Parametric Equations, Tangent Lines, & Arc Length

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 10.1 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

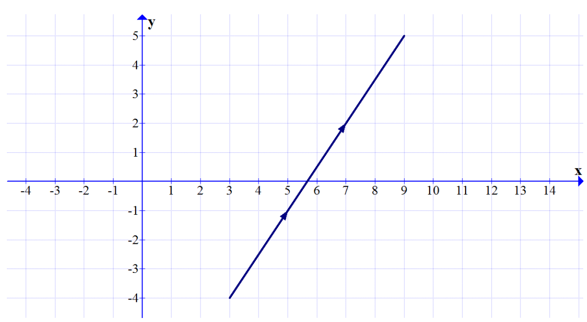
- Be able to sketch a parametric curve by eliminating the parameter, and indicate the orientation of the curve.
- Given a curve and an orientation, know how to find parametric equations that generate the curve.
- Without eliminating the parameter, be able to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at a given point on a parametric curve.
- Be able to find the arc length of a smooth curve in the plane described parametrically.

PRACTICE PROBLEMS:

For problems 1-5, sketch the curve by eliminating the parameter. Indicate the direction of increasing t.

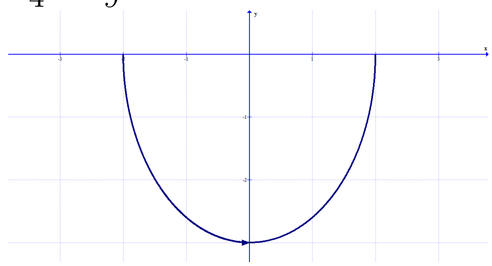
1. $\begin{cases} x = 2t + 3 \\ y = 3t - 4 \\ 0 \le t \le 3 \end{cases}$

$$y = \frac{3}{2}x - \frac{17}{2} \text{ from } (3, -4) \text{ to } (9, 5)$$



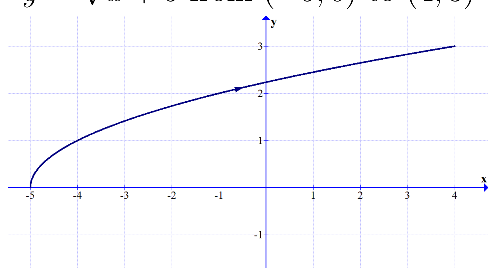
2. $\begin{cases} x = 2\cos t \\ y = 3\sin t \\ \pi \leq t \leq 2\pi \end{cases}$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \text{ from } (-2, 0) \text{ to } (2, 0)$$



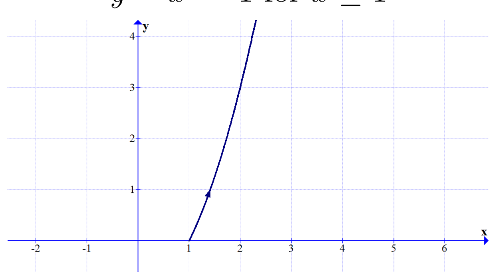
3. $\begin{cases} x = t - 5 \\ y = \sqrt{t} \\ 0 \leq t \leq 9 \end{cases}$

$$y = \sqrt{x+5} \text{ from } (-5, 0) \text{ to } (4, 3)$$



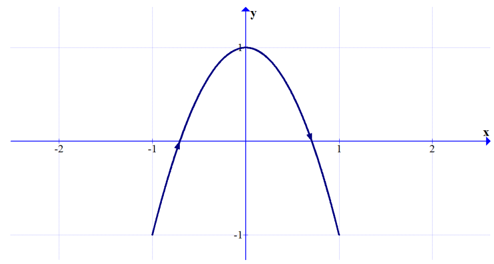
4. $\begin{cases} x = \sec t \\ y = \tan^2 t \\ 0 \leq t < \dfrac{\pi}{2} \end{cases}$

$$y = x^2 - 1 \text{ for } x \geq 1$$



5. $\begin{cases} x = \sin t \\ y = \cos(2t) \\ -\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2} \end{cases}$

$y = 1 - 2x^2$ from $(-1, -1)$ to $(1, -1)$



For problems 6-10, find parametric equations for the given curve. (For each, there are many correct answers; only one is provided.)

