The Definite Integral

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 5.5 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to evaluate the definite integral of a function over a given interval using geometry.
- Be familiar with the interpretation of the definite integral of a function over an interval as the net signed area between the graph of the function and the x-axis.
- Know how to use linearity properties of the definite integral to evaluate scalar multiples, sums, and differences of integrable functions.

PRACTICE PROBLEMS:

For problems 1 & 2, use the given values of a and b to express the given limit as a definite integral. Do not evaluate the limits or integrals.

1. $\displaystyle\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \frac{1}{1+(x_k^*)^2} \Delta x_k$, $a=-1$, $b=1$.

$$\boxed{\int_{-1}^{1} \frac{1}{1+x^2}\, dx}$$

2. $\displaystyle\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \cos(x_k^*) \Delta x_k$, $a=0$, $b=\pi$.

$$\boxed{\int_{0}^{\pi} \cos x\, dx}$$

For problems 3-9, sketch the region whose net signed area is represented by the given definite integral. Evaluate the given integral using an appropriate formula from geometry.

3. $\displaystyle\int_0^7 (x+1)\, dx$

$$\boxed{\frac{63}{2}}$$

4. $\displaystyle\int_{-7}^{7} x\,dx$

$\boxed{0}$

5. $\displaystyle\int_{-1}^{4} 6\,dx$

$\boxed{30}$

6. $\displaystyle\int_{-4}^{2} |x-1|\,dx$

$\boxed{13}$

7. $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx$

$\boxed{2\pi}$

8. $\displaystyle\int_{-2}^{0} \left(3x + 5\sqrt{4-x^2}\right)dx$

$\boxed{-6+5\pi\text{; Detailed Solution: Here}}$

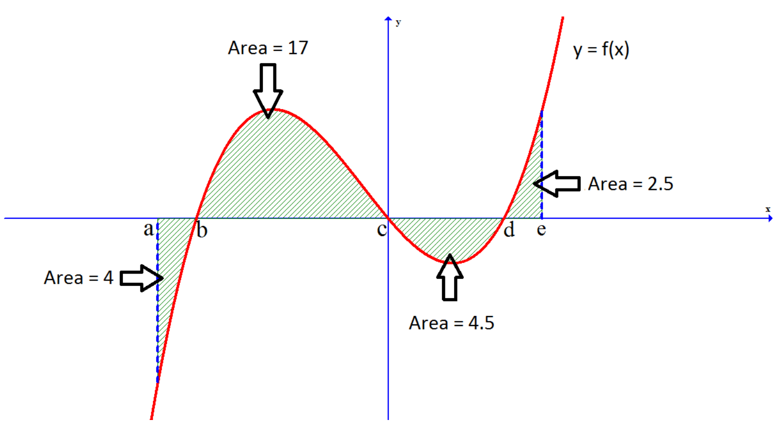
9. $\displaystyle\int_{4}^{8} \sqrt{8x-x^2}\,dx$

$\boxed{4\pi}$

10. Let $f(x) = \begin{cases} 3x & \text{if} \quad x \le 2 \\ 6 & \text{if} \quad x > 2 \end{cases}$. Compute $\displaystyle\int_{-1}^{5} f(x)\,dx$.

$\boxed{\dfrac{45}{2}\text{; Detailed Solution: Here}}$

11. For each of the following, use the areas shown to evaluate the given definite integral.



(a) $\displaystyle\int_b^c f(x)\,dx$

$\boxed{17}$

(b) $\displaystyle\int_c^d f(x)\,dx$

$\boxed{-4.5}$

(c) $\displaystyle\int_a^e f(x)\,dx$

$\boxed{11}$

(d) $\displaystyle\int_b^a f(x)\,dx$

$\boxed{4}$

12. Again consider the graph of $y = f(x)$ shown in problem 11. Compute $\displaystyle\int_a^e |f(x)|\,dx$ and $\displaystyle\left|\int_a^e f(x)\,dx\right|$. Which is larger?

$\boxed{\displaystyle\int_a^e |f(x)|\,dx = 28;\ \left|\int_a^e f(x)\,dx\right| = 11;\ \int_a^e |f(x)|\,dx \text{ is larger.}}$

13. Suppose that $\displaystyle\int_{-1}^3 f(x)\,dx = 6$ and $\displaystyle\int_{-1}^3 g(x)\,dx = -8$. Compute $\displaystyle\int_{-1}^3 (f(x) + 4g(x))\,dx$.

