Visualising the Geometry of Batik

Chapter 1: The Kissing Circles

Figure 1. Portrait of René Descartes, French Philosopher and Mathematician. Wikipedia

“I think therefore I am”, or in Latin “*Cogito ergo sum”* is one of the most famous philosophies from the French philosopher René Descartes. He argues that while everything else could be doubted – sensory perception, physical reality etc, the act of doubting presupposes a thinking entity. This quote showcases Descartes' brilliance in logical thinking, using syllogism in his chain of thought, and makes him a very prominent figure in philosophy. But did you know that his genius is not only shown in philosophy, but also in mathematics?

As a mathematician, he was also a prolific writer. When we are in school, we are introduced to coordinate system called “*Cartersian Coordinate”*. This coordinate is named after him, whose contribution in Math especially in analytical geometry earned him the honour to be named as one of the most used coordinate systems in the world..

In this article, we are going to expand upon one of his theorems to build a new Batik pattern. The name of the theory is the theory of Kissing Circles. This theory was found in his letter correspondence with princess Elisabeth of the Palatinate. She was a princess and a student of Descartes. In that letter, Descartes discussed the problem that has pondered for millenia, the problem of four mutually tangent (kissing) circles.

By using his theorem, we will be able to create a circle fractal named Apollonian Gasket. The core idea that this author proposes is by decorating the circle with batik Kawung, we can create a visually unique new batik Motif as seen in Figure 2 below.

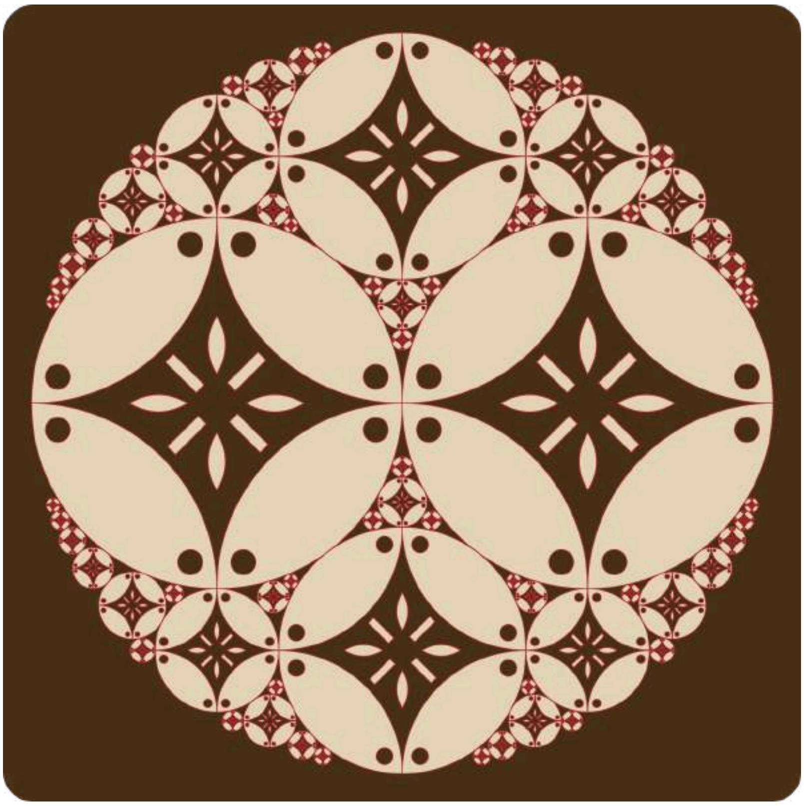


Figure 2. Kawung motif adorning Apollonian Gasket

The following text will briefly discuss what Batik Kawung is from a geometrical point of view, its mathematical formula, and how to draw it manually using a compass and ruler, as well as how to draw it using SVG code. Afterward, we will learn Descartes' theorem and how to apply it to create new motifs. In the final section, we will apply that motif to create a necklace pendant for jewellery.

Batik Kawung

Geometry

Batik Kawung is one of the oldest motifs known in Java. The motifs are found adorning many statues in Indonesian temples. For example, there is one that adorns Mahakala statue found in Singosari temple as shown in figure 3. It is also found in statues of Ganesha, Nandishvara, and Durga Mahishasuramardini. The popularity of the Kawung motif has stood the test of time. Even now, in the 21st century, Kawung is widely used in Indonesian fabric, building architecture, and even in popular culture. If you pay attention, you might even see it in the opening credit of 2013 Studio Ghibli animated movies, the *Tales of Princess Kaguya*. Knowing how popular this motif is, it is interesting to find out more about the geometry of this motif.

Figure 3. Kawung motif adorning sarong that Mahakala wears. Wikipedia

There are many ways the Kawung motif can be defined. In this chapter, we will use the definition of circular Kawung. Circular Kawung can be defined as a shape formed by the intersection of four circles, each with the same radius, positioned above (0, -r), below (0, r), to the left(-r, 0) , and to the right (r, 0) of its origin (0,0) as shown in Figure 4.

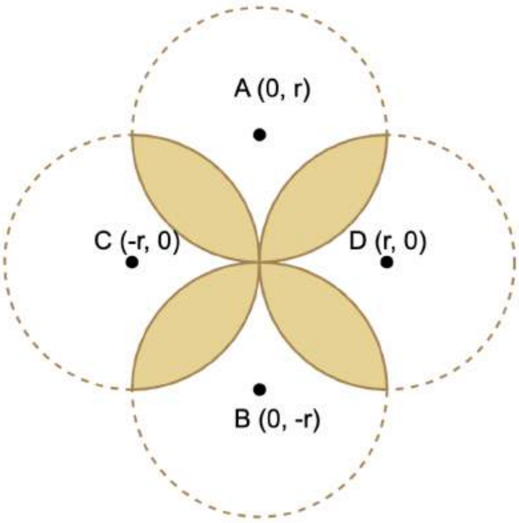


Figure 4. Kawung motif as intersection of 4 circles.

Manual Drawing Steps

Knowing how to sketch Batik Kawung manually using a pencil, a compass, and paper is a relatively straightforward process. Given a square ABCD, we can draw a circle centered at O with a radius equal to half of the square's side length. Next, we draw a quarter circle centered at vertex A. We then draw a quarter circle at vertices B, C, and D as shown in figure 5 below.

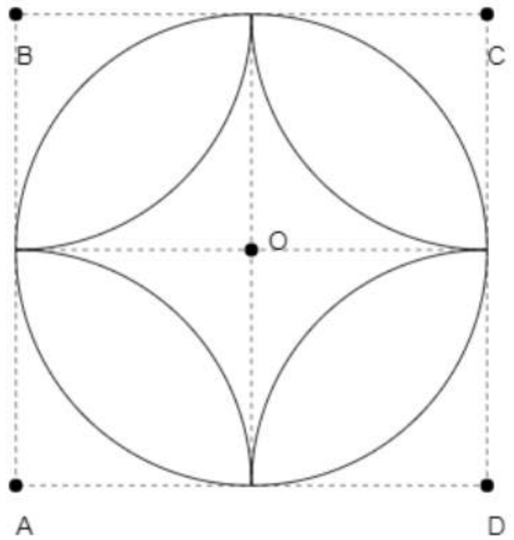


Figure 5. How to draw a Kawung using a compass and pencil

Using the steps mentioned above, we are able to draw the main motive *"klowongan"* of Batik Kawung. To make it more visually pleasing, batik artists usually add secondary motives and background motives. Secondary motive is usually called "*isen-isen".* Isen-isen is the decorative fillings found within the main patterns. These additional motifs are used to fill the spaces between the main motifs to enhance the overall aesthetic appeal of the fabric. For batik Kawung, usually isen cecek are used. The illustration of this concept can be found in the figure 6 below.

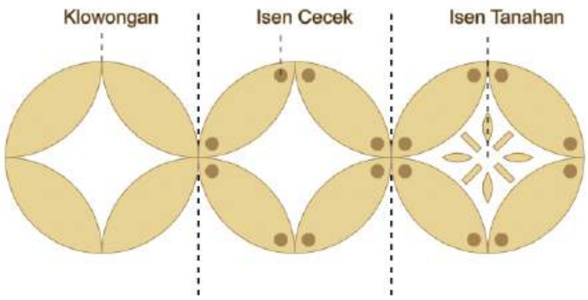


Figure 6. Main motive, secondary motive, and background filling of Batik Kawung

EXERCISE:

1. Using only a compass and ruler and pencil, draw 30 pieces of Batik Kawung.
2. Search the internet and learn the variety of *isen-isen* that is usually used in Batik Kawung. Adorn batik kawung you draw in the 1st exercise using isen-isen you just learned.
3. Try to color the batik you just drew. Find inspiration on the internet. Some artists color the batik using a uniform color, while others are coloring it with two different colours that make the final appearance like checkerboard.

Digital Drawing Steps

If we only want to draw hand written batik, then it is enough to only know how to draw it manually using the steps mentioned above. However, if we want to use computers to generate digital patterns for digital printing or 3D modelling design, we need to represent Batik in a way that computers understand. One of those ways is to represent Kawung in Scalable Vector Graphic (SVG).

Scalable Vector Graphics (SVG) is an XML-based markup language for describing two-dimensional based vector graphics. The benefit of using SVG is that the image can be rendered at any size and it will still look clean. The other benefit is that SVG format is easily readable by 3D modelling CAD programs. For instance, if we want to make a necklace pendant with motif Kawung, we can import an SVG file of Kawung and extrude it in a 3D modelling CAD program.

To draw Kawung using Scalable Vector Graphic, we can write the following code in a text editor.

```
<g transform="translate(x_origin, y_origin)”>
     <path d="M0 0 A R R 0 0 1 R R A R R 0 0 1 0 0"> </path>
     <path d="M0 0 A R R 0 0 1 R R A R R 0 0 1 0 0"
transform="rotate(90)"> </path>
     <path d="M0 0 A R R 0 0 1 R R A R R 0 0 1 0 0"
transform="rotate(180)"> </path>
     <path d="M0 0 A R R 0 0 1 R R A R R 0 0 1 0 0"
transform="rotate(270)"> </path>
</g>
```

The explanation of the code mentioned above are as follows:

First, we group four path elements inside a group tag, we then change the coordinate of the group to the origin we wanted. Second, we draw a path that describes the leaf of the Kawung. To describe the leaf, we use attribute d in the path and use the following command.

M0 0 → Move cursor to the origin (0,0)

A R R 0 0 1 R R → Using Arc command to draw an arc from (0,0) to end coordinate (R,R).

A R R 0 0 1 0 0 → Using Arc command draw an arc from (R,R) back to starting coordinate (0,0)

The same arc command is then used three times, the resulting path is then rotated 90 degrees to get the shape of Kawung.

EXERCISE:

1. Find the documentation of SVG Arc command on the internet. Play around with the parameter and see what happened to the resulting shape.
2. Find the documentation of SVG. Find how to give fill colour and stroke colour. Can you figure out how to make the same drawing as Figure 6?
3. Play around with SVG, be creative and make some interesting Kawung patterns.

