Chapter 3.4: Trigonometric Identities

Expected Skills:

- Be able to derive Pythagorean Identities relating tangent/secant or cotangent/cosecant from $\sin^2\theta + \cos^2\theta = 1$.
- Given the identities $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ be able to derive the double angle formulas and power reducing formulas (as described in the course notes).
- Be able to use the Law of Cosines to relate the sides lengths of a triangle with one of the angles.

Practice Problems:

1. Find the exact values of each of the following:

   (a) $\sin 15°$ and $\cos 15°$.

   $$\boxed{\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4} \text{ and } \cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4}}$$

   (b) $\sin 165°$ and $\cos 165°$.

   $$\boxed{\sin 165° = \frac{\sqrt{6}-\sqrt{2}}{4} \text{ and } \cos 165° = -\frac{\sqrt{6}+\sqrt{2}}{4}}$$

   (c) $\sin 195°$ and $\cos 195°$.

   $$\boxed{\sin 195° = \frac{\sqrt{2}-\sqrt{6}}{4} \text{ and } \cos 195° = -\frac{\sqrt{6}+\sqrt{2}}{4}}$$

2. Express $\cos\alpha\cos\beta$ in terms of $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$.
   Hint: write out the sum and difference identities for cosine and combine them appropriately.

   $$\boxed{\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]}$$

3. Express $\sin\alpha\sin\beta$ in terms of $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$.

   $$\boxed{\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]}$$

4. Express $\sin\alpha\cos\beta$ in terms of $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$.

   $$\boxed{\frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]}$$

5. Suppose $\tan\alpha = \dfrac{3}{4}$, $\tan\beta = 8$ where $0 < \alpha < \dfrac{\pi}{2}$ and $0 < \beta < \dfrac{\pi}{2}$. Evaluate $\sin(\alpha+\beta)$.

$\boxed{\dfrac{7}{\sqrt{65}}}$

6. Derive the following identity: $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

We will begin on the left hand side, applying a series of trigonometric identities until we arrive at the desired conclusion:

$$
\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} && \text{By definition of tangent.}\\
&= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} && \text{By the sum identities for sine and cosine.}\\
&= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \cdot \frac{\frac{1}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}}\\
&= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\
&= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}
\end{aligned}
$$

7. Suppose $\tan\alpha = \dfrac{3}{4}$ and $\tan\beta = 8$. Use the result of the previous exercise to evaluate $\tan(\alpha+\beta)$.

$\boxed{-\dfrac{7}{4}}$

8. Suppose $\cos\theta = \dfrac{3}{5}$ and $\sin\theta < 0$. Evaluate each of the following:

(a) $\sin(2\theta)$

$\boxed{-\dfrac{24}{25}}$

(b) $\cos(2\theta)$

$\boxed{-\dfrac{7}{25}}$

(c) $\tan(2\theta)$

$\boxed{\dfrac{24}{7}}$

9. Rewrite $\sin^4 \theta$ as an equivalent expression which does not have any trigonometric functions with powers greater than 1.

$\boxed{\frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta}$

10. One hand of a very large clock is 3 feet long and the other is 4 feet long.

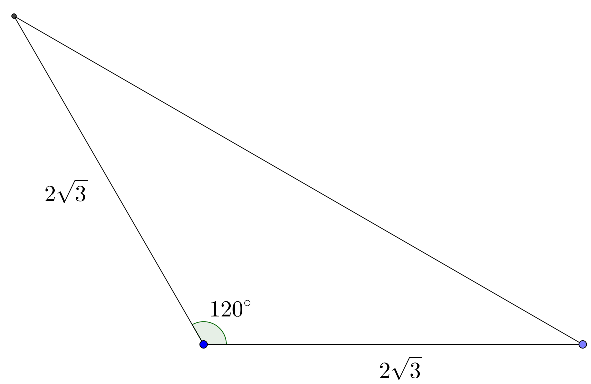
    (a) What is the distance between their tips at the moment when the clock strikes 3:00 pm?

    $\boxed{\text{5 feet}}$

    (b) What is the distance between their tips at the moment when the clock strikes 1:00 pm?

    $\boxed{\sqrt{25 - 12\sqrt{3}} \text{ feet}}$

11. Consider the following triangle:



    (a) Calculate the area of this triangle using the following theorem:

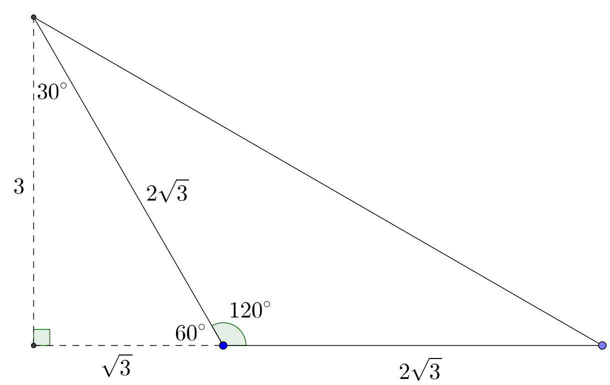
    Heron's Formula: The area of a triangle with sides of length a, b, and c is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$

Using the Law of Cosines, the missing side of the triangle has length 6. So, it follows that $a = 2\sqrt{3}$, $b = 2\sqrt{3}$, $c = 6$, and $s = 2\sqrt{3} + 3$. Appealing to Heron's Formula gives:

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{\left(2\sqrt{3}+3\right)\left(2\sqrt{3}+3-2\sqrt{3}\right)\left(2\sqrt{3}+3-2\sqrt{3}\right)\left(2\sqrt{3}+3-6\right)} \\ &= \sqrt{\left(2\sqrt{3}+3\right)(3)(3)\left(2\sqrt{3}-3\right)} \\ &= 3\sqrt{\left(2\sqrt{3}+3\right)\left(2\sqrt{3}-3\right)} \\ &= 3\sqrt{3} \end{aligned}$$

(b) Calculate the area of this triangle using the formula $A = \frac{1}{2}bh$.

Dropping a perpendicular from the upper vertex forms a 30-60-90 triangle. Since the hypotenuse has length $2\sqrt{3}$, we must scale all other sides of the triangle accordingly, as shown in the following figure:



Thus,

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}\left(2\sqrt{3}\right)(3) \\ &= 3\sqrt{3} \end{aligned}$$