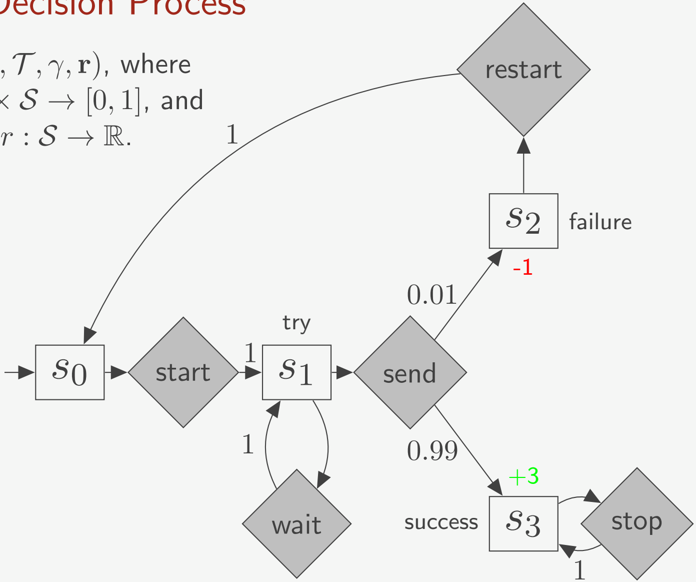
Variational Inference for Inverse Reinforcement Learning with Gaussian Processes

Paulius Dilkas

24th March 2019

Markov Decision Process

$\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{T}, \gamma, \mathbf{r})$, where
$\mathcal{T} : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to [0, 1]$, and
$\mathbf{r} \in \mathbb{R}^{|\mathcal{S}|}$ or $r : \mathcal{S} \to \mathbb{R}$.



Source: P. Dilkas. Nondeterministic Bigraphs and Their Use in Modelling Movement. 2018

Inverse Reinforcement Learning (COLT 1998)

Learning agents for uncertain environments (extended abstract)

Stuart Russell*
Computer Science Division
University of California
Berkeley, CA 94720
russell@cs.berkeley.edu

Inverse Reinforcement Learning Problem

Given:

- $\mathcal{M} \setminus \{\mathbf{r}\}$,
- $\mathcal{D} = \{\zeta_i\}_{i=1}^N$, where $\zeta_i = \{(s_{i,1}, a_{i,1}), \ldots, (s_{i,T}, a_{i,T})\}$,
- features $\mathbf{X} \in \mathbb{R}^{|\mathcal{S}| \times d}$,

find $\mathbf{r}$.

Value Iteration

Standard MDP

$$V_{\mathbf{r}}(s) := r(s) + \gamma \max_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} \mathcal{T}(s, a, s') V_{\mathbf{r}}(s')$$

Linearly Solvable / Maximum Causal Entropy MDP

$$V_{\mathbf{r}}(s) := \log \sum_{a \in \mathcal{A}} \exp \left(r(s) + \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s, a, s') V_{\mathbf{r}}(s') \right)$$

Under the Maximum Entropy Model...

$$p(\mathcal{D} \mid \mathbf{r}) = \prod_{i=1}^{N} \prod_{t=1}^{T} p(a_{i,t} \mid s_{i,t}) = \exp\left(\sum_{i=1}^{N} \sum_{t=1}^{T} Q_{\mathbf{r}}(s_{i,t}, a_{i,t}) - V_{\mathbf{r}}(s_{i,t}) \right)$$

where

$$Q_{\mathbf{r}}(s, a) = r(s) + \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s, a, s') V_{\mathbf{r}}(s')$$

Reward Function as a Gaussian Process

Automatic Relevance Determination Kernel

For any two states $\mathbf{x}_i, \mathbf{x}_j \in \mathbb{R}^d$,

$$k_{\boldsymbol{\lambda}}(\mathbf{x}_i, \mathbf{x}_j) = \lambda_0 \exp\left(-\frac{1}{2}(\mathbf{x}_i - \mathbf{x}_j)^\intercal \mathbf{\Lambda}(\mathbf{x}_i - \mathbf{x}_j) - \mathbb{1}[i \neq j]\sigma^2 \operatorname{tr}(\mathbf{\Lambda})\right)$$

where $\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$, $\sigma^2 = 10^{-2}/2$,

$$\mathbb{1}[b] = \begin{cases} 1 & \text{if } b \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

Reward Function as a Gaussian Process

Inducing Points

- $m \ll |\mathcal{S}|$ states,
- their features $\mathbf{X}_\mathbf{u}$
- and rewards $\mathbf{u}$.

