

An Brief Introduction to PDEs

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A very, *very* brief introduction to partial differential equations and some methods of solution.

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1 Terminology



A differential equation involving the partial derivatives of a function is known as a *partial differential equation* (PDE). For example:

$$\phi_{xx} + 2\phi_{xy} - y^2\phi_x = \sin(xy) \quad \text{or} \quad \phi_x + 2\phi_y = 0$$

are two differential equations for a function $\phi \equiv \phi(x, y)$. In general, a PDE for $\phi \equiv \phi(x, y)$ can be written as:

$$\mathcal{F}(x, y, \phi, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yy}, \dots) = 0$$

i.e. as a function of the *independent variables* x, y and the dependent variables $\phi(x, y)$ including all of its partial derivatives $\phi_x, \phi_y, \phi_{xx}$, etc. If the function \mathcal{F} above is *linear*, then the PDE is called *linear*. That is, if we write the PDE in terms of a *differential operator* \mathcal{L} acting on ϕ : $\mathcal{L}\phi = g(x, y)$ where g is some function independent of ϕ , then the PDE is linear if \mathcal{L} satisfies:

$$\mathcal{L}(\lambda\phi) = \lambda\mathcal{L}\phi \quad \text{and} \quad \mathcal{L}(\phi_1 + \phi_2) = \mathcal{L}\phi_1 + \mathcal{L}\phi_2$$

for any $\lambda \in \mathbb{R}$ and function $\phi_i \equiv \phi_i(x, y)$. If $g = 0$ in the PDE $\mathcal{L}\phi = g(x, y)$ then the PDE is *homogeneous*; otherwise it is *inhomogeneous*. As with ODEs, linearity of a homogeneous PDE allows one to create a *superposition* of solutions:

$$\mathcal{L}u_1 = 0 \quad \text{and} \quad \mathcal{L}u_2 = 0 \quad \implies \quad \mathcal{L}(au_1 + bu_2) = 0$$

and so $au_1 + bu_2$ is a solution for any $a, b \in \mathbb{R}$. This is extended to the inhomogeneous PDE since the addition of a homogeneous solution of a PDE to an inhomogeneous PDE is still a solution:

$$\underbrace{\mathcal{L}u_1 = g}_{\text{inhomogeneous solution}} \quad \text{and} \quad \underbrace{\mathcal{L}u_2 = 0}_{\text{homogeneous solution}} \quad \implies \quad \mathcal{L}(au_1 + bu_2) = a\mathcal{L}u_1 + b\mathcal{L}u_2 = g + 0 = g.$$

Consider the following first-order homogeneous PDE:

$$\phi_x = e^{2x}$$

for some function $\phi \equiv \phi(x, y)$. As with ODEs, we can solve this equation using (partial) integration:

$$\phi = \int e^{2x} dx = \frac{1}{2}e^{2x} + C(y)$$

in terms of an arbitrary *function* $C(y)$. As with ODEs, we can check this result by (partially) differentiating the solution with respect to x . The difference between PDEs and ODEs is now transparent: whereas the solution to an ODE contains arbitrary *constants of integration*, the solution to a PDE, in general, contains arbitrary *functions*. These are fixed by additional conditions on the solution that arise from the physical problem at hand. These are often split into two different classes: *initial conditions* and *boundary conditions*. Most physical problems we consider shall be *evolutionary* and depend upon time t . In general, they start from some initial time (usually $t = 0$) and so we must express the value of the function, and possibly some of its derivatives, at this time: these are called *initial conditions*. This should be contrasted with *boundary conditions* which specify the spatial region in which the PDE is valid. The most important boundary conditions are:

Dirichlet condition: ϕ is specified on the boundary.

Neumann condition: The derivative of ϕ normal to the boundary is specified.

The number of initial and boundary conditions one must specify in order to define the problem depend upon the *order* of the PDE in the temporal and spatial variables respectively. We shall illustrate these remarks in the subsequent chapters when we analyse three particular PDEs: the wave equation, the heat equation and Laplace's equation.

2 First-order PDEs and the Method of Characteristics



We begin our examination of partial differential equations by considering *quasi-linear* first-order PDES:

$$a(t, x, u) \frac{\partial u}{\partial t} + b(t, x, u) \frac{\partial u}{\partial x} = c(t, x, u) \quad \text{or} \quad a(t, x, u) u_t + b(t, x, u) u_x = c(t, x, u) \quad (1)$$

using the subscript notation introduced previously, for some function $u \equiv u(t, x)$ and arbitrary functions a, b, c . The term quasi-linear refers to PDEs whose highest-order terms appear only as individual terms multiplied by lower order terms. There are two specific subclasses:

$$\text{Semi-linear:} \quad a(t, x) u_t + b(t, x) u_x + d(t, x) = c(t, x, u)$$

$$\text{Linear:} \quad a(t, x) u_t + b(t, x) u_x = c(t, x) u + d(t, x).$$

Therefore *semi-linear* PDEs are those whose highest-order terms are linear; whereas a linear PDE conforms to description in the previous chapter. First-order PDEs occur often in physics and engineering, as they are intimately related to *conservation laws* which are fundamental to our description and understanding of the physical world.

