Parametric Equations, Tangent Lines, & Arc Length

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 10.1 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to sketch a parametric curve by eliminating the parameter, and indicate the orientation of the curve.
- Given a curve and an orientation, know how to find parametric equations that generate the curve.
- Without eliminating the parameter, be able to find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at a given point on a parametric curve.
- Be able to find the arc length of a smooth curve in the plane described parametrically.

PRACTICE PROBLEMS:

For problems 1-5, sketch the curve by eliminating the parameter. Indicate the direction of increasing t.

1. $\begin{cases} x = 2t + 3 \\ y = 3t - 4 \\ 0 \le t \le 3 \end{cases}$

2. $\begin{cases} x = 2\cos t \\ y = 3\sin t \\ \pi \le t \le 2\pi \end{cases}$

3. $\begin{cases} x = t - 5 \\ y = \sqrt{t} \\ 0 \le t \le 9 \end{cases}$

4. $\begin{cases} x = \sec t \\ y = \tan^2 t \\ 0 \le t < \dfrac{\pi}{2} \end{cases}$

5. $\begin{cases} x = \sin t \\ y = \cos(2t) \\ -\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2} \end{cases}$

For problems 6-10, find parametric equations for the given curve. (For each, there are many correct answers; only one is provided.)

6. A horizontal line which intersects the y-axis at $y = 2$ and is oriented rightward from $(-1, 2)$ to $(1, 2)$.

7. A circle or radius 4 centered at the origin, oriented clockwise.

8. A circle of radius 5 centered at $(1, -2)$, oriented counter-clockwise.

9. The portion of $y = x^3$ from $(-1, -1)$ to $(2, 8)$, oriented upward.

10. The ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$, oriented counter-clockwise.

For problems 11-13, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the given point without eliminating the parameter.

11. The curve $\begin{cases} x = 3\sin(3t) \\ y = \cos(3t) \\ 0 < t < 2\pi \end{cases}$ at $t = \pi$

12. The curve $\begin{cases} x = t^2 \\ y = 3t - 2 \\ t \geq 0 \end{cases}$ at $t = 1$

13. The curve $\begin{cases} x = 2\tan t \\ y = \sec t \\ 0 \leq t \leq \dfrac{\pi}{3} \end{cases}$ at $t = \dfrac{\pi}{4}$

14. Consider the curve described parametrically by $\begin{cases} x = \sqrt{t} \\ y = \sqrt[3]{t} + 1 \\ t \geq 0 \end{cases}$

    (a) Compute $\left.\dfrac{dy}{dx}\right|_{t=64}$ without eliminating the parameter.

    (b) Eliminate the parameter and verify your answer for part (a) using techniques from differential calculus.

    (c) Compute an equation of the line which is tangent to the curve at the point corresponding to $t = 64$.

15. Consider the curve described parametrically by $\begin{cases} x = 2\cos t \\ y = 4\sin t \\ 0 \le t \le 2\pi \end{cases}$

   (a) Compute $\left.\dfrac{dy}{dx}\right|_{t=\pi/4}$ without eliminating the parameter.

   (b) Eliminate the parameter and verify your answer for part (a) using techniques from differential calculus.

   (c) Compute an equation of the line which is tangent to the curve at the point corresponding to $t = \dfrac{\pi}{4}$.

   (d) At which value(s) of t will the tangent line to the curve be horizontal?

For problems 16-18, compute the length of the given parametric curve.

16. The curve described by $\begin{cases} x = t \\ y = \dfrac{2}{3}t^{3/2} \\ 0 \le t \le 4 \end{cases}$

17. The curve described by $\begin{cases} x = e^t \\ y = \dfrac{2}{3}e^{3t/2} \\ \ln 2 \le t \le \ln 3 \end{cases}$

18. The curve described by $\begin{cases} x = \dfrac{1}{2}t^2 \\ y = \dfrac{1}{3}t^3 \\ 0 \le t \le \sqrt{3} \end{cases}$

19. Compute the lengths of the following two curves:

$$C_1(t) = \begin{cases} x = \cos t \\ y = \sin t \\ 0 \le t \le 2\pi \end{cases} \qquad C_2(t) = \begin{cases} x = \cos(3t) \\ y = \sin(3t) \\ 0 \le t \le 2\pi \end{cases}$$

   Explain why the lengths are not equal even though both curves coincide with the unit circle.

20. This problem describes how you can find the area between a parametrically defined curve and the x-axis.

The Main Idea: Recall that if $y = f(x) \geq 0$, then the area between the curve and the x-axis on the interval $[a, b]$ is $\displaystyle\int_a^b f(x)dx = \int_a^b y\,dx$. Now, suppose that the same curve is described parametrically by $x = x(t)$, $y = y(t)$ for $t_0 \leq t \leq t_1$ and that the curve is traversed exactly once on this interval. Then, $A = \displaystyle\int_a^b y\,dx = \int_{t_0}^{t_1} y(t)x'(t)\,dt$.

Consider the curve $\begin{cases} x = \sin t \\ y = \cos(2t) \\ -\dfrac{\pi}{4} \leq t \leq \dfrac{\pi}{4} \end{cases}$

(a) Compute the area between the graph of the given curve and the x-axis by evaluating $A = \displaystyle\int_{t_0}^{t_1} y(t)x'(t)\,dt$.

(b) After eliminating the parameter to express the curve as an explicitly defined function ($y = f(x)$), calculate the area by evaluating $A = \displaystyle\int_a^b f(x)\,dx$.