Length of a Plane Curve (Arc Length)

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 6.4 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to find the arc length of a smooth curve in the plane described as a function of x or as a function of y.

PRACTICE PROBLEMS:

For problems 1-3, compute the exact arc length of the curve over the given interval.

1. $y = 4x^{\frac{3}{2}} - 1$ from $x = \dfrac{1}{12}$ to $x = \dfrac{2}{9}$

   $\boxed{\dfrac{19}{54}}$

2. $y = \dfrac{x^2}{2} - \dfrac{\ln(x)}{4}$ for $2 \le x \le 4$

   $\boxed{6 + \frac{1}{4}\ln 2;\ \text{Detailed Solution: Here}}$

3. $y = \dfrac{2}{3}(x^2 - 1)^{3/2}$ for $1 \le x \le 3$

   $\boxed{\dfrac{46}{3}}$

4. Consider the curve defined by $y = \sqrt{4 - x^2}$ for $0 \le x \le 2$.

   (a) Compute the arc length on the interval $[0, t]$ for $0 \le t < 2$. (Your arc length will depend on t.)

   $\boxed{2\sin^{-1}\left(\dfrac{t}{2}\right)}$

   (b) Use your answer from part (a) to compute the arc length on the interval $[0, 2]$. (Hint: You will need to introduce a limit.)

   $\boxed{\pi}$

(c) Confirm your answer from part (b) by using geometry.

On the interval $[0, 2]$, the curve is $\frac{1}{4}$ of a circle with a radius of 2. So, the length should be $\frac{1}{4}$ of the circumference; that is, Length $= \frac{1}{4} \cdot 2\pi r\Big|_{r=2} = \frac{1}{4} \cdot 2\pi(2) = \pi$.

5. Consider $F(x) = \int_1^x \sqrt{t^2 - 1}\,dt$. Compute the arc length on $[1, 3]$

4; Detailed Solution: Here

6. Consider the curve defined by $f(x) = \ln x$ on $\left[1, e^3\right]$

(a) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to x.

$$L = \int_1^{e^3} \sqrt{1 + \frac{1}{x^2}}\,dx$$

(b) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to y.

$$L = \int_0^3 \sqrt{1 + e^{2y}}\,dy$$

7. Consider the curve defined by $f(x) = \tan x$ on $\left[-\frac{\pi}{3}, \frac{\pi}{4}\right]$

(a) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to x.

$$L = \int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} \sqrt{1 + \sec^4 x}\,dx$$

(b) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to y.

$$L = \int_{-\sqrt{3}}^{1} \sqrt{1 + \frac{1}{(1 + y^2)^2}}\,dy$$

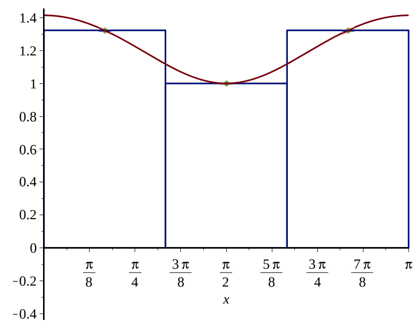
8. Consider the curve defined by $y = \sin x$ for $0 \le x \le \pi$.

(a) Set up but do not evaluate an integral which represents the length of the curve.

$$\int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx$$

(b) Estimate the value of your integral from part (a) by using a Midpoint Approximation with three rectangles of equal width.

Below is the graph of $y = \sqrt{1+\cos^2 x}$ on the interval $[0, \pi]$ along with three rectangles of equal width whose heights were determined by the function value at the midpoint of each resulting subinterval.

