Partial Derivatives

SUGGESTED REFERENCE MATERIAL:

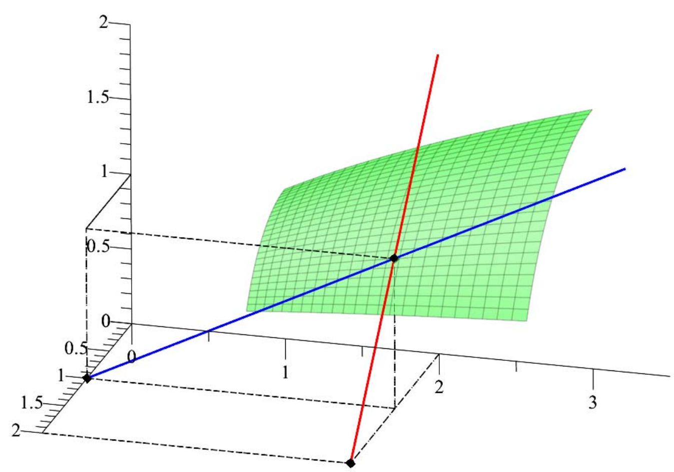
As you work through the problems listed below, you should reference Chapter 13.3 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to compute first-order and second-order partial derivatives.
- Be able to perform implicit partial differentiation.
- Be able to solve various word problems involving rates of change, which use partial derivatives.

PRACTICE PROBLEMS:

1. A portion of the surface defined by $z = f(x, y)$ is shown below.



Use the tangent lines in this figure to calculate the values of the first order partial derivatives of f at the point $(1, 2)$.

$\boxed{f_x(1, 2) = -1;\ f_y(1, 2) = \frac{1}{2}}$

For problems 2-9, find all first order partial derivatives.

2. $f(x, y) = (3x - y)^5$

$$\boxed{f_x(x, y) = 15(3x - y)^4;\ f_y(x, y) = -5(3x - y)^4}$$

3. $f(x, y) = e^x \sin y$

$$\boxed{f_x(x, y) = e^x \sin y;\ f_y(x, y) = e^x \cos y}$$

4. $f(x, y) = \tan^{-1}(4x - 7y)$

$$\boxed{f_x(x, y) = \frac{4}{1 + (4x - 7y)^2};\ f_y(x, y) = -\frac{7}{1 + (4x - 7y)^2}}$$

5. $f(x, y) = x \cos(x^2 + y^2)$

$$\boxed{f_x(x, y) = \cos(x^2 + y^2) - 2x^2 \sin(x^2 + y^2);\ f_y(x, y) = -2xy \sin(x^2 + y^2)}$$

6. Let $f(x, y, z) = \sqrt{x^2 - 2y + 3z^2}$. Compute $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, and $\dfrac{\partial f}{\partial z}$.

$$\boxed{\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 - 2y + 3z^2}};\ \frac{\partial f}{\partial y} = \frac{-1}{\sqrt{x^2 - 2y + 3z^2}};\ \frac{\partial f}{\partial z} = \frac{3z}{\sqrt{x^2 - 2y + 3z^2}}}$$

7. Let $w = \dfrac{4z}{x^2 + y^2}$. Compute $\dfrac{\partial w}{\partial x}$, $\dfrac{\partial w}{\partial y}$, and $\dfrac{\partial w}{\partial z}$.

$$\boxed{\frac{\partial w}{\partial x} = -\frac{8xz}{(x^2 + y^2)^2};\ \frac{\partial w}{\partial y} = -\frac{8yz}{(x^2 + y^2)^2};\ \frac{\partial w}{\partial z} = \frac{4}{x^2 + y^2}}$$

8. Consider $f(x, y, z) = \dfrac{xy}{x^2 + z^2}$. Determine $\dfrac{\partial f}{\partial x}(-1, 1, 2)$, $\dfrac{\partial f}{\partial y}(-1, 1, 2)$, and $\dfrac{\partial f}{\partial z}(-1, 1, 2)$.

