Chapter 3.9: Inverse Trigonometric Functions

Expected Skills:

- Be able to specify the domain and range of $\sin^{-1}(x)$, $\cos^{-1}(x)$, and $\tan^{-1}(x)$. Also be able to graph these functions.
- Be able to evaluate an inverse trigonometric function at a ratio which is related to the common angles of $0° - 30° - 45° - 60° - 90°$.
- Be able to evaluate limits involving inverse trigonometric functions.
- Be able to differentiate $\sin^{-1}(x)$, $\cos^{-1}(x)$, and $\tan^{-1}(x)$. Also be able to use the derivative to solve application problems.

Practice Problems:

1. For each of the following functions, state the domain and the range.

   (a) $f(x) = \sin^{-1} x$

   $\boxed{\text{Domain: } [-1, 1], \text{ Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$

   (b) $f(x) = \cos^{-1} x$

   $\boxed{\text{Domain: } [-1, 1], \text{ Range: } [0, \pi]}$

   (c) $f(x) = \tan^{-1} x$

   $\boxed{\text{Domain: } (-\infty, \infty), \text{ Range: } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

2. Evaluate each of the following. (Do not use a calculator. And remember the ranges from problem 1.)

   (a) $\arcsin \frac{\sqrt{3}}{2}$

   $\boxed{\frac{\pi}{3}}$

   (b) $\arcsin \left(-\frac{\sqrt{3}}{2}\right)$

   $\boxed{-\frac{\pi}{3}}$

(c) $\arccos \frac{\sqrt{3}}{2}$

$\boxed{\frac{\pi}{6}}$

(d) $\arccos \left(-\frac{\sqrt{3}}{2}\right)$

$\boxed{\frac{5\pi}{6}}$

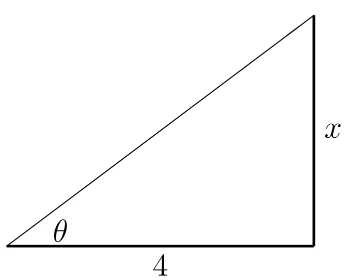
(e) $\arctan \frac{\sqrt{3}}{3}$

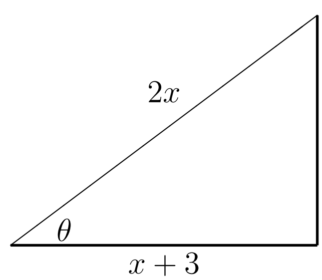
$\boxed{\frac{\pi}{6}}$

(f) $\arctan \left(-\frac{\sqrt{3}}{3}\right)$

$\boxed{-\frac{\pi}{6}}$

3. Use an inverse trigonometric function to express θ as a function of x:

(a) 

(b) 

$\boxed{\text{(a) } \theta = \tan^{-1}\left(\frac{x}{4}\right), \text{ (b) } \theta = \cos^{-1}\left(\frac{x+3}{2x}\right)}$

4. Find the exact value of each expression.

(a) $\sin\left(\tan^{-1}\left(\frac{3}{4}\right)\right)$

$\boxed{\frac{3}{5}}$

(b) $\sec\left(\arctan\left(-\frac{3}{5}\right)\right)$

$\boxed{\frac{\sqrt{34}}{5}}$

(c) $\sin\left(\arccos\left(-\frac{2}{3}\right)\right)$

$\boxed{\frac{\sqrt{5}}{3}}$

(d) $\csc\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

$\boxed{2}$

5. Find the exact value of each expression. Remember the ranges from problem (1)!

(a) $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

$\boxed{\frac{\pi}{3}}$

(b) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

$\boxed{\frac{\pi}{3}}$

(c) $\cos^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right)$

$\boxed{\frac{\pi}{4}}$

(d) $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$

$\boxed{\frac{\pi}{4}}$

(e) $\tan^{-1}\left(\tan\left(\frac{\pi}{6}\right)\right)$

$\boxed{\frac{\pi}{6}}$

(f) $\tan^{-1}\left(\tan\left(\frac{5\pi}{6}\right)\right)$

$\boxed{-\frac{\pi}{6}}$

6. For each of the following, find all solutions in the interval $[0, 2\pi]$. Give the exact values, not decimal approximations.