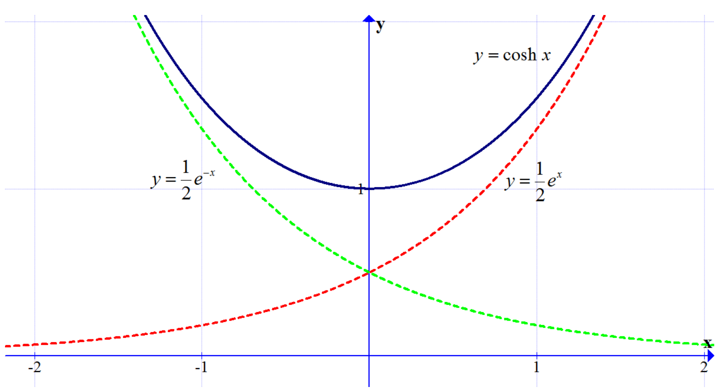


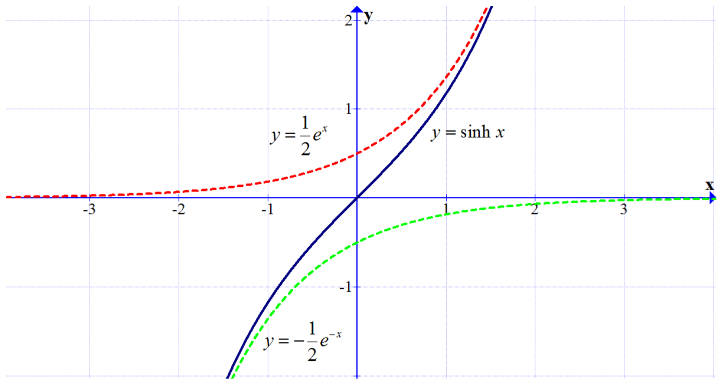
Using these rectangles, $\displaystyle\int_0^{\pi} \sqrt{1+\cos^2 x}\, dx \approx \frac{\pi}{3}\left(1+\sqrt{7}\right)$

9. Recall the definitions of Hyperbolic Sine & Hyperbolic Cosine from Math 121:

Hyperbolic Sine	Hyperbolic Cosine
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\cosh x = \dfrac{e^x + e^{-x}}{2}$

The sketches of $y = \cosh x$ and $y = \sinh x$ are shown below. The dashed curves are called "Curvilinear Asymptotes," which describe the end behavior of the functions.





(a) Show that $\cosh^2 x - \sinh^2 x = 1$

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= (\cosh x + \sinh x)(\cosh x - \sinh x)\\ &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)\\ &= (e^x)(e^{-x})\\ &= 1\end{aligned}$$

(b) Verify that $f(x) = \sinh x$ is an odd function. (**Hint:** Recall an odd function satisfies the identity $f(-x) = -f(x)$.)

To verify that a function is off, we check that $f(-x) = -f(x)$. We compute by appealing to the definition of $\sinh x$ from above.

$$\begin{aligned}\sinh(-x) &= \frac{e^{-x} - e^{-(-x)}}{2}\\ &= \frac{e^{-x} - e^{x}}{2}\\ &= -\left(\frac{e^x - e^{-x}}{2}\right)\\ &= -\sinh x\end{aligned}$$

Thus, $f(x) = \sinh x$ is odd.

(c) Show that $\frac{d}{dx}(\sinh x) = \cosh x$ and deduce that $\int \cosh x\,dx = \sinh x + C$.

$$\begin{aligned}
\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right) \\
&= \frac{1}{2}e^x + \frac{1}{2}e^{-x} \\
&= \frac{e^x + e^{-x}}{2} \\
&= \cosh x
\end{aligned}$$

Thus, as a result, $\int \cosh x\,dx = \sinh x + C$. (We could have also verified this integration formula by integrating the given definition of $\cosh x$.)

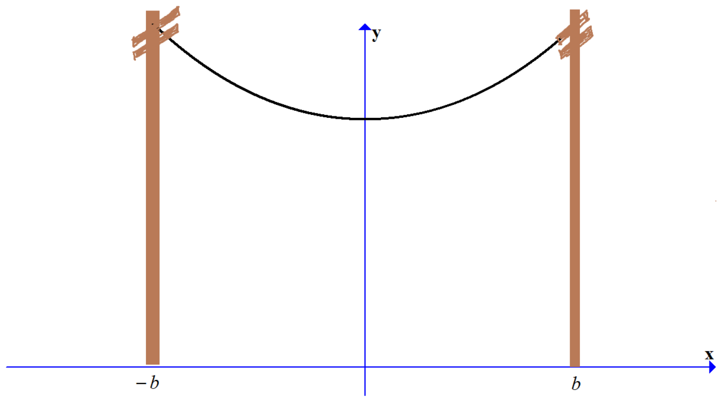
(d) Show that $\frac{d}{dx}(\cosh x) = \sinh x$ and deduce that $\int \sinh x\,dx = \cosh x + C$.

$$\begin{aligned}
\frac{d}{dx}(\cosh x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right) \\
&= \frac{1}{2}e^x - \frac{1}{2}e^{-x} \\
&= \frac{e^x - e^{-x}}{2} \\
&= \sinh x
\end{aligned}$$

Thus, as a result, $\int \sinh x\,dx = \cosh x + C$. (We could have also verified this integration formula by integrating the given definition of $\sinh x$.)

(e) A telephone wire which is supported only by two telephone poles will sag under its own weight and form the shape of a **catenary** as shown below.

Consider a telephone wire that is supported by two poles (one at $x = b$ and the other at $x = -b$), as in the diagram below.



The shape of the sagging wire can be modeled by $y = a \cosh\left(\frac{x}{a}\right)$, where $a > 0$ and $-b \le x \le b$. What is the length of the wire?

$$\boxed{A = 2a \sinh\left(\frac{b}{a}\right)}$$