

Variational Inference for Inverse Reinforcement Learning with Gaussian Processes: Supplementary Material

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1 Preliminaries

For any matrix \mathbf{A} , we will use either $A_{i,j}$ or $[\mathbf{A}]_{i,j}$ to denote the element of \mathbf{A} in row i and column j .

In this paper, all references to measurability are with respect to the Lebesgue measure. Similarly, whenever we consider the existence of an integral, we use the Lebesgue definition of integration.

Lemma 1.1 (Derivatives of probability distributions).

1. $\frac{\partial q(\mathbf{u})}{\partial \mathbf{m}} = q(\mathbf{u}) \frac{1}{2} (\mathbf{S}^{-1} + \mathbf{S}^{-\top})(\mathbf{u} - \mathbf{m})$.
2. $\frac{\partial q(\mathbf{u})}{\partial \mathbf{S}} = -\frac{1}{2} \mathbf{S}^{-\top} q(\mathbf{u}) + \frac{1}{2} q(\mathbf{u}) \mathbf{S}^{-\top} (\mathbf{u} - \mathbf{m})(\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-\top}$.
3. $\frac{\partial q(\mathbf{r})}{\partial \lambda_0} = q(\mathbf{r}) \frac{1}{2} \text{tr} \left((\mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1} \mathbf{u} \mathbf{u}^\top \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-\top} - \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1})_{\lambda_0} \mathbf{K}_{\mathbf{u},\mathbf{u}} \right)$.
4. For $i = 1, \dots, d$,

$$\frac{\partial q(\mathbf{r})}{\partial \lambda_i} = q(\mathbf{r}) \frac{1}{2} \text{tr} \left((\mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1} \mathbf{u} \mathbf{u}^\top \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-\top} - \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1}) \mathbf{L}_{\mathbf{u},\mathbf{u}} \right),$$

where

$$[\mathbf{L}_{\mathbf{u},\mathbf{u}}]_{j,k} = k \lambda (\mathbf{x}_{\mathbf{u},j}, \mathbf{x}_{\mathbf{u},k}) \left(-\frac{1}{2} (x_{\mathbf{u},j,i} - x_{\mathbf{u},k,i})^2 - \mathbb{1}[j \neq k] \sigma^2 \right).$$

Proof.

1.

$$\begin{aligned} \frac{\partial q(\mathbf{u})}{\partial \mathbf{m}} &= q(\mathbf{u}) \frac{\partial}{\partial \mathbf{m}} \left[-\frac{1}{2} (\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-1} (\mathbf{u} - \mathbf{m}) \right] \\ &= q(\mathbf{u}) \left(-\frac{1}{2} \right) (\mathbf{S}^{-1} + \mathbf{S}^{-\top})(\mathbf{u} - \mathbf{m}) \frac{\partial}{\partial \mathbf{m}} [\mathbf{u} - \mathbf{m}] \\ &= q(\mathbf{u}) \frac{1}{2} (\mathbf{S}^{-1} + \mathbf{S}^{-\top})(\mathbf{u} - \mathbf{m}). \end{aligned}$$

2.

$$\begin{aligned}
\frac{\partial q(\mathbf{u})}{\partial \mathbf{S}} &= \frac{\partial}{\partial \mathbf{S}} \left[\frac{1}{(2\pi)^{m/2} |\mathbf{S}|^{1/2}} \exp \left(-\frac{1}{2} (\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-1} (\mathbf{u} - \mathbf{m}) \right) \right] \\
&= \frac{\partial}{\partial \mathbf{S}} \left[\frac{1}{(2\pi)^{m/2} |\mathbf{S}|^{1/2}} \right] \exp \left(-\frac{1}{2} (\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-1} (\mathbf{u} - \mathbf{m}) \right) \\
&\quad + \frac{1}{(2\pi)^{m/2} |\mathbf{S}|^{1/2}} \frac{\partial}{\partial \mathbf{S}} \left[\exp \left(-\frac{1}{2} (\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-1} (\mathbf{u} - \mathbf{m}) \right) \right] \\
&= \frac{1}{(2\pi)^{m/2}} \frac{\partial}{\partial \mathbf{S}} \left[\frac{1}{|\mathbf{S}|^{1/2}} \right] \exp \left(-\frac{1}{2} (\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-1} (\mathbf{u} - \mathbf{m}) \right) \\
&\quad - \frac{1}{2} q(\mathbf{u}) \frac{\partial}{\partial \mathbf{S}} [(\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-1} (\mathbf{u} - \mathbf{m})].
\end{aligned}$$

The two remaining derivatives can be taken with the help of *The Matrix Cookbook* [4]:

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{S}} \left[\frac{1}{|\mathbf{S}|^{1/2}} \right] &= -\frac{1}{2} |\mathbf{S}|^{-3/2} \frac{\partial |\mathbf{S}|}{\partial \mathbf{S}} = -\frac{1}{2} |\mathbf{S}|^{-3/2} |\mathbf{S}| \mathbf{S}^{-\top} = -\frac{1}{2 |\mathbf{S}|^{1/2}} \mathbf{S}^{-\top}, \\
\frac{\partial}{\partial \mathbf{S}} [(\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-1} (\mathbf{u} - \mathbf{m})] &= -\mathbf{S}^{-\top} (\mathbf{u} - \mathbf{m}) (\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-\top}.
\end{aligned}$$

Plugging them back in gives

$$\frac{\partial q(\mathbf{u})}{\partial \mathbf{S}} = -\frac{1}{2} \mathbf{S}^{-\top} q(\mathbf{u}) + \frac{1}{2} q(\mathbf{u}) \mathbf{S}^{-\top} (\mathbf{u} - \mathbf{m}) (\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-\top}.$$

3.

$$\frac{\partial q(\mathbf{r})}{\partial \lambda_0} = q(\mathbf{r}) \frac{\partial}{\partial \lambda_0} \left[-\frac{1}{2} \mathbf{u}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{u} - \frac{1}{2} \log |\mathbf{K}_{\mathbf{u}, \mathbf{u}}| \right] = q(\mathbf{r}) \frac{1}{2} \text{tr} \left((\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{u} \mathbf{u}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} - \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}) \frac{\partial \mathbf{K}_{\mathbf{u}, \mathbf{u}}}{\partial \lambda_0} \right)$$

by Rasmussen and Williams [5], where

$$\frac{\partial \mathbf{K}_{\mathbf{u}, \mathbf{u}}}{\partial \lambda_0} = \frac{1}{\lambda_0} \mathbf{K}_{\mathbf{u}, \mathbf{u}}.$$

