Physics Informed, not Physics Enforced

Many PIML methods are "encouraging" physics, rather than "enforcing" it

Post-training verification required of key properties Lyapunov, passivity, invariance, etc

Verification & Analysis in Control

System Dynamics and/or Controlled Systems

- Stability / dissipativity / region of attraction
- Lipschitz properties
- Robustness
- Safety properties (constraint satisfaction)

Many people active in the area…

- Stability / dissipativity / region of attraction → *[Zhang et al, 2014], [Engelken et al, 2020], [Vogt et al, 2022], [Batuhan et al, 2019], [Drgona et al, 2022], [Abate et al, 2021], [Grüne et al, 2021], [Chen et al, 2021], etc*
- Lipschitz properties → *[Scaman et al, 2018], [Chakrabarty et al, 2019], [Pauli et al, 2022], [Fazlyab et al, 2022], [Erichson et al, 2021], etc*
- Robustness → *[Wang et al, 2016], [Gehr et al, 2018], [Jia et al, 2020], [Tjeng et al, 2019], etc*
- Safety properties (constraint satisfaction) → *[Zhang et al, 2022], [Robey et al, 2020], [Karg et al, 2020], etc*

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- Conservative over-bounding
  - Lipschitz constants
  - GP bounds
- Sampling-based methods
- Function positivity / global optimziation
  - Mixed-integer programming
  - SOS / Semi-definite programming

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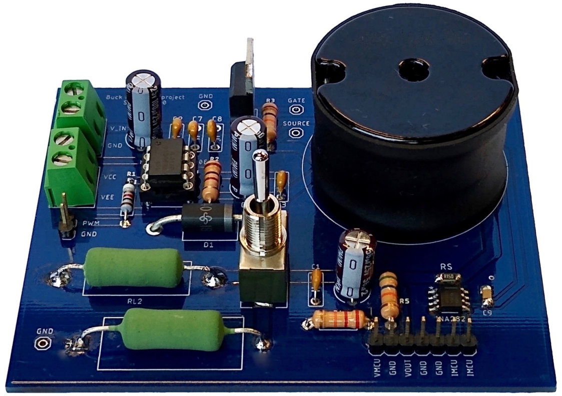
Standard MPC Design Process

DC-DC Buck Converter

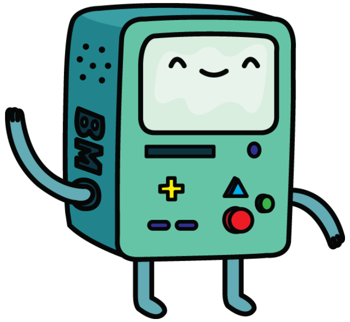


Average dynamics over a switching period, then linearize

$$
\begin{aligned}
\min_{X,U} \quad & \sum_{t=0}^{N-1} \left(\|x_t - x_{\text{eq}}\|_Q^2 + \|u_t - u_{\text{eq}}\|_R^2 \right) + \|x_N - x_{\text{eq}}\|_P^2 \\
\text{s.t.} \quad & \forall t = 0, \ldots, N-1 \\
& x_{t+1} = Ax_t + Bu_t \\
& \begin{bmatrix} i_L^{\min} \\ v_O^{\min} \end{bmatrix} \leq x_t \leq \begin{bmatrix} i_L^{\max} \\ v_O^{\max} \end{bmatrix} \\
& u^{\min} \leq u_t \leq u^{\max} \\
& x_N \in \mathcal{X}_N \\
& x_0 = x(0)
\end{aligned}
$$

State Estimator + Offset-free Tracking MPC

Control frequency of 10 kHz!



