Chapter 4.2 (Part 2) & 4.3 Practice Problems

EXPECTED SKILLS:

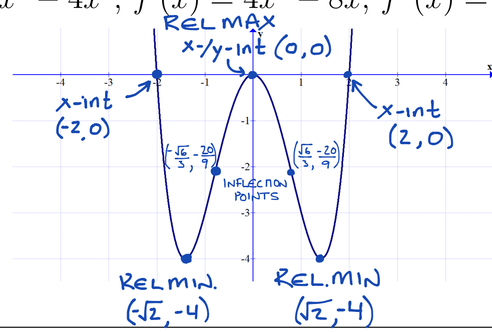
- Be able to use the extrema along with the end behavior (i.e. dominant term) of polynomials to sketch the graph of polynomial functions.
- Know how to determine if the graph of a function has a cusp or vertical tangent line at a point (i.e. the function is not differentiable at that point). And, be able to use this information, along with extrema, intercepts, and asymptotes, to sketch the graph of a function.

PRACTICE PROBLEMS:

For problems 1-12, sketch the given functions. Label the coordinates of all critical points, inflection points, x-intercepts, y-intercepts, and holes. Also label all horizontal asymptotes and vertical asymptotes

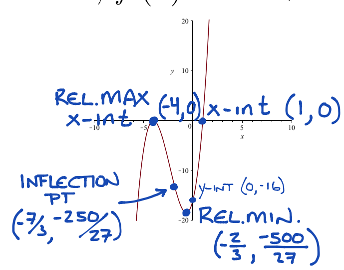
1. $f(x) = x^2(x^2 - 4)$

$$f(x) = x^4 - 4x^2;\ f'(x) = 4x^3 - 8x;\ f''(x) = 12x^2 - 8$$



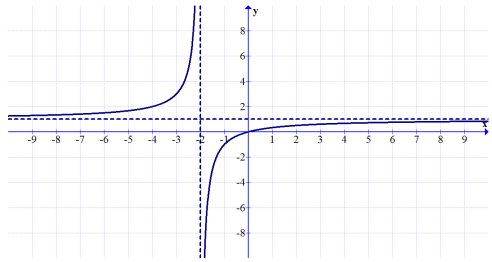
2. $f(x) = x^3 + 7x^2 + 8x - 16$
   (HINT: $f(1) = 0$)

$$f(x) = x^3 + 7x^2 + 8x - 16;\ f'(x) = 3x^2 + 14x + 8;\ f''(x) = 6x + 14$$



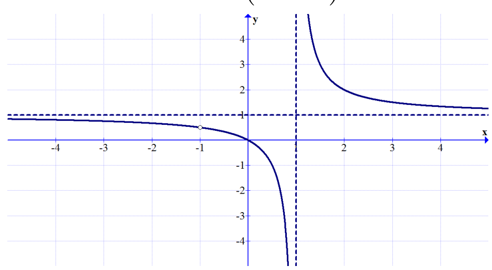
3. $f(x) = \dfrac{x}{x+2}$

$$f(x) = \frac{x}{x+2};\ f'(x) = \frac{2}{(x+2)^2};\ f''(x) = -\frac{4}{(x+2)^3}$$



4. $f(x) = \dfrac{x^2+x}{x^2-1}$

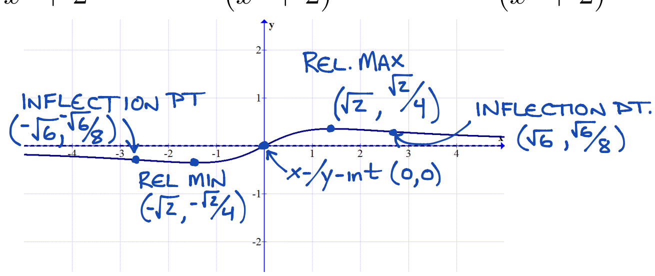
$$f(x) = \frac{x^2+x}{x^2-1};\ f'(x) = -\frac{1}{(x-1)^2};\ f''(x) = \frac{2}{(x-1)^3}$$



