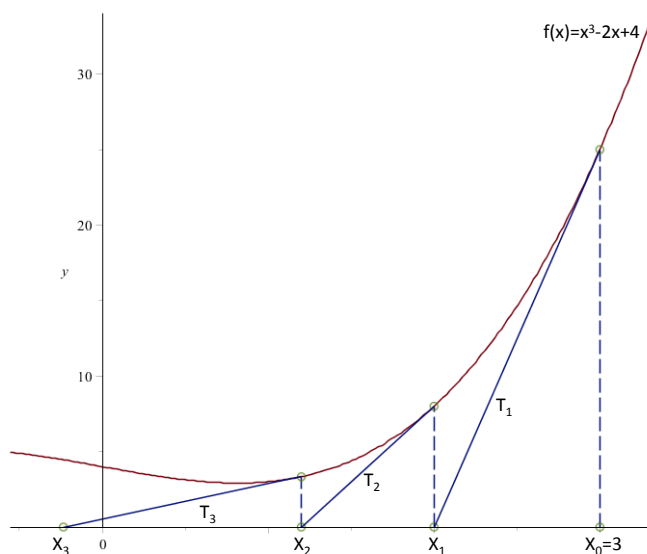# Chapter 1.2: Newton's Method

Expected Skills:

- Be able to apply Newton's Method to approximate a solution to $f(x) = 0$.

- Be able to use different stopping procedures to exit the Newton's Method algorithm, as described in the notes.

Practice Problems:

1. The following diagram shows $f(x) = x^3 - 2x + 4$ and the first 3 iterations of Newton's Method when initialized with $x_0 = 3$.

(a) Start at $x_0 = 3$ and do the following.

- Find an equation of T_1, the tangent line to the graph at x_0? Find the x-intercept of T_1 and call this value x_1.

$$\boxed{T_1 : y = 25x - 50. \ x_1 = 2}$$

- Find an equation of T_2, the tangent line to the graph at x_1? Find the x-intercept of T_2 and call this value x_2.

$$\boxed{T_2 : y = 10x - 12. \ x_2 = 1.2}$$

- Find an equation of T_3, the tangent line to the graph at x_2? Find the x-intercept of T_3 and call this value x_3.

$$\boxed{T_3 : y = \frac{58}{25}x - \frac{68}{125}. \ x_3 = -\frac{34}{145} \approx -.2344827586}$$

(b) After 3 iterations of Newton's Method, what is a approximate solution to $x^3 - 2x + 4 = 0$?

$$x_3 = -\frac{34}{145} \approx -.2344827586$$

(c) It turns out that the only real solution to $x^3 - 2x + 4 = 0$ is $x = -2$. If you were to perform another iteration, would you move closer to or farther away from the actual solution? Explain.

You would move further away. The tangent line to the graph of $f(x)$ at x_3 will intersect the positive x-axis.

(d) It turns out that we would need 22 iterations of Newton's Method to satisfy either of the following stopping criteria:

- Let $\epsilon = 0.1$. If $|f(x_k)| < \epsilon$, stop. Else, perform another iteration.
- let $\epsilon = 0.1$. If $|x_k - x_{k-1}| < \epsilon$, stop. Else, perform another iteration.

Perhaps initializing the algorithm with a different choice of x_0 will perform better. Choose $x_0 = 1$ and perform Newton's Method until your resulting estimate is "reasonably close" to the actual solution of $x = -2$. What is your approximation and how many iterations did you perform?

As you can see in the following diagram, it will only take one iteration of Newton's Method to obtain the exact solution of $x = 2$. Notice that the initialization of the algorithm can have a big impact on the number of iterations. Also, if there were multiple solutions to the equation, the initial guess could determine which of the solutions is reached.
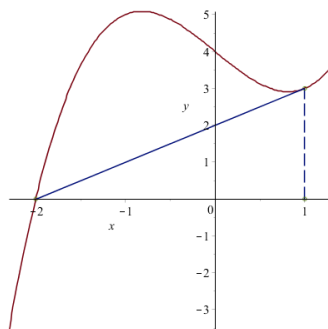
2. Use Newton's Method to estimate a solution to $x^3 + 7x - 5 = 0$ using the stopping procedures listed below. Initialize the algorithm with $x_0 = 3$. In each case, what is an estimate of the desired solution?

(a) Stop after 3 iterations.

The solution is approximately 0.693829439.

(b) Let $\epsilon = 0.1$. Use the stopping algorithm: "If $|f(x_k)| < \epsilon$, stop. Else, perform another iteration."

The solution is approximately 0.671232257.

(c) Again let $\epsilon = 0.1$. Use the stopping algorithm: "If $|x_k - x_{k-1}| < \epsilon$, stop. Else, perform another iteration."

