Dimensional Analysis

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Abstract

Here are a brief set of gapped lecture notes discussing foundational principles of Dimensional Analysis. The primary focus is on applications of the Buckingham Π Theorem to determine the functional dependence of some governing equations in physics.

Contents

Introduction

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The techniques of dimensional analysis are basic in the theory and practice of mathematical modelling. A good grasp of the possible relationships and comparative magnitudes among the various dimensioned parameters nearly always leads to a better understanding of the problem and sometimes points the way toward approximations and solutions. In these notes, we will briefly introduce some of the basic concepts from this topic. Along with several examples, a statement of the fundamental result in dimensional analysis, *the Buckingham Pi theorem*, is presented. A proof of this theorem can be found in the appendix.

There are many good books on this topic, so if you wish to delve deeper into dimensional analysis, I suggest you check out the following:

- J.C. Gibbings, 2011. *Dimensional Analysis*, Springer Science & Business Media.
- M. H. Holmes, 2009. *Introduction to the Foundations of Applied Mathematics*, Springer New York.
- D. S. Lemons, 2017, *A Student's Guide to Dimensional Analysis*, Cambridge University Press
- J. D. Logan, 2013, *Applied Mathematics*, John Wiley & Sons.

1 Foundations of Dimensional Homogeneity

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We begin with a reminder of the basic tenets of dimensional analysis before moving onto the Buckingham Π Theorem.

1.1 Dimensions and Units

Models in applied mathematics describe physical properties, so the variables must be physically quantifiable. There are two aspects to this: the quantity's **dimension** and the quantity's **unit**. These can be thought of as follows:

Dimension: describes the type of the quantity;

Unit: provide a means of assigning a numerical value to the quantity.

Example 1–1

Consider the distance between two points.

The dimension of this quantity is *length* which is unique. However, there are many possible units that can be used: metres, kilometres, nanometres, miles, furlongs, yards, etc

The units used will often be specific to the problem at hand and it is generally possible to convert between different units. The dimensions, however, will always stay the same.

Definition

The dimension of a quantity is denoted by using a square brackets notation (e.g. $[\text{length}] = \mathbf{L}$).

In order to compare two quantities, they need to be of the same dimension. For example, it makes sense to compare two lengths (are they the same, is one longer, etc), but it does not make sense to compare quantities of different dimension (is this length the same as that mass). Two quantities must be of the same dimension to be added (or subtracted) together.

The *Système International d'unité* has produced a set of internationally recognised units, called the *SI system*, which is used almost universally. There are seven such units, and they are specified in the table below:

The seven "base" SI units

Dimension	SI Unit
$[\text{Length}] = \mathbf{L}$	metre (m)
$[\text{Time}] = \mathbf{T}$	second (s)
$[\text{Mass}] = \mathbf{M}$	kilogram (kg)
$[\text{Temperature}] = \boldsymbol{\theta}$	Kelvin (K)
$[\text{Electric current}] = \mathbf{I}$	ampere (A)
$[\text{Amount of substance}] = \mathbf{N}$	mole (mol)
$[\text{Luminosity}] = \mathbf{J}$	candela (cd)

All other units you may have heard of are derived from these seven “base” units. Some examples are set out in the table below. There are, of course, many others.

Some examples of units derived from the seven "base" SI units

Dimension	SI Unit	SI base units
[Speed] $= \mathbf{L}\,\mathbf{T}^{-1}$	m s^{-1}	m s^{-1}
[Force] $= \mathbf{M}\,\mathbf{L}\,\mathbf{T}^{-2}$	Newton (N)	kg m s^{-2}
[Energy] $= \mathbf{M}\,\mathbf{L}^2\,\mathbf{T}^{-2}$	Joule (J)	N m or kg m^2 s^{-2}
[Pressure] $= \mathbf{M}\,\mathbf{L}^{-1}\,\mathbf{T}^{-2}$	Pascal (Pa)	N m^{-2} or kg m^{-1} s^{-2}

Perhaps the important quantities in our analysis will be the dimensionless ones:

Definition

A quantity Π is dimensionless if $[\Pi] = \mathbf{M}^0\,\mathbf{L}^0\,\mathbf{T}^0\,\boldsymbol{\theta}^0 \cdots = 1$.