Descartes Theorem

Descartes theorem is crucial to the development of Apollonian Gasket. Apollonian Gasket is a type of circle fractal, in which every circle is tangent to three other circles. We are going to substitute the circle with a Kawung shape to make a pattern author named as Apollonian Kawung Gasket. Descartes theorem is needed to find the radius, and the coordinate of the circle center. Figure 8 illustrates the application of Descartes' theorem in crafting the Apollonian Kawung pattern.

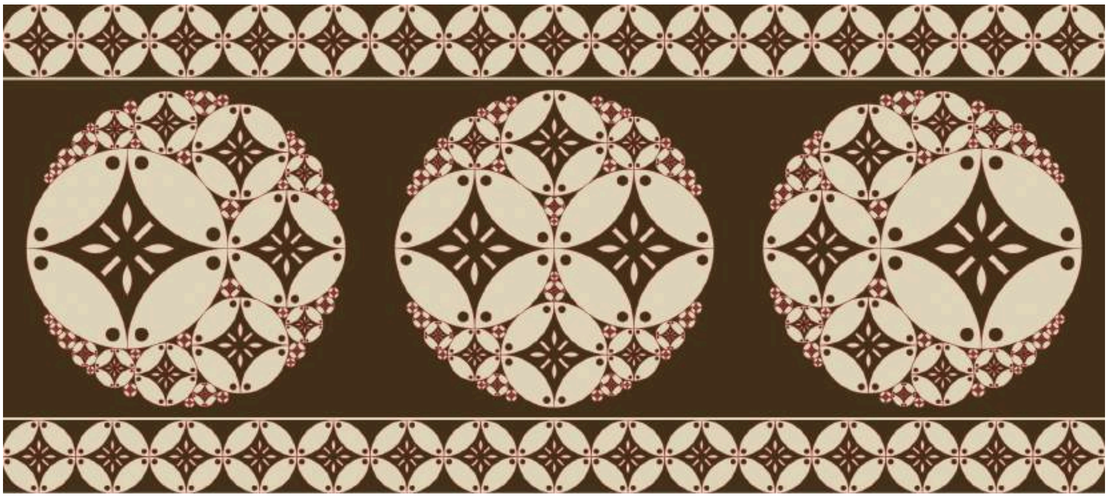


Figure 8. Application of Descartes Theorem

Descartes theorem is used to get the radius and location of the circle center, given that we know three others mutually tangent circle position and radius. For illustration, please refer to Figure 9. The first circle C1, is a circle located at position (0,0) and radius r1 let's say 140. The second one is located at (-70,0) and has radius r2 = 70. The third circle is located at (70,0) and has radius r3 = 70. Using Descartes' theorem we can find the radius of the fourth circle and the location of the circle centre.

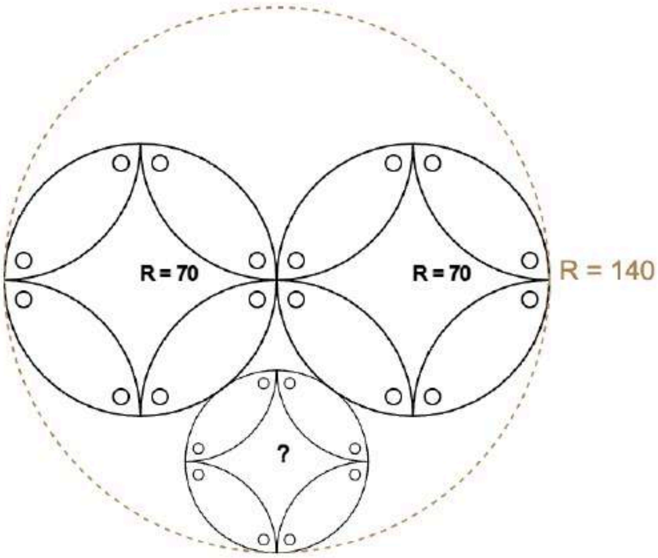


Figure 9. Descartes Theorem Illustration

To solve this problem, we can use the following formula:

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2)$$

Where k is the circle curvature:

$$k = \pm \frac{1}{r}$$

The value of k is positive for externally tangent circles. For circles that internally circumscribe other circles, the value is negative. In this case, for C1 the value is negative, while for circle 2,3, and 4 the value is positive. We can rewrite the equation above to get the value of the fourth circle curvature.

$$k_4 = k_1 + k_2 + k_3 \pm 2\sqrt{k_1 k_2 + k_2 k_3 + k_3 k_1}$$

EXERCISE:

1. Find the value of k_4
2. Given the image in the figure 10. Find the value of k4 (curvature of circle in the middle (Hint: straight line can be treated as a circle with infinite radius.

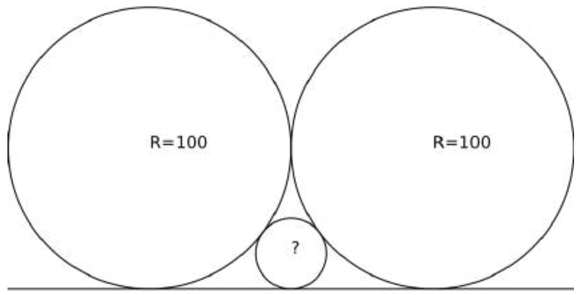


Figure 10. Descartes Theorem Illustration

To describe a circle completely, we need only the radius, but also the circle origin. To find the circle origin, we can use an extension of Descartes theorem that uses complex numbers. Complex number is an element of of number system that extends the real numbers with imaginary units i that satisfy the equation $i^1 = -1$. Every complex number z can be expressed in form $z = a + b\,i$. Comlex Descartes theorem can be stated as shown below. To get the x coordinate of the center, we get the real part of the complex number. To get the y coordinate of the circle's center, we use the imaginary part of the complex number.

$$z_4 = \frac{k_1 z_1 + k_2 z_2 + k_3 z_3 \pm 2\sqrt{k_1 k_2 z_1 z_2 + k_2 k_3 z_2 z_3 + k_3 k_1 z_3 z_1}}{k_4}$$

Optional: Complex Number

We can use computers to solve Descartes theorem, but there is merit in understanding the underlying Math. Feel free to skip this part if you are not into Math. As we can see, to use the formula we need to know how to do addition, multiplication, division, and taking the square root of a complex number.

Addition and subtraction of complex numbers is pretty straightforward. The operation is similar to vector operation, just sum the real part with the real part, and sum imaginary part with imaginary part. Geometrically, addition and subtraction of complex numbers is translation in the complex plane.

Multiplication is pretty similar with normal multiplication, with the exception that $i^1 =- 1$. Geometrically, multiplication can be stretching the number by certain modulus and rotating by a certain angle.

EXERCISE:

Given figure 10, we have the following equation:

$z_1 = \ -100 \ + \ 100i, k_1 \ = \frac{1}{100}$

$z_2 = \ 100 \ + \ 100i, k_2 \ = \frac{1}{100}$

$z_3 = \ 0 \ + \ 0i, \ k_3 \ = \ 0$ (straight line can be considered as circle with infinite radius)

1. Calculate value of k_4
2. Calculate the value of z_4
3. Given $z_1, z_2, and\ z_4$ calculate value of other circle ($z_5\ and\ k_5$) mutually tangent to these three circles

Recursion

To draw intricate patterns, we will apply Descartes theorem multiple times. To do it manually is a laborious task. We can use recursion to make it more efficient. Recursion solves problems by using functions that call themselves in the function. In this case, we will create a recursive function that will find the fourth circle given 3 circles, and only will stop if the radius of the newly created circle is less than a threshold. Below pseudocode to do it:

```
circleList = []

def recursiveDescartes(circle1, circle2, circle3):
    circle4 = calculate(circle1, circle2, circle3)
    circleList.append(circle4)
    if circle4.r > threshold:
        recursiveDescartes(circle1, circle2, circle4)
        recursiveDescartes(circle2, circle3, circle4)
        recursiveDescartes(circle1, circle3, circle4)
```

Fibonacci Tiling

To create an interesting pattern, besides using Descartes theorem we can also use the Fibonacci sequence. The Fibonacci sequence is named after 12th century Mathematician Leonardo Bonacci / Leonardo of Pisa. He is most well known for introducing arabic numeral system (that we also use nowadays) to western europe and for introducing the concept of fibonacci sequence in his book Liber Abaci

The fibonacci sequence can be defined as a sequence in which each number is the sum of the two preceding ones. Some of the few first Fibonacci sequences are: 0, 1, 1, 2, 3, 5, 8, 13, 21. We can use the following formula to define the Fibonacci sequence.

$$F_0 = 0, F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2};\ for\ n > 1$$

Geometrically, the Fibonacci sequence can be used to create a spiral. In the image below, we will create a Fibonacci spiral decorated with Batik Kawung as shown in Figure 11 below. Each Batik Kawung leaf's radius follows a fibonacci sequence. The leaf in the centre has radius 21, the next one has radius 13, the one after that has radius 8 and so on.

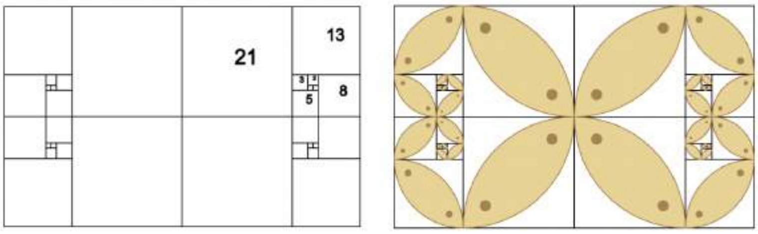


Figure 11. Fibonacci spiral decorated with batik Kawung pattern.

Application

Using Descartes theorem we learned in this chapter, we can create various products. For example, we can create clothes with various designs. Below are some images that you can create using theory we learn in this chapter. Figure 12 is an image created using variants of Ford circles arranged in a way that we see Kawung-like patterns emerge in the background. Figure 13 is image created by applying Descartes theorem recursively with 3 initial circles configured in triangular pattern

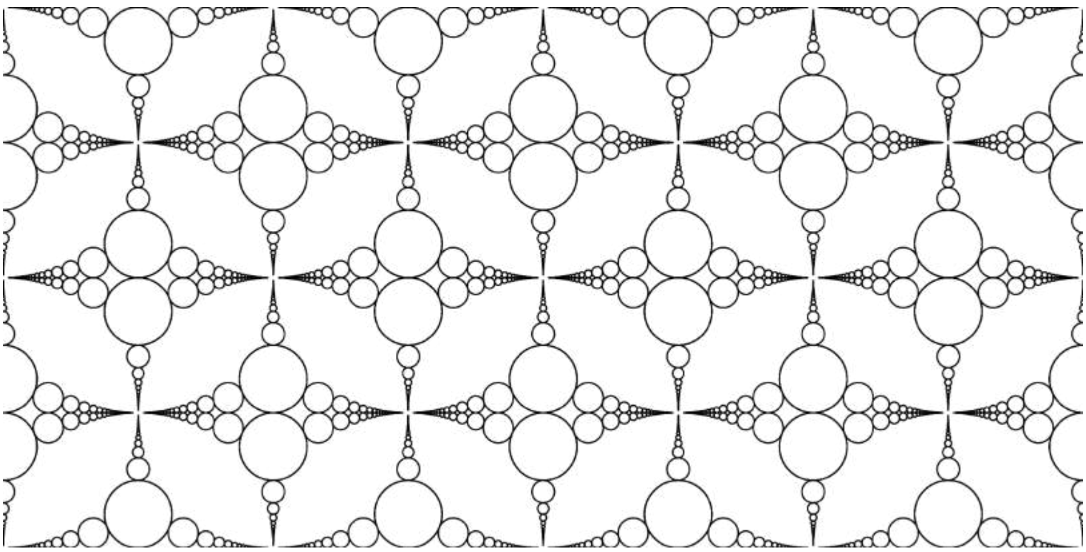


Figure 12. Ford Apollonian Kawung

Figure 13. Variant of Apollonian Gasket

Not only can we create 2D designs, we can also create 3D products such as necklace pendants. We can import a 2D svg pattern into a 3D CAD program and create a 3D model by extruding the 2D pattern. Figure 13 illustrates a sample necklace pendant ready to be casted by a jewellery shop. We can also make our design to a ring as shown in Figure 14. To make a ring, we first create a 2D design in rectangular shape. We transform it, do a 360 degree circular bend to the rectangle to make a ring shape.

