is *decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbf{N}$
A sequences is *strictly* increasing or decreasing if equality never holds.
A sequence is *monotone* if it is increasing or decreasing.

Define subsequences

2.2 Convergence

A sequence, $\{a_n\}$ is convergent if $\exists\ \ell \in \mathbf{R}$ such that $\forall \epsilon > 0\ \exists N$ such that $|a_n - \ell| < \epsilon$ $\forall n \geq N$. ℓ is called the limit of $\{a_n\}$.

In words, this means that a sequence is convergent if for any positive number epsilon we can pick a point in the sequence sufficiently far out such that all elements of the sequence after that point are within ϵ of ℓ. ϵ could be any positive number, but the idea is that as ϵ becomes arbitrarily small, we can find points of the sequence that are arbitrarily close to ℓ.

In general, proofs of convergence will follow a challenge-response format where given an ϵ you construct an N such that the criterion holds.

Define $\limsup$ and $\liminf$

2.2.1 Any convergent sequence is bounded

Theorem: Any convergent sequence is bounded.

Proof: Let $\{a_n\}$ be a convergent sequence with limit L. Then there exists an N such that for $n \geq N$, $|a_n - L| < \epsilon$, which implies $a_n < L + \epsilon$ for $n \geq N$. Because N is finite, we then know that $a_n \leq \max(a_1, a_2, \ldots, a_{N-1}, L + \epsilon)$ for some N and ϵ. Thus $\{a_n\}$ is bounded. ■

2.2.2 A bounded, monotone sequence converges

Theorem: A bounded, monotone sequence converges

Proof: Assume a sequence, $\{a_n\}$, is increasing (WLOG) and bounded, and let $\ell = \sup S$ where $S = \{a_n \mid n \in \mathbf{N}\}$ (i.e. the set of elements of the sequence). We claim that

$\lim_{n\to\infty} a_n = \ell$, or equivalently that $\{a_n\}$ converges to the limit ℓ.

Let $\epsilon > 0$. Then it must be that $\ell - \epsilon$ is *not* and upper bound because ℓ is the supremum of S. Thus there exists $a_N \in S$ such that $a_N > \ell - \epsilon$. This implies that for all $n > N$, $a_n \geq a_N > \ell - \epsilon$.

On the other hand, because all a_n are members of S and ℓ is the supremum, $a_n < \ell + \epsilon$ for all $n \geq N$. Thus for $n \geq N$, $a_n > \ell - \epsilon$ and $a_n < \ell + \epsilon$, which implies that $|a_n - \ell| < \epsilon$. Therefore by the definition of convergence, the $\{a_n\}$ converges. ■

2.2.3 The Squeeze Theorem

Prove the Squeeze Theorem

2.3 The Bolzano–Weierstrass Theorem

The Bolzano-Weierstrass Theorem is of great importance to analysis and states that any bounded sequence has a convergent subsequence.

This is not immediately obvious because the sequence $a_n = (-1)^n$ is bounded but does not converge. However, if we take the subsequences of $n_i = 2n$ or $n_i = 2n - 1$ for $n \in \mathbf{N}$ then we have only the odd or even terms of $\{a_n\}$. Those subsequences consist of only 1 and -1, respectively, and are thus convergent. We will now make this intuition more formal.

Theorem: Any bounded sequence has a convergent subsequence.

Proof: Let $\{a_n\}$ be a bounded subsequence. Then there exists a $l, u \in \mathbf{R}$ such that $l \leq a_n \leq u$ for all $n \in \mathbf{N}$. Then we know that $a_n \in [l, u]$ for all positive n. Now consider the bisection of this interval into two, giving the intervals:

$$\left[l, \frac{l+u}{2}\right], \left[\frac{l+u}{2}, u\right]$$

Because there are infinitely many terms in $\{a_n\}$, one or both of these intervals must contain infinitely many terms of $\{a_n\}$. Pick one such interval and label it I_1, with its endpoints labeled l_1 and u_1.

Now repeat this process for I_1, bisecting it into two closed intervals, picking one subinterval which contains infinitely many members of $\{a_n\}$, and labelling its endpoints l_2 and u_2. Because there are infintely many elements in $\{a_n\}$ it is possible to pick a sequence of closed intervals, I_n such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ where the width of I_n is $\frac{u-l}{2^n}$. Additionally, each of these intervals contains infinitely many elements of $\{a_n\}$.

Now choose a positive integer n_1 such that $a_{n_1} \in I_1$. Because I_2 contains infinitely many elements of $\{a_n\}$, there exists a positive integer n_2 such that $n_2 > n_1$ and $a_{n_2} \in I_2$. Continue picking elements of $\{a_n\}$ in this way to construct a subsequence, $\{a_{n_i}\}$, such that $a_{n_i} \in I_i$ for all n_i. We will show that $\{a_{n_i}\}$ converges.

Ending 1 We know that there must be one element, x in all I_i. Let $\epsilon > 0$ and pick an interval, I_N such that the width of I_N, $\frac{u-l}{2^N}$ is less than epsilon, and pick an element, a_{n_K} such that $a_{n_K} \in I_N$. x must be in this interval, and by construction $a_{n_i} \in I_N$ for all $n_i > n_K$. Thus $|a_{n_i} - x| < \epsilon$ for $n_i > n_K$ and $\{a_{n_i}\}$ converges to x. ■

Ending 2 Consider the sequence of upper bounds on these intervals $u_1, u_2, \ldots$, and note that they are bounded and decreasing and therefore converge to some limit, U. Similarly, the lower bounds converge to some limit L. Because the width of interval n is $\frac{u-l}{2^n}$, $\lim_{n\to\infty} u_n - l_n = 0$. Finally we know that $u_i \geq a_{n_i} \geq l_i$ for all i, so by the squeeze theorem $\{a_{n_i}\}$ converges.

Make the alternate ending of BW more rigorous

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