Casimir Stress by Euler-Maclaurin Summation

Dr Timothy J. Walton

(July 25, 2021)

Contents

1. The Euler-Maclaurin Summation Formula 1

2. The Casimir Energy 1

3. The Casimir Pressure 3

A. Vacuum Expectation Values 8

1. The Euler-Maclaurin Summation Formula

Definition. *Given a function f of (at least) class C^m $(m \in \mathbb{Z}^+)$, the Euler-Maclaurin summation formula is a mathematical identity relating the sum of f to its integral and is given by:*

$$\sum_{k=n}^{N} f(k) - \int_n^N f(x)\,dx = \frac{1}{2}\Big(f(n) + f(N)\Big) + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!}\Big(f^{(2r-1)}(N) - f^{(2r-1)}(n)\Big) + \frac{1}{(2m+1)!}\int_n^N P_{2m+1}(x) f^{(2m+1)}(x)\,dx \tag{1}$$

where $\{B_k\}$ denote the set of Bernoulli numbers, $\{P_k(x)\} = \{B_k(x-[x])\}$ denote the set of associated periodic Bernoulli polynomials and $n, N \in \mathbb{N}$.

2. The Casimir Energy

The spectrum of modes within a perfectly conducting box of dimensions $L_x \times L_y \times L_z$ with vacuum interior is given

$$\frac{\omega_N}{c_0} = \sqrt{\frac{\pi^2 n_x^2}{L_x^2} + \frac{\pi^2 n_y^2}{L_y^2} + \frac{\pi^2 n_z^2}{L_z^2}}\,,$$

where $n_x, n_y, n_z \in \mathbb{Z}^+$ and $N \equiv n_x, n_y, n_z$. We introduce the non-dimensional Ω_N by

$$\Omega_N \equiv \frac{\omega_N L_z}{c_0} = \pi\sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2}\,, \tag{2}$$

The vacuum energy of the system is then a sum over all possible modes:

$$\mathcal{E} \;=\; \frac{\hbar}{2}\sum_N \omega_N \;=\; \frac{\hbar c_0}{2L_z}\sum_N \Omega_N .$$

This expression is infinite, so must be regularised:

$$\mathcal{E}(s) \;=\; \frac{\hbar c_0}{2L_z}\sum_N \Omega_N\, W_s(\Omega_N) \;=\; \frac{\hbar\pi c_0}{2L_z}\sum_N \sqrt{\frac{n_x^2L_z^2}{L_x^2}+\frac{n_y^2L_z^2}{L_y^2}+n_z^2}\; W_s\left(\sqrt{\frac{n_x^2L_z^2}{L_x^2}+\frac{n_y^2L_z^2}{L_y^2}+n_z^2}\right).$$

in terms of the regulator $W_s(\Omega_N)$ which is a smooth function of $s \geq 0$ and Ω_N with $W_0(\Omega_N) = 1$ for all Ω_N. To analyse the geometry of the Casimir effect, we consider the limit $L_x, L_y \to \infty$, so that n_x, n_y become continuous and we may write

$$\mathcal{E}(s) \;=\; \frac{\hbar\pi c_0}{2L_z}\sum_{n_z=0}^{\infty}\int_{n_x=0}^{\infty} dn_x \int_{n_y=0}^{\infty} dn_y \sqrt{\frac{n_x^2L_z^2}{L_x^2}+\frac{n_y^2L_z^2}{L_y^2}+n_z^2}\; W_s\left(\sqrt{\frac{n_x^2L_z^2}{L_x^2}+\frac{n_y^2L_z^2}{L_y^2}+n_z^2}\right).$$

Introducing the notation

$$r_x \;=\; \frac{n_xL_z}{L_x} \qquad \text{and} \qquad r_y \;=\; \frac{n_yL_z}{L_y}, \tag{3}$$

we may write this as

$$\mathcal{E}(s) \;=\; L_xL_y\cdot\frac{\hbar\pi c_0}{2L_z^3}\sum_{n_z=0}^{\infty}\int_{n_x=0}^{\infty} dr_x \int_{n_y=0}^{\infty} dr_y \sqrt{r_x^2+r_y^2+n_z^2}\; W_s\left(\sqrt{r_x^2+r_y^2+n_z^2}\right).$$