1.2 The Principle of Dimensional Homogeneity (PDH)

The PDH is a fundamental rule in physics and engineering which states that any valid physical equation must be *dimensionally consistent*. Specifically:

- **Addition and Subtraction**: You can only add or subtract physical quantities that have the same dimensions. For example, you can add two lengths (metres + metres) but not a length and a time (metres + seconds).
- **Multiplication and Division**: multiplication and division dimensions is permitted according to normal arithmetic rules.
- **Equality**: The dimensions of the entire expression on the Left-Hand Side (LHS) of an equation must be identical to the dimensions of the entire expression on the Right-Hand Side (RHS).
- **Terms in an Equation**: In an equation with multiple terms added or subtracted, every single term must have the same overall dimension. For example, in $A + B = C$, the dimension of A, B, and C must all be the same.

It is just worth stating explicitly that we are able to manipulate both dimensions and units in the same way that we manipulate numbers and powers in equations, so that, for example, $\mathbf{L}$ divided by $\mathbf{L}$ results in 1 (which, of course, is dimensionless - the dimensions have cancelled each other out). Similarly $\mathbf{M}^2 \times \mathbf{M}$ results in $\mathbf{M}^3$.

Example 1–2

Assuming that Newton's Law of Universal Gravitation:

$$F = G\frac{m_1 m_2}{r^2}$$

is dimensionally homogeneous, find the dimensions and units of the gravitational constant, G.

The dimensions of mass are $\mathbf{M}$, the dimensions of length are $\mathbf{L}$: and the dimensions of force are $\mathbf{M}\,\mathbf{L}\,\mathbf{T}^{-2}$. Hence, assuming that the equation is dimensionally homogeneous, we have:

$$\frac{\mathbf{M}\,\mathbf{L}}{\mathbf{T}^2} = [G]\,\frac{\mathbf{M}\,\mathbf{M}}{\mathbf{L}^2} \quad \Longrightarrow \quad [G] = \frac{\mathbf{M}\,\mathbf{L}}{\mathbf{T}^2} \times \frac{\mathbf{L}^2}{\mathbf{M}^2} = \mathbf{L}^3\,\mathbf{M}^{-1}\,\mathbf{T}^{-2}.$$

The dimensions of G correspond to the SI units m^3 kg^{-1} s^{-2}.

1.3 The Buckingham Π Theorem

A fundamental insight in dimensional analysis was to realise all physical laws can ultimately be expressed as functional relationships between *dimensionless* quantities. This culminated in the *Buckingham Π Theorem* which provides a rigorous mathematical framework for performing dimensional analysis.

> **Theorem** (Buckingham Π Theorem)
>
> If a physically meaningful equation involves n dimensional variables, $Q_1, Q_2, \ldots, Q_n$, and these variables can be described using k fundamental (base) dimensions, then the equation can be reformulated in terms of $(n-k)$ independent dimensionless products, $\Pi_1, \Pi_2, \ldots, \Pi_{n-k}$:
>
> $$f(Q_1, Q_2, \ldots, Q_n) = 0 \quad \Longrightarrow \quad \Phi(\Pi_1, \Pi_2, \ldots, \Pi_{n-k}) = 0.$$

A proof of this statement can be found in the appendix. Here, our motivation is on *using* the theorem to understand the functional relationship between certain physical quantities in some scenario. We shall follow the procedure outlined below:

1. **List Variables**: Identify the n physical variables governing the problem.
2. **List Dimensions**: Determine the k base dimensions required (e.g., **M**, **L**, **T**).
3. **Determine Π Groups:** The number of groups is $n-k$.
4. **Select Repeating Variables**: Choose k variables $\{R_1, R_2, \ldots, R_k\}$ that are dimensionally independent and collectively contain all k base dimensions.
5. **Form Π Groups:** Create each Π term by multiplying a non-repeating variable (Q_j) by a power product of the repeating variables ($R_1^A R_2^B \cdots R_k^K$).

$$\Pi_j = Q_j R_1^A R_2^B \cdots R_k^K, \quad j = 1, 2, \ldots, (n-k)$$

The number of Π groups will determine the functional relationship between the dimensionless variables. Note that if we have two Π groups:

$$\Phi(\Pi_1, \Pi_2) = 0$$

then we may solve for one group, say Π_1 so that the relationship can be expressed as

$$\Pi_1 = g(\Pi_2) \quad \text{for some function } g.$$

This means the value of Π_1 varies depending on the value of Π_2. If there is only one Pi group (Π_1), the functional relationship becomes:

$$\Phi(\Pi_1) = 0.$$

This implies that Π_1 must satisfy some root condition of Φ and hence Π_1 must equal one of the specific, fixed values that satisfy the equation. Since the relationship must hold for all possible conditions of the physical system (all scales, materials, velocities, etc., as long as the underlying physics is the same), this fixed value cannot vary and therefore must be a constant:

$$\Pi_1 = \text{constant}.$$

The Buckingham Π Theorem is best understood through examples, to which we devote the remainder of the two lectures.