Figure 13. Apollonian necklace pendant.

Figure 14. Fibonacci Kawung Ring

EXERCISE:

1. Draw a design using Apollonian Kawung gasket.
2. The Apollonian Kawung gasket can differ based on the first three circles. Play around with the radius and coordinate of the first three circles and make your own version of apollonian kawung gasket.

Chapter 2: The Dancing Triangle

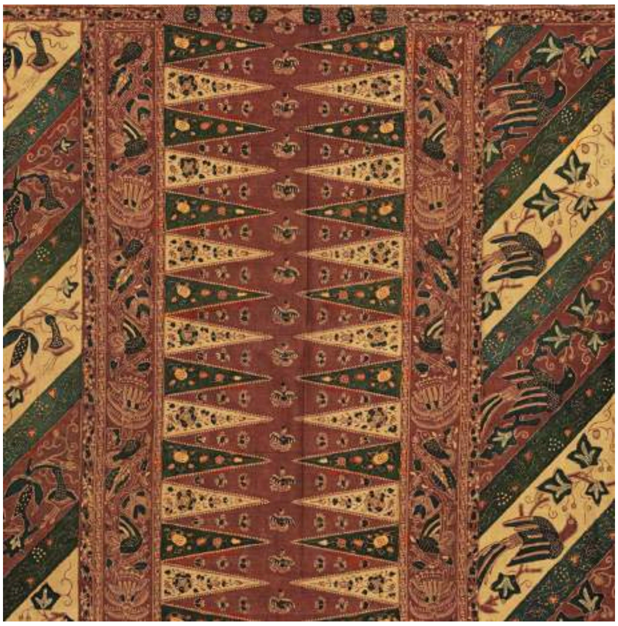


Figure 15. Sarong featuring Batik Tumpal pattern purchased by Siam King. Made by the Semarang workshop owned by Carolina Josephina von Franquemont. Wikipedia

When the Dutch arrived in Java in the 16th century to trade spices, they not only bought spices from the locals but also sold Ceramics from China and fabric from India. The Indian fabric is adorned with a triangle pattern that Javanese people really like, which later will influence the creation of Batik Tumpal.
Batik Tumpal is mainly characterised by isosceles triangles. The triangles are then decorated with a secondary pattern. The pattern is usually a flower pattern or wave-like pattern. In this chapter, we will create a general formula to generate Batik Tumpal that is not limited to isosceles triangles, but any triangle.

Batik Tumpal

Geometry

The main idea of this chapter is how to define Batik Tumpal geometrically. First, we are going to create a triangle as a border. Next, we create a secondary ornament that is tangent to the border. Lastly, we tessellate the triangles to create an intricate pattern that is visually appealing. Sample of the work we will create is shown in Figure 16.

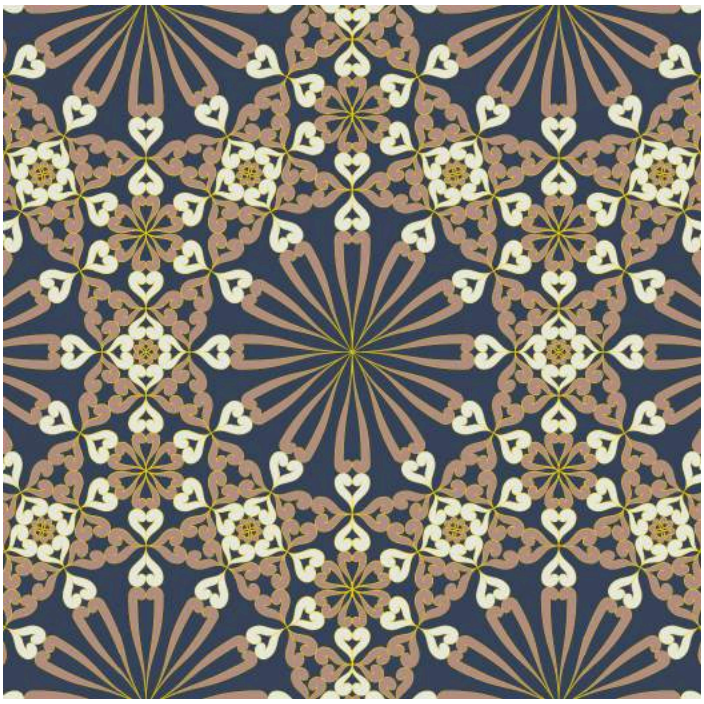


Figure 16. Batik Tumpal arranged in semi regular tessellation

In order to create an image like Figure 16, we need to define how to apply a secondary motive to the Batik Tumpal triangle border. We are going to create a wave spiral pattern that is tangent to the triangle's border. See figure 17 for the illustration. Given triangle ABC, we want to locate the location of the spiral centre in which the spiral will be tangent to line AB and line BC.

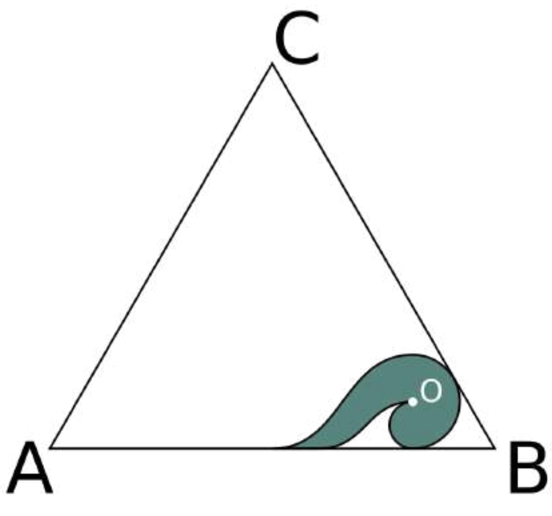


Figure 17. Locating centre of the spiral O and its R is essential for Batik Tumpal creation

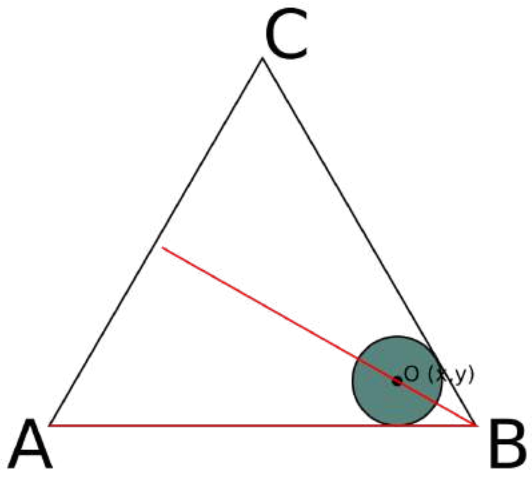


Figure 18. ABO is angle that pass the circle origin

To get the location of the centre spiral, we can use the following steps. First, we know y is equal to R since the circle is tangent to line AB. To get x (horizontal distance from point B to circle centre), we can use trigonometry formula $x = \frac{R}{tan(<ABO)}$. To get the angle <ABO, we can use a theorem that states that if two tangent lines meet, then <ABO is half of angle <ABC as shown in Figure 18.

Once we get the centre coordinate, creating a spiral is a trivial problem. We can create a semicircle with radius 0.5 R and another semicircle with radius R to create the spiral head. To complete the spiral, we then draw the spiral tail to the centre of the triangle.

Manual Drawing Steps

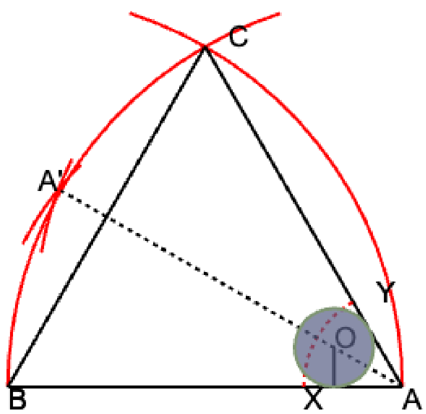


Figure 19. How to Draw Wave Spiral Pattern

Here is a step by step guide to draw Batik Tumpal wave spiral pattern:

1. Given a line AB, use your compass to draw and arc with length AB centered at A.
2. Draw an arc using the same compass length, centered at B. This should make an intersection point with the previous arc. The intersection point is called C. Line A, B, and C will make an equilateral triangle.
3. Draw a small arc centered at point A that passes through line AB and line AC. It met line AB in point X and met line AC in point Y.
4. Draw an arc centered at X
5. Draw another arc centered at Y. This should make an intersection with the previous arc at point A'. Line AA' is a line that divides angle BAC equally.
6. We can draw any circle on line AA with radius equal to distance to line AB, it is guaranteed that it will also be tangent to line AC.
7. After we locate the spiral center, we can proceed to make the spiral head and the spiral tail.

Digital Drawing Steps

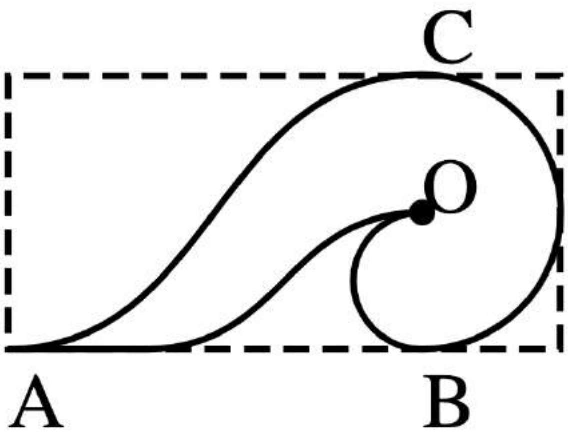


Figure 20. How to draw Batik Tumpal digitally

How to draw SVG of spiral wave pattern. Let's assume the origin (0,0) is at point A. and O (Xc, Yc) is the centre of the spiral.

1. To draw segment AC (tail of the spiral), we need to draw it using Bezier Curve. If we are using Cubic Bezier Curve with two control points we can use the following command. `<path d= "M0 0  C0.5Xc 0 0.5Xc 2R Xc 2R"> </path>`.
2. To draw a semicircle CB, we can use the Arc command as follows: `<path d="AR R 0 0 1 Xc 0"> </path>`.
3. To draw a semicircle OB, we can use the Arc command as follows: `<path d="A0.5R 0.5R 0 0 1 Xc R"> </path>`.

