Improper Integrals

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 7.8 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Given an improper integral, which either has an infinite interval of integration or an infinite discontinuity, be able to evaluate it using a limit.
- Know how to determine if such an integral converges (and if so, what it converges to) or diverges.

PRACTICE PROBLEMS:

For problems 1-13, evaluate each improper integral or show that it diverges.

1. $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$

   $\boxed{\infty}$

2. $\displaystyle\int_{-\infty}^{3} \frac{3x}{x^2+1}\, dx$

   $\boxed{-\infty}$

3. $\displaystyle\int_1^{\infty} e^{-x}\, dx$

   $\boxed{\dfrac{1}{e}}$

4. $\displaystyle\int_1^{\infty} xe^{-3x^2}\, dx$

   $\boxed{\dfrac{1}{6}e^{-3}}$

5. $\displaystyle\int_0^{4} \frac{1}{x^{2/3}}\, dx$

   $\boxed{3\sqrt[3]{4}}$

6. $\displaystyle\int_4^{\infty} \frac{1}{(x-2)^3}\,dx$

$\boxed{\dfrac{1}{8}}$

7. $\displaystyle\int_2^{6} \frac{1}{\sqrt{x-2}}\,dx$

$\boxed{4}$

8. $\displaystyle\int_0^{2} \frac{2}{\sqrt{4-x^2}}\,dx$

$\boxed{\pi}$

9. $\displaystyle\int_0^{\infty} \frac{1}{x^2+4x+5}\,dx$ (Hint: Complete the square)

$\boxed{\dfrac{\pi}{2} - \tan^{-1}(2)}$

10. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt[3]{\cos x}}\,dx$

$\boxed{\dfrac{3}{2}}$

11. $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\sec^2 x\,dx$

$\boxed{\infty}$

12. $\displaystyle\int_0^{9} \frac{1}{\sqrt[3]{(x-1)^2}}\,dx$

$\boxed{9}$

13. $\displaystyle\int_0^{1} \frac{1}{x\ln x}\,dx$

$\boxed{-\infty}$

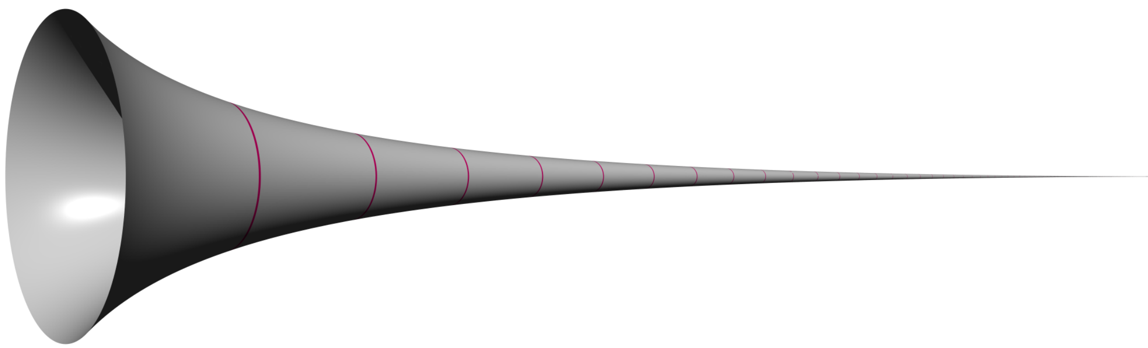
14. Find the value of the constant k so that $\displaystyle\int_{-\infty}^{\infty} \frac{k}{1+x^2}\,dx = 1$.

$\boxed{\dfrac{1}{\pi}}$

15. Compute the exact area between the graph of $y = \dfrac{4}{x^2 - 1}$ and the x-axis for $x \geq 7$.

$\boxed{2 \ln \left(\frac{4}{3} \right)}$

16. Consider Gabriel's Horn (shown below) which is formed by revolving the curve $y = \dfrac{1}{x}$ on $[1, \infty)$ around the x-axis.



Show that the volume within this horn is finite.