Example 1–3

Determine the period T of a simple pendulum.

1. **Variables**: The motion of the pendulum depends upon the following variables: the period T, the length ℓ of the pendulum, the mass M of bob and the acceleration g due to gravity. Thus, there are $n = 4$ variables.

2. **Dimensions**: The dimensions of the variables are:

$$[T] = \mathbf{T}, \quad [\ell] = \mathbf{L}, \quad [M] = \mathbf{M}, \quad [g] = \mathbf{L}\,\mathbf{T}^{-2}.$$

   Thus there are $k = 3$ base dimensions of the variables: **M**, **L** and **T**.

3. **Number of Π Groups**: $n - k = 4 - 3 = 1$.

4. **Repeating Variables**: We choose $k = 3$ variables that are dimensionally independent and contain the base dimensions of the variables. Let $R_1 = \ell$, $R_2 = g$ and $R_3 = M$. These are dimensionally independent.

5. **Form Π Groups**: There is only one group Π_1 to form:

$$\Pi_1 = T\,\ell^a\,g^b\,M^c$$

   We must have $[\Pi_1] = 1$ yielding:

$$1 = \mathbf{T}\,\mathbf{L}^a\,(\mathbf{L}\,\mathbf{T}^{-2})^b\,\mathbf{M}^c \quad \Longrightarrow \quad 1 = \mathbf{T}^{1-2b}\,\mathbf{L}^{a+b}\,\mathbf{M}^c$$

In order for this to be dimensionally coherent, we must have

$$\begin{cases} \mathbf{T}: & 1 - 2b = 0 \\ \mathbf{L}: & a + b = 0 \\ \mathbf{M}: & c = 0 \end{cases} \quad \Longrightarrow \quad a = -b = -\frac{1}{2}, \; c = 0$$

This leads to the resulting Π group:

$$\Pi_1 = T\,\ell^{-\frac{1}{2}}\,g^{\frac{1}{2}}\,M^0 \quad \Longrightarrow \quad \Pi_1 = T\sqrt{\frac{g}{\ell}}$$

Since there is only one Π group, the general function $\Phi(\Pi_1) = 0$ implies Π_1 must be a constant, C:

$$T\sqrt{\frac{g}{\ell}} = C \quad \Longrightarrow \quad T = C\sqrt{\frac{\ell}{g}}$$

.

Dimensional analysis correctly yields the functional dependence of the period of the pendulum on its length and gravity, and correctly shows the period is independent of mass: a non-intuitive result confirmed by theory. The constant $C = 2\pi$ must be found theoretically or experimentally.

1.4 Case Study: The Casimir Effect

The Casimir effect demonstrates that the vacuum is not empty; it exhibits a measurable attractive force between two uncharged, parallel, conducting plates separated by a distance d. The force originates from the quantum zero-point energy of the electromagnetic field. Since this phenomenon is both quantum and relativistic, the relevant fundamental constants are the reduced Planck constant ($\hbar$) and the speed of light (c). We seek the scaling of the Casimir pressure (P_C), which is the *force per unit area*.

We proceed as we have before using the Buckingham Pi Theorem procedure.

1. **Variables**: As outlined above, there are $n = 4$ variables: P_C, d, c and $\hbar$.

2. **Dimensions**: The dimensions of the variables are:

$$\begin{aligned} \text{Casimir pressure density, } P_C\text{:} \quad & [P_C] = \mathbf{M}\,\mathbf{L}^{-1}\,\mathbf{T}^{-2} \\ \text{Plate separation distance, } d\text{:} \quad & [d] \;=\; \mathbf{L} \\ \text{Speed of light, } c\text{:} \quad & [c] \;=\; \mathbf{L}\,\mathbf{T}^{-1} \\ \text{Reduced Planck constant, } \hbar\text{:} \quad & [\hbar] \;=\; \mathbf{M}\,\mathbf{L}^{2}\,\mathbf{T}^{-1} \end{aligned}$$

   There are $k = 3$ base dimensions: **M**, **L** and **T**.