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_{(x,y,z)=(-1,1,2)} = \frac{3}{25};\ \left.\frac{\partial f}{\partial y}\right|_{(x,y,z)=(-1,1,2)} = -\frac{1}{5};\ \left.\frac{\partial f}{\partial z}\right|_{(x,y,z)=(-1,1,2)} = \frac{4}{25}}$$

9. Suppose $f(x, y, z) = z^2 \sin(2xy)$. Compute $f_x\left(4, \frac{\pi}{3}, 1\right)$, $f_y\left(4, \frac{\pi}{3}, 1\right)$, and $f_z\left(4, \frac{\pi}{3}, 1\right)$.

$$\boxed{f_x\left(4, \frac{\pi}{3}, 1\right) = -\frac{\pi}{3},\ f_y\left(4, \frac{\pi}{3}, 2\right) = -4,\ f_z\left(4, \frac{\pi}{3}, 2\right) = \sqrt{3}}$$

For problems 10-11, find all values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.

10. $f(x, y) = 4x^2 + y^2 - 8xy + 4x + 6y - 10$

$$\boxed{(x, y) = \left(\frac{7}{6}, \frac{5}{3}\right)}$$

11. $f(x,y) = x^2 + 4y^2 - 3xy + 3$

$\boxed{(x,y) = (0,0)}$

For problems 12-13, compute all second partial derivatives.

12. $z = x^2y - y^3x^4$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = 2y - 12x^2y^3;\ \frac{\partial^2 z}{\partial y \partial x} = 2x - 12x^3y^2;\ \frac{\partial^2 z}{\partial x \partial y} = 2x - 12x^3y^2;\ \frac{\partial^2 z}{\partial y^2} = -6x^4y}$$

13. $f(x,y) = \ln(x^2 + 3y)$

$$\boxed{\begin{array}{l} f_{xx}(x,y) = \dfrac{-2x^2+6y}{(x^2+3y)^2};\ f_{xy}(x,y) = -\dfrac{6x}{(x^2+3y)^2}; \\ f_{yx}(x,y) = -\dfrac{6x}{(x^2+3y)^2};\ f_{yy}(x,y) = -\dfrac{9}{(x^2+3y)^2} \end{array}}$$

14. Consider the surface $S : z = x^2 + 3y^2$.

    (a) Find the slope of the tangent line to the curve of intersection of the surface S and the plane $y = 1$ at the point $(1, 1, 4)$.

    2; Detailed Solution: Here

    (b) Find a set of parametric equations for the tangent line whose slope you computed in part (a).

    There are many possible parameterizations. One possibility is $x = 1 + t$, $y = 1$, $z = 4 + 2t$. Detailed Solution: Here

    (c) Find the slope of the tangent line to the curve of intersection of the surface S and the plane $x = 1$ at the point $(1, 1, 4)$.

    6; Detailed Solution: Here

    (d) Find a set of parametric equations for the tangent line whose slope you computed in part (b).

    There are many possible parameterizations. One possibility is $x = 1$, $y = 1 + t$, $z = 4 + 6t$. Detailed Solution: Here

    (e) Find an equation of the tangent plane to the surface S at the point $(1, 1, 4)$. (Hint: The tangent plane contains both of tangent lines from parts (b) and (d).)

    $-2(x-1) - 6(y-1) + 1(z-4) = 0$; Detailed Solution: Here

15. Consider a closed rectangular box.

    (a) Find the instantaneous rate of change of the volume with respect to the width, w, if the length, l, and height, h, are held constant at the instant when $l = 3$, $w = 7$, and $h = 6$.

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(b) Find the instantaneous rate of change of the surface area with respect to the height, h, if the length, l, and width, w, are held constant at the instant when $l = 3$, $w = 7$, and $h = 6$.

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16. Use implicit partial differentiation to compute the slope of the surface $x^2 + 4y^2 - 36z^2 = -19$ in the x-direction at the points $(1, 2, 1)$ and $(1, 2, -1)$.

$$\boxed{\left.\frac{\partial z}{\partial x}\right|_{(x,y,z)=(1,2,1)} = \frac{1}{36};\ \left.\frac{\partial z}{\partial x}\right|_{(x,y,z)=(1,2,-1)} = -\frac{1}{36}}$$

17. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x\cos(y^2 + z^2) = 3yz$.