4. The derivation is the same as above, except

$$\frac{\partial \mathbf{K}_{\mathbf{u}, \mathbf{u}}}{\partial \lambda_i} = \mathbf{L}_{\mathbf{u}, \mathbf{u}},$$

where

$$\begin{aligned}
[\mathbf{L}_{\mathbf{u}, \mathbf{u}}]_{j, k} &= \frac{\partial}{\partial \lambda_i} k_\lambda(\mathbf{x}_{\mathbf{u}, j}, \mathbf{x}_{\mathbf{u}, k}) \\
&= k_\lambda(\mathbf{x}_{\mathbf{u}, j}, \mathbf{x}_{\mathbf{u}, k}) \frac{\partial}{\partial \lambda_i} \left[-\frac{1}{2} (\mathbf{x}_{\mathbf{u}, j} - \mathbf{x}_{\mathbf{u}, k})^\top \boldsymbol{\Lambda} (\mathbf{x}_{\mathbf{u}, j} - \mathbf{x}_{\mathbf{u}, k}) - \mathbb{1}[j \neq k] \sigma^2 \text{tr}(\boldsymbol{\Lambda}) \right] \\
&= k_\lambda(\mathbf{x}_{\mathbf{u}, j}, \mathbf{x}_{\mathbf{u}, k}) \frac{\partial}{\partial \lambda_i} \left[-\frac{1}{2} \sum_{l=1}^d \lambda_l (x_{\mathbf{u}, j, l} - x_{\mathbf{u}, k, l})^2 - \mathbb{1}[j \neq k] \sigma^2 \sum_{l=1}^d \lambda_l \right] \\
&= k_\lambda(\mathbf{x}_{\mathbf{u}, j}, \mathbf{x}_{\mathbf{u}, k}) \left(-\frac{1}{2} (x_{\mathbf{u}, j, i} - x_{\mathbf{u}, k, i})^2 - \mathbb{1}[j \neq k] \sigma^2 \right).
\end{aligned}$$

□

1.1 Linear Algebra and Numerical Analysis

Definition 1.2 (Norms). For any finite-dimensional vector $\mathbf{x} = (x_1, \dots, x_n)^\top$, its *maximum norm* is

$$\|\mathbf{x}\|_\infty = \max_i |x_i|$$

whereas its *taxicab* (or *Manhattan*) *norm* is

$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|.$$

Let \mathbf{A} be a matrix. For any vector norm $\|\cdot\|_p$, we can also define its *induced norm* for matrices as

$$\|\mathbf{A}\|_p = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}.$$

In particular, for $p = \infty$, we have

$$\|\mathbf{A}\|_\infty = \max_i \sum_j |A_{i,j}|.$$

Lemma 1.3 (Perturbation Lemma [3]). *Let $\|\cdot\|$ be any matrix norm, and let \mathbf{A} and \mathbf{E} be matrices such that \mathbf{A} is invertible and $\|\mathbf{A}^{-1}\|\|\mathbf{E}\| < 1$, then $\mathbf{A} + \mathbf{E}$ is invertible, and*

$$\|(\mathbf{A} + \mathbf{E})^{-1}\| \leq \frac{\|\mathbf{A}^{-1}\|}{1 - \|\mathbf{A}^{-1}\|\|\mathbf{E}\|}.$$

2 Proofs

We primarily think of rewards as a vector $\mathbf{r} \in \mathbb{R}^{|\mathcal{S}|}$, but sometimes we use a function notation $r(s)$ to denote the reward of a particular state $s \in \mathcal{S}$. The functional notation is purely a notational convenience.

MDP values are characterised by both a state and a reward function/vector. In order to prove the next theorem, we think of the value function as $V : \mathcal{S} \rightarrow \mathbb{R}^{|\mathcal{S}|} \rightarrow \mathbb{R}$, i.e., V takes a state $s \in \mathcal{S}$ and returns a function $V(s) : \mathbb{R}^{|\mathcal{S}|} \rightarrow \mathbb{R}$ that takes a reward vector $\mathbf{r} \in \mathbb{R}^{|\mathcal{S}|}$ and returns a value of the state s , $V_{\mathbf{r}}(s) \in \mathbb{R}$. The function $V(s)$ computes the values of all states and returns the value of state s .

For any vector \mathbf{x} , we write $\mathbb{R}_d[\mathbf{x}]$ to denote a vector space of polynomials with degree at most d , where variables are elements of \mathbf{x} , and coefficients are in \mathbb{R} .

Proposition 2.1. *MDP value functions $V(s) : \mathbb{R}^{|\mathcal{S}|} \rightarrow \mathbb{R}$ (for $s \in \mathcal{S}$) are Lebesgue measurable.*

Proof sketch. For any reward vector $\mathbf{r} \in \mathbb{R}^{|\mathcal{S}|}$, the collection of converged value functions $\{V_{\mathbf{r}}(s) \mid s \in \mathcal{S}\}$ satisfy

$$V_{\mathbf{r}}(s) = \log \sum_{a \in \mathcal{A}} \exp \left(r(s) + \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s, a, s') V_{\mathbf{r}}(s') \right)$$

for all $s \in \mathcal{S}$. Let $s_0 \in \mathcal{S}$ be an arbitrary state. In order to prove that $V(s_0)$ is measurable, it is enough to show that for any $\alpha \in \mathbb{R}$, the set

$$\left\{ \mathbf{r} \in \mathbb{R}^{|\mathcal{S}|} \mid \begin{aligned} &V_{\mathbf{r}}(s_0) \in (-\infty, \alpha); \\ &V_{\mathbf{r}}(s) \in \mathbb{R} \text{ for all } s \in \mathcal{S} \setminus \{s_0\}; \\ &V_{\mathbf{r}}(s) = \log \sum_{a \in \mathcal{A}} \exp \left(r(s) + \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s, a, s') V_{\mathbf{r}}(s') \right) \text{ for all } s \in \mathcal{S} \end{aligned} \right\}$$

is measurable. Since this set can be constructed in Zermelo-Fraenkel set theory *without* the axiom of choice, it is measurable [2], which proves that $V(s)$ is a measurable function for any $s \in \mathcal{S}$. \square

Proposition 2.2. *If the initial values of the MDP value function satisfy the following bound, then the bound remains satisfied throughout value iteration:*

$$|V_{\mathbf{r}}(s)| \leq \frac{\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|}{1 - \gamma}. \quad (1)$$

Proof. We begin by considering (1) without taking the absolute value of $V_{\mathbf{r}}(s)$, i.e.,

$$V_{\mathbf{r}}(s) \leq \frac{\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|}{1 - \gamma}, \quad (2)$$

and assuming that the initial values of $\{V_{\mathbf{r}}(s) \mid s \in \mathcal{S}\}$ already satisfy (2). For each $s \in \mathcal{S}$, the value of $V_{\mathbf{r}}(s)$ is updated via this rule:

$$V_{\mathbf{r}}(s) := \log \sum_{a \in \mathcal{A}} \exp \left(r(s) + \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s, a, s') V_{\mathbf{r}}(s') \right).$$

