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# Variational Inference for Inverse Reinforcement Learning with Gaussian Processes

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30th March 2019



H. Kretzschmar, M. Spies, C. Sprunk and W. Burgard, "Socially compliant mobile robot navigation via inverse reinforcement learning", *I. J. Robotics Res.*, 2016



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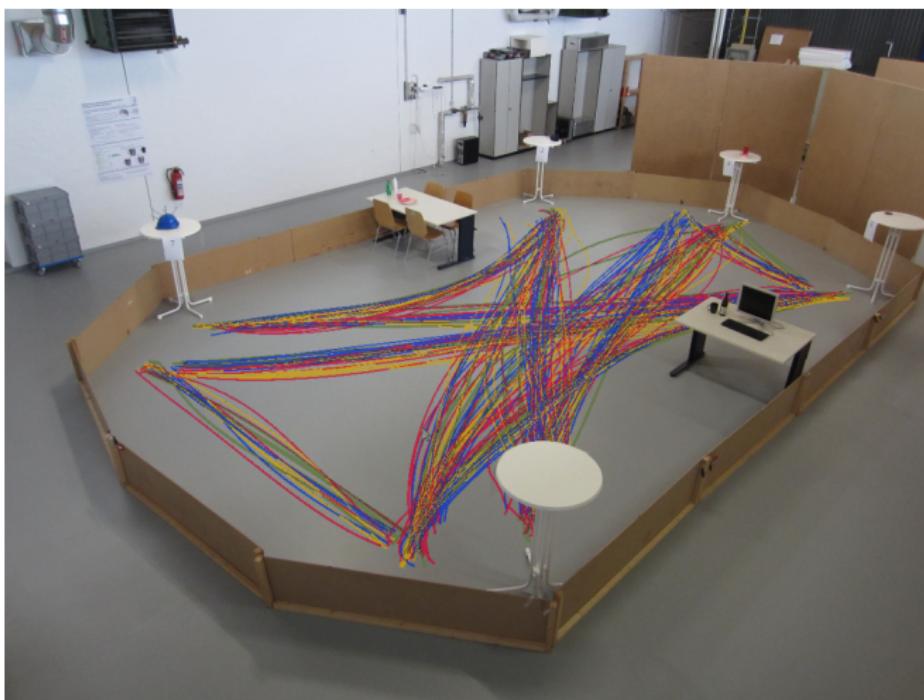
H. Kretzschmar, M. Spies, C. Sprunk and W. Burgard, "Socially compliant mobile robot navigation via inverse reinforcement learning", *I. J. Robotics Res.*, 2016

# Inverse Reinforcement Learning (IRL)

## Model (MDP)



## Demonstrations



H. Kretzschmar, M. Spies, C. Sprunk and W. Burgard, "Socially compliant mobile robot navigation via inverse reinforcement learning", *I. J. Robotics Res.*, 2016

## Definition (Markov Decision Process)

An MDP is a set  $\mathcal{M} = \{\mathcal{S}, \mathcal{A}, \mathcal{T}, \gamma, \mathbf{r}\}$  that consists of:

- states  $\mathcal{S}$
- actions  $\mathcal{A}$
- transition function  $\mathcal{T} : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \rightarrow [0, 1]$
- discount factor  $\gamma \in [0, 1)$
- reward function/vector  $\mathbf{r} \in \mathbb{R}^{|\mathcal{S}|}$  (or  $r : \mathcal{S} \rightarrow \mathbb{R}$ )

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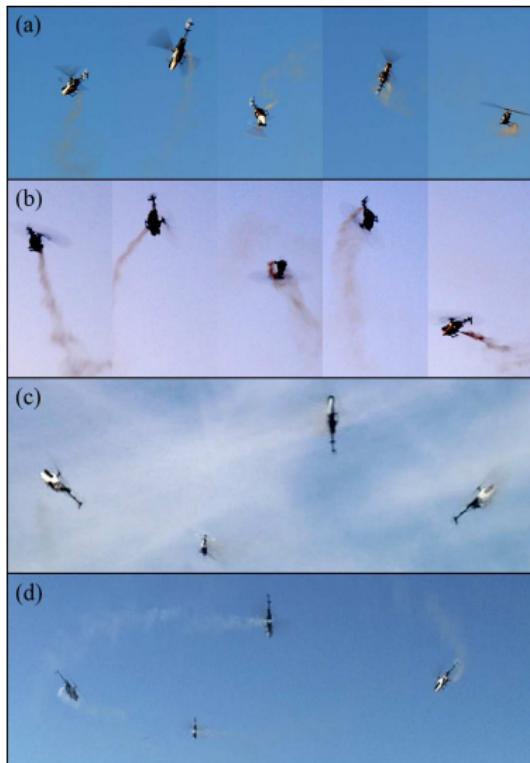
## Definition (Inverse Reinforcement Learning (Russell 1998))