The technique that we shall employ to solve first-order PDEs - the method of characteristics - is used to solve quasi-linear PDEs and hence can also be employed for the various subclasses. It has a nice geometric interpretation that we attempt to describe below:

A solution to (1) is of the form $u = u(t, x)$: a surface in \mathbb{R}^3 known as the *integral surface* of the PDE. The surface is subject to some initial condition:

$$u(t_0(s), x_0(s)) = F(t_0(s), x_0(s))$$

in terms of some arbitrary function F and s -parametrised curve $(t_0(s), x_0(s))$ in the tx -plane which we shall call the *anchor curve*. This terminology is not widespread, but is a useful concept¹. This curve can be “lifted” to the integral surface and defines the initial curve:

$$(t_0(s), x_0(s), u_0(s)) = (t_0(s), x_0(s), u(t_0(s), x_0(s))) = (t_0(s), x_0(s), F(t_0(s), x_0(s))).$$

Consider now a vector which is *normal* to the surface at each point. It can be shown² that a normal vector to any point on the integral surface is given by:

$$\mathbf{n} = (u_t, u_x, -1).$$

Notice now that can rewrite (1) in terms of \mathbf{n} :

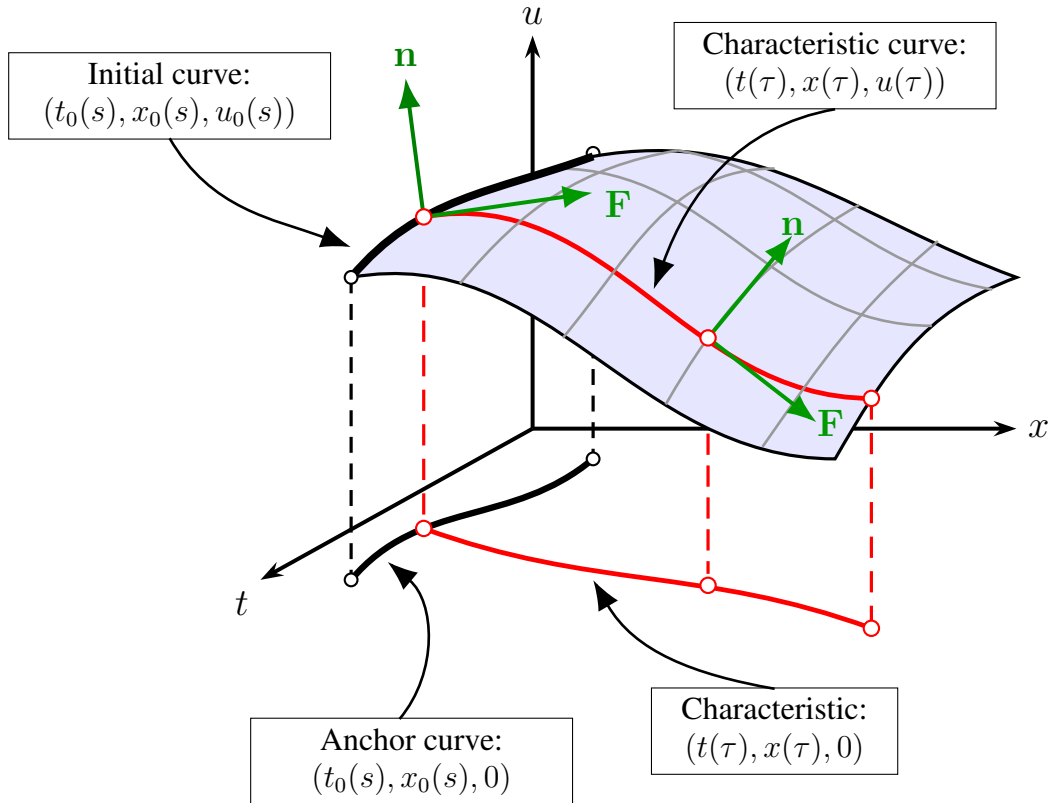
$$\begin{aligned} a(t, x, u) u_t + b(t, x, u) u_x - c(t, x, u) &= 0 \implies \begin{pmatrix} a(t, x, u) \\ b(t, x, u) \\ c(t, x, u) \end{pmatrix} \cdot \begin{pmatrix} u_t \\ u_x \\ -1 \end{pmatrix} = 0 \\ &\implies \mathbf{F} \cdot \mathbf{n} = 0 \end{aligned}$$

¹I came across the idea in the book: “Partial Differential Equations: Theory and Solved Problems” by T.Hillen, I.E. Leonard and H. Van Roessel (Wiley, 2012): an interesting and useful read.

²Given any surface in \mathbb{R}^3 : $u = f(t, x)$ or $g(t, x, u) = f(t, x) - u = 0$, the vector $(\nabla g)(t, x, u) = (\partial_t g, \partial_x g, \partial_u g) = (f_t, f_x, -1)$ is normal to every point on the surface.

in terms of the vector $\mathbf{F} \equiv (a(t, x, u), b(t, x, u), c(t, x, u))$. This says that \mathbf{F} is perpendicular to \mathbf{n} at *every* point on the integral surface. That is, \mathbf{F} is *tangent* to *every* point on the integral surface. Given such a vector field, we can find the family of *integral curves* to which the vector field is tangent to and the integral surface, the solution to our PDE which we are trying to describe, is made up of these integral curves. If (t_0, x_0, u_0) is a point lying on the integral surface of the PDE and $(t(\tau), x(\tau), u(\tau))$ a τ -parametrised curve that passes through (t_0, x_0, u_0) at $\tau = 0$, then it can be shown that the conditions for the curve to remain on the integral surface (i.e. to be an integral curve of \mathbf{F}) are:

$$\frac{dt}{d\tau} = a(t(\tau), x(\tau), u(\tau)), \quad \frac{dx}{d\tau} = b(t(\tau), x(\tau), u(\tau)), \quad \frac{du}{d\tau} = c(t(\tau), x(\tau), u(\tau)). \quad (2)$$



