Sequences

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Find the general term of a sequence.
- Determine whether a sequence converges, and if so, what it converges to. This may require techniques such as L'Hopital's Rule and The Squeeze Theorem.

PRACTICE PROBLEMS:

For problems 1 – 8, rewrite the sequence by placing the general term inside braces.

1. $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \ldots$

   $\left\{\frac{1}{4^n}\right\}_{n=1}^{+\infty}$. Note that any integer can be used as the lower index for a sequence. So you could describe this sequence with $\left\{\frac{1}{4^{n+1}}\right\}_{n=0}^{+\infty}$ or even $\left\{\frac{1}{4^{n+10}}\right\}_{n=-9}^{+\infty}$, although admittedly it is weird to use -9 as a starting index unless there is some compelling reason to do so. It is recommended to use either 0 or 1 as a starting index in most cases.

2. $\frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, -\frac{1}{256}, \ldots$

   $\boxed{\left\{(-1)^{n+1}\frac{1}{4^n}\right\}_{n=1}^{+\infty}}$

3. $0, 1, 2^3, 3^4, 4^5, \ldots$

   $\boxed{\left\{n^{n+1}\right\}_{n=0}^{+\infty}}$

4. $3, 2, 1, 0, -1, -2, -3, -4, -5, \ldots$

   $\boxed{\{4-n\}_{n=1}^{+\infty}}$

5. $1, \frac{1}{e}, e^2, \frac{1}{e^3}, e^4, \frac{1}{e^5}, \ldots$

   $\boxed{\left\{e^{(-1)^n n}\right\}_{n=0}^{+\infty}}$

6. $\dfrac{1}{1\cdot 2}, \dfrac{1}{1\cdot 2\cdot 3\cdot 4}, \dfrac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}, \dfrac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}, \ldots$

   $\boxed{\left\{\dfrac{1}{(2n)!}\right\}_{n=1}^{+\infty}}$

7. $\dfrac{3}{1\cdot 2}, \dfrac{5}{1\cdot 2\cdot 3\cdot 4}, \dfrac{7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}, \dfrac{9}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}, \ldots$

   $\boxed{\left\{\dfrac{2n+1}{(2n)!}\right\}_{n=1}^{+\infty}}$

8. $0, 1, 0, -1, 0, 1, 0, -1, 0, 1 \ldots$ [Hint: Think about a trigonometric function.]

   $\left\{\sin\left(\dfrac{\pi}{2}n\right)\right\}_{n=0}^{+\infty}$ or $\left\{-\cos\left(\dfrac{\pi}{2}n\right)\right\}_{n=1}^{+\infty}$ or $\left\{\cos\left(\dfrac{\pi}{2}n\right)\right\}_{n=-1}^{+\infty}$. There are other possibilities as well.

9. For each of the sequences in problems 1 – 8, determine if the sequence converges, and if so, what it converges to. If it diverges, determine if the general term approaches $+\infty$, $-\infty$, or neither.

   The sequence in problem:

   #1 converges to 0.

   #2 converges to 0.

   #3 diverges to $+\infty$.

   #4 diverges to $-\infty$.

   #5 diverges. The even-numbered terms converge to 0, but the odd-numbered terms diverge to $+\infty$

   #6 converges to 0.

   #7 converges to 0.

   #8 diverges. The odd-numbered terms converge to 0 (in fact, they are all equal to 0), but the even-numbered terms diverge since they oscillate between 1 and -1.

   Detailed Solution: Here

For problems 10 – 35, determine if the sequence converges, and if so, what it converges to. If it diverges, determine if the general term approaches $+\infty$, $-\infty$, or neither.

10. $\{5\}_{n=1}^{+\infty}$

    Converges to 5.

11. $\{5n\}_{n=0}^{+\infty}$

$\boxed{\text{Diverges to } +\infty.}$

12. $\left\{5-5n^3\right\}_{n=1}^{+\infty}$

$\boxed{\text{Diverges to } -\infty.}$

13. $\left\{\dfrac{4n-3n^5}{2n^5+4n^3+n^2+5}\right\}_{n=1}^{+\infty}$

$\boxed{\text{Converges to } -\dfrac{3}{2}.\ \text{See \underline{Limits at Infinity Review} problem \#2.}}$

14. $\left\{(-1)^n\ \dfrac{n^3+n^2+n+1}{n^3+1}\right\}_{n=1}^{+\infty}$

$\boxed{\parbox{0.9\textwidth}{The odd-numbered terms converge to -1, but the even-numbered terms converge to 1, so the sequence diverges.}}$

15. $\left\{\dfrac{n^4-3n^3-2n}{4n^2+19}\right\}_{n=0}^{+\infty}$

$\boxed{\text{Diverges to } +\infty.}$

16. $\left\{\dfrac{1-10n^2}{n^2-4n^3}\right\}_{n=1}^{+\infty}$

$\boxed{\text{Converges to } 0.}$

17. $\left\{(-1)^{n+1}\ \dfrac{1-10n^2}{n^2-4n^3}\right\}_{n=1}^{+\infty}$

$\boxed{\text{Converges to } 0.}$

18. $\left\{\dfrac{\sqrt{4+3n^2}}{2+7n}\right\}_{n=1}^{+\infty}$

$\boxed{\text{Converges to } \dfrac{\sqrt{3}}{7}.\ \text{See \underline{Limits at Infinity Review} problem \#3.}}$

