The Gradient & Directional Derivatives

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 13.6 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to compute a gradient vector, and use it to compute a directional derivative of a given function in a given direction.
- Be able to use the fact that the gradient of a function $f(x, y)$ is perpendicular (normal) to the level curves $f(x, y) = k$ and that it points in the direction in which $f(x, y)$ is increasing most rapidly.

PRACTICE PROBLEMS:

For problems 1-3, compute the directional derivative of f at the point P in the direction of $\overrightarrow{v}$.

1. $f(x, y) = x^4 - y^4$; $P(0, -2)$; $\overrightarrow{v} = \dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}$

   $\boxed{\dfrac{32}{\sqrt{2}}}$

2. $f(x, y) = y \sin x$; $P\left(\dfrac{\pi}{2}, 1\right)$; $\overrightarrow{v} = \langle 1, -1 \rangle$

   $\boxed{-\dfrac{1}{\sqrt{2}}}$

3. $f(x, y, z) = e^x \cos(yz)$ at $P = (1, \pi, 0)$, $\overrightarrow{v} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

   $\boxed{-\dfrac{2e}{\sqrt{14}}}$

4. Find the directional derivative of $g(x, y, z) = z \ln(x + y)$ at $P(0, 1, -2)$ in the direction from P to $Q(1, 3, 2)$.

   $\boxed{-\dfrac{6}{\sqrt{21}}}$; Detailed Solution: Here

5. Find the directional derivative of $f(x,y) = \dfrac{y^2}{x+y}$ at the point $(-1,-1)$ in the direction of a vector which makes a counterclockwise angle $\theta = \dfrac{\pi}{4}$ with the positive x-axis.

$\boxed{\dfrac{\sqrt{2}}{4}}$

6. Suppose $f(x,y) = \tan(xy)$. Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(1,\pi) = 0$.

$\boxed{\mathbf{u} = \left\langle \dfrac{1}{\sqrt{\pi^2+1}}, -\dfrac{\pi}{\sqrt{\pi^2+1}} \right\rangle \text{ or } \mathbf{u} = \left\langle -\dfrac{1}{\sqrt{\pi^2+1}}, \dfrac{\pi}{\sqrt{\pi^2+1}} \right\rangle\text{; Detailed Solution: Here}}$

7. Suppose that $f(x,y,z)$ is a differentiable function. Let $f_x(1,1,2) = 5$, $f_y(1,1,2) = -1$, and $f_z(1,1,2) = 0$. What is the directional derivative of $f(x,y,z)$ at $(1,1,2)$ in the direction of $\overrightarrow{a} = \langle -3, 0, 4\rangle$?

$\boxed{-3}$

8. Suppose $D_{\mathbf{u}}f(3,-2) = 1$ and $D_{\mathbf{v}}f(3,-2) = 2$ where $\mathbf{u} = \dfrac{4}{5}\mathbf{i} + \dfrac{3}{5}\mathbf{j}$ and $\mathbf{v} = -\dfrac{4}{5}\mathbf{i} + \dfrac{3}{5}\mathbf{j}$. Compute $f_x(3,-2)$ and $f_y(3,-2)$.

$\boxed{f_x(3,-2) = -\dfrac{5}{8};\ f_y(3,-2) = \dfrac{5}{2}}$

For problems 9-11, find the gradient of f at the given point.

9. $f(x,y) = 3xy - y^2x^3$ at $(1,-1)$

$\boxed{\nabla f(1,-1) = -6\mathbf{i} + 5\mathbf{j}}$

10. $f(x,y) = \cos(2x - y^2)$ at $(\pi/4, 0)$

$\boxed{\nabla f\left(\dfrac{\pi}{4}, 0\right) = \langle -2, 0\rangle}$

11. $f(x,y,z) = 4xyz - y^2z^3 + 4z^3y$ at $(2,3,1)$

$\boxed{\nabla f(2,3,1) = 12\mathbf{i} + 6\mathbf{j} + 33\mathbf{k}}$

12. For each of the following, determine the maximum value of the directional derivative at the given point as well as a unit vector in the direction in which the maximum value occurs.

   (a) $g(x,y) = e^{xy^2}$; $P(1,3)$

   The maximum value of the directional derivative of g at P is $e^9\sqrt{117}$ which occurs in the direction of $\mathbf{u} = \left\langle \dfrac{9}{\sqrt{117}}, \dfrac{6}{\sqrt{117}} \right\rangle$.

(b) $w = \sqrt{4 - x^2 - y^2 - z^2}$; $P(1, -1, 0)$

The maximum value of the directional derivative of w at P is 1 which occurs in the direction of $\mathbf{u} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$.

