Dot Product & Projections

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 11.3 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Know how to compute the dot product of two vectors.
- Be able to use the dot product to find the angle between two vectors; and, in particular, be able to determine if two vectors are orthogonal.
- Know how to compute the direction cosines of a vector.
- Be able to decompose vectors into orthogonal components. And, know how to compute the orthogonal projection of one vector onto another.

PRACTICE PROBLEMS:

1. For each of the following, compute $\overrightarrow{u} \cdot \overrightarrow{v}$ based on the given information.

   (a) $\overrightarrow{u} = \langle 3, -1 \rangle$; $\overrightarrow{v} = \langle 2, -5 \rangle$

   (b) $\overrightarrow{u} = \langle 4, -5, 1 \rangle$; $\overrightarrow{v} = \langle 3, 6, -1 \rangle$

   (c) $\overrightarrow{u} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$; $\overrightarrow{v} = 9\mathbf{i} - 2\mathbf{j}$;

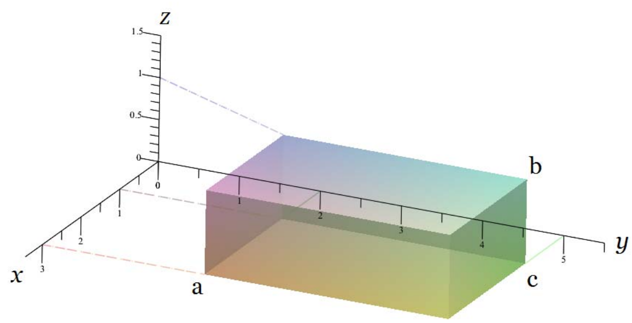
   (d) $\|\overrightarrow{u}\| = 3$; $\|\overrightarrow{v}\| = 4$; the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$ is $\dfrac{\pi}{4}$

2. Explain why the operation $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ does not make sense.

3. Determine whether the angle between $\overrightarrow{v} = \langle 1, 2, 3 \rangle$ and $\overrightarrow{w} = \langle -6, 4, -1 \rangle$ is acute, obtuse, or right. Explain.

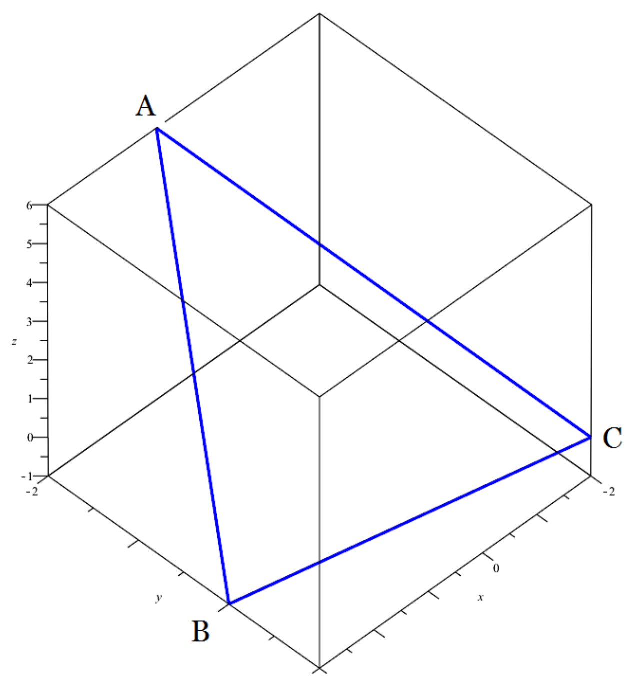
4. Give an example of a vector which is orthogonal to both $\overrightarrow{v} = \langle 1, 1, 1 \rangle$ and $\overrightarrow{w} = \langle 2, 0, 4 \rangle$. (Hint: Set up a system of equations.)

5. Consider the triangle with vertices a, b, and c.



Use vectors to compute the angle between the diagonal which extends from vertex a to vertex b and the line segment which extends from vertex a to vertex c. (Verify your answer with HW 11.1 #3c.)

6. Consider the triangle, shown below,with vertices $A(1, -2, 6)$, $B(3, 0, -1)$, and $C(-2, 1, 0)$.



Compute all three angles of the triangle.