NOTE: There is a hole in the graph at the point $\left(-1, \dfrac{1}{2}\right)$

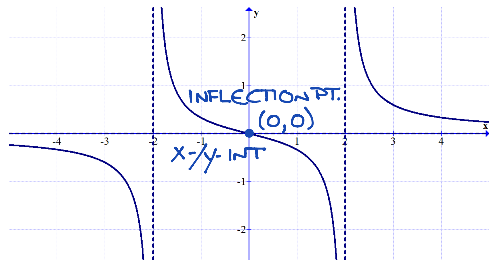
5. $f(x) = \dfrac{x}{x^2+2}$

$$f(x) = \frac{x}{x^2+2};\ f'(x) = \frac{2-x^2}{(x^2+2)^2};\ f''(x) = \frac{2x(x^2-6)}{(x^2+2)^3}$$



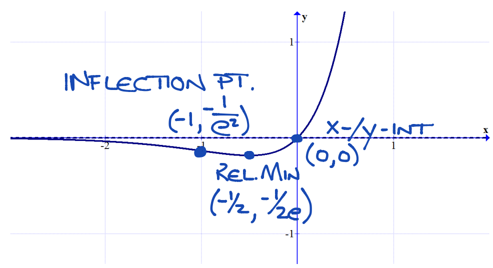
6. $f(x) = \dfrac{x}{x^2-4}$

$$f(x) = \frac{x}{x^2-4};\ f'(x) = -\frac{x^2+4}{(x^2-4)^2};\ f''(x) = \frac{2x(x^2+12)}{(x^2-4)^3}$$



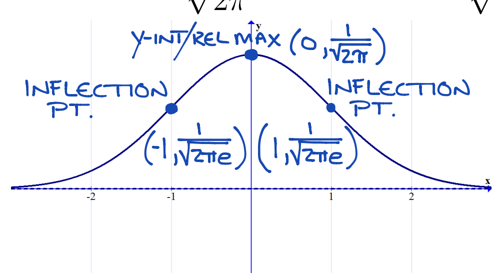
7. $f(x) = xe^{2x}$

$$f(x) = xe^{2x};\ f'(x) = e^{2x}(2x+1);\ f''(x) = 4e^{2x}(x+1)$$



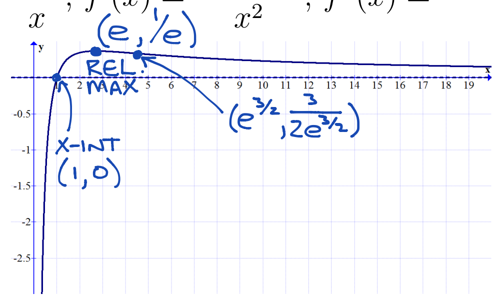
8. $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2};\ f'(x) = -\frac{x}{\sqrt{2\pi}}e^{-x^2/2};\ f''(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}(x^2-1)$$



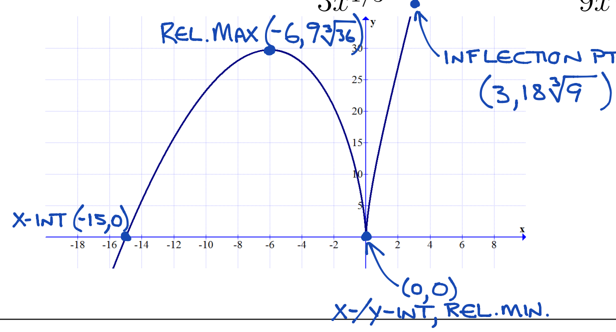
9. $f(x) = \dfrac{\ln x}{x}$

$$f(x) = \frac{\ln x}{x};\ f'(x) = \frac{1-\ln x}{x^2};\ f''(x) = \frac{-3+2\ln x}{x^3}$$