4. To draw segment OA (tail of the spiral), we need to draw it using Bezier Curve as follows. `<path d= "C0.5Xc R 0.5Xc 0 0 0"> </path>.`

Tessellation

From the wall of Alhambra palace in Spain to Ming Dynasty brocade in China, tessellation has been used in art. The use of tessellation can be found in many pieces of art including batik. For instance, batik Kawung is usually arranged in a square grid tessellation. But the potential of tessellation is not limited to only square grid, there are so much more. In this chapter, we will discuss how to apply the tessellation principle to make a visually appealing Batik Tumpal configuration.

Tessellation has been studied for centuries. One of the earliest documented studies of tessellation can be found in the work of Johannes Keppler, a German astronomer, mathematician and philosopher. In the 16th century, he wrote about regular and semi-regular tessellation, which are coverings of a plane with regular polygon.

The key properties of tessellations are:

1. All tiles in tessellation share at least 1 vertex.
2. The sum of interior angles around a vertex in tessellation is 360 degree.

Can we figure out what are the shapes that will tessellate? To study this we need a formula to calculate interior angles of regular polygons. We know from school that the total interior angle of a triangle is 180 degrees. But what about the other polygon? To answer this, we can divide those shapes into triangles. For example, a square can be divided into two triangles, so the sum of interior angles of squares are 360 degrees. The Pentagon for instance, can be divided into three triangles, thus having a sum of interior angles of 540 degrees. In general, formula for calculating the sum of interior angles is:

$$Sum\ of\ Interior\ Angles\ = (n - 2) * 180$$

To get the size of the interior angle, we can use the following equation:

$$Size\ of\ Interior\ Angles\ = \frac{Sum\ of\ Interior\ Angles}{n}$$

Regular Polygon	**Sum of Interior Angles**	**Interior Angles**
Triangles	180	60
Square	360	90

Pentagon	540	108
Hexagon	720	120
Heptagon	900	128.57
Octagon	1080	135
Nonagon	1260	140
Decagon	1440	144
Hendecagon	1620	147.27
Dodecagon	1800	150

EXERCISE:

1. How many regular tessellations are there? What are the shapes that construct the tessellations? (Hint: find shapes that have an internal angle a factor of 360. Eg: Square has an internal angle of 90 degrees. 90 degrees is factor of 360)
2. How many semi regular tessellations are there? What are the shapes that construct the tessellations? (Hint: find a combination of shapes that has a total internal angle of 360. Eg: Square + Hexagon + Dodecagon total internal angle is 360. Square has 90 degree, hexagon 120 degree, and dodecagon has an internal angle of 150 degree.
3. Beside regular and semi tessellations, what are the other types of tessellations? Give at least another three types of tessellation.

In this chapter we will be introduced with vertex configuration. Vertex configuration is how we name tessellations based on what polygons meet at each vertex. The naming rules are as follows:

1. Find the polygon with the least number of sides.
2. Find the longest consecutive run of this polygon
3. Indicate the number of sides of this regular polygon.
4. Proceeding in a clockwise or counter-clockwise order, indicate the number of sides of each polygon as you see them in arrangement

Tumpal is arranged in 3 regular tessellations and 8 semi-regular tessellations. Regular tessellation tiles the spaces using 1 shape. While in the semi-regular tessellation, one or more shapes are used. Shapes we are going to use are triangles, squares, hexagons, octagons, and dodecagons. We are going to decorate this shape with batik Tumpal as illustrated in Figure 21.

Figure 21. Regular polygon tiles that are decorated with Batik Tumpal pattern

These tiles are then used to create the 11 uniform tessellations. The first one, shown in Figure 22 is equilateral triangle tessellation that has 3.3.3.3.3.3 vertex configuration. It means that six equilateral triangles meet at each vertex. It has the Schaffli symbol of {3, 6}.

Figure 22. Triangular tiling of Batik Tumpal. Triangular tiling is one of the most widely used tiling. It is commonly used as a backdrop. It has vertex configuration 3.3.3.3.3.3 and Schläfli Symbol {3,6}

Figure 23. Square tiling of Batik Tumpal. Square tiling is the most common form tiling. We can find it in chess boards, checker boards, and many other applications. It has vertex configuration 4.4.4.4. Schlafli symbol {4,4}

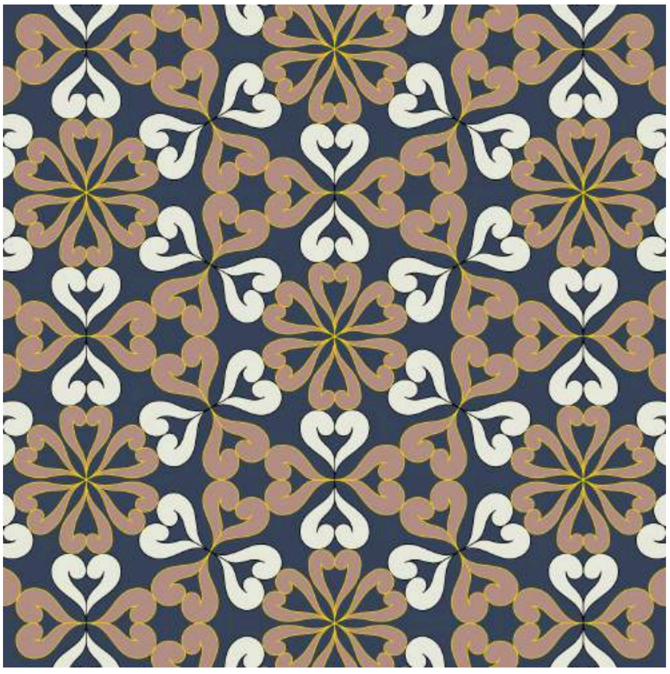


Figure 24. Hexagonal tiling of Batik Tumpal. This tiling is naturally found in nature as shown in bees honeycomb or in atomic structure of graphite. It has Vertex configuration 6.6.6. Schlafli symbol {6,3}

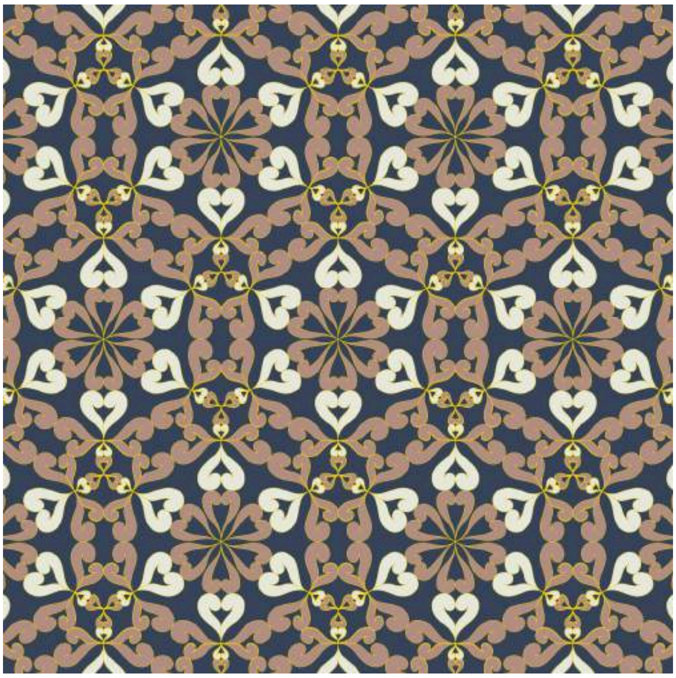


Figure 25. Trihexagonal tiling. This tiling has long been documented. Johannes Keppler, father of astronomy, wrote about this tiling in his 1619 book *Harmonices Mundi*. This tiling is also called *Kagome* tiling by the Japanese. This type of tiling has long been used in Japanese basketry. Vertex configuration 3.6.3.6

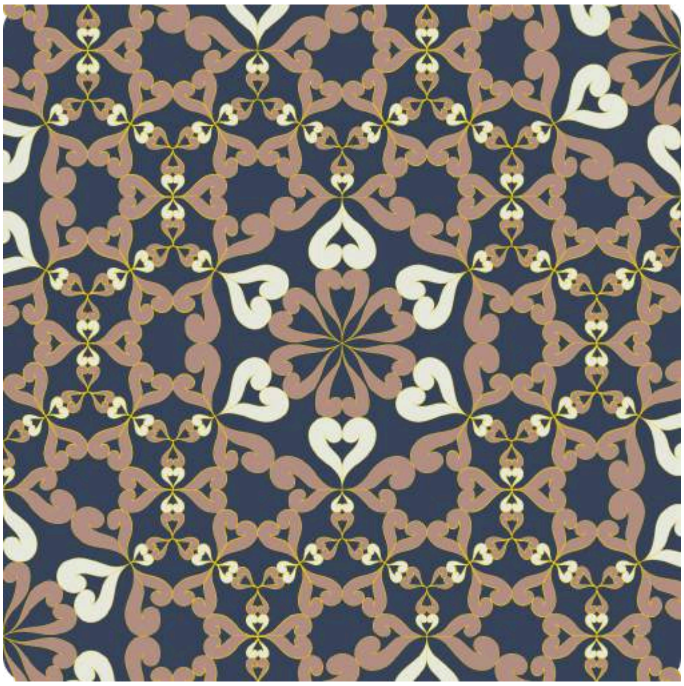


Figure 26. Batik Tumpal arranged in Snub Trihexagonal Tiling. It has vertex configuration 3.3.3.3.6

Figure 26. Batik Tumpal arranged in Elongated Triangular Tiling. It has vertex configuration 3.3.3.4.4

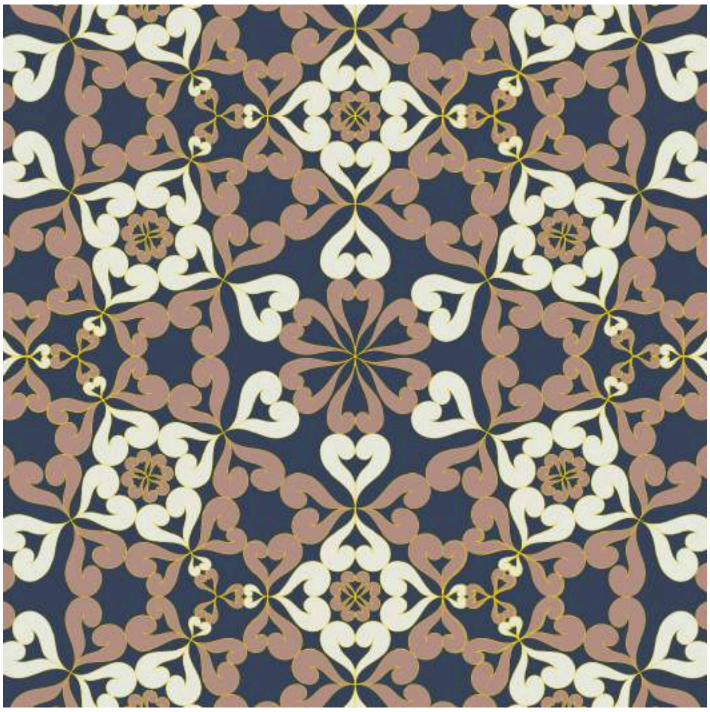


Figure 27. Batik Tumpal arranged in Rhombi Trihexagonal Tiling. Rhombitrihexagonal tiling is widely used in art. It is featured in the Temple of Diana in Nimes, France and in the Roman floor mosaic in Castel di Guido. It has vertex configuration 3.4.6.4

