

# Triple Integrals

## SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapters 14.5 & 14.6 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

## EXPECTED SKILLS:

- Be able to set up and evaluate triple integrals over rectangular boxes.
- Know how to set up and evaluate triple integrals over more general regions by using Theorem 14.5.2 (projecting the solid onto the  $xy$ -plane), as well as by projecting the solid onto the  $xz$ - or  $yz$ -planes.
- Be able to set up and evaluate triple integrals in spherical and cylindrical coordinates. Also, be able to convert integrals from rectangular coordinates to these other coordinate systems, remembering that  $dV = r \, dz \, dr \, d\theta = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$ .

## PRACTICE PROBLEMS:

1. Evaluate the following triple integrals.

(a)  $\int_1^3 \int_0^1 \int_0^z y e^{-z^3} \, dy \, dz \, dx$

$$\boxed{\frac{1}{3} \left( 1 - \frac{1}{e} \right)}$$

(b)  $\iiint_G z \sqrt{y} \, dV$ , where  $G$  is the solid enclosed by  $z = y$ ,  $y = x^2$ ,  $y = 4$ , and the  $xy$ -plane.

$$\boxed{64}$$

2. Consider the region  $G$  in 3-space which is enclosed by  $z = 0$ ,  $z = 1$ ,  $x = 0$ ,  $y = x$ , and  $y = 1 - x$ . For each of the following, set up  $\iiint_G f(x, y, z) \, dV$  with the indicated order of integration. Sketching  $G$  may be helpful.

(a)  $dz \, dy \, dx$

$$\boxed{\int_0^{1/2} \int_x^{1-x} \int_0^1 f(x, y, z) \, dz \, dy \, dx}$$

(b)  $dz dx dy$

$$\int_0^{1/2} \int_0^y \int_0^1 f(x, y, z) dz dx dy + \int_{1/2}^1 \int_0^{1-y} \int_0^1 f(x, y, z) dz dx dy$$

(c)  $dy dz dx$

$$\int_0^{1/2} \int_0^1 \int_x^{1-x} f(x, y, z) dy dz dx$$

(d)  $dy dx dz$

$$\int_0^1 \int_0^{1/2} \int_x^{1-x} f(x, y, z) dy dx dz$$

(e)  $dx dz dy$

$$\int_0^{1/2} \int_0^1 \int_0^y f(x, y, z) dx dz dy + \int_{1/2}^1 \int_0^1 \int_0^{1-y} f(x, y, z) dx dz dy$$

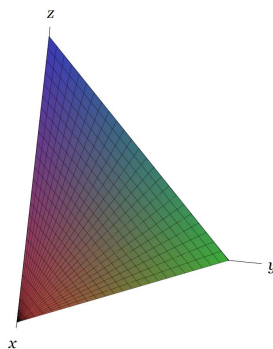
(f)  $dx dy dz$

$$\int_0^1 \int_0^{1/2} \int_0^y f(x, y, z) dx dy dz + \int_0^1 \int_{1/2}^1 \int_0^{1-y} f(x, y, z) dx dy dz$$

3. Consider the integral  $I = \int_0^7 \int_0^{3-\frac{3}{7}x} \int_0^{21-3x-7y} 1 dz dy dx$ .

(a) Sketch a solid whose volume is equivalent to the value of  $I$ .

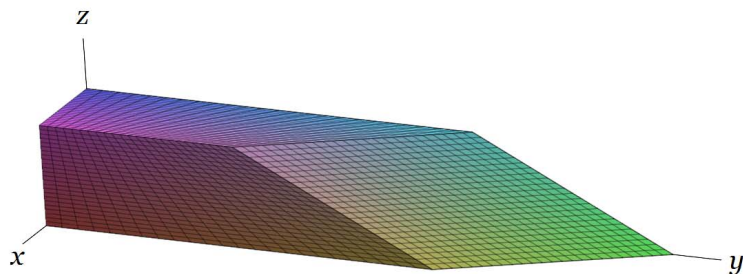
The integral represents the volume of the solid in the first octant which is enclosed by the plane  $3x + 7y + z = 21$  and the coordinate axes.



(b) Reverse the order of integration to  $dy dz dx$ .

$$\int_0^7 \int_0^{21-3x} \int_0^{3-\frac{3}{7}x-\frac{1}{7}z} 1 dy dz dx$$

4. The solid below is enclosed by  $x = 0$ ,  $x = 1$ ,  $y = 0$ ,  $z = 0$ ,  $z = 1$ , and  $2x + y + 2z = 6$ .



- (a) Set up a triple integral or triple integrals with the order of integration as  $dydx dz$  which represent(s) the volume of the solid.

$$\int_0^1 \int_0^1 \int_0^{6-2x-2z} 1 \, dy \, dx \, dz$$

- (b) Set up a triple integral or triple integrals with the order of integration as  $dzdydx$  which represent(s) the volume of the solid.

$$\int_0^1 \int_0^{4-2x} \int_0^1 1 \, dz \, dy \, dx + \int_0^1 \int_{4-2x}^{6-2x} \int_0^{3-x-\frac{1}{2}y} 1 \, dz \, dy \, dx$$

5. Use a triple integral to calculate the volume of the solid which is bounded by  $z = 3 - x^2$ ,  $z = 2x^2$ ,  $y = 0$ , and  $y = 1$ .

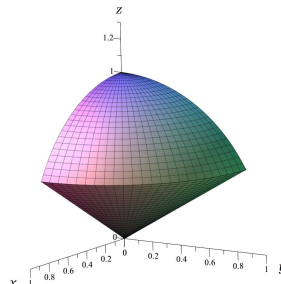
4; Detailed Solution: [Here](#)

6. Use a triple integral to calculate the volume of the solid which is bounded by  $z = y + 4$ ,  $z = 0$ , and  $x^2 + y^2 = 4$ .