Note that both \log and \exp are increasing functions, $\gamma > 0$, and the \mathcal{T} function gives a probability (a non-negative number). Thus

$$\begin{aligned} V_{\mathbf{r}}(s) &\leq \log \sum_{a \in \mathcal{A}} \exp \left(r(s) + \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s, a, s') \frac{\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|}{1 - \gamma} \right) \\ &= \log \sum_{a \in \mathcal{A}} \exp \left(r(s) + \frac{\gamma(\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|)}{1 - \gamma} \sum_{s' \in \mathcal{S}} \mathcal{T}(s, a, s') \right) \\ &= \log \sum_{a \in \mathcal{A}} \exp \left(r(s) + \frac{\gamma(\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|)}{1 - \gamma} \right) \end{aligned}$$

by the definition of \mathcal{T} . Then

$$\begin{aligned} V_{\mathbf{r}}(s) &\leq \log \left(|\mathcal{A}| \exp \left(r(s) + \frac{\gamma(\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|)}{1 - \gamma} \right) \right) \\ &= \log \left(\exp \left(\log |\mathcal{A}| + r(s) + \frac{\gamma(\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|)}{1 - \gamma} \right) \right) \\ &= \log |\mathcal{A}| + r(s) + \frac{\gamma(\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|)}{1 - \gamma} \\ &= \frac{\gamma(\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|) + (1 - \gamma)(\log |\mathcal{A}| + r(s))}{1 - \gamma} \\ &\leq \frac{\gamma(\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|) + (1 - \gamma)(\log |\mathcal{A}| + \|\mathbf{r}\|_{\infty})}{1 - \gamma} \\ &= \frac{\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|}{1 - \gamma} \end{aligned}$$

by the definition of $\|\mathbf{r}\|_{\infty}$.

The proof for

$$V_{\mathbf{r}}(s) \geq \frac{\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|}{\gamma - 1} \quad (3)$$

follows the same argument until we get to

$$\begin{aligned} V_{\mathbf{r}}(s) &\geq \frac{\gamma(\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|) + (\gamma - 1)(\log |\mathcal{A}| + r(s))}{\gamma - 1} \\ &\geq \frac{\gamma(\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|) + (\gamma - 1)(-\log |\mathcal{A}| - \|\mathbf{r}\|_{\infty})}{\gamma - 1} \\ &= \frac{\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|}{\gamma - 1}, \end{aligned}$$

where we use the fact that $r(s) \geq -\|\mathbf{r}\|_\infty - 2 \log |\mathcal{A}|$. Combining (2) and (3) gives (1). \square

Theorem 2.3 (The Lebesgue Dominated Convergence Theorem [6]). *Let (X, \mathcal{M}, μ) be a measure space and $\{f_n\}$ a sequence of measurable functions on X for which $\{f_n\} \rightarrow f$ pointwise a.e. on X and the function f is measurable. Assume there is a non-negative function g that is integrable over X and dominates the sequence $\{f_n\}$ on X in the sense that*

$$|f_n| \leq g \text{ a.e. on } X \text{ for all } n.$$

Then f is integrable over X and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

Lemma 2.4. *Let $c : \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^m \rightarrow (a, b) \subset \mathbb{R}$ be an arbitrary bounded function. Then, for $i = 0, \dots, d$,*

$$\left. \frac{\partial q(\mathbf{r})}{\partial \lambda_i} \right|_{\lambda_i = c(\mathbf{r}, \mathbf{u})}$$

has upper and lower bounds of the form $q(\mathbf{r})d(\mathbf{u})$, where $d(\mathbf{u}) \in \mathbb{R}_2[\mathbf{u}]$.

Proof. Remember that

$$\frac{\partial q(\mathbf{r})}{\partial \lambda_i} = q(\mathbf{r}) \frac{1}{2} \text{tr} \left((\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{u} \mathbf{u}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-\top} - \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}) \frac{\partial \mathbf{K}_{\mathbf{u}, \mathbf{u}}}{\partial \lambda_i} \right)$$

by Lemma 1.1. We begin by producing constant upper and lower bounds for the elements of

$$\left. \frac{\partial \mathbf{K}_{\mathbf{u}, \mathbf{u}}}{\partial \lambda_i} \right|_{\lambda_i = c(\mathbf{r}, \mathbf{u})}.$$

If $i = 0$, then each element of $\frac{\partial \mathbf{K}_{\mathbf{u}, \mathbf{u}}}{\partial \lambda_0}$ is of the form

$$\exp \left(-\frac{1}{2} (\mathbf{x}_j - \mathbf{x}_k)^\top \mathbf{\Lambda} (\mathbf{x}_j - \mathbf{x}_k) - \mathbb{1}[j \neq k] \sigma^2 \text{tr}(\mathbf{\Lambda}) \right),$$

i.e., without λ_0 , so

$$\left. \frac{\partial \mathbf{K}_{\mathbf{u}, \mathbf{u}}}{\partial \lambda_0} \right|_{\lambda_0 = c(\mathbf{r}, \mathbf{u})} = \frac{\partial \mathbf{K}_{\mathbf{u}, \mathbf{u}}}{\partial \lambda_0}$$

is already independent of \mathbf{r} and \mathbf{u} —there is no need for any bounds.

If $i > 0$, then each element of $\frac{\partial \mathbf{K}_{\mathbf{u}, \mathbf{u}}}{\partial \lambda_i}$ is a constant multiple of $k_{\mathbf{\Lambda}}(\mathbf{x}_j, \mathbf{x}_k)$, for some \mathbf{x}_j and \mathbf{x}_k . Since $k_{\mathbf{\Lambda}}(\mathbf{x}_j, \mathbf{x}_k)$ is a decreasing function of λ_i , and $c(\mathbf{r}, \mathbf{u}) > a$,

$$\begin{aligned} k_{\mathbf{\Lambda}}(\mathbf{x}_j, \mathbf{x}_k)|_{\lambda_i = c(\mathbf{r}, \mathbf{u})} &= \lambda_0 \exp \left(-\frac{1}{2} c(\mathbf{r}, \mathbf{u}) (x_{j,i} - x_{k,i})^2 - \mathbb{1}[j \neq k] \sigma^2 c(\mathbf{r}, \mathbf{u}) \right. \\ &\quad \left. - \sum_{n \in \{1, \dots, d\} \setminus \{i\}} \frac{1}{2} \lambda_n (x_{j,n} - x_{k,n})^2 + \mathbb{1}[j \neq k] \sigma^2 \lambda_n \right) \\ &< \lambda_0 \exp \left(-\frac{1}{2} a (x_{j,i} - x_{k,i})^2 - \mathbb{1}[j \neq k] \sigma^2 a \right. \\ &\quad \left. - \sum_{n \in \{1, \dots, d\} \setminus \{i\}} \frac{1}{2} \lambda_n (x_{j,n} - x_{k,n})^2 + \mathbb{1}[j \neq k] \sigma^2 \lambda_n \right), \end{aligned}$$

which gives an upper bound on each element of

$$\left. \frac{\partial \mathbf{K}_{\mathbf{u}, \mathbf{u}}}{\partial \lambda_i} \right|_{\lambda_i = c(\mathbf{r}, \mathbf{u})}.$$

A similar line of reasoning establishes lower bounds as well.