Figure 28. Batik Tumpal arranged in Truncated Hexagonal Tiling. It has vertex configuration 3.12.12

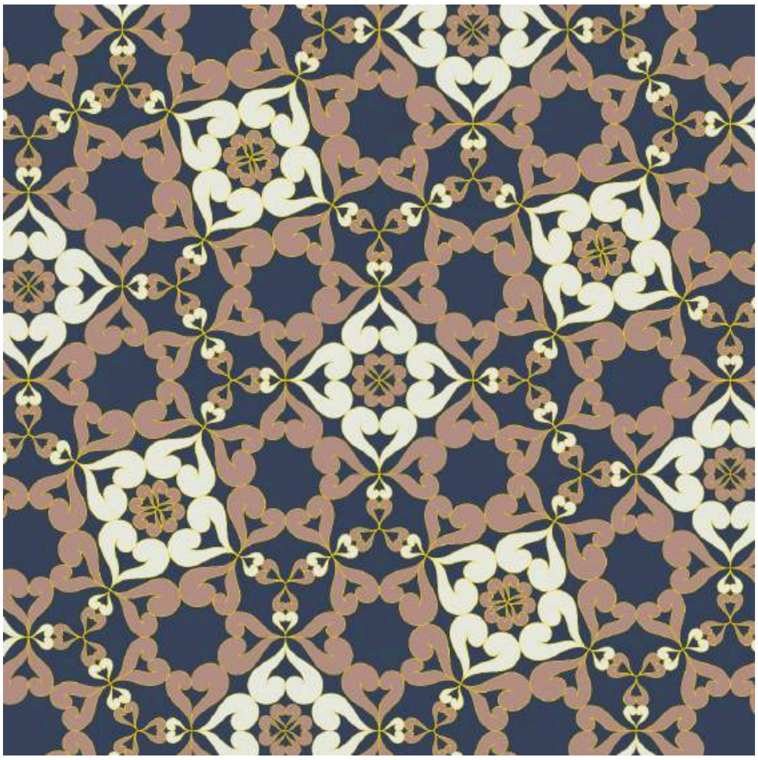


Figure 29. Batik Tumpal arranged in Snub Square Tiling. It has vertex configuration 3.3.4.3.4

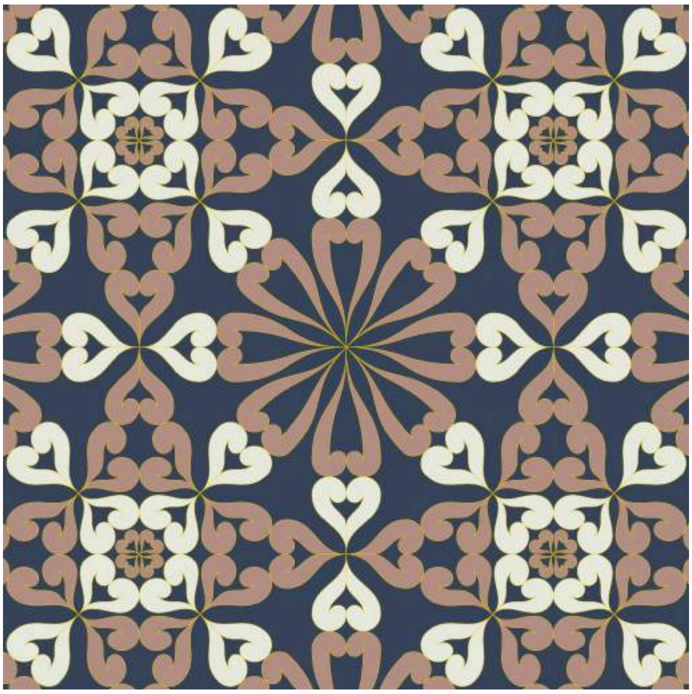


Figure 30. Batik Tumpal arranged in Truncated Square Tiling. It has vertex configuration 4.8.8

2 - Uniform Tiling

Using combinations of uniform tiling we have discussed in previous text, we can create more intricate patterns. Grünbaum and Shephard enumerated the full list of 20 2-uniform tilings in Tilings and patterns, 1987. Below are some examples of those tilings.

Figure 31. Batik Tumpal arranged in combination of 3.3.3.3.6 and 3.3.6.6 tessellation

Figure 32. Batik Tumpal arranged in combination of 3.3.3.3.3.3 and 3.3.6.6 tessellation

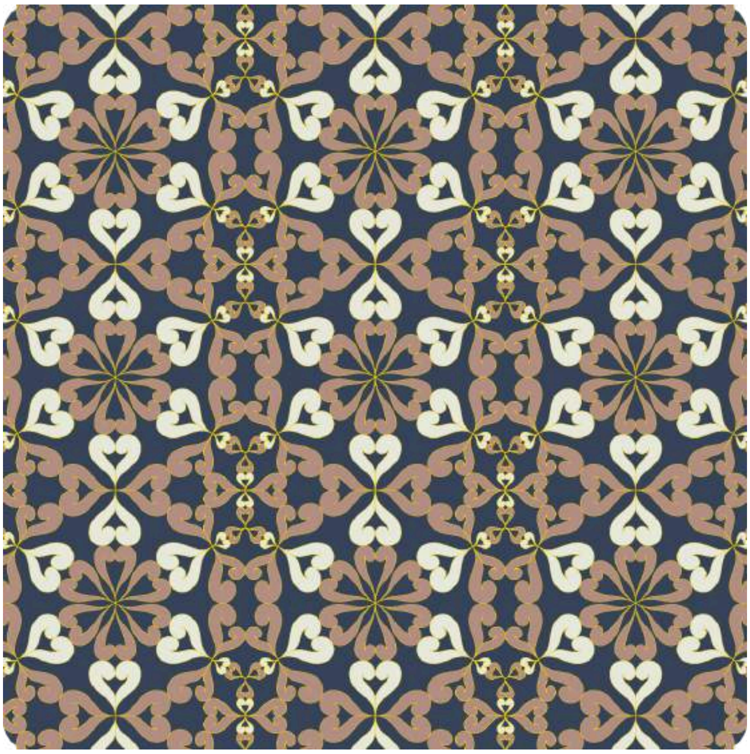


Figure 33. Batik Tumpal arranged in combination of 3.6.3.6 and 3.3.6.6

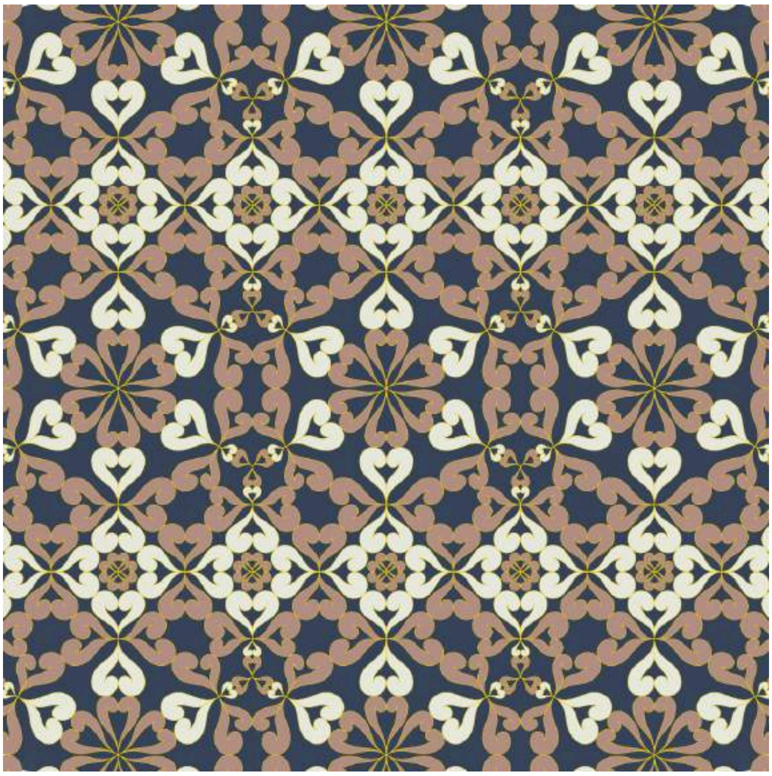


Figure 34. Batik Tumpal arranged in combination of 3.6.3.6 and 3.4.4.6

Application

Using the knowledge we learn in this chapter, we can apply the design to create clothes or jewellery. Figure 31 illustrates the design application of Batik Tumpal if it is made into a ring.

Figure 31. Batik Tumpal Ring

Chapter 3: The Hugging Curves

Bezier Batik Kawung

“The pure mathematician, like a musician, is a free creator of his world of ordered beauty” – Bertrand Russel, British philosopher

In the first chapter, we defined Kawung as a shape made by intersecting four circles. Now we are going to define a more generalised formula, which we can use to create any number of leaves Kawung. In order to do so, we need to understand the concept of Bezier Curve. Bezier curve is a parametric curve used in computer graphics and related fields. Using the Bezier curve, by using sets of control points we can define any path shape we want, including Batik Kawung.

To describe a shape using Bezier Curve, we need at least 3 points, start, mid, and end point. To make a Kawung leaf, we can make two Bezier curves. For the first curve, we start at the **origin**, the **midpoint between corner 1 and 2**, and **corner 2**. For the second curve, we start at corner 2, midpoint between corner 2 and corner 3, and then back to origin. See figure 32 for the illustration. We describe the leaf as a bezier curve with the first curve at (0,0, (0,-100), and (100,-100), and then the 2nd curve at (100,-100), (100,0), to (0,0). We then rotated it four times at 90 degrees to make the Kawung shape.

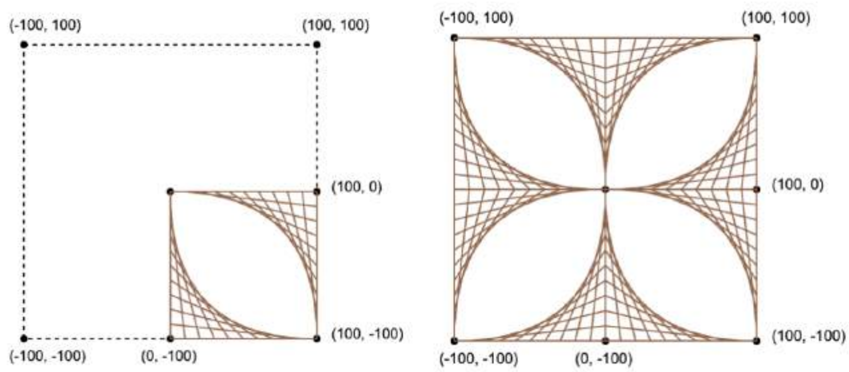


Figure 32. Four leaves Batik Kawung was created using the Bezier Curve.

The benefits of using Bezier Curve to define Batik Kawung is that the formula is applicable for any n number of leaves. As long as we know the origin, the midpoint between vertices, and the vertices coordinated we can create the kawung leaf. Figure 33 illustrates how to create three leaves kawung using the Bezier curve.

