Chapter 2.5 Practice Problems

EXPECTED SKILLS:

- Know the derivatives of the 6 elementary trigonometric functions.
- Be able to use these derivatives in the context of word problems.

PRACTICE PROBLEMS:

1. Fill in the given table:

$f(x)$	$f'(x)$
$\sin x$	
$\cos x$	
$\tan x$	
$\cot x$	
$\sec x$	
$\csc x$	

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

2. Use the definition of the derivative to show that $\dfrac{d}{dx}(\cos x) = -\sin x$

   Hint: $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$
\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \left(\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right) \\
&= \lim_{h\to 0} \left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right) \\
&= (\cos x)(0) - (\sin x)(1) \\
&= -\sin x
\end{aligned}
$$

3. Use the quotient rule to show that $\frac{d}{dx}(\cot x) = -\csc^2 x$.

$$\boxed{\begin{aligned} \frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\ &= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} \\ &= -\csc^2 x \end{aligned}}$$

4. Use the quotient rule to show that $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

$$\boxed{\begin{aligned} \frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\ &= \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x} \\ &= -\frac{\cos x}{\sin^2 x} \\ &= -\frac{1}{\sin x}\frac{\cos x}{\sin x} \\ &= -\csc x \cot x \end{aligned}}$$

5. Evaluate $\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{3} + h\right) - \tan\left(\frac{\pi}{3}\right)}{h}$ by interpreting the limit as the derivative of a function at a particular point.

$$\boxed{\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{3} + h\right) - \tan\left(\frac{\pi}{3}\right)}{h} = \frac{d}{dx}(\tan x)\bigg|_{x=\frac{\pi}{3}} = \sec^2\left(\frac{\pi}{3}\right) = 4}$$

For problems 6-14, differentiate

6. $f(x) = 2\cos x + 4\sin x$

$\boxed{-2\sin x + 4\cos x}$

7. $f(x) = 5\cos x + \cot x$

$\boxed{-5\sin x - \csc^2 x}$

8. $g(x) = 4\csc x + 2\sec x$

$\boxed{-4\csc(x)\cot(x) + 2\sec(x)\tan(x)}$

9. $f(x) = \sin x \cos x$

$\boxed{\cos^2 x - \sin^2 x}$

10. $f(x) = \dfrac{\sin^2 x}{\cos x}$

$\boxed{2\sin x + \sin x \tan^2 x}$

11. $f(x) = x^3 \sin x$

$\boxed{3x^2 \sin x + x^3 \cos x}$

12. $f(x) = \sec^2 x + \tan^2 x$

$\boxed{4\sec^2(x)\tan(x)}$

13. $f(x) = \dfrac{x + \sec x}{1 + \cos x}$

$\boxed{\dfrac{1 + 2\tan x + \cos x + \sec(x)\tan(x) + x\sin x}{(1+\cos x)^2}}$

For problems 14-17, compute $\dfrac{d^2y}{dx^2}$

14. $f(x) = \tan x$

$\boxed{2\sec^2 x \tan x}$

15. $f(x) = \sin x$

$\boxed{-\sin x}$

16. $f(x) = \cos^2 x$

$\boxed{2\sin^2 x - 2\cos^2 x}$

17. $f(x) = \sin^2 x + \cos^2 x$

$\boxed{0}$

For problems 18-19, find all values of x in the interval $[0, 2\pi]$ where the graph of the given function has horizontal tangent lines.

18. $f(x) = \sin x \cos x$

    $\boxed{\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}}$

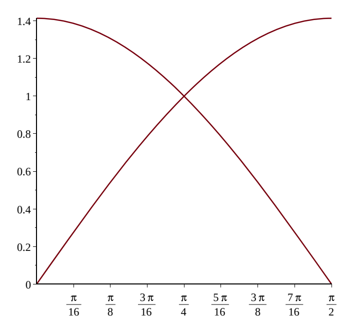
19. $g(x) = \csc x$

    $\boxed{\dfrac{\pi}{2}, \dfrac{3\pi}{2}}$

20. Compute an equation of the line which is tangent to the graph of $f(x) = \dfrac{\cos x}{x}$ at the point where $x = \pi$.

    $\boxed{y = \dfrac{1}{\pi^2}x - \dfrac{2}{\pi}}$

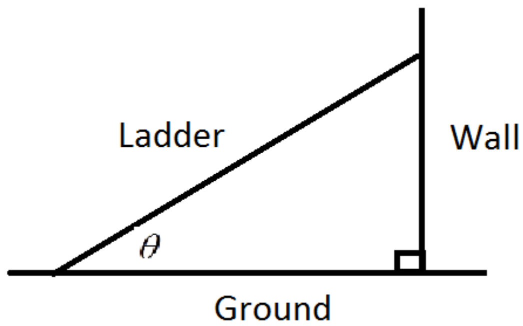
21. Consider the graphs of $f(x) = \sqrt{2}\cos(x)$ and $g(x) = \sqrt{2}\sin(x)$ shown below on the interval $\left[0, \dfrac{\pi}{2}\right]$.



Show that the graphs of $f(x)$ and $g(x)$ intersect at a right angle when $x = \dfrac{\pi}{4}$. (Hint: Show that the tangent lines to f and g at $x = \dfrac{\pi}{4}$ are perpendicular to each other.)

$f'\left(\dfrac{\pi}{4}\right) = -1$ and $g'\left(\dfrac{\pi}{4}\right) = 1$. So, the tangent lines to f and g at $x = \dfrac{\pi}{4}$ are perpendicular to one another since the product of their slopes is -1.

22. A 15 foot ladder leans against a vertical wall at an angle of θ with the horizontal, as shown in the figure below. The top of the ladder is h feet above the ground. If the ladder is pushed towards the wall, find the rate at which h changes with respect to θ at the instant when $\theta = 30°$. Express your answer in **feet/degree**.



$$\boxed{\frac{dh}{d\theta} = \frac{15\sqrt{3}}{2} \text{ ft/radian} = \frac{\pi\sqrt{3}}{24} \text{ ft/degree}}$$

23. Use the Intermediate Value Theorem to show that there is at least one point in the interval $(0, 1)$ where the graph of $f(x) = \sin x - \frac{1}{3}x^3$ will have a horizontal tangent line.

$f'(x) = \cos x - x^2$. Firstly, notice that $f'(x)$ is continuous for all x; therefore, it is continuous for all x in $[0, 1]$. Secondly, notice that $f'(0) = 1 > 0$ and $f'(1) = \cos(1) - 1 < 0$. Thus, the Intermediate Value Theorem states there is at least one x_0 in the interval $(0, 1)$ with $f'(x_0) = 0$. In other words, there is at least one x_0 in $(0, 1)$ where $f(x)$ will have a horizontal tangent line.

24. **Multiple Choice:** At how many points on the interval $[-\pi, \pi]$ is the tangent line to the graph of $y = 2x + \sin x$ parallel to the secant line which passes through the graph endpoints of the interval?

(a) 0
(b) 1
(c) 2
(d) 3
(e) None of these

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