Given:

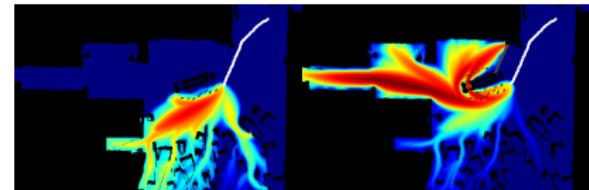
- $\mathcal{M} \setminus \{\mathbf{r}\}$ ,
- demonstrations  $\mathcal{D} = \{\zeta_i\}_{i=1}^N$ , where  $\zeta_i = \{(s_{i,t}, a_{i,t})\}_{t=1}^T$ ,
- features  $\mathbf{X} \in \mathbb{R}^{|\mathcal{S}| \times d}$ ,

find  $\mathbf{r}$ .

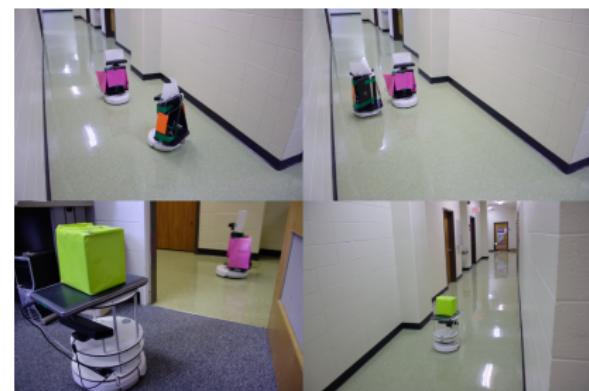
# Other Applications



P. Abbeel, A. Coates, M. Quigley and A. Y. Ng, "An application of reinforcement learning to aerobatic helicopter flight", in *NIPS*, 2006

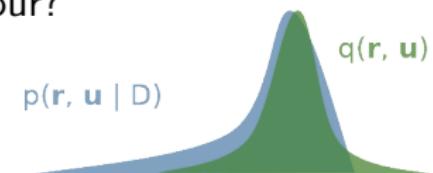


B. D. Ziebart, N. D. Ratliff, G. Gallagher, C. Mertz, K. M. Peterson, J. A. Bagnell, M. Hebert, A. K. Dey and S. S. Srinivasa, "Planning-based prediction for pedestrians", in *IROS*, 2009

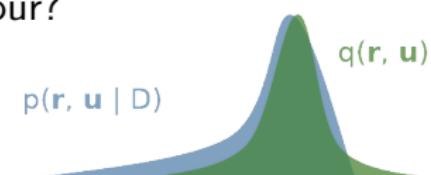


K. D. Bogert and P. Doshi, "Multi-robot inverse reinforcement learning under occlusion with interactions", in *AAMAS*, 2014

- Has the model learned optimal behaviour?
- Can it recognise its own weak spots?
- Solution: variational inference (VI)



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- Can it recognise its own weak spots?
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## Outline for the rest of the talk

- Maximum causal entropy and stochastic policies
- Reward function as a Gaussian process (GP)
- Variational approximation of the posterior distribution
- Theoretical results: how can we compute the gradient?
- Empirical results: does it work?
- Further work: what comes next?

# Maximum Causal Entropy

## Standard MDP

$$V_r(s) := r(s) + \gamma \max_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} T(s, a, s') V_r(s')$$

## Maximum Causal Entropy MDP<sup>1</sup>

$$V_r(s) := \log \sum_{a \in \mathcal{A}} \exp \left( r(s) + \gamma \sum_{s' \in \mathcal{S}} T(s, a, s') V_r(s') \right)$$

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<sup>1</sup>B. D. Ziebart, J. A. Bagnell and A. K. Dey, “Modeling interaction via the principle of maximum causal entropy”, in *ICML*, 2010.

