Chapter 4.1: Natural Logarithm & The Number e

Expected Skills:

- Be able to specify the domain and range of $f(x) = e^x$ and $f(x) = \ln x$.
- Be able to graph $f(x) = e^x$ and $f(x) = \ln x$, labeling all intersections with the coordinate axes and all asymptotes.
- Be able to solve equations involving the natural logarithm or exponential function.
- Be able to evaluate limits involving the exponential function of the natural log function.
- Be able to differentiate the exponential function or the natural log function; also, be able to solve application problems such as tangent line, rates of change, local/absolute extrema, and curve sketching.
- Be able to perform logarithmic differentiation.

Practice Problems:

Algebraic Questions

1. Approximate each of the following quantities using 3 rectangles of equal width and left endpoints, as described in the lecture notes. Determine whether your approximation is an over approximation or an under approximation.

   (a) $\ln 4$

   $\boxed{\text{1.833333333; over approximation}}$

   (b) $\ln 6$

   $\boxed{\text{2.676282051; over approximation}}$

2. Approximate each of the following quantities using 3 rectangles of equal width and right endpoints, as described in the lecture notes. Determine whether your approximation is an over approximation or an under approximation.

   (a) $\ln 4$

   $\boxed{\text{1.083333333; under approximation}}$

   (b) $\ln 6$

   $\boxed{\text{1.287393162; under approximation}}$

3. Evaluate each of the following without using a calculator.

   (a) $\ln 1$

   $\boxed{0}$

   (b) $\ln e$

   $\boxed{1}$

   (c) $\ln(e^2)$

   $\boxed{2}$

   (d) $\ln \sqrt[3]{e}$

   $\boxed{\frac{1}{3}}$

   (e) $e^{\ln 7}$

   $\boxed{7}$

   (f) e^0

   $\boxed{1}$

4. Use the properties of logarithms to expand (as much as possible) the expression as a sum, difference, and/or constant multiple of logarithms. (Assume that all variables are positive.)

   (a) $\ln\left(5x^2\sqrt{y}\right)$

   $\boxed{\ln 5 + 2\ln x + \frac{1}{2}\ln y}$

   (b) $\ln \dfrac{x^3}{y^2 z^4}$

   $\boxed{3\ln x - 2\ln y - 4\ln z}$

   (c) $\ln \sqrt[4]{x^3(x^2+3)}$

   $\boxed{\frac{3}{4}\ln x + \frac{1}{4}\ln\left(x^2+3\right)}$

5. Use the properties of logarithms to condense the expression to the logarithm of a single quantity.

   (a) $\ln 2 + \ln x$

   $\boxed{\ln(2x)}$

   (b) $3\ln x + 4\ln y - 4\ln z$

   $\boxed{\ln\left(\dfrac{x^3y^4}{z^4}\right)}$

6. Solve the given equation for x. Where appropriate, you may leave your answers in logarithmic form.

   (a) $e^x + 5 = 60$

   $\boxed{x = \ln 55}$

   (b) $11e^x + 5 = 60$

   $\boxed{x = \ln 5}$

   (c) $\left(3^{x-5}\right) - 4 = 11$

   $\boxed{x = \dfrac{\ln 15}{\ln 3} + 5}$

   (d) $\ln x - \ln(x+1) = 2$

   $\boxed{\text{No Solution}}$

   (e) $\dfrac{1 + \ln x}{2} = 0$

   $\boxed{x = e^{-1}}$

7. The equation $Q(t) = 30e^{-4t}$ gives the mass (in grams) of a radioactive element that will **remain** from some initial quantity after t hours of radioactive decay.

   (a) How many grams were there initially?

   $\boxed{30 \text{ grams}}$

   (b) How long will it take for 40% of the element to **decay**? You may leave your answer in logarithmic form.

   $\boxed{-\frac{1}{4}\ln\frac{3}{5} \text{ hours}}$

8. In a research experiment the population of a certain species is given by $P(t) = 15(7^t)$, where t is the number of weeks since the beginning of the experiment.

   (a) How large was the population at the beginning of the experiment?

   $\boxed{15}$

   (b) How long will it take for the population to reach 300? You may leave your answer in logarithmic form.

   $\boxed{\dfrac{\ln 20}{\ln 7} \text{ weeks}}$

Limit & Continuity Questions

For problems 9-18, evaluate the following limits by first making an appropriate substitution. If the limit does not exist, write DNE, $+\infty$, or $-\infty$ (whichever is most appropriate).

