Chapter 3.3 Practice Problems

EXPECTED SKILLS:

- Know how to compute the derivatives of exponential functions.
- Be able to compute the derivatives of the inverse trigonometric functions, specifically, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ and $\sec^{-1} x$.
- Know how to apply logarithmic differentiation to compute the derivatives of functions of the form $(f(x))^{g(x)}$, where f and g are non-constant functions of x.

PRACTICE PROBLEMS:

For problems 1-16, differentiate. In some cases it may be better to use logarithmic differentiation.

1. $y = e^{6x}$

   $\boxed{6e^{6x}}$

2. $g(x) = xe^{2x}$

   $\boxed{e^{2x} + 2xe^{2x}}$

3. $f(x) = 5^{x^2}$

   $\boxed{2x\ln(5)5^{x^2}}$

4. $y = e^x \cos x$

   $\boxed{-e^x \sin x + e^x \cos x}$

5. $g(x) = e^{x^2(x-1)}$

   $\boxed{e^{x^2(x-1)}(3x^2 - 2x)}$

6. $f(x) = \dfrac{1 - e^{2x}}{1 - e^x}$

   $\boxed{e^x}$

7. $f(x) = \dfrac{\ln x}{e^x + 3x}$

   $$\boxed{\frac{e^x + 3x - x\ln(x)e^x - 3x\ln(x)}{x(e^x + 3x)^2}}$$

8. $f(x) = \ln(e^x + 5)$

$\boxed{\dfrac{e^x}{e^x + 5}}$

9. $y = x^{x^2}$

$\boxed{x^{x^2}(x + 2x \ln x)}$

10. $f(x) = e^{\cos^2 2x + \sin^2 2x}$

$\boxed{0}$

11. $h(x) = \exp\left(\dfrac{1}{1 - \ln x}\right)$

$\boxed{\dfrac{1}{x(1 - \ln x)^2} \exp\left(\dfrac{1}{1 - \ln x}\right)}$

12. $f(x) = (\ln x)^{e^x}$

$\boxed{(\ln x)^{e^x}\left(\dfrac{e^x}{x \ln x} + e^x \ln(\ln x)\right)}$

13. $y = \cos^{-1}(3x)$

$\boxed{-\dfrac{3}{\sqrt{1 - 9x^2}}}$

14. $y = \arcsin(x^2)$

$\boxed{\dfrac{2x}{\sqrt{1 - x^4}}}$

15. $y = \dfrac{\arctan(e^x)}{x^3}$

$\boxed{\dfrac{xe^x - 3\tan^{-1}(e^x) - 3e^{2x}\tan^{-1}(e^x)}{x^4(1 + e^{2x})}}$

16. $y = x^{\cos x}$

$\boxed{x^{\cos x}\left(\dfrac{\cos x}{x} - \sin x \ln x\right)}$

17. Compute an equation of the line which is tangent to the graph of $y = e^{3x}$ at the point where $x = \ln 2$.

$\boxed{y - 8 = 24(x - \ln 2)}$

18. Compute an equation of the line which is tangent to the graph of $f(x) = \cos^{-1} x$ at the point where $x = \dfrac{1}{2}$.

$$\boxed{y = -\frac{2}{\sqrt{3}}x + \frac{\pi + \sqrt{3}}{3}}$$

19. Find all value(s) of x at which the tangent lines to the graph of $f(x) = \tan^{-1}(4x)$ are perpendicular to the line which passes through $(0, 1)$ and $(2, 0)$.

$$\boxed{x = \pm\frac{1}{4}}$$

20. Find a linear function $T_1(x) = mx + b$ which satisfies both of the following conditions:

- $T_1(x)$ has the same y-intercept as $f(x) = e^{2x}$.
- $T_1(x)$ has the same slope as $f(x) = e^{2x}$ at the y-intercept.

$$\boxed{y = 2x + 1}$$

21. Compute an equation of the line which is tangent to the curve $e^{xy^2} + y = x^4$ at $(-1, 0)$.

$$\boxed{y = -4x - 4}$$

22. The equation $y'' + 5y' - 6y = 0$ is called a differential equation because it involves an unknown function y and its derivatives. Find the value(s) of the constant A for which $y = e^{Ax}$ satisfies this equation.

$$\boxed{A = -6 \text{ and } A = 1}$$

23. Evaluate $\displaystyle\lim_{h \to 0} \frac{\sin^{-1}\left(\frac{\sqrt{3}}{2} + h\right) - \frac{\pi}{3}}{h}$ by interpreting the limit as the derivative of a function a particular point.

$$\boxed{\lim_{h \to 0} \frac{\sin^{-1}\left(\frac{\sqrt{3}}{2} + h\right) - \frac{\pi}{3}}{h} = \frac{d}{dx}(\sin^{-1}(x)\Big|_{x=\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{1 - x^2}}\Bigg|_{x=\frac{\sqrt{3}}{2}} = 2}$$

24. **Multiple Choice:** Which of the following is the equation of the tangent line to the graph of $f(x) = \tan^{-1}(2x)$ at the point where $x = 0$?

(a) $y = x$

(b) $y = x + 1$

(c) $y = x - 1$

(d) $y = 2x$

(e) $y = 2x - 1$

$\boxed{\text{D}}$

25. **Multiple Choice:** Consider the curve defined implicitly by $\sin x = e^y$ for $0 < x < \pi$. What is $\dfrac{dy}{dx}$ in terms of x?

(a) $-\tan x$
(b) $-\cot x$
(c) $\cot x$
(d) $\tan x$
(e) $\csc x$

$\boxed{\text{C}}$

26. Consider the following two hyperbolic functions:

Hyperbolic Sine	Hyperbolic Cosine
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\cosh x = \dfrac{e^x + e^{-x}}{2}$

(a) Compute $\lim\limits_{x\to\infty} \sinh x$

$\boxed{+\infty}$

(b) Compute $\lim\limits_{x\to-\infty} \sinh x$

$\boxed{-\infty}$

(c) Compute $\lim\limits_{x\to\infty} \cosh x$

$\boxed{+\infty}$

(d) Compute $\lim\limits_{x\to-\infty} \cosh x$

$\boxed{+\infty}$

(e) Compute the x and y intercepts, if any, for $y = \sinh x$.

$\boxed{\text{The } x \text{ and } y \text{ intercept of } y = \sinh x \text{ is } (0,0).}$

(f) Compute the x and y intercepts, if any, for $y = \cosh x$.

$\boxed{y = \cosh x \text{ has a } y\text{-intercept of } (0,1)\text{; but, it does not have any } x \text{ intercepts.}}$

(g) Solve $\sinh x = 1$ for x.

$\boxed{x = \ln\left(1 + \sqrt{2}\right)}$

(h) Show that $\cosh^2 x - \sinh^2 x = 1$

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= (\cosh x + \sinh x)(\cosh x - \sinh x) \\
&= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right) \\
&= (e^x)(e^{-x}) \\
&= 1
\end{aligned}$$

(i) Show that $\dfrac{d}{dx}(\sinh x) = \cosh x$

$$\begin{aligned}
\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right) \\
&= \frac{1}{2}e^x + \frac{1}{2}e^{-x} \\
&= \frac{e^x + e^{-x}}{2} \\
&= \cosh x
\end{aligned}$$

(j) Show that $\dfrac{d}{dx}(\cosh x) = \sinh x$

$$\begin{aligned}
\frac{d}{dx}(\cosh x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right) \\
&= \frac{1}{2}e^x - \frac{1}{2}e^{-x} \\
&= \frac{e^x - e^{-x}}{2} \\
&= \sinh x
\end{aligned}$$