# Reward Function as a Gaussian Process

## Automatic Relevance Determination Kernel

For any two states  $\mathbf{x}_i, \mathbf{x}_j \in \mathbb{R}^d$ ,

$$k_{\lambda}(\mathbf{x}_i, \mathbf{x}_j) = \lambda_0 \exp \left( -\frac{1}{2} (\mathbf{x}_i - \mathbf{x}_j)^T \boldsymbol{\Lambda} (\mathbf{x}_i - \mathbf{x}_j) - \mathbb{1}[i \neq j] \sigma^2 \text{tr}(\boldsymbol{\Lambda}) \right)$$

where  $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_d)$ ,  $\sigma^2 = 10^{-2}/2$ ,

$$\mathbb{1}[b] = \begin{cases} 1 & \text{if } b \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

# Reward Function as a Gaussian Process

## Inducing Points

- $m \ll |\mathcal{S}|$  states,
- their features  $\mathbf{X}_u$
- and rewards  $\mathbf{u}$ .

## The GP Then Gives...

- Kernel/covariance matrices:  $\mathbf{K}_{u,u}$ ,  $\mathbf{K}_{r,u}$ ,  $\mathbf{K}_{r,r}$
- Prior probabilities:
  - $p(\mathbf{u}) = \mathcal{N}(\mathbf{u}; \mathbf{0}, \mathbf{K}_{u,u})$
  - $p(\mathbf{r} | \mathbf{u}) = \mathcal{N}(\mathbf{r}; \mathbf{K}_{r,u}^T \mathbf{K}_{u,u}^{-1} \mathbf{u}, \mathbf{K}_{r,r} - \mathbf{K}_{r,u}^T \mathbf{K}_{u,u}^{-1} \mathbf{K}_{r,u})$

# Variational Approximation

## Previous Work

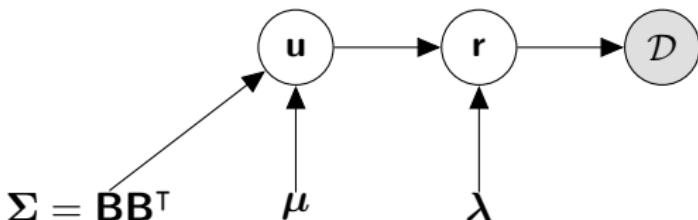
- Levine et al. (2011) assume that  $\mathbf{r} = \mathbf{K}_{\mathbf{r}, \mathbf{u}}^T \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{u}$  and maximise the likelihood
- Jin et al. (2017) add more assumptions and use a deep GP model
- Wulfmeier et al. (2015) use a neural network

$$p(\mathbf{r}, \mathbf{u} | \mathcal{D}) = \frac{p(\mathcal{D} | \mathbf{r}) p(\mathbf{r} | \mathbf{u}) p(\mathbf{u})}{p(\mathcal{D})}$$

can be approximated with  $q(\mathbf{r}, \mathbf{u}) = q(\mathbf{r} | \mathbf{u})q(\mathbf{u})$ , where

- $q(\mathbf{r} | \mathbf{u}) = p(\mathbf{r} | \mathbf{u})$
- $q(\mathbf{u}) = \mathcal{N}(\mathbf{u}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$

# Variational Approximation



Goal: **minimise** the *Kullback-Leibler divergence*:

$$D_{\text{KL}}(q(\mathbf{r}, \mathbf{u}) \parallel p(\mathbf{r}, \mathbf{u} \mid \mathcal{D})) = \mathbb{E}_{q(\mathbf{r}, \mathbf{u})} [\log q(\mathbf{r}, \mathbf{u}) - \log p(\mathbf{r}, \mathbf{u} \mid \mathcal{D})]$$

Equivalently, **maximise** the *evidence lower bound*:

$$\begin{aligned} \mathcal{L} &= \mathbb{E}_{q(\mathbf{r}, \mathbf{u})} [\log p(\mathcal{D}, \mathbf{r}, \mathbf{u}) - \log q(\mathbf{r}, \mathbf{u})] \\ &= \mathbf{t}^T \mathbf{K}_{\mathbf{r}, \mathbf{u}}^T \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \boldsymbol{\mu} - \textcolor{red}{\mathbb{E}[\nu]} - D_{\text{KL}}(q(\mathbf{u}) \parallel p(\mathbf{u})) \end{aligned}$$

where

$$\nu = \sum_{i=1}^N \sum_{t=1}^T V_{\mathbf{r}}(s_{i,t}) - \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s_{i,t}, a_{i,t}, s') V_{\mathbf{r}}(s').$$

