it, or including that which we, $2c$ cycles before, deleted from it, until the (2^{k-1})st combination, which corresponds to the empty set plus element k.

Proof of (1). Since the binary representation of 2^{k-1} is a 1 bit followed by $(k - 1)$ zeros, the kth element is included on cycle 2^{k-1}. The kth element will remain until the binary number 11 followed by $(k - 1)$ zeros appears. This will be on cycle number $(2^k + 2^{k-1}) > (2^k - 1)$. Thus, all combinations from 2^{k-1} through $(2^k - 1)$ will include the kth element.

Proof of (2). Since $(2^{k-1} + c) + (2^{k-1} - c) = 2^k$, the binary representations of $(2^{k-1} + c)$ and $(2^{k-1} - c)$ correspond in all their low-order zeros, and the low-order 1, in which they also correspond. The bit above the 1 must differ in the two numbers, due to the binary carry. Thus, $B(2^{k-1} + c) = -B(2^{k-1} - c)$.

To complete the proof by induction, we may note, by Table 1, that the algorithm has generated all combinations for $k \leqq 4$.

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Generation of Permutations by Addition

By John R. Howell

1. Introduction. Suppose one wishes to generate the $k!$ permutations of k distinct marks. Representing these k marks by 0, 1, 2, $\cdots$, $(k - 1)$ written side by side to form the "digits" of a base k integer, then the repeated addition of 1 will generate integers whose "digits" represent permutations of k marks. Many numbers are also generated which are not permutations. D. H. Lehmer [2] states that this so-called addition method can be made more efficient by adding more than 1 to each successive integer.

2. Method. In this note, we show that the correct number greater than 1 to add to this integer is a multiple of $(k - 1)$ radix k.

Lemma 1. *The arithmetic difference radix k between an integer composed of mutually unlike digits and another integer composed of a permutation of the same digits is a multiple of* $(k - 1)$.

Considering the process of "casting out nines," it is obvious that the two integers are congruent mod $(k - 1)$. Hence, their difference is zero mod $(k - 1)$.

The method seems to have two advantages. First, one can generate all $k!$ permutations in lexicographic order. Second, all permutations "between" two given permutations can be obtained. The process can be made to be cyclic if upon obtaining $(k - 1)$, $\cdots$, 0 one takes the next permutation to be 0, 1, $\cdots$, $(k - 1)$.

3. Example. Suppose we wish to generate the 4! permutations of 4 marks. Representing these 4 marks by 0, 1, 2 and 3, we add 3 radix 4 to 0123 to get 0132. Continuing this process we get the 4! permutations desired. The array below shows

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the first 16 numbers generated by this process. An asterisk marks each integer whose digits represent a required permutation. The other integers were rejected because of the occurrence of repeated digits.

Sequence	*Integer*	*Sequence*	*Integer*
1	0123*	9	0303
2	0132*	10	0312*
3	0201	11	0321*
4	0210	12	0330
5	0213*	13	0333
6	0222	14	1002
7	0231*	15	1011
8	0300	16	1020

4. Adaptation to a Computer. In a computer such as the IBM 7090 where convert instructions are available it is easy to do radix k arithmetic. Otherwise one could simulate the process by adding 9 digit-wise and testing the resulting sum for having unique digits each one of which is one of the original k digits.

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Multiple Quadrature with Central Differences on One Line

By Herbert E. Salzer

Abstract. The coefficients A_{2m}^n in the n-fold quadrature formulas for the stepwise integration of (1) $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$, at intervals of h, namely, for n even, (2) $\delta^n y_0 = h^n \sum_{m=1}^{10} (1 + A_{2m}^n \delta^{2m}) f_0 + \cdots$, for n odd, (3) $\mu\delta^n y_0 = h^n \sum_{m=1}^{10} (1 + A_{2m}^n \delta^{2m}) f_0 + \cdots$, are tabulated exactly for $n = 1(1)6$, $m = 1(1)10$. They were calculated from the well-known symbolic formulas (4) $\delta^n y = (\delta/D)^n f$, (5) $(\delta/D)^n = (\delta h/2 \sinh^{-1}(\delta/2))^n$ and (6) $\mu = (1 + \delta^2/4)^{1/2} = 1 + \frac{\delta^2}{8} - \frac{\delta^4}{128} + \frac{\delta^6}{1024} -$

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