3. **Number of Π Groups**: $n - k \;=\; 4 - 3 \;=\; 1$.

4. **Repeating Variables**: We choose $k = 3$ variables that are dimensionally independent and contain the base dimensions of the variables. Let $R_1 = d$, $R_2 = c$ and $R_3 = \hbar$. These are dimensionally independent.

5. **Form Π Groups**: We have

$$\Pi_1 \;=\; P_C\; d^p\; c^q\; \hbar^r$$

   for some p, q, r. We must have $[\Pi_1] = 1$ and hence:

$$1 \;=\; (\mathbf{M}\,\mathbf{L}^{-1}\,\mathbf{T}^{-2})\,\mathbf{L}^{a}\;(\mathbf{L}\,\mathbf{T}^{-1})^{b}\;(\mathbf{M}\,\mathbf{L}^{2}\,\mathbf{T}^{-1})^{c} \;=\; \mathbf{M}^{1+c}\,\mathbf{T}^{-2-b-c}\,\mathbf{L}^{-1+a+b+2c}$$

   This yields the equations:

$$\left\{\begin{aligned} \mathbf{M}: &\quad 1 + c &= 0 \\ \mathbf{T}: &\quad -2 - b - c &= 0 \\ \mathbf{L}: &\quad -1 + a + b + 2c &= 0 \end{aligned}\right. \quad\Longrightarrow\quad a \;=\; 4,\; b \;=\; -1,\; c \;=\; -1.$$

   This results in the Π group:

$$\Pi_1 \;=\; P_C\, d^4\, c^{-1}\, \hbar^{-1} \;=\; \frac{P_C\; d^4}{\hbar\; c}$$

   Since there is only one Π group, it must be equal to a dimensionless constant, α.

$$\frac{P_C d^4}{\hbar c} \;=\; \alpha \quad\Longrightarrow\quad P_C = \alpha\,\frac{\hbar\; c}{d^4}$$

Dimensional analysis predicts that the Casimir pressure scales rigorously with d^{-4}. Full quantum electrodynamic calculation yields the constant $C = -\dfrac{\pi^2}{240}$, where the negative sign indicates an attractive (inward) pressure.

2 Advanced Applications of the Buckingham Π Theorem

We continue considering the Buckingham Π theorem, though in this lecture we will consider examples with more than one Π group.

2.1 Case Study: Drag Force on a Sphere

The drag force F_D on a sphere depends on its diameter L, velocity V, the fluid density ρ, and the dynamic viscosity μ. This is the foundation of fluid dynamic similarity.

1. **Variables**: As described above, there are $n = 5$ variables: F_D, L, V, ρ and μ.
2. **Dimensions**: The dimensions of the variables are:

$$\begin{aligned} \text{Drag force, } F_D: &\quad [F_D] = \mathbf{M\,L\,T}^{-2} \\ \text{Diameter, } L: &\quad [L] \;=\; \mathbf{L} \\ \text{Velocity, } V: &\quad [V] \;=\; \mathbf{L\,T}^{-1} \\ \text{Fluid density, } \rho: &\quad [\rho] \;=\; \mathbf{M\,L}^{-3} \\ \text{Dynamic viscosity, } \mu: &\quad [\mu] \;=\; \mathbf{M\,L}^{-1}\,\mathbf{T}^{-1} \end{aligned}$$

   There are $k = 3$ base dimensions: **M**, **L** and **T**.
3. **Number of Π Groups**: $n - k \;=\; 5 - 3 \;=\; 2$.
4. **Repeating Variables**: We choose $k = 3$ variables that are dimensionally independent and contain the base dimensions of the variables. Let $R_1 = \rho$, $R_2 = V$ and $R_3 = L$.
5. **Form Π Groups**: We have two groups Π_1, Π_2. We form Π_1 using F_D:

$$\Pi_1 \;=\; F_D\,\rho^a\,V^b\,L^c$$

   for some a, b, c. We must have $[\Pi_1] = 1$ and hence:

$$1 \;=\; (\mathbf{M\,L\,T}^{-2})\;(\mathbf{M\,L}^{-3})^a\;(\mathbf{L\,T}^{-1})^b\;\mathbf{L}^c \;=\; \mathbf{M}^{1+a}\;\mathbf{T}^{-2-b}\;\mathbf{L}^{1-3a+b+c}$$

