Chapter 4.8 Practice Problems

EXPECTED SKILLS:

- Understand the hypotheses and conclusion of Rolle's Theorem or the Mean Value Theorem.
- Be able to find the value(s) of "c" which satisfy the conclusion of Rolle's Theorem or the Mean Value Theorem.

PRACTICE PROBLEMS:

1. For each of the following, verify that the hypotheses of Rolle's Theorem are satisfied on the given interval. Then find all value(s) of c in that interval that satisfy the conclusion of the theorem.

   (a) $f(x) = x^2 - 4x - 11$; $[0, 4]$

   > $f(x)$ is a polynomial; so, it is continuous and differentiable everywhere on $(-\infty, \infty)$. In particular, it is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Finally, notice that $f(0) = f(4) = -11$. Thus, all of the hypotheses of Rolle's Theorem are satisfied and there exists a c in $(0, 4)$ with $f'(c) = 0$. In particular, $c = 2$.

   (b) $f(x) = \sin x$; $[0, 2\pi]$

   > $f(x)$ is continuous and differentiable everywhere on $(-\infty, \infty)$. In particular, it is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$. Finally, notice that $f(0) = f(2\pi) = 0$. Thus, all of the hypotheses of Rolle's Theorem are satisfied and there exists a c in $(0, 2\pi)$ with $f'(c) = 0$. In particular, c is either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

2. Let $f(x) = \dfrac{1}{x^2}$

   (a) Show that there is no point c in the interval $(-1, 1)$ such that $f'(c) = 0$, even though $f(-1) = f(1) = 1$.

   > $f'(x) = -\dfrac{2}{x^3}$ which is never 0. Thus, there does not exist a c in $(-1, 1)$ with $f'(c) = 0$.

   (b) Explain why the result from part (a) does not contradict Rolle's Theorem.

   > $f(x)$ is not continuous at $x = 0$ which is in $[-1, 1]$, so Rolle's Theorem doesn't apply.

3. For each of the following, verify that the hypotheses of the Mean Value Theorem are satisfied on the given interval. Then find all value(s) of c in that interval that satisfy the conclusion of the theorem.

   (a) $f(x) = x^2 - 4x$; $[1, 5]$

   $f(x)$ is a polynomial; so, it is continuous and differentiable everywhere on $(-\infty, \infty)$. In particular, it is continuous on $[1, 5]$ and differentiable on $(1, 5)$. Thus, all of the hypotheses of the Mean Value Theorem are satisfied and there exists a c in $(1, 5)$ with $f'(c) = \dfrac{f(5) - f(1)}{5 - 1}$. In particular, $c = 3$.

   (b) $f(x) = x - \cos x$; $[0, 2\pi]$

   x is a polynomial and is, therefore, continuous and differentiable everywhere on $(-\infty, \infty)$. $\cos x$ is also continuous and differentiable everywhere on $(-\infty, \infty)$. So, since the difference of continuous functions is continuous and the difference of differentiable functions is differentiable, we have that $f(x)$ is continuous and differentiable everywhere on $(-\infty, \infty)$. In particular, it is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$. Thus, all of the hypotheses of the Mean Value Theorem are satisfied and there exists a c in $(0, 2\pi)$ with $f'(c) = \dfrac{f(2\pi) - f(0)}{2\pi - 0}$. In particular, $c = \pi$.

4. Let $f(x) = x^{2/3}$

   (a) Show that there is no point c in $(-8, 1)$ such that $f'(c)$ will be equal to the slope of the secant line through $(-8, f(-8))$ and $(1, f(1))$.

   It can be shown that the slope of the secant line which passes through $(-8, f(-8))$ and $(1, f(1))$ is $-\dfrac{1}{3}$. And, $f'(x) = \dfrac{2}{3x^{1/3}}$. However, the only solution to $f'(x) = -\dfrac{1}{3}$ is -8, which is not in $(-8, 1)$.

   (b) Explain why the result from part (a) does not contradict the Mean Value Theorem.

   $f(x)$ is not differentiable at $x = 0$ which is in $(-8, 1)$. Thus, the Mean Value Theorem does not apply.

