Qualifying Exam - Theme 2
Resurgent Transseries

Subject: Mathematical Physics

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Abstract: Physical observables like the energy spectrum and the scattering amplitude are given by a perturbative power series. This kind of series generally diverges for every non-zero value of the coupling constant, although keeping only a few terms provides well agreement with experimental data. The explanation to this issue comes from the theory of the resurgent transseries, which describes non-perturbative expansions generated by asymptotically divergent series. In this work, we will present the formal definition of the asymptotic series; the Borel summation process for asymptotically divergent series and the Stokes phenomena (which originate the transseries). Also, we will exemplify the occurrence of the resurgent transseries by the calculation of the perturbative and non-perturbative energy spectrum of a particle in a double well potential.

Keywords: Resurgent Transseries, Borel Summation, Quartic potential, Stokes Phenomena.

Contents

1 Introduction

The study of transseries is motivated by the fact that the physical observables, which usually appears in Quantum Mechanics and Quantum Field theory such that the energy spectrum and scattering amplitudes, are given by power series of a constant parameter (commonly called "coupling constant"). This kind of series generally diverges for every non-zero value of the coupling constant, although keeping only a few terms provides well agreement with experimental data. For example, consider the electron g-factor g_e which characterizes its magnetic moment, in the framework of relativistic quantum mechanics it is given by [1]:

$$\begin{aligned}\frac{g_e-2}{2} &= \frac{\alpha}{2\pi} - (0.328\,478\,965\ldots)\left(\frac{\alpha}{\pi}\right)^2 + (1.176\,11\ldots)\left(\frac{\alpha}{\pi}\right)^3 - (1.434\ldots)\left(\frac{\alpha}{\pi}\right)^4 + \mathcal{O}\left(\alpha^5\right)\\ &= 0.001\,159\,652\,140\,(5)(4)(27)\,,\end{aligned}\tag{1}$$

where $\alpha = e^2/4\pi \approx 1/137$ is the fine structure constant. The measurement of g-factor yields an experimental value for the above mentioned quantity, which is [2]

$$\left.\frac{g_e-2}{2}\right|_{exp} = 0.001\,159\,652\,181\,28(2)\,,\tag{2}$$

which shows an excellent agreement with the theoretical value, despite the series used in calculation (1) keeps only $\mathcal{O}\left(\alpha^4\right)$. The fact that measurable quantities are described by infinite series in the perturbative approach, is one evidence that there are some non-perturbative information that must be counted. It is precisely this non-perturbative information, when is possible to be encountered, that is given by a resurgent transseries.

Now, you might be questioning: how is it possible to associate a number to a divergent series? The answer to this question was found by Émile Borel in 1899 [3], when he developed the method which is now called *Borel summation*. This process consists in a linear map acting on a power series that produces a new series which has finite non-zero convergence radius. After that, the series can be analytically continued into a complex function, notwithstanding, this function may present branch cuts and singularities [4]. If this function has no singularities, then the deal of associating a number to the series was complete. On the other hand, if the Borel transform have singularities along some direction θ, this direction is known as Stokes line and this is just when the resurgent transseries becomes relevant. The asymptotic behavior of some functions may vary in different regions of the complex plane, bounded by these Stokes lines. This aspect is known today as the Stokes phenomenon in honor to its discoverer Sir G. G. Stokes, which investigated this topic in 1864 [5], hundreds of years before the developments of the transseries theory. Roughly speaking, the Stokes phenomena is the the fact that the nonperturbative content in the transseries, which is exponentially suppressed when compared to the perturbative part, grow into the dominant term when the Stokes line in the complex plane is reached.

The origin of the transseries remounts various parts of mathematics like the model theory, computer algebra and surreal numbers [6]. The reason of the complete series, including the non-perturbative sector, being called transseries is that it transcends the usual power series format of perturbation theories. Usually, the transseries posses terms proportional to the exponential and logarithm of the coupling constant. As an illustration, let us consider the transseries for the energy spectrum of a quantum mechanics system, which has the form

$$\begin{aligned}E^{(N)}(g) =& \sum_{k=0}^{\infty} g^k E_k^{(N)} + \frac{1}{g^{N+1/2}} e^{-\frac{S}{g}}\left(\varepsilon_N^{(0)} + \varepsilon_N^{(1)} g + \ldots\right) + \\ &+ \left(\frac{1}{g^{N+1/2}}\right)^2 e^{-\frac{2S}{g}} \ln\left(\pm\frac{1}{g}\right)\left(\xi_N^{(0)} + \xi_N^{(1)} g + \ldots\right) + \ldots\,.\end{aligned}\tag{3}$$

Here, S, $E_k^{(N)}$, $\varepsilon_N^{(0)}$, $\varepsilon_N^{(1)}$ …, $\xi_N^{(0)}$, $\xi_N^{(1)}$ … are parameters which can be determined in the solution of the quantum mechanics problem in question, g is the coupling constant and the first term is the usual perturbative series. On the other hand, the name *resurgent* comes from J. Écale [7], which said that some functions "resurrect" or surge up from their singularities. In fact, the complete non-perturbative information of a given observable, can be extracted from the large order behavior of the perturbative divergent series [4]. Hence, to find the "resurgence" of the transseries, one needs only to know the asymptotic behavior of the perturbative computation of an observable in terms of some coupling constant. This concept also make sense in the context of the instanton[1] analysis [4].

