Implicit Differentiation

As you work through the problems listed below, you should reference Chapter 3.1 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to solve for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ using implicit differentiation, i.e., without first solving for y.

PRACTICE PROBLEMS:

For problems 1 & 2, solve each equation for y to express y as an explicit function of x. Then find $\frac{dy}{dx}$.

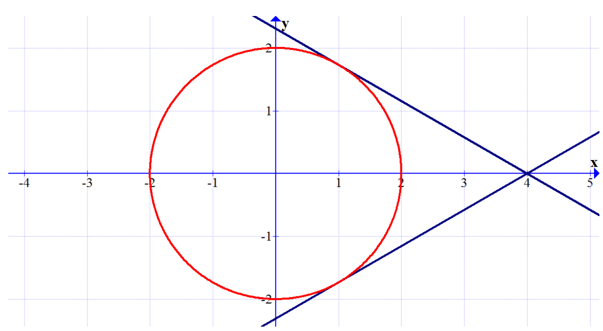
1. $yx + 2x = 6$

   $\boxed{y = \frac{6-2x}{x} \text{ for } x \neq 0; \ \frac{dy}{dx} = -6x^{-2}}$

2. $3x + 12xy + 4y = 0$

   $\boxed{y = -\frac{3x}{12x+4} \text{ for } x \neq -\frac{1}{3}; \ \frac{dy}{dx} = \frac{-3}{4(3x+1)^2}}$

3. Consider the circle $x^2 + y^2 = 4$, shown below.



   (a) By first expressing the circle as two separate explicit functions of x, compute the slope of the tangent line to the circle at each point where $x = 1$.

   $\boxed{\left.\frac{dy}{dx}\right|_{(x,y)=\left(1,\sqrt{3}\right)} = -\frac{1}{\sqrt{3}} \text{ and } \left.\frac{dy}{dx}\right|_{(x,y)=\left(1,-\sqrt{3}\right)} = \frac{1}{\sqrt{3}}}$

(b) By using implicit differentiation, compute the slope of the tangent line to the circle at each point where $x = 1$.

$$\boxed{\left.\frac{dy}{dx}\right|_{(x,y)=\left(1,\sqrt{3}\right)} = -\frac{1}{\sqrt{3}} \text{ and } \left.\frac{dy}{dx}\right|_{(x,y)=\left(1,-\sqrt{3}\right)} = \frac{1}{\sqrt{3}}}$$

(c) Find the point of intersection of the lines which are tangent to the circle when $x = 1$.

$(4, 0)$; Video Solution: http://www.youtube.com/watch?v=I_07fHrtkMw

For problems 4-8, use implicit differentiation to find $\frac{dy}{dx}$.

4. $x^2y = 9$

$$\boxed{\frac{dy}{dx} = \frac{-2y}{x}}$$

5. $xy^2 + y^3 = 6$

$\frac{dy}{dx} = \frac{-y}{2x+3y}$; Video Solution: http://www.youtube.com/watch?v=UGWa6cYZyLY

6. $\frac{1-y^2}{1-2x} = x$

$$\boxed{\frac{dy}{dx} = \frac{4x-1}{2y}}$$

7. $y\cos x + y^2x = 3x$

$$\boxed{\frac{dy}{dx} = \frac{3-y^2+y\sin x}{2xy+\cos x}}$$

8. $x^2 + y^3 = 10$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{3y^2}}$$

For problem 9-10, compute $\frac{d^2y}{dx^2}$ in terms of x and y

9. $2x^2 - 3y^2 = 4$

$\frac{d^2y}{dx^2} = -\frac{8}{9y^3}$; Video Solution: http://www.youtube.com/watch?v=P7EvTVQ07yw

10. $y + \sin y = x$

$\boxed{\dfrac{d^2y}{dx^2} = \dfrac{\sin y}{(1 + \cos y)^3}}$

For problems 11-12, find the equation of the line tangent to the curve at the given point.

11. $x^2 + y^2 = 10$ at $(1, 3)$

$\boxed{y = \dfrac{-x}{3} + \dfrac{10}{3}}$

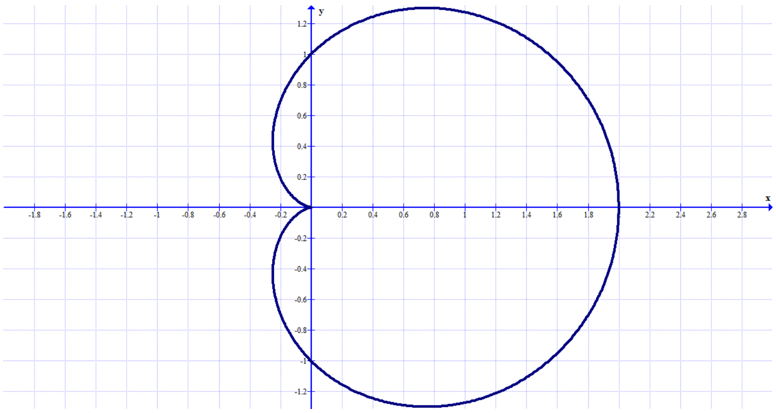
12. $\dfrac{1 - xy}{1 - 5x} = 2x$ at $(1, 9)$

$\boxed{y = 9x}$

13. Consider the ellipse given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a and b are positive real numbers. Use implicit differentiation to compute the slope of the line which is tangent to the curve at (x_0, y_0).

