Planes

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 11.6 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

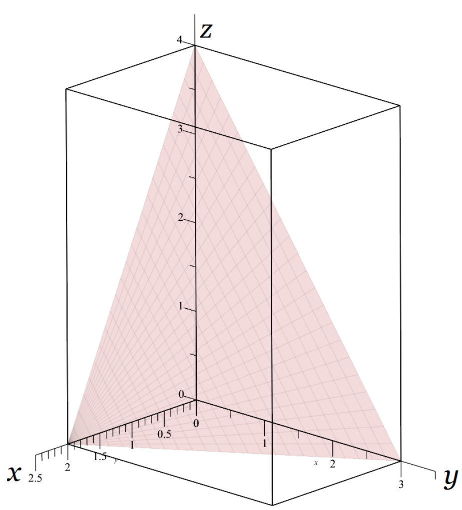
EXPECTED SKILLS:

- Be able to find the equation of a plane that satisfies certain conditions by finding a point on the plane and a vector normal to the plane.
- Know how to find the parametric equations of the line of intersection of two (non-parallel) planes.
- Be able to find the (acute) angle of intersection between two planes.

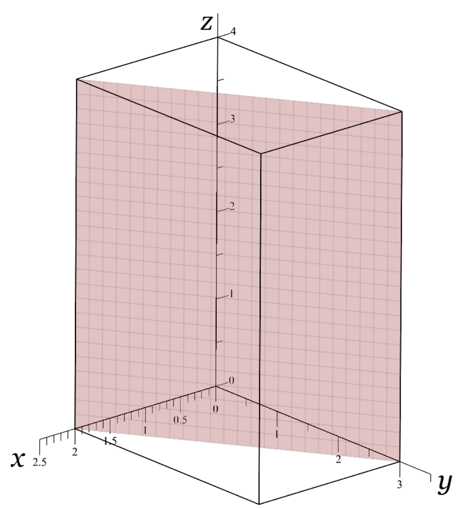
PRACTICE PROBLEMS:

1. For each of the following, find an equation of the plane indicated in the figure.

(a)



(b)



$\boxed{\text{(a) } 6x + 4y + 3z = 12;\ \text{(b) } 3x + 2y = 6}$

For problems 2-6, determine whether the following are parallel, perpendicular, or neither.

2. Plane $P_1 : 5x - 3y + 4z = -1$ and plane $P_2 : 2x - 2y - 4z = 9$

$\boxed{\text{The planes are perpendicular.}}$

3. Plane $P_1 : 3x - 2y + z = -3$ and plane $P_2 : 5x + y - 6z = 8$

   $\boxed{\text{The planes are neither parallel nor perpendicular; Detailed Solution: Here}}$

4. Plane $P_1 : 3x - 2y + z = -3$ and plane $P_2 : -6x + 4y - 2z = 1$

   $\boxed{\text{The planes are parallel.}}$

5. Plane $P : 5x - 3y + 4z = -1$ and line $\overrightarrow{\ell}(t) = \langle 2 + 2t, 3 - 2t, 5 - 4t \rangle$

   $\boxed{\text{The plane and the line are parallel.}}$

6. Plane $P : 5x - 3y + 4z = -1$ and line $\overrightarrow{\ell}(t) = \left\langle 2 + \frac{5}{2}t, 3 - \frac{3}{2}t, 5 + 2t \right\rangle$

   $\boxed{\text{The plane and the line are perpendicular.}}$

7. Give an example of a plane, P, and a line, L, which are neither parallel nor perpendicular to each other.

   Suppose your line has the form $\overrightarrow{\ell}(t) = \overrightarrow{\ell_0} + t\overrightarrow{v}$ and that your plane has $\overrightarrow{n}$ as a normal vector. Then all possible answers are those for which $\overrightarrow{v} \not\parallel \overrightarrow{n}$ (i.e., $\overrightarrow{v} \neq c\overrightarrow{n}$ for any scalar c) and $\overrightarrow{v} \not\perp \overrightarrow{n}$ (i.e., $\overrightarrow{v} \cdot \overrightarrow{n} \neq 0$). The first condition ensures that L and P are not perpendicular; the second condition ensures that L and P are not parallel.

For problems 8-13, find an equation of the plane which satisfies the given conditions.

8. The plane which passes through the point $P(1, 2, 3)$ and which has a normal vector of $\mathbf{n} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$.

   $\boxed{4(x - 1) - 2(y - 2) + 6(z - 3) = 0}$

9. The plane which passes through $P(-2, 0, 1)$ and is perpendicular to the line $\overrightarrow{\ell}(t) = \langle 1, 2, 3 \rangle + t\langle 3, -2, 2 \rangle$.