The GP Then Gives Gives...

- Kernel/covariance matrices: $\mathbf{K}_{\mathbf{u},\mathbf{u}}$, $\mathbf{K}_{\mathbf{r},\mathbf{u}}$, $\mathbf{K}_{\mathbf{r},\mathbf{r}}$
- Prior probabilities:
  - $p(\mathbf{u}) = \mathcal{N}(\mathbf{u}; \mathbf{0}, \mathbf{K}_{\mathbf{u},\mathbf{u}})$
  - $p(\mathbf{r} \mid \mathbf{u}) = \mathcal{N}(\mathbf{r}; \mathbf{K}_{\mathbf{r},\mathbf{u}}^\intercal \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1} \mathbf{u}, \mathbf{K}_{\mathbf{r},\mathbf{r}} - \mathbf{K}_{\mathbf{r},\mathbf{u}}^\intercal \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1} \mathbf{K}_{\mathbf{r},\mathbf{u}})$

Variational Inference

Previous Work

- Levine et al. (2011) assume that $\mathbf{r} = \mathbf{K}_{\mathbf{r},\mathbf{u}}^\intercal \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1} \mathbf{u}$ and maximise the likelihood
- Jin et al. (2017) add more assumptions and use a deep GP model
- Wulfmeier et al. (2015) use a neural network

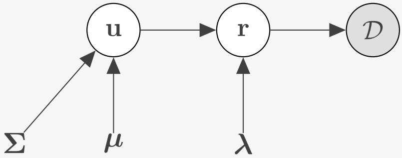
What about posterior probabilities?

$$p(\mathbf{r}, \mathbf{u} \mid \mathcal{D}) = \frac{p(\mathcal{D} \mid \mathbf{r}) p(\mathbf{r} \mid \mathbf{u}) p(\mathbf{u})}{p(\mathcal{D})}$$

Solution: approximate $p(\mathbf{r}, \mathbf{u} \mid \mathcal{D})$ with $q(\mathbf{r}, \mathbf{u}) = q(\mathbf{r} \mid \mathbf{u}) q(\mathbf{u})$, where

- $q(\mathbf{r} \mid \mathbf{u}) = p(\mathbf{r} \mid \mathbf{u})$
- $q(\mathbf{u}) = \mathcal{N}(\mathbf{u}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$

Variational Inference



Goal: minimise the *Kullback-Leibler divergence*:

$$\begin{aligned} D_{\mathrm{KL}}(q(\mathbf{r},\mathbf{u}) \parallel p(\mathbf{r},\mathbf{u} \mid \mathcal{D})) &= \mathbb{E}_{(\mathbf{r},\mathbf{u})\sim q(\mathbf{r},\mathbf{u})}[\log q(\mathbf{r},\mathbf{u}) - \log p(\mathbf{r},\mathbf{u} \mid \mathcal{D})] \\ &= \mathbb{E}_{(\mathbf{r},\mathbf{u})\sim q(\mathbf{r},\mathbf{u})}[\log q(\mathbf{r},\mathbf{u}) - \log p(\mathcal{D},\mathbf{r},\mathbf{u})] \\ &+ \mathbb{E}_{(\mathbf{r},\mathbf{u})\sim q(\mathbf{r},\mathbf{u})}[\log p(\mathcal{D})] \end{aligned}$$

Equivalently, maximise the *evidence lower bound*:

$$\mathcal{L} = \mathbb{E}_{(\mathbf{r},\mathbf{u})\sim q(\mathbf{r},\mathbf{u})}[\log p(\mathcal{D},\mathbf{r},\mathbf{u}) - \log q(\mathbf{r},\mathbf{u})]$$

Mathematical Preliminaries

Vector norms	Matrix norms
$\|\mathbf{x}\|_1 = \sum_i \|x_i\|$	$\|\mathbf{A}\|_p = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$
$\|\mathbf{x}\|_\infty = \max_i \|x_i\|$	$\|\mathbf{A}\|_\infty = \max_i \sum_j \|A_{i,j}\|$

Lemma (Perturbation Lemma)

Let $\|\cdot\|$ be any matrix norm, and let $\mathbf{A}$ and $\mathbf{E}$ be matrices such that $\mathbf{A}$ is invertible and $\|\mathbf{A}^{-1}\|\|\mathbf{E}\| < 1$, then $\mathbf{A}+\mathbf{E}$ is invertible, and

$$\|(\mathbf{A}+\mathbf{E})^{-1}\| \leq \frac{\|\mathbf{A}^{-1}\|}{1-\|\mathbf{A}^{-1}\|\|\mathbf{E}\|}.$$

Theoretical Results

Seeing V as $V : \mathcal{S} \to \mathbb{R}^{|\mathcal{S}|} \to \mathbb{R}$...