We will present in Section 2 the theoretical foundations, which consists in the formal definition of the asymptotic series; the Borel summation process for asymptotically divergent series and the Stokes phenomena (which originate the transseries). In Section 3 will exemplify the occurrence of the resurgent transseries by the calculation of the perturbative and non-perturbative energy spectrum for the quantum mechanics of a particle in a double well potential following the Uniform WKB method [8]. Finally, we will discuss the relation between the perturbative and non perturbative physics.

[1]Instantons are solutions of the equations of motions in classical field theory, on a Euclidean spacetime. They also appear in the path integral formulation of the quantum field theory as the leading quantum corrections to the classical solutions.

2 Asymptotic series

2.1 Definition

In this section we will present the Poincaré definition of a formal asymptotic series [9] and exemplify this concept by a simple case from mathematical physics. Given a complex valued function $f(z)$, with an power series of the form:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots , \tag{4}$$

denoting as S_n the n-th partial sum, for a fixed n, in the limit $|z| \to 0$ which a established direction ϕ, we should have:

$$\boxed{\lim_{|z| \to 0} |z|^{-n}[f(z) - S_n(z)] = 0 , \quad (n \text{ fixed}) .} \tag{5}$$

This means that, for big values of $|z|$ with a fixed phase $e^{i\phi}$ and a given n, the partial sum express the function in the sense that S_n approach $f(z)$ as well as much as required (the rest goes to zero like $|z|^{n+1}$) in the direction of the angle ϕ. If the function does not admit a series expansion (i. e., if S_n doesn't converge to $f(z)$ as $n \to \infty$), then, we have:

$$\boxed{\lim_{n \to \infty} |z|^{-n}[f(z) - S_n(z)] = \infty , \quad (|\mathrm{z}| \text{ fixed}) .} \tag{6}$$

If a expansion obey the two prerequisites (5) and (6), hence, this series is called an asymptotic series. Generally, in the cases of physical interest the coefficients of the expansion grows factorially:

$$a_n \sim n! . \tag{7}$$

It is interest to note that is possible to find an optimal n, which is usually denoted by N^*, such that the asymptotic series provides the greatest approximation to $f(z)$, starting to move away for $n > N^*$. This phenomenon is called *optimal truncation* and naturally, as soon as $|z|$ increases, $N^* \to \infty$. Thereafter, we can write

$$f(z) \approx \sum_{n=0}^{N^*} a_n z^n . \tag{8}$$

Here, the symbol '≈' denotes that the series does not converge to the function, moreover, the relation between the expansion and the function obey the aforementioned properties. Usually, for the asymptotic expansion it is valid that $|a_n z^n| \leq |a_{n+1} z^{n+1}|$ at the moment of the optimal truncation.

As an example, let us consider the Stieltjes Integral, defined for small values of x by:

$$T(x) = \int_0^{+\infty} \frac{e^{-t}}{1 + xt} dt . \tag{9}$$

It's power series can be obtained considering the sum of the geometric series, such that,

$$\begin{aligned} T(x) &= \int_0^{+\infty} e^{-t} \sum_{n=0}^{\infty} (-1)^n x^n t^n dt \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \Gamma(n+1) , \quad |x| < 1 , \end{aligned} \tag{10}$$

where $\Gamma(n+1) = n!$ is the Gamma function. There are two remarkable points in this example: the first is that the procedure above is formally incorrect, since the summation of the geometric series require that $|xt| < 1$, while the integral was carried out from 0 to $+\infty$ in the variable t. Hence, the function $T(x)$ does not admits a power series expansion. The second comment is that the coefficients of the expansion grows factorially, so the series have zero radius of convergence for any value of x. In spite of all this, the numerical values of $T(x)$ can be calculated by the series since only a few terms are kept. Therefore, the series in (10) is in deed an asymptotic series accordingly to the Poincaré formal definition presented above. As we mentioned, to find the optimal truncation is necessary that the next term of the sum be greater than the previous:

$$n!\,|x^n| \le (n+1)!\,\left|x^{n+1}\right| \,, \tag{11}$$

or

$$|x|^{-1} \le n+1\,. \tag{12}$$

Then, in this case the optimal truncation occurs at

$$\boxed{n = |x|^{-1} - 1 \equiv N^*\,,} \tag{13}$$

for example, for $x = 1/10$ which implies $N^* = 9$, the exact integral (9) gives $T(0.1) = 0.91563$ and the partial sum result $S_9(0.1) = 0.91545$. The precision can be enhanced considering smaller values of x.

Truncating the asymptotic series with represents the Stieltjes Integral until the order of N^* we have an exponentially small error. To proof this, let us consider that, around the optimal truncation point, the rest of the sum is at least of the order of the N*-th term:

$$|R_{N^*}(x)| \approx N^*!x^{N^*} \approx N^*!(N^*)^{-N^*} \approx \sqrt{N^*}e^{-N^*}\,. \tag{14}$$

To the final approximation we have used the Stirling expansion, which is the leading order terms from the asymptotic expansion of the Gamma function. In conclusion, replacing $x^{-1} \sim N^*$, we have

$$|R_{N^*}(x)| \approx \frac{e^{-1/x}}{\sqrt{x}}\,, \tag{15}$$

this means that the error in the calculation the function $T(x)$ by its truncated asymptotic series is exponentially small, just as we wanted to demonstrate.

2.2 Borel Transform

As we mentioned in the introduction, the proposes of the Borel transform is to associate a number to a asymptotic divergent series. Moreover, it is possible to find a function which relates the perturbative series of some observable $f \equiv \langle F \rangle$ to a number, for each value of the coupling constant. Starting from the perturbative asymptotic series

$$f\,(g) = \sum_{n=0}^{\infty} c_n g^{n+1}\,, \tag{16}$$

where g is the coupling constant and the coefficients grow as $c_n \sim n!$, following [4] we define the Borel transform as a linear map acting on the formal power series such that

$$\mathcal{B}[f](k) = \sum_{n=0}^{\infty} \frac{c_n}{n!}k^n\,. \tag{17}$$

This new series $\mathcal{B}[f](t)$ typically posses a finite radius of convergence for $k \in \mathbb{C}$. The Borel summation is the analytical continuation of the new series obtained by the Borel transform. Since this function will be defined on the complex plane, is necessary to chose some direction θ along which $\mathcal{B}[f](t)$ has no singularities, then the Borel summation is defined as

$$\mathcal{S}_\theta f(g) = \int_0^{e^{i\theta}\infty} \mathcal{B}[f](k)e^{-k/g}dk\,. \tag{18}$$

If the original series in equation (16) is an asymptotic series, then $\mathcal{B}\,[f]\,(k)$ will have singularities. Furthermore, calling θ the direction in which the Borel summation present these singularities, then, the Borel summation will not be well defined in this direction. This direction defines the Stokes line and the singularities generates ambiguity, when different sectors of the complex plane are considered. To be specific, we need to define lateral Borel summation which avoid the singularity by the right and the left. In this case, two different contours of integration must be used and in general each contour result in a different value for the summation. Moreover, the lateral Borel summations of our original asymptotic series are connected via the so-called Stokes automorphism,

$$\mathcal{S}_{\theta^+} = \mathcal{S}_{\theta^-} \circ \mathfrak{S}_\theta \,, \tag{19}$$

which is an operator. Actually, the Stokes automorphism $\mathfrak{S}_\theta$ is related to the *alien derivative*, which appears in alien calculus, but we will not present a detailed discussion about this topic here. For a formal definition of this operator and a review about alien calculus in the context of transseries, see [4], specially the appendix A. For now, it is important to know that the Stokes automorphism describe the Stokes phenomenon: starting from a transseries which includes the original perturbative series (16), when the Stokes line is crossed, identified by the multi-values of the lateral Borel summations, then a series of exponentially suppressed terms will appear. These nonperturbative content may eventually grow and take over the dominance of the transseries, depending in which sector of the coupling constant complex plane we are interested.

To exemplify the Borel summation, let us consider the inverse process of we have done in equation (10) i.e. starting from the divergent series, we will recover the function of Stieltjes in equation (9) via Borel summation. Defining the function

$$f(x) = \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n n! x^{n+1} \,, \tag{20}$$

Its Borel transform will be:

$$\mathcal{B}[f](k) = \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n k^n \,. \tag{21}$$

The Borel summation $\mathcal{S}f(g)$ along the direction $\theta = 0$ is given by

$$\mathcal{S}f(x) = \frac{1}{x}\int_0^{+\infty}\sum_{n=0}^{\infty}(-1)^n k^n e^{-k/x} dk = \int_0^{\infty}\frac{e^{-u}}{1+xu}du = T(x) \,, \tag{22}$$

where we did the change of variables $u \to k/x$ and we have used again the sum of the geometric series. Moreover, let's see graphically the comparison between the partial sums $S_n(x)$ calculated using $f(x)$ in (20) and the exact function $T(x)$:

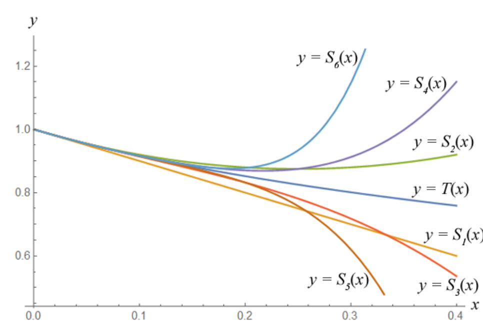

Figure 1: Graphic of the Stieltjes Integral $T(x)$ and the partial sums $y = S_n(x)$ for $n = 1, 2, 3, 4, 5, 6$.

In Figure 1 we can observe that the partial sums coincide well with the exact function for small values of x, with play the role of the coupling constant in this example. Further more, adding more terms the partial sums starts to put away from the exact value, as expected since there is an optimal truncation. The above procedure was applied to a series with zero radius of convergence, but also works for a series with finite and non null radius of convergence.

With the purpose of illustrate the necessity of lateral Borel summations, it is interesting to consider the Borel summation of the series in equation (16) but without the alternating signal

$$f(g) = \sum_{n=0}^{\infty} n! g^n \,. \tag{23}$$

Applying the same steps aforementioned, one can find that

$$\mathcal{S}_\theta f(g) = \int_0^{e^{i\theta}\infty} \frac{e^{-t}}{1-gt} dt\,. \tag{24}$$

Considering $f(g)$ as a function of the complex variable g, follows that this function posses a brunch in the real positive axis (direction $\theta = 0$). This can be seen in the integral of equation (24), for the variable t is evaluated from 0 to $+\infty$, therefore, any value of the real and positive part of g will be generate a divergence. By the other hand, if we look to the function $\frac{e^{-t}}{1-gt}$, depending on the complex variable t and for a given g fixed, this function presents only a simple pole in $t = \frac{1}{g}$. This poles characterizes the Stokes line and the Borel sum is not well defined along $\theta = 0$, it is necessary to consider lateral Borel summations. Calculating only the imaginary part of the integral (24), using the contour of integration γ' in Figure 2, in the limit of $\epsilon \to 0$, we have:

$$\mathrm{Im}[\mathcal{S}_{\theta^+} f] = \int_\pi^0 \frac{e^{-(\frac{1}{g}+\epsilon e^{i\phi})} i\epsilon e^{i\phi} d\phi}{1 - g(\frac{1}{g} + \epsilon e^{i\phi})} = \frac{i\pi}{g} e^{-1/g}\,. \tag{25}$$

If the pole is skirted under, then, the integral results in a negative imaginary part

$$\mathrm{Im}[\mathcal{S}_{\theta^-} f] = -\frac{i\pi}{g} e^{-1/g}\,, \tag{26}$$

as we expected, the values of the lateral Borel summations is different but related.

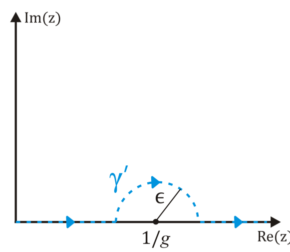

Figure 2: Contour of integration with avoid the pole by above.

Despite we have avoided the pole using the contour γ' shown in Figure 2, the same result must be found if we chose depart from the branch by a small angle θ, as in the contours C^+ and C^- depicted in Figure 3. The name "lateral Borel summation" makes more sense using this kind of contour, although the γ' contour makes the calculation more easy in this case.

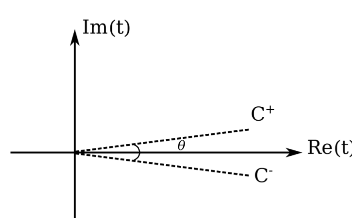

Figure 3: Contours C^+ and C^- for the lateral Borel summations $\mathcal{S}_{\theta^\pm} f(g)$.

In the next section we will find the transseries for the double-well potential to show in a practical and well known example from Quantum Mechanics the theoretical framework presented above. We will see that the non perturbative terms raise naturally from the perturbative series (resurgence) and the transseries posses an multi-valued imaginary part due to the presence of singularities.

3 Transseries for the double well potential

3.1 Perturbative part

In this section we will obtain the perturbative part of the energy spectrum of a particle in a double well potential, given by the following expression:

$$U(x) = x^2(1+gx)^2 \,. \tag{27}$$

We will follow the Uniform WKB method [8] to find the perturbative solution of the Schrödinger's equation, with differs from the standard perturbations theory presented in under-graduation courses. By the change o variables $y = gx$ in the stationary Schrödinger's equation one can find that:

$$-g^4\frac{d^2}{dy^2}\psi(y) + y^2(1+y)^2\psi(y) = g^2E\psi(y) \,, \tag{28}$$

where

$$V(y) = y^2(1+y)^2 \tag{29}$$

will be the mathematical expression for the potential which we will use from now on. The Uniform WKB method starts from an ansatz for the wave function which apparently turns the problem in a more complicated one, because transforms the linear ODE (28) in a non-linear equation. Therefore, the non-linear ODE can be solved iteratively as we will see next. The Uniform WKB ansatz is

$$\psi_\nu(y) = \frac{D_\nu\left(\frac{1}{g}u(y)\right)}{\sqrt{u'(y)}} \,, \tag{30}$$

where D_ν is the parabolic cylinder function [10], which satisfies the differential equation:

$$\frac{d^2}{dz^2}D_\nu(z) + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right)D_\nu(z) = 0 \,, \tag{31}$$

and the boundary conditions

$$\begin{aligned} D_\nu(0) &= \frac{\sqrt{\pi}}{2^{-\frac{\nu}{2}}\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)} \,, \\ D'_\nu(0) &= \frac{\sqrt{\pi}}{2^{-\frac{1}{2}-\frac{\nu}{2}}\Gamma\left(-\frac{\nu}{2}\right)} \,. \end{aligned} \tag{32}$$

The parabolic cylinder function have the following integral representation

$$D_\nu(z) = \sqrt{\frac{2}{\pi}}e^{z^2/4}\int_0^\infty t^\nu e^{-t^2/2}\cos\left(zt - \nu\frac{\pi}{2}\right)dt \,, \quad \mathrm{Re}(\nu) > -1 \,. \tag{33}$$

The function $D_\nu(z)$ also may be write in a power series with a pre-factor $e^{-\frac{z^2}{4}}$, with have a finite radius of convergence in all complex plane:

$$D_\nu(z) = e^{-\frac{z^2}{4}}\left(1 + (-\nu)\frac{z^2}{2!} + (-\nu)(2-\nu)\frac{z^4}{4!} + \dots\right) \,. \tag{34}$$

See in Figure 4 the plot of $D_\nu(x)$ for some values of ν.

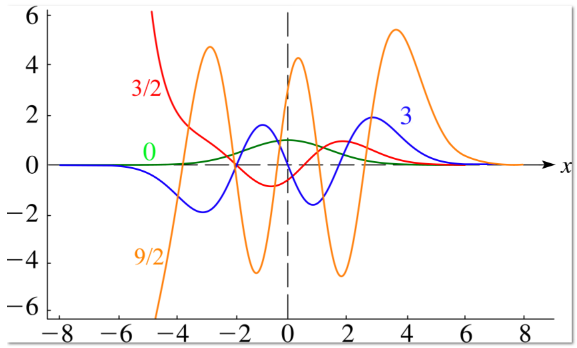

Figure 4: Graphic of the parabolic cylinder function $D_\nu(x)$ for real x and $\nu = 0,\, 3/2,\, 3,\, 9/2$.