19. $\left\{e^{1/n}\right\}_{n=1}^{+\infty}$

$\boxed{\text{Converges to } 1.}$

20. $\left\{\dfrac{e^{-n}}{n^{-2}}\right\}_{n=1}^{+\infty}$

$\boxed{\text{Converges to } 0.}$

21. $\left\{\dfrac{e^{n}-e^{-n}}{e^{n}+e^{-n}}\right\}_{n=1}^{+\infty}$

Converges to 1. See Limits at Infinity Review problem #5.; Detailed Solution: Here

22. $\left\{\dfrac{e^{\sqrt{n}}}{n}\right\}_{n=1}^{+\infty}$

$\boxed{\text{Diverges to } +\infty.}$

23. $\left\{e^{n}\sin(e^{-n})\right\}_{n=1}^{+\infty}$

$\boxed{\text{Converges to } 1.}$

24. $\left\{e^{n}\pi^{-n}\right\}_{n=1}^{+\infty}$

$\boxed{\text{Converges to } 0.}$

25. $\left\{\ln\left(\dfrac{1}{n}\right)\right\}_{n=1}^{+\infty}$

$\boxed{\text{Diverges to } -\infty.}$

26. $\left\{\dfrac{\ln(6n)}{\ln(2n)}\right\}_{n=1}^{+\infty}$

Converges to 1.; Detailed Solution: Here

27. $\left\{\ln(n+2)-\ln(3n+5)\right\}_{n=0}^{+\infty}$

$\boxed{\text{Converges to } \ln\left(\dfrac{1}{3}\right).}$

28. $\left\{\sqrt{n^{2}+8n-5}-n\right\}_{n=1}^{+\infty}$

Converges to 4. See Limits at Infinity Review problem #4.; Detailed Solution: Here

29. $\left\{\sqrt{n^{2}-n}+n\right\}_{n=0}^{+\infty}$

$\boxed{\text{Diverges to } +\infty.}$

30. $\left\{\sqrt{n^2-n}-n\right\}_{n=1}^{+\infty}$

$\boxed{\text{Converges to } -\frac{1}{2}.}$

31. $\left\{\frac{\cos n}{n}\right\}_{n=1}^{+\infty}$

Converges to 0. See Limits at Infinity Review problem #7.

32. $\left\{\arccos\left(\frac{n^2}{3n-n^2}\right)\right\}_{n=1}^{+\infty}$

Converges to π.

33. $\left\{\arctan\left(\frac{1}{n}\right)-\arctan(n)\right\}_{n=1}^{+\infty}$

Converges to $-\frac{\pi}{2}$. See Limits at Infinity Review problem #8.

34. $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{+\infty}$

Converges to e. See Limits at Infinity Review problem #9.

35. $\left\{(1+3^n)^{1/n}\right\}_{n=1}^{+\infty}$

Converges to 3. See Limits at Infinity Review problem #10.; Detailed Solution: Here

36. $\left\{\left(\frac{4}{n}\right)^{2/n}\right\}_{n=1}^{+\infty}$

Converges to 1.

37. Consider the sequence $\sqrt{30}, \sqrt{30+\sqrt{30}}, \sqrt{30+\sqrt{30+\sqrt{30}}}, \ldots$

   (a) Define the sequence recursively.

   $\boxed{a_1=\sqrt{30}, a_{n+1}=\sqrt{30+a_n} \text{ for integers } n\geq 1.}$

   (b) Assuming the sequence converges to some limit L, find L.

   $\boxed{L=6.}$

38. Consider the sequence $\{a_n\}_{n=1}^{+\infty}$ that has the following recursive definition: $a_{n+1}=10-a_n$ for integers $n\geq 1$.

(a) Assuming the sequence converges to some limit L, find L.

$\boxed{L = 5.}$

(b) How must a_1 be defined to ensure that the sequence converges? Justify your answer.

If $a_1 = 5$, the sequence is $5, 5, 5, 5, ...$, which clearly converges to 5. If $a_1 \neq 5$, say $a_1 = K (K \neq 5)$, then the sequence oscillates bewteen K and $10 - K$, e.g. if $a_1 = 3$ we have $3, 7, 3, 7, 3, 7, ...$ Such a sequence diverges.

39. The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, ...$ begins with two 1's and thereafter each term in the sequence is the the sum of previous two terms.

(a) Define the Fibonacci sequence recursively.

$\boxed{a_1 = 1, a_2 = 1, a_{n+2} = a_n + a_{n+1} \text{ for integers } n \geq 1.}$

(b) Clearly the Fibonacci sequence diverges to $+\infty$, but consider the ratio of successive terms $\dfrac{a_{n+1}}{a_n}$ for $n \geq 1$, i.e

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \dots$$

Assuming this "ratio sequence" converges to some limit L, find L.

$L = \dfrac{1+\sqrt{5}}{2}$. This number is known as the Golden Ratio.

Fibonacci Fun Fact: The Fibonacci Sequence can be described without recursion as

$$\left\{ \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \right\}_{n=1}^{+\infty},$$

and with some clever factoring it can be shown from this non-recursive definition that the limit of the "ratio sequence" is $L = \dfrac{1+\sqrt{5}}{2}$.

Detailed Solution: Here