Relative and Absolute Extrema

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 13.8 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to use partial derivatives to find critical points (possible locations of maxima or minima).
- Know how to use the Second Partials Test for functions of two variables to determine whether a critical point is a relative maximum, relative minimum, or a saddle point.
- Be able to solve word problems involving maxima and minima.
- Know how to compute absolute maxima and minima on closed regions.

PRACTICE PROBLEMS:

For problems 1-10, identify all critical points of the given function. Then, classify each as the location of a relative maximum, relative minimum, or saddle point.

1. $g(x, y) = x^2 + y^2 - 3x - 4y + 6$

   $\boxed{\text{Relative minimum at } \left(\frac{3}{2}, 2\right)}$

2. $f(x, y) = x^2 + 4y^2 - 4y - 2$

   $\boxed{\text{Relative minimum at } \left(0, \frac{1}{2}\right)}$

3. $g(x, y) = 4x^2 - 3y^2 + 8x - 9y - 4$

   $\boxed{\text{Saddle point at } \left(-1, -\frac{3}{2}\right)}$

4. $f(x, y) = x^3 - 3x + y^2 - 6y$

   $\boxed{\text{Relative minimum at } (1, 3)\text{; Saddle point at } (-1, 3)}$

5. $h(x, y) = x^2 - 5xy + y^2$

   $\boxed{\text{Saddle point at } (0, 0)}$

6. $f(x,y) = 3x + y^2 - e^x$

   $\boxed{\text{Saddle point at } (\ln 3, 0)}$

7. $f(x,y) = x^6 + y^6$

   $\boxed{\text{Relative minimum at } (0,0)}$

8. $f(x,y) = x^2y - 6y^2 - 3x^2$

   $\boxed{\text{Relative maximum at } (0,0)\text{; Saddle points at } (6,3) \text{ and } (-6,3)\text{; Detailed Solution: Here}}$

9. $f(x,y) = x^3 - 3xy + \frac{1}{2}y^2$

   $\boxed{\text{Relative minimum at } (3,9)\text{; Saddle point at } (0,0)}$

10. $f(x,y) = \frac{1}{3}x^3 - 2x + x^2 + 2xy + y^2$

    $\boxed{\text{Relative minimum at } \left(\sqrt{2}, -\sqrt{2}\right)\text{; Saddle point at } \left(-\sqrt{2}, \sqrt{2}\right)}$

11. Consider $h(x,y) = 3\sqrt{x^2 + y^2} + 6$

    (a) Explain why the Second Partials Test may not be used to locate the relative extrema/saddle points of $h(x,y)$.

    $(0,0)$ is the only critical point of $h(x,y)$; but, the Second Partials Test does not apply because $h(x,y)$ does not have continuous second partial derivatives in any disk centered at this critical point.

    (b) Locate all relative maxima, relative minima, and saddle points, if any.

    $\boxed{\text{Relative minimum at } (0,0)}$

For problems 12-15, find the absolute extrema of the given function on the specified region R.

12. $f(x,y) = 5 - 4y - 2x$; R is the closed triangular region in the xy-plane with vertices $(3,0)$, $(0,1)$, and $(1,2)$.

    $\boxed{\text{Absolute minimum of } -5 \text{ at } (1,2)\text{; Absolute maximum of 1 at } (0,1)}$

13. $f(x,y) = x^2 - 4xy + 5y^2 - 8y$; R is the closed triangular region with vertices $(0,0)$, $(3,0)$, and $(3,3)$.

    $\boxed{\text{Absolute minimum of } -11 \text{ at } (3,2)\text{; Absolute maximum of 9 at } (3,0)}$

14. $g(x, y) = x^2 - y^2 - 2x$; R: is the closed region in the xy-plane bounded by the graphs $y = x^2$ and $y = 4$.

Absolute minimum of -17 at $(1, 4)$; Absolute maximum of 2 at $(-1, 1)$;
Detailed Solution: Here

15. $f(x, y) = x^2 + xy + y^2$; R is the closed square region defined by $-1 \le x \le 1$, $-1 \le y \le 1$.

Absolute minimum of 0 at $(0, 0)$; Absolute maximum of 3 at $(1, 1)$ and $(-1, -1)$

16. A closed rectangular box with a volume of 16 ft^3 is made from two kinds of materials. The top and bottom are made of material costing \$0.10 per square foot and the sides are made of material costing \$0.05 per square foot. Find the dimensions of the box so the cost is minimized.

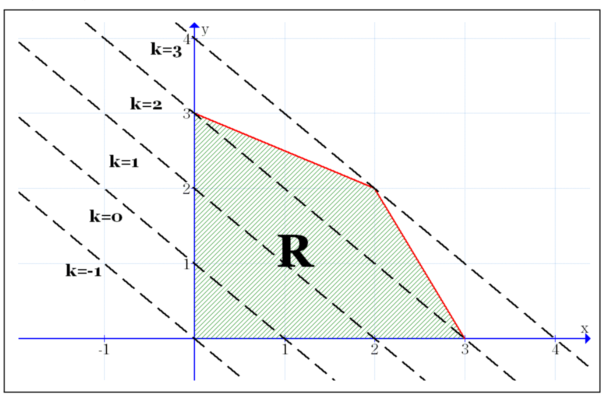
2ft $\times$ 2ft $\times$ 4ft; Detailed Solution: Here

17. Determine the dimensions of a rectangular box, open at the top, which has a volume of 32ft^3 and requires the least amount of material for construction.

4ft $\times$ 4ft $\times$ 2ft

18. Consider the region R which satisfies all of the following constraints: $x \ge 0$, $y \ge 0$, $x + 2y \le 6$, $2x + y \le 6$.

(a) On the same set of axes, sketch R. Also sketch the level curves $f(x, y) = k$ of $f(x, y) = x + y - 1$ for $k = -1, 0, 1, 2, 3$.



(b) At which point will $f(x, y)$ achieve an absolute maximum value? And, what is this maximum value?

f achieves its maximum at $(2, 2)$. The maximum value is 3.

(c) At which point will $f(x, y)$ achieve an absolute minimum value? And, what is this minimum value?

f achieves its minimum at $(0, 0)$. The maximum value is -1.