STM32L476:

- 80 MHz clock
- 128 kB of RAM
- 1 MB of flash

25-dimensional QP with five parameters

Control Approximation via Deep NN

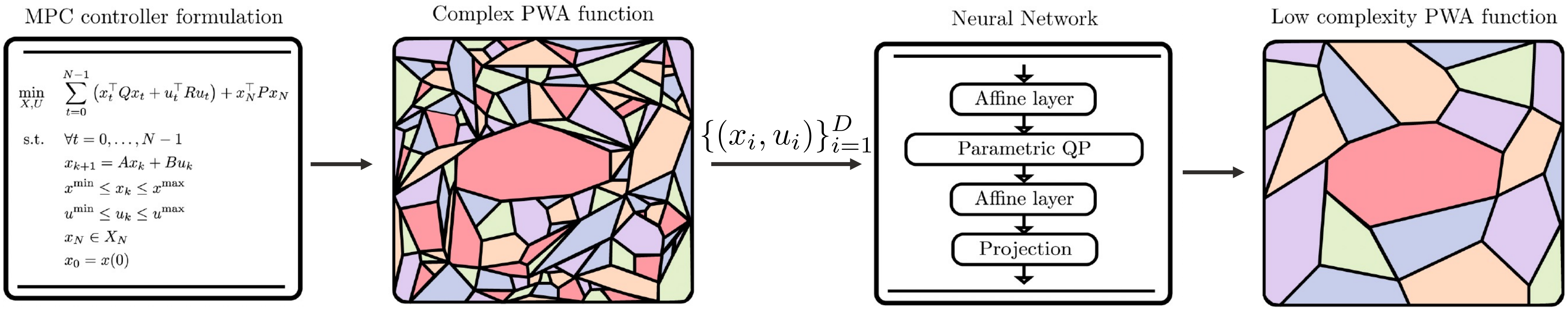
$$z^\star(x) = \min \|Lz + Fx + f\|^2 + \epsilon\|z\|^2$$
$$\text{s.t. } z \geq 0$$
$$\hat{u}(x) = Gz^\star(x) + g$$

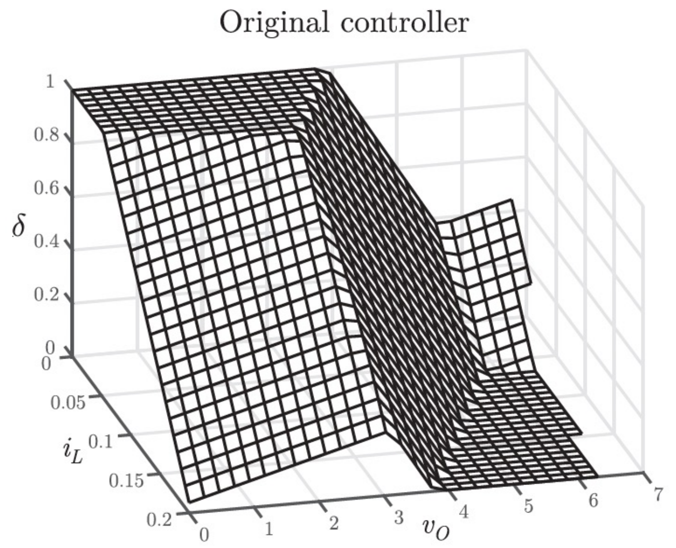
Choose parametric QP structure
→ Approximation of the dual of a standard linear MPC controller

Affine layer	$y_1 = Fx + f$	F, f
QP layer	$y_2 = \min_{z \geq 0} \|Lz + y_1\|^2 + \epsilon\|z\|^2$	L
Affine layer	$y_3 = Gy_2 + g$	G, g
Saturation layer	$\hat{u} = \text{sat}(y_3)$	

Optimal solution is differentiable [Amos et al, 2017], [Agrawal et al, 2019]