   (a) $(\sin x - 1)(4\sin x - 3) = 0$

   $\boxed{\frac{\pi}{2},\ \arcsin\left(\frac{3}{4}\right),\ \pi - \arcsin\left(\frac{3}{4}\right)}$

   (b) $3\tan x = 1$

   $\boxed{\tan^{-1}\left(\frac{1}{3}\right),\ \pi + \tan^{-1}\left(\frac{1}{3}\right)}$

   (c) $5\cos^2 x + 11\cos x + 2 = 0$

   Notice that solving this equation reduces to solving $\cos x = -\frac{1}{5}$. So, there are solutions in both quadrants II and III. The reference angle is $\arccos\left(\frac{1}{5}\right)$. Thus, the two solutions of the given equation are $\pi - \arccos\left(\frac{1}{5}\right)$ and $\pi + \arccos\left(\frac{1}{5}\right)$.

   Alternatively, one could find the angle in the second quadrant by calculating $\arccos\left(-\frac{1}{5}\right)$. Then, the angle in the third quadrant is $2\pi - \arccos\left(-\frac{1}{5}\right)$

   (d) $3\tan x = -1$

   $\boxed{\pi + \tan^{-1}\left(-\frac{1}{3}\right),\ 2\pi + \tan^{-1}\left(-\frac{1}{3}\right)}$

7. Evaluate the following limits. If a limit does not exist, write $+\infty$, $-\infty$, or DNE.

   (a) $\lim_{x\to\infty} \arccos\left(\frac{-x^2}{x^2+3x}\right)$

   $\boxed{\pi}$

   (b) $\lim_{x\to 0} \arctan\left(\frac{1}{x^2}\right)$

   $\boxed{\frac{\pi}{2}}$

   (c) $\lim_{h\to 0} \frac{\sin^{-1}\left(\frac{\sqrt{3}}{2}+h\right) - \frac{\pi}{3}}{h}$

   (**Hint:** Interpreting the limit as the derivative of a function a particular point.)

   $$\boxed{\lim_{h\to 0} \frac{\sin^{-1}\left(\frac{\sqrt{3}}{2}+h\right) - \frac{\pi}{3}}{h} = \frac{d}{dx}(\sin^{-1}(x))\Big|_{x=\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{1-x^2}}\Big|_{x=\frac{\sqrt{3}}{2}} = 2}$$

8. Calculate $\dfrac{dy}{dx}$

   (a) $y = \left(\tan^{-1} x\right)^3$

   $$\boxed{\frac{3\left(\tan^{-1} x\right)^2}{1+x^2}}$$

   (b) $y = 3x^2 \sin^{-1}(4x)$

   $$\boxed{\frac{12x^2}{\sqrt{1-16x^2}} + 6x\sin^{-1}(4x)}$$

9. Compute an equation of the line which is tangent to the graph of $f(x) = \cos^{-1} x$ at the point where $x = \dfrac{1}{2}$.

   $$\boxed{y = -\frac{2}{\sqrt{3}}x + \frac{\pi + \sqrt{3}}{3}}$$

10. Find all value(s) of x at which the tangent lines to the graph of $f(x) = \tan^{-1}(4x)$ are perpendicular to the line which passes through $(0,1)$ and $(2,0)$.

   $$\boxed{x = \pm\frac{1}{4}}$$

11. Let $f(x) = \arctan x^2$.

   (a) Find all intervals on which $f(x)$ is increasing and those on which $f(x)$ is decreasing.

   $\boxed{\text{Decreasing on } (-\infty, 0)\text{; Increasing on } (0, \infty)}$

   (b) Locate all local extrema. Express each as an ordered pair (x, y).