$16\pi$

7. The integral  $\int_0^{\pi/2} \int_0^{\pi/3} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$  is given in spherical coordinates. Sketch a solid whose volume is represented by the value of this integral.

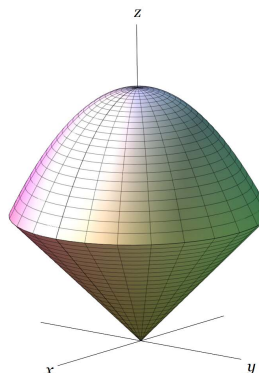
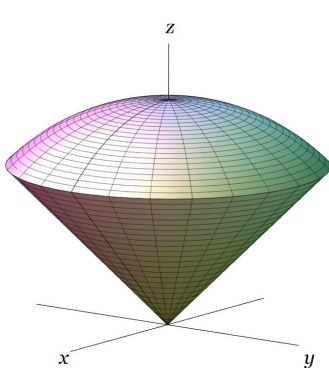
The integral can be interpreted as the volume of the solid in the first octant which is contained within the sphere  $x^2 + y^2 + z^2 = 1$ , the cone  $z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$ , and the planes  $x = 0$  and  $y = 0$ .



8. Mario and Luigi run a concession stand in Wario Stadium, “Missed Jump Creamery,” and their top selling product is ice cream cones. Mario’s ice cream cones can be modeled by the solid which is bounded above by  $x^2 + y^2 + z^2 = 4$  and is bounded below by  $z = \sqrt{x^2 + y^2}$ . Luigi’s ice cream cones can be modeled by the solid which is bounded above by  $z = 2 - x^2 - y^2$  and bounded below by  $z = \sqrt{x^2 + y^2}$ .

Mario’s Ice Cream Cone

Luigi’s Ice Cream Cone



(Images are not to scale)

- (a) Set up a triple integral in rectangular coordinates which represents the volume of Mario’s ice cream cones.

$$V_M = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 1 \, dz \, dy \, dx$$

- (b) Set up a triple integral in cylindrical coordinates which represents the volume of Mario’s ice cream cones.

$$V_M = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

- (c) Set up a triple integral in spherical coordinates which represents the volume of Mario's ice cream cones.

$$V_M = \int_0^{\pi/4} \int_0^{2\pi} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

- (d) Set up a triple integral in rectangular coordinates which represents the volume of Luigi's ice cream cones.

$$V_L = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{2-x^2-y^2} 1 \, dz \, dy \, dx$$

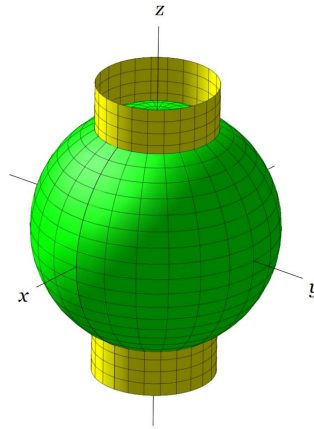
- (e) Set up a triple integral in cylindrical coordinates which represents the volume of Luigi's ice cream cones.

$$V_L = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r \, dz \, dr \, d\theta$$

- (f) Determine whose ice cream cones have the larger volume.

$$V_M = \frac{16\pi}{3} \left(1 - \frac{\sqrt{2}}{2}\right) \approx 4.907; \quad V_L = \frac{5\pi}{6} \approx 2.618; \quad \text{So, Mario's ice cream cones have the larger volume.}$$

9. Consider the surfaces  $x^2 + y^2 + z^2 = 16$  and  $x^2 + y^2 = 4$ , shown below.



- (a) Set up a triple integral in cylindrical coordinates which can be used to calculate the volume of the solid which is inside of  $x^2 + y^2 + z^2 = 16$  but outside of  $x^2 + y^2 = 4$ .

$$V = \int_0^{2\pi} \int_2^4 \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta$$

- (b) Set up a triple integral in spherical coordinates which can be used to calculate the volume of the solid which is inside of  $x^2 + y^2 + z^2 = 16$  but outside of  $x^2 + y^2 = 4$ .

$$V = \int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{2 \csc \phi}^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

- (c) Calculate the volume of the solid which is inside of  $x^2 + y^2 + z^2 = 16$  but outside of  $x^2 + y^2 = 4$  by evaluating one of your integrals from parts (a) or (b).

$$32\pi\sqrt{3}$$

10. Convert the integral  $\int_0^4 \int_{-\sqrt{4-(y-2)^2}}^{\sqrt{4-(y-2)^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz \, dz \, dx \, dy$  from rectangular coordinates to cylindrical coordinates.

$$\int_0^\pi \int_0^{4 \sin \theta} \int_0^{\sqrt{4-r^2}} r^3 z \cos \theta \sin \theta \, dz \, dr \, d\theta$$

11. Consider the integral  $\int_{-3}^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} (x^2 + y^2) \, dz \, dx \, dy$ .

- (a) Convert the given integral from rectangular coordinates to cylindrical coordinates.

$$\int_{-\pi/2}^{\pi/2} \int_0^3 \int_0^{\sqrt{9-r^2}} r^3 \, dz \, dr \, d\theta; \text{ Detailed Solution: } [Here](#)$$

- (b) Convert the given integral from rectangular coordinates to spherical coordinates.

$$\int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_0^3 \rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi; \text{ Detailed Solution: } [Here](#)$$