Practice Exam 2

1. A point has coordinates $x = -2$ and $y = -2$. Find its polar coordinates r and θ.
Answer: $r = \sqrt{8} = 2\sqrt{2}$. $\theta = \pi + \pi/4 = 5\pi/4$ or $\theta = -3\pi/4$ (give or take any integer multiple of 2π).

2. (a) Eliminate the parameter to find a Cartesian equation of the curve $x = 1 - t^2$, $y = 2t$.
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.
Answer: (a) Solve for t from the second equation, $t = y/2$. Plug into first, $x = 1 - (y/2)^2 = 1 - y^2/4$.
(b) I will omit the actual sketch, but the curve is a parabola with vertex $(1, 0)$, opening to the left.

3. Find the length of the curve $y = \sqrt{x^3}$ from the point $(0, 0)$ to the point $(4, 8)$.
Answer: $\frac{8}{27}(10^{3/2} - 1)$.

4. Find the surface area obtained by rotating the curve $y = 4 - x^2$, $0 \le x \le 2$, around the y-axis.
Answer: $\frac{\pi}{6}(17^{3/2} - 1)$.

5. Use the Comparison Theorem to determine whether the integral is convergent or divergent:
(a) $\displaystyle\int_2^\infty \frac{x+3}{x^2 - \sqrt{x}}\,dx$;
(b) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^3+3}}\,dx$
Answer: (a) divergent by comparison to $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ (this integral diverges and the given one is greater than it).
(b) convergent by comparison to $\displaystyle\int_2^\infty \frac{1}{x^{3/2}}\,dx$ (this integral converges and the given one is smaller than it).

6. (a) Set up, but **do not evaluate**, an integral for the length of the curve

$$x = e^{3t}, \quad y = \sqrt{t}, \quad \text{where } 1 \le t \le 2.$$

(b) Also set up an integral for the surface area obtained by rotating this curve about the y-axis.
Answer: (a) $\displaystyle\int_1^2 \sqrt{9e^{6t} + 1/(4t)}\,dt$
(b) $\displaystyle\int_1^2 2\pi e^{3t}\sqrt{9e^{6t} + 1/(4t)}\,dt$

7. Find the area of the region that lies inside the polar curve $r = 3$ and outside the polar curve $r = 3 - \cos\theta$.
Answer: $6 - \pi/4$. (integrate between $-\pi/2$ and $\pi/2$).

8. Evaluate the improper integral, if it is convergent.
(a) $\displaystyle\int_1^\infty xe^{-x^2}\,dx$;
(b) $\displaystyle\int_0^1 x^2 \ln x\,dx$.
Answer: (a) $1/(2e)$, (b) $-1/9$.

9. Find an equation of the tangent line to the parametric curve

$$x = 4t - \sin(2t), \quad y = 1 - \cos(2t)$$

at the point where $t = \pi/4$.
Answer: $y = 1 + \frac{1}{2}(x - \pi + 1)$.

10. Find a Cartesian equation for the polar curve $r = 2\sin\theta + 2\cos\theta$, and identify this curve.
Answer: $x^2+y^2 = 2y+2x$, which after completing the squares becomes $(x-1)^2+(y-1)^2 = 2$, and is identified as the circle of radius $\sqrt{2}$ with center $(1, 1)$.

11. Sketch the polar curve $r = \dfrac{1}{1 + \cos\theta}$.
Answer: The curve looks like a parabola opening to the left (and in fact, it **is** a parabola opening to the left.)
Instead of providing a sketch, I'll explain how to draw it. As θ increases from 0 to 2π, the following happens: the denominator $1+\cos\theta$ starts off from 2, goes down to its minimal value 0 at $\theta = \pi$, then goes back to 2 when θ reaches 2π. Accordingly, the function $r = \dfrac{1}{1 + \cos\theta}$ starts off at $1/2$, grows to infinity as θ approaches π and then comes down to $1/2$.
Having established this, we sketch the curve as follows. Place the pencil at distance $1/2$ to the right of the origin, on the positive part of the x-axis. Move it counterclockwise around the origin while increasing the distance (r) from the origin. You don't reach the negative part of the x-axis, since r increases indefinitely as θ approaches π. In the bottom half of the plane, where θ is between π and 2π, the polar distance r decreases from infinity to $1/2$, as the curve returns to its starting point.