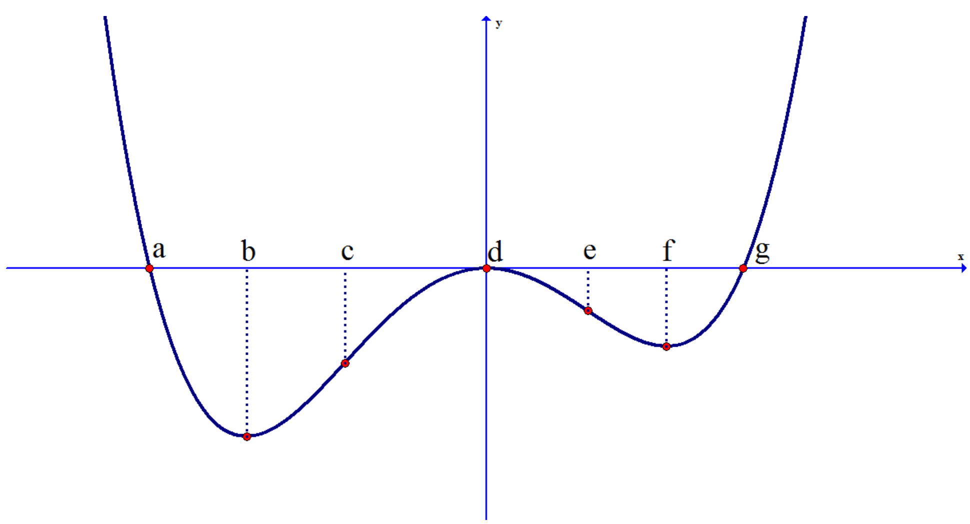
Chapter 4.1 & 4.2 (Part 1) Practice Problems

EXPECTED SKILLS:

- Understand how the signs of the first and second derivatives of a function are related to the behavior of the function.
- Know how to use the first and second derivatives of a function to find intervals on which the function is increasing, decreasing, concave up, and concave down.
- Be able to find the critical points of a function, and apply the First Derivative Test and Second Derivative Test (when appropriate) to determine if the critical points are relative maxima, relative minima, or neither
- Know how to find the locations of inflection points.

PRACTICE PROBLEMS:

1. Consider the graph of $y = f(x)$, shown below.



(a) Determine the interval(s) where $f(x)$ is increasing.

$\boxed{(b, d) \cup (f, \infty)}$

(b) Determine the interval(s) where $f(x)$ is decreasing.

$\boxed{(-\infty, b) \cup (d, f)}$

(c) Determine the interval(s) where $f(x)$ is concave up.

$\boxed{(-\infty, c) \cup (e, \infty)}$

(d) Determine the interval(s) where $f(x)$ is concave down.

$\boxed{(c, e)}$

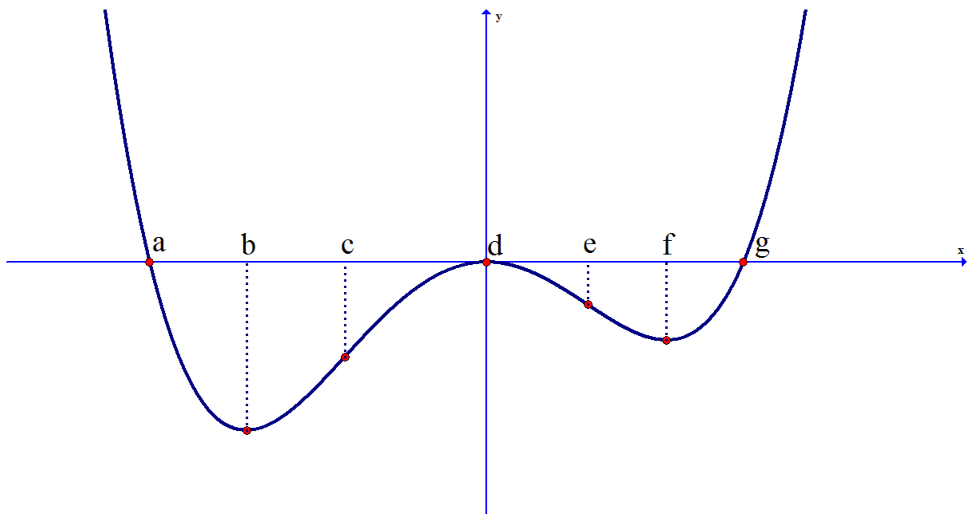
(e) Determine the value(s) of x where $f(x)$ has relative (local) extrema. Classify each as the location of a relative maximum or a relative minumum.

