The Standard Normal distribution

21.2

Introduction

Mass-produced items should conform to a specification. Usually, a mean is aimed for but due to random errors in the production process we set a tolerance on deviations from the mean. For example if we produce piston rings which have a target mean inside diameter of 45 mm then realistically we expect the diameter to deviate slightly from this value. The deviations from the mean value are often modelled very well by the Normal distribution. Suppose we decide that diameters in the range 44.95 mm to 45.05 mm are acceptable, then what proportion of the output is satisfactory? In this Block we shall see how to use the normal distribution to answer questions like this.

Prerequisites

Before starting this Block you should . . .

① be familiar with the basic properties of probability

② be familiar with continuous random variables

Learning Outcomes

After completing this Block you should be able to . . .

✓ recognise the shape of the frequency curve for the standard normal distribution

✓ calculate probabilities using the standard normal distribution

✓ recognise key areas under the frequency curve

Learning Style

To achieve what is expected of you . . .

☞ allocate sufficient study time

☞ briefly revise the prerequisite material

☞ attempt *every* guided exercise and most of the other exercises

1. A Key Transformation

A normal distribution is, perhaps, the most important example of a continuous random variable. The probability density function of a normal distribution is

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(a-\mu)^2}{2\sigma^2}}$$

This curve is always 'bell-shaped'with the centre of the bell located at the value of μ. The depth of the bell is controlled by the value of σ. As with all normal distribution curves it is symmetrical about the centre and decays exponentially as $x \to \pm\infty$. As with any probability density function the area under the curve is equal to 1. See Figure 1.

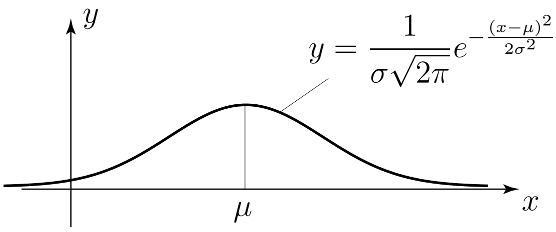


Figure 1

A normal distribution is completely defined by specifying its mean (the value of μ) and its variance (the value of σ^2). The normal distribution with mean μ and variance σ^2 is written $N(\mu, \sigma^2)$. Hence the disribution $N(20, 25)$ has a mean of 20 and a standard deviation of 5; remember that the second "coordinate" is the variance and the variance is the square of the standard deviation.

The standard normal distribution curve.

At this stage we shall, for simplicity, consider what is known as a standard normal distribution which is obtained by choosing particularly simple values for μ and σ.

Key Point

The **standard normal distribution** has a mean of zero and a variance of one.

In Figure 2 we show the graph of the standard normal bistribution which has probability density function $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

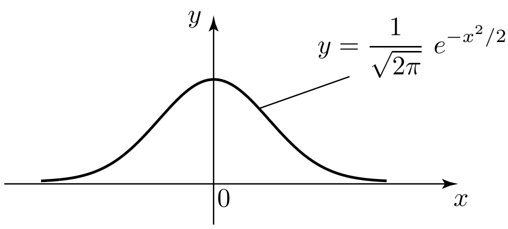


Figure 2. The standard normal distribution curve

The result which makes the standard normal distribution so important is as follows:

Key Point

If the behaviour of a continuous random variable X is described by the distribution $N(\mu, \sigma^2)$ then the behaviour of the random variable $Z = \frac{X-\mu}{\sigma}$ is described by the standard normal distribution $N(0, 1)$.

We call Z the standardised normal variable.

Example If the random variable X is described by the distribution $N(45, 0.000625)$ then what is the transformation to obtain the standardised normal variable?

Solution

Here, $\mu = 45$ and $\sigma^2 = 0.000625$ so that $\sigma = 0.025$. Hence $Z = (X - 45)/0.025$ is the required transformation.

Example When the random variable X takes values between 44.95 and 45.05, between which values does the random variable Z lie?

Solution

When $X = 45.05$, $Z = \frac{45.05-45}{0.025} = 2$
When $X = 44.95$, $Z = \frac{44.95-45}{0.025} = -2$
Hence Z lies between -2 and 2.