We may change variables using a polar coordinate representation $(r_x = R\sin\theta, r_y = R\cos\theta)$ with $R^2 = r_x^2 + r_y^2$ and $dr_x\, dr_y = R\, dR\, d\theta$ so that

$$\begin{aligned}\frac{\mathcal{E}}{L_xL_y} \;&=\; \frac{\hbar\pi c_0}{2L_z^3}\sum_{n_z=0}^{\infty}\int_{R=0}^{\infty}\int_{\theta=0}^{\pi/2} R\, dR\, d\theta\, \sqrt{R^2+n_z^2}\; W_s\left(\sqrt{R^2+n_z^2}\right)\\ &=\; \frac{\hbar\pi^2 c_0}{4L_z^3}\sum_{n_z=0}^{\infty}\int_{R=0}^{\infty} R\, dR\, \sqrt{R^2+n_z^2}\; W_s\left(\sqrt{R^2+n_z^2}\right)\end{aligned}$$

Introducing the change of variable $U^2 = R^2 + n_z^2$ with $U\, dU = R\, dR$, we have

$$\frac{\mathcal{E}}{L_xL_y} \;=\; \frac{\hbar\pi^2 c_0}{4L_z^3}\sum_{n_z=0}^{\infty}\int_{U=n_z}^{\infty} U^2 W_s(U)\, dU \;=\; \frac{\hbar\pi^2 c_0}{4L_z^3}\sum_{n_z=0}^{\infty} f_s(n_z),$$

where

$$f_s(n_z) \;\equiv\; \int_{U=n_z}^{\infty} U^2 W_s(U)\, dU.$$

If we choose the regulator $W_s(U) = e^{-sU}$ then

$$f_s(n_z) \;=\; \int_{U=n_z}^{\infty} U^2 e^{-sU}\, dU$$

We may then use the Euler-Maclaurin summation formula (1) in order to evaluate $\mathcal{E}(s)$. With $m = 2$, we have

$$\begin{aligned}
\sum_{n_z=0}^{\infty} f_s(n_z) - \int_0^{\infty} f_s(x)\,dx &= \frac{1}{2}\Big(f_s(0) + f_s(\infty) \Big) + \frac{B_2}{2}\left(f_s'(\infty) - f_s'(0)\right) + \frac{B_4}{24}\left(f_s'''(\infty) - f'''(0)\right) \\
&\quad + \frac{1}{120}\int_0^{\infty} P_5(x) f_s'''''(x)\,dx \\
&= \frac{1}{2}\int_0^{\infty} U^2 e^{-sU}\,dU - \frac{1}{360} + O(s^2) \\
&= \frac{1}{s^3} - \frac{1}{360} + O(s^2)
\end{aligned}$$

using

$$\frac{d}{da} f_s(a) \;=\; \frac{d}{da}\int_a^{\infty} U^2 e^{-sU}\,dU \;=\; -a^2 e^{-sa}$$

and

$$\begin{aligned}
\int_0^{\infty} U^2 e^{-sU}\,dU &= -\frac{U^2}{s} e^{-sU}\bigg|_{U=0}^{\infty} + \frac{2}{s}\int_{U=n_z}^{\infty} U e^{-sU}\,dU \\
&= \frac{2}{s}\left(-\frac{U}{s} e^{-sU}\bigg|_{U=0}^{\infty} + \frac{1}{s}\int_{U=n_z} e^{-sU}\,dU \right) \\
&= \frac{2}{s^2}\left(\frac{e^{-sU}}{s}\bigg|_{U=n_z}^{\infty} \right) = \frac{2}{s^2}\cdot\frac{1}{s} = \frac{2}{s^3}.
\end{aligned}$$

The terms of order s^2 occur from the integral involving the periodic Bernoulli functions and so will vanish upon renormalization. Thus, we define the regularized energy density

$$\frac{\mathcal{E}_{\text{reg}}(s)}{L_x L_y} \;=\; \frac{\hbar \pi^2 c_0}{4 L_z^3}\left(\sum_{n_z=0}^{\infty} f_s(n_z) - \int_0^{\infty} f_s(x)\,dx - \frac{1}{s^3} \right)$$

and the renormalized energy density is

$$\mathcal{U}_{\text{ren}} \;\equiv\; \frac{\mathcal{E}_{\text{reg}}(0)}{L_x L_y} \;=\; -\frac{\hbar \pi^2 c_0}{1440 L_z^3}.$$