6. A horizontal line which intersects the y-axis at $y = 2$ and is oriented rightward from $(-1, 2)$ to $(1, 2)$.

$$\boxed{\begin{cases} x = t \\ y = 2 \\ -1 \le t \le 1 \end{cases}}$$

7. A circle or radius 4 centered at the origin, oriented clockwise.

$$\boxed{\begin{cases} x = 4\sin t \\ y = 4\cos t \\ 0 \le t \le 2\pi \end{cases}}$$

8. A circle of radius 5 centered at $(1, -2)$, oriented counter-clockwise.

$$\boxed{\begin{cases} x = 5\cos t + 1 \\ y = 5\sin t - 2 \\ 0 \le t \le 2\pi \end{cases} \text{; Detailed Solution: Here}}$$

9. The portion of $y = x^3$ from $(-1, -1)$ to $(2, 8)$, oriented upward.

$$\boxed{\begin{cases} x = t \\ y = t^3 \\ -1 \le t \le 2 \end{cases}}$$

10. The ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$, oriented counter-clockwise.

$$\boxed{\begin{cases} x = 2\cos t \\ y = 4\sin t \\ 0 \le t \le 2\pi \end{cases}}$$

For problems 11-13, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the given point without eliminating the parameter.

11. The curve $\begin{cases} x = 3\sin(3t) \\ y = \cos(3t) \\ 0 < t < 2\pi \end{cases}$ at $t = \pi$

$$\boxed{\left.\frac{dy}{dx}\right|_{t=\pi} = 0;\ \left.\frac{d^2y}{dx^2}\right|_{t=\pi} = \frac{1}{9}}$$

12. The curve $\begin{cases} x = t^2 \\ y = 3t - 2 \\ t \ge 0 \end{cases}$ at $t = 1$

$$\boxed{\left.\frac{dy}{dx}\right|_{t=1} = \frac{3}{2};\ \left.\frac{d^2y}{dx^2}\right|_{t=1} = -\frac{3}{4}}$$

13. The curve $\begin{cases} x = 2\tan t \\ y = \sec t \\ 0 \le t \le \frac{\pi}{3} \end{cases}$ at $t = \dfrac{\pi}{4}$

$\boxed{\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{\sqrt{2}}{4},\ \left.\frac{d^2y}{dx^2}\right|_{t=\pi/4} = \frac{\sqrt{2}}{16}\text{; Detailed Solution: Here}}$

14. Consider the curve described parametrically by $\begin{cases} x = \sqrt{t} \\ y = \sqrt[3]{t} + 1 \\ t \ge 0 \end{cases}$

 (a) Compute $\left.\dfrac{dy}{dx}\right|_{t=64}$ without eliminating the parameter.

$$\boxed{\left.\frac{dy}{dx}\right|_{t=64} = \frac{1}{3}}$$

(b) Eliminate the parameter and verify your answer for part (a) using techniques from differential calculus.

The curve is equivalent to $y = x^{2/3} + 1$, $x \geq 0$. And, $t = 64$ corresponds to $x = 8$. Thus, $\left.\frac{dy}{dx}\right|_{t=64} = \left.\frac{dy}{dx}\right|_{x=8} = \frac{1}{3}$

(c) Compute an equation of the line which is tangent to the curve at the point corresponding to $t = 64$.

$$y - 5 = \frac{1}{3}(x - 8)$$

15. Consider the curve described parametrically by $\begin{cases} x = 2\cos t \\ y = 4\sin t \\ 0 \leq t \leq 2\pi \end{cases}$

(a) Compute $\left.\frac{dy}{dx}\right|_{t=\pi/4}$ without eliminating the parameter.

