

# Jacobian & Change of Variables

## SUGGESTED REFERENCE MATERIAL:

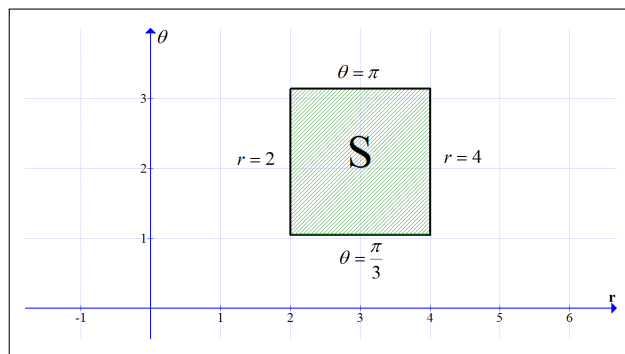
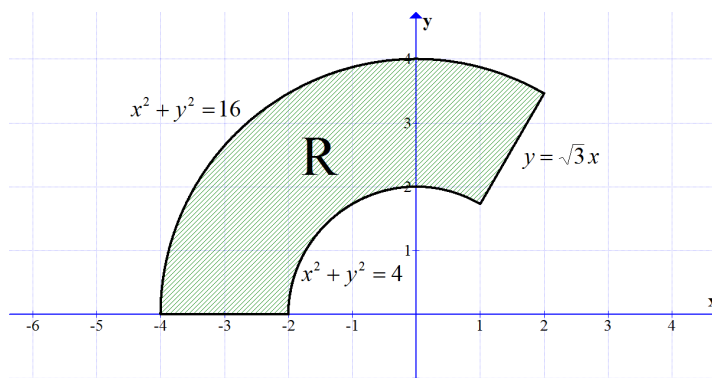
As you work through the problems listed below, you should reference Chapter 14.7 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

## EXPECTED SKILLS:

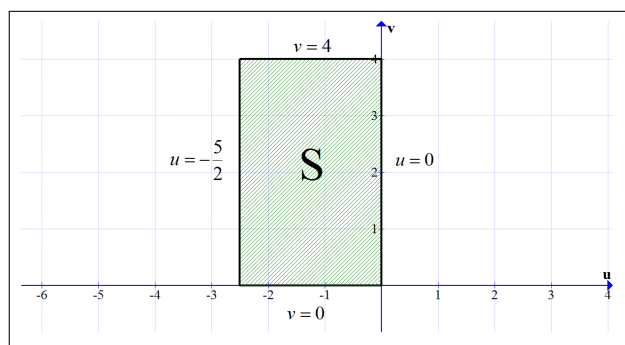
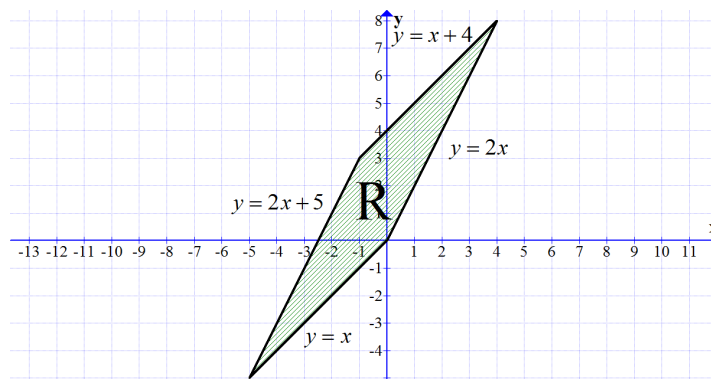
- Be able to use the change of variable formulas (14.7.2 and 14.7.4) for converting an integral from rectangular coordinates to another coordinate system by changing the integrand, the region of integration, and including the Jacobian factor.

## PRACTICE PROBLEMS:

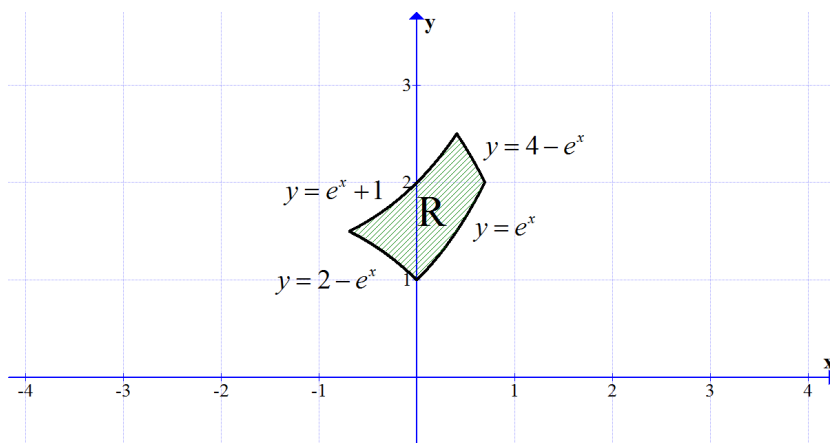
1. For each of the following, sketch the image of the region under the given transformation.
  - (a)  $R$  is the region shown below and the transformation is:  $x = r \cos \theta$ ,  $y = r \sin \theta$ , with  $r \geq 0$  and  $0 \leq \theta \leq 2\pi$ .



(b)  $R$  is the region shown below and the transformation is:  $x = 2u + v$ ,  $y = 2u + 2v$ .