The solution is approximately 0.671232257.

3. Estimate $\sqrt{3}$ using Newton's Method. Initialize your search with $x_0 = 1$ and use the stopping procedures listed below. In each case, what is the estimated value of $\sqrt{3}$ and how many iterations were required? (Hint: find the positive value of x such that $x^2 = 3$.)

(a) Stop after 3 iterations.

The solution is approximately 1.732142857.

(b) Let $\epsilon = 0.1$. Use the stopping algorithm: "If $|f(x_k)| < \epsilon$, stop. Else, perform another iteration."

The solution is approximately 1.75 after 2 iterations.

(c) Again let $\epsilon = 0.1$. Use the stopping algorithm: "If $|x_k - x_{x-1}| < \epsilon$, stop. Else, perform another iteration."

The solution is approximately 1.732142857 after 3 iterations.

4. Newton's Method will fail to when trying to solve $x^{1/3} = 0$.

(a) Fill in the following table:

k	x_k	$|f(x_k)|$	$|x_k - x_{k-1}|$
0	1		$-$
1			
2			
3			
4			
5			

| k	x_k	$|f(x_k)|$	$|x_k - x_{k-1}|$
0	1	1	$-$
1	-2	1.25992105	3
2	4	1.587401052	6
3	-8	2	12
4	16	2.5198421	24
5	-32	3.174802104	48

(b) From one iteration to the next, what is happening to $|f(x_k)|$? What is an interpretation of this?

From one iteration to the next $|f(x_k)|$ gets larger. As a result, the y-values of $y = x^{1/3}$ are getting farther away from 0 (which is the opposite of what we need).

(c) From one iteration to the next, what is happening to $|x_k - x_{k-1}|$)? What is an interpretation of this?

From one iteration to the next $|x_k - x_{k-1}|$ gets larger. As a result, the approximations are getting farther apart (which is the opposite of what we need).

(d) Sketch $y = x^{1/3}$ and the results from the first 3 iterations of Newton's Method.
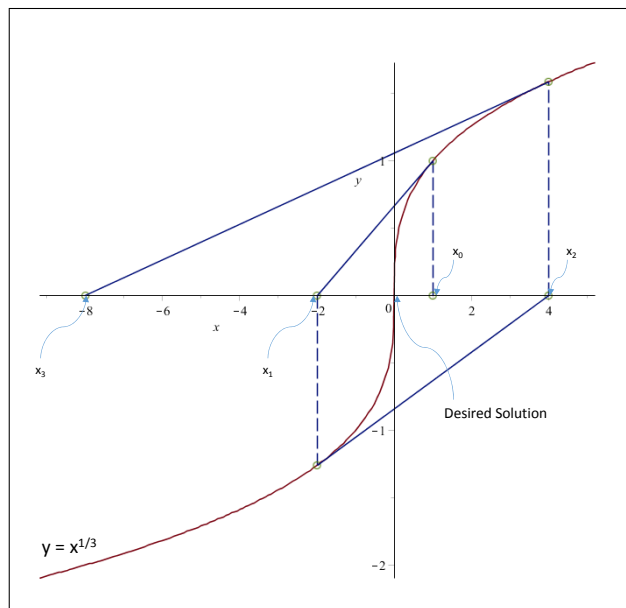
5. The **Secant Method** is a modification of Newton's Method in which an approximation of the derivative is used rather than the derivative itself. Consider the following diagram:
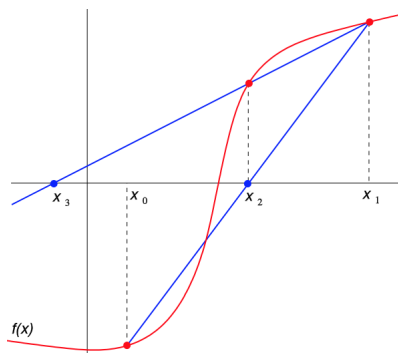
The algorithm starts by choosing x_0 and x_1 "close" to the actual solution to $f(x) = 0$. To get the next estimate, x_2, one finds the x-intercept of the secant line through

$(x_0, f(x_0))$ and $(x_1, f(x_1))$. If this estimate is sufficiently good, the algorithm terminates. Else, the process is repeated where x_3 is the x-intercept of the secant line through $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Determine the general update formula for x_{x+1} if you know x_k and x_{k-1}.

Note: This method can sometimes be faster than Newton's Method since we only have to compute the function value rather than both the function and derivative values. It is also similar to the False position method discussed in question 6 of the Bisection Method HW; but, unlike the False position method, in the secant method the solution need not remain bracketed by the endpoints x_{k-1} and x_k. So, there can be convergence issues.

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$