9. $\lim\limits_{x\to\infty} e^x$

   $\boxed{+\infty}$

10. $\lim\limits_{x\to-\infty} e^x$

    $\boxed{0}$

11. $\lim\limits_{x\to-\infty} \left(\dfrac{1}{e^x}\right)$

    $\boxed{+\infty}$

12. $\lim\limits_{x\to\infty} e^{1/x}$

    $\boxed{1}$

13. $\lim\limits_{x\to\infty} \left(\dfrac{7}{e^x - 8}\right)$

    $\boxed{0}$

14. $\lim\limits_{x\to-\infty} \left(\dfrac{7}{e^x - 8}\right)$

    $\boxed{-\dfrac{7}{8}}$

15. $\lim\limits_{x\to 0^+} \ln x$

    $\boxed{-\infty}$

16. $\lim\limits_{x\to\infty} \ln x$

    $\boxed{+\infty}$

17. $\lim\limits_{x\to\infty} \left(\dfrac{\ln 6x}{\ln 2x}\right)$

    $\boxed{1}$

18. $\lim\limits_{x\to\infty} [\ln(x+2) - \ln(3x+5)]$

    $\boxed{\ln\left(\dfrac{1}{3}\right)}$

For problems 19-22, evaluate the following limits by first making an appropriate substitution. If the limit does not exist, write DNE, $+\infty$, or $-\infty$ (whichever is most appropriate).

19. $\lim\limits_{x\to\infty} \left(e^x \sin\left(e^{-x}\right)\right)$

    $\boxed{1}$

20. $\lim\limits_{x\to1} \left(\dfrac{\sin\left(\ln x^5\right)}{\ln x}\right)$

    $\boxed{5}$

21. $\lim\limits_{x\to\frac{\pi}{2}^+} e^{\sec x}$

    $\boxed{0}$

22. $\lim\limits_{x\to0^+} \tan^{-1}\left(\ln x\right)$

    $\boxed{-\dfrac{\pi}{2}}$

Derivative of the Natural Logarithmic Function

For problems 23-35, calculate $\dfrac{dy}{dx}$.

23. $y = \ln\left(x^2\right)$

    $\boxed{\dfrac{2}{x}}$

24. $y = \dfrac{1}{\ln(3x)}$

    $\boxed{-\dfrac{1}{x[\ln(3x)]^2}}$

25. $y = x^2 \ln x$

    $\boxed{x + 2x\ln x}$

26. $y = \ln\left(\dfrac{1}{x}\right)$

    $\boxed{-\dfrac{1}{x}}$

27. $y = \ln(x^2+1)^2$

$\boxed{\dfrac{4x}{x^2+1}}$

28. $y = \left[\ln(x^2+1)\right]^2$

$\boxed{\dfrac{4x\ln(x^2+1)}{x^2+1}}$

29. $y = \sqrt{\ln 2x}$

$\boxed{\dfrac{1}{2x\sqrt{\ln(2x)}}}$

30. $y = \tan(\ln x)$

$\boxed{\dfrac{1}{x}\sec^2(\ln x)}$

31. $y = \ln(\ln x)$

$\boxed{\dfrac{1}{x\ln x}}$

32. $y = \ln|\sec x|$

$\boxed{\tan x}$

33. $y = \ln|\sec x + \tan x|$

$\boxed{\sec x}$

34. $y = \ln(x^x)$

$\boxed{1+\ln(x)}$

35. $y = \ln\left(\dfrac{2x+1}{\sqrt{x}(3x-4)^{10}}\right)$

$\boxed{\dfrac{2}{2x+1} - \dfrac{1}{2x} - \dfrac{30}{3x-4}}$

36. Use logarithmic differentiation to calculate $\dfrac{dy}{dx}$ if $y = \dfrac{2x+1}{\sqrt{x}(3x-4)^{10}}$

$\boxed{\dfrac{2x+1}{\sqrt{x}(3x-4)^{10}}\left(\dfrac{2}{2x+1} - \dfrac{1}{2x} - \dfrac{30}{3x-4}\right)}$

37. Let $y = x^{x^2}$. Use logarithmic differentiation to calculate $\dfrac{dy}{dx}$.

$\boxed{x^{x^2}(x + 2x\ln x)}$

38. Let $y = x^{\cos x}$. Use logarithmic differentiation to calculate $\dfrac{dy}{dx}$.

$\boxed{x^{\cos x}\left(\dfrac{\cos x}{x} - \sin x \ln x\right)}$

39. Compute an equation of the line which is tangent to the graph of $f(x) = \ln(x^2 - 3)$ at the point where $x = 2$.

$\boxed{y = 4x - 8}$

40. Find the value(s) of x at which the tangent line to the graph of $y = \ln(x^2 + 11)$ is perpendicular to $y = -6x + 5$.

$\boxed{x = 1 \text{ and } x = 11}$

41. Find the value(s) of x at which the tangent line to the graph of $y = -\ln x$ passes through the origin.

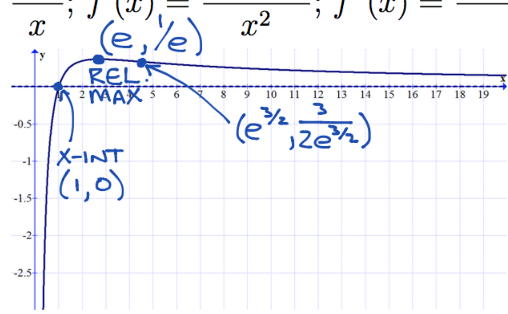
$\boxed{x = e}$

42. Calculate $\dfrac{d^2y}{dx^2}$ if $y = \ln(3x^2 + 2)$.

$\boxed{\dfrac{12 - 18x^2}{(3x^2+2)^2}}$

43. Sketch $f(x) = \dfrac{\ln x}{x}$. Label the coordinates of all critical points, inflection points, x-intercepts, y-intercepts, and holes. Also label all horizontal asymptotes and vertical asymptotes.

$$f(x) = \frac{\ln x}{x};\ f'(x) = \frac{1 - \ln x}{x^2};\ f''(x) = \frac{-3 + 2\ln x}{x^3}$$



44. **Multiple Choice:** Let $y = \ln(\cos x)$. Which of the following is $\dfrac{dy}{dx}$?

