# Vector Valued Functions

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapters 12.1 & 12.2 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to find the domain of vector-valued functions.
- Be able to describe, sketch, and recognize graphs of vector-valued functions (parameterized curves).
- Know how to differentiate vector-valued functions. And, consequently, be able to find the tangent line to a curve (as a vector equation or as a set of parametric equations).
- Be able to determine angles between tangent lines.
- Know how to use differentiation formulas involving cross-products and dot products.
- Be able to evaluate indefinite and definite integrals of vector-valued functions as well as solve vector initial-value problems.

PRACTICE PROBLEMS:

1. For each of the following, determine the domain of the given function.

   (a) $\mathbf{r}(t) = t^2\ \mathbf{i} + \sqrt{1-t}\,\mathbf{j} - \dfrac{1}{t}\ \mathbf{k}$

   $\boxed{(-\infty, 0) \cup (0, 1]}$

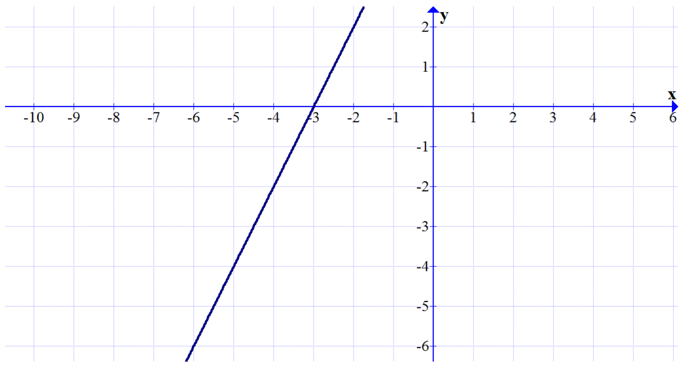
   (b) $\mathbf{r}(t) = \left\langle \ln(t+1), \dfrac{1}{e^t - 2}, t \right\rangle$

   $\boxed{(-1, \ln 2) \cup (\ln 2, \infty)}$

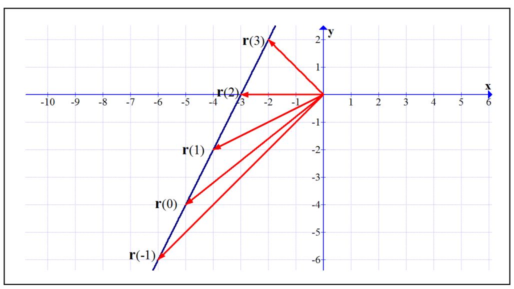
   (c) $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 5\mathbf{k}$

   $\boxed{(-\infty, \infty)}$

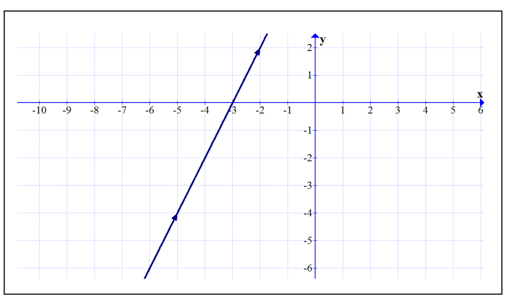
2. Consider the curve $C : \mathbf{r}(t) = \langle -5 + t, -4 + 2t \rangle$, shown below.




(a) Sketch the following position vectors: $\mathbf{r}(-1)$, $\mathbf{r}(0)$, $\mathbf{r}(1)$, $\mathbf{r}(2)$, and $\mathbf{r}(3)$.



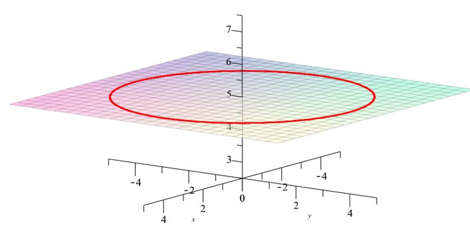

(b) Indicate the orientation of the curve (i.e., the direction or increasing $t$).




3. Sketch the following vector valued functions. Also, describe the curve in words.

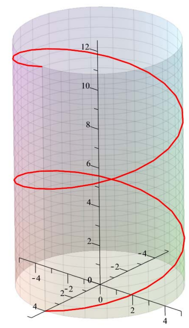
(a) $\overrightarrow{r}(t) = \langle 4\cos t, 4\sin t, 5 \rangle$, $0 \leq t \leq 4\pi$

The curve is a circle in the $z = 5$ plane which has a radius of 4 and a center at $(0, 0, 5)$, traversed twice counterclockwise.



