Differential Equations & Separation of Variables

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 8.2 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to verify that a given function is a solution to a differential equation.

- Be able to solve first-order separable equations by using the technique of separation of variables.

- Be able to solve initial-value problems for first-order separable equations.

PRACTICE PROBLEMS:

1. Verify that $y = x^2 + 1$ is a solution to the differential equation $y - \dfrac{dy}{dx} = (x-1)^2$.

2. Find the value(s) of the constant A for which $y = e^{Ax}$ is a solution to the differential equation $y'' + 5y' - 6y = 0$.

For problems 3-7, use separation of variables to solve the given differential equation. Express your solution as an explicit function of x.

3. $\dfrac{dy}{dx} = \dfrac{x^2 - 1}{y^2}$

4. $\dfrac{dy}{dx} - x(y^2 + 1) = 0$

5. $\dfrac{dy}{dx} - \sqrt{xy}\ln x = 0$

6. $y' = yx^2$

7. $\dfrac{dy}{dx} - e^{-y}\sec^2 x = 0$

For problems 8-9, use separation of variables to solve the given differential equation. You may leave your solution as an implicitly defined function.

8. $\dfrac{dy}{dx} = xy^3$

9. $\dfrac{dy}{dx} = \dfrac{1}{(x^2 - 5x + 6)y}$

For problems 10-11, find the solution of the differential equation which satisfies the initial condition.

10. $\dfrac{dy}{dx} = \dfrac{x^2 - 2}{y}$, $y(0) = 1$

11. $\dfrac{dy}{dx} = \dfrac{\ln x}{xy^2}$, $y(e) = 1$

12. Find the general solution of the differential equation $\left(\dfrac{\sqrt{x}}{2+y}\right)\dfrac{dy}{dx} = 1$, for $x \neq 0$.
    Express the solution as an explicit function of x.

13. Find an equation of the curve that passes through the point $(0, 1)$ and whose slope at (x, y) is xe^y.

14. Suppose that a ball is moving along a straight line through a resistive medium in such a way that its velocity $v = v(t)$ decreases at a rate that is twice the square root of the velocity. Suppose that at time $t = 3$ seconds, the velocity of the ball is 16 meters/second.

    (a) Set up and solve an initial value problem whose solution is $v = v(t)$. Express your solution as an explicit function of t.

    (b) At what time does the ball come to a complete stop?

15. **Definition:** A quantity $y = y(t)$ is said to follow an **Exponential Decay Model** if it decreases at a rate which is proportional to the amount of the quantity present.

    (a) Assuming that the initial amount the quantity present is y_0, we arrive at the following initial value problem:

    $$\begin{cases} \dfrac{dy}{dt} = -ky \quad (k > 0) \\[2mm] y(0) = y_0 \end{cases}$$

    Show that the solution to this initial value problem is $y(t) = y_0 e^{-kt}$.

    (b) **Definition:** The time required for the original amount of the quantity to reduce by half is called the **half-life**.

    Compute the half life of a quantity which follow the Exponential Decay Model from part a.

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For problems 16-18, use the results from problem 15.

16. Carbon 14, an isotope of Carbon, is radioactive and decays at a rate proportional to the amount present (and therefore follows an exponential decay model). Its half life is 5730 years. If there were 20 grams of Carbon 14 originally present, how much would be left after 2000 years?

17. A scientist wants to determine the half life of a certain radioactice substance. She determines that in exactly 5 days an 80 milligram sample decays to 10 milligrams. Based on this data, what is the half life?

18. An unknown amount of a radioactive substance is being studied. After two days, the mass is 15 grams. After eight days, the mass is 9 grams. Assume exponential decay.

    (a) How much of the substance was there initially?
    (b) What is the half-life of the substance?

19. Newton's Law of Cooling states that the rate at which an object cools (or warms) is proportional to the difference in temperature between the object and the surrounding medium. Suppose an object has an initial temperature T_0 is placed in a room with a temperature of T_R. If $T(t)$ represents the temperature of the object at time t, then Newton's Law of Cooling can be expressed with the following initial value problem:

$$\begin{cases} \dfrac{dT}{dt} = -k(T - T_R), \quad (k > 0) \\[2mm] T(0) = T_0 \end{cases}$$

    (a) Use separation of variables to show that $T(t) = (T_0 - T_R)e^{-kt} + T_R$ is the particular solution to this initial value problem.

    (b) An object is taken from 350° oven has a temperature of 350° and is left to cool in a room that is 65°. After 1 hour, the temperature drops to 200°. Determine $T(t)$, the temperature at time T.

    (c) For the object described in part (b), determine the temperature after 2 hours.