# The Gradient & Directional Derivatives

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 13.6 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to compute a gradient vector, and use it to compute a directional derivative of a given function in a given direction.
- Be able to use the fact that the gradient of a function $f(x, y)$ is perpendicular (normal) to the level curves $f(x, y) = k$ and that it points in the direction in which $f(x, y)$ is increasing most rapidly.

PRACTICE PROBLEMS:

**For problems 1-3, compute the directional derivative of $f$ at the point $P$ in the direction of $\vec{v}$.**

1. $f(x, y) = x^4 - y^4$; $P(0, -2)$; $\vec{v} = \dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}$

2. $f(x, y) = y \sin x$; $P\left(\dfrac{\pi}{2}, 1\right)$; $\vec{v} = \langle 1, -1 \rangle$

3. $f(x, y, z) = e^x \cos(yz)$ at $P = (1, \pi, 0)$, $\vec{v} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

4. Find the directional derivative of $g(x, y, z) = z \ln(x + y)$ at $P(0, 1, -2)$ in the direction from $P$ to $Q(1, 3, 2)$.

5. Find the directional derivative of $f(x, y) = \dfrac{y^2}{x + y}$ at the point $(-1, -1)$ in the direction of a vector which makes a counterclockwise angle $\theta = \dfrac{\pi}{4}$ with the positive $x$-axis.

6. Suppose $f(x, y) = \tan(xy)$. Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(1, \pi) = 0$.

7. Suppose that $f(x, y, z)$ is a differentiable function. Let $f_x(1, 1, 2) = 5$, $f_y(1, 1, 2) = -1$, and $f_z(1, 1, 2) = 0$. What is the directional derivative of $f(x, y, z)$ at $(1, 1, 2)$ in the direction of $\vec{a} = \langle -3, 0, 4 \rangle$?

8. Suppose $D_{\mathbf{u}}f(3, -2) = 1$ and $D_{\mathbf{v}}f(3, -2) = 2$ where $\mathbf{u} = \dfrac{4}{5}\mathbf{i} + \dfrac{3}{5}\mathbf{j}$ and $\mathbf{v} = -\dfrac{4}{5}\mathbf{i} + \dfrac{3}{5}\mathbf{j}$. Compute $f_x(3, -2)$ and $f_y(3, -2)$.

**For problems 9-11, find the gradient of $f$ at the given point.**

9. $f(x, y) = 3xy - y^2x^3$ at $(1, -1)$

10. $f(x, y) = \cos(2x - y^2)$ at $(\pi/4, 0)$

11. $f(x, y, z) = 4xyz - y^2z^3 + 4z^3y$ at $(2, 3, 1)$

12. For each of the following, determine the maximum value of the directional derivative at the given point as well as a unit vector in the direction in which the maximum value occurs.

    (a) $g(x, y) = e^{xy^2}$; $P(1, 3)$
    (b) $w = \sqrt{4 - x^2 - y^2 - z^2}$; $P(1, -1, 0)$

13. The temperature at the point $(x, y, z)$ in a room is $T(x, y, z) = \dfrac{xz}{x^2 + y^2}$. Find the direction in which the temperature increases most rapidly at the point $(-3, 4, 1)$.

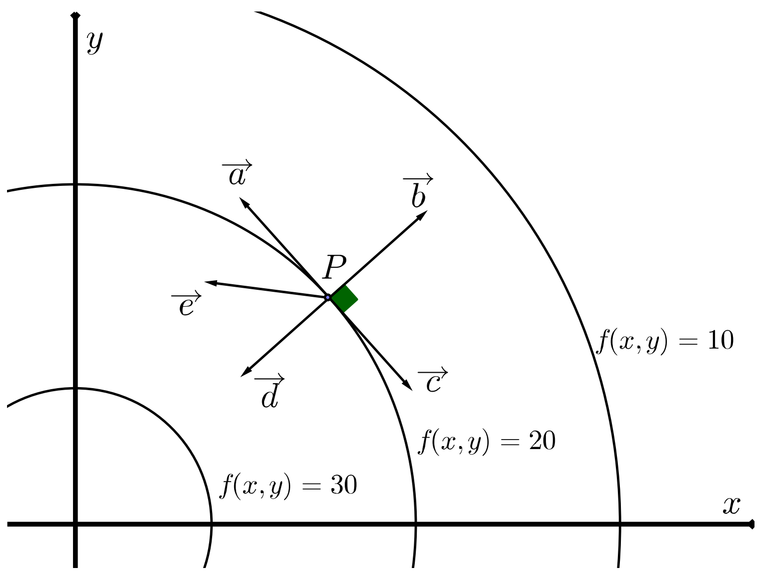
14. Compute a unit vector in the direction in which $f(x, y, z) = x^3yz^2$ decreases most rapidly at $P(2, -1, 1)$; and, find the rate of change of $f$ at $P$ in that direction.

**For problems 15-16, sketch the level curve of $f(x, y)$ which passes through the given point $P$. Then draw the gradient of $f$ at $P$ on the same axes.**

15. $f(x, y) = 20 - 5x + y$; $P = (3, 5)$

16. $f(x, y) = x^2 + y^2$; $P\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

17. The graph shown below depicts some level curves of an unspecified function $f(x, y)$.




Which of the vectors is most likely to be $\nabla f$ at $P$? Explain your reasoning.