(b) $\vec{r}(t) = \langle 4\cos t, 4\sin t, t\rangle$, $0 \le t \le 4\pi$.

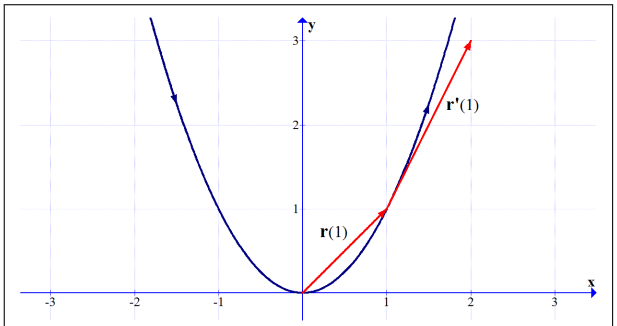
The curve is a helix on the cylinder $x^2 + y^2 = 16$ which climbs from the point $(4, 0, 0)$ to the point $(4, 0, 4\pi)$



4. Consider $\mathbf{r}(t) = \langle t, t^2\rangle$

(a) Sketch $\mathbf{r}(t)$ and indicate the direction of increasing $t$.

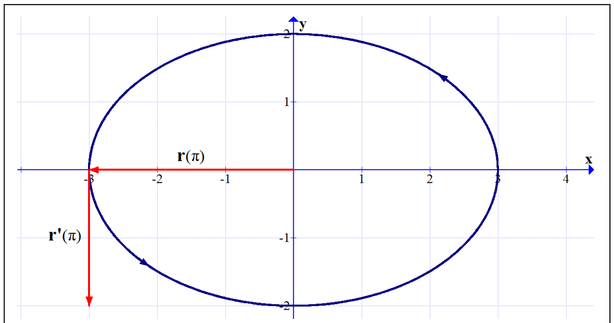
(b) On your sketch, draw $\mathbf{r}(1)$ and $\mathbf{r}'(1)$.




5. Consider $\mathbf{r}(t) = \langle 3\cos t, 2\sin t\rangle$

(a) Sketch $\mathbf{r}(t)$ and indicate the direction of increasing $t$.

(b) On your sketch, draw $\mathbf{r}(\pi)$ and $\mathbf{r}'(\pi)$.



6. For each of the following, find an equation of the line which is tangent to the given curve at the indicated point.

   (a) $\mathbf{r}(t) = \left\langle \ln t, 2\sqrt{t}, t^2 \right\rangle$ at $(x, y, z) = (0, 2, 1)$

   $\boxed{\overrightarrow{\ell}(t) = \langle 0, 2, 1\rangle + t\langle 1, 1, 2\rangle}$

   (b) $\mathbf{r}(t) = \langle \sin t, \cos t, \tan t\rangle$ when $t = \pi$

   $\boxed{\overrightarrow{\ell}(t) = \langle 0, -1, 0\rangle + t\langle -1, 0, 1\rangle}$

7. Find all points on the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ where its tangent line is parallel to the vector $2\mathbf{i} + 8\mathbf{j} + 24\mathbf{k}$.

   The tangent line will be parallel to the given vector when $t = 2$ which corresponds to the point $(x, y, z) = (2, 4, 8)$; Detailed Solution: Here

8. The following vector valued functions describe the paths of two bugs flying in space.

$$\mathbf{r_1}(t) = \langle t^2, 2t + 3, t^2\rangle$$
$$\mathbf{r_2}(t) = \langle 5t - 6, t^2, 9\rangle$$

   At some moment in time, the two bugs collide.

   (a) Determine the moment in time when the bugs collide as well as the location in space where the bugs collide.

   The bugs intersect when $t = 3$. This corresponds to the point $(x, y, z) = (9, 9, 9)$. Detailed Solution: Here

   (b) What is the angle between their paths at the point of collision?