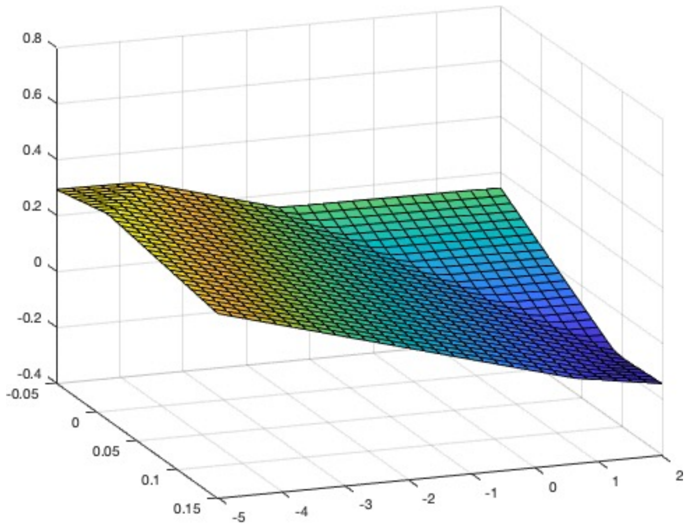
Differentiable Programming → Approximate Explicit MPC





91% complexity reduction

Training : 35min



(2021) Embedded PWM Predictive Control of DC-DC Power Converters Via Piecewise-Affine Neural Networks. IEEE OJIES.
E.T. Maddalena, M.W.F. Specq, V.L. Wisniewski, C.N. Jones.

Verification of Approximate Controllers $\leftrightarrow$ Function Positivity

- Verify that the approximation error is nowhere greater than γ

$$0 \leq \gamma - \max_{x} \|u^{\star}(x) - \hat{u}(x)\|_{\infty}$$

Optimal control law

Approximate control law

Verification → Proving non-negativity of a function

Verification of Approximate Controllers $\leftrightarrow$ Function Positivity

Verify that the approximation error is nowhere greater than γ

$$0 \leq \gamma - \max_{x} \|u^\star(x) - \hat{u}(x)\|_\infty$$

$$\begin{aligned}
\text{s.t. } u^\star(x_0) = \arg\min \sum_{i=0}^{N} l(x_i, u_i) \\
\text{s.t. } \forall i = 0, \dots, N-1 \\
x_{i+1} = Ax_i + Bu_i \\
x_i \in X,\ u_i \in U \\
x_N \in X_N,\ x_0 = x
\end{aligned}$$

Optimal control law

$$\begin{aligned}
z^\star(x) = \min \|Lz + Fx + f\|^2 + \varepsilon\|z\|^2 \\
\text{s.t. } z \geq 0 \\
\hat{u}(x) = Gz^\star(x) + g
\end{aligned}$$

Approximate control law

Verification → Proving non-negativity of a function

MILP-Representable Functions

A function $\psi : X \to V$ with domain $X \subseteq \mathbb{R}^n$ and range $V \subseteq \mathbb{R}^m$ is MILP representable if there exists a polyhedral set $P \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^c \times \mathbb{R}^b$ such that $v = \psi(x)$ if and only if there exists $z \in \mathbb{R}^c$ and $\beta \in \{0,1\}^b$ such that $(x, v, z, \beta) \in P$.

MILP representable functions are dense in the family of continuous functions on a compact domain

Verification Problem

$$
\begin{aligned}
\min\ & \tau \\
\text{s.t. } & x_0 \in X \\
& v_1 = \psi_1(x_0) \\
& v_2 = \psi_2(x_0) \\
& \tau = f(x_0, v_1, v_2)
\end{aligned}
$$

Mixed-Integer Optimization Problem

$$
\begin{aligned}
\min\ & \tau \\
\text{s.t. } & x_0 \in X \\
& (x_0, v_1, z_1, \beta_1) \in P_1 \\
& (x_0, v_2, z_2, \beta_2) \in P_2 \\
& (v_1, v_2, \tau, z_3, \beta_3) \in P_f \\
& \beta_i \in \{0,1\}^{b_i}
\end{aligned}
$$

R.Schwan, C.N. Jones and D.Kuhn, Stability Verification of Neural Network Controllers using Mixed-Integer Programming, IEEE Transactions on Automatic Control. 2023, To appear

