Trigonometric Substitution

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 7.4 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to evaluate integrals that involve particular expressions (see Table 7.4.1) by making the appropriate trigonometric substitution.
- Know how to evaluate integrals that involve quadratic expressions by first completing the square and then making the appropriate substitution.

PRACTICE PROBLEMS:

For problems 1-12, evaluate the given integral. Notice that it may not be necessary to use a trigonometric substitution for all problems.

1. $\displaystyle\int \sqrt{3-x^2}\,dx$

   $\boxed{\dfrac{3}{2}\arcsin\left(\dfrac{x}{\sqrt{3}}\right)+\dfrac{1}{2}x\sqrt{3-x^2}+C}$

2. $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

   $\boxed{\dfrac{1}{2}\tan^{-1}x+\dfrac{x}{2(x^2+1)}+C}$

3. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$

   $\boxed{\arcsin\left(\dfrac{x}{2}\right)+C}$

4. $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

   $\boxed{-\dfrac{1}{4}\sqrt{1-4x^2}+C}$

5. $\displaystyle\int \frac{x^2}{\sqrt{1-2x^2}}\,dx$

   $\boxed{\dfrac{1}{4\sqrt{2}}\arcsin\left(\sqrt{2}x\right)-\dfrac{1}{4}x\sqrt{1-2x^2}+C}$; Detailed Solution: Here

6. $\displaystyle\int_1^{\sqrt{3}} x\sqrt{x^2+1}\,dx$

$\boxed{\dfrac{1}{3}(8-2\sqrt{2})}$

7. $\displaystyle\int_{\sqrt{2}}^{2} \frac{\sqrt{4-x^2}}{x^2}\,dx$

$\boxed{1-\dfrac{\pi}{4}}$

8. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+16}}\,dx$

$\boxed{-\dfrac{\sqrt{x^2+16}}{16x}+C}$

9. $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

$\boxed{\sqrt{3}-\dfrac{\pi}{3}}$

10. $\displaystyle\int_{-\sqrt{5}}^{\sqrt{15}} \frac{1}{x^2+5}\,dx$

$\boxed{\dfrac{7\pi}{12\sqrt{5}}}$

11. $\displaystyle\int \frac{1}{4x^2-2x+17/4}\,dx$

$\boxed{\dfrac{1}{4}\arctan\left(x-\dfrac{1}{4}\right)+C}$

12. $\displaystyle\int \frac{1}{\sqrt{-x^2+4x-3}}\,dx$

$\boxed{\sin^{-1}(x-2)+C}$

13. Compute the area enclosed within the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$.

$\boxed{6\pi}$

14. Let R be the region in the xy-plane which is enclosed by $y = \dfrac{1}{x^2+1}$, $y = 0$, $x = 0$ and $x = 1$. Calculate the volume of the solid which results from revolving R around the x-axis. (Hint: see number 2 above.)

$\boxed{\frac{\pi}{4}\left(\frac{\pi}{2}+1\right)\text{; Detailed Solution: Here}}$

15. Compute the length of the curve $y = x^2$ on the interval $\left[-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right]$. (Hint: See problem 25 (a) in the "Trigonometric Integrals (Chapter 7.3)" homework.)

$\boxed{\frac{\sqrt{2}}{4} + \frac{1}{4}\ln\left|\sqrt{2}+1\right| + \frac{\sqrt{3}}{2} - \frac{1}{4}\ln\left|2-\sqrt{3}\right|}$

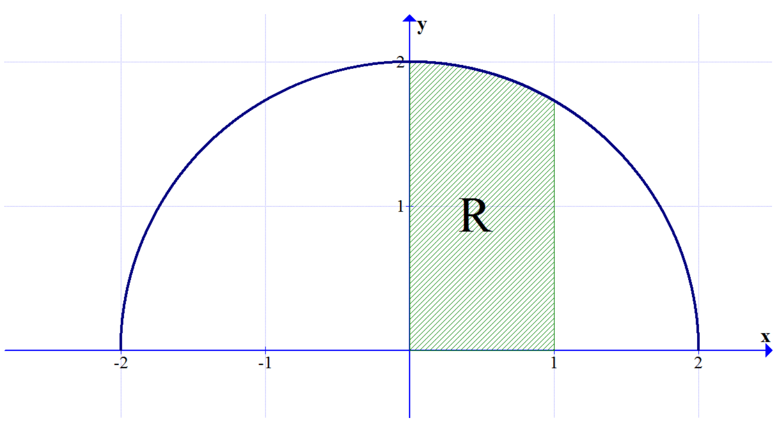
16. (a) Evaluate $\displaystyle\int \frac{\sqrt{x^2+1}}{x}\,dx$. (Hint: $\displaystyle\int \frac{\sec^3\theta}{\tan\theta}\,d\theta = \sec\theta - \ln|\csc\theta + \cot\theta| + C$)

$\boxed{\sqrt{x^2+1} - \ln\left|\frac{\sqrt{x^2+1}+1}{x}\right| + C}$

(b) Compute the length of the curve $y = \ln x$ on the interval $[1, 3]$. (Hint: Use part a.)

$\boxed{\sqrt{10} - \ln\left(\frac{\sqrt{10}+1}{3}\right) - \sqrt{2} + \ln\left(\sqrt{2}+1\right)}$

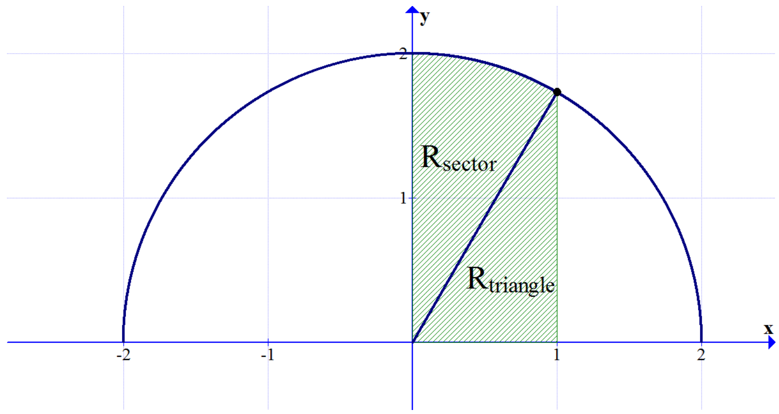
17. Consider the region R which is enclosed by $y = \sqrt{4-x^2}$, $y = 0$, $x = 0$, and $x = 1$, in the first quadrant.



(a) By evaluating an appropriate integral, compute the area of R.

$\boxed{A = \int_0^1 \sqrt{4-x^2}\,dx = \frac{\pi}{3} + \frac{\sqrt{3}}{2}}$

(b) Verify your answer geometrially by combining the area of the sector and the area of the triangle, shown below.



$$\boxed{R_{\text{Triangle}} = \frac{1}{2}bh = \frac{1}{2}(1)\left(\sqrt{3}\right) = \frac{\sqrt{3}}{2};\ R_{\text{Sector}} = \frac{1}{2}r^2\theta = \frac{1}{2}(2)^2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}}$$