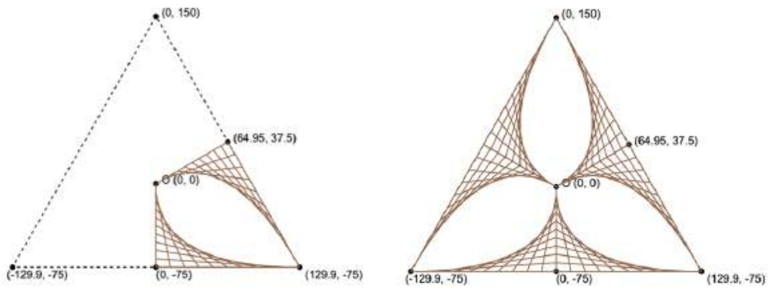


Figure 32. Three leaves Batik Kawung was created using the Bezier Curve.

Geometry

The general formula to create any n - number of Kawung can be described as follows: Let assume we have a circle at origin (0,0) with radius r. The formula to get the vertices and the midpoint are:

$$P_i(r) = r * \frac{2\pi}{n} * i$$

$$x_i(r) = r * cos(\frac{2\pi}{n} * i)$$

$$y_i(r) = r * sin(\frac{2\pi}{n} * i)$$

$$x_{mp} = \frac{x_{n-1} + x_n}{2};\ y_{mp} = \frac{y_{n-1} + y_n}{2}$$

Manual Drawing Steps

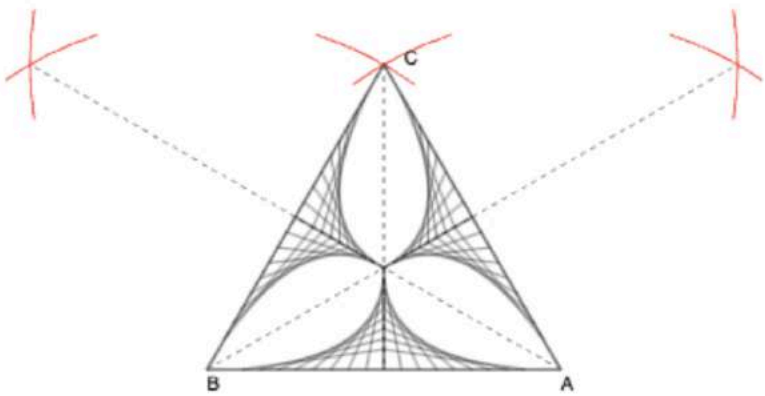


Figure 33. Step by Step guide on how to draw three leaves Kawung

Here is a step - by-step guide on how to draw three leaves Kawung. Given a triangle ABC, draw an angular bisector of angle BAC and draw a line. Next, draw an angular bisector of ACB, and draw a line. Third, draw an angular bisector of angle CBA and draw a line. The intersection of the line is the centre of the triangle. Now we have located the centre of the triangle, the next step is to draw a bezier curve using the **origin , midpoint, and vertex**.

We can draw a Bezier Curve using a series of lines. Let's say we want to create a Kawung Leaf from the origin to vertex A.

First, we located midpoint AB.

Second, we create a line from centre to midpoint AB. Next, we divide it into 10 segments with equal length.

Third, we create a line from midpoint AB to vertex A. Next, we divide it into ten segments with equal length.

Lastly, we connect the point from the second step to the points we create in the third step, as illustrated in Figure 33.

EXERCISE:

1. Draw four leaves kawung using Bezier Curve. You can draw a line between (1,0) to (10,1). Next, draw a line between point (2,0) and (10,2). Third, draw a line between (3,0) and (10,3). Continue this step until the end of the line. Draw a line between point (0,1) and (10,1), between point (0,2) and (10,2), and so on until the end of line. Voila, a kawung leaf will emerge from just a series of lines.

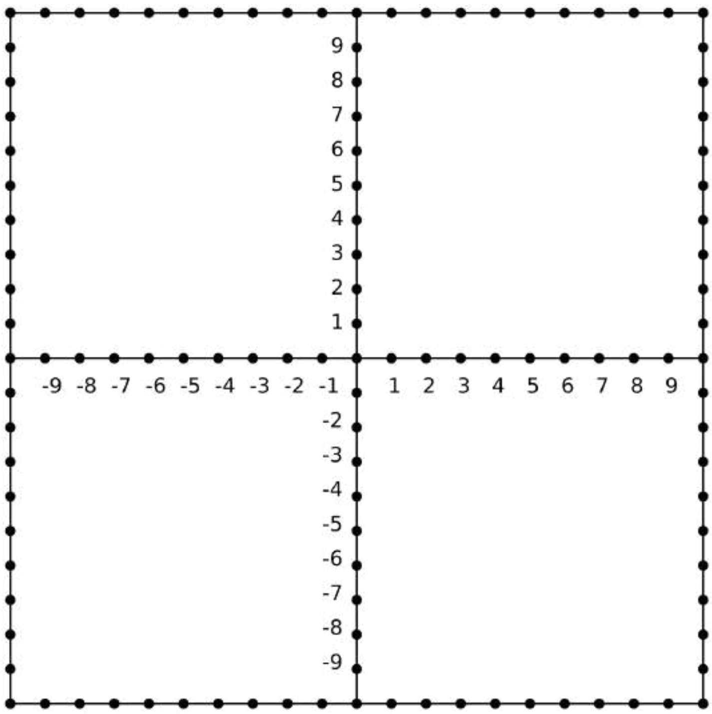


Figure 34. Four Leaves Bezier Kawung Drawing Exercise

2. Draw three leaves Kawung. Can you figure out how to connect the dots to make the leaves shapes appear?

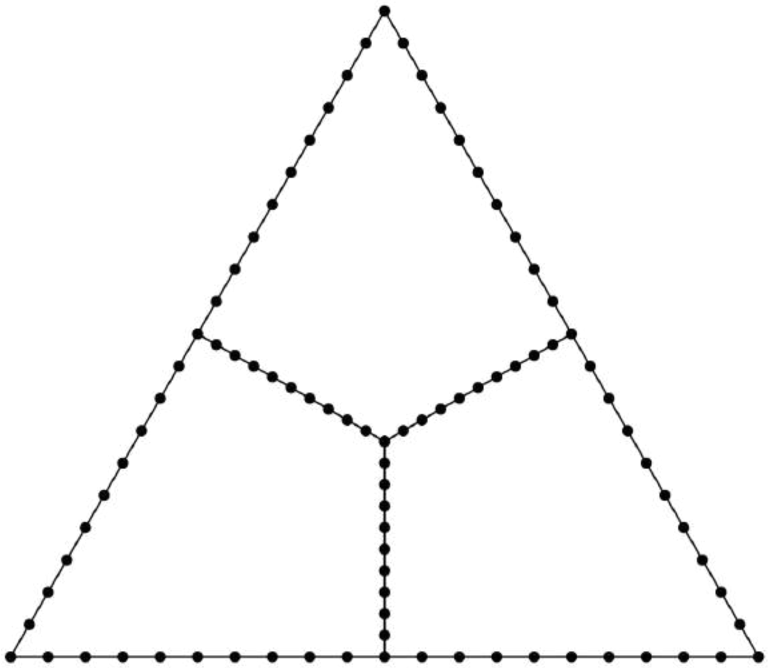


Figure 35. Three Leaves Bezier Kawung Drawing Exercise

Digital Drawing Steps

To draw n-leaf bezier curve Kawung, we can use path element that start with the origin, draw quadratic Bezier curve using midpoint between vertices as control point, and vertex as the end point as shown in the code below:

```
<g transform="translate(x_origin,y_origin)">
    <path d="M 0 0 Q x_mp1 y_mp1, x1 y1, Q x_mp2 y_mp2 0 0"/>
    <path d="M 0 0 Q x_mp1 y_mp1, x1 y1, Q x_mp2 y_mp2 0 0"
transform="rotate(90)"/>
    <path d="M 0 0 Q x_mp1 y_mp1, x1 y1, Q x_mp2 y_mp2 0 0"
transform="rotate(180)"/>
    <path d="M 0 0 Q x_mp1 y_mp1, x1 y1, Q x_mp2 y_mp2 0 0"
transform="rotate(270)"/>
</g>
```

Arbitrary Shape Kawung

Kawung doesn't have to be constructed using vertices from regular polygons. Using the same principle, we can also create Kawung from any arbitrary shape. First, we determine the location of the center of a shape by averaging every x and y coordinate. Second, we create a bezier curve from the center to one of the vertices through one of the midpoints.

Let's say we have an arbitrary pentagon with vertices at (0,0), (97,0), (114, 70), (70, 114), and (0,97). To get the coordinate of the center we can use the average of the vertices.

$$x_c = \frac{x_1 + x_2 + ... + x_{n-1} + x_n}{n} ; y_c = \frac{y_1 + y_2 + ... + y_{n-1} + y_n}{n}$$

After that, we can get the midpoint between vertices to define the Bezier curve. If we repeat that for all vertices, we can get the same Kawung shape as shown in Figure 36.

$$x_{mp} = \frac{x_{n-1} + x_n}{2}; \ y_{mp} = \frac{y_{n-1} + y_n}{2}$$

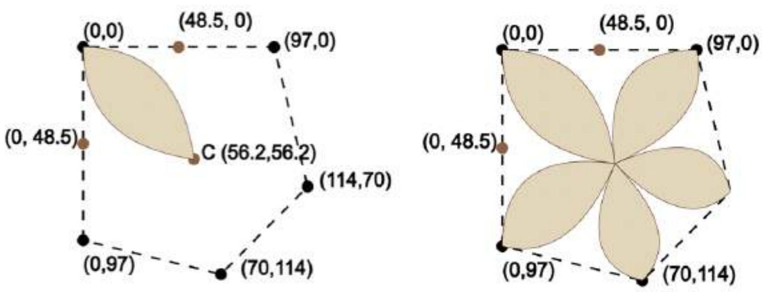


Figure 36. Arbitrary Polygon Kawung Shape

Semi - Regular Tessellation

Using the concept we have discussed, we can create tessellation of kawung using regular tiling as we did with Batik Tumpal in chapter 2.