This is the renormalized energy density for one of the modes. There is an equal energy density for the other mode and so the total energy density is simply

$$\mathcal{U}_{\text{TOT}} \;=\; 2\,\mathcal{U}_{\text{ren}} \;=\; -\frac{\hbar \pi^2 c_0}{720 L_z^3}.$$

3. The Casimir Pressure

In this section, we shall consider only the TE modes (an analogous approach works for the TM modes also). The pre-potential for the TE modes in our perfectly conducting box are given by

$$\phi_N^{\text{TE}} \;=\; \mathcal{N}_N^{\text{TE}} \cos(k_x x)\,\cos(k_y y)\sin(k_z z)\,dz \tag{4}$$

in terms of the normalization constant $\mathcal{N}_N^{\mathsf{TE}}$ and where

$$k_x = \frac{\pi n_x}{L_x}, \qquad k_y = \frac{\pi n_y}{L_y} \qquad \text{and} \qquad k_z = \frac{\pi n_z}{L_z}.$$

Given the pre-potential, a mode of the TE component of the (spatial) potential 1-form is given by

$$A_N^{\mathsf{TE}} = \# d\phi_N^{\mathsf{TE}}$$

yielding a mode of the operator-valued potential 1-form

$$\widehat{A}_N^{\mathsf{TE}} = \widehat{a}_N \, A_N^{\mathsf{TE}} \, e^{i\omega_N t} + \widehat{a}_N^{\dagger} \, \overline{A_N^{\mathsf{TE}}} \, e^{-i\omega_N t} = \left(\widehat{a}_N \, e^{i\omega_N t} + \widehat{a}_N^{\dagger} \, e^{-i\omega_N t} \right) \# d\phi_N^{\mathsf{TE}}$$

since A_N^{TE} is real and we have introduced the creation and annihilation operators $\widehat{a}_N^{\dagger}, \widehat{a}_N$ respectively satisfying

$$\left[\widehat{a}_N, \, \widehat{a}_M^{\dagger} \right] = \delta_{NM} \qquad \text{and} \qquad \widehat{a}_N | \, 0 \, \rangle = 0 \tag{5}$$

where $| , 0 \, \rangle$ denotes the vacuum state with $\langle \, 0 \, | \, 0 \, \rangle = 1$. The fields are obtained from $\widehat{A}_N^{\mathsf{TE}}$:

$$\widehat{e}_N^{\mathsf{TE}} = i\omega_N \widehat{A}_N^{\mathsf{TE}} \qquad \text{and} \qquad \widehat{b}_N^{\mathsf{TE}} = \# d \widehat{A}_N^{\mathsf{TE}}.$$

In order to normalize the modes, we equate the vacuum expectation value of the energy to $1/2\hbar\omega_N$ on a mode-by-mode basis. The vacuum energy is given by

$$E_N \equiv \int_{\mathcal{V}} \langle \, 0 \, | \, \widehat{E}_{NM} \, | \, 0 \, \rangle$$

where $\mathcal{V}$ denotes the volume of the perfectly conducting box and in terms of the operator-valued energy density 3-form

$$\widehat{E}_{NM} \equiv \frac{\epsilon_0}{2} \left(\widehat{e}_N^{\mathsf{TE}} \wedge \# \overline{\widehat{e}_M^{\mathsf{TE}}} + c_0^2 \widehat{b}_N^{\mathsf{TE}} \wedge \# \overline{\widehat{b}_M^{\mathsf{TE}}} \right).$$

The fields in the perfectly conducting box for the TE modes are given using the pre-potential ϕ_N^{TE} (4) and yield the vacuum expectation of the operator-valued energy density 3-form[1]

$$\langle \, 0 \, | \, \widehat{E}_{NM} \, | \, 0 \, \rangle = \frac{\epsilon_0}{2} \left(\omega_N^2 \, A_N^{\mathsf{TE}} \wedge \# A_N^{\mathsf{TE}} + c_0^2 \# d A_N^{\mathsf{TE}} \wedge d A_N^{\mathsf{TE}} \right).$$