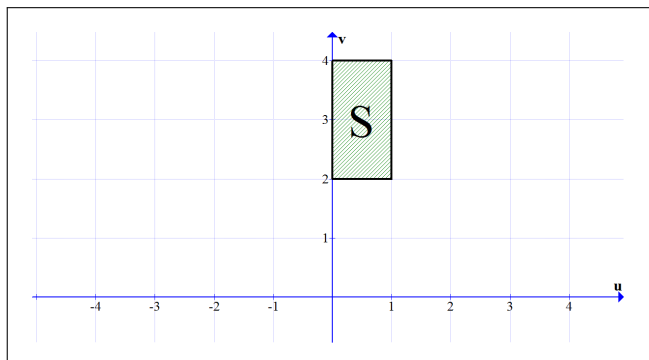
2. Consider the region  $R$  shown below in the  $xy$ -plane.



(a) Find a transformation that would map a rectangular region  $S$  in the  $uv$  plane to the region  $R$  in the  $xy$  plane. (You may express your answer as  $u = u(x, y)$  and  $v = v(x, y)$ .)

$$\boxed{u = y - e^x, v = y + e^x}$$

- (b) Sketch the region  $S$  from part (a) whose image under your transformation is  $R$ .



- (c) Solve for  $x$  and  $y$  in terms of  $u$  and  $v$ . In doing so, you are finding the inverse transformation. (You may express your answer as  $x = x(u, v)$ ,  $y = y(u, v)$ .)

$$x = \ln \left( \frac{1}{2}(v - u) \right) \text{ (where } v > u), y = \frac{1}{2}(u + v)$$

3. For each of the following transformations, compute the Jacobian  $\frac{\partial(x, y)}{\partial(u, v)}$ .

(a)  $x = u + 2v$ ,  $y = -3u + v$

$$\boxed{7}$$

(b)  $x = ve^u$ ,  $y = ve^{-u}$

$$\boxed{2v}$$

(c)  $x = e^u \cos v$  and  $y = e^u \sin v$ .

$$\boxed{e^{2u}}$$

4. Consider the transformation  $x = u + v$ ,  $y = v - u$ .

(a) Compute the Jacobian  $\frac{\partial(x, y)}{\partial(u, v)}$ .

$$\boxed{2}$$

- (b) Solve the  $u$  and  $v$  in terms of  $x$  and  $y$ . (That is, find the inverse transformation.)

$$\boxed{u = \frac{1}{2}(x - y), v = \frac{1}{2}(x + y)}$$

- (c) Compute the Jacobian  $\frac{\partial(u, v)}{\partial(x, y)}$  for the transformation that you found in part (b).

$$\boxed{\frac{1}{2}}$$

5. Verify that the Jacobian  $\frac{\partial(x, y)}{\partial(r, \theta)}$  for the transformation  $x = r \cos \theta$ ,  $y = r \sin \theta$  is  $r$ .

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r$$

6. Verify that the Jacobian  $\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}$  for the transformation  $x = \rho \cos \theta \sin \phi$ ,  $y = \rho \sin \theta \sin \phi$ ,  $z = \rho \cos \phi$  is  $\rho^2 \sin \phi$ .

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \cos \theta \sin \phi & -\rho \sin \theta \cos \phi & \rho \cos \theta \cos \phi \\ \sin \theta \sin \phi & \rho \cos \theta \sin \phi & \rho \sin \theta \cos \phi \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} = \dots = \rho^2 \sin \phi$$

7. Use the transformation  $u = 3x + y$ ,  $v = 6x - y$  to evaluate  $\iint_R \frac{3x + y}{6x - y} dA$  where  $R$  is the region enclosed by  $3x + y = 2$ ,  $3x + y = -3$ ,  $6x - y = 1$ ,  $6x - y = e$ .

$$-\frac{5}{18}$$

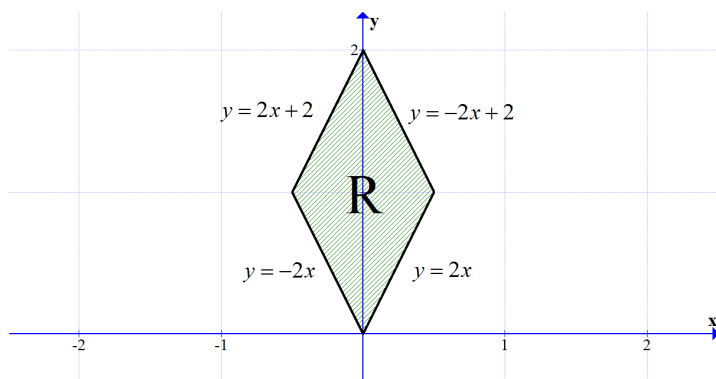
8. Use the transformation  $u = x - y$ ,  $v = x + y$  to evaluate  $\iint_R e^{(x-y)^2} dA$  where  $R$  is the region in the  $xy$  plane which is enclosed by  $y = 0$ ,  $y = -x$ , and  $y = x - 1$ .

$$\frac{1}{4}(e - 1); \text{ Detailed Solution: } [Here](#)$$

9. Use the transformation  $x = au$ ,  $y = bv$ , and  $z = cw$  to evaluate  $\iiint_G dV$ , where  $G = \{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$ . Note:  $a$ ,  $b$  and  $c$  are real, positive constants.

$$\frac{4\pi}{3}abc; \text{ This is the volume of the ellipsoid } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

10. Use the transformation  $u = 2x + y$ ,  $v = -2x + y$  to compute the area of the region  $R$ , shown below.



1

11. Compute the area of the region  $R$  described in exercise 2.

$6 \ln 2 - 3 \ln 3$