   This yields the equations:

$$\begin{cases} \mathbf{M}: & 1 + a \;=\; 0 \\ \mathbf{T}: & -2 - b \;=\; 0 \\ \mathbf{L}: & 1 - 3a + b + c \;=\; 0 \end{cases} \quad\Longrightarrow\quad a \;=\; -1,\; b \;=\; 2,\; c \;=\; -2.$$

   This results in the Π group

$$\Pi_1 \;=\; F_D\,\rho^{-1}\,V^{-2}\,L^{-2} \;=\; \frac{F_D}{\rho\,V^2\,L^2}$$

   We now repeat the process and form Π_2 using μ:

$$\Pi_2 \;=\; \mu\,\rho^a\,V^b\,L^c$$

   for some a, b, c. We must have $[\Pi_2] = 1$ and hence:

$$1 \;=\; (\mathbf{M\,L}^{-1}\;\mathbf{T}^{-1})\;(\mathbf{M\,L}^{-3})^a\;(\mathbf{L\,T}^{-1})^b\;\mathbf{L}^c \;=\; \mathbf{M}^{1+a}\;\mathbf{T}^{-1-b}\;\mathbf{L}^{-1-3a+b+c}$$

This yields the equations:

$$\begin{cases} \mathbf{M}: & 1+a \;=\; 0 \\ \mathbf{T}: & -1-b \;=\; 0 \\ \mathbf{L}: & -1-3a+b+c \;=\; 0 \end{cases} \implies a = -1,\; b = -1,\; c = -1.$$

This results in the Π group:

$$\Pi_2 = \mu\,\rho^{-1}\,V^{-1}\,L^{-1} = \frac{\mu}{\rho\,V\,L}$$

The physical law for the drag force must be expressible as a relationship between these two dimensionless groups:

$$\Phi(\Pi_1, \Pi_2) = 0 \implies \Pi_1 = g(\Pi_2) \implies \frac{F_D}{\rho\,V^2\,L^2} = g\left(\frac{\mu}{\rho\,V\,L}\right)$$

for some function g.

The first Π group (Π_1) is proportional to the standard *drag coefficient* C_D:

$$C_D = \frac{F_D}{\frac{1}{2}\rho\,V^2\,A}$$

where $A \propto L^2$ is the reference area and the second Π group (Π_2) is the inverse of the *Reynolds number* Re:

$$\mathrm{Re} = \frac{\rho\,V\,L}{\mu}.$$

Thus, the functional relationship between the Π groups may be written:

$$C_D = h(\mathrm{Re})$$

for some function h and demonstrates the *Principle of Dynamic Similarity*: two geometrically similar flow systems are dynamically equivalent if their Reynolds numbers (Re) are equal. This is the core concept used in testing scaled models in fluid dynamics.

2.2 Case Study: Planck's Radiation Law

Planck's Law describes the spectral energy density $u(\nu, T)$ of electromagnetic radiation within a black body in thermal equilibrium. It is dependent on the frequency (ν) and the absolute temperature (T) and describes the radiation energy per unit volume contained within a unit frequency interval. It is also fundamentally dependent on three universal constants: the speed of light (c), Planck's constant (h), and the Boltzmann constant (k_B).

The relationship can be stated as:

$$u(\nu, T) = f(\nu, T, c, h, k_B).$$

While Buckingham's Π Theorem cannot derive the full functional form of Planck's Radiation Law (as this requires quantum concepts), it can successfully determine the correct dimensional structure and the independent dimensionless groups upon which the law must depend.

1. **Variables**: As described above, there are $n = 6$ physical variables: u, ν, T, c, h and k_B.

