Trigonometric Integral

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 7.3 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Know antiderivatives for all six elementary trigonometric functions.
- Be able to evaluate integrals that involve powers of sine, cosine, tangent, and secant by using appropriate trigonometric identities.

PRACTICE PROBLEMS:

1. Fill in the following table

$\int \sin x\, dx =$	$\boxed{-\cos x + C}$
$\int \cos x\, dx =$	$\boxed{\sin x + C}$
$\int \tan x\, dx =$	$\boxed{\ln\lvert\sec x\rvert + C}$
$\int \cot x\, dx =$	$\boxed{\ln\lvert\sin x\rvert + C}$
$\int \sec x\, dx =$	$\boxed{\ln\lvert\sec x + \tan x\rvert + C}$
$\int \csc x\, dx =$	$\boxed{-\ln\lvert\csc x + \cot x\rvert + C}$

2. $\displaystyle\int_{\pi/4}^{\pi/3} \cot 2x dx$

$\boxed{\dfrac{1}{4}\ln 3 - \dfrac{1}{2}\ln 2}$

<u>Powers of Sines & Cosines:</u> For each of the following, evaluate the given integral.

3. $\displaystyle\int \sin(x)\cos^3(x)\, dx$

$\boxed{-\dfrac{1}{4}\cos^4 x + C}$

4. $\displaystyle\int \sin^3(x)\cos^4(x)\,dx$

$\boxed{\dfrac{1}{7}\cos^7 x - \dfrac{1}{5}\cos^5 x + C}$

5. $\displaystyle\int \sqrt{\sin x}\cos^3(x)\,dx$

$\boxed{\dfrac{2}{3}(\sin x)^{3/2} - \dfrac{2}{7}(\sin x)^{7/2} + C}$

6. $\displaystyle\int \sin^2 x\,dx$

$\boxed{\dfrac{x}{2} - \dfrac{1}{4}\sin(2x) + C}$

7. $\displaystyle\int \sin^3(bx)\,dx$, where b is a non-zero constant

$\boxed{\dfrac{1}{3b}\cos^3(bx) - \dfrac{1}{b}\cos(bx) + C\text{; Detailed Solution: Here}}$

8. $\displaystyle\int \sin^2 x\cos^2 x\,dx$

$\boxed{\dfrac{x}{8} - \dfrac{1}{32}\sin(4x) + C}$

9. $\displaystyle\int_{\pi/4}^{\pi/2} \cos^3 x\,dx$

$\boxed{\dfrac{2}{3} - \dfrac{5\sqrt{2}}{12}}$

10. $\displaystyle\int \cos^4 5x\,dx$

$\boxed{\dfrac{3}{8}x + \dfrac{1}{20}\sin(10x) + \dfrac{1}{160}\sin(20x) + C}$

11. Consider the trigonmetric identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$

    (a) Use this identity to derive an identity for $\sin(A-B)$ in terms of $\sin A$, $\cos A$, $\sin B$, and $\cos B$.

    $\boxed{\sin(A-B) = \sin A\cos B - \cos A\sin B}$

(b) Use the given identity and your answer for part (a) to derive the following identity:

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

Adding the given identity to the identity from part (a) and then dividing both sides by 2 yields the desired result.

12. Consider the trigonmetric identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(a) Use this identity to derive an identity for $\cos(A-B)$ in terms of $\sin A$, $\cos A$, $\sin B$, and $\cos B$.

$\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B}$

(b) Use the given identity and your answer for part (a) to derive the following identity:

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

Adding the given identity to the identity from part (a) and then dividing both sides by 2 yields the desired result.

(c) Use the given identity and your answer for part (a) to derive the following identity:

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

Subtracting the given identity from the identity from part (a) and then dividing both sides by 2 yields the desired result.

For problems 13-16, use an appropriate identity from problem 11 or 12 to evaluate the given integral.