7. Let $\overrightarrow{v} = \langle 1, 2 \rangle$ and $\overrightarrow{b} = \langle -3, 4 \rangle$.

   (a) Find the vector component of $\overrightarrow{v}$ along $\overrightarrow{b}$ and the vector component of $\overrightarrow{v}$ orthogonal to $\overrightarrow{b}$.

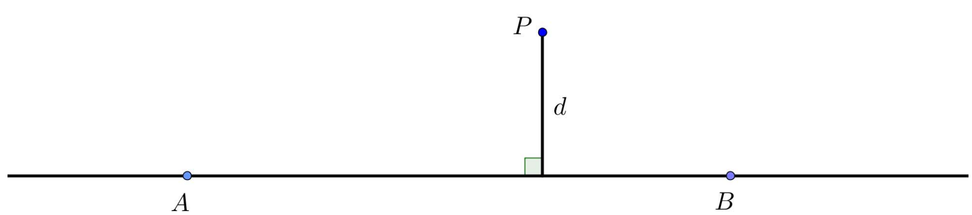
   (b) Sketch $\overrightarrow{v}$, $\overrightarrow{b}$, and the vector components that you found in part (a).

8. Express $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ as the sum of a vector parallel to $\mathbf{b} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and a vector perpendicular to $\mathbf{b}$

9. Suppose that $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. Under what condition will $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$? Explain.

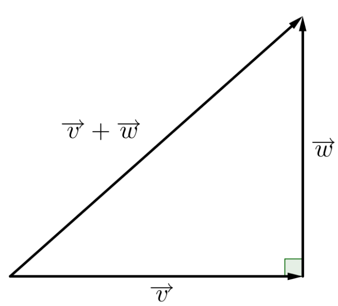
10. The following questions deal with finding the distance from a point to a line:

    (a) Given three points A, B, and P in 2-space or 3-space as shown in the picture below, describe two different ways that you could use the dot product to calculate the distance, d, between the point P and the line which contains A and B.



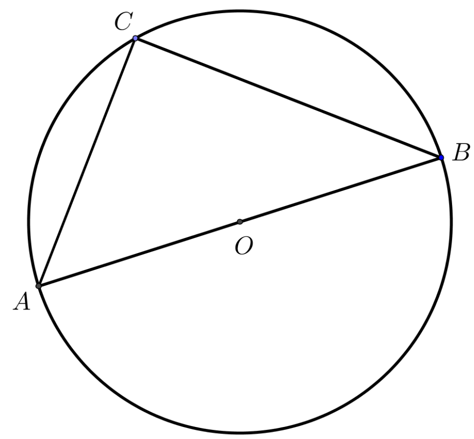
    (b) Use one of your methods from part (a) to compute the distance from the point $P(5, 3, 0)$ to the line containing $A(1, 0, 1)$ and $B(2, 3, 1)$.

11. Consider the triangle shown below which is formed by vectors $\mathbf{v}$ and $\mathbf{w}$.



    Prove Pythagorean's Theorem $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$. (Hint: Use properties of the dot product to expand $\|\mathbf{v} + \mathbf{w}\|^2$.)

12. Let A and B be endpoints of a diameter of a circle with a radius of r . And, suppose that C is any other point on the circle, as shown below.



Prove that triangle ABC is a right triangle. (Hint: Express each of $\overrightarrow{CA}$ and $\overrightarrow{CB}$ as the combination of $\overrightarrow{CO}$ and some other vector.)

13. Let $\overrightarrow{v}$ and $\overrightarrow{w}$ be vectors, either both in $\mathbb{R}^2$ or in $\mathbb{R}^3$. Prove the Cauchy-Schwarz Inequality: $|\overrightarrow{v} \cdot \overrightarrow{w}| \leq \|\overrightarrow{v}\|\|\overrightarrow{w}\|$.

14. Let $\overrightarrow{v} = \langle 1, 2, 3 \rangle$.

    (a) Compute the direction cosines of $\overrightarrow{v}$.

    (b) Verify that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, where α, β, and γ be the angles between $\overrightarrow{v}$ and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively.