Unit 3 Group Work
PCHA 2022-23 / Dr. Kessner

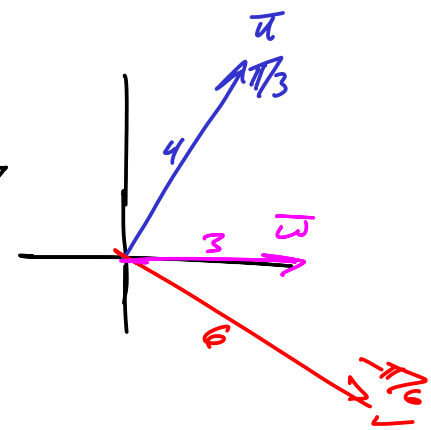
Name / Pledge: KEY

Partner(s):

You can use your notes and/or textbook. No calculator. Have fun!

1. Suppose you have the following vectors:

$$\vec{u} = \left\langle 2, 2\sqrt{3} \right\rangle = 4\left\langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \right\rangle$$
$$\vec{v} = \left\langle 3\sqrt{3}, -3 \right\rangle = 6\left\langle \tfrac{\sqrt{3}}{2}, -\tfrac{1}{2} \right\rangle$$
$$\vec{w} = \langle 3, 0 \rangle$$



Calculate the following:

a) $|\vec{u}|$ $= 4$

b) $|\vec{v}|$ $= 6$

c) Unit vector in the direction of $\vec{v}$.

$$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{6}\langle 3\sqrt{3}, -3 \rangle = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$$

d) Angle between $\vec{u}$ and $\vec{v}$.

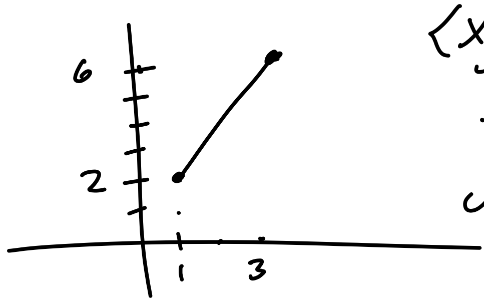
$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{\langle 2, 2\sqrt{3}\rangle \cdot \langle 3\sqrt{3}, -3\rangle}{4\cdot 6} = 0 \Rightarrow \theta = \pi/2$$

e) Angle between $\vec{u}$ and $\vec{w}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{w}}{|\vec{u}||\vec{w}|} = \frac{\langle 2, 2\sqrt{3}\rangle \cdot \langle 3, 0\rangle}{4\cdot 3} = \frac{6}{12} = \frac{1}{2}$$

$$\rightarrow \theta = \pi/3$$

2. a) Parametrize the line segment from $(1, 2)$ to $(3, 6)$.

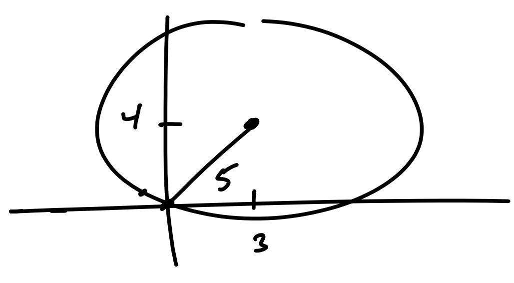


$$\langle x, y\rangle = \langle 1,2\rangle + t\langle 2,4\rangle$$
$$x(t) = 1+2t$$
$$y(t) = 2+4t$$

b) Parametrize the line segment from $(3, 6)$ to $(1, 2)$ (same points, opposite direction).

$$\langle x, y\rangle = \langle 3,6\rangle + t\langle -2,-4\rangle$$
$$x(t) = 3-2t$$
$$y(t) = 6-4t$$

c) Parametrize the circle with center $(3, 4)$ and radius 5.



$$\langle x, y\rangle = \langle 3,4\rangle + \langle 5\cos t, 5\sin t\rangle$$
$$x(t) = 3+5\cos t$$
$$y(t) = 4+5\sin t$$

d) Parametrize the same circle, but make the period = 6.

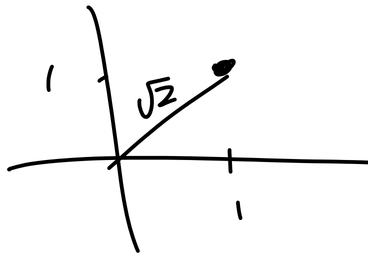
$$x(t) = 3+5\cos\left(\frac{2\pi}{6}t\right)$$
$$y(t) = 4+5\sin\left(\frac{2\pi}{6}t\right)$$

3. Find all polar coordinates of the following (rectangular) points:

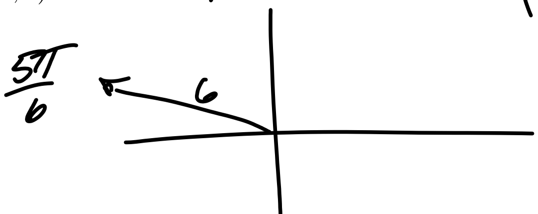
a) (1, 1)

$\left(\sqrt{2}, \frac{\pi}{4} + 2k\pi\right)$ $(k \in \mathbb{Z})$

$\left(-\sqrt{2}, \frac{5\pi}{4} + 2k\pi\right)$



b) $(-3\sqrt{3}, 3)$ $= 6\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$



$\langle 6, \frac{5\pi}{6} + 2\pi k \rangle$

$\langle -6, \frac{11\pi}{6} + 2\pi k \rangle$

Convert the following equations from rectangular to polar coordinates:

c) $3x + 4y = 5$

$3r\cos\theta + 4r\sin\theta = 5$

$r(3\cos\theta + 4\sin\theta) = 5$

$r = \frac{5}{3\cos\theta + 4\sin\theta}$

d) $x^2 + y^2 = 25$

$r = 5$

Convert from polar to rectangular:

e) $r = -5\sin\theta$

$r^2 = -5r\sin\theta$

$x^2 + y^2 = -5y$

$x^2 + \left(y^2 + 5y + \frac{25}{4}\right) = \frac{25}{4}$

$x^2 + \left(y + \frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2$

f) $r = 5\csc\theta$

$r = \frac{5}{\sin\theta}$

$r\sin\theta = 5$

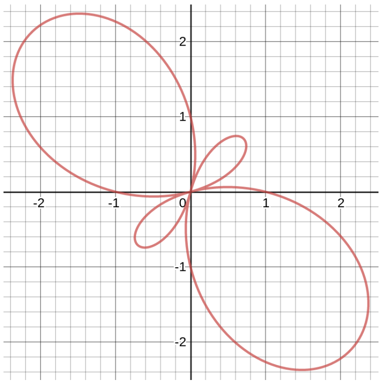
$y = 5$

4. Analyze the graph of the polar function $r = 1 - 2\sin 2\theta$:

1) Find the max $|r|$ values and θ values where they occur.

2) State and prove any symmetry relations.

3) **Challenge:** What is going on at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$?



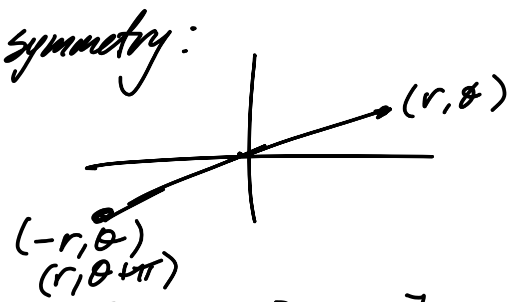
$r = 1 - 2\sin 2\theta$

max $|r| = 3$ when $\sin 2\theta = -1$

$2\theta = \frac{3\pi}{2} + 2\pi k$

$\theta = \frac{3\pi}{4} + \pi k$

origin symmetry:



check: $r \overset{?}{=} 1 - 2\sin[2(\theta+\pi)]$

$= 1 - 2\sin(2\theta + 2\pi)$

$= 1 - 2\sin 2\theta$ ✓

5. For each of the following 2x2 matrices, determine whether it is invertible, and if so, find the inverse matrix and the determinant of the inverse.

$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ $\det A = 9 \Rightarrow$ invertible $A^{-1} = \frac{1}{9}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}$

$\det A^{-1} = 1/9$

$B = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$ $\det B = -4 \Rightarrow$ invertible $B^{-1} = \frac{1}{-4}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

$\det B^{-1} = -\frac{1}{4}$

$C = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ $\det C = -4 \Rightarrow$ invertible $C^{-1} = \frac{1}{-4}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$

$\det C^{-1} = -1/4$

$D = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\det D = 0$ not invertible

Let $E = \begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix}$. Find E^{-1}. Verify that $EE^{-1} = I$.

$\det E = -1$ $E^{-1} = \frac{1}{-1}\begin{pmatrix} 4 & -5 \\ -5 & 6 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 5 & -6 \end{pmatrix}$

$EE^{-1} = \begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} -4 & 5 \\ 5 & -6 \end{pmatrix} = \begin{pmatrix} 25-24 & 0 \\ 0 & 25-24 \end{pmatrix} = I = E^{-1}E$ ✓

Use the inverse matrix you found to solve the following linear systems:

$6x + 5y = 1$
$5x + 4y = 0$

$E\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = E^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -4 & 5 \\ 5 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

$6x + 5y = 0$
$5x + 4y = 1$

$\begin{pmatrix} x \\ y \end{pmatrix} = E^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = E^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 5 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \end{pmatrix}$

$6x + 5y = 1$
$5x + 4y = 2$

6. Consider the following system of linear equations:

$$\begin{aligned} x + 3z &= 4 \\ -x - 2z &= -3 \\ y - 2z &= -1 \end{aligned}$$

a. Write the linear system as a matrix equation.

$$\underbrace{\begin{pmatrix} 1 & 0 & 3 \\ -1 & 0 & -2 \\ 0 & 1 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$$

b. Calculate the determinant of the matrix to verify that the matrix is invertible.

$$\det A = 1 \begin{vmatrix} 0 & -2 \\ 1 & -2 \end{vmatrix} + 0 \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + 3 \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 2 - 3$$

$$= -1 \quad (\text{invertible})$$

c. Find the inverse matrix and use it to solve the system.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$R_1 + R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

R_{23}

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$-3R_3 + R_1$

$2R_3 + R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 0 \\ 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

where the right-hand block is A^{-1}.

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -3 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -8+9+0 \\ 8-6-1 \\ 4-3+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$