Integrating this over the whole box $\mathcal{V}$ we obtain

$$\begin{aligned} E_N = \int_{\mathcal{V}} \langle \, 0 \, | \, \widehat{E}_{NM} \, | \, 0 \, \rangle &= \frac{1}{8} \epsilon_0 (\mathcal{N}_N^{\mathsf{TE}})^2 c_0^2 L_x L_y L_z (k_x^2 + k_y^2) \left(k_x^2 + k_y^2 + k_z^2 \right) \\ &= \frac{1}{8} \epsilon_0 (\mathcal{N}_N^{\mathsf{TE}})^2 L_x L_y L_z (k_x^2 + k_y^2) \omega_N^2. \end{aligned}$$

Thus, by equating this to $1/2\hbar\omega_N$

$$E_N = \int_{\mathcal{V}} \langle \, 0 \, | \, \widehat{E}_{NM} \, | \, 0 \, \rangle = \frac{1}{2} \hbar \omega_N$$

[1] See (A) for further details of vacuum expectation values

we are able to normalize the modes. We have

$$(\mathcal{N}_N^{\mathsf{TE}})^2 = \frac{4\hbar}{\epsilon_0 L_x L_y L_z (k_x^2 + k_y^2)\omega_N}. \tag{6}$$

Having normalized the modes, we may now calculate the force at a face of the box by using the operator-valued stress 2-forms

$$\sigma_{NM}^K = -\frac{\epsilon_0}{2}\left(\widehat{e}_N^{\mathsf{TE}}(K) \wedge \#\overline{\widehat{e}_M^{\mathsf{TE}}} + c^2 \#\widehat{b}_N^{\mathsf{TE}} \wedge \overline{\widehat{b}_M^{\mathsf{TE}}}(K) + \widehat{e}_N^{\mathsf{TE}} \wedge i_K \#\overline{\widehat{e}_M^{\mathsf{TE}}} - c^2 i_K \#\widehat{b}_N^{\mathsf{TE}} \wedge \overline{\widehat{b}_M^{\mathsf{TE}}}\right)$$

in terms of a Killing vector K. The vacuum expectation value of the force component in the K-direction due to one mode of the quantum electromagnetic fluctuations in the vacuum is given by

$$\mathcal{F}_N^K = \int_\Sigma \langle 0 \,|\, \sigma_{NM}^K \,|\, 0 \rangle.$$

The vacuum expectation value of the total force is then given by summing all these modes:

$$\mathcal{F}^K = \sum_N \mathcal{F}_N^K.$$

To compute the vacuum expectation value of the force on the xy-face at $z = L_z$ we use the Killing vector $K = \partial_z$. Using the fields as defined above, we obtain

$$\mathcal{F}_N^K = \frac{\epsilon_0 (\mathcal{N}_N^{\mathsf{TE}})^2 c^2 \pi^2 n_z^2 (k_x^2 + k_y^2) L_x L_y}{8 L_z^2}.$$

Substituting in for the normalization coefficient $\mathcal{N}_N^{\mathsf{TE}}$ from (6) yields

$$\mathcal{F}_N^K = \frac{\hbar \pi^2 c_0^2}{2 L_z^3} \frac{n_z^2}{\omega_N} = \frac{\hbar \pi^2 c_0}{2 L_z^2} \frac{n_z^2}{\Omega_N}.$$

using (2). Thus, the vacuum expectation value of the total force is given by

$$\mathcal{F}^K = \frac{\hbar \pi^2 c_0}{2 L_z^2} \sum_N \frac{n_z^2}{\Omega_N}.$$