Combining the bounds with the observation that every element of $\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{u} \mathbf{u}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-\top}$ is in $\mathbb{R}_2[\mathbf{u}]$ gives the required result. \square

Remark. In order to find $\frac{\partial q(\mathbf{u})}{\partial t}$, where t is the i th element of the vector \mathbf{m} , we can find $\frac{\partial q(\mathbf{u})}{\partial \mathbf{m}}$ and simply take the i th element. A similar line of reasoning applies to matrices as well. Thus, we only need to consider derivatives with respect to \mathbf{m} and \mathbf{S} .

Lemma 2.5. *Let $c : \mathbb{R}^{|S|} \times \mathbb{R}^m \rightarrow (a, b) \subset \mathbb{R}$ be an arbitrary bounded function. Then, for $i = 1, \dots, m$, every element of*

$$\left. \frac{\partial q(\mathbf{u})}{\partial \mathbf{m}} \right|_{m_i = c(\mathbf{r}, \mathbf{u})}$$

has upper and lower bounds of the form $q(\mathbf{u})d(\mathbf{u})$, where $d(\mathbf{u}) \in \mathbb{R}_1[\mathbf{u}]$.

Proof. Using Lemma 1.1,

$$\left. \frac{\partial q(\mathbf{u})}{\partial \mathbf{m}} \right|_{m_i = c(\mathbf{r}, \mathbf{u})} = q(\mathbf{u}) \frac{1}{2} (\mathbf{S}^{-1} + \mathbf{S}^{-\top})(\mathbf{u} - \mathbf{c}(\mathbf{r}, \mathbf{u})),$$

where $\mathbf{c}(\mathbf{r}, \mathbf{u}) = (m_1, \dots, m_{i-1}, c(\mathbf{r}, \mathbf{u}), m_{i+1}, \dots, m_m)^\top$. Since $c(\mathbf{r}, \mathbf{u})$ is bounded and $\mathbf{S}^{-1} + \mathbf{S}^{-\top}$ is a constant matrix, we can use the bounds on $c(\mathbf{r}, \mathbf{u})$ to manufacture both upper and lower bounds on

$$\left. \frac{\partial q(\mathbf{u})}{\partial \mathbf{m}} \right|_{m_i = c(\mathbf{r}, \mathbf{u})}$$

of the required form. \square

Lemma 2.6. *Let $i, j = 1, \dots, m$, and let $\epsilon > 0$ be arbitrary. Furthermore, let*

$$c : \mathbb{R}^{|S|} \times \mathbb{R}^m \rightarrow (S_{i,j} - \epsilon, S_{i,j} + \epsilon) \subset \mathbb{R}$$

be a function with a codomain arbitrarily close to $S_{i,j}$. Then every element of

$$\left. \frac{\partial q(\mathbf{u})}{\partial \mathbf{S}} \right|_{S_{i,j} = c(\mathbf{r}, \mathbf{u})}$$

has upper and lower bounds of the form $q(\mathbf{u})d(\mathbf{u})$, where $d(\mathbf{u}) \in \mathbb{R}_2[\mathbf{u}]$.

Proof. Using Lemma 1.1,

$$\left. \frac{\partial q(\mathbf{u})}{\partial \mathbf{S}} \right|_{S_{i,j} = c(\mathbf{r}, \mathbf{u})} = -\frac{1}{2} \mathbf{C}(\mathbf{r}, \mathbf{u})^{-\top} + \frac{1}{2} \mathbf{C}(\mathbf{r}, \mathbf{u})^{-\top} (\mathbf{u} - \mathbf{m})(\mathbf{u} - \mathbf{m})^\top \mathbf{C}(\mathbf{r}, \mathbf{u})^{-\top},$$

where

$$[\mathbf{C}(\mathbf{r}, \mathbf{u})]_{k,l} = \begin{cases} c(\mathbf{r}, \mathbf{u}) & \text{if } (k, l) = (i, j), \\ S_{k,l} & \text{otherwise.} \end{cases}$$

We can also express $\mathbf{C}(\mathbf{r}, \mathbf{u})$ as $\mathbf{C}(\mathbf{r}, \mathbf{u}) = \mathbf{S} + \mathbf{E}(\mathbf{r}, \mathbf{u})$, where

$$[\mathbf{E}(\mathbf{r}, \mathbf{u})]_{k,l} = \begin{cases} c(\mathbf{r}, \mathbf{u}) - S_{i,j} & \text{if } (k, l) = (i, j), \\ 0 & \text{otherwise.} \end{cases}$$

We begin by establishing upper and lower bounds on $\mathbf{C}(\mathbf{r}, \mathbf{u})^{-1}$. For this we use the maximum norm $\|\cdot\|_\infty$ on both vectors and matrices. We can apply Lemma 1.3 to \mathbf{S} and $\mathbf{E}(\mathbf{r}, \mathbf{u})$ since

$$\|\mathbf{E}(\mathbf{r}, \mathbf{u})\|_\infty = \max_k \sum_l |[\mathbf{E}(\mathbf{r}, \mathbf{u})]_{k,l}| = |c(\mathbf{r}, \mathbf{u}) - S_{i,j}| < \epsilon$$

can be made arbitrarily small so that $\|\mathbf{S}^{-1}\|_\infty \|\mathbf{E}(\mathbf{r}, \mathbf{u})\|_\infty < 1$. Then $\mathbf{C}(\mathbf{r}, \mathbf{u})$ is invertible, and

$$\|\mathbf{C}(\mathbf{r}, \mathbf{u})^{-1}\|_\infty \leq \frac{\|\mathbf{S}^{-1}\|_\infty}{1 - \|\mathbf{S}^{-1}\|_\infty \|\mathbf{E}(\mathbf{r}, \mathbf{u})\|_\infty} < \frac{\|\mathbf{S}^{-1}\|_\infty}{1 - \|\mathbf{S}^{-1}\|_\infty \epsilon},$$

which means that

$$\max_k \sum_l |[\mathbf{C}(\mathbf{r}, \mathbf{u})^{-1}]_{k,l}| < \frac{\|\mathbf{S}^{-1}\|_\infty}{1 - \|\mathbf{S}^{-1}\|_\infty \epsilon},$$

i.e., for any row k and column l ,

$$|[\mathbf{C}(\mathbf{r}, \mathbf{u})^{-1}]_{k,l}| < \frac{\|\mathbf{S}^{-1}\|_\infty}{1 - \|\mathbf{S}^{-1}\|_\infty \epsilon},$$

which bounds all elements of $\mathbf{C}(\mathbf{r}, \mathbf{u})^{-1}$ as required. Since every element of $(\mathbf{u} - \mathbf{m})(\mathbf{u} - \mathbf{m})^\top$ is in $\mathbb{R}_2[\mathbf{u}]$, and the elements of $\mathbf{C}(\mathbf{r}, \mathbf{u})^{-1}$ are bounded, the desired result follows. \square

Our main theorem is a specialised version of an integral differentiation result by Chen [1].