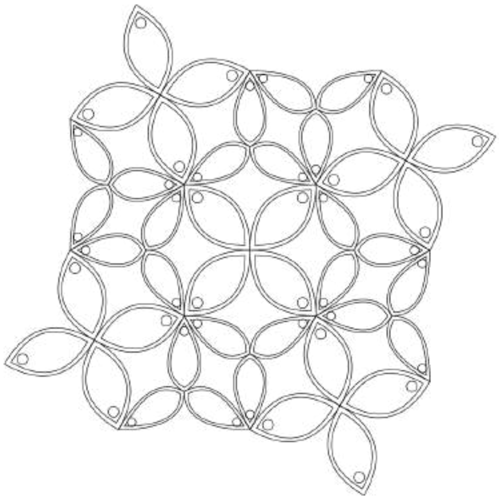


Figure 36. Batik Kawung arranged in Snub Square Tiling. It has vertex configuration 3.3.4.3.4

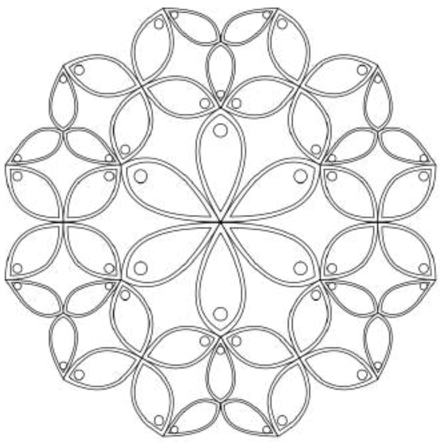


Figure 37. Batik Kawung arranged in Rhombi Trihexagonal Tiling. It has vertex configuration 3.4.6.4

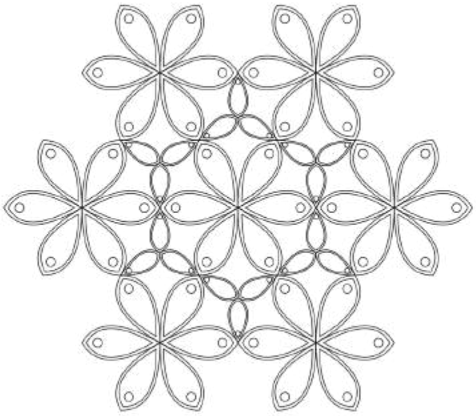


Figure 38. Batik Kawung is arranged in Trihexagonal tiling. It has vertex configuration 3.6.3.6

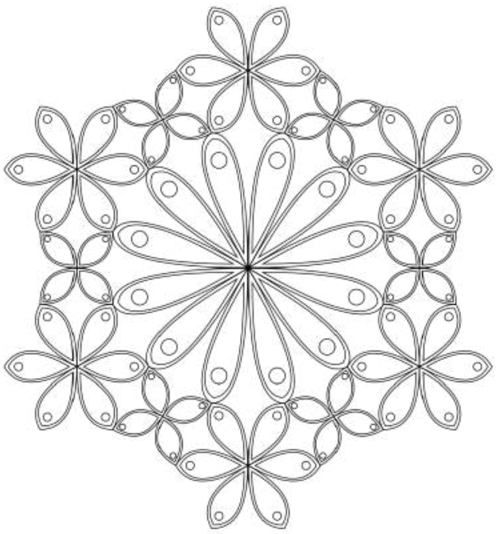


Figure 38. Batik Kawung is arranged in truncated trihexagonal tiling. It has vertex configuration 4.6.12

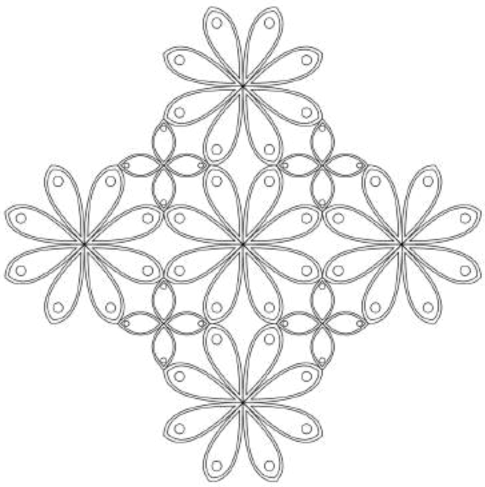


Figure 39. Batik Kawung arranged in Truncated Square Tiling. It has vertex configuration 4.8.8

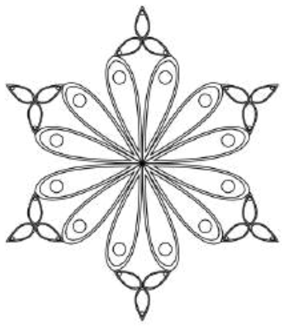


Figure 40. Batik Kawung arranged in Truncated Hexagonal Tiling. It has vertex configuration 3.12.12

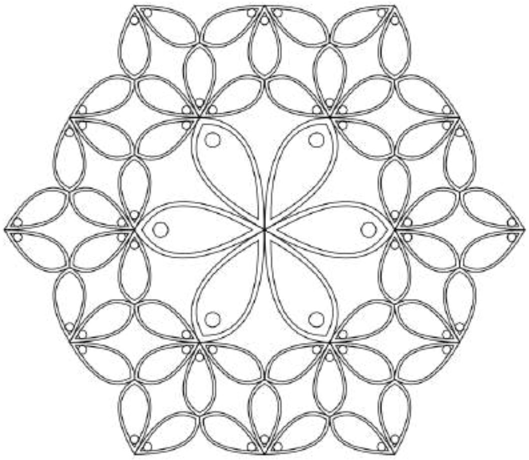


Figure 41. Batik Kawung arranged in Snub Trihexagonal Tiling. It has vertex configuration 3.3.3.3.6

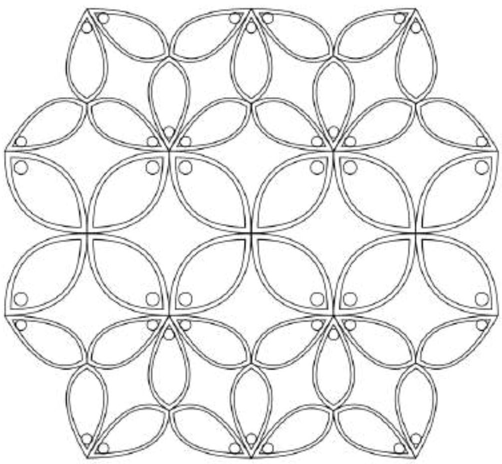
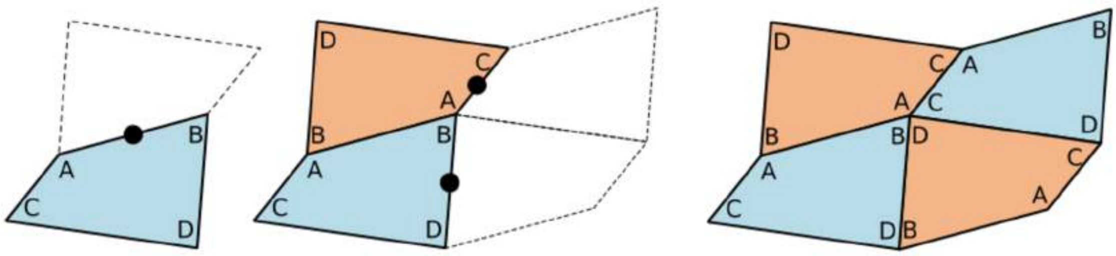


Figure 42.. Batik Kawung arranged in Elongated Triangular Tiling. It has vertex configuration 3.3.3.4.4

Tessellation Conjecture

There are few conjectures about tessellations that are interesting to ponder. The first one is that squares, rectangles, parallelogram, trapezoid, kites, any arbitrary quadrilateral tessellate plane. This is because of two key properties of quadrilaterals.

- The sum of interior angles of a quadrilateral are 360 degrees.
- Rotating by 180˚ about the midpoint of one of its sides, and then repeatedly using the midpoints of other sides will make that quadrilateral tessellate.



Hyperbolic Tessellation

So far we have shown pattern and tessellation in Euclidean geometry. Now we are going to explore a completely different geometry, hyperbolic geometry. The work presented here is inspired by the Dutch great artist, M.C Escher, mainly his work on Circle Limit. You can browse the web to see his work. To understand his work more solidly, we need to understand different types of plane, hyperbolic planes.

The key differences between Euclidean, hyperbolic, and Elliptic plane can be found in the sum of all interior angle of triangles:

- In the Euclidean plane, the sum of all interior angles is equal to 180 degrees.
- In the Elliptic plane, the sum of all interior angles is greater than 180 degrees.
- In the Hyperbolic plane, the sum of all interior angles is less than 180 degrees

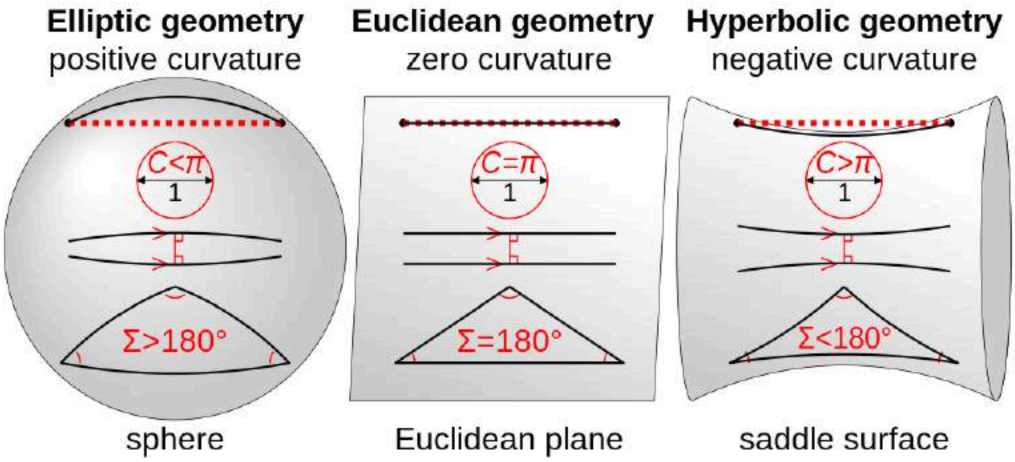


Figure XX. Difference between Elliptic, Euclidean, and Hyperbolic geometry. Wikipedia

We are going to create tessellation in the hyperbolic plane. But the question is, how do we know what type of n - gon will tessellate? At each vertex, how many k polygons meet each other? Which {*n*,*k*} tessellation will work? To answer this question, we can use the following formula:

$$If\ \frac{1}{n} + \frac{1}{k} = \frac{1}{2},\ tessellation\ in\ Euclidean\ plane$$

$$If\ \frac{1}{n} + \frac{1}{k} < \frac{1}{2},\ tessellation\ in\ hyperbolic\ \ plane$$

$$If\ \frac{1}{n} + \frac{1}{k} > \frac{1}{2},\ tessellation\ in\ hyperbolic\ \ plane$$

Proof

In Euclidean plane, we know from chapter 2 that sum of interior angle of a regular polygon can be described as:

$$Sum\ of\ Interior\ Angle\ =\ (n - 2)\ 180^{\circ}$$

For a tessellation {*n*,*k*}, there are *k* regular polygons at each vertex. So the angle at each vertex is 360˚/*k*. Since a regular *n*-gon has *n* equal angles, each being 360˚/*k*, therefore the angle sum is *n*360˚/*k*.

$$Sum\ of\ Interior\ Angle\ =\ n\ \frac{360^{\circ}}{k}$$

If we substitute first equation to second equation we get this formula:

$$(n - 2)\ 180^{\circ} = n\ \frac{360^{\circ}}{k}$$

$$(n - 2)\frac{180^{\circ}}{360^{\circ}} = \frac{n}{k}$$

$$(n - 2)\frac{1}{2} = \frac{n}{k}$$

$$\frac{n}{2} - 1 = \frac{n}{k}$$

$$\frac{1}{2} - \frac{1}{n} = \frac{1}{k}$$

$$\frac{1}{2} = \frac{1}{k} + \frac{1}{n}$$

Exercise:

1. Check whether {6,3} tessellation is on the Euclidean, Elliptic, or Hyperbolic plane!
2. Check whether {5,4} tessellation is on the Euclidean, Elliptic, or Hyperbolic plane!