Once again this expression is infinite so needs to be regularized:

$$\mathcal{F}^K(s) = \frac{\hbar \pi^2 c_0}{2 L_z^2} \sum_N \frac{n_z^2}{\Omega_N} W_s(\Omega_N)$$

in terms of some regulating function $W_s(\Omega_N)$. Expanding this out we have

$$\mathcal{F}^K(s) = \frac{\hbar \pi c_0}{2 L_z^2} \sum_N \frac{n_z^2}{\sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2}} W_s\left(\sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2}\right)$$

using (2). As before, we consider the limit $L_x, L_y \to \infty$ so that n_x, n_y become continuous and we may write

$$\mathcal{F}^K(s) = \frac{\hbar\pi c_0}{2L_z^2} \sum_{n_z=0}^{\infty} \int_{n_x=0}^{\infty} dn_x \int_{n_y=0}^{\infty} dn_y \frac{n_z^2}{\sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2}} W_s\left(\sqrt{\frac{n_x^2 L_z^2}{L_x^2} + \frac{n_y^2 L_z^2}{L_y^2} + n_z^2}\right)$$

Using the r_x, r_y notation (3), we may write this as

$$\mathcal{F}^K(s) = L_x L_y \cdot \frac{\hbar\pi c_0}{2L_z^4} \sum_{n_z=0}^{\infty} \int_{r_x=0}^{\infty} dr_x \int_{r_y=0}^{\infty} dr_y \frac{n_z^2}{\sqrt{r_x^2 + r_y^2 + n_z^2}} W_s\left(\sqrt{r_x^2 + r_y^2 + n_z^2}\right).$$

Introducing a polar coordinate representation $(r_x = R\sin\theta, r_y = R\cos\theta)$ with $R^2 = r_x^2 + r_y^2$ and $dr_x\, dr_y = R\, dR\, d\theta$ we obtain

$$\begin{aligned}
\mathcal{F}^K(s) &= L_x L_y \cdot \frac{\hbar\pi c_0}{2L_z^4} \sum_{n_z=0}^{\infty} \int_{R=0}^{\infty} \int_{\theta=0}^{\pi/2} R\, dR\, d\theta \frac{n_z^2}{\sqrt{R^2 + n_z^2}} W_s\left(\sqrt{R^2 + n_z^2}\right) \\
&= L_x L_y \cdot \frac{\hbar\pi^2 c_0}{4L_z^6} \sum_{n_z=0}^{\infty} \int_{R=0}^{\infty} R\, dR \frac{n_z^2}{\sqrt{R^2 + n_z^2}} W_s\left(\sqrt{R^2 + n_z^2}\right).
\end{aligned}$$

Introducing the change of variable $U^2 = R^2 + n_z^2$ with $U\, dU = R\, dR$, we have

$$\frac{\mathcal{F}^K(s)}{L_x L_y} = \frac{\hbar\pi^2 c_0}{4L_z^4} \sum_{n_z=0}^{\infty} \int_{U=n_z}^{\infty} n_z^2 W_s(U)\, dU = \frac{\hbar\pi^2 c_0}{4L_z^4} \sum_{n_z=0}^{\infty} g_s(n_z)$$

where

$$g_s(n_z) \equiv \int_{U=n_z}^{\infty} n_z^2 W_s(U)\, dU = \int_{U=n_z}^{\infty} n_z^2 e^{-sU}\, dU,$$

where we have chosen the same regulator as before: $W_s(U) = e^{-sU}$. This is now in a form suitable to use Euler-Maclaurin (1). With $m = 2$, we have

$$\begin{aligned}
\sum_{n_z=0}^{\infty} g_s(n_z) - \int_0^{\infty} g_s(x)\, dx &= \frac{1}{2}\Big(g_s(0) + g_s(\infty)\Big) + \frac{B_2}{2}\left(g_s'(\infty) - g_s'(0)\right) + \frac{B_4}{24}\left(g_s'''(\infty) - g'''(0)\right) \\
&\quad + \frac{1}{120}\int_0^{\infty} P_5(x) g_s'''''(x)\, dx \\
&= -\frac{1}{120} + O(s^2) \\
&= -\frac{1}{120} + O(s^2)
\end{aligned}$$

using

$$\frac{d}{da} g_s(a) = \frac{d}{da}\int_a^{\infty} a^2 e^{-sU}\, dU = \int_a^{\infty} 2a\, e^{-sU}\, dU - a^2 e^{-sa}$$

and

$$\int_{U=0}^{\infty} e^{-sU}\,dU \quad = \quad -\frac{e^{-sU}}{s}\Bigg|_{U=0}^{\infty} = \frac{1}{s}.$$