   $\cos^{-1}\left(\dfrac{42}{\sqrt{76}\sqrt{61}}\right)$; Detailed Solution: Here

9. Prove the following theorem:

   **Theorem:** *If $\overrightarrow{r}(t)$ is a differentiable vector valued function in 2-space or 3-space, and if $\|\overrightarrow{r}(t)\|$ is constant for all $t$, then $\overrightarrow{r}(t) \cdot \overrightarrow{r}'(t) = 0$. That is, $\overrightarrow{r}(t)$ and $\overrightarrow{r}'(t)$ are orthogonal vectors for all $t$.*

   (Hint: $\|\overrightarrow{r}(t)\|^2 = \overrightarrow{r}(t) \cdot \overrightarrow{r}(t)$)

Suppose $\|\overrightarrow{r}(t)\| = k$, where $k$ is constant. Then:

$$\begin{aligned}
\|\overrightarrow{r}(t)\|^2 &= k^2 \\
\overrightarrow{r}(t) \cdot \overrightarrow{r}(t) &= k^2 \\
\frac{d}{dt}\left[\overrightarrow{r}(t) \cdot \overrightarrow{r}(t)\right] &= \frac{d}{dt}\left(k^2\right) \\
\overrightarrow{r}(t) \cdot \overrightarrow{r}'(t) + \overrightarrow{r}'(t) \cdot \overrightarrow{r}(t) &= 0 \\
2\left[\overrightarrow{r}(t) \cdot \overrightarrow{r}'(t)\right] &= 0 \\
\overrightarrow{r}(t) \cdot \overrightarrow{r}'(t) &= 0
\end{aligned}$$

And, the result is proven.

10. Explain why the following calculation is incorrect:

$$\frac{d}{dt}\left[\mathbf{r_1}(t) \times \mathbf{r_2}(t)\right] = \mathbf{r_1}(t) \times \mathbf{r_2}'(t) + \mathbf{r_2}(t) \times \mathbf{r_1}'(t)$$

The order of the terms matters when dealing with cross products. The correct derivative statement is:

$$\frac{d}{dt}\left[\mathbf{r_1}(t) \times \mathbf{r_2}(t)\right] = \mathbf{r_1}(t) \times \mathbf{r_2}'(t) + \mathbf{r_1}'(t) \times \mathbf{r_2}(t)$$

11. Evaluate the following integrals.

(a) $\displaystyle\int \left[(2t+1)^5\mathbf{i} - \frac{1}{t}\mathbf{j}\right] dt$

$\left(\frac{1}{12}(2t+1)^6 + c_1\right)\mathbf{i} - (\ln|t| + c_2)\mathbf{j}$; i.e., $\left\langle \frac{1}{12}(2t+1)^6, -\ln|t| \right\rangle + \overrightarrow{c}$

(b) $\displaystyle\int \langle \sin t, \cos t, \tan t\rangle\, dt$

$(-\cos t + c_1)\mathbf{i} + (\sin t + c_2)\mathbf{j} + (\ln|\sec t| + c_3)\mathbf{k}$; or, equivalently, $\langle -\cos t, \sin t, \ln|\sec t|\rangle + \overrightarrow{c}$

(c) $\displaystyle\int_0^{\ln 3} \left[e^t\mathbf{i} + e^{2t}\mathbf{j}\right] dt$

$\langle 2, 4\rangle$

12. Evaluate $\displaystyle\int_0^{2\pi} \|\mathbf{r}'(t)\|\, dt$ if $\mathbf{r}(t) = \langle 3\cos t, 3\sin t\rangle$. Interpret your answer geometrically.

$6\pi$. The given curve represents a circle centered at the origin with a radius of 3; this integral gives the arc length (circumference) of the circle.

13. Solve the following vector initial value problems: $\begin{cases} \dfrac{d\mathbf{r}}{dt} = e^{-t}\mathbf{i} + 3t^2\mathbf{j} \\ \mathbf{r}(0) = 2\mathbf{i} - 8\mathbf{j} \end{cases}$

$\boxed{\mathbf{r}(t) = \langle -e^{-t} + 3, t^3 - 8 \rangle}$

14. A particle moves through 3-space in such a way that its velocity is $\mathbf{v}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. If the particle's initial position at time $t = 0$ is $(1, 2, 3)$, what is the particle's position when $t = 1$? (Hint: set up an initial value problem.)

The position of the particle at time $t = 1$ is $(x, y, z) = \left(\frac{3}{2}, \frac{7}{3}, \frac{13}{4}\right)$. Detailed Solution: Here

15. Suppose that $C : \mathbf{r}(t)$ is a smooth vector valued function in 2-space or 3-space defined for $a \leq t \leq b$. We define the **arc length function** by

$$s(t) = \int_{t_0}^{t} \|\mathbf{r}'(u)\|\, du$$

This function gives the arc length for the part of $C$ between $\mathbf{r}(t_0)$ and $\mathbf{r}(t)$.

(a) Compute the arc length function for the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ which gives the length of the curve from $t_0 = 0$ to an arbitrary $t$.

