#6

$$\frac{dy}{dx} + y = \frac{1}{e^{2x} - 5e^x + 4}$$

$$\mu(x) = e^{\int 1 dx} = e^x \implies \text{multiply both sides by } e^x$$

$$e^x\left(\frac{dy}{dx} + y\right) = \frac{e^x}{e^{2x} - 5e^x + 4}$$

$$\frac{d}{dx}\left(e^x y\right) = \frac{e^x}{e^{2x} - 5e^x + 4}$$

$$e^x y = \int \frac{e^x}{e^{2x} - 5e^x + 4}\, dx$$

RHS: substitution and then partial fractions

$$u = e^x \Rightarrow du = e^x dx$$

$$\int \frac{1}{u^2 - 5u + 4}\, du$$

$$\frac{1}{u^2-5u+4} = \frac{1}{(u-4)(u-1)} = \frac{A}{u-4} + \frac{B}{u-1}$$

$$1 = A(u-1) + B(u-4)$$

$$u=1: \quad 1 = -3B \Rightarrow B = -\tfrac{1}{3}$$

$$u=4: \quad 1 = 3A \Rightarrow A = \tfrac{1}{3}$$

So $$\int \frac{1}{u^2-5u+4}\,du = \frac{1}{3}\ln|u-4| - \frac{1}{3}\ln|u-1|$$

$$= \frac{1}{3}\ln\left|\frac{u-4}{u-1}\right| = \frac{1}{3}\ln\left|\frac{e^x-4}{e^x-1}\right| + C$$

Thus $$e^x y = \frac{1}{3}\ln\left|\frac{e^x-4}{e^x-1}\right| + C$$

$$\boxed{y = \frac{1}{3}e^{-x}\ln\left|\frac{e^x-4}{e^x-1}\right| + Ce^{-x}}$$