$\boxed{\text{Relative max when } x = d\text{; Relative minima when } x = b \text{ and } x = f}$

(f) Determine the value(s) of x where $f(x)$ has an inflection point.

$\boxed{\text{Point of Inflection when } x = c \text{ and } x = e}$

2. The graph of **<u>the derivative</u>** of $y = f(x)$ is shown below.



(a) Determine the interval(s) where $f(x)$ is increasing.

$\boxed{(-\infty, a) \cup (g, \infty)}$

(b) Determine the interval(s) where $f(x)$ is decreasing.

$\boxed{(a, d) \cup (d, g)}$

(c) Determine the interval(s) where $f(x)$ is concave up.

$\boxed{(b, d) \cup (f, \infty)}$

(d) Determine the interval(s) where $f(x)$ is concave down.

$\boxed{(-\infty, b) \cup (d, f)}$

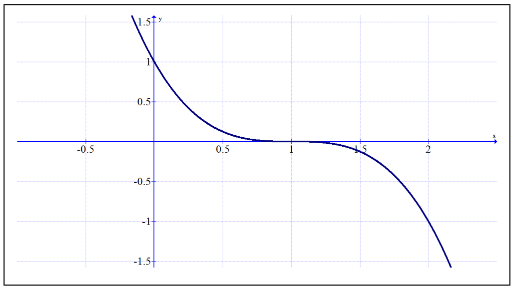
(e) Determine the value(s) of x where $f(x)$ has relative (local) extrema. Classify each as the location of a relative maximum or a relative minumum.

Relative maximum when $x = a$; Relative minimum when $x = g$; Neither a relative max nor a relative min at the critical point of $x = d$.

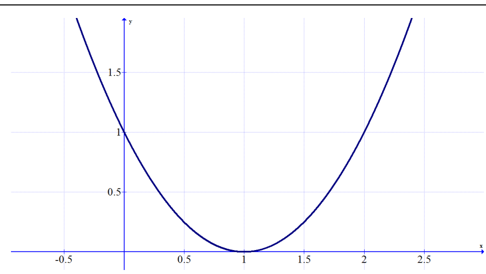
(f) Determine the value(s) of x where $f(x)$ has an inflection point.

$\boxed{\text{Points of inflection when } x = b,\ x = d \text{ and } x = f}$

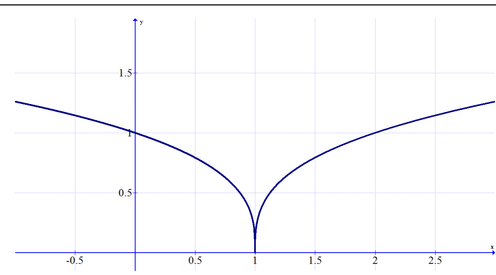
3. Sketch the graph of a continuous function, $y = f(x)$, which is decreasing on $(-\infty, \infty)$, has an inflection point at $x = 1$, and is concave down on $(1, \infty)$.



4. Sketch the graph of a continuous function, $y = f(x)$, which is decreasing on $(-\infty, 1)$, has a relative minimum at $x = 1$, and does not have any inflection points.

