The Fundamental Theorem of Calculus

SUGGESTED REFERENCE MATERIAL:

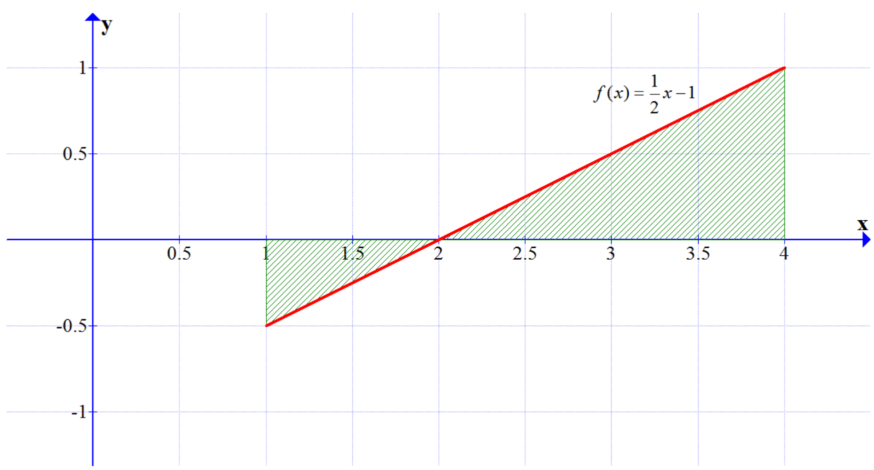
As you work through the problems listed below, you should reference Chapter 5.6 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to use one part of the Fundamental Theorem of Calculus (FTC) to evaluate definite integrals via antiderivatives.
- Know how to use another part of the FTC to compute derivatives of functions defined as integrals.

PRACTICE PROBLEMS:

1. Consider the graph of $f(x) = \dfrac{1}{2}x - 1$ on $[1, 4]$, shown below.



   (a) Use a definite intergal and the Fundamental Theorem of Calculus to compute the net signed area between the graph of $f(x)$ and the x-axis on the interval $[1, 4]$.

   $$\boxed{\int_1^4 \left(\frac{1}{2}x - 1\right) dx = \frac{3}{4}}$$

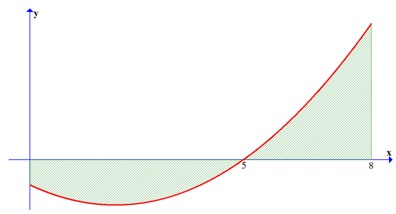
   (b) Verify your answer from part (a) by using appropriate formulae from geometry.

   $A_{\text{lower triangle}} = \dfrac{1}{4}$; $A_{\text{upper triangle}} = 1$;

   Thus, the value of the definite integral is $-A_{\text{lower triangle}} + A_{\text{upper triangle}} = \dfrac{3}{4}$

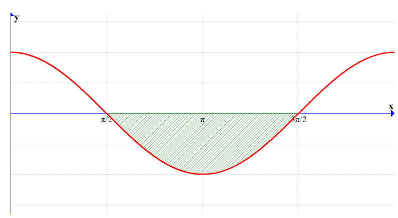
For problems 2-4, sketch a region whose net signed area is equivalent to the value of the given definite integral. Then evaluate the definite integral using any method.

2. $\displaystyle\int_0^8 (x^2 - 4x - 5)\,dx$



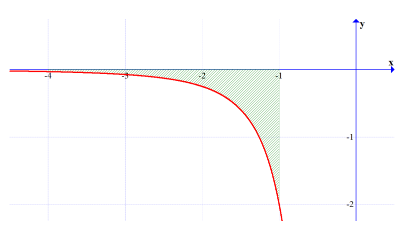
$$\int_0^8 (x^2 - 4x - 5)\,dx = \frac{8}{3}$$

3. $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx$



$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx = -2$$

4. $\displaystyle\int_{-4}^{-1} \frac{2}{x^3}\,dx$



$$\int_{-4}^{-1} \frac{2}{x^3}\,dx = -\frac{15}{16}$$

For problems 5-15, evaluate the given definite integral.

5. $$\int_{4}^{25} \frac{1}{x\sqrt{x}}\, dx$$

$$\boxed{\frac{3}{5}}$$

6. $$\int_{-e}^{-1} \frac{x+1}{x}\, dx$$

$\boxed{-2+e\text{; Detailed Solution: Here}}$

7. $$\int_{\ln 2}^{\ln 3} e^{2x}\, dx$$

$$\boxed{\frac{5}{2}}$$

8. $$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \csc(x)\cot(x)\, dx$$

$$\boxed{1-\frac{2}{\sqrt{3}}}$$

9. $$\int_{0}^{\sqrt{3}} \frac{3}{1+x^2}\, dx$$

$\boxed{\pi}$

10. $$\int_{-9}^{9} |x-5|\, dx$$

$\boxed{106}$

11. $$\int_{1}^{e^6} \frac{1}{10x}\, dx$$

$$\boxed{\frac{3}{5}}$$

12. $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$

$\boxed{\dfrac{\pi}{12}}$

13. $\displaystyle\int_0^{\pi} |\cos x|\,dx$

$\boxed{\text{2; Video Solution: https://www.youtube.com/watch?v=3M-TfaGLFnI}}$

14. $\displaystyle\int_0^3 f(x)\,dx$ if $f(x) = \begin{cases} x+5 & \text{if } x \leq 1 \\ 4x+2 & \text{if } x > 1 \end{cases}$

$\boxed{\dfrac{51}{2}}$

15. $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$. (HINT: Use a trigonometric identity first to rewrite the integrand.)

