7.5 # 14

$$\int \frac{x^5-3x^3+6}{x^3+x}\,dx$$ We begin with long division.

$$\begin{array}{r|l} & x^2-4 \\ \hline x^3+x & x^5+0x^4-3x^3+0x^2+0x+6 \\ & -(x^5+0x^4+x^3) \\ \hline & -4x^3+0x^2+0x+6 \\ & -(-4x^3+0x^2-4x) \\ \hline & 4x+6 \end{array}$$

So $$\int \frac{x^5-3x^3+6}{x^3+x}\,dx = \underbrace{\int (x^2-4)\,dx}_{\frac{1}{3}x^3-4x+C} + \underbrace{\int \frac{4x+6}{x^3+x}\,dx}_{\text{Partial Fractions}}$$

$$\frac{4x+6}{x^3+x} = \frac{4x+6}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$4x+6 = A(x^2+1)+(Bx+C)x$$

$x=0: \quad 6=A$

$$4x+6 = Ax^2+A+Bx^2+Cx = (A+B)x^2+Cx+A$$

$C=4$

$A+B=0 \implies B=-A=-6$

So $$\int \frac{4x+6}{x^3+x}\,dx = \int \frac{6}{x}\,dx + \int \frac{-6x+4}{x^2+1}\,dx$$

$$= \int \frac{6}{x}\,dx + \int \frac{-6x}{x^2+1}\,dx + \int \frac{4}{x^2+1}\,dx$$

$$= 6\ln|x| - 3\ln|x^2+1| + 4\arctan x + C$$

So $$\int \frac{x^5-3x^3+6}{x^3+x}\,dx = \frac{1}{3}x^3-4x+6\ln|x|-3\ln(x^2+1)+4\arctan x+C$$