Tessellation Generation Algorithm

To generate tessellation {p,q} of regular polygon, we can do the following steps:

1. Start with initial vertex (0,0) at the origin
2. For each vertex, in the list of vertices, do both of this thing: a) rotate around origin $\frac{2\pi}{q}$ and b) flip the x to -x and y to -y and do translation with length edge_length around the y coordinate
3. Repeat it over and over again until we generate enough vertices.
4. To get the edges (pair of vertices), we can compare the distance between two vertices. If the distance is equal to edge_length, then that pair of vertices should be connected.
5. Now that we have a list of edges, we now want to know the list of edges that will constitute the polygon. To achieve this, one thing we can do is represent the list of edges as an adjacency list. Then we can run an algorithm to check whether there is a cyclic path that n steps. If there is such a step, then that is the list of edges that we need.

Let's say we want to translate z as much as t. Translation formula that works in the Euclidean, Elliptic, and Hyperbolic geometry can be defined as:

$$\frac{(z+t)}{1 - curvature * conjugate(t) * z}$$

where curvature = 0 for Euclidean, -1 for hyperbolic, and 1 for elliptical plane. To get distance, we can use the following formula:

- Planar distance for euclidean
- 2 atan(chord distance) for spherical

- 2 atanh(chord distance) for hyperbolic

Now that we already have the dataset of the polygons and its corresponding coordinates, we are now ready to create tessellations. The hyperbolic plane can not be metrically represented in the Euclidean plane, but Poincaré described ways that it can be conformally represented in the Euclidean plane. One of those is to represent the hyperbolic plane as the points inside a disk.

Poincaré disk model, also called the conformal disk model, is a model of 2-dimensional hyperbolic geometry in which all points are inside the unit disk, and straight lines are either circular arcs contained within the disk that are orthogonal to the unit circle or diameters of the unit circle

Below are some examples of Batik Kawung tiling in the Poincare disk.

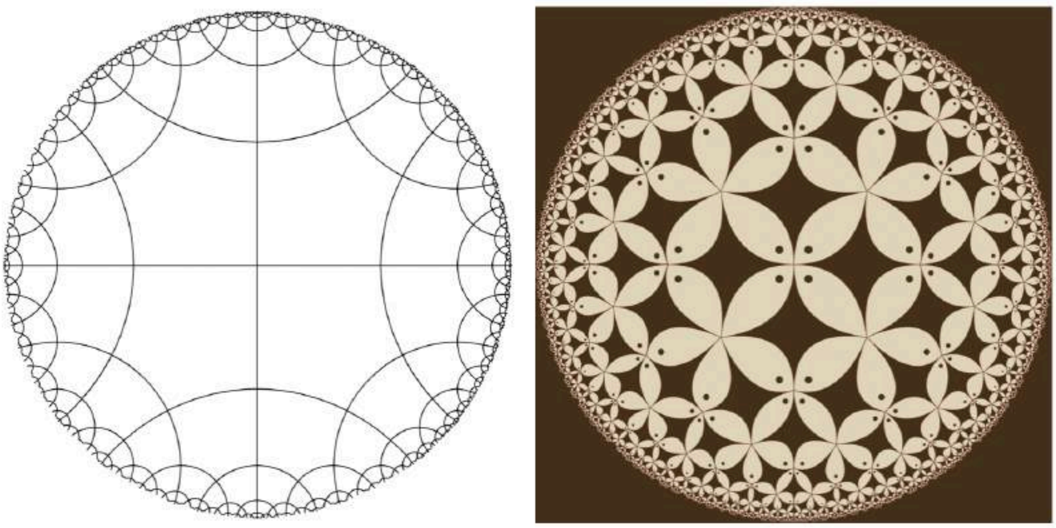


Figure xx. {5, 4} Kawung Tessellation inside Poincare Disk

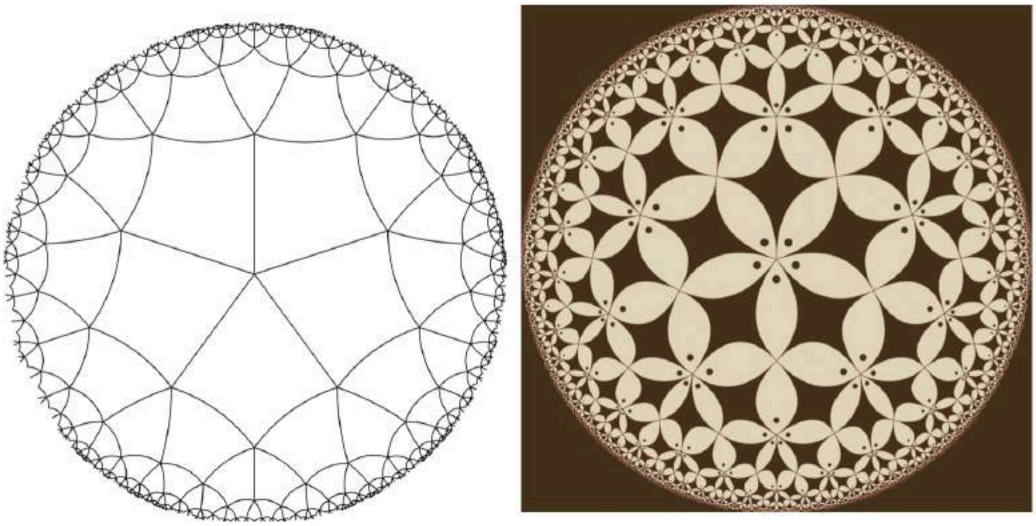


Figure xx {4, 5} Kawung Tessellation inside Poincare Disk

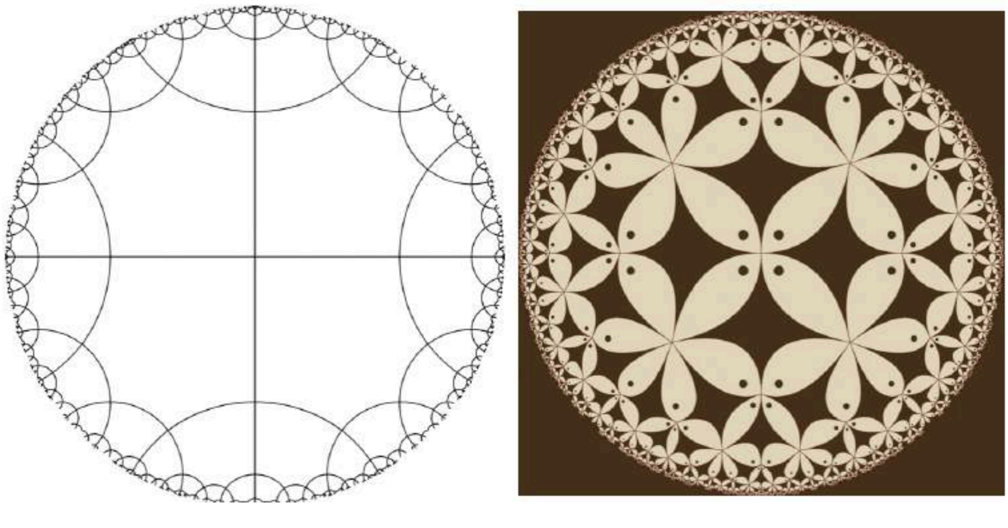


Figure {6, 4} Kawung Tessellation inside Poincare Disk

So far we have made tessellation inside a circular Poincare disk. We can also make hyperbolic tessellation in square borders. Chamberlain Fong in his paper "Analytical Methods for Squaring the Disc", has proposed a method to map from circular disk to square. Below are formula to map circular disk to square:

$$x \; = \; Re\,(\frac{1-i}{-K_e}F(cos^{-1}(\frac{1+i}{\sqrt{2}}\,(u \; + \; vi),\; \frac{1}{\sqrt{2}})) + 1$$

$$y \; = \; Im\,(\frac{1-i}{-K_e}F(cos^{-1}(\frac{1+i}{\sqrt{2}}\,(u \; + \; vi),\; \frac{1}{\sqrt{2}})) - 1$$

$$K_e = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - \frac{1}{2}\sin^2 t}} \approx 1.854$$

Below are some examples of hyperbolic tessellation mapped to squares.

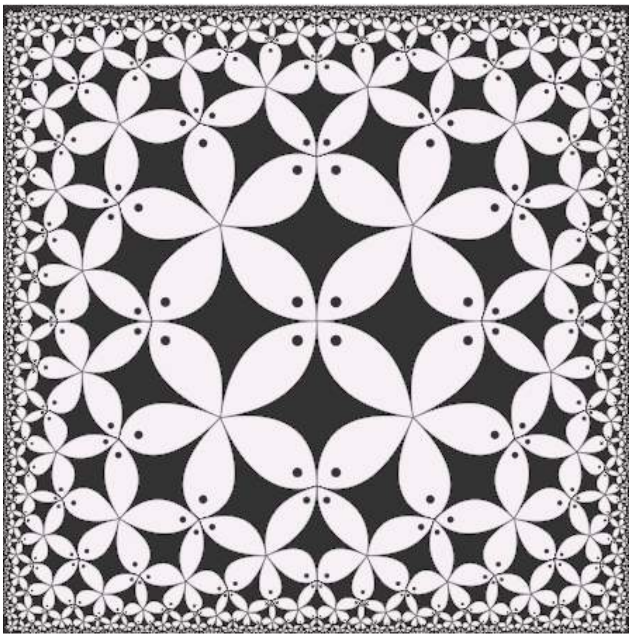


Figure xx {5,4} tessellations mapped from poincare disk to square

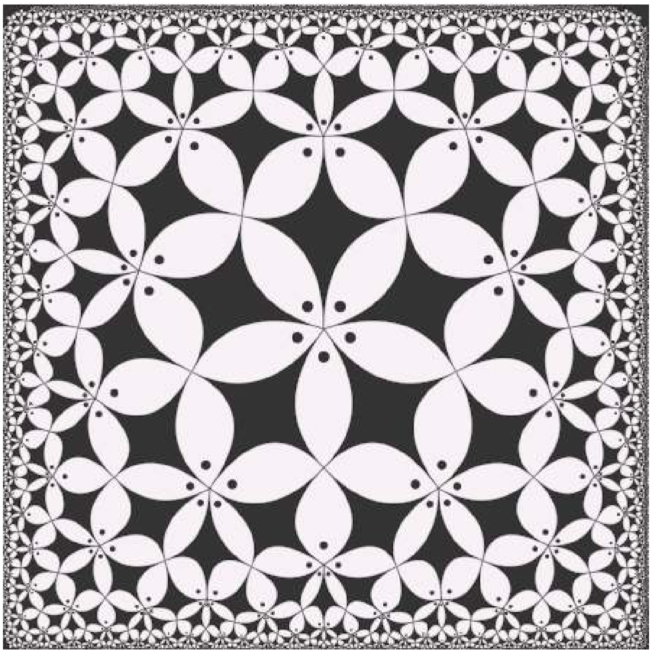


Figure xx {4,5} tessellations mapped from poincare disk to square

Application

References:

https://mathstat.slu.edu/escher/index.php/Tessellations_by_Polygons