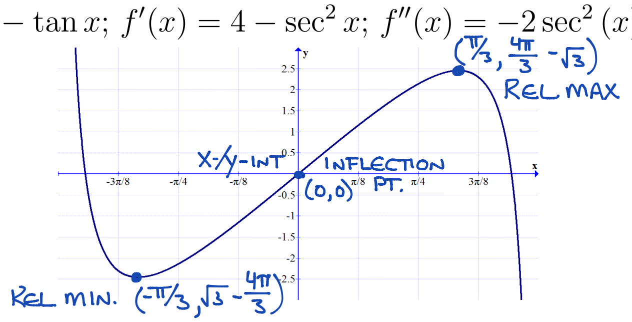
10. $f(x) = x^{2/3}(x+15)$

$$f(x) = x^{2/3}(x+15);\ f'(x) = \frac{5(x+6)}{3x^{1/3}};\ f''(x) = \frac{10(x-3)}{9x^{4/3}}$$



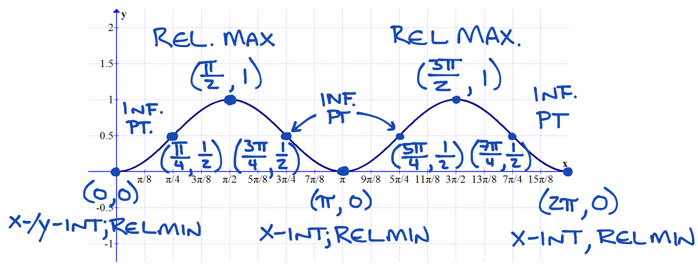
11. $f(x) = 4x - \tan x$ on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$$f(x) = 4x - \tan x;\ f'(x) = 4 - \sec^2 x;\ f''(x) = -2\sec^2(x)\tan(x)$$

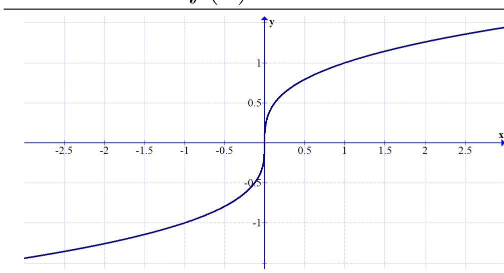
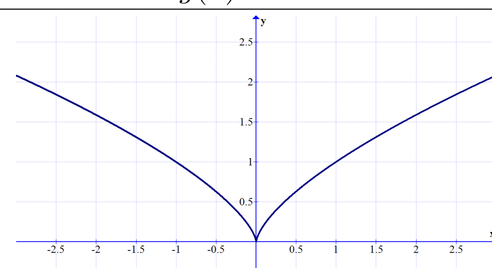


12. $f(x) = \sin^2(x)$ on $[0, 2\pi]$

$$f(x) = \sin^2(x);\ f'(x) = 2\sin x \cos x;\ f''(x) = 4\cos^2 x - 2$$



13. Consider the graphs of $f(x) = x^{1/3}$ and $g(x) = x^{2/3}$. $x_0 = 0$ is a critical point for both $f(x)$ and $g(x)$ since 0 is in the domain of each function but $f'(0)$ and $g'(0)$ are both undefined. How does the behavior of $f(x)$ differ from that of $g(x)$ at this critical point?

$f(x) = x^{1/3}$	$g(x) = x^{2/3}$
	

In both cases, the graph has a vertical tangent line at $x = 0$. However, $g(x)$ has a cusp at $x = 0$. Also, $g(x)$ has a relative (local) minimum at this critical point, whereas $f(x)$ does not.

14. Consider a general quadratic curve $f(x) = ax^2 + bx + c$, where $a \neq 0$. Show that $f(x)$ cannot have any inflection points.

$f''(x) = 2a$ which is always defined and never 0, since $a \neq 0$. So, if $a > 0$, $f(x)$ is always concave up; and, if $a < 0$, $f(x)$ is always concave down.

15. Consider a general quartic curve $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$.

   (a) What is the largest number of distinct inflection points that $f(x)$ could have?
   $\boxed{2}$

   (b) What condition on the coefficients a, b, c, d, and e is necessary for the number of distinct inflection points to be maximized?
   a, b, and c must satisfy $6b^2 - 16ac > 0$; d and e can be any real number.