Proposition

MDP value functions $V(s) : \mathbb{R}^{|\mathcal{S}|} \to \mathbb{R}$ (for $s \in \mathcal{S}$) are Lebesgue measurable.

Proposition

If the initial values of the MDP value function satisfy the following bound, then the bound remains satisfied throughout value iteration:

$$|V_{\mathbf{r}}(s)| \leq \frac{\|\mathbf{r}\|_\infty + \log|\mathcal{A}|}{1-\gamma}.$$

Theoretical Results

Theorem

Whenever the derivative exists,

$$\frac{\partial}{\partial t} \iint V_{\mathbf{r}}(s) q(\mathbf{r} \mid \mathbf{u}) q(\mathbf{u}) \, d\mathbf{r} \, d\mathbf{u} = \iint \frac{\partial}{\partial t} [V_{\mathbf{r}}(s) q(\mathbf{r} \mid \mathbf{u}) q(\mathbf{u})] \, d\mathbf{r} \, d\mathbf{u},$$

where t is any scalar part of $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, or $\boldsymbol{\lambda}$.

A Note on Polynomials

Definition

Let $\mathbb{R}_d[\mathbf{x}]$ denote the vector space of polynomials with degree at most d, where variables are elements of $\mathbf{x}$, and coefficients are in $\mathbb{R}$.

Example

$$\begin{aligned}\mathbb{R}_2[\mathbf{x}] \supset \{&2x_1^2 + \pi x_2,\\ &x_1x_2,\\ &-3x_1 + 1,\\ &0\}\end{aligned}$$

Helpful Lemmas

Lemma

$$\int \|\mathbf{r}\|_\infty q(\mathbf{r} \mid \mathbf{u}) \, d\mathbf{r} \leq a + \|\mathbf{K}_{\mathbf{r},\mathbf{u}}^\intercal \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1} \mathbf{u}\|_1,$$

where a is a constant independent of $\mathbf{u}$.

Lemma

Let $c : \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^m \to (a, b) \subset \mathbb{R}$ be an arbitrary bounded function. Then, for $i = 0, \ldots, d$,

$$\left. \frac{\partial q(\mathbf{r} \mid \mathbf{u})}{\partial \lambda_i} \right|_{\lambda_i = c(\mathbf{r},\mathbf{u})}$$

has upper and lower bounds of the form $q(\mathbf{r} \mid \mathbf{u}) d(\mathbf{u})$, where $d(\mathbf{u}) \in \mathbb{R}_2[\mathbf{u}]$.

Helpful Lemmas

Lemma

Let $c : \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^m \to (a, b) \subset \mathbb{R}$ *be an arbitrary bounded function. Then, for* $i = 1, \ldots, m$*, every element of*

$$\left.\frac{\partial q(\mathbf{u})}{\partial \boldsymbol{\mu}}\right|_{\mu_i = c(\mathbf{r}, \mathbf{u})}$$

has upper and lower bounds of the form $q(\mathbf{u})d(\mathbf{u})$*, where* $d(\mathbf{u}) \in \mathbb{R}_1[\mathbf{u}]$.

Helpful Lemmas

Lemma

Let $i, j = 1, \ldots, m$, *and let* $\epsilon > 0$ *be arbitrary. Furthermore, let*

$$c : \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^m \to (\Sigma_{i,j} - \epsilon, \Sigma_{i,j} + \epsilon) \subset \mathbb{R}$$

be a function with a codomain arbitrarily close to $\Sigma_{i,j}$. *Then every element of*

$$\left.\frac{\partial q(\mathbf{u})}{\partial \mathbf{\Sigma}}\right|_{\Sigma_{i,j} = c(\mathbf{r}, \mathbf{u})}$$

has upper and lower bounds of the form $q(\mathbf{u}) d(\mathbf{u})$, *where* $d(\mathbf{u}) \in \mathbb{R}_2[\mathbf{u}]$.