Now do this exercise

The random variable X follows a normal distribution with mean 1000 and variance 100. When X takes values between 1005 and 1010, between which values does the standardised normal variable Z lie?

Answer

2. Probabilities and the standard normal distribution

Since the standard normal distribution is used so frequently tables have been produced to help us calculate probabilities. This table is located at the end of this block. It is based upon the following diagram:

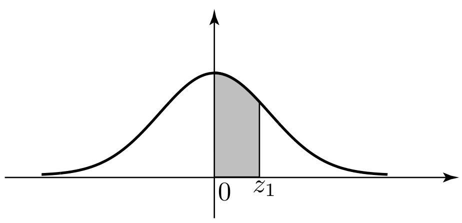


Figure 3

Since the total area under the curve is equal to 1 it follows from the symmetry in the curve that the area under the curve in the region $x > 0$ is equal to 0.5. In Figure 3 the shaded area is the probability that Z takes values between 0 and z_1.
When we 'look-up' a value in the table we obtain the value of the shaded area.

Example What is the probability that Z takes values between 0 and 1.9?

Solution

The second column headed '0' is the one to choose and its entry in the row neginning '1.9' is 4713. This is to be read as 0.4713 (we omitted the 0 in each entry for clarity) The interpretation is that the probability that Z takes values between 0 and 1.9 is 0.4713.

Example What is the probability that Z takes values between 0 and 1.96?

Solution

This time we want the column headed '6' and the row beginning 1.9.

The entry is 4750 so that the required probability is 0.4750.

Example What is the probability that Z takes values between 0 and 1.965?

Solution

There is no entry corresponding to 1.965 so we take the average of the values for 1.96 and 1.97. (This linear interpolation is not strictly correct but is acceptable).

The two values are 4750 and 4756 with an average of 4753. Hence the required probability is 0.4753.

Now do this exercise

What are the probabilities that Z takes values between

(i) 0 and 2 (ii) 0 and 2.3 (iii) 0 and 2.33 (iv) 0 and 2.333?

Answer

Note from the table that as Z increases from 0 the entries increase, rapidly at first and then more slowly, toward 5000 i.e. a probability of 0.5. This is consistent with the shape of the curve. After $Z = 3$ the increase is quite slow so that we tabulate entries for values of Z rising by 0.1 instead of 0.01 as in the rest of the table.

3. Calculating other probabilities

In this section we see how to calculate probabilities represented by areas other than those of the type shown in Figure 3.

Case 1

Figure 4 illustrates what we do if both Z values are positive. By using the properties of the standard normal distribution we can organise matters so that any required area is always of 'standard form'.

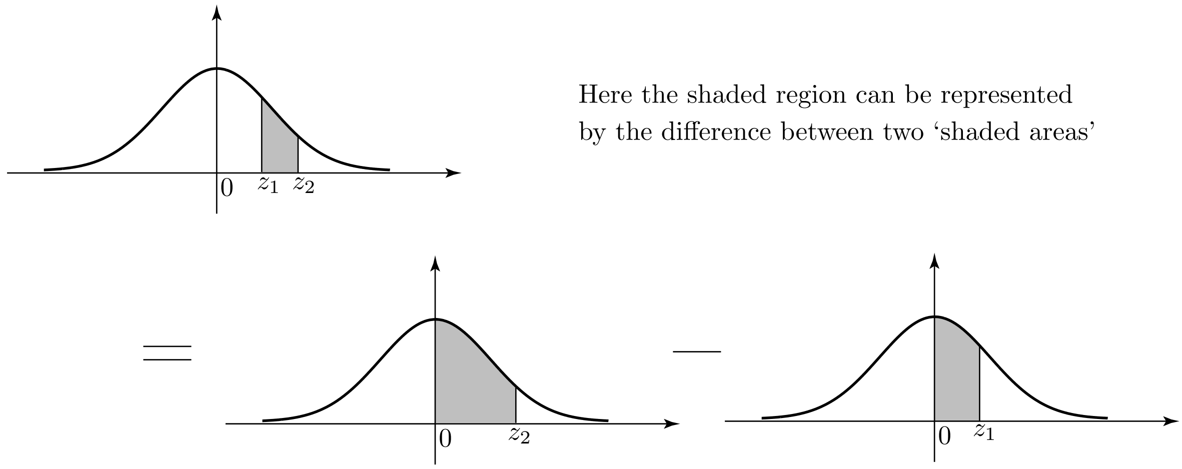


Figure 4

Example Find the probability that Z takes values bwteen 1 and 2.

