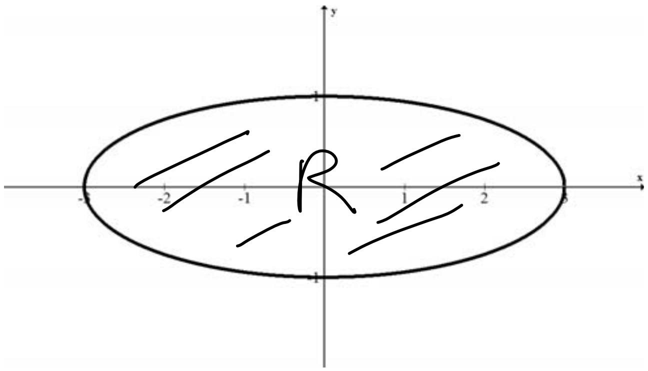
14.2 #12

$$x + 2y + z = 4 \implies z = 4 - x - 2y$$



$$\frac{x^2}{9} + y^2 = 1$$

$$\implies x = \pm\sqrt{9 - 9y^2} = \pm 3\sqrt{1-y^2}$$

So volume is $\displaystyle\int_{-1}^{1}\int_{-3\sqrt{1-y^2}}^{3\sqrt{1-y^2}} (4 - x - 2y)\,dx\,dy$

[Note: $\displaystyle\int_{-3}^{3}\int_{-\sqrt{1-\frac{x^2}{9}}}^{\sqrt{1-\frac{x^2}{9}}} (4-x-2y)\,dy\,dx$ would work just as well.]

$$\int_{-1}^{1}\left[4x\Big|_{-3\sqrt{1-y^2}}^{3\sqrt{1-y^2}} - \frac{1}{2}x^2\Big|_{-3\sqrt{1-y^2}}^{3\sqrt{1-y^2}} - 2yx\Big|_{-3\sqrt{1-y^2}}^{3\sqrt{1-y^2}} \right] dy$$

$$= \int_{-1}^{1} \left(24\sqrt{1-y^2} - 0 - 12y\sqrt{1-y^2} \right) dy$$

$$= \underbrace{24\int_{-1}^{1}\sqrt{1-y^2}}_{\text{geometry}} - \underbrace{12\int_{-1}^{1} y\sqrt{1-y^2}\, dy}$$

geometry

$24\left(\frac{1}{2}\right)\pi(1)^2$

$u = 1-y^2 \Rightarrow du = -2y\, dy$

$y = 1 \Rightarrow u = 0$ } $\Rightarrow$ limits of integration are the same, so the result is 0

$y = -1 \Rightarrow u = 0$

$$= 12\pi - 0 = \boxed{12\pi}$$

Comment : $\int_{-1}^{1}\sqrt{1-y^2}\, dy$ can be solved using trigonometric substitution ($y = \sin\theta$, etc.) but using geometry is much easier.