These are known as the *characteristic equations* and define the *characteristic curve* of the PDE. The *projection* of the characteristic curve into the xy -plane are the *characteristics*. The characteristic equations are accompanied by the initial conditions³:

$$t(0) = t_0(s), \quad x(0) = x_0(s), \quad u(0) = u_0(s) \quad (3)$$

which stipulate the integral curves (i.e. the characteristics) must start on the initial curve. Since each characteristic starts at a different point on the initial curve (a different value of s), we have a *one-parameter family of characteristics* $(t_s(\tau), x_s(\tau), u_s(\tau)) = (t(\tau, s), x(\tau, s), u(\tau, s))$: a relation that represents a parametrised surface in \mathbb{R}^3 . Thus, we have reduced our problem of solving (1) to solving the system of first-order ODEs (2) subject to the initial conditions (3). This is known as the *Cauchy problem for quasilinear PDEs* and one can consider the characteristic curves as propagating forward from the initial curve, independently of one another, and “knitting” together to form the integral surface.

³The initial conditions are chosen here to be at $\tau = 0$, but they may be at any $\tau = \tau_0 \in \mathbb{R}$.

The heuristic argument above can be used to form an algorithm for solving quasilinear PDEs. We outline the method in the following examples:

Example

Consider the PDE system:

$$\begin{cases} u_t + c u_x = 0 \\ u(0, x) = f(x) \end{cases}$$

in terms of a constant $c \in \mathbb{R}$ and arbitrary function f . The anchor curve is $(0, x)$: the x -axis, which is s -parametrised by the curve $(0, s)$. This gives initial curve $(0, s, f(s))$ on the integral surface in \mathbb{R}^3 . Denoting τ -derivatives using a prime (i.e. $df/d\tau = f'(\tau)$ for any function f), the characteristic equations and their solution are:

$$\begin{cases} t'(\tau) = 1 \\ x'(\tau) = c \\ u'(\tau) = 0 \end{cases} \implies \begin{cases} t(\tau) = \tau + C_1 \\ x(\tau) = c\tau + C_2 \\ u(\tau) = C_3 \end{cases}$$

in terms of the constants of integration $C_1, C_2, C_3 \in \mathbb{R}$. These constants are fixed by the matching the characteristic curve to the initial curve at $\tau = 0$:

$$\begin{cases} t(0) = 0 \\ x(0) = s \\ u(0) = f(s) \end{cases} \implies \begin{cases} t(\tau) = \tau \\ x(\tau) = c\tau + s \\ u(\tau) = f(s) \end{cases}$$

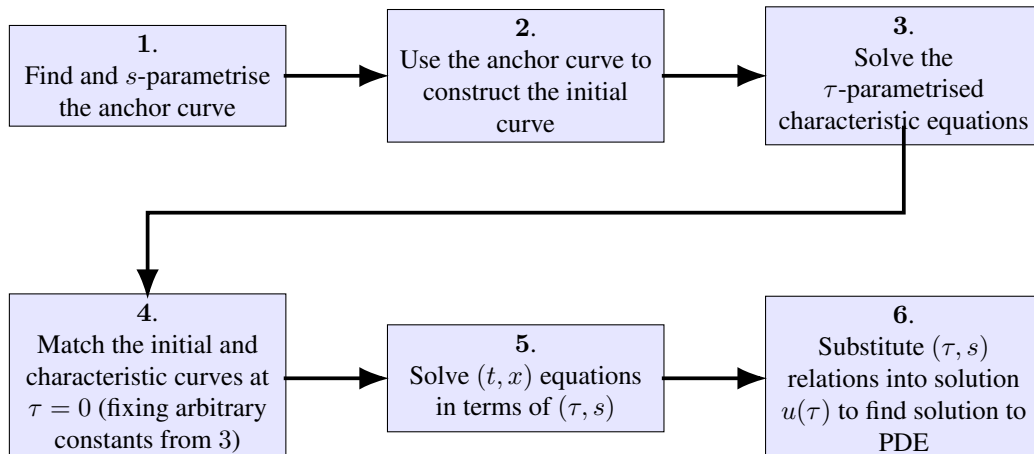
The first two equations relate (t, x) to (τ, s) and can be inverted:

$$\tau = t \quad \text{and} \quad s = x - c\tau = x - ct$$

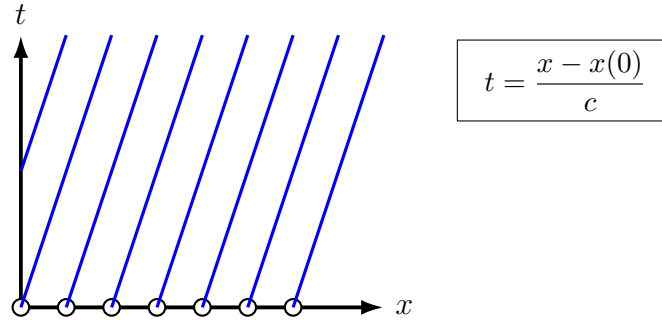
yielding solution to the PDE:

$$u(\tau) = f(s) = f(x - ct).$$

It can be checked that this is indeed a solution by plugging the result back in the partial differential equation. As can be seen from the example, the general method to solve a quasi-linear PDE is as follows:



To help further understand a solution, it is useful to examine the characteristics. For the example above, these are given by the τ -parametrised curves $(t, x) = (\tau, c\tau + s)$ or $s = x - ct$. Notice that the solution $u = f(x - ct)$ is constant along this characteristics. The equation for the characteristics can be rearranged as $t = (x - s)/c$ or $t = (x - x(0))/c$ as $x(0) = s$ is the initial value of x on the characteristic curves for this problem. Thus, plotting the characteristics:



The characteristics given the direction upon which the initial curve propagates (the circles indicate particular positions on the s -parametrised initial curve). For the example above, we see they move forward in time to the right (increasing x) with “speed” given by the gradient $1/c$. One can see that the initial profile, as specified by the initial condition $u(0, x) = f(x)$ will be parallel propagated along the characteristics without spreading.

