Differentiating and Integrating Power Series

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Know (i.e. memorize) the Maclaurin series for $e^x, \sin x$ and $\cos x$. Algebraically manipulate these series expansions, as well as other given power series expansions, to form new expansions.
- Differentiate and integrate power series expansions term-by-term.
- Use a series expansion to approximate an integral to some specified accuracy.

PRACTICE PROBLEMS:

1. Confirm that $\dfrac{d}{dx}(e^x) = e^x$ by differentiating the Maclaurin series for e^x term-by-term.
2. Recall that the Maclaurin series for e^x converges to e^x for all real numbers x. It can be shown (in a complex analysis course) that this convergence holds for any complex number as well. Based on this fact, use the Maclaurin series for $e^x, \sin x$, and $\cos x$ to prove Euler's Formula:
$$e^{ix} = \cos x + i \sin x,$$
where i is the imaginary number with the property $i^2 = -1$ (and thus $i^3 = -i$, $i^4 = 1$, etc).
3. The purpose of this problem is to find the Maclaurin series for $\arctan x$. If we attempt to take successive derivatives of $\arctan x$ the computation becomes unpleasant rather quickly (try it if you want). Here is a simpler alternative.
   (a) Find the Maclaurin series for $\frac{1}{1-x}$.
   (b) Replace x in part (a) with the appropriate quantity to obtain the Maclaurin series for $\frac{1}{1+x^2}$.
   (c) Integrate the answer in part (b) term-by-term to obtain the Maclaurin series for $\arctan x$.
4. Find the first four nonzero terms of the Maclaurin series for $f(x) = e^{(x^2)} \arctan x$ by multiplying the Maclaurin series of the factors. See the previous problem for the Maclaurin series for $\arctan x$.
5. Consider the function $f(x) = \sin x \cos x$.

(a) Find the first three nonzero terms of the Maclaurin series for $f(x)$ by multiplying the Maclaurin series of the factors.

(b) Confirm your answer in part (a) by using the trigonometric identity $\sin 2x = 2 \sin x \cos x$.

6. Find the first three nonzero terms of the Maclaurin series for $f(x) = \tan x$ by performing a long division on the Maclaurin series for $\sin x$ and $\cos x$.

7. Use the result in #6 to find the first three nonzero terms of the Maclaurin series for $f(x) = \sec^2 x$.

8. Use a Maclaurin series to approximate $\displaystyle\int_0^1 \cos(x^2)\ dx$ to four decimal-place accuracy.

9. Use a Maclaurin series to approximate $\displaystyle\int_0^1 \arctan(x^2)\ dx$ to two decimal-place accuracy. See problem #3 for the Maclaurin series for $\arctan x$.