Polynomial Approximations of Functions

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Find and use the local linear and local quadratic approximations of a function $f(x)$ at a specified $x = x_0$.
- Determine the Maclaurin polynomials of various degrees for a function $f(x)$, and use sigma notation to write the n-th Maclaurin polynomial.
- Determine the Taylor polynomials of various degrees for a function $f(x)$ at a specified $x = x_0$, and use sigma notation to write the n-th Taylor polynomial.

PRACTICE PROBLEMS:

1. Consider the function $f(x) = \sqrt{x}$.

   (a) Find the local linear approximation $p_1(x)$ and the local quadratic approximation $p_2(x)$ to $f(x)$ at $x = 4$.

   $p_1(x) = 2 + \dfrac{1}{4}(x-4)$

   $p_2(x) = 2 + \dfrac{1}{4}(x-4) - \dfrac{1}{64}(x-4)^2$

   Note that $p_1(x)$ and $p_2(x)$ are just the 1st and 2nd Taylor polynomials for $f(x)$ about $x = 4$.

   (b) Approximate $\sqrt{4.1}$ using your answers in part (a).

   $p_1(4.1) = 2 + \dfrac{1}{4}(4.1-4) = \dfrac{81}{40} = 2.025$

   $p_2(4.1) = 2 + \dfrac{1}{4}(4.1-4) - \dfrac{1}{64}(4.1-4)^2 = \dfrac{81}{40} - \dfrac{1}{6400} = 2.02484375$

   Calculator: $\sqrt{4.1} \approx 2.024845673$.

For problems 2 – 4, use the appropriate local linear and local quadratic approximations to approximate the following values.

2. $\sin 0.1$

   $p_1(x) = p_2(x) = x$, so $\sin 0.1 \approx 0.1$
   Calculator: $\sin 0.1 \approx 0.09983341665$

3. $\sqrt[3]{28}$

$p_1(x) = 3 + \frac{1}{27}(x-27)$, so $\sqrt[3]{28} \approx p_1(28) = \frac{82}{27} = 3.\overline{037}$

$p_2(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2$, so $\sqrt[3]{28} \approx p_2(28) = \frac{82}{27} - \frac{1}{2187} \approx 3.03657979$

Calculator: $\sqrt[3]{28} \approx 3.036588972$.

4. $\tan 44°$

$p_1(x) = 1 + 2\left(x - \frac{\pi}{4}\right)$, so $\tan 44° \approx p_1\left(\frac{\pi}{4} - \frac{\pi}{180}\right) = 1 - \frac{\pi}{90} \approx 0.965093415$

$p_2(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2$, so

$\tan 44° \approx p_2\left(\frac{\pi}{4} - \frac{\pi}{180}\right) = 1 - \frac{\pi}{90} + \frac{2\pi^2}{(180)^2} \approx 0.9657026498$

Calculator: $\tan 44° \approx 0.9656887747$.

5. Suppose that the values of $f(x)$ and its first four derivatives at $x = 0$ are as follows:

$f(0) = 5 \quad f'(0) = -2 \quad f''(0) = 0 \quad f'''(0) = -1 \quad f^{(4)}(0) = 12$

Based on this information, list out as many Maclaurin polynomials for $f(x)$ as possible.

$p_0(x) = 5$

$p_1(x) = p_2(x) = 5 - 2x$

$p_3(x) = 5 - 2x - \frac{1}{6}x^3$

$p_4(x) = 5 - 2x - \frac{1}{6}x^3 + \frac{1}{2}x^4$

6. Find the 4th Maclaurin polynomial $p_4(x)$ for the function $f(x) = 2x^4 - x^3 + 6$.

$p_4(x) = 2x^4 - x^3 + 6$. Why does it make sense that $p_4(x) = f(x)$?

For problem 7, find the Macluarin polynomials $p_0(x), p_1(x), p_2(x), p_3(x)$, and $p_4(x)$. Then write the n-th Maclaurin polynomial $p_n(x)$ using sigma notation.

7. $f(x) = \ln(1 + x)$

$p_0(x) = 0$

$p_1(x) = x$

$p_2(x) = x - \frac{1}{2}x^2$

$p_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

$p_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

$p_n(x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k}$ [Does this series look familiar? Try plugging in $x = 1$.]

For problems 8 & 9, find the Taylor polynomials $p_0(x), p_1(x), p_2(x), p_3(x)$, and $p_4(x)$ about $x = x_0$. Then write the n-th Taylor polynomial $p_n(x)$ at $x = x_0$ using sigma notation.

8. $f(x) = \dfrac{1}{1-x}$; $x_0 = 2$

$p_0(x) = -1$

$p_1(x) = -1 + (x-2)$

$p_2(x) = -1 + (x-2) - (x-2)^2$

$p_3(x) = -1 + (x-2) - (x-2)^2 + (x-2)^3$

$p_4(x) = -1 + (x-2) - (x-2)^2 + (x-2)^3 - (x-2)^4$

$p_n(x) = \sum_{k=0}^{n} (-1)^{k+1} (x-2)^k$

9. $f(x) = e^{2x}$; $x_0 = \ln 3$

$$p_0(x) = 9$$

$$p_1(x) = 9 + 18(x - \ln 3)$$

$$p_2(x) = 9 + 18(x - \ln 3) + 18(x - \ln 3)^2$$

$$p_3(x) = 9 + 18(x - \ln 3) + 18(x - \ln 3)^2 + 12(x - \ln 3)^3$$

$$p_4(x) = 9 + 18(x - \ln 3) + 18(x - \ln 3)^2 + 12(x - \ln 3)^3 + 6(x - \ln 3)^4$$

$$p_n(x) = \sum_{k=0}^{n} \frac{2^k(9)}{k!}(x - \ln 3)^k$$