Note: It can be shown the surface area of this horn is infinite. Thus, it appears that the horn can be filled with a finite amount of paint; but, there is not enough to paint the inside of the surface for a coating of uniform thickness. This is called **The Paradox of Gabriel's Horn**.

$\boxed{V = \pi \text{ cubic units}}$

17. Determine the values of the constant p for which the following integral will converge and those for which it will diverge.

$$\int_1^\infty \frac{1}{x^p}\, dx$$

(Hint: Consider two cases – when $p = 1$ and when $p \neq 1$.)

$\boxed{\text{Converges to } \frac{1}{p-1} \text{ if } p > 1\text{; Diverges if } p \leq 1\text{; Detailed Solution: Here}}$

18. Determine the values of the constant p for which the following integral will converge and those for which it will diverge.

$$\int_0^1 \frac{1}{x^p}\,dx$$

(Hint: Consider two cases – when $p = 1$ and when $p \neq 1$.)

$\boxed{\text{Converges to } \dfrac{1}{1-p} \text{ if } p < 1\text{; Diverges if } p \geq 1}$

19. Consider the Gamma Function: $\Gamma(\alpha) = \displaystyle\int_0^\infty x^{\alpha-1}e^{-x}\,dx$, which is defined for $\alpha > 0$.

   (a) Compute $\Gamma(1)$.

   $\boxed{1}$

   (b) Compute $\Gamma(2)$.

   $\boxed{1}$

   (c) Compute $\Gamma(3)$.

   $\boxed{2}$

20. If $f(x)$ is a continuous for $x \geq 0$, the **Laplace Transform** of $f(x)$ is given by:

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty f(x)e^{-sx}\,dx$$

and the domain of $\mathcal{L}\{f(x)\}(s)$ is the set consisting of all numbers s for which the integral converges. Laplace Transforms are useful for solving differential equations

   (a) Compute the Laplace Transform of $f(x) = 1$ and state its domain.

   $\boxed{\mathcal{L}\{1\}(s) = \dfrac{1}{s} \text{ for } s > 0}$

   (b) Compute the Laplace Transform of $f(x) = e^x$ and state its domain.

   $\boxed{\mathcal{L}\{e^x\}(s) = \dfrac{1}{1-s} \text{ for } s > 1}$

   (c) Compute the Laplace Transform of $f(x) = x$ and state its domain.

   $\boxed{\mathcal{L}\{x\}(s) = \dfrac{1}{s^2} \text{ for } s > 0}$

21. **Definition:** In probability theory, the **probability density function (pdf)** of a random variable X is a function $f(x)$ from which we can compute the probability that X lies in the interval $[a, b]$ as follows:

$$P(a \le X \le b) = \int_a^b f(x)\, dx$$

And, in order for $f(x)$ to be a valid pdf, it must satisfy the following:

- $f(x) \ge 0$ for all values of x
- $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$

(a) Verify that $f(x) = \begin{cases} 2e^{-2x} & x \ge 0 \\ 0 & x < 0 \end{cases}$ is a valid probability density function.

> Notice that $f(x) > 0$ for all $x \ge 0$ because $e^{-2x} > 0$; and $f(x) = 0$ for $x < 0$. Thus, $f(x) \ge 0$ for all x. Also,
>
> $$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{\infty} 2e^{-2x}\, dx = 1$$
>
> Thus, $f(x)$ is a valid pdf.

(b) Using the density function in part a, compute $P(0 \le X \le 1)$.

$$\boxed{\int_0^1 2e^{-2x}\, dx = 1 - \frac{1}{e^2}}$$

(c) **Definition:** The **cumulative distribution function (CDF)** for a continuous random variable X is defined as:

$$F(t) = P(X \le t) = \int_{-\infty}^{t} f(x)\, dx$$

This CDF describes the accumulation of probability up to the real number t. Compute the CDF for the random variable X which has the density function from part a.

> The CDF is $F(t) = \displaystyle\int_{-\infty}^{t} f(x)\, dx = \int_0^t 2e^{-2x}\, dx = 1 - e^{-2t}$ for $t \ge 0$ and 0 for $t < 0$.