Example

Consider the PDE system:

$$\begin{cases} (t + x) u_t + x u_x = u + 1 \\ u(0, x) = x^2. \end{cases}$$

The anchor curve is $(0, x)$: the x -axis, which is s -parametrised by the curve $(0, s)$. This gives initial curve $(0, s, s^2)$ on the integral surface in \mathbb{R}^3 . The characteristic equations and their solution are:

$$\left. \begin{aligned} t'(\tau) &= t + x \\ x'(\tau) &= x \\ u'(\tau) &= u + 1 \end{aligned} \right\} \implies \begin{cases} t(\tau) = C_1 \tau e^\tau + C_2 e^\tau \\ x(\tau) = C_1 e^\tau \\ u(\tau) = C_3 e^\tau - 1 \end{cases}$$

in terms of the constants of integration $C_1, C_2, C_3 \in \mathbb{R}$. Note the ODE system is *coupled*, in general requiring simultaneous solution, but these can be solved *sequentially* to yield a solution (starting with the $x(\tau)$ ode). The constants of integration are fixed by the matching the characteristic curve to the initial curve at $\tau = 0$:

$$\left. \begin{aligned} t(0) &= 0 \\ x(0) &= s \\ u(0) &= s^2 \end{aligned} \right\} \implies \begin{cases} t(\tau) = s\tau e^\tau \\ x(\tau) = s e^\tau \\ u(\tau) = (1 + s^2)e^\tau - 1 \end{cases}$$

The first two equations relate (t, x) to (τ, s) . Using

$$\frac{t}{x} = \frac{s\tau e^\tau}{s e^\tau} = \tau \quad \text{and} \quad s = x e^{-\tau} = x e^{-\frac{t}{x}}$$

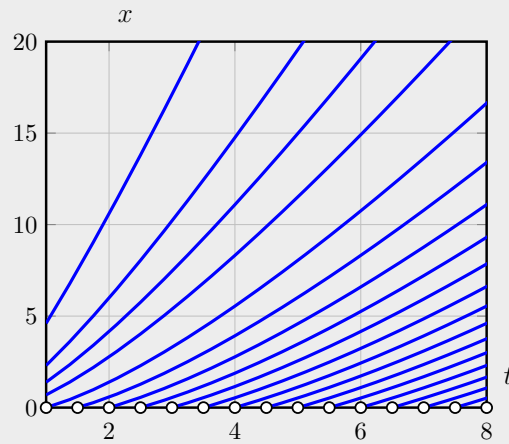
we can invert the relations:

$$\tau = \frac{t}{x} \quad \text{and} \quad s = xe^{-\frac{t}{x}}.$$

These yield solution to the PDE:

$$u(\tau) = (1 + s^2)e^\tau - 1 = (1 + x^2e^{-\frac{2t}{x}})e^{\frac{t}{x}} - 1 = x^2e^{-\frac{t}{x}} + e^{\frac{t}{x}} - 1.$$

The characteristics of this PDE are given by the τ -parametrised curve $(t, x) = (s\tau e^\tau, se^\tau)$ or $s = xe^{-t/x}$ yielding $t = x \ln(x/s) = x \ln(x/x(0))$ since $x(0) = s$ on the initial curve.



One can see from these characteristics how the initial profile will spread as it is propagated along the characteristics of the PDE.

3 Second-order PDEs and their Classification



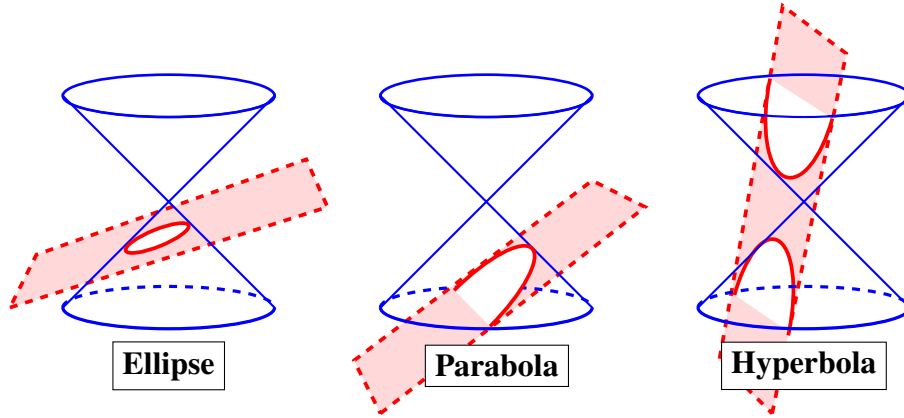
With analogy to a general second-order *algebraic* equation in the plane:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

in terms of arbitrary constants A, B, \dots, F , we consider the general form of a linear second-order PDE in terms of two independent variables x, y

$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} + D\phi_x + E\phi_y + F\phi = G$$

where A, B, \dots, G are, in general, functions of x and y (if G is also a function of ϕ then the PDE is called *quasi-linear*).



