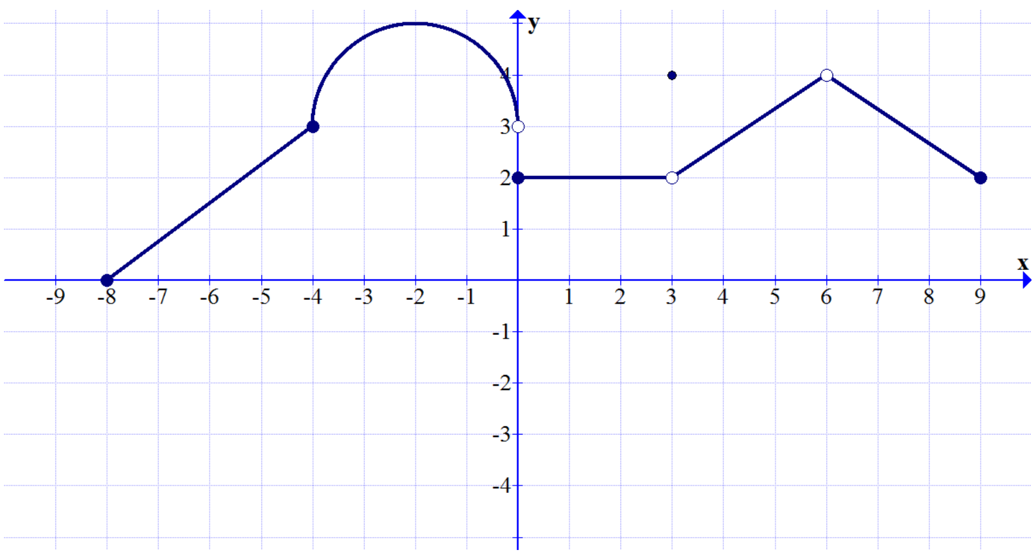
Chapter 1.5 Practice Problems

EXPECTED SKILLS:

- Know what it means for a function to be continuous at a specific value and on an interval.
- Find values where a function is not continuous; specifically, you should be able to do this for polynomials, rational functions, exponential and logarithmic functions, and other elementary functions.
- Determine the values for which a piecewise function is discontinuous, if any such values exist.
- Use the Intermediate Value Theorem to show the existence of a solution to an equation.

PRACTICE PROBLEMS:

Use the graph of $f(x)$, shown below, to answer questions 1-3



1. For which values of x is $f(x)$ discontinuous?

   $\boxed{f(x) \text{ is discontinuous when } x = 0,\ x = 3, \text{ and } x = 6.}$

2. At each point of discontinuity, explain why $f(x)$ is discontinuous.

   At $x = 0$, $f(x)$ is discontinuous because $\lim\limits_{x\to 0} f(x)$ DNE.
   At $x = 3$, $f(x)$ is discontinuous because $\lim\limits_{x\to 3} f(x) \neq f(3)$.
   At $x = 6$, $f(x)$ is discontinuous because $f(6)$ is undefined

3. Determine whether $f(x)$ is continuous on the given interval. If not, explain why.

(a) $[-8, -4]$

$\boxed{\text{Yes}}$

(b) $[-8, 0]$

$\boxed{\text{No because } \lim_{x \to 0^-} f(x) \neq f(0)}$

(c) $[-8, 0)$

$\boxed{\text{Yes}}$

(d) $[-2, 1]$

$\boxed{\text{No because } \lim_{x \to 0} f(x) \text{ DNE}}$

(e) $(3, 6)$

$\boxed{\text{Yes}}$

(f) $[3, 6)$

$\boxed{\text{No because } \lim_{x \to 3^+} f(x) \neq f(3)}$

(g) $(6, 9]$

$\boxed{\text{Yes}}$

(h) $[6, 9]$

$\boxed{\text{No because } f(6) \text{ is undefined}}$

4. For each of the following, sketch the graph of a function, $y = f(x)$, which satisfies the given characteristic. (There are many possible answers for each)

(a) $f(x)$ is continuous everywhere except at $x = 1$.

$\boxed{\text{Any graph for which either } f(1) \text{ is undefined or } \lim_{x \to 1} f(x) \text{ DNE or } \lim_{x \to 1} f(x) \neq f(1)}$

(b) $f(x)$ is continuous everywhere except at $x = -2$ where the $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x)$.

