Chapter 1.6 Practice Problems

EXPECTED SKILLS:

- Know where the trigonometric and inverse trigonometric functions are continuous.
- Be able to use $\lim_{x \to 0} \frac{\sin x}{x} = 1$ or $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ to help find the limits of functions involving trigonometric expressions, when appropriate.
- Understand the squeeze theorem and be able to use it to compute certain limits.

PRACTICE PROBLEMS:

Evaluate the following limits. If a limit does not exist, write DNE, $+\infty$, or $-\infty$ (whichever is most appropriate).

1. $\lim_{x \to \frac{\pi}{4}} \sin(2x)$

   $\boxed{1}$

2. $\lim_{\theta \to \pi} (\theta \cos \theta)$

   $\boxed{-\pi}$

3. $\lim_{x \to 0^+} \csc x$

   $\boxed{+\infty}$

4. $\lim_{x \to \frac{\pi}{2}^+} \tan x$

   $\boxed{-\infty}$

5. $\lim_{x \to \frac{\pi}{2}^-} \tan x$

   $\boxed{+\infty}$

6. $\lim_{x \to \frac{\pi}{4}} \sec x$

   $\boxed{\sqrt{2}}$

7. $\lim_{x \to 0} \left(\frac{\sin x}{3x} \right)$

   $\boxed{\frac{1}{3}}$

8. $\lim_{x \to 0} \left(\frac{\sin 3x}{3x} \right)$

$\boxed{1}$

9. $\lim_{x \to 0} \left(\frac{\sin x}{|x|} \right)$

$\boxed{\text{DNE}}$

10. $\lim_{x \to 0} \left(\frac{1 - \cos x}{4x} \right)$

$\boxed{0}$

11. $\lim_{x \to 0^-} \left(\frac{\cos x}{x} \right)$

$\boxed{-\infty}$

12. $\lim_{x \to 0} \left(\frac{\sin 2x}{x} \right)$

$\boxed{2}$

13. $\lim_{x \to 0} \left(\frac{\tan 2x}{x} \right)$

$\boxed{2}$

14. $\lim_{x \to 0} \left(\frac{1 - 3\cos x}{3x} \right)$

$\boxed{\text{DNE}}$

15. $\lim_{x \to \infty} \arccos \left(\frac{-x^2}{x^2 + 3x} \right)$

$\boxed{\pi}$

16. $\lim_{x \to 0} \left(\frac{3x^2}{1 - \cos^2 x} \right)$

$\boxed{3}$

17. $\lim_{x \to 0} \left(\frac{\tan 5x}{\sin 9x} \right)$

$\boxed{\frac{5}{9}}$

18. **Multiple Choice:** Evaluate $\lim\limits_{x\to 0} \dfrac{\tan^2 x}{x^2}$

   (a) -1
   (b) 0
   (c) 1
   (d) $-\infty$
   (e) $+\infty$

   $\boxed{\text{c}}$

For problems 19-23, evaluate the following limits by first making an appropriate substition. If the limit does not exist, write DNE, $+\infty$, or $-\infty$ (whichever is most appropriate).

19. $\lim\limits_{x\to\infty} \left(e^x \sin\left(e^{-x}\right)\right)$

    $\boxed{1}$

20. $\lim\limits_{x\to 1} \left(\dfrac{\sin\left(\ln x^5\right)}{\ln x}\right)$

    $\boxed{5}$

21. $\lim\limits_{x\to \frac{\pi}{2}^+} e^{\sec x}$

    $\boxed{0}$

22. $\lim\limits_{x\to 0} \sin\left(\dfrac{1}{x^2}\right)$

    $\boxed{\text{DNE}}$

23. $\lim\limits_{x\to 0^+} \tan^{-1}\left(\ln x\right)$

    $\boxed{-\dfrac{\pi}{2}}$

For problem 24-28, determine the value(s) of x where the given function is continuous.