13. The temperature at the point (x, y, z) in a room is $T(x, y, z) = \dfrac{xz}{x^2 + y^2}$. Find the direction in which the temperature increases most rapidly at the point $(-3, 4, 1)$.

$$\boxed{\frac{7}{625}\mathbf{i} + \frac{24}{625}\mathbf{j} - \frac{3}{25}\mathbf{k}}$$

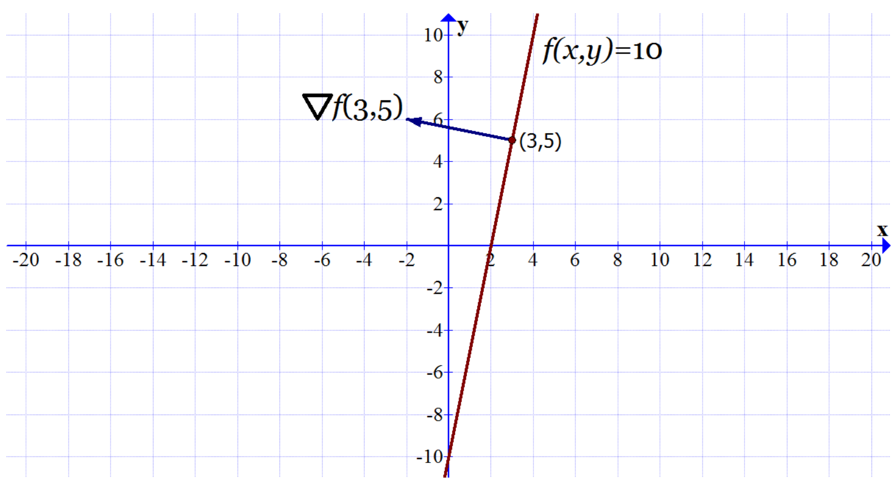
14. Compute a unit vector in the direction in which $f(x, y, z) = x^3yz^2$ decreases most rapidly at $P(2, -1, 1)$; and, find the rate of change of f at P in that direction.

The direction in which f decreases most rapidly is $\mathbf{u} = \left\langle \frac{3}{\sqrt{29}}, -\frac{2}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right\rangle$. And, the rate of change in this direction is $-4\sqrt{29}$. Detailed Solution: Here

For problems 15-16, sketch the level curve of $f(x, y)$ which passes through the given point P. Then draw the gradient of f at P on the same axes.

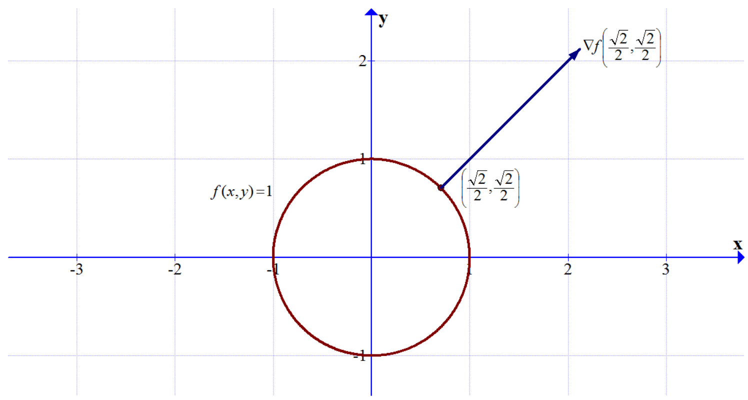
15. $f(x, y) = 20 - 5x + y$; $P = (3, 5)$

Point P is on the level curve $f(x, y) = 10$, i.e., $y = 5x - 10$; $\nabla f(3, 5) = \langle -5, 1 \rangle$.

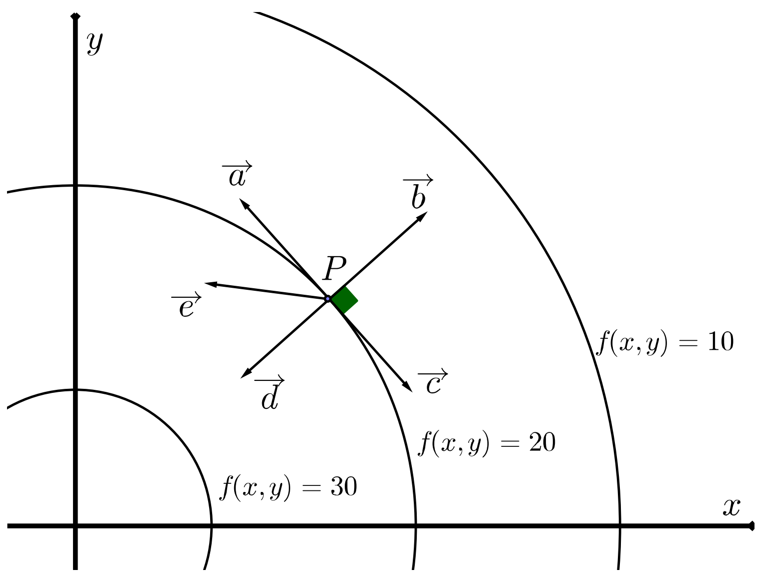


16. $f(x,y) = x^2 + y^2$; $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Point P is on the level curve $f(x,y) = 1$, i.e., $x^2+y^2 = 1$; $\nabla f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left\langle \sqrt{2}, \sqrt{2} \right\rangle$.



17. The graph shown below depicts some level curves of an unspecified function $f(x,y)$.



Which of the vectors is most likely to be ∇f at P? Explain your reasoning.

$\vec{d}$. $\nabla f(P)$ should point in the direction of greatest increase and it should be normal to point P on the level curve of $f(x,y)$.