As with the algebraic equation, such a PDE can be classified by the value of the discriminant $\Delta(x, y) \equiv B^2 - 4AC$:

$$\begin{aligned} \Delta(x, y) > 0: & \quad \text{the PDE is } \textit{hyperbolic} \\ \Delta(x, y) = 0: & \quad \text{the PDE is } \textit{parabolic} \\ \Delta(x, y) < 0: & \quad \text{the PDE is } \textit{elliptic}. \end{aligned}$$

Notice that this implies the character of a PDE is fully determined by the coefficients of the second derivatives: it has nothing to do with the lower derivative terms. Since the discriminant is a function of x and y , this classification applies to *a specific point* (x, y) *on the domain* where the PDE is valid. In particular, this means *a PDE may change classification from one region to another*. If A, B, C are constants, this cannot occur and the classification remains the same throughout the domain where the PDE is valid.

It can be shown any PDE of a particular classification can be reduced to canonical form:

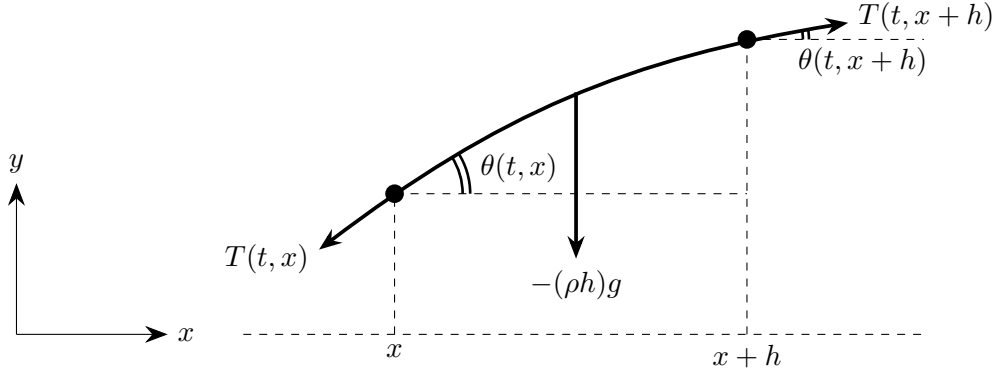
$$\begin{aligned} \text{Hyperbolic:} & \quad \phi_{xx} - \phi_{yy} + \dots = 0 \\ \text{Parabolic:} & \quad \phi_{xx} + \dots = 0 \quad \text{or} \quad \phi_{yy} + \dots = 0 \\ \text{Elliptic:} & \quad \phi_{xx} + \phi_{yy} + \dots = 0 \end{aligned}$$

where the ‘dots’ represent terms with derivatives of lower orders. We shall consider a PDE of each type in the following three sections, ones associated with the leading terms in each classification.

4 The Wave Equation



Consider a taut string with constant mass density ρ . Let $\phi(t, x)$ denote the small vertical displacement of the string; we shall neglect the horizontal displacement. The forces acting on a small section of the string of length h are tension T and gravity $-(\rho h)g$:



Since there is no motion in the horizontal direction, the forces balance:

$$T(t, x) \cos(\theta(t, x)) = T(t, x+h) \cos(\theta(t, x+h)) \implies \frac{T(t, x+h) \cos(\theta(t, x+h)) - T(t, x) \cos(\theta(t, x))}{h}$$

or in the limit as $h \rightarrow 0$:

$$\frac{\partial}{\partial x} (T(t, x) \cos(\theta(t, x))) = 0 \implies T(t, x) \cos(\theta(t, x)) = T_0(t). \quad (4)$$

In the vertical direction we apply Newton's Second Law:

$$\begin{aligned} \rho h \phi_{tt}(t, x) &= T(t, x+h) \sin(\theta(t, x+h)) - T(t, x) \sin(\theta(t, x)) - \rho h g \\ \implies \phi_{tt}(t, x) &= \frac{T(t, x+h) \sin(\theta(t, x+h)) - T(t, x) \sin(\theta(t, x))}{\rho h} - g \end{aligned}$$

or, in the limit as $h \rightarrow 0$:

$$\phi_{tt}(t, x) = \frac{1}{\rho} \frac{\partial}{\partial x} (T(t, x) \sin(\theta(t, x))) - g.$$

Using (4), this can be written

$$\phi_{tt}(t, x) = \frac{T_0(t)}{\rho} \frac{\partial}{\partial x} (\tan(\theta(t, x))) - g.$$

Geometrically it is clear that $\tan(\theta(t, x)) = \phi_x(t, x)$ and so

$$\phi_{tt}(t, x) = \frac{T_0(t)}{\rho} \phi_{xx}(t, x) - g.$$

Neglecting the force due to gravity and assume $T_0(t) \equiv T_0$ is a constant, then this can be written:

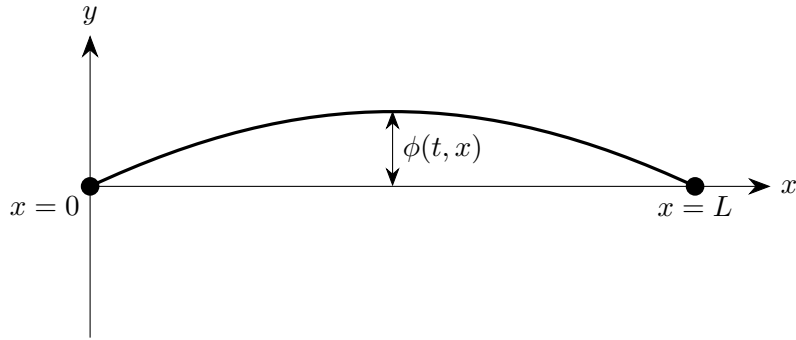
$$\phi \equiv \phi(t, x) : \quad \phi_{tt} = c^2 \phi_{xx} \quad \text{where} \quad c \equiv \sqrt{\frac{T_0}{\rho}}$$

