Chapter 3.6: Limits & Continuity of Trig. Functions

Expected Skills:

- Know where the trigonometric functions are continuous and be able to evaluate basic trigonometric limits.
- Be able to use $\lim_{x \to 0} \frac{\sin x}{x} = 1$ or $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ to help find the limits of functions involving trigonometric expressions, when appropriate.
- Understand the squeeze theorem and be able to use it to compute certain limits.

Practice Problems:

Evaluate the following limits using the squeeze theorem.

1. Let $f(x)$ be a function which satisfies $5x - 6 \leq f(x) \leq x^2 + 3x - 5$ for all $x \geq 0$. Compute $\lim_{x \to 1} f(x)$.

2. $\lim_{x \to \infty} \frac{x + \cos x}{3x + 1}$

For problems 3-19, evaluate the given limit. If a limit does not exist, write DNE, $+\infty$, or $-\infty$ (whichever is most appropriate).

3. $\lim_{x \to \frac{\pi}{4}} \sin(2x)$

4. $\lim_{\theta \to \pi} (\theta \cos \theta)$

5. $\lim_{x \to 0^+} \csc x$

6. $\lim_{x \to \frac{\pi}{2}^+} \tan x$

7. $\lim_{x \to \frac{\pi}{2}^-} \tan x$

8. $\lim_{x \to \frac{\pi}{4}} \sec x$

9. $\lim_{x \to 0} \left(\frac{\sin x}{3x} \right)$

10. $\lim_{x \to 0} \left(\frac{\sin 3x}{3x} \right)$

11. $\lim_{x \to 0} \left(\frac{\sin x}{|x|} \right)$

12. $\lim_{x \to 0} \left(\frac{1 - \cos x}{4x} \right)$

13. $\lim_{x \to 0^-} \left(\frac{\cos x}{x} \right)$

14. $\lim_{x \to 0} \left(\frac{\sin 2x}{x} \right)$

15. $\lim_{x \to 0} \left(\frac{\tan 2x}{x} \right)$

16. $\lim_{x \to 0} \left(\frac{1 - 3\cos x}{3x} \right)$

17. $\lim_{x \to 0} \left(\frac{3x^2}{1 - \cos^2 x} \right)$

18. $\lim_{x \to 0} \left(\frac{\tan 5x}{\sin 9x} \right)$

For problems 19-20, evaluate the given limit by making an appropriate substitution (change of variables). If a limit does not exist, write DNE, $+\infty$, or $-\infty$ (whichever is most appropriate).

19. $\lim_{x \to 8} \frac{\sin(x-8)}{x^2 - 64}$

20. $\lim_{x \to \infty} x \sin\left(\frac{2}{x} \right)$

For problem 21-23, determine the value(s) of x where the given function is continuous.

21. $f(x) = \csc x$

22. $f(x) = \frac{1}{1 - 2\cos x}$ on $[0, 2\pi]$

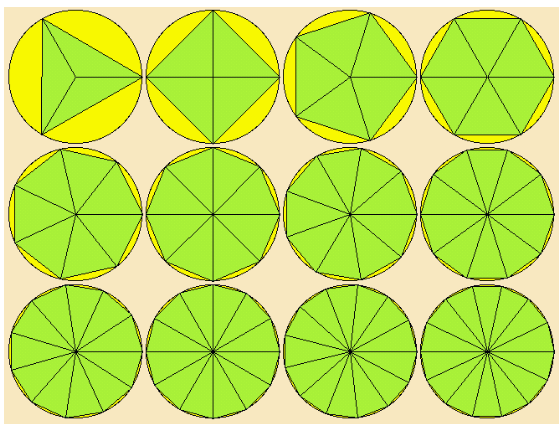
23. $f(x) = \begin{cases} \cos x & \text{if} \quad x < \frac{\pi}{4} \\ \sin x & \text{if} \quad x \geq \frac{\pi}{4} \end{cases}$

24. Find all non-zero value(s) of k so that $f(x) = \begin{cases} \dfrac{3\sin(kx)}{x} & \text{if} \quad x > 0 \\ 6k^2 + 5x & \text{if} \quad x \leq 0 \end{cases}$ is continuous at $x = 0$.

25. Use the Intermediate Value Theorem to prove that there is at least one solution to $\cos x = x^2$ in $(0, 1)$.

26. Let x be a fixed real number. Compute $\lim\limits_{h \to 0} \dfrac{\sin(x+h) - \sin x}{h}$. (Hint: The identity $\sin(A+B) = \sin A \cos B + \cos A \sin B$ will be useful.)

27. A Regular Polygon is a polygon that is equiangular (all angles are equal in measure) and equilateral (all sides have the same length). The diagram below shows several regular polygons inscribed within a circle of radius r.



   (a) Let A_n be the area of a regular n-sided polygon inscribed within a circle of radius r. Divide the polygon into n congruent triangles each with a central angle of $\dfrac{2\pi}{n}$ radians, as shown in the diagram above for several different values of n. Show that $A_n = \dfrac{1}{2} r^2 \sin\left(\dfrac{2\pi}{n}\right) n$.

   (b) What can you conclude about the area of the n-sided polygon as the number of sides of the polygon, n, approaches infinity? In other words, compute $\lim\limits_{n \to \infty} A_n$.