Theorem 2.7. *Whenever the derivative exists,*

$$\frac{\partial}{\partial t} \iint V_{\mathbf{r}}(s) q(\mathbf{r}) q(\mathbf{u}) d\mathbf{r} d\mathbf{u} = \iint \frac{\partial}{\partial t} [V_{\mathbf{r}}(s) q(\mathbf{r}) q(\mathbf{u})] d\mathbf{r} d\mathbf{u},$$

where t is any scalar part of \mathbf{m} , \mathbf{S} , or λ .

Proof. Let

$$\begin{aligned} f(\mathbf{r}, \mathbf{u}, t) &= V_{\mathbf{r}}(s) q(\mathbf{r}) q(\mathbf{u}), \\ F(t) &= \iint f(\mathbf{r}, \mathbf{u}, t) d\mathbf{r} d\mathbf{u}, \end{aligned}$$

and fix the value of t . Let $(t_n)_{n=1}^\infty$ be any sequence such that $\lim_{n \rightarrow \infty} t_n = t$, but $t_n \neq t$ for all n . We want to show that

$$F'(t) = \lim_{n \rightarrow \infty} \frac{F(t_n) - F(t)}{t_n - t} = \iint \frac{\partial f}{\partial t} \Big|_{(\mathbf{r}, \mathbf{u}, t)} d\mathbf{r} d\mathbf{u}. \quad (4)$$

We have

$$\frac{F(t_n) - F(t)}{t_n - t} = \iint \frac{f(\mathbf{r}, \mathbf{u}, t_n) - f(\mathbf{r}, \mathbf{u}, t)}{t_n - t} d\mathbf{r} d\mathbf{u} = \iint f_n(\mathbf{r}, \mathbf{u}) d\mathbf{r} d\mathbf{u},$$

where

$$f_n(\mathbf{r}, \mathbf{u}) = \frac{f(\mathbf{r}, \mathbf{u}, t_n) - f(\mathbf{r}, \mathbf{u}, t)}{t_n - t}.$$

Since

$$\lim_{n \rightarrow \infty} f_n(\mathbf{r}, \mathbf{u}) = \frac{\partial f}{\partial t} \Big|_{(\mathbf{r}, \mathbf{u}, t)},$$

(4) follows from Theorem 2.3 as soon as we show that both f and f_n are measurable and find a non-negative integrable function g such that for all n , \mathbf{r} , \mathbf{u} ,

$$|f_n(\mathbf{r}, \mathbf{u})| \leq g(\mathbf{r}, \mathbf{u}).$$

The MDP value function is measurable by Proposition 2.1. The result of multiplying or adding measurable functions (e.g., probability density functions (PDFs)) to a measurable function is still measurable. Thus, both f and f_n are measurable.

It remains to find g . For the time being, we can assume that t is a parameter of $q(\mathbf{r})$. Then

$$|f_n(\mathbf{r}, \mathbf{u})| = |V_{\mathbf{r}}(s)| \left| \frac{q(\mathbf{r})|_{t=t_n} - q(\mathbf{r})}{t_n - t} \right| q(\mathbf{u})$$

since PDFs are non-negative. An upper bound for $|V_{\mathbf{r}}(s)|$ is given by Proposition 2.2, while

$$\frac{q(\mathbf{r})|_{t=t_n} - q(\mathbf{r})}{t_n - t} = \frac{\partial q(\mathbf{r})}{\partial t} \Big|_{t=c(\mathbf{r}, \mathbf{u})}$$

for some function $c : \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^m \rightarrow (\min\{t, t_n\}, \max\{t, t_n\})$ due to the mean value theorem (since q is a continuous and differentiable function of t , regardless of the specific choices of q and t).

Let $\epsilon > 0$ be arbitrary. Then, for sufficiently large n , $|t_n - t| < \epsilon$, and thus

$$|c(\mathbf{r}, \mathbf{u}) - t| < \epsilon. \quad (5)$$

We can rearrange the inequality to produce bounds on $c(\mathbf{r}, \mathbf{u})$ and $|c(\mathbf{r}, \mathbf{u})|$ that will be useful later:

$$\begin{aligned} t - \epsilon &< c(\mathbf{r}, \mathbf{u}) < t + \epsilon, \\ |c(\mathbf{r}, \mathbf{u})| &< \max\{|t - \epsilon|, |t + \epsilon|\}. \end{aligned} \quad (6)$$

We then have that

$$|f_n(\mathbf{r}, \mathbf{u})| \leq \frac{\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|}{1 - \gamma} \left| \frac{\partial q(\mathbf{r})}{\partial t} \right|_{t=c(\mathbf{r}, \mathbf{u})} q(\mathbf{u}).$$

The bound is clearly non-negative and measurable. It remains to show that it is also integrable. Let $c > 0$ be as given by either Lemma 2.4 or Lemma 2.5. Then

$$\frac{\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|}{1 - \gamma} \left| \frac{\partial q(\mathbf{r})}{\partial t} \right|_{t=c(\mathbf{r}, \mathbf{u})} q(\mathbf{u}) \leq \frac{\|\mathbf{r}\|_{\infty} + \log |\mathcal{A}|}{1 - \gamma} c q(\mathbf{r}) q(\mathbf{u}).$$

Since $\|\mathbf{r}\|_{\infty} \leq \|\mathbf{r}\|_1$, and

$$\iint \|\mathbf{r}\|_1 q(\mathbf{r}) q(\mathbf{u}) d\mathbf{r} d\mathbf{u} = \sum_{i=1}^{|\mathcal{S}|} \mathbb{E}[|r_i|]$$

results in a finite number, we get the required result. \square

3 Derivatives of the Evidence Lower Bound

3.1 $\partial/\partial \mathbf{m}$

We begin by removing terms independent of \mathbf{m} :

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mathbf{m}} &= -\frac{1}{2} \frac{\partial}{\partial \mathbf{m}} [\mathbf{m}^{\top} \mathbb{E}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}] \mathbf{m}] + \frac{\partial}{\partial \mathbf{m}} [\mathbf{t}^{\top} \mathbb{E}[\mathbf{K}_{\mathbf{r}, \mathbf{u}}^{\top} \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}] \mathbf{m}] \\ &\quad - \sum_{i=1}^N \sum_{t=1}^T \frac{\partial}{\partial \mathbf{m}} \mathbb{E}[V_{\mathbf{r}}(s_{i,t})] - \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s_{i,t}, a_{i,t}, s') \frac{\partial}{\partial \mathbf{m}} \mathbb{E}[V_{\mathbf{r}}(s')]. \end{aligned}$$