# Mathematical Preliminaries

Vector norms

$$\|\mathbf{x}\|_1 = \sum_i |x_i|$$

$$\|\mathbf{x}\|_\infty = \max_i |x_i|$$

Matrix norms

$$\|\mathbf{A}\|_p = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{Ax}\|_p}{\|\mathbf{x}\|_p}$$

$$\|\mathbf{A}\|_\infty = \max_i \sum_j |A_{i,j}|$$

## Lemma (Perturbation Lemma)

Let  $\|\cdot\|$  be any matrix norm, and let  $\mathbf{A}$  and  $\mathbf{E}$  be matrices such that  $\mathbf{A}$  is invertible and  $\|\mathbf{A}^{-1}\| \|\mathbf{E}\| < 1$ , then  $\mathbf{A} + \mathbf{E}$  is invertible, and

$$\|(\mathbf{A} + \mathbf{E})^{-1}\| \leq \frac{\|\mathbf{A}^{-1}\|}{1 - \|\mathbf{A}^{-1}\| \|\mathbf{E}\|}.$$

# Theoretical Results

Seeing  $V$  as  $V : \mathcal{S} \rightarrow \mathbb{R}^{|\mathcal{S}|} \rightarrow \mathbb{R}$ ...

## Proposition

*MDP value functions  $V(s) : \mathbb{R}^{|\mathcal{S}|} \rightarrow \mathbb{R}$  (for  $s \in \mathcal{S}$ ) are Lebesgue measurable.*

## Proposition

*If the initial values of the MDP value function satisfy the following bound, then the bound remains satisfied throughout value iteration:*

$$|V_{\mathbf{r}}(s)| \leq \frac{\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|}{1 - \gamma}.$$

# Theoretical Results

## Theorem

*Whenever the derivative exists,*

$$\frac{\partial}{\partial t} \iint V_r(s) q(\mathbf{r} \mid \mathbf{u}) q(\mathbf{u}) d\mathbf{r} d\mathbf{u} = \iint \frac{\partial}{\partial t} [V_r(s) q(\mathbf{r} \mid \mathbf{u}) q(\mathbf{u})] d\mathbf{r} d\mathbf{u},$$

*where  $t$  is any scalar part of  $\mu$ ,  $\Sigma$ , or  $\lambda$ .*

# A Note on Polynomials

## Definition

Let  $\mathbb{R}_d[\mathbf{x}]$  denote the vector space of polynomials with degree at most  $d$ , where variables are elements of  $\mathbf{x}$ , and coefficients are in  $\mathbb{R}$ .

## Example

$$2x_1^2 + \pi x_2 \in \mathbb{R}_2[\mathbf{x}]$$

$$x_1 x_2 \in \mathbb{R}_2[\mathbf{x}]$$

$$-3x_1 + 1 \in \mathbb{R}_2[\mathbf{x}]$$

$$0 \in \mathbb{R}_2[\mathbf{x}]$$

# Helpful Lemmas

## Lemma

$$\int \|\mathbf{r}\|_\infty q(\mathbf{r} \mid \mathbf{u}) d\mathbf{r} \leq a + \|\mathbf{K}_{\mathbf{r}, \mathbf{u}}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{u}\|_1,$$

where  $a$  is a constant independent of  $\mathbf{u}$ .

## Lemma

Let  $c : \mathbb{R}^{|S|} \times \mathbb{R}^m \rightarrow (a, b) \subset \mathbb{R}$  be an arbitrary bounded function.  
Then, for  $i = 0, \dots, d$ ,

$$\frac{\partial q(\mathbf{r} \mid \mathbf{u})}{\partial \lambda_i} \Big|_{\lambda_i=c(\mathbf{r}, \mathbf{u})}$$

has upper and lower bounds of the form  $q(\mathbf{r} \mid \mathbf{u})d(\mathbf{u})$ , where  $d(\mathbf{u}) \in \mathbb{R}_2[\mathbf{u}]$ .

# Helpful Lemmas

## Lemma

Let  $c : \mathbb{R}^{|S|} \times \mathbb{R}^m \rightarrow (a, b) \subset \mathbb{R}$  be an arbitrary bounded function.  
Then, for  $i = 1, \dots, m$ , every element of

$$\frac{\partial q(\mathbf{u})}{\partial \mu} \Big|_{\mu_i=c(\mathbf{r}, \mathbf{u})}$$

has upper and lower bounds of the form  $q(\mathbf{u})d(\mathbf{u})$ , where  
 $d(\mathbf{u}) \in \mathbb{R}_1[\mathbf{u}]$ .

