Differentiating and Integrating Power Series

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Know (i.e. memorize) the Maclaurin series for $e^x, \sin x$ and $\cos x$. Algebraically manipulate these series expansions, as well as other given power series expansions, to form new expansions.
- Differentiate and integrate power series expansions term-by-term.
- Use a series expansion to approximate an integral to some specified accuracy.

PRACTICE PROBLEMS:

1. Confirm that $\dfrac{d}{dx}(e^x) = e^x$ by differentiating the Maclaurin series for e^x term-by-term.

$$\begin{aligned}
\frac{d}{dx}(e^x) &= \frac{d}{dx}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right] \\
&= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
&= e^x.
\end{aligned}$$

2. Recall that the Maclaurin series for e^x converges to e^x for all real numbers x. It can be shown (in a complex analysis course) that this convergence holds for any complex number as well. Based on this fact, use the Maclaurin series for $e^x, \sin x$, and $\cos x$ to prove Euler's Formula:

$$e^{ix} = \cos x + i \sin x,$$

where i is the imaginary number with the property $i^2 = -1$ (and thus $i^3 = -i$, $i^4 = 1$, etc).

Since $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$, we have

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \ldots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + \ldots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \\
&= \cos x + i \sin x
\end{aligned}$$

Note: Recall that Euler's formula was useful when solving second order linear homogeneous differential equations with constant coefficients, specifcally for when the characteristic equation had complex roots.

3. The purpose of this problem is to find the Maclaurin series for $\arctan x$. If we attempt to take successive derivatives of $\arctan x$ the computation becomes unpleasant rather quickly (try it if you want). Here is a simpler alternative.

   (a) Find the Maclaurin series for $\frac{1}{1-x}$.

   $1 + x + x^2 + x^3 + x^4 + \ldots = \displaystyle\sum_{k=0}^{\infty} x^k$. See Convergence of Taylor Series #5.

   (b) Replace x in part (a) with the appropriate quantity to obtain the Maclaurin series for $\frac{1}{1+x^2}$.

   If we replace x in $\frac{1}{1-x}$ with $-x^2$ we get $\frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$.

   So the Maclaurin series for $\dfrac{1}{1+x^2}$ is

   $$1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \ldots = 1 - x^2 + x^4 - x^6 + \ldots = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

   (c) Integrate the answer in part (b) term-by-term to obtain the Maclaurin series for $\arctan x$.

   We know that $\displaystyle\int \frac{1}{1+x^2} = \arctan x + C$.

   So $\arctan x = \displaystyle\int (1 - x^2 + x^4 - x^6 + \ldots)\, dx - C = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right] - C.$

   Since $\arctan(0) = 0$, we have $C = 0$, and therefore $\arctan x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1}}{2k+1}$.

4. Find the first four nonzero terms of the Maclaurin series for $f(x) = e^{(x^2)} \arctan x$ by multiplying the Maclaurin series of the factors. See the previous problem for the Maclaurin series for $\arctan x$.

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, we have $e^{(x^2)} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots$

So

$$\begin{aligned} e^{(x^2)} \arctan x &= \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right) \\ &= (1)(x) + \left[(1)\left(-\frac{x^3}{3}\right) + (x^2)(x)\right] + \left[(1)\left(\frac{x^5}{5}\right) + (x^2)\left(-\frac{x^3}{3}\right) + \left(\frac{x^4}{2!}\right)(x)\right] \\ &+ \left[(1)\left(-\frac{x^7}{7}\right) + (x^2)\left(\frac{x^5}{5}\right) + \left(\frac{x^4}{2!}\right)\left(-\frac{x^3}{3!}\right) + \left(\frac{x^6}{3!}\right)(x)\right] + \ldots \\ &= x + \frac{2}{3}x^3 + \frac{11}{30}x^5 + \frac{2}{35}x^7 + \ldots \end{aligned}$$

5. Consider the function $f(x) = \sin x \cos x$.

   (a) Find the first three nonzero terms of the Maclaurin series for $f(x)$ by multiplying the Maclaurin series of the factors.

   $x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \ldots$

   (b) Confirm your answer in part (a) by using the trigonometric identity $\sin 2x = 2\sin x \cos x$.

   $$\sin x \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots\right) = x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \ldots$$

6. Find the first three nonzero terms of the Maclaurin series for $f(x) = \tan x$ by performing a long division on the Maclaurin series for $\sin x$ and $\cos x$.

   $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$; Detailed Solution: Here

7. Use the result in #6 to find the first three nonzero terms of the Maclaurin series for $f(x) = \sec^2 x$.

   $$\sec^2 x = \frac{d}{dx}(\tan x) = \frac{d}{dx}\left[x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots\right] = 1 + x^2 + \frac{2}{3}x^4 + \ldots$$

8. Use a Maclaurin series to approximate $\int_0^1 \cos(x^2)\, dx$ to four decimal-place accuracy.

$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k}}{(2k)!}$, so $\cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{4k}}{(2k)!}$.

Thus $\int_0^1 \cos(x^2) \ dx = \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{4k+1}}{(2k)!(4k+1)} \Bigg|_0^1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!(4k+1)}$,

which is an alternating series that converges to $S = \int_0^1 \cos(x^2) \ dx$.

Thus $|S - s_n| < a_{n+1}$, where s_n is the n-th partial sum and $a_k = \frac{1}{(2k)!(4k+1)}$.

For four decimal-place accuracy, we want

$$a_{n+1} = \frac{1}{(2n+2)!(4n+5)} \leq 0.00005.$$
$$(2n+2)!(4n+5) \geq 20,000$$

Now for $n = 2 : (6!)(13) < 20,000$ and for $n = 3 : (8!)(17) > 20,000$.
So we want the 3-rd partial sum s_3.

Therefore $\int_0^1 \cos(x^2) \ dx \approx \sum_{k=0}^{3} \frac{(-1)^k}{(2k)!(4k+1)} = 1 - \frac{1}{(2!)(5)} + \frac{1}{(4!)(9)} - \frac{1}{(6!)(13)}$.

9. Use a Maclaurin series to approximate $\int_0^1 \arctan(x^2) \ dx$ to two decimal-place accuracy. See problem #3 for the Maclaurin series for $\arctan x$.

$$\int_0^1 \arctan(x^2) \ dx \approx \sum_{k=0}^{4} \frac{(-1)^k}{(2k+1)(4k+3)} = \frac{1}{3} - \frac{1}{(3)(7)} + \frac{1}{(5)(11)} - \frac{1}{(7)(15)} + \frac{1}{(9)(19)}.$$