Here

$$\begin{aligned}\frac{\partial}{\partial \mathbf{m}}[\mathbf{m}^\top \mathbb{E}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}] \mathbf{m}] &= (\mathbb{E}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}] + \mathbb{E}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}]^\top) \mathbf{m}, \\ \frac{\partial}{\partial \mathbf{m}}[\mathbf{t}^\top \mathbb{E}[\mathbf{K}_{\mathbf{r}, \mathbf{u}}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}] \mathbf{m}] &= \mathbf{t}^\top \mathbb{E}[\mathbf{K}_{\mathbf{r}, \mathbf{u}}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial \mathbf{m}} \mathbb{E}[V_{\mathbf{r}}(s)] &= \frac{\partial}{\partial \mathbf{m}} \iiint V_{\mathbf{r}}(s) p(\mathbf{r} | \boldsymbol{\lambda}, \mathbf{X}_{\mathbf{u}}, \mathbf{u}) \mathcal{N}(\mathbf{u}; \mathbf{m}, \mathbf{S}) q(\boldsymbol{\lambda}) d\mathbf{r} d\mathbf{u} d\boldsymbol{\lambda} \\ &= \iiint V_{\mathbf{r}}(s) p(\mathbf{r} | \boldsymbol{\lambda}, \mathbf{X}_{\mathbf{u}}, \mathbf{u}) \frac{\partial}{\partial \mathbf{m}} \mathcal{N}(\mathbf{u}; \mathbf{m}, \mathbf{S}) q(\boldsymbol{\lambda}) d\mathbf{r} d\mathbf{u} d\boldsymbol{\lambda},\end{aligned}\tag{7}$$

where Substituting it back into (7) gives

$$\begin{aligned}\frac{\partial}{\partial \mathbf{m}} \mathbb{E}[V_{\mathbf{r}}(s)] &= \frac{1}{2} \iiint V_{\mathbf{r}}(s) (\mathbf{S}^{-1} + \mathbf{S}^{-\top})(\mathbf{u} - \mathbf{m}) p(\mathbf{r} | \boldsymbol{\lambda}, \mathbf{X}_{\mathbf{u}}, \mathbf{u}) \mathcal{N}(\mathbf{u}; \mathbf{m}, \mathbf{S}) q(\boldsymbol{\lambda}) d\mathbf{r} d\mathbf{u} d\boldsymbol{\lambda} \\ &= \frac{1}{2} \mathbb{E}[V_{\mathbf{r}}(s) (\mathbf{S}^{-1} + \mathbf{S}^{-\top})(\mathbf{u} - \mathbf{m})].\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \mathbf{m}} &= -\frac{1}{2} (\mathbb{E}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}] + \mathbb{E}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}]^\top) \mathbf{m} + \mathbf{t}^\top \mathbb{E}[\mathbf{K}_{\mathbf{r}, \mathbf{u}}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}] \\ &\quad - \frac{1}{2} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E}[V_{\mathbf{r}}(s_{i,t}) (\mathbf{S}^{-1} + \mathbf{S}^{-\top})(\mathbf{u} - \mathbf{m})] \\ &\quad - \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s_{i,t}, a_{i,t}, s') \mathbb{E}[V_{\mathbf{r}}(s') (\mathbf{S}^{-1} + \mathbf{S}^{-\top})(\mathbf{u} - \mathbf{m})].\end{aligned}$$

3.2 $\partial/\partial \mathbf{S}$

Similarly to the previous section,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \mathbf{S}} &= \frac{1}{2} \frac{\partial}{\partial \mathbf{S}} \log |\mathbf{S}| - \frac{1}{2} \frac{\partial}{\partial \mathbf{S}} \text{tr}[\mathbb{E}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}] \mathbf{S}] \\ &\quad - \sum_{i=1}^N \sum_{t=1}^T \frac{\partial}{\partial \mathbf{S}} \mathbb{E}[V_{\mathbf{r}}(s_{i,t})] - \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s_{i,t}, a_{i,t}, s') \frac{\partial}{\partial \mathbf{S}} \mathbb{E}[V_{\mathbf{r}}(s')],\end{aligned}$$

where

$$\frac{\partial}{\partial \mathbf{S}} \log |\mathbf{S}| = \mathbf{S}^{-\top},$$

and

$$\frac{\partial}{\partial \mathbf{S}} \text{tr}[\mathbb{E}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}] \mathbf{S}] = \mathbb{E}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}]^\top$$

by *The Matrix Cookbook* [4]. Then

$$\frac{\partial}{\partial \mathbf{S}} \mathbb{E}[V_{\mathbf{r}}(s)] = \iiint V_{\mathbf{r}}(s) q(\mathbf{r}) \frac{\partial}{\partial \mathbf{S}} \mathcal{N}(\mathbf{u}; \mathbf{m}, \mathbf{S}) q(\boldsymbol{\lambda}) d\mathbf{r} d\mathbf{u} d\boldsymbol{\lambda},$$

where and

$$\begin{aligned}\frac{\partial}{\partial \mathbf{S}} \mathbb{E}[V_{\mathbf{r}}(s)] &= \frac{1}{2} \iiint V_{\mathbf{r}}(s) (\mathbf{S}^{-\top}(\mathbf{u} - \mathbf{m})(\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-\top} - \mathbf{S}^{-\top}) q(\mathbf{r}) \mathcal{N}(\mathbf{u}; \mathbf{m}, \mathbf{S}) q(\boldsymbol{\lambda}) d\mathbf{r} d\mathbf{u} d\boldsymbol{\lambda} \\ &= \frac{1}{2} \mathbb{E}[V_{\mathbf{r}}(s) (\mathbf{S}^{-\top}(\mathbf{u} - \mathbf{m})(\mathbf{u} - \mathbf{m})^\top \mathbf{S}^{-\top} - \mathbf{S}^{-\top})].\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \mathbf{S}} &= \frac{1}{2} \mathbf{S}^{-\top} - \frac{1}{2} \mathbb{E}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1}]^{\top} - \frac{1}{2} \sum_{i=1}^N \sum_{t=1}^T \mathbb{E}[V_{\mathbf{r}}(s_{i,t}) (\mathbf{S}^{-\top} (\mathbf{u} - \mathbf{m}) (\mathbf{u} - \mathbf{m})^{\top} \mathbf{S}^{-\top} - \mathbf{S}^{-\top})] \\ &\quad - \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s_{i,t}, a_{i,t}, s') \mathbb{E}[V_{\mathbf{r}}(s') (\mathbf{S}^{-\top} (\mathbf{u} - \mathbf{m}) (\mathbf{u} - \mathbf{m})^{\top} \mathbf{S}^{-\top} - \mathbf{S}^{-\top})].\end{aligned}$$

