The Fundamental Theorem of Calculus

SUGGESTED REFERENCE MATERIAL:

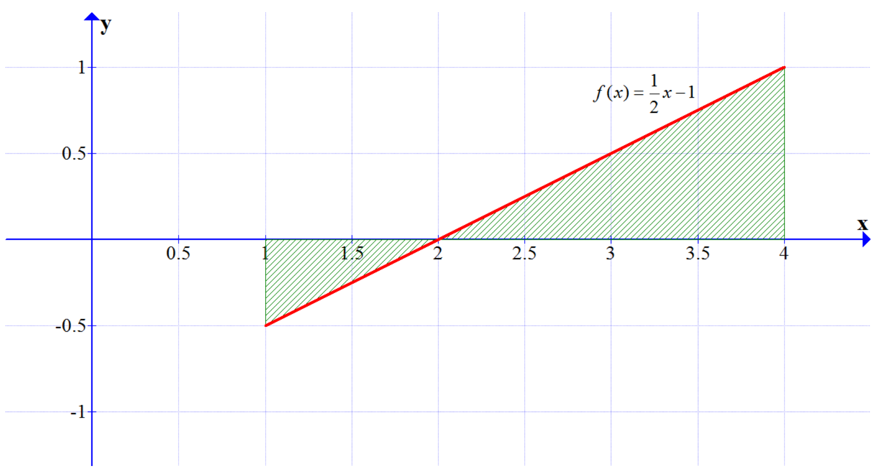
As you work through the problems listed below, you should reference Chapter 5.6 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to use one part of the Fundamental Theorem of Calculus (FTC) to evaluate definite integrals via antiderivatives.
- Know how to use another part of the FTC to compute derivatives of functions defined as integrals.

PRACTICE PROBLEMS:

1. Consider the graph of $f(x) = \dfrac{1}{2}x - 1$ on $[1, 4]$, shown below.



   (a) Use a definite intergal and the Fundamental Theorem of Calculus to compute the net signed area between the graph of $f(x)$ and the x-axis on the interval $[1, 4]$.

   (b) Verify your answer from part (a) by using appropriate formulae from geometry.

For problems 2-4, sketch a region whose net signed area is equivalent to the value of the given definite integral. Then evaluate the definite integral using any method.

2. $\displaystyle\int_0^8 (x^2 - 4x - 5)\,dx$

3. $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx$

4. $\displaystyle\int_{-4}^{-1} \frac{2}{x^3}\,dx$

For problems 5-15, evaluate the given definite integral.

5. $\displaystyle\int\limits_4^{25} \frac{1}{x\sqrt{x}}\,dx$

6. $\displaystyle\int_{-e}^{-1} \frac{x+1}{x}\,dx$

7. $\displaystyle\int_{\ln 2}^{\ln 3} e^{2x}\,dx$

8. $\displaystyle\int\limits_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \csc(x)\cot(x)\,dx$

9. $\displaystyle\int\limits_0^{\sqrt{3}} \frac{3}{1+x^2}\,dx$

10. $\displaystyle\int\limits_{-9}^{9} |x-5|\,dx$

11. $\displaystyle\int\limits_1^{e^6} \frac{1}{10x}\,dx$

12. $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$

13. $\displaystyle\int_0^{\pi} |\cos x|\, dx$

14. $\displaystyle\int_0^3 f(x)\, dx$ if $f(x) = \begin{cases} x+5 & \text{if } x \leq 1 \\ 4x+2 & \text{if } x > 1 \end{cases}$

15. $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\, dx$. (HINT: Use a trigonometric identity first to rewrite the integrand.)

16. $\boxed{\textbf{Definitions:}}$ If an object moves along a straight line with position function $s(t)$, its velocity function is $v(t) = s'(t)$. Then:

- The displacement from time t_1 to time t_2 is the net change of position of the particle during the time period from t_1 to t_2 and is calculated by evaluating $\displaystyle\int_{t_1}^{t_2} v(t)\, dt$.
- The total distance traveled from time t_1 to time t_2 is calculated by evaluating $\displaystyle\int_{t_1}^{t_2} |v(t)|\, dt$.

Assume that a particle is moving along a straight line such that its velocity at time t is $v(t) = t^2 - 6t + 5$ (meters per second).

(a) Compute the displacement of the particle during the time period $0 \leq t \leq 6$.

(b) Compute the total distance traveled by the particle during the time period $0 \leq t \leq 6$.

17. The following Riemann Sum was derived by dividing an interval $[a, b]$ into n subintervals of equal width and then choosing x_k^* to be the right endpoint of each subinterval.

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \left(1 + \frac{4}{n}k\right) \frac{4}{n}$$

(a) What is the interval, $[a, b]$?

(b) Convert the Riemann Sum to an equivalent definite integral.

(c) Using the definite integral from part (b) and part of the Fundamental Theorem of Calculus, evaluate the limit.

18. Explain what is wrong with the following calculation:

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_{x=-1}^{x=1} = -1 - (1) = -2$$

For problems 19-22, use part of the Fundamental Theorem of Calculus to compute the indicated derivative.

19. $\dfrac{d}{dx}\displaystyle\int_{2}^{x} \ln(t)\,dt$

20. $\dfrac{d}{dx}\displaystyle\int_{x}^{10} e^{t^2}\,dt$

21. $\dfrac{d}{dx}\displaystyle\int_{\pi}^{3x^2} \cos t\,dt$

22. $\dfrac{d}{dx}\displaystyle\int_{2}^{e^x} \ln(t)\,dt$

23. Consider $F(x) = \displaystyle\int_4^x \sqrt[3]{t^2+11}\,dt$. Compute each of the following:

    (a) $F(4)$

    (b) $F'(4)$

    (c) $F''(4)$

24. Let $F(x) = \displaystyle\int_1^x t\ln t\,dt$, for $x > 0$.

    (a) Find the open interval(s) on which $F(x)$ is increasing and those on which $F(x)$ is decreasing.

    (b) Find all points (x, y) where the graph of $F(x)$ has a local (relative) maximum or a local (relative) minimum.

    (c) Find the interval(s) on which $F(x)$ is concave up and those on which $F(x)$ is concave down.

    (d) Determine the x-value(s) of each inflection point of $F(x)$.