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = -2$$

(b) Eliminate the parameter and verify your answer for part (a) using techniques from differential calculus.

The curve is equivalent to the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 1$. And, $t = \frac{\pi}{4}$ corresponds to the point $(x, y) = (\sqrt{2}, 2\sqrt{2})$. Thus, you can use implicit differentiation and $\left.\frac{dy}{dx}\right|_{t=\pi/4} = \left.\frac{dy}{dx}\right|_{(x,y)=\left(\sqrt{2},2\sqrt{2}\right)} = -2$

(c) Compute an equation of the line which is tangent to the curve at the point corresponding to $t = \frac{\pi}{4}$.

$$y - 2\sqrt{2} = -2\left(x - \sqrt{2}\right)$$

(d) At which value(s) of t will the tangent line to the curve be horizontal?

$$t = \frac{\pi}{2} \text{ and } t = \frac{3\pi}{2}$$

For problems 16-18, compute the length of the given parametric curve.

16. The curve described by $\begin{cases} x = t \\ y = \frac{2}{3}t^{3/2} \\ 0 \leq t \leq 4 \end{cases}$

$$\boxed{-\frac{2}{3}+\frac{10\sqrt{5}}{3}}$$

17. The curve described by $\begin{cases} x = e^t \\ y = \dfrac{2}{3}e^{3t/2} \\ \ln 2 \le t \le \ln 3 \end{cases}$

$\boxed{-2\sqrt{3}+\frac{16}{3}}$; Detailed Solution: Here

18. The curve described by $\begin{cases} x = \dfrac{1}{2}t^2 \\ y = \dfrac{1}{3}t^3 \\ 0 \le t \le \sqrt{3} \end{cases}$

$$\boxed{\frac{7}{3}}$$

19. Compute the lengths of the following two curves:

$$C_1(t) = \begin{cases} x = \cos t \\ y = \sin t \\ 0 \le t \le 2\pi \end{cases} \qquad C_2(t) = \begin{cases} x = \cos(3t) \\ y = \sin(3t) \\ 0 \le t \le 2\pi \end{cases}$$

Explain why the lengths are not equal even though both curves coincide with the unit circle.

> The length of $C_1(t)$ is 2π and the length of $C_2(t) = 6\pi$. Notice that $C_2(t)$ is the just curve $C_1(t)$ traversed three times.

20. This problem describes how you can find the area between a parametrically defined curve and the x-axis.

The Main Idea: Recall that if $y = f(x) \ge 0$, then the area between the curve and the x-axis on the interval $[a, b]$ is $\int_a^b f(x)dx = \int_a^b y\,dx$. Now, suppose that the same curve is described parametrically by $x = x(t)$, $y = y(t)$ for $t_0 \le t \le t_1$ and that the curve is traversed exactly once on this interval. Then, $A = \int_a^b y\,dx = \int_{t_0}^{t_1} y(t)x'(t)\,dt$.

Consider the curve $\begin{cases} x = \sin t \\ y = \cos(2t) \\ -\dfrac{\pi}{4} \leq t \leq \dfrac{\pi}{4} \end{cases}$

(a) Compute the area between the graph of the given curve and the x-axis by evaluating $A = \displaystyle\int_{t_0}^{t_1} y(t)x'(t)\,dt$.

$$\boxed{A = \int_{-\pi/4}^{\pi/4} \cos(2t)\cos t\,dt = \frac{2\sqrt{2}}{3}}$$

(b) After eliminating the parameter to express the curve as an explicitly defined function ($y = f(x)$), calculate the area by evaluating $A = \displaystyle\int_a^b f(x)\,dx$.

$$\boxed{A = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(1 - 2x^2\right)\,dx = \frac{2\sqrt{2}}{3}}$$