which is known as the *wave equation*. The number c is an important parameter defining the speed of the wave. It can be shown that the same equation arises in different physical settings, such as longitudinal or torsional stress waves in a rod. It is easily generalised to higher dimensions:

$$\begin{aligned}
\text{In two spatial dimensions, } \phi \equiv \phi(t, x, y): \quad & \phi_{tt} = c^2 (\phi_{xx} + \phi_{yy}) \\
\text{In three spatial dimensions, } \phi \equiv \phi(t, x, y, z): \quad & \phi_{tt} = c^2 \underbrace{(\phi_{xx} + \phi_{yy} + \phi_{zz})}_{\nabla^2 \phi}
\end{aligned}$$

in terms of the *Laplacian* operator ∇^2 . Such equations appear in the study of acoustic waves, water waves, electromagnetic waves and others.

Since the wave equation contains second-order derivatives in both space x and time t , we require two boundary conditions and two initial conditions to determine the solution for all $t > 0$. We consider the vertical motion $\phi(t, x)$ of a plucked string at time t and position x whose ends at $x = 0$ and $x = L$ are held fixed:



The motion of the string is governed by the wave equation in one spatial dimension:

$$\phi_{tt} = c^2 \phi_{xx} \quad (0 < x < L, t > 0)$$

where we have specified the domain of validity of the PDE. The boundary conditions of the problem are the specification that the solution must vanish at the endpoints, since the string is held fixed and unable to move vertically at these points:

$$\phi(t, 0) = 0 \quad \text{and} \quad \phi(t, L) = 0 \quad (t \geq 0).$$

Note that these conditions must hold *for all time* $t > 0$. The initial conditions of the problem are the specification of the initial position of the string and its initial velocity. For example:

$$\phi(0, x) = \sin\left(\frac{\pi x}{L}\right) \quad \text{and} \quad \phi_t(0, x) = 0 \quad (0 \leq x \leq L)$$

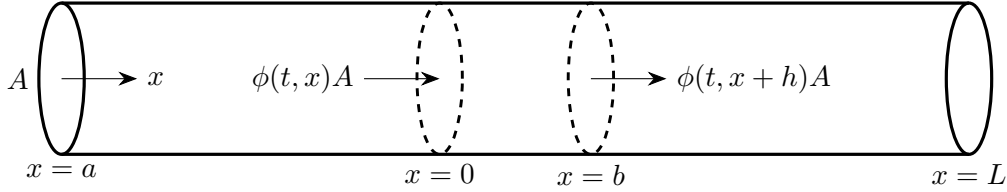
specifies a string held at rest with initial shape corresponding to half a sine-wave. Notice that the initial conditions and boundary conditions ‘overlap’ at $t = 0, x = 0$ and $t = 0, x = L$, so it must be checked they are compatible at this points. The wave equation will then determine the motion of the string once it is released. Together, we have the following *initial-value boundary-value problem*:

$$\left\{ \begin{array}{ll} \phi_{tt} = c^2 \phi_{xx} & (0 < x < L, t > 0) \\ \phi(t, 0) = 0 & (t \geq 0) \\ \phi(t, L) = 0 & (t \geq 0) \\ \phi(0, x) = \sin\left(\frac{\pi x}{L}\right) & (0 \leq x \leq L) \\ \phi_t(0, x) = 0 & (0 \leq x \leq L) \end{array} \right.$$

5 The Heat Equation



Consider a one-dimensional rod of length L and constant cross-sectional area A :



Consider a small section of the rod between $x = a$ and $x = b$. The heat energy Q of the section is given by

$$Q = \int_{x=a}^b e(t, x) A dx$$

where $e(t, x)$ is the thermal energy density of the rod. Due to the conservation of energy, the rate of change of thermal energy through a section of the rod between $x = a$ and $x = b$ is due to the energy flow through the ends:

$$\begin{array}{ccccc} \text{Rate of change of} & = & \text{Heat energy flowing across} & - & \text{Heat energy generated} \\ \text{heat energy} & & \text{boundaries per unit time} & & \text{inside per unit time} \end{array} .$$

We introduce the *heat flux*: the amount of thermal energy per unit time flowing per unit surface area. Consequently, the heat energy flowing per unit time across the boundaries of the section of the rod is $\phi(t, a)A - \phi(t, b)A$. Thus:

$$\frac{d}{dt} \int_{x=a}^b e(t, x) dx = \phi(t, a) - \phi(t, b)$$

after cancellation of the constant A . Assuming a, b to be constants then the derivative can be interchanged with the integral:

$$\frac{d}{dt} \int_{x=a}^b e(t, x) dx = \int_{x=a}^b \frac{\partial e}{\partial t} dx.$$

Furthermore, noting that

$$\phi(t, a) - \phi(t, b) = - \int_{x=a}^b \frac{\partial \phi}{\partial x} dx$$

yields

$$\int_{x=a}^b \left(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} \right) dx = 0.$$

Since this integral must vanish for arbitrary a, b then the integrand itself must vanish (see the next chapter for further discussion of this) and hence

$$\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} = 0 \implies \frac{\partial e}{\partial t} = - \frac{\partial \phi}{\partial x}.$$

The thermal energy density is related to the temperature of the rod by $e(t, x) = c\rho T(t, x)$ where c is the specific heat of the rod and ρ is the constant mass density of the rod. Along with Fourier's Law relating the heat flux to the *gradient* of the temperature of the rod:

$$\phi(t, x) = -K \frac{\partial T}{\partial x}$$

where K is the (constant) thermal conductivity of the rod, we obtain

$$c\rho(x) \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \implies T_t = \kappa T_{xx} \quad \text{where} \quad \kappa = \frac{K}{c\rho}.$$