$\boxed{-26}$

14. Suppose that $\displaystyle\int_0^8 f(x)\,dx = 3$ and $\displaystyle\int_4^8 f(x)\,dx = 10$. Compute $\displaystyle\int_0^4 f(x)\,dx$.

$\boxed{-7}$

15. Suppose that $\displaystyle\int_{-2}^9 f(x)\,dx = 4$ and $\displaystyle\int_{-2}^6 f(x)\,dx = 11$. Compute $\displaystyle\int_9^6 f(x)\,dx$.

7; Video Solution: https://www.youtube.com/watch?v=IBZspjDFQmY

16. Express each of the following in terms of $\displaystyle\int_0^\pi \sin x\,dx$. **Do not evaluate any of the integrals.** Hint: Draw a graph and consider the net signed area.

(a) $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx$.

$\boxed{\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx = -\int_0^\pi \sin x\,dx}$

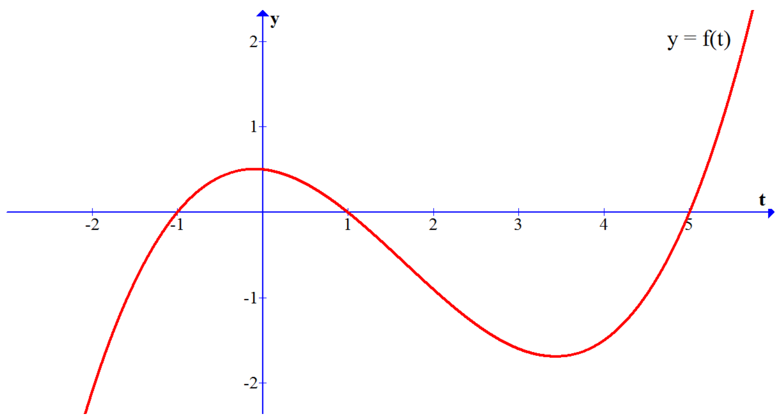
(b) $\displaystyle\int_{\frac{3\pi}{2}}^{2\pi} \cos x\, dx.$

$$\boxed{\int_{\frac{3\pi}{2}}^{2\pi} \cos x\, dx = \frac{1}{2}\int_0^{\pi} \sin x\, dx}$$

(c) $\displaystyle\int_{\pi}^{\frac{3\pi}{2}} \sin x\, dx.$

$$\boxed{\int_{\pi}^{\frac{3\pi}{2}} \sin x\, dx = -\frac{1}{2}\int_0^{\pi} \sin x\, dx}$$

17. Suppose $F(x) = \displaystyle\int_0^x f(t)\, dt$, where $f(t)$ is shown below.



Arrange the following quantities in order from least to greatest. $F(0)$, $F(1)$, $F(5)$, $F(1) - F(5)$, $F(5) - F(1)$

$$\boxed{F(5) - F(1) < F(5) < F(0) < F(1) < F(1) - F(5)}$$

18. The following Riemann Sum was derived by dividing an interval $[a, b]$ into n subintervals of equal width and then choosing x_k^* to be the right endpoint of each subinterval.

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \left(1 + \frac{4}{n}k\right) \frac{4}{n}$$

(a) What is the interval, $[a, b]$?

$\boxed{\text{If we consider } f(x) = x\text{, then the interval is } [1, 5].}$

(b) Convert the Riemann Sum to an equivalent definite integral.

$$\boxed{\lim_{n\to+\infty} \sum_{k=1}^{n} \left(1 + \frac{4}{n}k\right) \frac{4}{n} = \int_1^5 x\, dx}$$

(c) Using the definite integral from part (b) and an appropriate formula from geometry, evaluate the limit.

$\boxed{12}$

NOTE: In number 18, there are many correct answers. For example, we could have considered $f(x) = 1 + x$. In that case, $[a, b] = [0, 4]$ and $\lim_{n\to+\infty} \sum_{k=1}^{n} \left(1 + \frac{4}{n}k\right) \frac{4}{n} = \int_0^4 (1 + x)\, dx$. The value of this definite integral is also 12.