13. $\displaystyle\int \sin(2x)\cos\left(\frac{x}{2}\right)dx$

$$\boxed{-\frac{1}{5}\cos\left(\frac{5x}{2}\right) - \frac{1}{3}\cos\left(\frac{3x}{2}\right) + C}$$

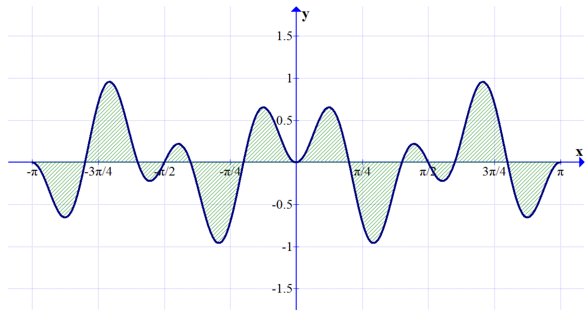
14. $\displaystyle\int \cos(3x)\cos(4x)\,dx$

$$\boxed{\frac{1}{2}\sin x + \frac{1}{14}\sin(7x) + C}$$

15. $\displaystyle\int \sin(5x)\cos(2x)\,dx$

$$\boxed{-\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C}$$

16. The graph of $f(x) = \sin 2x \sin 5x$ on the interval $[-\pi, \pi]$ is shown below.



Compute the net signed area between the graph of $f(x)$ and the x-axis on the interval $[-\pi, \pi]$

$\boxed{0}$

Powers of Tangents & Secants: For each of the following, evaluate the given integral.

17. $\displaystyle\int \tan^2 3x \, dx$

$$\boxed{-x + \frac{1}{3}\tan(3x) + C}$$

18. $\displaystyle\int_0^{\pi/4} \tan^3(x)\sec^3(x)\, dx$

$$\boxed{\frac{2}{15}\left(1+\sqrt{2}\right)}$$

19. $\displaystyle\int \tan(x)\sec^3(x) dx$

$$\boxed{\frac{1}{3}\sec^3 x + C}$$

20. $\displaystyle\int \tan^3(x)\sec^4(x)\, dx$

$$\boxed{\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C}$$

21. $\displaystyle\int \tan^5(2x)\sec^2(2x)\, dx$

$$\boxed{\frac{1}{12}\tan^6(2x) + C}$$

22. $\displaystyle\int \tan(x)\sec^{5/2}(x)\,dx$

$\boxed{\dfrac{2}{5}\sec^{5/2}x + C}$; Detailed Solution: Here

23. $\displaystyle\int \sec^4 x\,dx$

$\boxed{\dfrac{1}{3}\tan^3 x + \tan x + C}$

24. Consider $\displaystyle\int_{\pi/2}^{\pi} \sec x\,dx$

(a) Explain why this integral is improper.

The integral is improper because $\sec x$ has an infinite discontinuity at $x = \dfrac{\pi}{2}$ which is the lower limit of integration.

(b) Evaluate the given integral. If it diverges, explain why.

The integral diverges because $\displaystyle\int_{\pi/2}^{\pi} \sec x\,dx = -\infty$

25. (a) Use integration by parts to evaluate $\displaystyle\int \sec^3(x)\,dx$. (Hint: $\sec^3 x = \sec^2 x \sec x$ and $\tan^2 x = \sec^2 x - 1$)

$\boxed{\dfrac{1}{2}\sec(x)\tan(x) + \dfrac{1}{2}\ln|\sec x + \tan x| + C}$

(b) Use part (a) to evaluate $\displaystyle\int \tan^2(x)\sec(x)\,dx$

$\boxed{\dfrac{1}{2}\sec(x)\tan(x) - \dfrac{1}{2}\ln|\sec x + \tan x| + C}$

26. Let R be the region bounded between the graphs of $y = \sin x$ and $y = \cos x$ on the interval $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$.

(a) Compute the area of R.

$\boxed{\sqrt{2} - 1}$

(b) Compute the volume of the solid which results from revolving R around the x-axis.

$\boxed{\dfrac{\pi}{2}}$

27. Find the length of the curve $y = \ln(\sin x)$ on the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

$\boxed{2\ln\left(2+\sqrt{2}\right) - \ln 2\text{; Detailed Solution: Here}}$