The constant κ is called the *thermal diffusivity* of the rod and this PDE is known as the *heat equation* or, more generally, the *diffusion equation* (as it applies to other physical processes involving quantities that 'spread out'). It differs from the wave equation simply by only having a first-order derivative in t and not a second-order derivative. As with the wave equation, it is easily generalised to higher dimensions:

$$\begin{aligned} \text{In two spatial dimensions, } \phi \equiv \phi(t, x, y): \quad & \phi_t = \kappa (\phi_{xx} + \phi_{yy}) \\ \text{In three spatial dimensions, } \phi \equiv \phi(t, x, y, z): \quad & \phi_t = \kappa \nabla^2 \phi \end{aligned}$$

Returning to our one-dimensional rod, since the heat equation contains a single derivative in time t , only a single initial condition is required to determine the solution. Such a condition specifies the initial temperature distribution of the rod at time $t = 0$. Boundary conditions are also required at each of the rod ($x = 0$ and $x = L$) and the appropriate condition depends upon the physical mechanism in effect. For example, we may want to prescribe the temperature at one end: $T(t, 0) = 10$ implies that the temperature at $x = 0$ is held at 10° for all time $t > 0$, perhaps due to being in contact with a thermal bath; and we may wish to describe the other end as being (perfectly insulated): $T_x(0, t) = 0$ since T_x is related to the heat flow via Fourier's Law. In such a case, there is no heat flow through this end of the rod. Thus, the heat equation is also an initial-value boundary-value problem of the form:

$$\left\{ \begin{array}{ll} \phi_t = \kappa^2 \phi_{xx} & (0 < x < L, t > 0) \\ \phi(t, 0) = T_0 & (t \geq 0) \\ \phi(t, L) = T_L & (t \geq 0) \\ \phi(0, x) = x^2 & (0 \leq x \leq L) \end{array} \right.$$

in terms of two constants T_1, T_2 specifying the constant temperatures of the end points at $x = 0$ and $x = L$ respectively.

6 Laplace's Equation



Consider the heat and wave equations in two spatial dimensions:

$$\phi_{xx} + \phi_{yy} = \kappa \phi_t \quad \text{and} \quad \phi_{xx} + \phi_{yy} = c^2 \phi_{tt}$$

respectively. If either equation is in a *steady* or *stationary state* (i.e. independent of time) then $\phi_t = \phi_{tt} = 0$ and they both reduce to *Laplace's equation*:

$$\phi_{xx} + \phi_{yy} = 0.$$

In three dimensions this becomes

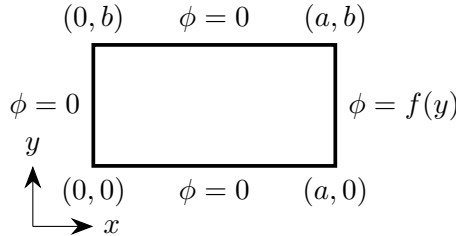
$$\nabla^2 \phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = 0.$$

Functions ϕ which satisfy Laplace's equation are called harmonic. The inhomogeneous version of Laplace's equation:

$$\nabla^2 \phi = f(t, x, y, z)$$

is known as *Poisson's equation*. These equations appear in many physical processes including electrostatics, steady fluid flow, Brownian motion and Newtonian gravity.

Since Laplace's equation only involves spatial derivatives of second-order, it is a *boundary-value problem*. For example, consider the following physical problem: a thin rectangular plate has its edges fixed at temperatures zero on three sides and $f(y)$ on the remaining side:



Its lateral sides are then insulated and it is allowed to stand for a “long” time with the edges maintained at the aforementioned boundary temperatures. To find the temperature distribution in the plate, which is now in a steady state, we solve Laplace's equation on the rectangle:

$$\left\{ \begin{array}{ll} \phi_{xx} + \phi_{yy} = 0 & (0 < x, y < L) \\ \phi(0, y) = 0 & (0 \leq y \leq b) \\ \phi(a, y) = f(y) & (0 \leq y \leq b) \\ \phi(x, 0) = 0 & (0 \leq x \leq a) \\ \phi(x, b) = 0 & (0 \leq x \leq a). \end{array} \right.$$

7 Methods of Solution



In the final section we shall discuss to particular methods of solution: *separation of variables* and the *Laplace transform*. Our aim will be rather modest: rather than solve the PDE we shall aim to reduce it to a system of ODEs which, from a previous chapter, we have methods to solve (though we will indicate briefly how to proceed). We shall demonstrate the methods through two examples.

Example

Consider the PDE system (a rescaled version of the wave equation on a unit string):

$$\begin{cases} \phi_{tt} = \phi_{xx} & (0 < x < 1, t > 0) \\ \phi(t, 0) = 0 & (t \geq 0) \\ \phi(t, 1) = 0 & (t \geq 0) \\ \phi(0, x) = f(x) & (0 \leq x \leq 1) \\ \phi_t(0, x) = 0 & (0 \leq x \leq 1) \end{cases}$$

in terms of some known function $f(x)$. The method of *separation of variables* attempts to find a solution to a PDE by writing the unknown function of two (or more) variables by a product of functions of a single variable. For this example, we assume:

$$\phi(t, x) = T(t)X(x)$$

in terms of the non-zero functions T, X to be found. Substitution of this into the PDE yields:

$$\phi_{tt} = \phi_{xx} \implies T_{tt}X = TX_{xx} \implies \frac{T_{tt}}{T} = \frac{X_{xx}}{X}$$

after division of both sides by XT . We have now separated variables: the left-hand side of the final expression is purely a function of x whereas the right-hand side is purely a function of t . Differentiating with respect to t gives:

$$\frac{d}{dt} \left(\frac{T_{tt}}{T} \right) = 0 \implies \frac{T_{tt}}{T} = \text{constant}.$$