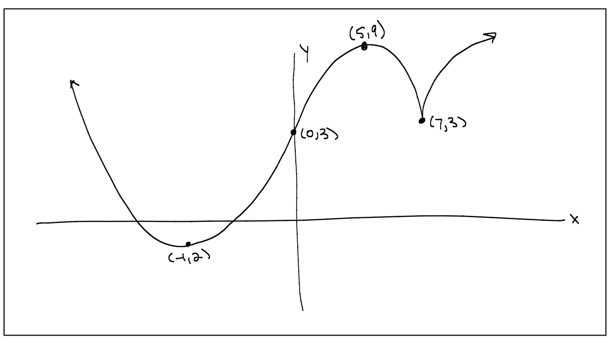


or



5. Sketch the graph of a continuous function $y = f(x)$ which satisfies all of the following conditions:

- Domain of $f(x)$ is $(-\infty, \infty)$
- $f(-1) = -2$, $f(0) = f(7) = 3$, and $f(5) = 9$
- $f'(x) < 0$ on $(-\infty, -1) \cup (5, 7)$ and $f'(x) > 0$ on $(-1, 0) \cup (0, 5) \cup (7, \infty)$
- $f''(x) < 0$ on $(0, 7) \cup (7, \infty)$ and $f''(x) > 0$ on $(-\infty, 0)$



6. Consider the function that you sketched in question 5. At which value(s) of x must $f'(x) = 0$? At which value(s) of x must $f'(x)$ fail to exist?

$\boxed{f'(x) = 0 \text{ when } x = -1 \text{ and } x = 5;\ f'(x) \text{ DNE when } x = 7}$

For problems 7-15, calculate each of the following:

(a) The intervals on which $f(x)$ is increasing
(b) The intervals on which $f(x)$ is decreasing
(c) The intervals on which $f(x)$ is concave up
(d) The intervals on which $f(x)$ is concave down
(e) All points of inflection. Express each as an ordered pair (x, y)

7. $f(x) = x^3 - 2x + 3$

$$\boxed{\text{a.}\left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right);\ \text{b.}\ \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right);\ \text{c.}\ (0, \infty);\ \text{d.}\ (-\infty, 0);\ \text{e.}\ (0, 3)}$$

8. $f(x) = \dfrac{x}{x-2}$

$\boxed{\text{a. none; b. } (-\infty, 2) \cup (2, \infty);\ \text{c. } (2, \infty);\ \text{d. } (-\infty, 2);\ \text{e. none}}$

9. $f(x) = \sin x$ on $[0, 2\pi]$

$$\boxed{\text{a.}\ \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right);\ \text{b.}\ \left(\frac{\pi}{2}, \frac{3\pi}{2}\right);\ \text{c.}\ (\pi, 2\pi);\ \text{d.}\ (0, \pi);\ \text{e.}\ (\pi, 0)}$$

10. $f(x) = (4x - 1)^4$

$$\boxed{\text{a.}\left(\frac{1}{4}, \infty\right);\ \text{b.}\ \left(-\infty, \frac{1}{4}\right);\ \text{c.}\ \left(-\infty, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \infty\right);\ \text{d. none; e. none}}$$

11. $f(x) = xe^x$

$$\boxed{\text{a.}\ (-1, \infty);\ \text{b.}\ (-\infty, -1);\ \text{c.}\ (-2, \infty);\ \text{d.}\ (-\infty, -2);\ \text{e.}\ \left(-2, -\frac{2}{e^2}\right)}$$

12. $f(x) = \arctan(2x)$

$\boxed{\text{a.}\ (-\infty, \infty);\ \text{b. none; c.}\ (-\infty, 0);\ \text{d.}\ (0, \infty);\ \text{e.}\ (0, 0)}$

13. $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$\boxed{\text{a.}\ (-\infty, 0);\ \text{b.}(0, \infty);\ \text{c.}\ (-\infty, -1) \cup (1, \infty);\ \text{d.}\ (-1, 1);\ \text{e.}\ \left(-1, \frac{1}{\sqrt{2\pi}e^{1/2}}\right) \text{ and } \left(1, \frac{1}{\sqrt{2\pi}e^{1/2}}\right)}$$

