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$$\frac{4}{k^2+4k+3} = \frac{A}{k+1} + \frac{B}{k+3}$$

$$4 = A(k+3) + B(k+1)$$

$k=-3$: $4 = -2B \Rightarrow B=-2$

$k=-1$: $4 = 2A \Rightarrow A=2$

So $$\sum_{k=0}^{\infty} \frac{4}{k^2+4k+3} = \sum_{k=0}^{\infty} \left(\frac{2}{k+1} - \frac{2}{k+3} \right)$$

Partial sums:

$$S_0 = \frac{2}{1} - \frac{2}{3}$$

$$S_1 = \left(\frac{2}{1} - \frac{2}{3}\right) + \left(\frac{2}{2} - \frac{2}{4}\right) = \frac{2}{1} + \frac{2}{2} - \frac{2}{3} - \frac{2}{4}$$

$$S_2 = \left(\frac{2}{1} - \cancel{\frac{2}{3}}\right) + \left(\frac{2}{2} - \frac{2}{4}\right) + \left(\cancel{\frac{2}{3}} - \frac{2}{5}\right) = \frac{2}{1} + \frac{2}{2} - \frac{2}{4} - \frac{2}{5}$$

$$S_3 = \left(\frac{2}{1} - \cancel{\frac{2}{3}}\right) + \left(\frac{2}{2} - \cancel{\frac{2}{4}}\right) + \left(\cancel{\frac{2}{3}} - \frac{2}{5}\right) + \left(\cancel{\frac{2}{4}} - \frac{2}{6}\right) = \frac{2}{1} + \frac{2}{2} - \frac{2}{5} - \frac{2}{6}$$

$$\vdots$$

$$S_n = \frac{2}{1} + \frac{2}{2} - \frac{2}{n+2} - \frac{2}{n+3}$$

$$\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \left(\frac{2}{1} + \frac{2}{2} - \cancel{\frac{2}{n+2}}^{\,0} - \cancel{\frac{2}{n+3}}^{\,0}\right) = 2 + 1 = 3$$

So the sum of the series is 3.