# Helpful Lemmas

## Lemma

Let  $i, j = 1, \dots, m$ , and let  $\epsilon > 0$  be arbitrary. Furthermore, let

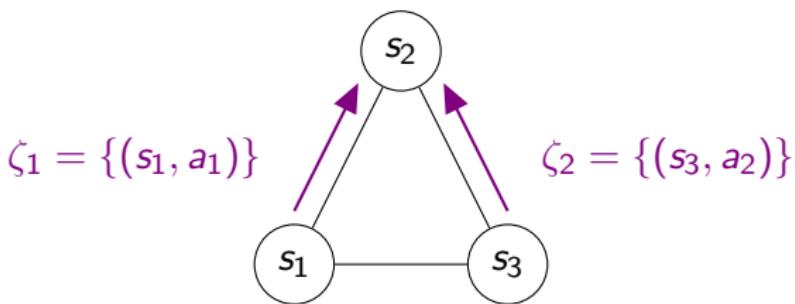
$$c : \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^m \rightarrow (\Sigma_{i,j} - \epsilon, \Sigma_{i,j} + \epsilon) \subset \mathbb{R}$$

be a function with a codomain arbitrarily close to  $\Sigma_{i,j}$ . Then every element of

$$\left. \frac{\partial q(\mathbf{u})}{\partial \Sigma} \right|_{\Sigma_{i,j}=c(\mathbf{r}, \mathbf{u})}$$

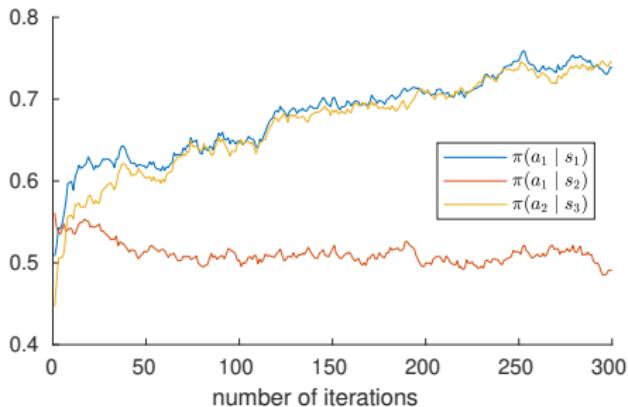
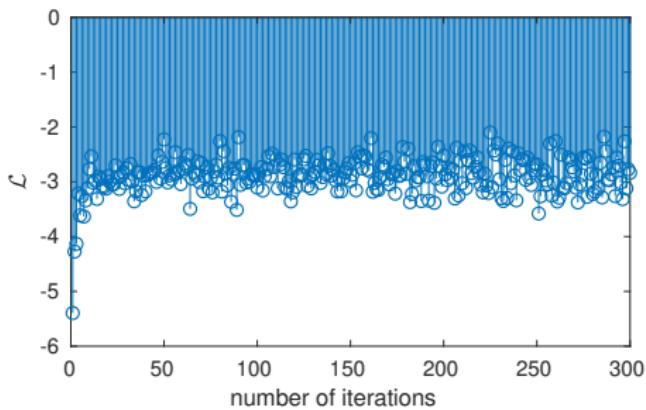
has upper and lower bounds of the form  $q(\mathbf{u})d(\mathbf{u})$ , where  $d(\mathbf{u}) \in \mathbb{R}_2[\mathbf{u}]$ .