Solution

Using the table

$P(Z = z_2)$ i.e. $P(Z = 2)$ is 0.4772

$P(Z = z_1)$ i.e. $P(Z = 1)$ is 0.3413.

Hence $P(1 < Z < 2) = 0.4772 - 0.3413 = 0.1359$

(Remember that with a continuous distribution, $P(Z = 1)$ is meaningless so that $P(1 \leq Z \leq 2)$ is also 0.1359.

Case 2

The following diagram illustrates the procedure to be followed when finding probabilities of the form $P(Z > z_1)$.

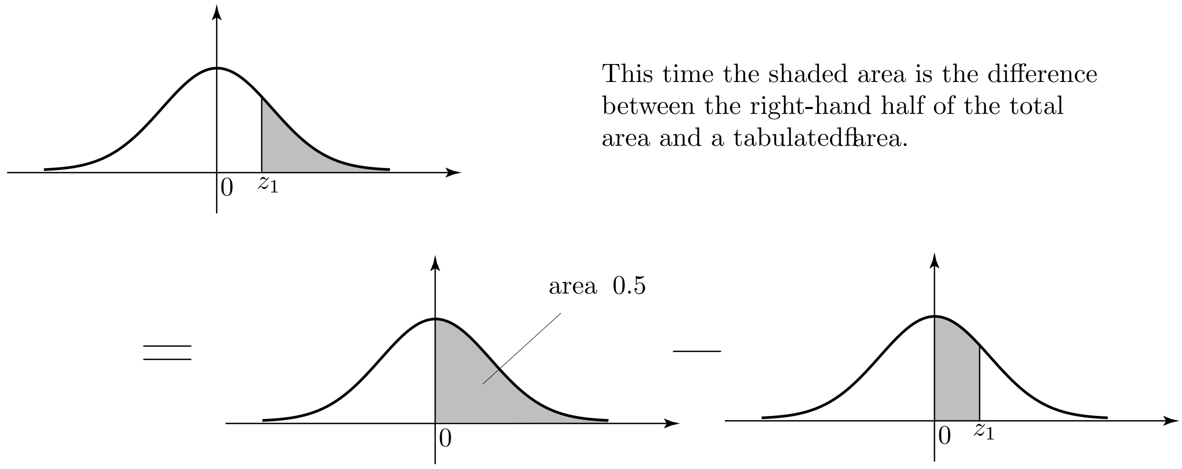


Figure 5

Example What is the probability that $Z > 2$?

Solution

$P(0 < Z < 2) = 0.4772$ (from the table)

Hence the probability is $0.5 - 0.4772 = 0.0228$.

Case 3

Here we consider the procedure to be followed when calculating probabilities of the form $P(Z < z_1)$. Here the shaded area is the sum of the left-hand half of the total area and a 'standard'area.