   $\boxed{3(x + 2) - 2y + 2(z - 1) = 0}$

10. The plane which passes through points $A(1, 2, 3)$, $B(2, -1, 5)$ and $C(-1, 3, 3)$.

    $\boxed{-2(x - 1) - 4(y - 2) - 5(z - 3) = 0}$

11. The plane which passes through $A(1, 2, 3)$ and is parallel to the plane $3x - 5y + z = 2$.

    $\boxed{3(x - 1) - 5(y - 2) + 1(z - 3) = 0}$

12. The plane which passes through $A(-2, 1, 5)$ and is perpendicular to the line of intersection of $P_1 : 3x + 2y - z = 5$ and $P_2 : -y + z = 7$.

    $\boxed{1(x + 2) - 3(y - 1) - 3(z - 5) = 0;\ \text{Detailed Solution: Here}}$

13. The plane which contains the point $A(-2,-1,3)$ and which contains the line $L: x = 1+t, y = 3-2t, z = 4t$.

$\boxed{2(x+2) - 3(y+1) - 2(z-3) = 0}$

14. Consider the planes $P_1 : x + y + z = 7$ and $P_2 : 2x + 4z = 6$.

(a) Compute an equation of the line of intersection of P_1 and P_2.

$\boxed{\text{One parametric equation of the line of intersection is } L: x = 3-2t, y = 4+t, z = t}$

(b) Compute an equation of the plane which passes through the point $A(1,2,3)$ and contains the line of intersection of P_1 and P_2.

$\boxed{5(x-1) + 4(y-2) + 6(z-3) = 0}$

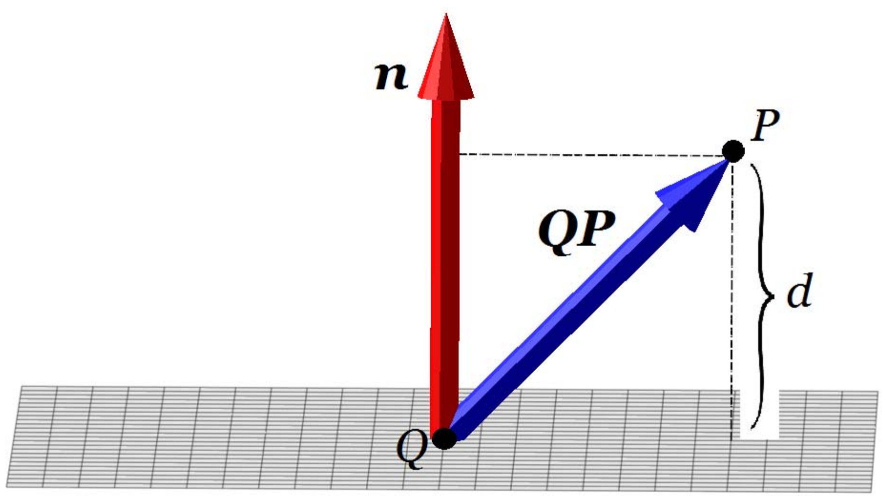
15. Find the acute angle of intersection of $P_1 : 3x - 2y + 5z = 0$ and $P_2 : -x - y + 2z = 3$.

$$\boxed{\cos^{-1}\left(\frac{9}{\sqrt{38}\sqrt{6}}\right)}$$

16. Find the acute angle of intersection of $P_1 : 3x - 2y - 5z = 0$ and $P_2 : -x - y + 2z = 3$.

$$\boxed{\pi - \cos^{-1}\left(\frac{-11}{\sqrt{38}\sqrt{6}}\right); \text{ Detailed Solution: Here}}$$

17. Consider the plane which passes through the point Q and whose normal vectors are parallel to $\mathbf{n}$. And, let P be another point in space, as illustrated below.



(a) Show that the distance between the point P and the given plane is $d = \dfrac{|\mathbf{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$.

$$\boxed{d = \|\text{Proj}_{\mathbf{n}}\mathbf{QP}\| = \left\| \left(\frac{\mathbf{QP} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n} \right\| = \frac{|\mathbf{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|^2}\|\mathbf{n}\| = \frac{|\mathbf{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|}}$$

(b) Use this method to compute the distance between the point $P(2, -1, 4)$ and the plane $x + 2y + 3z = 5$.

$$\boxed{d = \frac{7}{\sqrt{14}}}$$

18. Consider planes $P_1 : 2x - 4y + 5z = -2$ and $P_2 : x - 2y + \frac{5}{2}z = 5$.

(a) Verify that P_1 and P_2 are parallel.

$\mathbf{n_1} = \langle 2, -4, 5 \rangle$ is normal to plane P_1.
$\mathbf{n_2} = \left\langle 1, -2, \frac{5}{2} \right\rangle$ is normal to plane P_2.
Since $\mathbf{n_1} = 2\mathbf{n_2}$, we have that $\mathbf{n_1}$ and $\mathbf{n_2}$ are parallel. And, because these normal vectors are parallel, the planes P_1 and P_2 are parallel, too.

(b) Compute the distance between P_1 and P_2. (Hint: See the previous problem.)

$d = \frac{12}{\sqrt{45}}$; Detailed Solution: Here