14. $f(x) = \dfrac{\ln x}{x}$

$\boxed{\text{a. } (0, e)\text{; b. } (e, \infty)\text{; c. } (e^{3/2}, \infty)\text{; d. } (0, e^{3/2})\text{; e. } \left(e^{3/2}, \dfrac{3}{2e^{3/2}}\right)}$

15. $f(x) = 2x + 3x^{2/3}$

$\boxed{\text{a. } (-\infty, -1) \cup (0, \infty)\text{; b. } (-1, 0)\text{; c. none; d. } (-\infty, 0) \cup (0, \infty)\text{; e. none}}$

For problems 16-20, compute the critical points of the given function. Then use the First Derivative Test to determine all relative (local) extrema. Express each extremum as an ordered pair (x, y).

16. $f(x) = x^2 - 16$

$\boxed{\text{Relative min at } (0, -16)}$

17. $f(x) = (2x + 3)^3$

$\boxed{\text{Critical Point at } -\dfrac{3}{2}\text{, No relative extrema}}$

18. $f(x) = \dfrac{3x}{x^2 + 1}$

$\boxed{\text{Relative max at } \left(1, \dfrac{3}{2}\right)\text{; Relative min at } \left(-1, -\dfrac{3}{2}\right)}$

19. $f(x) = e^x - x$

$\boxed{\text{Relative min at } (0, 1)}$

20. $f(x) = x^3 - x^5$

Relative maximum at $\left(\sqrt{\dfrac{3}{5}}, \left(\dfrac{2}{5}\right) \cdot \left(\dfrac{3}{5}\right)^{3/2}\right)$

Relative minimum at $\left(-\sqrt{\dfrac{3}{5}}, -\left(\dfrac{2}{5}\right) \cdot \left(\dfrac{3}{5}\right)^{3/2}\right)$

Critical point at $(0, 0)$, which is neither a relative max nor a relative min

For problems 21-22, use the Second Derivative Test to determine the relative (local) extrema. Express each as an ordered pair (x, y).

21. $f(x) = \sin(3x)$ on $[0, \pi]$

$\boxed{\text{Relative maxima at } \left(\dfrac{\pi}{6}, 1\right) \text{ and } \left(\dfrac{5\pi}{6}, 1\right)\text{; Relative minimum at } \left(\dfrac{\pi}{2}, -1\right)}$

22. $f(x) = \sec(3x)$ on $[0, \pi]$

$\boxed{\text{Relative minima at } (0,1) \text{ and } \left(\frac{2\pi}{3}, 1\right)\text{; Relative maxima at } \left(\frac{\pi}{3}, -1\right) \text{ and } (\pi, -1)}$

For problems 23-27, determine the critical points. Classify each as a relative extremum, relative minimum, or neither. Express all relative extrema as ordered pairs (x, y).

23. $f(x) = \sin^2 x$ on $[0, 2\pi]$

$\boxed{\begin{array}{l}\text{Relative minima at } (0,0),\ (\pi,0), \text{ and } (2\pi,0); \\ \text{Relative maxima at } \left(\frac{\pi}{2}, 1,\right) \text{ and } \left(\frac{3\pi}{2}, 1\right)\end{array}}$

24. $f(x) = \dfrac{x^3}{3} + x^2 + x + 3$

$\boxed{\text{No relative extrema}}$

25. $f(x) = xe^x$

$\boxed{\text{Relative minimum at } \left(-1, -\frac{1}{e}\right)}$

26. $f(x) = 2x + 3x^{2/3}$

$\boxed{\text{Relative maximum at } (-1,1)\text{; Relative minimum at } (0,0)}$

27. $f(x) = \dfrac{\ln x}{x}$

$\boxed{\text{Relative Maximum at } \left(e, \frac{1}{e}\right)}$

HINT: For problems 25-27, it may be helpful to use your work from earlier in the assignment.