Chapter 4.3: Hyperbolic Functions

Expected Skills:

- Be able to define $\sinh x$ and $\cosh x$ in terms of exponential functions.
- Be able to determine the domain, range, and graph of $\sinh x$ and $\cosh x$.
- Be able to justify properties and solve equations involving the hyperbolic functions.
- Be able to compute limits and derivatives involving the hyperbolic functions.

Practice Problems:

1. Consider $f(x) = \sinh x$.

   (a) Compute $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$

   $\boxed{\lim\limits_{x\to\infty} f(x) = +\infty;\ \lim\limits_{x\to-\infty} f(x) = -\infty}$

   (b) Determine whether the graph of $f(x)$ has any curvilinear asymptotes.

   $\boxed{y = \frac{1}{2}e^x \text{ and } y = -\frac{1}{2}e^{-x}}$

   (c) Compute the x and y intercepts of $f(x)$.

   $\boxed{\text{The } x \text{ and } y \text{ intercept of } y = \sinh x \text{ is } (0,0).}$

   (d) Solve $\sinh x = 1$ for x.

   $\boxed{x = \ln\left(1+\sqrt{2}\right)}$

   (e) Show that $\frac{d}{dx}(\sinh x) = \cosh x$

$$\boxed{\begin{aligned}
\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right) \\
&= \frac{1}{2}e^x + \frac{1}{2}e^{-x} \\
&= \frac{e^x + e^{-x}}{2} \\
&= \cosh x
\end{aligned}}$$

(f) Find all intervals on which $f(x)$ is increasing and those on which $f(x)$ is decreasing.

$\boxed{\text{Increasing on } (-\infty, \infty)\text{; never decreasing.}}$

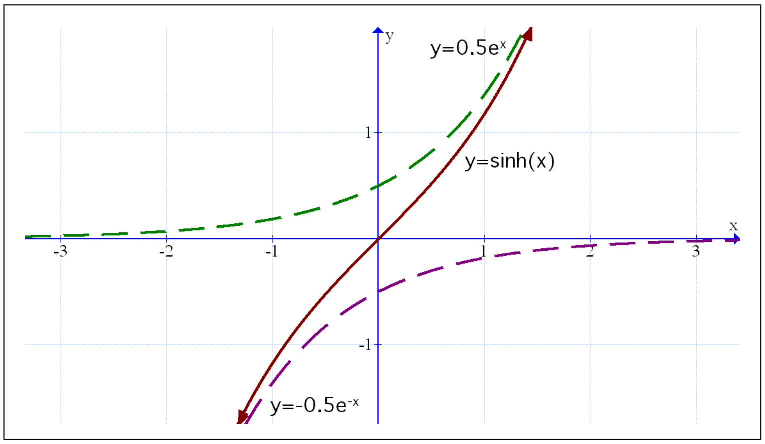
(g) Find all intervals on which $f(x)$ is concave up and those on which $f(x)$ is concave down.

$\boxed{\text{Concave up on } (0, \infty)\text{; concave down on } (-\infty, 0).}$

(h) Determine the coordinates of all local extrema (max/min) and all inflection points.

$\boxed{\text{No max/min; Inflection point at } (0, 0).}$

(i) Sketch the graph of $f(x)$



2. In order to verify the identity $\sinh 2x = 2\sinh x \cosh x$ compute the following by appealing to the appropriate definitions.

(a) $\sinh 2x$

$$\boxed{\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}}$$

(b) $2\sinh x \cosh x$

$$\boxed{2\sinh x \cosh x = 2\left(\frac{e^{x} + e^{-x}}{2}\right)\left(\frac{e^{x} - e^{-x}}{2}\right) = \frac{e^{2x} - e^{-2x}}{2}}$$

3. We define the **Hyperbolic Tangent** function to be $f(x) = \tanh x = \dfrac{\sinh x}{\cosh x}$.

(a) Express $f(x)$ in terms of exponential functions.

$$\boxed{\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}}$$

(b) What is the domain of $f(x)$?

$\boxed{(-\infty, \infty)}$

(c) Compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

$\boxed{\lim_{x\to\infty} f(x) = 1 \text{ and } \lim_{x\to-\infty} f(x) = -1}$

(d) Determine whether the graph of $f(x)$ has any curvilinear asymptotes.

$\boxed{\text{The graph has horizontal asymptotes } y = 1 \text{ and } y = -1.}$

(e) Find the coordinates of all x and y intercepts of $f(x)$.

$\boxed{(0, 0)}$

(f) Find all x for which $f(x) = \frac{1}{2}$.

$\boxed{\frac{1}{2}\ln 3}$

4. Compute an equation of the line which is tangent to $f(x) = \cosh x$ at the point where $x = \ln 2$.

$\boxed{y - \frac{5}{4} = \frac{3}{4}(x - \ln 2)}$

5. Suppose $y = x\cosh x$. Compute $\frac{d^2y}{dx^2}$.

$\boxed{\frac{d^2y}{dx^2} = 2\sinh(x) + x\cosh(x)}$