$\boxed{1 - \dfrac{\pi}{4}}$

16. **Definitions:** If an object moves along a straight line with position function $s(t)$, its velocity function is $v(t) = s'(t)$. Then:

- The displacement from time t_1 to time t_2 is the net change of position of the particle during the time period from t_1 to t_2 and is calculated by evaluating $\displaystyle\int_{t_1}^{t_2} v(t)\,dt$.
- The total distance traveled from time t_1 to time t_2 is calculated by evaluating $\displaystyle\int_{t_1}^{t_2} |v(t)|\,dt$.

Assume that a particle is moving along a straight line such that its velocity at time t is $v(t) = t^2 - 6t + 5$ (meters per second).

(a) Compute the displacement of the particle during the time period $0 \leq t \leq 6$.

$\boxed{-6 \text{ meters}}$

(b) Compute the total distance traveled by the particle during the time period $0 \leq t \leq 6$.

$\boxed{\dfrac{46}{3} \text{ meters}}$

17. The following Riemann Sum was derived by dividing an interval $[a, b]$ into n subintervals of equal width and then choosing x_k^* to be the right endpoint of each subinterval.

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \left(1 + \frac{4}{n}k\right) \frac{4}{n}$$

   (a) What is the interval, $[a, b]$?

   $\boxed{\text{If we consider } f(x) = x, \text{ then the interval is } [1, 5]}$

   (b) Convert the Riemann Sum to an equivalent definite integral.

   $$\boxed{\lim_{n\to+\infty} \sum_{k=1}^{n} \left(1 + \frac{4}{n}k\right) \frac{4}{n} = \int_1^5 x\, dx}$$

   (c) Using the definite integral from part (b) and part of the Fundamental Theorem of Calculus, evaluate the limit.

   $\boxed{12}$

NOTE: In number 17, we could have considered $f(x) = 1 + x$. In that case, $[a, b] = [0, 4]$ and $\lim_{n\to+\infty} \sum_{k=1}^{n} \left(1 + \frac{4}{n}k\right) \frac{4}{n} = \int_0^4 (1 + x)\, dx$. The value of this definite integral is also 12.

18. Explain what is wrong with the following calculation:

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = -\frac{1}{x}\Bigg|_{x=-1}^{x=1} = -1 - (1) = -2$$

$\boxed{\text{$f(x) = \frac{1}{x^2}$ is not continuous at $x = 0$ which is in $[-1, 1]$; so, the FTC does not immediately apply.}}$

For problems 19-22, use part of the Fundamental Theorem of Calculus to compute the indicated derivative.

19. $\displaystyle \frac{d}{dx} \int_2^x \ln(t)\, dt$

   $\boxed{\ln(x)}$

20. $\displaystyle \frac{d}{dx} \int_x^{10} e^{t^2}\, dt$

   $\boxed{-e^{x^2}}$

21. $\dfrac{d}{dx}\displaystyle\int_{\pi}^{3x^2} \cos t \, dt$

$\boxed{6x\cos\left(3x^2\right)}$

22. $\dfrac{d}{dx}\displaystyle\int_{2}^{e^x} \ln(t)\, dt$

$\boxed{xe^x}$

23. Consider $F(x) = \displaystyle\int_4^x \sqrt[3]{t^2+11}\, dt$. Compute each of the following:

   (a) $F(4)$

   $\boxed{0}$

   (b) $F'(4)$

   $\boxed{3}$

   (c) $F''(4)$

   $\boxed{\dfrac{8}{27}}$

24. Let $F(x) = \displaystyle\int_1^x t\ln t\, dt$, for $x > 0$.

   (a) Find the open interval(s) on which $F(x)$ is increasing and those on which $F(x)$ is decreasing.

   $\boxed{F(x) \text{ is increasing on } (1,\infty) \text{ and is decreasing on } (0,1).}$

   (b) Find all points (x, y) where the graph of $F(x)$ has a local (relative) maximum or a local (relative) minimum.

   $\boxed{F(x) \text{ has a local minimum at } (1,0) \text{ and does not have any local maxima.}}$

   (c) Find the interval(s) on which $F(x)$ is concave up and those on which $F(x)$ is concave down.

   $\boxed{F(x) \text{ is concave down on } \left(0,\frac{1}{e}\right) \text{ and is concave up on } \left(\frac{1}{e},\infty\right)}$

   (d) Determine the x-value(s) of each inflection point of $F(x)$.

   $\boxed{F(x) \text{ has an inflection point when } x = \frac{1}{e}}$