(a) $(\ln x)(-\sin x) + (\cos x)(\ln x)$
(b) $-\tan x$
(c) $\cot x$
(d) $\sec x$
(e) $\dfrac{1}{\ln(\cos x)}$

$\boxed{\text{B}}$

45. **Multiple Choice:** Let $h(x) = \ln[(f(x))^2 + 1]$. Suppose that $f(1) = -1$ and $f'(1) = 1$. Find $h'(1)$.

(a) -2
(b) -1
(c) 0
(d) 1
(e) 2

$\boxed{\text{B}}$

46. Consider the triangle formed by the tangent line to the graph of $y = -\ln x$ at the point $P(t, -\ln t)$, the horizontal line which passes through P, and the y-axis. Find a function $A(t)$ which gives the area of this triangle.

$\boxed{A(t) = \frac{1}{2}t}$

Derivative of the Exponential Function

For problems 47-57, differentiate.

47. $y = e^{6x}$

$\boxed{6e^{6x}}$

48. $g(x) = xe^{2x}$

$\boxed{e^{2x} + 2xe^{2x}}$

49. $y = e^{x}\cos x$

$\boxed{-e^{x}\sin x + e^{x}\cos x}$

50. $g(x) = e^{x^2(x-1)}$

$\boxed{e^{x^2(x-1)}(3x^2 - 2x)}$

51. $f(x) = \dfrac{1 - e^{2x}}{1 - e^x}$

$\boxed{e^x}$

52. $f(x) = \dfrac{\ln x}{e^x + 3x}$

$\boxed{\dfrac{e^x + 3x - x\ln(x)e^x - 3x\ln(x)}{x(e^x + 3x)^2}}$

53. $f(x) = \ln(e^x + 5)$

$\boxed{\dfrac{e^x}{e^x + 5}}$

54. $f(x) = e^{\cos^2 2x + \sin^2 2x}$

$\boxed{0}$

55. $h(x) = \exp\left(\dfrac{1}{1 - \ln x}\right)$

$\boxed{\dfrac{1}{x(1 - \ln x)^2}\exp\left(\dfrac{1}{1 - \ln x}\right)}$

56. $f(x) = (\ln x)^{e^x}$

$\boxed{(\ln x)^{e^x}\left(\dfrac{e^x}{x\ln x} + e^x\ln(\ln x)\right)}$

57. $y = \dfrac{\arctan(e^x)}{x^3}$

$\boxed{\dfrac{xe^x - 3\tan^{-1}(e^x) - 3e^{2x}\tan^{-1}(e^x)}{x^4(1 + e^{2x})}}$

58. Compute an equation of the line which is tangent to the graph of $y = e^{3x}$ at the point where $x = \ln 2$.

$\boxed{y - 8 = 24(x - \ln 2)}$

59. Compute an equation of the line which is tangent to the curve $e^{xy^2} + y = x^4$ at $(-1, 0)$.

$\boxed{y = -4x - 4}$

60. Find a linear function $T_1(x) = mx + b$ which satisfies both of the following conditions:

- $T_1(x)$ has the same y-intercept as $f(x) = e^{2x}$.
- $T_1(x)$ has the same slope as $f(x) = e^{2x}$ at the y-intercept.

$\boxed{y = 2x + 1}$

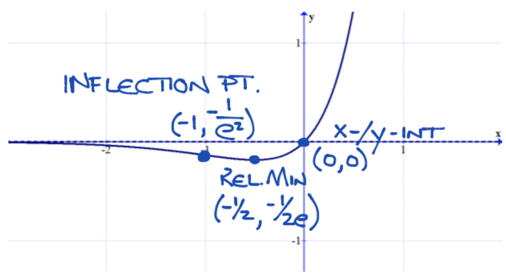
61. The equation $y'' + 5y' - 6y = 0$ is called a differential equation because it involves an unknown function y and its derivatives. Find the value(s) of the constant A for which $y = e^{Ax}$ satisfies this equation.

$\boxed{A = -6 \text{ and } A = 1}$

62. Sketch the given functions. Label the coordinates of all critical points, inflection points, x-intercepts, y-intercepts, and holes. Also label all horizontal asymptotes and vertical asymptotes.

(a) $f(x) = xe^{2x}$

$$f(x) = xe^{2x};\ f'(x) = e^{2x}(2x+1);\ f''(x) = 4e^{2x}(x+1)$$



(b) $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2};\ f'(x) = -\frac{x}{\sqrt{2\pi}} e^{-x^2/2};\ f''(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}(x^2 - 1)$$