3.3 $\partial/\partial\alpha_j$

We begin in the usual way:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \alpha_j} &= -\frac{1}{2} \frac{\partial}{\partial \alpha_j} \mathbb{E}[\text{tr}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-2}]] - \frac{1}{2} \frac{\partial}{\partial \alpha_j} \mathbb{E}[\text{tr}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{S}]] - \frac{1}{2} \frac{\partial}{\partial \alpha_j} \mathbb{E}[\mathbf{m}^{\top} \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{m}] - \frac{1}{2} \frac{\partial}{\partial \alpha_j} \mathbb{E}[\log |\mathbf{K}_{\mathbf{u}, \mathbf{u}}|] \\ &\quad + \frac{\partial}{\partial \alpha_j} \mathbb{E}[\mathbf{t}^{\top} \mathbf{K}_{\mathbf{r}, \mathbf{u}}^{\top} \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{m}] + \frac{\partial}{\partial \alpha_j} [\alpha_j + \log \Gamma(\alpha_j) + (1 - \alpha_j) \psi(\alpha_j)] \\ &\quad - \sum_{i=1}^N \sum_{t=1}^T \frac{\partial}{\partial \alpha_j} \mathbb{E}[V_{\mathbf{r}}(s_{i,t})] - \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s_{i,t}, a_{i,t}, s') \frac{\partial}{\partial \alpha_j} \mathbb{E}[V_{\mathbf{r}}(s')].\end{aligned}$$

First,

$$\frac{\partial}{\partial \alpha_j} [\alpha_j + \log \Gamma(\alpha_j) + (1 - \alpha_j) \psi(\alpha_j)] = 1 + \psi(\alpha_j) - \psi(\alpha_j) + (1 - \alpha_j) \psi'(\alpha_j) = 1 + (1 - \alpha_j) \psi'(\alpha_j)$$

by the definition of ψ . The remaining terms can all be treated in the same way, as they all contain expectations of scalar functions that are independent of α_j , and α_j only occurs in $\Gamma(\lambda_j; \alpha_j, \beta_j)$. Thus we can work with an abstract function as follows:

$$\begin{aligned}\frac{\partial}{\partial \alpha_j} \mathbb{E}[f(k_{\boldsymbol{\lambda}}, \mathbf{r})] &= \frac{\partial}{\partial \alpha_j} \iiint f(k_{\boldsymbol{\lambda}}, \mathbf{r}) q(\boldsymbol{\lambda}) q(\mathbf{r}) q(\mathbf{u}) d\boldsymbol{\lambda} d\mathbf{r} d\mathbf{u} \\ &= \iiint f(k_{\boldsymbol{\lambda}}, \mathbf{r}) q(\lambda_0) \cdots q(\lambda_{j-1}) \frac{\partial}{\partial \alpha_j} \left[\frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \lambda_j^{\alpha_j-1} \right] e^{-\beta_j \lambda_j} \\ &\quad q(\lambda_{j+1}) \cdots q(\lambda_d) q(\mathbf{r}) q(\mathbf{u}) d\boldsymbol{\lambda} d\mathbf{r} d\mathbf{u}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial}{\partial \alpha_j} \left[\frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \lambda_j^{\alpha_j-1} \right] &= \frac{\frac{\partial}{\partial \alpha_j} [\beta_j^{\alpha_j} \lambda_j^{\alpha_j-1}] \Gamma(\alpha_j) - \beta_j^{\alpha_j} \lambda_j^{\alpha_j-1} \Gamma'(\alpha_j)}{(\Gamma(\alpha_j))^2} \\ &= \frac{\beta_j^{\alpha_j} \lambda_j^{\alpha_j-1} \frac{\partial}{\partial \alpha_j} [\alpha_j \log \beta_j + (\alpha_j - 1) \log \lambda_j] \Gamma(\alpha_j) - \beta_j^{\alpha_j} \lambda_j^{\alpha_j-1} \Gamma'(\alpha_j)}{(\Gamma(\alpha_j))^2} \\ &= \frac{\beta_j^{\alpha_j} \lambda_j^{\alpha_j-1} (\log \beta_j + \log \lambda_j) \Gamma(\alpha_j) - \beta_j^{\alpha_j} \lambda_j^{\alpha_j-1} \Gamma'(\alpha_j)}{(\Gamma(\alpha_j))^2} \\ &= \frac{\beta_j^{\alpha_j} \lambda_j^{\alpha_j-1}}{\Gamma(\alpha_j)} \left(\log \beta_j + \log \lambda_j - \frac{\Gamma'(\alpha_j)}{\Gamma(\alpha_j)} \right),\end{aligned}$$

which means that

$$\begin{aligned}
\frac{\partial}{\partial \alpha_j} \mathbb{E}[f(k_{\boldsymbol{\lambda}}, \mathbf{r})] &= \iiint f(k_{\boldsymbol{\lambda}}, \mathbf{r}) q(\lambda_0) \cdots q(\lambda_{j-1}) \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \lambda_j^{\alpha_j-1} e^{-\beta_j \lambda_j} \left(\log \beta_j + \log \lambda_j - \frac{\Gamma'(\alpha_j)}{\Gamma(\alpha_j)} \right) \\
&\quad q(\lambda_{j+1}) \cdots q(\lambda_d) q(\mathbf{r}) q(\mathbf{u}) d\boldsymbol{\lambda} d\mathbf{r} d\mathbf{u} \\
&= \iiint f(k_{\boldsymbol{\lambda}}, \mathbf{r}) \left(\log \beta_j + \log \lambda_j - \frac{\Gamma'(\alpha_j)}{\Gamma(\alpha_j)} \right) q(\boldsymbol{\lambda}) q(\mathbf{r}) q(\mathbf{u}) d\boldsymbol{\lambda} d\mathbf{r} d\mathbf{u} \\
&= \mathbb{E} \left[f(k_{\boldsymbol{\lambda}}, \mathbf{r}) \left(\log \beta_j + \log \lambda_j - \frac{\Gamma'(\alpha_j)}{\Gamma(\alpha_j)} \right) \right] \\
&= \left(\log \beta_j - \frac{\Gamma'(\alpha_j)}{\Gamma(\alpha_j)} \right) \mathbb{E}[f(k_{\boldsymbol{\lambda}}, \mathbf{r})] + \mathbb{E}[f(k_{\boldsymbol{\lambda}}, \mathbf{r}) \log \lambda_j].
\end{aligned}$$

