Length of a Plane Curve (Arc Length)

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 6.4 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to find the arc length of a smooth curve in the plane described as a function of x or as a function of y.

PRACTICE PROBLEMS:

For problems 1-3, compute the exact arc length of the curve over the given interval.

1. $y = 4x^{\frac{3}{2}} - 1$ from $x = \dfrac{1}{12}$ to $x = \dfrac{2}{9}$
2. $y = \dfrac{x^2}{2} - \dfrac{\ln(x)}{4}$ for $2 \le x \le 4$
3. $y = \dfrac{2}{3}(x^2 - 1)^{3/2}$ for $1 \le x \le 3$
4. Consider the curve defined by $y = \sqrt{4 - x^2}$ for $0 \le x \le 2$.
   (a) Compute the arc length on the interval $[0, t]$ for $0 \le t < 2$. (Your arc length will depend on t.)
   (b) Use your answer from part (a) to compute the arc length on the interval $[0, 2]$. (Hint: You will need to introduce a limit.)
   (c) Confirm your answer from part (b) by using geometry.
5. Consider $F(x) = \displaystyle\int_1^x \sqrt{t^2 - 1}\, dt$. Compute the arc length on $[1, 3]$
6. Consider the curve defined by $f(x) = \ln x$ on $\left[1, e^3\right]$
   (a) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to x.
   (b) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to y.

7. Consider the curve defined by $f(x) = \tan x$ on $\left[-\dfrac{\pi}{3}, \dfrac{\pi}{4}\right]$

   (a) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to x.

   (b) Set up but do not evaluate an integral which represents the length of the curve by integrating with respect to y.

8. Consider the curve defined by $y = \sin x$ for $0 \le x \le \pi$.

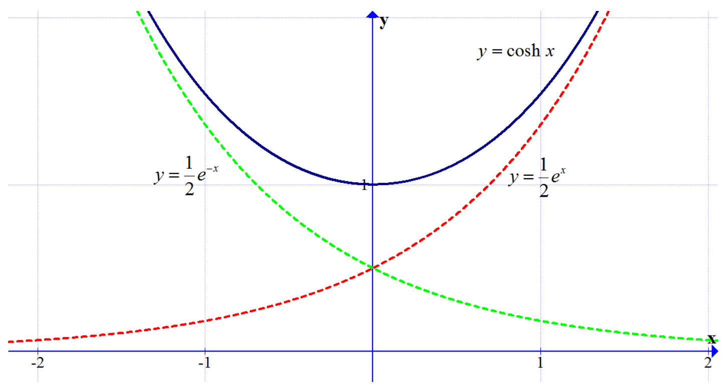
   (a) Set up but do not evaluate an integral which represents the length of the curve.

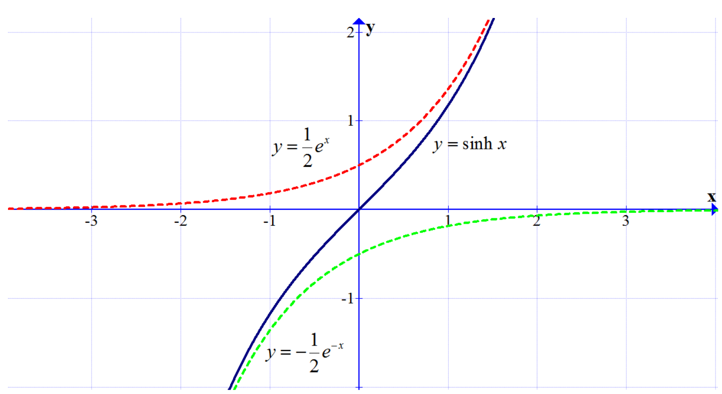
   (b) Estimate the value of your integral from part (a) by using a Midpoint Approximation with three rectangles of equal width.

9. Recall the definitions of Hyperbolic Sine & Hyperbolic Cosine from Math 121:

Hyperbolic Sine	Hyperbolic Cosine
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\cosh x = \dfrac{e^x + e^{-x}}{2}$

The sketches of $y = \cosh x$ and $y = \sinh x$ are shown below. The dashed curves are called "Curvilinear Asymptotes," which describe the end behavior of the functions.





(a) Show that $\cosh^2 x - \sinh^2 x = 1$

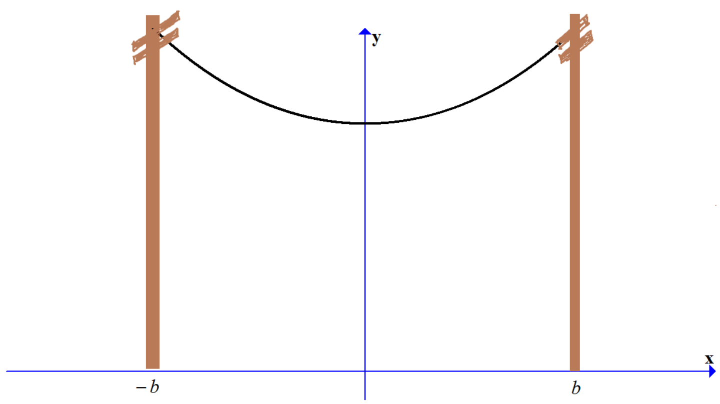
(b) Verify that $f(x) = \sinh x$ is an odd function. (**Hint:** Recall an odd function satisfies the identity $f(-x) = -f(x)$.)

(c) Show that $\dfrac{d}{dx}(\sinh x) = \cosh x$ and deduce that $\displaystyle\int \cosh x\, dx = \sinh x + C$.

(d) Show that $\dfrac{d}{dx}(\cosh x) = \sinh x$ and deduce that $\displaystyle\int \sinh x\, dx = \cosh x + C$.

(e) A telephone wire which is supported only by two telephone poles will sag under its own weight and form the shape of a **catenary** as shown below.

Consider a telephone wire that is supported by two poles (one at $x = b$ and the other at $x = -b$), as in the diagram below.



The shape of the sagging wire can be modeled by $y = a\cosh\left(\dfrac{x}{a}\right)$, where $a > 0$ and $-b \le x \le b$. What is the length of the wire?