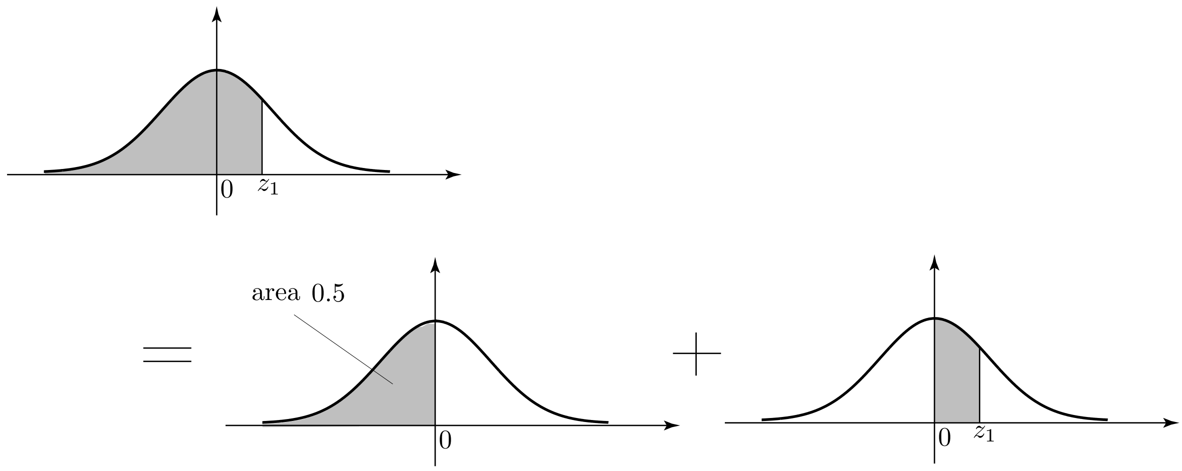


Figure 6

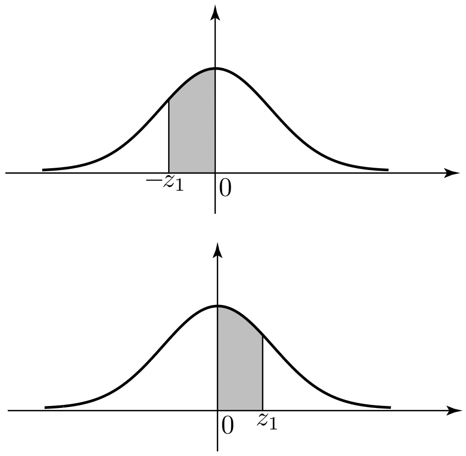
Example What is the probability that $Z < 2$?

Solution

$P(Z > 2) = 0.5 + 0.4772 = 0.9772.$

Case 4

Here we consider what needs to be done when calculating probabilities of the form $P(-z_1 < Z < 0)$ where z_1 is positive. This time we make use of the symmetry in the standard normal distribution curve.



by symmetry this shaded area is equal in value to the one above.

Figure 7

Example What is the probability that $-2 < Z < 0$?

Solution

The area is equal to that corresponding to $P(0 < Z < 2) = 0.4772.$

Case 5

Finally we consider probabilities of the form $P(-z_2 < Z < z_1)$. Here we use the sum property and the symmetry property.

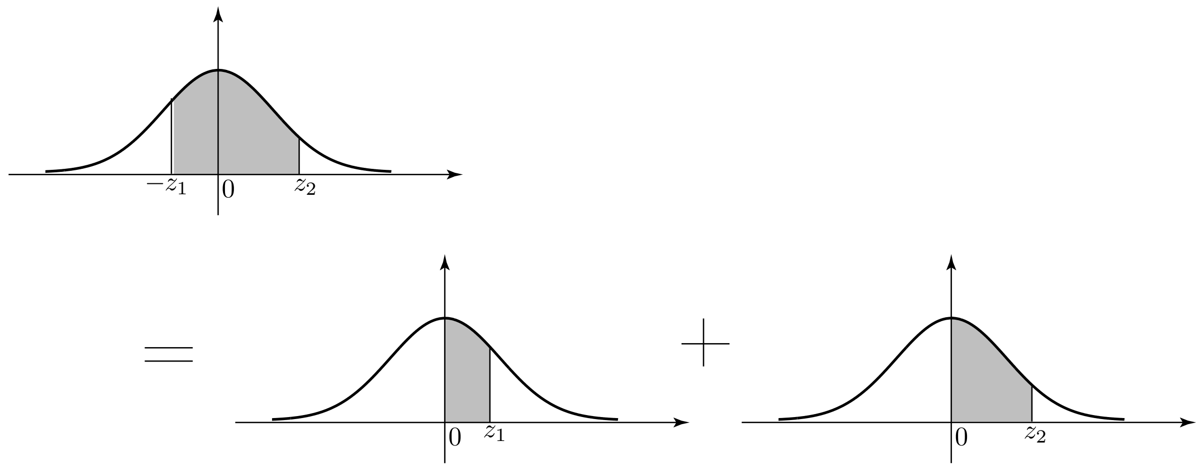


Figure 8

Example What is the probability that $-1 < Z < 2$?