24. $f(x) = \csc x$

    $\boxed{f(x) \text{ is continuous for all } x \neq \pi k\text{, where } k \text{ is any integer.}}$

25. $f(x) = e^{\sin x}$

$\boxed{f(x) \text{ is always continuous.}}$

26. $f(x) = \dfrac{1}{1 - 2\cos x}$ on $[0, 2\pi]$

$\boxed{f(x) \text{ is continuous for all } x \text{ in } [0, 2\pi] \text{ except for } x = \dfrac{\pi}{3} \text{ and } x = \dfrac{5\pi}{3}}$

27. $f(x) = \sin^{-1} x$

$\boxed{f(x) \text{ is continuous on its domain of } [-1, 1]}$

28. $f(x) = \begin{cases} \cos x & \text{if } \ x < \dfrac{\pi}{4} \\ \\ \sin x & \text{if } \ x \geq \dfrac{\pi}{4} \end{cases}$

$\boxed{f(x) \text{ is always continuous.}}$

29. Find all non-zero value(s) of k so that $f(x) = \begin{cases} \dfrac{3\sin(kx)}{x} & \text{if } \ x > 0 \\ \\ 6k^2 + 5x & \text{if } \ x \leq 0 \end{cases}$ is continuous at $x = 0$.

$\boxed{k = \dfrac{1}{2}}$

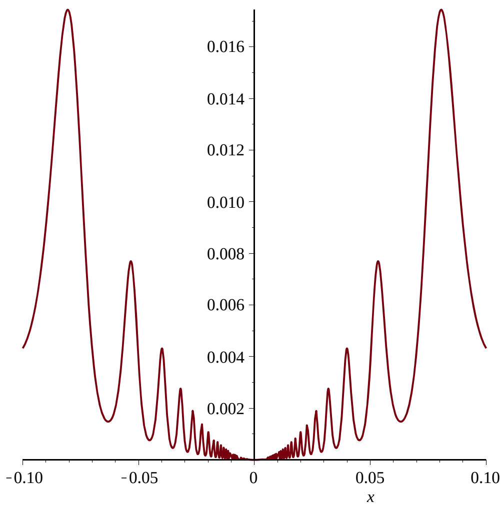
30. Use the Intermediate Value Theorem to prove that there is at least one solution to $\cos x = x^2$ in $(0, 1)$.

> Let $f(x) = \cos(x) - x^2$. Since $f(x)$ is continuous on $(-\infty, \infty)$, it is also continuous on $[0, 1]$. Notice that $f(0) = 1 > 0$ and $f(1) = \cos(1) - 1 < 0$. Thus, the Intermediate Value Theorem states that there must be some c in $(0, 1)$ such that $f(c) = 0$. i.e., there must be at least one c in $(0, 1)$ such that $\cos(c) - c^2 = 0 \implies \cos(c) = c^2$, as desired.

31. Let $f(x)$ be a function which satisfies $5x - 6 \leq f(x) \leq x^2 + 3x - 5$ for all $x \geq 0$. Compute $\lim\limits_{x \to 1} f(x)$.

$\boxed{-1}$

32. The graph of $f(x) = x^2 e^{\cos(1/x)}$ is shown below on $[-0.1, 0.1]$:



Make a conjecture about $\lim_{x\to 0} f(x)$ and then use the Squeeze Theorem to show this is true.

Claim: $\lim_{x\to 0} f(x) = 0$
Proof: We can bound $f(x) = x^2 e^{\cos(1/x)}$ above by ex^2 and below by $e^{-1}x^2$, both of which approach 0 as x approaches 0. Thus, by the squeeze theorem, $f(x) \to 0$ as well when $x \to 0$.

33. Let x be a fixed real number. Compute $\lim_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h}$. (Hint: The identity $\sin(A+B) = \sin A \cos B + \cos A \sin B$ will be useful.)

$\boxed{\cos x}$