# Experimental Scenario

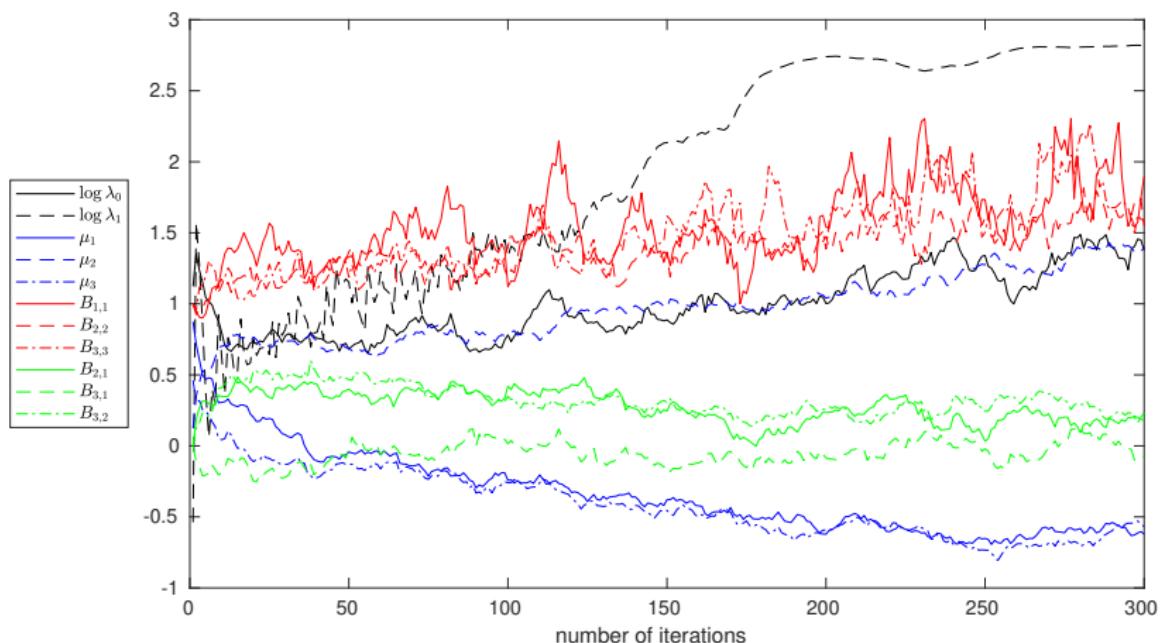


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# Convergence

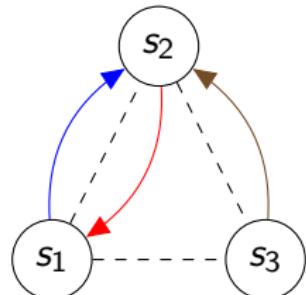
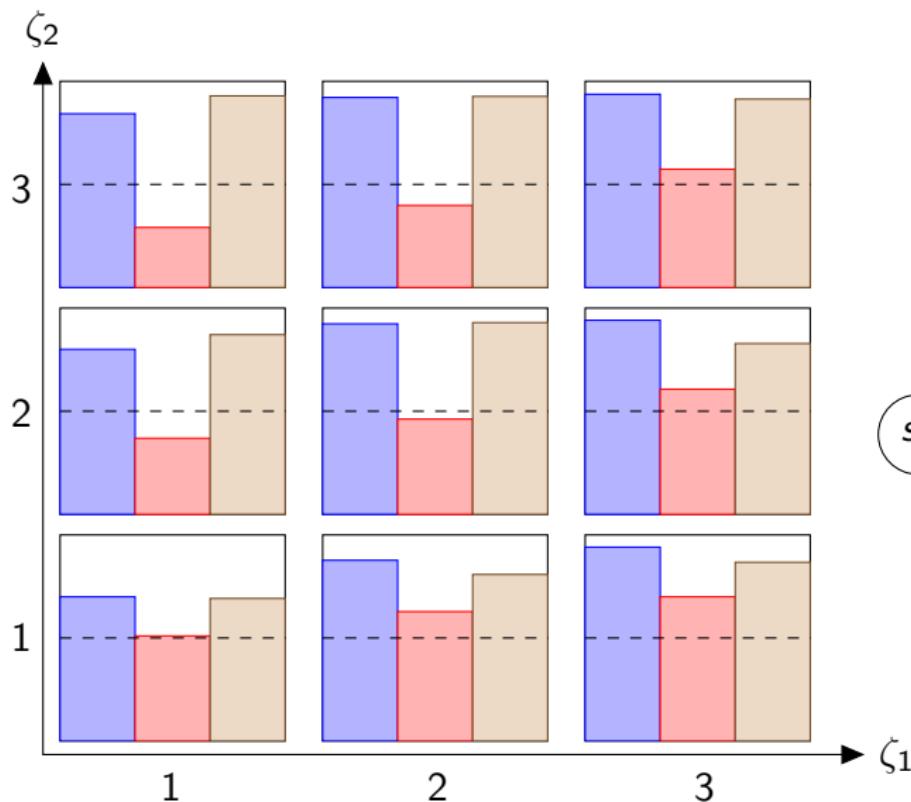