$\boxed{\left.\dfrac{dy}{dx}\right|_{(x,y)=(x_0,y_0)} = -\dfrac{b^2 x_0}{a^2 y_0}}$

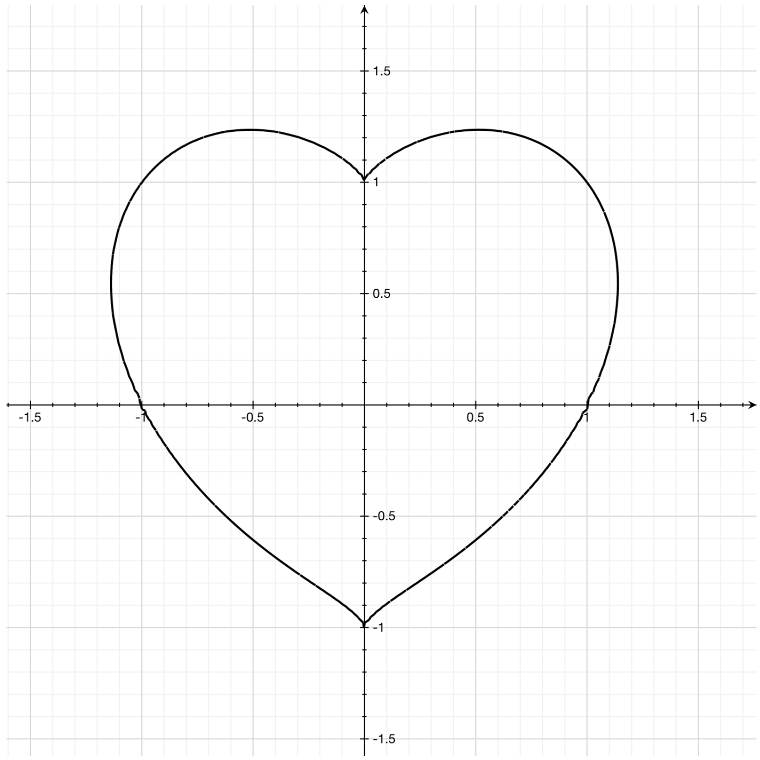
14. The set of ordered pairs (x, y) which satisfy the equation $(x^2 + y^2 - x)^2 = x^2 + y^2$ form the curve shown below, called a <u>cardioid</u>.



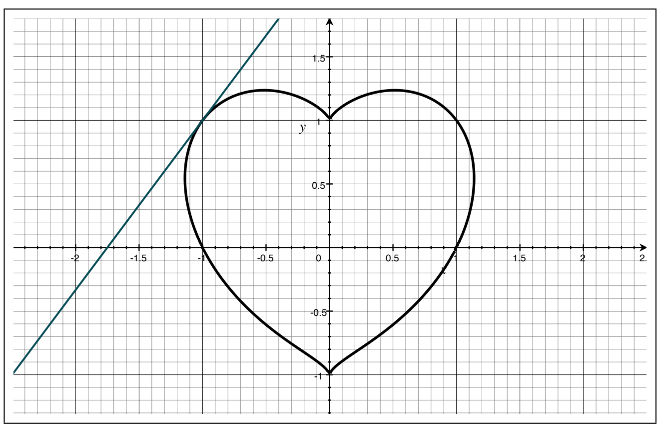
Let L_1 be the line which is tangent to the curve at the point $(0, 1)$ and let L_2 be the line which is tangent to the curve at the point $(0, -1)$. At which point in the xy-plane do L_1 and L_2 intersect?

$\boxed{(-1, 0)}$

15. The curve below is the graph of $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$.



(a) Sketch the tangent line to to graph at the point $(-1, 1)$.



(b) Find an equation of line which is tangent to the graph at the point $(-1, 1)$.

Pro-tip: Plug in $(-1, 1)$ after applying $\frac{d}{dx}$ to both sides of the equation but before solving for $\frac{dy}{dx}$.

$\boxed{y = \frac{4}{3}x + \frac{7}{3}}$