

Partial Fraction Decomposition

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 7.5 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to recognize an improper rational function, and perform the necessary long division to turn it into a proper rational function.
- Know how to write down the partial fraction decomposition for a proper rational function, compute the unknown coefficients in the partial fractions, and integrate each partial fraction.

PRACTICE PROBLEMS:

For problems 1-3, write out the partial fraction decomposition. (Do not solve for the numerical values of the coefficients.)

1. $\frac{2x + 3}{(x - 2)(x - 5)}$

2. $\frac{6}{x^2(x^2 - 9)}$

3. $\frac{5x^4 - 1}{x(x - 2)(x^2 + x + 1)^2}$

For problems 4 & 5, use the given partial fraction decomposition to evaluate the integral.

4. $\int \frac{11x^2 - 28x + 20}{(2x + 1)(x - 3)^2} dx$

Hint: $\frac{11x^2 - 28x + 20}{(2x + 1)(x - 3)^2} = \frac{3}{2x + 1} + \frac{4}{x - 3} + \frac{5}{(x - 3)^2}$

5. $\int \frac{11x^2 + 7x - 15}{(3x - 4)(x^2 + 1)} dx$

Hint: $\frac{11x^2 + 7x - 15}{(3x - 4)(x^2 + 1)} = \frac{5}{3x - 4} + \frac{2x + 5}{x^2 + 1}$

For problems 6-17, evaluate the given integral.

6. $\int \frac{4}{x^2 - 1} dx$

Partial Fraction Decomposition: $\frac{2}{x - 1} - \frac{2}{x + 1}$

7. $\int \frac{4x - 1}{x^2 - 5x + 6} dx$

Partial Fraction Decomposition: $\frac{11}{x - 3} - \frac{7}{x - 2}$
--

8. $\int \frac{x^2}{x^2 + 1} dx$

Partial Fraction Decomposition: $1 - \frac{1}{x^2 + 1}$

9. $\int \frac{3x^2 - 4}{x + 1} dx$

Partial Fraction Decomposition: $3x - 3 - \frac{1}{x + 1}$
--

10. $\int \frac{4x - 1}{2x^2 - 18x + 36} dx$

Partial Fraction Decomposition: $\frac{23}{6} \left(\frac{1}{x - 6} \right) - \frac{11}{6} \left(\frac{1}{x - 3} \right)$

11. $\int \frac{1}{x^2(x - 1)^2} dx$

Partial Fraction Decomposition: $\frac{2}{x} + \frac{1}{x^2} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}$

12. $\int \frac{2x + 3}{(x - 3)(x + 1)^2} dx$

Partial Fraction Decomposition: $\frac{9}{16} \left(\frac{1}{x - 3} \right) - \frac{9}{16} \left(\frac{1}{x + 1} \right) - \frac{1}{4} \left(\frac{1}{(x + 1)^2} \right)$
--

13. $\int \frac{x^4 - 4x^2 + 5}{x^3 - 4x} dx$

Partial Fraction Decomposition: $x - \frac{5}{4} \left(\frac{1}{x} \right) + \frac{5}{8} \left(\frac{1}{x + 2} \right) + \frac{5}{8} \left(\frac{1}{x - 2} \right)$
--

$$14. \int \frac{x^5 - 3x^3 + 6}{x^3 + x} dx$$

Partial Fraction Decomposition: $x^2 - 4 + \frac{6}{x} - \frac{6x - 4}{x^2 + 1}$
--

$$15. \int \frac{x^3 - 6x^2 + 3x - 17}{x^2 + 3} dx$$

Partial Fraction Decomposition: $x - 6 + \frac{1}{x^2 + 3}$

$$16. \int \frac{2x^2}{(x-1)^3} dx$$

Partial Fraction Decomposition: $\frac{2}{x-1} + \frac{4}{(x-1)^2} + \frac{2}{(x-1)^3}$

$$17. \int \frac{4x^2 - 4x + 2}{x^2 - x} dx$$

Partial Fraction Decomposition: $4 - \frac{2}{x} + \frac{2}{x-1}$

For problems 18-19, evaluate the given integral by making substitution that transforms the problem into integrating a rational function.

$$18. \int \frac{\sin x}{\cos^2 x + 6 \cos x + 5} dx$$

$$19. \int \frac{e^{5x}}{e^{4x} - 1} dx$$

20. By the end of this problem, you will know the antiderivatives of $\sec x$. Observe the following:

$$\begin{aligned} \int \sec x \, dx &= \int \frac{1}{\cos x} \, dx \\ &= \int \frac{\cos x}{\cos^2 x} \, dx \\ &= \int \frac{\cos x}{1 - \sin^2 x} \, dx \end{aligned}$$

(a) Use the substitution $u = \sin x$ to convert the given integral to an integral of a rational function.

- (b) Use partial fractions to evaluate your integral from part (a). Show that the antiderivatives have the form $\frac{1}{2} \ln \left| \frac{u+1}{u-1} \right| + C$

Notice that substituting back in for u yields:

$$\begin{aligned} \int \sec x \, dx &= \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{(\sin x + 1)(\sin x + 1)}{(\sin x - 1)(\sin x + 1)} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{(\sin x + 1)^2}{\sin^2 x - 1} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{(\sin x + 1)^2}{\cos^2 x} \right| + C \\ &= \ln \left| \frac{\sin x + 1}{\cos x} \right| + C \\ &= \ln |\tan x + \sec x| + C \end{aligned}$$