$\boxed{s = \sqrt{2}t}$

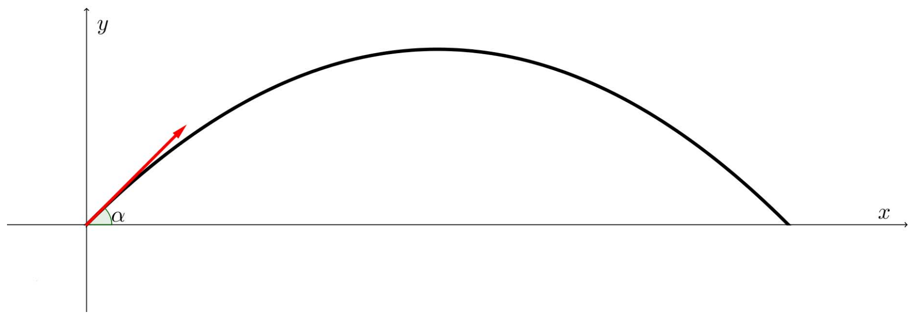
(b) Use your answer from part (a) to reparameterize the helix with respect to arc length. (In other words, express the curve $C$ as $\mathbf{r}(s)$.)

$$\boxed{\mathbf{r}(s) = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}}$$

(c) Compute $\mathbf{r}'(s)$ and $\|\mathbf{r}'(s)\|$

$\mathbf{r}'(s) = -\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$ and $\|\mathbf{r}'(s)\| = 1$
In fact, whenever a curve is parameterized in terms of arc length, it can be shown using the chain rule that all tangent vectors will be unit tangent vectors.

16. From the ground, a projectile is shot upward at an angle of $\alpha$ with the horizontal, $\left(0 < \alpha < \frac{\pi}{2}\right)$, at an initial speed of $v_0$ meters/second, as demonstrated in the diagram below.




You should make the following assumptions:

- The mass of the object, $m$, is constant.
- The only force acting on the object after it is launched is the force of gravity, $g$. Ignore air resistance and assume that the force of gravity is constant.

(a) Set up an initial value problem which can be used to find $\mathbf{r}(t)$, a vector valued function that gives the position of the particle at time $t$.

$$\boxed{\begin{cases} \mathbf{a}(t) = \dfrac{d^2\mathbf{r}}{dt^2} = \langle 0, -g\rangle \\[2ex] \mathbf{v}(0) = \mathbf{r}'(0) = \langle v_0 \cos\alpha, v_0 \sin\alpha\rangle \\[2ex] \mathbf{r}(0) = \langle 0, 0\rangle \end{cases}}$$

(b) Solve your initial value problem from part (a) to determine $\mathbf{r}(t)$.

$$\boxed{\mathbf{r}(t) = \left\langle v_0(\cos\alpha)t, -\frac{1}{2}gt^2 + v_0(\sin\alpha)t\right\rangle}$$

(c) Verify that the trajectory of the projectile is a parabola.

We can express the trajectory of the projectile (from part b) parametrically:

$$\begin{cases} x = v_0(\cos\alpha)t \\ \\ y = -\dfrac{1}{2}gt^2 + v_0(\sin\alpha)t \end{cases}$$

Notice that if we solve the first equation for $t$, we get $t = \dfrac{x}{v_0\cos\alpha}$. (It was OK to do this division since we had some non-zero instantaneous speed $v_0$ and $\cos\alpha \neq 0$ for $0 < \alpha < \dfrac{\pi}{2}$). Then, plugging this into the second equation, we get:

$$\begin{aligned} y &= -\frac{1}{2}g\left(\frac{x}{v_0\cos\alpha}\right)^2 + v_0(\sin\alpha)\left(\frac{x}{v_0\cos\alpha}\right) \\ &= -\frac{g}{2(v_0\cos\alpha)^2}x^2 + (\tan\alpha)x \\ &= -Ax^2 + Bx \end{aligned}$$

where $A$ is the constant $\dfrac{g}{2(v_0\cos\alpha)^2}$ and $B$ is the constant $\tan\alpha$. Thus, the trajectory is parabolic.

(d) What is the flight time of the projectile?

We can find the value of $t$ for which the projectile returns to the ground by setting $y = 0$ in the parametric representation of the trajectory.

$$\begin{aligned} y &= 0 \\ -\frac{1}{2}gt^2 + v_0(\sin\alpha)t &= 0 \\ -t\left(\frac{1}{2}gt - v_0\sin\alpha\right) &= 0 \end{aligned}$$

which happens when $t = 0$ and when $t = \dfrac{2v_0\sin\alpha}{g}$

(e) What is the range of the projectile?