5. Consider $f(x) = x^3 - x^2$.

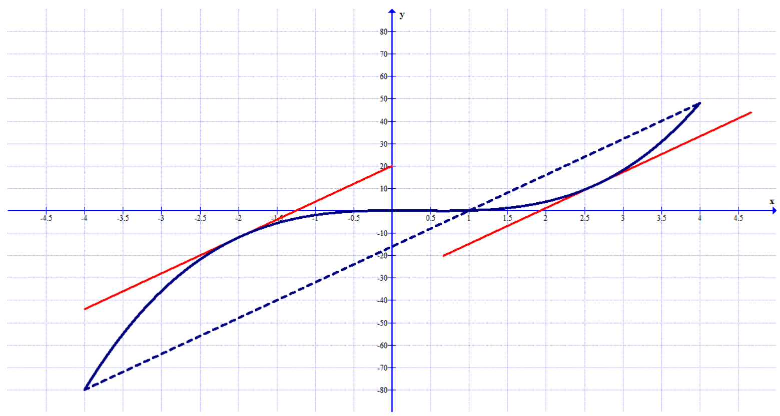
(a) Find the value(s) of c which satisfy the conclusion of the Mean Value Theorem on $[-4, 4]$.

$\boxed{c = -2 \text{ or } c = \frac{8}{3}}$

(b) At each value of c found in part (a), calculate an equation of the line which is tangent to the graph of $f(x)$.

$\boxed{y = 16x - 20;\ y = 16x - \frac{832}{27}}$

(c) On the axes provided below, sketch the tangent lines which you found in part (b).



6. Consider the quadratic function $f(x) = c_1x^2 + c_2x + c_3$, where $c_1 \neq 0$. Show that the number c in the conclusion of the mean value theorem is always the midpoint of the given interval $[a, b]$.

Since $f(x)$ is a polynomial, it is continuous and differentiable everywhere on $(-\infty, \infty)$. In particular, it is continuous on $[a, b]$ and differentiable on (a, b). Thus, there is a c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Notice that:

$$\begin{aligned}
\frac{f(b) - f(a)}{b - a} &= \frac{(c_1b^2 + c_2b + c_3) - (c_1a^2 + c_2a + c_3)}{b - a} \\
&= \frac{c_1(b^2 - a^2) + c_2(b - a)}{b - a} \\
&= c_1(b + a) + c_2
\end{aligned}$$

Finally, notice that $f'(x) = 2c_1x + c_2$. Setting this equal to $c_1(b + a) + c_2$ and solving for x yields $x = \frac{b + a}{2}$. Thus, the value of c in the conclusion of the MVT is $c = \frac{b + a}{2}$, which is the midpoint of the interval $[a, b]$

7. **Theorem:** Suppose that $f'(x) = 0$ for all x in some open interval I. Then, $f(x)$ is constant on the interval.

   Prove this theorem. (HINT: Consider any two numbers a and b in the interval I, where $a < b$. Show that $f(a) = f(b)$ on the interval I.)

   Pick any two numbers a and b in the interval I, where $a < b$. Since, by assumption, $f(x)$ is differentiable for all x in I, we have the following:

   - $f(x)$ is continuous on $[a, b]$
   - $f(x)$ is differentiable on (a, b)

   Therefore, by the Mean Value Theorem, there exists a number c in (a, b) such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

   But, $f'(x) = 0$ for all x in the interval I; so, in particular, $f'(c) = 0$. Thus, it follows that $f(b) - f(a) = 0 \implies f(b) = f(a)$. In other words, $f(x)$ is constant on the interval I.

8. **Definition:** A function $F(x)$ is an <u>antiderivative</u> of $f(x)$ if $\frac{d}{dx}[F(x)] = f(x)$. For example, since $\frac{d}{dx}\left[x^2 + 6\right] = 2x$, we say that $F(x) = x^2 + 6$ is an antiderivative of $f(x) = 2x$.

   (a) List some other antiderivatives of $2x$.

   All antiderivatives of $2x$ have the form $x^2 + C$, where C is any constant.

   (b) **Theorem:** Suppose $g'(x) = f'(x)$ for all x in an open interval I. Then, for some constant c, $g(x) = f(x) + c$ for all x in the interval I.

   Prove this theorem. (HINT: Define a new function $h(x) = g(x) - f(x)$ and appeal to the theorem in problem 7.)

   Define $h(x) = g(x) - f(x)$. Then, for all x in the interval I,

   $$h'(x) = g'(x) - f'(x) = 0$$

   By problem 7, we know that $h(x) = C$ for some constant C. And, it follows that $g(x) = f(x) + C$.

(c) Let $f(x) = \sin^{-1}(x)$ and $g(x) = -\cos^{-1}(x)$. Verify that $f'(x) = g'(x)$ and find the constant C such that $\sin^{-1}(x) = -\cos^{-1}(x) + C$.

$$f'(x) = g'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\sin^{-1}(x) = -\cos^{-1}(x) + \frac{\pi}{2} \text{ for all } x \text{ in } [-1, 1]$$