# Parameter Convergence

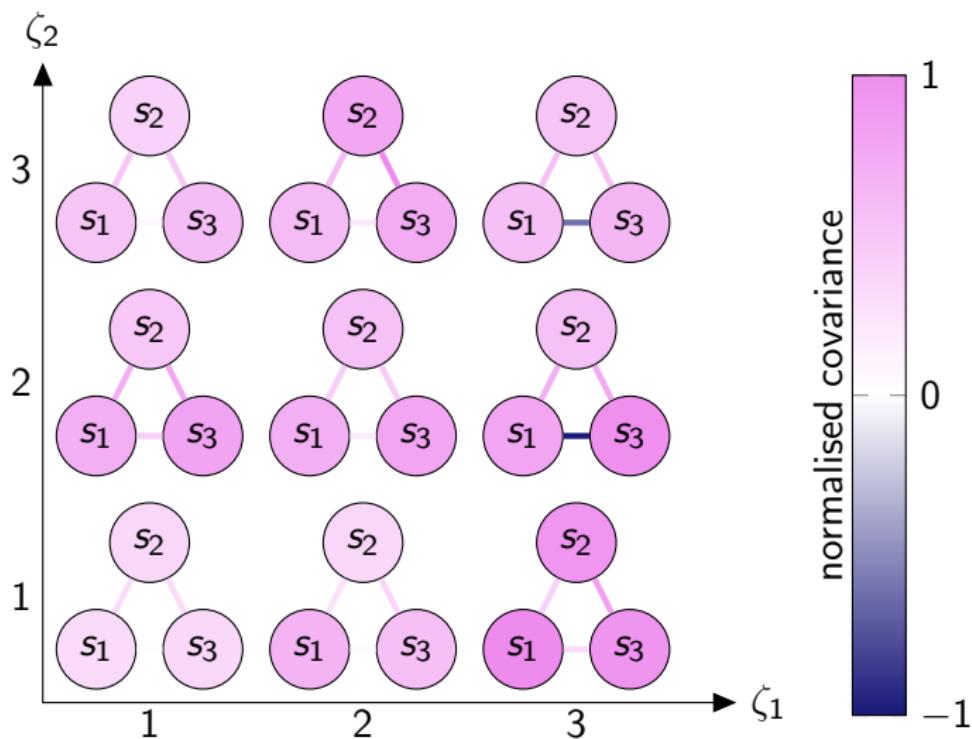


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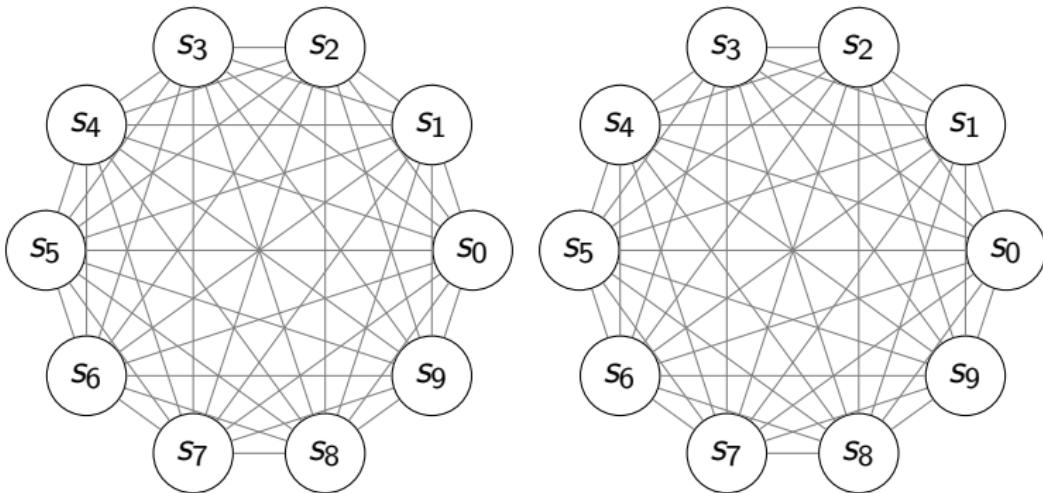
# Policy