Replacing $\psi(y)$ from equation (30) in the Schrödinger's equation, the aforementioned non-linear ODE for $u(y)$ is found:

$$V(y) - \frac{1}{4}u^2(u')^2 - g^2E + g^2\left(\nu + \frac{1}{2}\right)(u')^2 + \frac{g^4}{2}\sqrt{u'}\left(\frac{u''}{(u')^{3/2}}\right)' = 0\,. \tag{35}$$

This equation can be solved using the power series for the energy and the function $u(y)$ in terms of the "coupling constant" g^2

$$E = E_0 + g^2E_1 + g^4E_2 + \ldots \tag{36}$$

$$u(y) = u_0(y) + g^2u_1(y) + g^4u_2(y) + \ldots \tag{37}$$

Inserting this series in (35) and keeping only to zero order in g^2 we find $u_0(y)$, keeping until $\mathcal{O}(g^2)$ the function $u_1(y)$ and E_0 can be found, and so on. Thus, repeating this procedure all the perturbative series can be calculated iteratively, resulting for the energy spectrum until $\mathcal{O}\left(g^{10}\right)$:

$$\begin{aligned}
E(\nu, g^2) = {} & 2\left(\nu + \frac{1}{2}\right) - 2g^2\left[3\left(\nu + \frac{1}{2}\right)^2 + \frac{1}{4}\right] \\
& - 2g^4\left[17\left(\nu + \frac{1}{2}\right)^3 + \frac{19}{4}\left(\nu + \frac{1}{2}\right)\right] \\
& - 2g^6\left[\frac{375}{2}\left(\nu + \frac{1}{2}\right)^4 + \frac{459}{4}\left(\nu + \frac{1}{2}\right)^2 + \frac{131}{32}\right] \\
& - 2g^8\left[\frac{10689}{4}\left(\nu + \frac{1}{2}\right)^5 + \frac{23405}{8}\left(\nu + \frac{1}{2}\right)^3 + \frac{22709}{64}\left(\nu + \frac{1}{2}\right)\right] \\
& - 3g^{10}\left[29183\left(\nu + \frac{1}{2}\right)^6 + 50715\left(\nu + \frac{1}{2}\right)^4 + \frac{217663}{16}\left(\nu + \frac{1}{2}\right)^2 + \frac{10483}{32}\right] + \mathcal{O}(g^{12})\,.
\end{aligned} \tag{38}$$

It in important to note that (38) for $g = 0$ is the expression for the quantum harmonic oscillator $E = \nu + 1/2$ except for a factor of 2, which is present due to our ansatz. Similarly, for the function $u(y)$ which provides the wave function by (30), we have calculated up to order g^{10}, but we will show here only until $\mathcal{O}(g^4)$ because the expression is very cumbersome

$$u(y) = \sqrt{2}y\sqrt{1+\frac{2y}{3}} + g^2\left(\nu+\frac{1}{2}\right)\frac{\ln\left[\left(1+\frac{2y}{3}\right)(1+y)^2\right]}{\sqrt{2}y\sqrt{1+\frac{2y}{3}}} + \frac{g^4}{8\sqrt{6}y^3(1+y)^2(3+2y)^{3/2}}$$
$$\times\left\{y^3\left[80+y(109+38y)\right] + \left(\nu+\frac{1}{2}\right)^2\left(408y^4+1140y^3+768y^2-168y-192\right)\right.$$
$$\left.72(1+y)^2\ln\left[\left(1+\frac{2y}{3}\right)(1+y)^2\right] - 18(1+y)^2\ln^2\left[\left(1+\frac{2y}{3}\right)(1+y)^2\right]\right\} + \mathcal{O}(g^6)\,. \tag{39}$$

To recover the wave function $\psi^{(N)}(x)$ in the well know form from perturbation theory, it is necessary to do $\nu \to N$ (with N an integer positive), change the variable $y \to gx$ and expand the ansatz (30) in a power series of g^2:

$$\begin{aligned}\psi^{(N)}(x) &= \frac{D_N\left(\frac{1}{g}\left[u_0(g\,x)+g^2u_1(g\,x)+g^4u_2(g\,x)+\ldots\right]\right)}{\sqrt{(d/dx)\left[u_0(g\,x)+g^2u_1(g\,x)+g^4u_2(g\,x)+\ldots\right]/g}}\\ &= \frac{D_N(\sqrt{2}x)}{\sqrt{2}} + g^2\psi_1^{(N)}(x) + g^4\psi_2^{(N)}(x) + \ldots\end{aligned} \tag{40}$$

Using the following relation for $D_N(z)$ [10]:

$$D_N(z) = 2^{-N/2}e^{-\frac{z^2}{4}}\mathcal{H}_N\left(\frac{z}{\sqrt{2}}\right)\,,\quad N=0,1,2,\ldots \tag{41}$$

where $\mathcal{H}_N$ is the N-th Hermite polynomial, one can find

$$\psi^{(N)}(x) = \frac{2^{-N/2}}{\sqrt{2}}e^{-\frac{x^2}{2}}\mathcal{H}_N(x) + g^2\psi_1^{(N)}(x) + g^4\psi_2^{(N)}(x) + \ldots \tag{42}$$

The contribution from zero order in g^2 in equation (42) are the wave function for the non perturbed quantum harmonic oscillator.

We will see in the next section that the boundary condition provides a implicit relation between ν and g^2. This relation will lead us to the transseries for the energy spectrum in the form (3), which is encoded in the perturbative series (38).