2. **Dimensions**: The dimensions of the variables are:

$$\begin{aligned}
\text{Spectral energy density, } u\text{:} \quad & [u] \;=\; \left[\text{Energy} \cdot \text{Volume}^{-1}\, \text{Frequency}^{1}\right] \;=\; \mathbf{M}\,\mathbf{L}^{-1}\,\mathbf{T}^{-1} \\
\text{Frequency, } \nu\text{:} \quad & [\nu] \;=\; \mathbf{T}^{-1} \\
\text{Temperature, } T\text{:} \quad & [T] \;=\; \boldsymbol{\theta} \\
\text{Speed of light, } c\text{:} \quad & [c] \;=\; \mathbf{L}\,\mathbf{T}^{-1} \\
\text{Planck's constant, } h\text{:} \quad & [h] \;=\; \mathbf{M}\,\mathbf{L}^{2}\,\mathbf{T}^{-1} \\
\text{Boltzmann constant, } k_B\text{:} \quad & [k_B] \;=\; \mathbf{M}\,\mathbf{L}^{2}\,\mathbf{T}^{-2}\,\boldsymbol{\theta}^{-1}.
\end{aligned}$$

   There are $k = 4$ base dimensions: **M, L, T** and $\boldsymbol{\theta}$.

3. **Number of Π Groups**: $n - k \;=\; 6 - 4 \;=\; 2$.

4. **Repeating Variables**: We choose $k = 4$ variables that are dimensionally independent and contain the base dimensions of the variables. Let $R_1 = c$, $R_2 = h$, $R_3 = k_B$ and $R_4 = T$.

5. **Form Π Groups**: We have two groups Π_1, Π_2. We form Π_1 using u:

$$\Pi_1 \;=\; u\, c^a\, h^b\, k_B^c\, T^d$$

   for some a, b, c, d. We must have $[\Pi_1] = 1$ and hence:

$$\begin{aligned}
1 \;&=\; (\mathbf{M}\,\mathbf{L}^{-1}\,\mathbf{T}^{-1})\,(\mathbf{L}\,\mathbf{T}^{-1})^a\,(\mathbf{M}\,\mathbf{L}^{2}\,\mathbf{T}^{-1})^b\,(\mathbf{M}\,\mathbf{L}^{2}\,\mathbf{T}^{-2}\,\boldsymbol{\theta}^{-1})^c\,\boldsymbol{\theta}^d \\
&=\; \mathbf{M}^{1+b+c}\,\mathbf{T}^{-1-a-b-2c}\,\mathbf{L}^{-1+a+2b+2c}\,\boldsymbol{\theta}^{-c+d}
\end{aligned}$$

   This yields the equations:

$$\left\{\begin{aligned}
\mathbf{M}: \quad & 1 + b + c \;=\; 0 \\
\mathbf{T}: \quad & -1 - a - b - 2c \;=\; 0 \\
\mathbf{L}: \quad & -1 + a + 2b + 2c \;=\; 0 \\
\boldsymbol{\theta}: \quad & -c + d \;=\; 0
\end{aligned}\right. \quad \Longrightarrow \quad a \;=\; 3,\; b \;=\; 2,\; c \;=\; -3,\; d \;=\; -3.$$

   This results in the Π group

$$\Pi_1 \;=\; u\, c^3\, h^2\, k_B^{-3}\, T^{-3} \;=\; \frac{u\, h^2\, c^3}{k_B^3\, T^3}.$$

We now repeat the process and form Π_2 using ν:

$$\Pi_2 \,=\, \nu\, c^a\, h^b\, k_B^c\, T^d$$

for some a, b, c, d. We must have $[\Pi_2] = 1$ and hence:

$$\begin{aligned} 1 \,&=\, \mathbf{T}^{-1}\ \ (\mathbf{L}\,\mathbf{T}^{-1})^a\ (\mathbf{M}\,\mathbf{L}^2\,\mathbf{T}^{-1})^b\ (\mathbf{M}\,\mathbf{L}^2\,\mathbf{T}^{-2}\,\boldsymbol{\theta}^{-1})^c\,\boldsymbol{\theta}^d \\ &=\, \mathbf{M}^{1+b+c}\,\mathbf{T}^{-1-a-b-2c}\,\mathbf{L}^{-1+a+2b+2c}\,\boldsymbol{\theta}^{-c+d} \end{aligned}$$

This yields the equations:

$$\begin{cases} \mathbf{M}: & b+c \,=\, 0 \\ \mathbf{T}: & -1-a-b-2c \,=\, 0 \\ \mathbf{L}: & a+2b+2c \,=\, 0 \\ \boldsymbol{\theta}: & -c+d \,=\, 0 \end{cases} \quad\Longrightarrow\quad a \,=\, 0,\ b \,=\, 1,\ c \,=\, -1,\ d \,=\, -1.$$