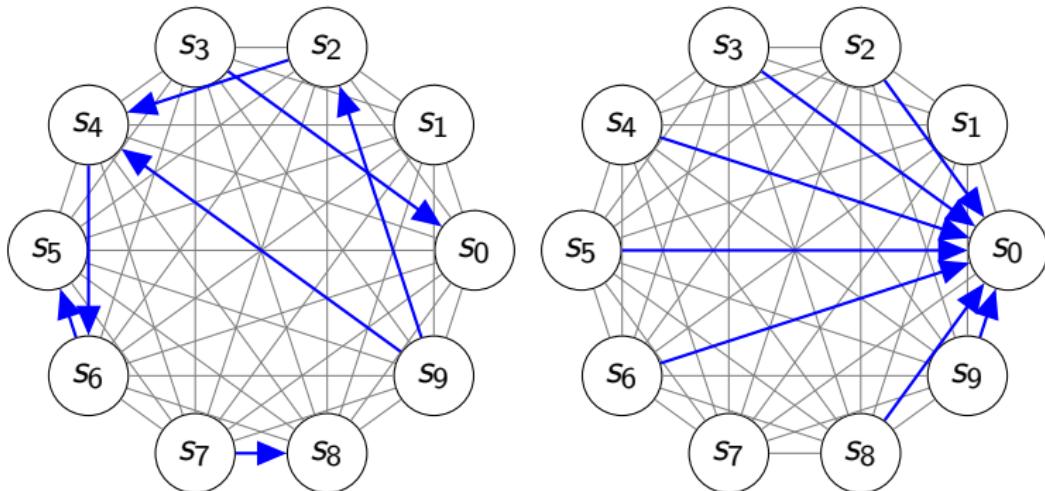
# Covariance: Attempt 1



# Covariance: Attempt 2 (with Cliques!)



# Covariance: Attempt 2 (with Cliques!)



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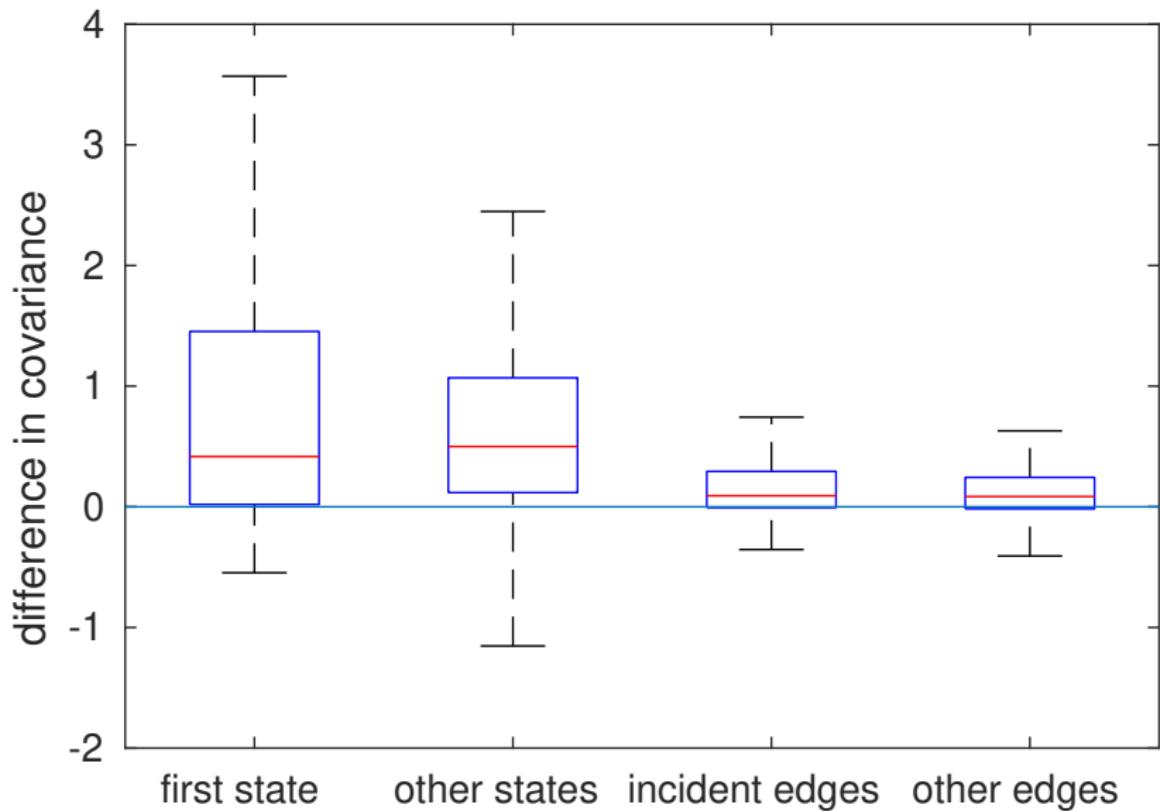
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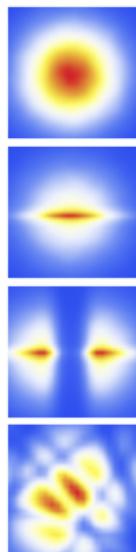
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## Covariance: Attempt 2 (with Cliques!)



# Further Work

- More flexible models using...
  - normalizing flows<sup>1</sup>
  - spectral kernels<sup>2</sup>
- Faster GP inference and MDP solving
- IRL in the context of reinforcement learning
- Interplay between rewards and stochastic/deterministic policies



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<sup>1</sup>D. J. Rezende and S. Mohamed, “Variational inference with normalizing flows”, in *ICML*, 2015.

<sup>2</sup>A. Wilson and R. Adams, “Gaussian process kernels for pattern discovery and extrapolation”, in *ICML*, 2013.