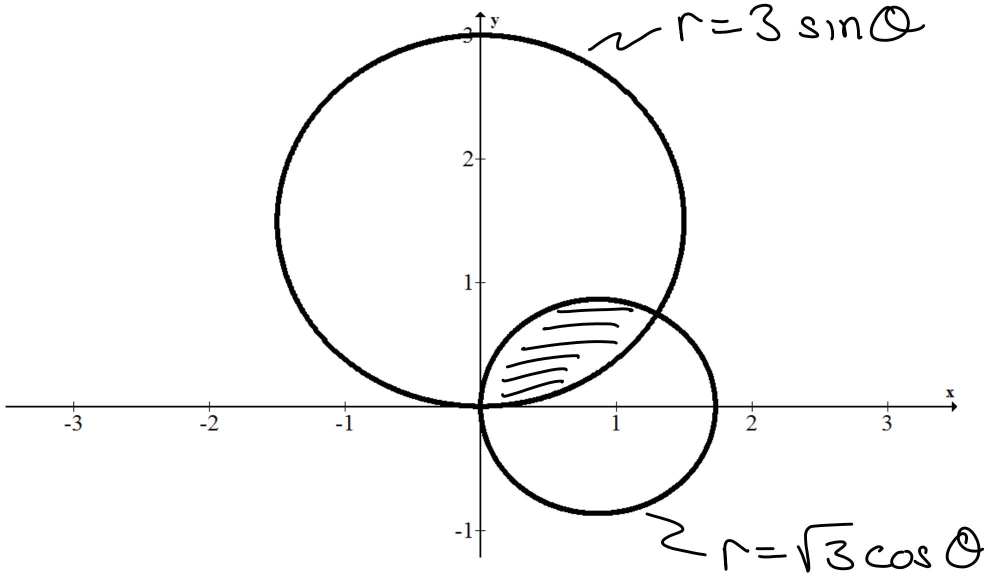
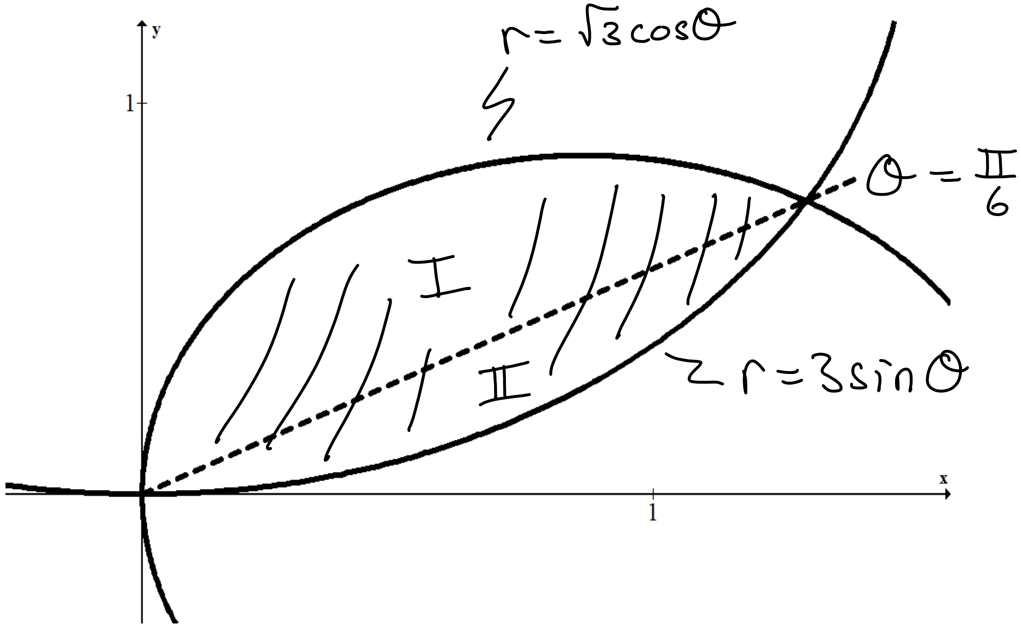
14.3 #8



$$3\sin\theta = \sqrt{3}\cos\theta$$

$$\tan\theta = \frac{\sqrt{3}}{3}$$

$$\theta = \frac{\pi}{6}$$



We need two sets of iterated integrals

$$\mathrm{I}: \int_{\pi/6}^{\pi/2} \int_{0}^{\sqrt{3}\cos\theta} 1\, r\, dr\, d\theta$$

$$\mathrm{II}: \int_{0}^{\pi/6} \int_{0}^{3\sin\theta} 1\, r\, dr\, d\theta$$

$$\text{I}: \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{0}^{\sqrt{3}\cos\theta} 1\, r\, dr\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2} r^2 \Big|_{0}^{\sqrt{3}\cos\theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3}{2}\cos^2\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3}{2}\cdot\frac{1}{2}(1+\cos 2\theta)\, d\theta = \frac{3}{4}\left(\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \frac{1}{2}\sin 2\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}\right)$$

$$= \frac{3}{4}\left(\left(\frac{\pi}{2}-\frac{\pi}{6}\right) + \frac{1}{2}\left(\sin\pi - \sin\frac{\pi}{3}\right)\right) = \frac{3}{4}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) = \frac{\pi}{4} - \frac{3\sqrt{3}}{16}$$

$$\text{II}: \int_{0}^{\frac{\pi}{6}} \int_{0}^{3\sin\theta} 1\, r\, dr\, d\theta = \int_{0}^{\frac{\pi}{6}} \frac{1}{2} r^2 \Big|_{0}^{3\sin\theta} d\theta = \int_{0}^{\frac{\pi}{6}} \frac{9}{2}\sin^2\theta\, d\theta$$

$$= \int_{0}^{\frac{\pi}{6}} \frac{9}{2}\cdot\frac{1}{2}(1-\cos 2\theta)\, d\theta = \frac{9}{4}\left(\theta \Big|_{0}^{\frac{\pi}{6}} - \frac{1}{2}\sin 2\theta \Big|_{0}^{\frac{\pi}{6}}\right)$$

$$= \frac{9}{4}\left(\left(\frac{\pi}{6}-0\right) - \frac{1}{2}\left(\sin\frac{\pi}{3} - \sin 0\right)\right) = \frac{9}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{3\pi}{8} - \frac{9\sqrt{3}}{16}$$

$$\boxed{\text{I} + \text{II} = \frac{5\pi}{8} - \frac{3\sqrt{3}}{4}}$$