Some Examples

Compositions

$$\psi_1 \circ \psi_2$$

Max / min

$$\psi(x) = \max\{\psi_1(x), \psi_2(x)\}$$

Piecewise Affine Functions

$$\psi(x) = A_i x + c_i \quad \forall x \in X_i,\ \forall i \in \mathcal{I}$$

Parametric convex QPs

$$\begin{aligned} \psi(x) = \arg\min_z\ & \frac{1}{2} z^T P z + (Qx + q)^T z \\ \text{s.t. } & Az = Bx + b \\ & Fz \leq Gx + g \end{aligned}$$

MILP-Representable Functions → Mixed-integer programming

Verification Problem

$$
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\text{s.t. } & x_0 \in X \\
& v_1 = \psi_1(x_0) \\
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& \tau = f(x_0, v_1, v_2)
\end{aligned}
$$

Various approximate control laws can be captured

- Deep neural networks
- Quadratic programs
- Simple saturations
- etc

Verification of various properties can be made

- Lyapnuov function
- Worst-case error
- Region of attraction
- etc

A number of people have developed similar formulations for polynomial control laws and SOS solvers

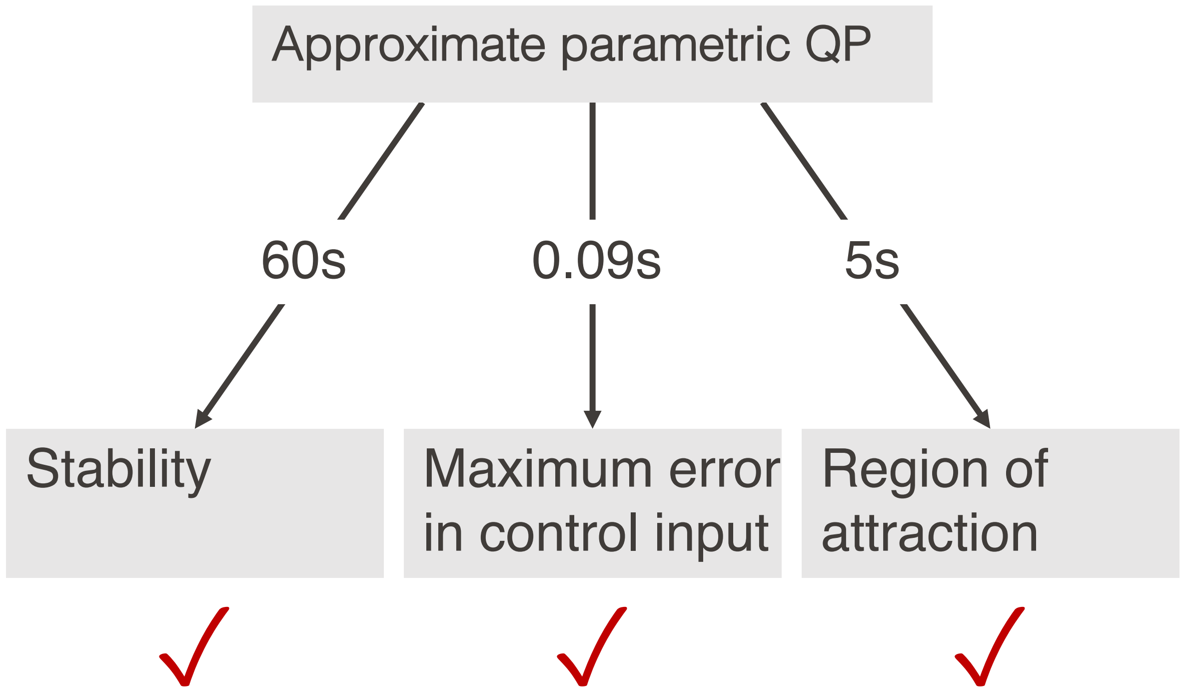
EVANQP

DOI 10.48550/arXiv.2206.13374 | Preprint arXiv | Grant NCCR Automation (51NF40180545)

EPFL Verifier for Approximate Neural Networks and QPs

https://github.com/PREDICT-EPFL/evanqp

Software tool to train approximate controllers using neural networks and/or parametric quadratic programs, and then to verify key properties.



A similar tool targeted at power systems: *[Venzke, Qu, Low, Chatzivasileiadis, 2020]*

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