With these results in mind, we can simplify the initial expression to

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha_j} &= 1 + (1 - \alpha_j) \psi'(\alpha_j) + \left(\log \beta_j - \frac{\Gamma'(\alpha_j)}{\Gamma(\alpha_j)} \right) \left(-\frac{1}{2} \mathbb{E}[\text{tr}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-2}]] - \frac{1}{2} \mathbb{E}[\text{tr}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{S}]] \right. \\
&\quad - \frac{1}{2} \mathbb{E}[\mathbf{m}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{m}] - \frac{1}{2} \mathbb{E}[\log |\mathbf{K}_{\mathbf{u}, \mathbf{u}}|] + \mathbb{E}[\mathbf{t}^\top \mathbf{K}_{\mathbf{r}, \mathbf{u}}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{m}] \\
&\quad \left. - \sum_{i=1}^N \sum_{t=1}^T \mathbb{E}[V_{\mathbf{r}}(s_{i,t})] - \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s_{i,t}, a_{i,t}, s') \mathbb{E}[V_{\mathbf{r}}(s')] \right) \\
&\quad - \frac{1}{2} \mathbb{E}[\text{tr}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-2}] \log \lambda_j] - \frac{1}{2} \mathbb{E}[\text{tr}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{S}] \log \lambda_j] - \frac{1}{2} \mathbb{E}[\mathbf{m}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{m} \log \lambda_j] \\
&\quad - \frac{1}{2} \mathbb{E}[\log |\mathbf{K}_{\mathbf{u}, \mathbf{u}}| \log \lambda_j] + \mathbb{E}[\mathbf{t}^\top \mathbf{K}_{\mathbf{r}, \mathbf{u}}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{m} \log \lambda_j] \\
&\quad - \sum_{i=1}^N \sum_{t=1}^T \mathbb{E}[V_{\mathbf{r}}(s_{i,t}) \log \lambda_j] - \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s_{i,t}, a_{i,t}, s') \mathbb{E}[V_{\mathbf{r}}(s') \log \lambda_j].
\end{aligned}$$

3.4 $\partial/\partial \beta_j$

Finally,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \beta_j} &= -\frac{1}{2} \frac{\partial}{\partial \beta_j} \mathbb{E}[\text{tr}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-2}]] - \frac{1}{2} \frac{\partial}{\partial \beta_j} \mathbb{E}[\text{tr}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{S}]] - \frac{1}{2} \frac{\partial}{\partial \beta_j} \mathbb{E}[\mathbf{m}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{m}] \\
&\quad - \frac{1}{2} \frac{\partial}{\partial \beta_j} \mathbb{E}[\log |\mathbf{K}_{\mathbf{u}, \mathbf{u}}|] + \frac{\partial}{\partial \beta_j} \mathbb{E}[\mathbf{t}^\top \mathbf{K}_{\mathbf{r}, \mathbf{u}}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{m}] - \frac{\partial}{\partial \beta_j} [\log \beta_j] \\
&\quad - \sum_{i=1}^N \sum_{t=1}^T \frac{\partial}{\partial \beta_j} \mathbb{E}[V_{\mathbf{r}}(s_{i,t})] - \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s_{i,t}, a_{i,t}, s') \frac{\partial}{\partial \beta_j} \mathbb{E}[V_{\mathbf{r}}(s')].
\end{aligned}$$

Similarly to the previous section, we can handle all derivatives of expectations in the same way:

$$\begin{aligned}
\frac{\partial}{\partial \beta_j} \mathbb{E}[f(k_{\boldsymbol{\lambda}}, \mathbf{r})] &= \frac{\partial}{\partial \beta_j} \iiint f(k_{\boldsymbol{\lambda}}, \mathbf{r}) q(\boldsymbol{\lambda}) q(\mathbf{r}) q(\mathbf{u}) d\boldsymbol{\lambda} d\mathbf{r} d\mathbf{u} \\
&= \iiint f(k_{\boldsymbol{\lambda}}, \mathbf{r}) q(\lambda_0) \cdots q(\lambda_{j-1}) \frac{\lambda_j^{\alpha_j-1}}{\Gamma(\alpha_j)} \frac{\partial}{\partial \beta_j} [\beta_j^{\alpha_j} e^{-\beta_j \lambda_j}] \\
&\quad q(\lambda_{j+1}) \cdots q(\lambda_d) q(\mathbf{r}) q(\mathbf{u}) d\boldsymbol{\lambda} d\mathbf{r} d\mathbf{u}.
\end{aligned}$$

Since

$$\frac{\partial}{\partial \beta_j} [\beta_j^{\alpha_j} e^{-\beta_j \lambda_j}] = \alpha_j \beta_j^{\alpha_j-1} e^{-\beta_j \lambda_j} - \beta_j^{\alpha_j} e^{-\beta_j \lambda_j} \lambda_j = \beta_j^{\alpha_j} e^{-\beta_j \lambda_j} \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right),$$

we have that

$$\begin{aligned}
\frac{\partial}{\partial \beta_j} \mathbb{E}[f(k_{\boldsymbol{\lambda}}, \mathbf{r})] &= \iiint f(k_{\boldsymbol{\lambda}}, \mathbf{r}) q(\lambda_0) \cdots q(\lambda_{j-1}) \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \lambda_j^{\alpha_j-1} e^{-\beta_j \lambda_j} \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right) \\
&\quad q(\lambda_{j+1}) \cdots q(\lambda_d) q(\mathbf{r}) q(\mathbf{u}) d\boldsymbol{\lambda} d\mathbf{r} d\mathbf{u} \\
&= \iiint f(k_{\boldsymbol{\lambda}}, \mathbf{r}) \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right) q(\boldsymbol{\lambda}) q(\mathbf{r}) q(\mathbf{u}) d\boldsymbol{\lambda} d\mathbf{r} d\mathbf{u} \\
&= \mathbb{E} \left[f(k_{\boldsymbol{\lambda}}, \mathbf{r}) \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right) \right] = \frac{\alpha_j}{\beta_j} \mathbb{E}[f(k_{\boldsymbol{\lambda}}, \mathbf{r})] - \mathbb{E}[f(k_{\boldsymbol{\lambda}}, \mathbf{r}) \lambda_j].
\end{aligned}$$

This gives us the final expression of $\frac{\partial \mathcal{L}}{\partial \beta_j}$:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \beta_j} &= -\frac{1}{\beta_j} - \frac{1}{2} \mathbb{E} \left[\text{tr}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-2}] \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right) \right] - \frac{1}{2} \mathbb{E} \left[\text{tr}[\mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{S}] \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right) \right] \\
&\quad - \frac{1}{2} \mathbb{E} \left[\mathbf{m}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{m} \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right) \right] - \frac{1}{2} \mathbb{E} \left[\log |\mathbf{K}_{\mathbf{u}, \mathbf{u}}| \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right) \right] \\
&\quad + \mathbb{E} \left[\mathbf{t}^\top \mathbf{K}_{\mathbf{r}, \mathbf{u}}^\top \mathbf{K}_{\mathbf{u}, \mathbf{u}}^{-1} \mathbf{m} \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right) \right] \\
&\quad - \sum_{i=1}^N \sum_{t=1}^T \mathbb{E} \left[V_{\mathbf{r}}(s_{i,t}) \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right) \right] - \gamma \sum_{s' \in \mathcal{S}} \mathcal{T}(s_{i,t}, a_{i,t}, s') \mathbb{E} \left[V_{\mathbf{r}}(s') \left(\frac{\alpha_j}{\beta_j} - \lambda_j \right) \right].
\end{aligned}$$

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