Call this constant λ then we have obtained:

$$\frac{T_{tt}}{T} = \frac{X_{xx}}{X} = \lambda \implies \begin{cases} T_{tt} = \lambda T \\ X_{xx} = \lambda X \end{cases}$$

and we have reduced the problem to a pair of coupled ODEs. To proceed, one must solve the second-order ODEs. Since the auxiliary equation for both is $M^2 = \lambda$ with solution $M = \pm\sqrt{\lambda}$, it is clear that one must analyse the cases $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$ separately. These lead to three distinct solutions:

$$\begin{aligned} \lambda < 0: \quad & \phi(t, x) = [A_1 \sin(\sqrt{\lambda}x) + B_1 \cos(\sqrt{\lambda}x)] [C_1 \sin(\sqrt{\lambda}t) + D_1 \cos(\sqrt{\lambda}t)] \\ \lambda = 0: \quad & \phi(t, x) = [A_2 + B_2x] [C_2 + D_2t] \\ \lambda > 0: \quad & \phi(t, x) = [A_3 \sinh(\sqrt{\lambda}x) + B_3 \cosh(\sqrt{\lambda}x)] [C_3 \sinh(\sqrt{\lambda}t) + D_3 \cosh(\sqrt{\lambda}t)]. \end{aligned}$$

However, they must satisfy both the initial conditions and the boundary conditions. The only class of solution that yield solutions that satisfy both the PDE and the initial conditions is $\lambda < 0$. Consequently, it is convention to write the constant $\lambda = -\alpha^2$ and hence the solution subject to the initial conditions becomes:

$$\phi(t, x) = A \sin(n\pi x) [B \sin(n\pi t) + C \cos(n\pi t)]$$

where $\alpha = n\pi$ for any integer n . The boundary condition $\phi_t(0, x) = 0$ implies $B = 0$ yielding

$$\phi(t, x) = A \sin(n\pi x) \cos(n\pi t)$$

is a solution for arbitrary A . In fact, since the PDE was linear:

$$\phi(t, x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cos(n\pi t) = A_1 \sin(\pi x) \cos(\pi t) + A_2 \sin(2\pi x) \cos(2\pi t) + \dots$$

is a solution to the PDE. We are yet to satisfy the final boundary condition $\phi(0, x) = f(x)$ for some known function $f(x)$. This is satisfied provided:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x).$$

In fact, it can be shown that any function can be expanded in terms of a linear combination of sines and cosines (so that any odd function f can be expanded as above). This is known as a *Fourier series* and is beyond the scope of the current course. Needless to say, the boundary condition can be satisfied and this condition determines the unknown constants A_n . Thus, we are able to obtain a unique solution to our PDE problem.

Example

Consider the PDE system (a rescaled version of the heat equation in a rod of unit length):

$$\begin{cases} \phi_t = \phi_{xx} & (0 < x < 1, t > 0) \\ \phi(t, 0) = 0 & (t \geq 0) \\ \phi(t, 1) = T_0 & (t \geq 0) \\ \phi(0, x) = 0 & (0 \leq x \leq 1) \end{cases}$$

in terms of some constant T_0 . We shall solve the system using Laplace transforms and in order to perform the transform of a function of two variables *with respect to time* t , we transform from t to s and consider the other variable as a parameter:

$$\begin{aligned} \mathcal{L}\{\phi(t, x)\} &= \Phi(s, x), & \mathcal{L}\{\phi_t(t, x)\} &= s\Phi(s, x) - \phi(0, x) \\ \mathcal{L}\{\phi_{xx}(t, x)\} &= \Phi_{xx}(s, x), & \mathcal{L}\{\phi_{xx}(t, x)\} &= \Phi_{xx}(s, x) \end{aligned}$$

where $s > 0$. Consequently, the Laplace transform of the PDE becomes:

$$\mathcal{L}\{\phi_t\} = \mathcal{L}\{\phi_{xx}\} \implies s\Phi(s, x) - \phi(0, x) = \Phi_{xx}(s, x) \implies s\Phi(s, x) = \Phi_{xx}(s, x).$$

where we have used the initial condition. Thus, we have reduced the problem to a second-order ODE in x . As $s > 0$, this has solution

$$\Phi(s, x) = A \sinh(\sqrt{s} x) + B \cosh(\sqrt{s} x).$$

We can fix the constants using the boundary conditions which must also be Laplace transformed:

$$\begin{cases} \mathcal{L}\{\phi(t, 0)\} = \mathcal{L}\{0\} \\ \mathcal{L}\{\phi(t, 1)\} = \mathcal{L}\{T_0\} \end{cases} \implies \begin{cases} \Phi(s, 0) = 0 \\ \Phi(s, 1) = \frac{T_0}{s}. \end{cases}$$

These boundary conditions for $\Phi(s, x)$ yield:

$$\Phi(s, x) = \frac{T_0 \sinh(\sqrt{s} x)}{s \sinh(\sqrt{s})}.$$

The solution to the PDE can then be found by taking the inverse Laplace transform:

$$\phi(t, x) = \mathcal{L}^{-1} \left\{ \frac{T_0 \sinh(\sqrt{s} x)}{s \sinh(\sqrt{s})} \right\}.$$

Of course, this last step is non-trivial and beyond the methods we have employed in this course.

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