First-Order Linear Equations (Integrating Factors)

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Be able to solve first-order linear equations by using the appropriate integrating factors.
- Be able to set up and solve application problems using integrating factors.

PRACTICE PROBLEMS:

For problems 1-6, use an integrating factor to solve the given differential equation. Express your answer as an explicit function of x.

1. $\dfrac{dy}{dx} - 4y = e^{5x}$
2. $\dfrac{dy}{dx} + 3x^2y = x^2$
3. $y' = x - 2y$
4. $\dfrac{dy}{dx} - y = \sin\left(e^{-x}\right)$
5. $y' + \dfrac{y}{x\ln x} = x$, for $x > 1$
6. $\dfrac{dy}{dx} + y = \dfrac{1}{e^{2x} - 5e^x + 4}$
7. Look at the First-Order Separable Equations practice problems $3 - 9$ and determine which ODE's, if any, are first-order linear equations. If there are any, solve them using integrating factors.

For problems 8-9, solve the initial value problem. Express your answer as an explicit function of x.

8. $\dfrac{dy}{dx} + \dfrac{1}{x}y = \dfrac{1}{x + x^3}$, for $x > 0$; $y(1) = 0$
9. $(\cos x)\dfrac{dy}{dx} + y\sin x = \sin x\cos x$, for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$; $y(0) = 5$

10. A tank intially contains 7 pounds of salt dissolved in 100 gallons of water. Then, salt water containing 3 pounds of salt per gallon enters the tank at a rate of 8 gallons per minute, and the mixed solution is drained from the tank at a rate of 8 gallons per minute. Let $y = y(t)$ be the amount of salt in the tank at time t.

    (a) Using this information, set up an initial value problem (IVP) whose solution is $y(t)$.

    (b) Using integrating factors, solve the IVP from part (a).

    (c) Using separation of variables, solve the IVP from part (a).

11. Suppose the saltwater solution in problem #10 is drained from the tank at a rate of 6 gallons per minute.

    (a) Set up an initial value problem (IVP) whose solution is $y(t)$. [Hint: The volume of saltwater is no longer a constant, but rather a function of t.]

    (b) Using integrating factors, solve the IVP from part (a).
    [Note that unlike problem #10 the ODE is no longer separable.]

    (c) Suppose that the tank has a capacity of 200 gallons. How much salt is in the tank when it reaches the point of overflowing?

12. Suppose that an object with mass m falls to the earth with a velocity $v = v(t)$ and is subjected to the force of gravity as well as air resistance (which is propotional to its velocity). Using Newton's Second Law it can be shown that

$$m\frac{dv}{dt} = -mg - kv$$

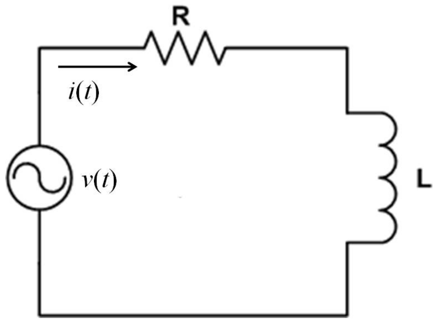
    where g is the acceleration due to gravity and k is some positive constant of proportionality.

    (a) Assuming that the object's initial velocity is v_0, set up an initial value problem (IVP) whose solution is $v(t)$.

    (b) Solve the IVP from part (a).

    (c) Evaluate $\lim_{t\to\infty} v(t)$.

13. Consider the simple electrical circuit shown below. An electromotive force (e.g. a generator) produces a voltage of $V(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t. The circuit also contains a resistor with a constant resistance of R ohms (Ω) and an inductor with a constant inductance of L henries (H). Such a circuit is called an <u>RL circuit</u>.



Using Ohm's Law and Kirchoff's Law it can be shown that

$$L\frac{dI}{dt} + RI = V(t)$$

Suppose that the RL circuit above has a resistance of 6 Ω and an inductance of 3 H. If a generator produces a variable voltage of $V(t) = 9\sin t$ and the initial current is $I(0) = 2$ A, find $I(t)$.

Hint: Recall to solve an integral of the form $\int e^x \sin x \, dx$, use integration by parts twice.