To find the range, we need to determine the $x$ coordinate at the time when the projectile returns to the ground. Specifically, the range is:

$$x\left(\frac{2v_0\sin\alpha}{g}\right) = v_0(\cos\alpha)\left(\frac{2v_0\sin\alpha}{g}\right) = \frac{v_0^2\sin(2\alpha)}{g}$$

(f) What angle $\alpha$ maximizes the range?

In part (e), we have already computed the range to be $\frac{v_0^2}{g}\sin(2\alpha)$. This is maximized when $\sin(2\alpha) = 1$; i.e., when $\alpha = \frac{\pi}{4}$

17. Suppose that $C : \mathbf{r}(t)$ is a curve in 2-space or 3-space and that $\|\mathbf{r}'(t)\| \neq 0$. We define the following vectors:

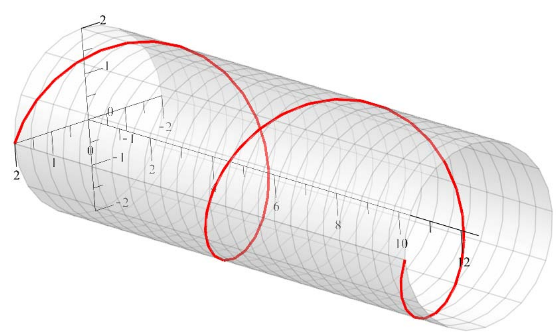
- The Unit Tangent Vector to $C$ at $t$ is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.
- The Principal Unit Normal Vector to $C$ at $t$ is $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.
- The Unit Binormal Vector to $C$ at $t$ is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

The coordinate system determined at the point $t$ by $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ is called the Frenet Frame or the **TNB** Frame.

(a) Explain why $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ are all mutually othogonal.

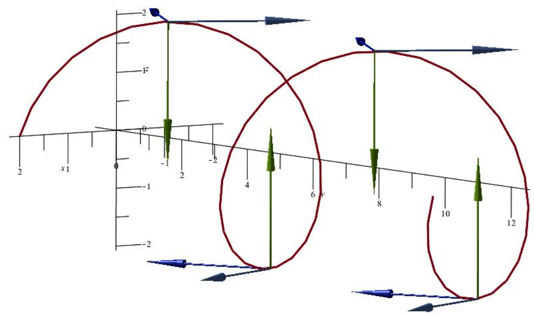
$\mathbf{N}(t) \perp \mathbf{T}(t)$ by problem 9. $\mathbf{B}(t) \perp \mathbf{N}(t)$ and $\mathbf{B}(t) \perp \mathbf{T}(t)$ because for any vectors in three space $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ and $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = 0$

(b) Consider the helix described by $\mathbf{r}(t) = \langle 2\cos t, t, 2\sin t\rangle$.



Compute the unit tangent, principal unit normal, and binormal vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$.

NOTE: Here is a sketch of the helix from problems 17b with the **TNB**-Frame (Frenet Frame) represented at four different points.



$$\boxed{\begin{array}{l}\mathbf{T}(t) = \left\langle -\frac{2}{\sqrt{5}}\sin t, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos t \right\rangle;\ \mathbf{N}(t) = \langle -\cos t, 0, -\sin t\rangle; \\ \mathbf{B}(t) = \left\langle -\frac{1}{\sqrt{5}}\sin t, -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\cos t \right\rangle\end{array}}$$

(c) **Definition:** *The plane determined by the unit tangent and normal vectors* $\mathbf{T}$ *and* $\mathbf{N}$ *at a point* $P$ *on a curve* $C$ *is called the* **osculating plane** *of* $C$ *at* $P$*. From the latin "Osculum," meaning to kiss, this is the plane that comes closest to containing the part of the curve near* $P$*.*

Compute an equation of the osculating plane of the helix from part (b) at the point which corresponds to $t = \pi$.

$\boxed{2y + z = 2\pi}$

(d) **Definition:** *The plane determined by the unit normal and binormal vectors* $\mathbf{N}$ *and* $\mathbf{B}$ *at a point* $P$ *on a curve* $C$ *is called the* **normal plane** *of* $C$ *at* $P$*. It consists of all lines that are orthogonal to the tangent vector* $\mathbf{T}$*.*

Compute an equation of the normal plane of the helix from part (b) at the point which corresponds to $t = \pi$.

$\boxed{y - 2z = \pi}$