Solution

$$\begin{aligned} P(-1 < Z < 0) = P(0 < Z < 1) &= 0.3413 \\ P(0 < Z < 2) &= 0.4772 \end{aligned}$$

Hence the required probability, $P(-1 < Z < 2)$ is 0.8185.

Other cases can be handed by a combination of the ideas already used.

Now do this exercise

Find the following probabilities.

(i)	$P(0 < Z < 1.5)$	(ii)	$P(Z > 1.8)$
(iii)	$P(1.5 < Z < 1.8)$	(iv)	$P(Z < 1.8)$
(v)	$P(-1.5 < Z < 0)$	(vi)	$P(Z < -1.5)$
(vii)	$P(-1.8 < Z < -1.5)$	(viii)	$P(-1.5 < Z < 1.8)$

(A simple sketch of the standard normal curve will help).

Answer

4. Confidence Intervals

We use probability models to make predictions in situations where there is not sufficient data available to make a definite statement. Any statement based on these models carries with it a **risk** of being proved incorrect by events.
Notice that the normal probability curve extends to infinity in both directions. **Theoretically** any value of the normal random variable is possible, although, of course, values far from the mean position, zero, are very unlikely.
Consider the diagram in Figure 9,

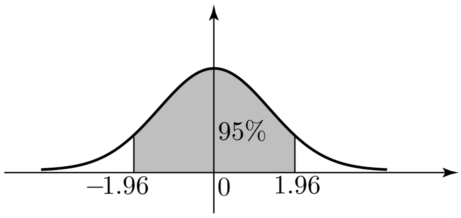


Figure 9

The shaded area is 95% of the total area. If we look at the entry in Table 1 corresponding to $Z = 1.96$ we see the value 4750. This means that the probability of Z taking a value between 0 and 1.96 is 0.475. By symmetry, the probability that Z takes a value between -1.96 and 0 is also 0.475. Combining these results we see that

$$P(-1.96 < Z < 1.96) = 0.95 \quad \text{or} \quad 95\%$$

We say that the 95% confidence interval for Z (about its mean of 0) is $(-1.96, 1.96)$. It follows that there is a 5% chance that Z lies outside this interval.

Now do this exercise

We wish to find the 99% confidence interval for Z about its mean, i.e. the value of z_1 in Figure 10

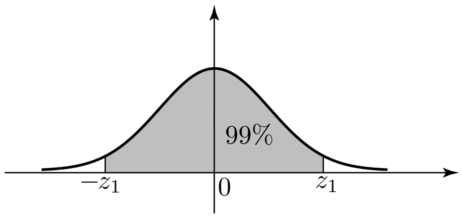


Figure 10 The shaded area is 99% of the total area.

First, note that 99% corresponds to a probability of 0.99.
Find z_1 such that

$$P(0 < Z < z_1) = \frac{1}{2} \times 0.99 = 0.495.$$

Answer

Now do this exercise

Now quote the 99% confidence interval.

Answer

Notice that the risk of Z lying outside this wider interval is reduced to 1%.

Now do this exercise

Find the value of Z

(i) which is exceeded on 5% of occasions

(ii) which is exceeded on 99% of occasions.