This results in the Π group:

$$\Pi_2 \,=\, \nu\, c^0\, h^1\, k_B^{-1}\, T^{-1} \,=\, \frac{h\,\nu}{k_B\,T}.$$

The physical law for the spectral energy density must be expressible as a relationship between these two dimensionless groups:

$$\Phi(\Pi_1, \Pi_2) \,=\, 0 \quad\Longrightarrow\quad \Pi_1 \,=\, g(\Pi_2) \quad\Longrightarrow\quad \frac{u\,h^2\,c^3}{k_B^3\,T^3} \,=\, g\!\left(\frac{h\,\nu}{k_B\,T}\right)$$

for some function g, or solving for the spectral energy density:

$$u(\nu, T) \,=\, \frac{k_B^3\,T^3}{h^2\,c^3}\, g\!\left(\frac{h\,\nu}{k_B\,T}\right)$$

The full derivation of Planck's Law, involving quantum statistics and boundary conditions for electromagnetic modes, shows that the function g is given by

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto g(x) \,=\, \frac{8\pi}{e^x - 1}. \end{aligned}$$

While the dimensional analysis above can only find the functional form of the relationship, it correctly isolates the dependence of the spectral energy density on the ratio of photon energy ($h\,\nu$) to thermal energy ($k_B\,T$).

2.3 Case Study: Stellar Luminosity

The primary goal of this analysis is to determine the dimensionless functional relationship governing a star's luminosity (L) and its dependence on fundamental physical constants and the star's properties. This relationship underpins the *Hertzsprung-Russell* (H-R) diagram and stellar structure models. We assume the star acts as a hot blackbody radiator powered by nuclear reactions, with its structure stabilised by the balance between gravity and internal pressure.

1. **Variables**: The stellar system is defined here are $n = 7$ physical variables: the star's luminosity L, mass M, radius R, surface temperature T and the fundamental constants G (the gravitational constant), σ (the Stefan-Boltzmann constant) and c (the speed of light).

2. **Dimensions**: The dimensions of the variables are:

$$\begin{aligned}
\text{Luminosity, } L: \quad [L] &= \mathbf{M}\,\mathbf{L}^2\,\mathbf{T}^{-3} \\
\text{Mass, } M: \quad [M] &= \mathbf{M} \\
\text{Radius, } R: \quad [R] &= \mathbf{L} \\
\text{Surface temperature, } T: \quad [T] &= \boldsymbol{\theta} \\
\text{Gravitational constant, } G: \quad [G] &= \mathbf{M}^{-1}\,\mathbf{L}^3\,\mathbf{T}^{-2} \\
\text{Stefan-Boltzmann constant, } \sigma: \quad [\sigma] &= \mathbf{M}\,\mathbf{T}^{-2}\,\boldsymbol{\theta}^{-1} \\
\text{Speed of light, } c: \quad [G] &= \mathbf{M}\,\mathbf{L}^2\,\mathbf{T}^{-1}.
\end{aligned}$$

   There are $k = 4$ base dimensions: **M**, **L**, **T** and $\boldsymbol{\theta}$.

3. **Number of Π Groups**: $n - k = 7 - 4 = 3$.

4. **Repeating Variables**: We choose $k = 4$ variables that are dimensionally independent and contain the base dimensions of the variables. Let $R_1 = R$, $R_2 = M$, $R_3 = T$ and $R_4 = c$.

5. **Form Π Groups**: We have three groups Π_1, Π_2 and Π_3. We form Π_1 using u:

   The physical law for the stellar luminosity must be expressible as a relationship between these three dimensionless groups:

$$\Phi(\Pi_1, \Pi_2, \Pi_3) = 0 \quad \Longrightarrow \quad \Pi_1 = g(\Pi_2, \Pi_3) \quad \Longrightarrow \quad \frac{L\,R}{M\,c^3} = g\left(\frac{G\,M}{R\,c^2}, \frac{\sigma\,R^3\,T^4}{M\,c^3}\right)$$

   for some function g, or solving for the luminosity:

$$L = \frac{M\,c^3}{R}\,g\left(\frac{G\,M}{R\,c^2}, \frac{\sigma\,R^3\,T^4}{M\,c^3}\right).$$

The derived functional relationship reveals the three dominant, universal factors that govern stellar luminosity:

1. **The Gravitational/Relativistic Parameter ($\Pi_2 = G\,M/R\,c^2$)**

   This term is proportional to the ratio of the star's Radius (R) to the *Schwarzschild radius* ($R_S = 2GM/c^2$). Π_2 quantifies the strength of the relativistic gravitational field and determines whether the star can be treated classically or if General Relativity is necessary. In a more complete theory, the mass-luminosity relationship ($L \propto M^3 - M^4$) is implicitly contained within the function g.