The only non-zero terms arise from $-B_4/24g'''(0)$ and the term involving the periodic Bernoulli functions, though these terms are of order s^2 and so vanish upon renormalization. Thus, we define the regularized total force density (pressure) to be

$$\frac{\mathcal{F}^K_{\mathsf{reg}}(s)}{L_x L_y} \quad = \quad \frac{\hbar\pi^2 c_0}{4L_z^4}\left(\sum_{n_z=0}^{\infty} g_s(n_z) - \int_{n_z=0}^{\infty} g_s(x)\,dx\right)$$

and the renormalized total force density (pressure) to be

$$\mathcal{P}^K_{\mathsf{ren}} \quad \equiv \quad \frac{\mathcal{F}^K_{\mathsf{reg}}(0)}{L_x L_y} = -\frac{\hbar\pi^2 c_0}{480L_z^4}.$$

This is only for the TE modes, but there is an analogous derivation to find the renormalized pressure for the TM modes, which is equal to that of the TE modes. Hence, the total pressure is

$$\mathcal{P}^K_{\mathsf{TOT}} \quad = \quad 2\mathcal{P}^K_{\mathsf{ren}} = -\frac{\hbar\pi^2 c_0}{240L_z^4}.$$

A. Vacuum Expectation Values

Let $\widehat{\Lambda}_{NM}$ denote an operator-valued p-form quadratic in the creation and annihilation operators $\widehat{a}_N, \widehat{a}^\dagger_M$. Specifically, let

$$\widehat{\Lambda}_{NM} \;=\; \widehat{\alpha}_N \wedge \widehat{\beta}_M$$

in terms of the operator-valued forms

$$\widehat{\alpha}_N \;=\; \left(\widehat{a}_N\, e^{i\omega_N t} + \widehat{a}^\dagger_N\, e^{-i\omega_N t}\right)\alpha_N \qquad \text{and} \qquad \widehat{\beta}_N \;=\; \left(\widehat{a}_N\, e^{i\omega_N t} + \widehat{a}^\dagger_N\, e^{-i\omega_N t}\right)\beta_N$$

where α_N, β_N are differential forms (whose degrees sum to p). Then

$$\begin{aligned}
\widehat{\Lambda}_{NM} &= \left(\widehat{a}_N\, e^{i\omega_N t} + \widehat{a}^\dagger_N\, e^{-i\omega_N t}\right)\left(\widehat{a}_M\, e^{i\omega_M t} + \widehat{a}^\dagger_M\, e^{-i\omega_M t}\right)\alpha_N \wedge \beta_M \\
&= \left(\widehat{a}_N \widehat{a}_M\, e^{i(\omega_N+\omega_M)t} + \widehat{a}^\dagger_N \widehat{a}_M\, e^{-i(\omega_N-\omega_M)t} + \widehat{a}_N \widehat{a}^\dagger_M\, e^{i(\omega_N-\omega_M)t} + \widehat{a}^\dagger_N \widehat{a}^\dagger_M\, e^{-i(\omega_N+\omega_M)t}\right)\alpha_N \wedge \beta_M.
\end{aligned}$$

From (5), we have

$$\begin{aligned}
\langle\, 0 \,|\, \widehat{a}_N \widehat{a}_M \,|\, 0 \,\rangle &= 0 & \qquad \langle\, 0 \,|\, \widehat{a}^\dagger_N \widehat{a}_M \,|\, 0 \,\rangle &= 0, \\
\langle\, 0 \,|\, \widehat{a}_N \widehat{a}^\dagger_M \,|\, 0 \,\rangle &= \delta_{NM} & \qquad \langle\, 0 \,|\, \widehat{a}^\dagger_N \widehat{a}^\dagger_M \,|\, 0 \,\rangle &= 0
\end{aligned}$$

where we have used the commutation relation (5) and the normalization of the vacuum state $\langle\, 0 \,|\, 0 \,\rangle = 1$. This gives

$$\langle\, 0 \,|\, \widehat{\Lambda}_{NM} \,|\, 0 \,\rangle \;=\; \delta_{NM}\, e^{i(\omega_N-\omega_M)t} \alpha_N \wedge \beta_M \;=\; \alpha_N \wedge \beta_N.$$