Answer

The Normal Probability Integral

$Z = \frac{x-\mu}{\sigma}$	0	1	2	3	4	5	6	7	8	9
0	0000	0040	0080	0120	0160	0199	0239	0279	0319	0359
.1	0398	0438	0478	0517	0577	0596	0636	0675	0714	0753
.2	0793	0832	0871	0909	0948	0987	1026	1064	1103	1141
.3	1179	1217	1255	1293	1331	1368	1406	1443	1480	1517
.4	1555	1591	1628	1664	1700	1736	1772	1808	1844	1879
.5	1915	1950	1985	2019	2054	2088	2123	2157	2190	2224
.6	2257	2291	2324	2357	2389	2422	2454	2486	2517	2549
.7	2580	2611	2642	2673	2703	2734	2764	2794	2822	2852
.8	2881	2910	2939	2967	2995	3023	3051	3078	3106	3133
.9	3159	3186	3212	3238	3264	3289	3315	3340	3365	3389
1.0	3413	3438	3461	3485	3508	3531	3554	3577	3599	3621
1.1	3643	3665	3686	3708	3729	3749	3770	3790	3810	3830
1.2	3849	3869	3888	3907	3925	3944	3962	3980	3997	4015
1.3	4032	4049	4066	4082	4099	4115	4131	4147	4162	4177
1.4	4192	4207	4222	4236	4251	4265	4279	4292	4306	4319
1.5	4332	4345	4357	4370	4382	4394	4406	4418	4429	4441
1.6	4452	4463	4474	4484	4495	4505	4515	4525	4535	4545
1.7	4554	4564	4573	4582	4591	4599	4608	4616	4625	4633
1.8	4641	4649	4656	4664	4671	4678	4686	4693	4699	4706
1.9	4713	4719	4726	4732	4738	4744	4750	4756	4761	4767
2.0	4772	4778	4783	4788	4793	4798	4803	4808	4812	4817
2.1	4821	4826	4830	4834	4838	4842	4846	4850	4854	4857
2.2	4861	4865	4868	4871	4875	4878	4881	4884	4887	4890
2.3	4893	4896	4898	4901	4904	4906	4909	4911	4913	4916
2.4	4918	4920	4922	4925	4927	4929	4931	4932	4934	4936
2.5	4938	4940	4941	4943	4946	4947	4948	4949	4951	4952
2.6	4953	4955	4956	4957	4959	4960	4961	4962	4963	4964
2.7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986
	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
	4987	4990	4993	4995	4997	4998	4998	4999	4999	4999

End of Block 21.2

The transformation is $Z = \frac{X-1000}{10}$.
when $X = 1005, Z = \frac{5}{10} = 0.5$
when $X = 1010,\ Z = \frac{10}{10} = 1$.
Hence Z lies between 0.5 and 1.

Back to the theory

(i) The entry is 4772; the probability is 0.4772.

(ii) The entry is 4803; the probability is 0.4803.

(iii) The entry is 4901; the probability is 0.4901.

(iv) The entry for 2.33 is 4901, that for 2.34 is 4904.

Linear interpolation gives a value of $4901 + 0.3(4904 - 4901)$ i.e. about 4903; the probability is 0.4903.

Back to the theory

(i) 0.4332 (direct from table)

(ii) $0.5 - 0.4641 = 0.0359$

(iii) $P(0 < Z < 1.8) - P(0 < Z < 1.5) = 0.4641 - 0.4332$
$= 0.0309$

(iv) $0.5 + 0.4641 = 0.9641$

(v) $P(-1.5 < Z < 0) = P(0 < Z < 1.5) = 0.4332$

(vi) $P(Z < -1.5) = P(Z > 1.5) = 0.5 - 0.4332 = 0.0668$

(vii) $P(-1.8 < Z < -1.5) = P(1.5 < Z < 1.8) = 0.0359$

(viii) $P(0 < Z < 1.5) + P(0 < Z < 1.8) = 0.8973$

Back to the theory

We look for a table value of 4950. The nearest we get is 4949 and 4951 corresponding to $Z = 2.57$ and $Z = 2.58$ respectively. We choose $Z = 2.58$

Back to the theory

$(-2.58,\ 2.58)$ or $-2.58 < Z < 2.58$.

Back to the theory

(i) The value is z_1, where $P(Z > z_1) = 0.05$. Hence $P(0 < Z < z_1) = 0.5 - 0.05 = 0.45$ This corresponds to a table entry of 4500. The nearest values are 4495 $(Z = 1.64)$ and 4505 $(Z = 1.65)$.

(ii) Values less than z_1 occur on 1% of occasions. By symmetry values greater than $(-z_1)$ occur on 1% of occasions so that $P(0 < z < -z_1) = 0.49$. The nearest table corresponding to 4900 is 4901 $(Z = 2.33)$.

Hence the required value is $z_1 = -2.33$.

Back to the theory