2. **The Radiation Parameter ($\Pi_3 = \sigma R^3\,T^4/M\,c^3$)**

   This term is closely related to the *Stefan-Boltzmann Law* ($L \propto R^2 \sigma T^4$). It represents the ratio of the energy radiated by the star's surface to the star's internal energy scale. The fundamental nature of Π_3 is to tie the energy transport mechanism (radiation) directly to the star's mass and size.

3. **The Characteristic Power Scale ($M\,c^3/R$)**

   This dimensional term sets the overall scale for the star's power output. This factor is known to be the fundamental unit of luminosity in relativistic physics, related to the maximum possible power output dictated by the central mass.

The power of this dimensional analysis lies in its ability to predict, without solving the complex differential equations of stellar structure, that the final solution must be a function of these specific three dimensionless combinations of mass, radius, and temperature. This forms the basis for all stellar modelling and the classification of stars on the H-R diagram.

A Proof of Buckingham's Π Theorem

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The proof of Buckingham's Π Theorem relies on representing the exponents of the fundamental dimensions as vectors and solving a system of linear homogeneous equations. Let $Q_1, Q_2, \ldots, Q_n$ denote a collection of n dimensional variables which can be written in terms of k fundamental dimensions, $D_1, D_2, \ldots, D_k$:

$$[Q_j] \;=\; D_1^{a_{1j}}\, D_2^{a_{2j}} \;\cdots\; D_k^{a_{kj}} \qquad (j = 1, 2, \ldots, n)$$

where $a_{ij} \in \mathbb{R}$ for $i = 1, 2, \ldots k$. We construct the $(k \times n)$ *dimensional matrix* $\mathcal{D}$, where the columns are the dimension vectors of the variables and the rows correspond to the fundamental dimensions:

$$\mathcal{D} \;=\; \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}.$$

The number of independent fundamental dimensions is defined by the *rank* of the matrix $\mathcal{D}$, denoted as rank$(\mathcal{D}) \;=\; k$.

A dimensionless Π group is formed by a product of the variables raised to unknown exponents:

$$\Pi \;=\; Q_1^{x_1}\, Q_2^{x_2} \;\cdots\; Q_n^{x_n}.$$

Collect the exponents into the vector $X \in \mathbb{R}^n$:

$$X \;=\; \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

For Π to be dimensionless, when it is expressed in terms of the k fundamental base dimensions the net exponent for each must of these be zero. This condition generates a system of k linear homogeneous equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \;&=\; 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \;&=\; 0 \\ &\;\vdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \;&=\; 0 \end{aligned}$$

In matrix form, this is the homogeneous system:

$$\mathcal{D}\, X \;=\; 0_n$$

where 0_n is the zero vector on $\mathbb{R}^n$. The set of all solutions X to this forms the *null space* (or *kernel*) of the matrix $\mathcal{D}$. Each independent vector in the null space corresponds to an independent dimensionless Π group. The *Rank-Nullity Theorem* states that for a $(k \times n)$ matrix $\mathcal{D}$:

$$\text{rank}(\mathcal{D}) + \text{nullity}(\mathcal{D}) \;=\; n.$$

We established that rank$(\mathcal{D}) = k$, the number of independent fundamental dimensions. The *nullity* of $\mathcal{D}$ is the dimension of the null space - that is, the number of linearly independent solution vectors X that satisfy the condition for dimensionless Π groups. Substituting rank$(\mathcal{D}) = k$ into the Rank-Nullity Theorem yields:

$$k + \text{nullity}(\mathcal{D}) \;=\; n \quad\Longrightarrow\quad \text{nullity}(\mathcal{D}) = n - k$$

Thus, the number of independent dimensionless Π groups that can be formed from the n variables is exactly the dimension of the null space, which is $n - k$. ■