$\boxed{\dfrac{\partial z}{\partial x} = \dfrac{\cos(y^2+z^2)}{3y + 2zx\sin(y^2+z^2)};\ \dfrac{\partial z}{\partial y} = \dfrac{-3z - 2xy\sin(y^2+z^2)}{3y + 2xz\sin(y^2+z^2)}\text{; Detailed Solution: Here}}$

18. **Laplace's Equation**, shown below, is a second order partial differential equation. In the study of heat conduction, the Laplace Equation is the steady state heat equation.

Laplace's Equation:

$$\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = 0$$

A function which satisfies Laplace's Equation is said to be **harmonic**.

(a) Verify that $f(x, y) = e^x \cos y$ is a harmonic function.

> You can verify by direct computation that $f_{xx}(x, y) = e^x \cos y$ and $f_{yy}(x, y) = -e^x \cos y$. Then, $f_{xx}(x, y) + f_{yy}(x, y) = e^x \cos y + (-e^x \cos y) = 0$. Thus, since $f(x, y)$ satisfies Laplace's Equation, it is a harmonic function.

(b) Suppose $u(x, y)$ and $v(x, y)$ are functions which have continuous mixed partial derivatives. Also, assume that $u(x, y)$ and $v(x, y)$ satisfy the **Cauchy Riemann Equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

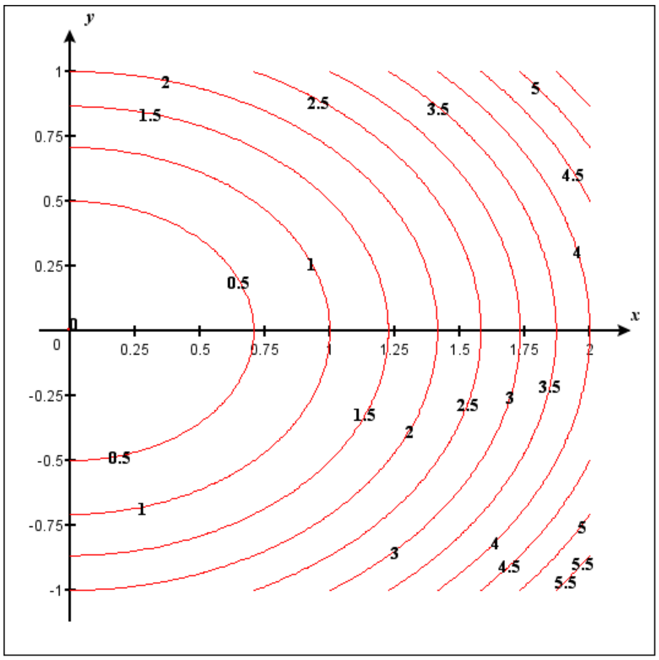
Verify that $u(x, y)$ and $v(x, y)$ are both harmonic functions.

We begin by showing that $u(x,y)$ is a harmonic function. To do so, we differentiate the first of the Riemann Equations with respect to x which yields $\dfrac{\partial^2 u}{\partial x \partial x} = \dfrac{\partial^2 v}{\partial x \partial y}$. And, we differentiate the second of the Cauch-Riemann Equations with respect to y which yields $\dfrac{\partial^2 u}{\partial y \partial y} = -\dfrac{\partial^2 v}{\partial y \partial x}$. Then,

$$\begin{aligned} \frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} &= \frac{\partial^2 v}{\partial x \partial y} + \left(-\frac{\partial^2 v}{\partial y \partial x}\right) \\ &= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} \\ &\text{by symmetry of mixed partial derivatives} \\ &= 0 \end{aligned}$$

So, since $u(x,y)$ satisfies Laplace's Equation, it is a harmonic function. A similar argument holds for $v(x,y)$.

19. The figure below shows some level curves of a function $z = f(x,y)$.



Use this to give an approximation for $\dfrac{\partial f}{\partial x}(1,0)$.

The slope is approximately 2. Note: You should use the level curve which passes through $(1,0)$ as well as one which is close to $(1,0)$ to estimate the slope.