   $\boxed{\text{Local minimum at } (0,0)\text{; No local maximum}}$

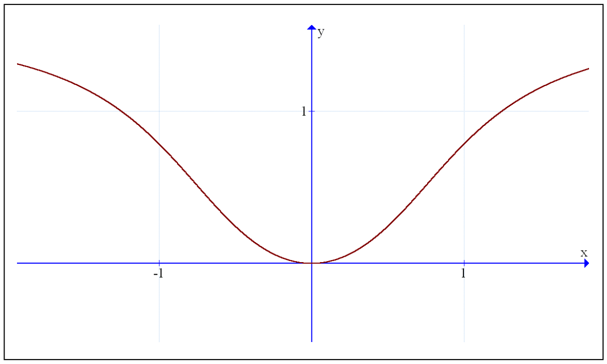
   (c) Find all intervals on which $f(x)$ is concave up and those on which $f(x)$ is concave down.

   $$\boxed{\text{Concave down on } \left(-\infty, -\frac{1}{\sqrt[4]{3}}\right) \cup \left(\frac{1}{\sqrt[4]{3}}, \infty\right)\text{; Concave up on} \left(-\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)}$$

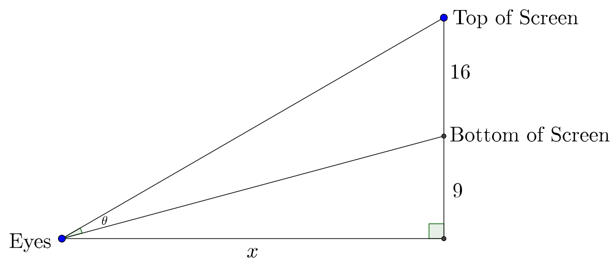
   (d) Locate all points of inflection. Express each as an ordered pair (x, y).

   $$\boxed{\text{Points of inflection } \left(-\frac{1}{\sqrt[4]{3}}, \frac{\pi}{6}\right) \text{ and } \left(\frac{1}{\sqrt[4]{3}}, \frac{\pi}{6}\right)}$$

(e) Sketch $f(x)$.

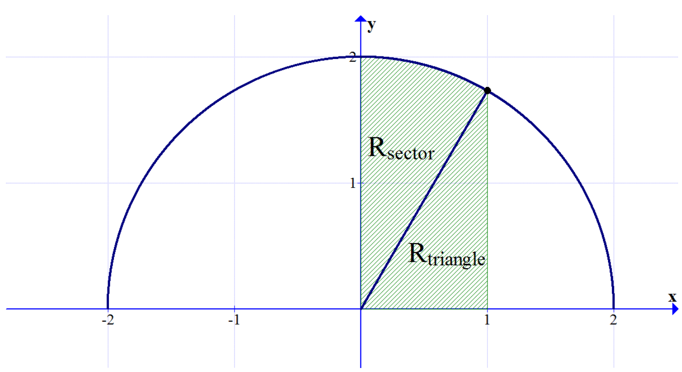


12. The screen at the front of a movie theater is 16 feet high and positioned 9 feet above eye level. How far away from the front of the room should you sit in order to have the "best" view ? (HINT: Find the largest possible angle θ in diagram shown below.)



$\boxed{\text{15 Feet}}$

13. Find the area of the shaded region by adding together the area of the sector and the area of the triangle.



$\boxed{\begin{array}{l} Area of R_{\text{Triangle}} = \frac{1}{2}bh = \frac{1}{2}(1)\left(\sqrt{3}\right) = \frac{\sqrt{3}}{2}; \text{ Area of } R_{\text{Sector}} = \frac{1}{2}r^2\theta = \frac{1}{2}(2)^2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}. \\ \text{Thus, the total area is } A = \frac{\sqrt{3}}{2} + \frac{\pi}{3}. \end{array}}$