3.2 Non-perturbative part

Now we will obtain the non perturbative part of the spectrum with consists in a resurgent transseries. The ground state wave function must be symmetric in respect to the central of the well $\left(y=-\frac{1}{2}\right)$ and the first excited state is anti-symmetric in respect to this point. Similarly, all the even (odd) states will be symmetric (anti-symmetric) respectively, thus the boundary conditions are:

$$\psi'_{\text{even}}\left(-\frac{1}{2}\right) = 0\,, \tag{43}$$
$$\psi_{\text{odd}}\left(-\frac{1}{2}\right) = 0\,. \tag{44}$$

Applying the condition (44) in the ansatz (30), we have:

$$D_\nu\left(\frac{u(-\frac{1}{2})}{g}\right) = 0\,, \tag{45}$$

where $g \to 0$ thus it is necessary to know the behavior of $D_\nu(z)$ when $z \to \infty$. We will make use of the asymptotic behavior of D_ν [10]:

$$D_\nu(z) \sim z^\nu\, e^{-z^2/4}F_1\left(z^2\right) + e^{\pm i\pi\nu}\frac{\sqrt{2\pi}}{\Gamma(-\nu)}z^{-1-\nu}\, e^{z^2/4}F_2\left(z^2\right)\,,\quad \frac{\pi}{2} < \pm\arg(z) < \pi \tag{46}$$

where,

$$F_1\left(z^2\right)=\sum_{k=0}^{\infty}\frac{\Gamma\left(k-\frac{\nu}{2}\right)\Gamma\left(k+\frac{1}{2}-\frac{\nu}{2}\right)}{\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)}\frac{1}{k!}\left(\frac{-2}{z^2}\right)^k, \tag{47}$$

$$F_2\left(z^2\right)=\sum_{k=0}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}+\frac{\nu}{2}\right)\Gamma\left(k+1+\frac{\nu}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\Gamma\left(1+\frac{\nu}{2}\right)}\frac{1}{k!}\left(\frac{2}{z^2}\right)^k. \tag{48}$$

Therefore, we have,

$$\left[\frac{u\left(-1/2\right)}{g}\right]^{\nu}e^{-u^2(-1/2)/4g^2}F_1\left(\frac{u^2\left(-1/2\right)}{g^2}\right)+e^{\pm i\pi\nu}\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\left[\frac{u\left(-1/2\right)}{g}\right]^{-1-\nu}e^{u^2(-1/2)/4g^2}F_2\left(\frac{u^2\left(-1/2\right)}{g^2}\right)=0\,. \tag{49}$$

Multiplying by $e^{-\frac{u_0^2(-1/2)}{2g^2}}$, where $u_0(y)$ is the first coefficient of the series expansion (37), after some algebra follows that:

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-\nu}=-\frac{e^{-\frac{u_0^2(-1/2)}{2g^2}}}{\sqrt{\pi g^2}}\left[\frac{u^2\left(-1/2\right)}{2}\right]^{\nu+\frac{1}{2}}\frac{F_1\left(\frac{u^2(-1/2)}{g^2}\right)}{F_2\left(\frac{u^2(-1/2)}{g^2}\right)}e^{-\frac{1}{2g^2}\left[u^2(-1/2)-u_0^2(-1/2)\right]}. \tag{50}$$

Defining:

$$\xi\equiv\frac{1}{\sqrt{\pi g^2}}e^{-\frac{u_0^2(-1/2)}{2g^2}}=\frac{1}{\sqrt{\pi g^2}}e^{-\frac{1}{6g^2}} \tag{51}$$

and

$$H_0\equiv\left[\frac{u^2\left(-1/2\right)}{2}\right]^{\nu+\frac{1}{2}}\frac{F_1\left(\frac{u^2(-1/2)}{g^2}\right)}{F_2\left(\frac{u^2(-1/2)}{g^2}\right)}e^{-\frac{1}{2g^2}\left[u^2(-1/2)-u_0^2(-1/2)\right]}, \tag{52}$$

we find an implicit relation between ν and g:

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-\nu}=-\xi H_0\left(\nu,g^2\right), \tag{53}$$

which will be useful to obtain the expression of ν as a function of g^2. The first step in the calculation of the transseries is to consider $\nu=N+\delta\nu$ in the boundary condition relation (53), with N a positive integer:

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-\nu}=\frac{1}{\Gamma(-N-\delta\nu)}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-N}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-\delta\nu}. \tag{54}$$

Using the expansion $w^{-\delta\nu}=e^{-\delta\nu\ln(w)}\approx 1-\delta\nu\ln(w)$, we have for the first term

$$\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-\delta\nu}=1-\delta\nu\ln\left(\frac{e^{\pm i\pi}2}{g^2}\right)+\mathcal{O}\left(\delta\nu^2\right). \tag{55}$$

The second term can be expanded considering the properties of the Gamma function such that

$$\Gamma(-N-\delta\nu)=\frac{\Gamma(-\delta\nu)}{(-N-\delta\nu)(-N-\delta\nu+1)\dots(-1-\delta\nu)},$$

so,

$$\begin{aligned}\frac{1}{\Gamma(-N-\delta\nu)}&=\frac{(-1)^N N!}{\Gamma(-\delta\nu)}\left(\frac{1+\delta\nu}{1}\right)\left(\frac{2+\delta\nu}{2}\right)\dots\left(\frac{N+\delta\nu}{N}\right)\\&=\frac{(-1)^N N!}{\Gamma(-\delta\nu)}\left[1+h_N\,\delta\nu+\mathcal{O}\left(\delta\nu^2\right)\right],\end{aligned} \tag{56}$$

where $h_N = \dfrac{1}{1} + \dfrac{1}{2} + \ldots + \dfrac{1}{N}$ is the N-th harmonic number. To obtain an expression for $\frac{1}{\Gamma(-\delta\nu)}$, let us consider the series expansion for $\ln\left[\Gamma(z)\right]$

$$\ln\left[\Gamma(z)\right] = -\ln z - \gamma z + \sum_{k=2}^{\infty} \frac{(-1)^k}{k}\zeta(k) z^k \,, \tag{57}$$

where γ is the Euler-Mascheroni constant and $\zeta(z)$ is the Riemann Zeta function. Equation (57) implies that:

$$\Gamma(z) = e^{-\ln z} e^{-\gamma z + \mathcal{O}\left(z^2\right)} \,, \tag{58}$$

thus, using the Taylor series of the exponential function one can find

$$\frac{1}{\Gamma(z)} = z\left[1 + \gamma z + \mathcal{O}\left(z^2\right)\right] \,. \tag{59}$$

Finally, replacing $z \to -\delta\nu$ we have the desired expression

$$\frac{1}{\Gamma(-\delta\nu)} = -\delta\nu + \gamma\delta\nu^2 + \mathcal{O}\left(\delta\nu^3\right) \,, \tag{60}$$

plugging this result in equation (61) follows that

$$\frac{1}{\Gamma(-N-\delta\nu)} = -(-1)^N N! \left[\delta\nu - \left(\gamma - h_N\right)\delta\nu^2 + \mathcal{O}\left(\delta\nu^3\right)\right] \,. \tag{61}$$

It is well know that the Digamma function $\psi(z) = \dfrac{d}{dz}\ln\Gamma(z)$ obeys the relation $\psi(N+1) = h_N - \gamma$, therefore, substituting (61) and (55) in equation (54), results:

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-\nu} = -(-1)^N N! \left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-N}\left\{\delta\nu - \left[\ln\left(\frac{e^{\pm i\pi}2}{g^2}\right) - \psi(N+1)\right]\delta\nu^2 + \mathcal{O}\left(\delta\nu^3\right)\right\} \,. \tag{62}$$

Now we will work on the right hand side of equation (53), considering the expansion of $H_0(\nu, g^2)$ in a power series of $\delta\nu$ and we will assume that exists a expansion of $\delta\nu$ in a power series of ξ such that:

$$\begin{aligned} H_0(\nu, g^2) &= H_0(N, g^2) + \frac{\partial}{\partial N}H_0(N, g^2)\,\delta\nu + \mathcal{O}(\delta\nu)^2 \,, \\ \delta\nu &= c_0 + c_1\xi + c_2\xi^2 + \mathcal{O}\xi^3 \,. \end{aligned}$$

Using (62) and comparing the coefficients of ξ in both sides of equation (53) we can calculate the constants c_0, c_1, ... It is important to note that the right hand side of equation (53) begins with ξ^1, thus $c_0 = 0$. In the next order, we have

$$\begin{aligned} -H_0\left(N, g^2\right)\xi - c_1\frac{\partial H_0\left(N, g^2\right)}{\partial N}\xi^2 + \mathcal{O}\left(\xi^3\right) = -(-1)^N N! &\left(\frac{e^{\pm i\pi}2}{g^2}\right)^{-N} \\ &\times\left\{c_1\xi + \left\{c_2 - \left[\ln\left(\frac{e^{\pm i\pi}2}{g^2}\right) - \psi(N+1)\right]c_1^2\right\}\xi^2 + \mathcal{O}\left(\xi^3\right)\right\} \end{aligned} \tag{63}$$

Matching the ξ^1 coefficients, results:

$$c_1 = (-1)^N \frac{H_0(N, g^2)}{N!}\left(\frac{e^{\pm i\pi}2}{g^2}\right)^N \,. \tag{64}$$

Following this procedure we calculated the coefficients of $\delta\nu$ until order ξ^5. Here we present the result only up to order ξ^3 because the higher order expressions are huge. Recording that $\nu = N + \delta\nu$, we have:

$$
\begin{aligned}
\nu =& N + \left(\frac{2}{g^2}\right)^N \frac{H_0}{N!}\xi + \left(\frac{2}{g^2}\right)^{2N} \frac{H_0}{(N!)^2}\xi^2 \left[H_0' + \left(\ln\left(\frac{e^{\pm i\pi}2}{g^2}\right) - \psi(N+1)\right) H_0\right] \\
&+ \left(\frac{2}{g^2}\right)^{3N} \frac{H_0}{6(N!)^3}\xi^3 \left\{18H_0H_0' \left[\ln\left(\frac{e^{\pm i\pi}2}{g^2}\right) - \psi(N+1)\right] + 3H_0H_0'' + 6H_0'^2\right. \\
&\left. + H_0^2 \left\{9\ln^2\left(\frac{e^{\pm i\pi}2}{g^2}\right) + 9\psi(N+1)\left[\psi(N+1) - 2\ln\left(\frac{e^{\pm i\pi}2}{g^2}\right)\right] - 3\psi^{(1)}(N+1) + \pi^2\right\}\right\} , \qquad (65)
\end{aligned}
$$

Replacing the series for ν (65) in the perturbative energy spectrum (38) we obtain the transseries for the N-th energy level

$$
\begin{aligned}
E(N,g^2) =& 2\left(N+\frac{1}{2}\right) - 2g^2\left[\left(N+\frac{1}{2}\right)^2 + \frac{1}{4}\right] - 2g^4\left[17\left(N+\frac{1}{2}\right)^3 + \frac{19}{4}\left(N+\frac{1}{2}\right)\right] \\
&- \frac{2^N e^{-\frac{1}{6g^2}}}{\sqrt{g^2\pi}N!}\left(\frac{e^{\pm i\pi}}{g^2}\right)^N \left[1 - 3g^2\left(1 + N + N^2 + (1+2N)^2 1.2972\right)\right] \\
&- \frac{2^{2N} e^{-\frac{2}{6g^2}}}{\sqrt{g^4\pi^2}(N!)^2}\left(\frac{e^{\pm i\pi}}{g^2}\right)^{2N} 8\left(1 + N + N^2 + (1+2N)^2 1.2972\right) \\
&\times \left\{-3g^2\left(1 + N + N^2 + (1+2N)^2 1.2972\right) + \left[\ln\left(\frac{2e^{\pm i\pi}}{g^2}\right) - \psi(N+1)\right]\right. \\
&\left.\times \left[1 - 3g^2\left(1 + N + N^2 + (1+2N)^2 1.2972\right)\right]\right\} + \dots . \qquad (66)
\end{aligned}
$$

This expression contains the perturbative and non perturbative information for the energy spectrum and it is in the form of equation (3) regardless our coupling constant for double well is g^2. It is important to note that the term $\ln\left(e^{\pm i\pi}\right)$ is imaginary pure, although the observable energy is obviously real. To solve this issue [8] claimed that the imaginary part which comes from the Borel summation of the perturbative series (38) cancels with the imaginary part of the transseries (66). Also, we have to note that, if we consider a complex coupling constant g^2 the condition $\mathrm{Re}\left(g^2\right) = 0$ provides a real expression for the transseries (66). By analogy with the saddle point analysis presented in [4] for a very similar quartic potential, we can see that the condition $\arg\left(g^2\right) = 0$ and $\arg\left(g^2\right) = \pi$ characterizes the Stokes line, where subleading exponentials start contributing to the series. Increasing the $\arg\left(g^2\right)$ these subleading terms grow and becomes of the same magnitude of the leading contributions when $\arg\left(g^2\right) = \pi/2$ is reached.

4 Discussion

We have presented a pedagogical introduction of the resurgent transseries theory explaining the origin and the necessity of the transseries, remarking how is it possible to associate a physical observable to an infinite perturbative series. For this purpose, we have started with the formal definition of a asymptotic divergent series, and explained how to associate a number with this series using the Borel summation process. As we saw, there are some functions that present singularities a long the direction of the Borel summation. These was our cases of interest, because they give rise to the necessity of the transseries and manifest the Stokes phenomenon.

To illustrate the theory of resurgent transseries, we shown how the perturbative series encodes all information about the non perturbative part for the energy spectrum, in the context of a quantum mechanics analysis of a particle in a double-well potential. To derive the nonperturbative content we used the uniform WKB method considering an ansatz for the wave function in terms of the Parabolic Cylinder functions. To sum up, the essential idea of this method is to use the asymptotic behavior of these especial functions in the boundary conditions and regard the energy levels label ν containing a non-integer part $\nu = N + \delta\nu$. Dealing with the expansion in powers of $\delta\nu$ of the boundary conditions equation, we calculated the resurgent trans series from the perturbative energy spectrum.

In spite of the transseries contributions (66) appears to be exponentially small compared to the perturbative part (since the coupling constant g^2 is small), it is important to note that the Stokes phenomenon may change the situation. Considering the g^2 as a complex quantity, we saw that there are some regions in the complex plane in which the transseries contribution becomes dominant. These different sectors of the complex plane are bounded by the Stokes line. Our purpose here was just to present this phenomenon in a comprehensive way using a well known example from Quantum Mechanics, the reader who is interested is encouraged to follow the cited references to go delve into the subject.

References

[1] M. Maggiore, *A Modern Introduction to Quantum Field Theory.* Oxford Master Series in Physics, OUP Oxford, 2004.

[2] "2018 codata value: electron g factor," *NIST*, 2019.

[3] E. Borel, "Mémoire sur les séries divergentes," *Annales scientifiques de l'École Normale Supérieure*, vol. 3e série, 16, pp. 9–131, 1899.

[4] I. Aniceto, G. Başar, and R. Schiappa, "A primer on resurgent transseries and their asymptotics," *Physics Reports*, vol. 809, pp. 1–135, 2019.

[5] G. G. Stokes, "On the discontinuity of arbitrary constants which appear in divergent developments," *Transactions of the Cambridge Philosophical Society*, vol. 10, p. 105, 1864.

[6] G. Edgar, "Transseries for beginners," *Real Analysis Exchange*, vol. 35, no. 2, pp. 253–310, 2010.

[7] J. Ecalle, *Les fonctions resurgentes. 1. Les algèbres de fonctions résurgentes.* Fonctions résurgentes, Univ. de Paris-Sud, Dép. de Mathématique, 1981.

[8] G. V. Dunne and M. Ünsal, "Uniform wkb, multi-instantons, and resurgent trans-series," *Physical Review D*, vol. 89, no. 10, p. 105009, 2014.

[9] G. B. Arfken and H. J. Weber, "Mathematical methods for physicists," 1999.

[10] "*NIST Digital Library of Mathematical Functions.*" http://dlmf.nist.gov/, Release 1.1.3 of 2021-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.