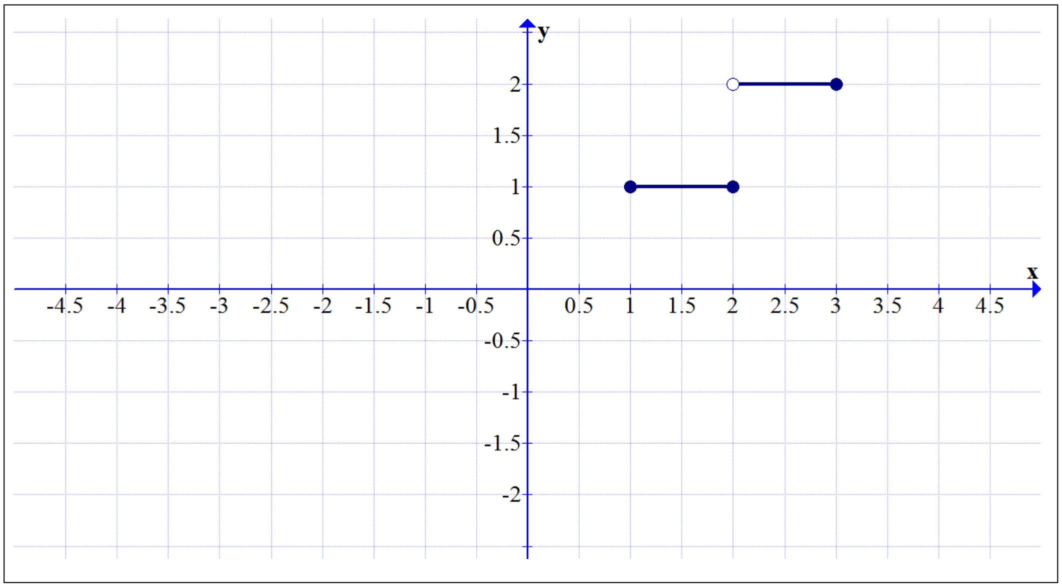
$\boxed{\text{Any graph for which either } f(-2) \text{ is undefined or } \lim_{x \to -2} f(x) \neq f(-2)}$

(c) $f(x)$ is continuous everywhere except at $x = 0$, where $f(0) = 2$.

$\boxed{\text{Any graph for which } \lim_{x \to 0} f(x) \text{ DNE or } \lim_{x \to 0} f(x) \neq 2}$

5. Sketch the graph of a function which satisfies the following criteria:

   - The domain of $f(x)$ is $[1, 3]$
   - $f(x)$ is continuous on $[1, 2]$ and $(2, 3]$.
   - $f(x)$ is not continuous on $[1, 3]$



For problems 6-15, determine the value(s) of x where the given function has a point of discontinuity, if any such values exist.

6. $f(x) = |x|$

   $\boxed{f(x) \text{ is always continuous}}$

7. $f(x) = x^2 - x - 5$

   $\boxed{f(x) \text{ is always continuous}}$

8. $f(x) = \dfrac{x}{x-1}$

   $\boxed{f(x) \text{ has a discontinuity when } x = 1}$

9. $f(x) = \sqrt[3]{x-1}$

   $\boxed{f(x) \text{ is always continuous}}$

10. $f(x) = \dfrac{x^2 + 3x - 10}{x - 7}$

    $\boxed{f(x) \text{ has discontinuity when } x = 7}$

11. $f(x) = \dfrac{x^2 - 4}{x - 2}$

$\boxed{f(x) \text{ has a discontinuity when } x = 2}$

12. $f(x) = \dfrac{1}{x^2 - 2} + \dfrac{x^3 - 1}{2x^2 - 1}$

$\boxed{f(x) \text{ has a discontinuity when } x = \sqrt{2},\ x = -\sqrt{2},\ x = \dfrac{\sqrt{2}}{2}, \text{ and } x = -\dfrac{\sqrt{2}}{2}}$

13. $f(x) = \begin{cases} x^2 - 1, & \text{if } x < 2 \\ \dfrac{3}{x - 1}, & \text{if } x \geq 2 \end{cases}$

$\boxed{f(x) \text{ is always continuous}}$

14. $f(x) = \begin{cases} 5 + \dfrac{1}{x}, & \text{if } x < -1 \\ 3x^2 + 2x + 3, & \text{if } x > -1 \end{cases}$

$\boxed{f(x) \text{ has a discontinuity when } x = -1}$

15. $f(x) = \begin{cases} x^2 - 3x + 4, & \text{if } x \leq 1 \\ x^4 - 4x^3 - 2x^2 + 6, & \text{if } x > 1 \end{cases}$

$\boxed{f(x) \text{ has discontinuity when } x = 1}$

16. Find the value(s) of k such that $f(x)$ is continuous everywhere:

$$f(x) = \begin{cases} x^2 - 7, & \text{if } x \leq 2 \\ 4x^3 - 3kx + 2, & \text{if } x > 2 \end{cases}$$

$\boxed{k = \dfrac{37}{6}}$

17. Find the value(s) of k and m such that $f(x)$ is continuous everywhere:

$$f(x) = \begin{cases} 2x + 8m, & \text{if } x \leq -2 \\ mx + k, & \text{if } -2 < x \leq 2 \\ -3x^2 + 8x - 2k, & \text{if } x > 2 \end{cases}$$

$\boxed{m = \dfrac{1}{2} \text{ and } k = 1}$

18. **Multiple Choice:** Where is $f(x) = \dfrac{\sqrt{x-2}}{x^2 - x}$ continuous?

   (a) $x \neq 0$ and $x \neq 1$
   (b) $x \leq 2$ where $x \neq 0$ and $x \neq 1$
   (c) $x \leq 2$
   (d) $x \geq 2$
   (e) $|x| > 2$

   $\boxed{\text{d}}$

19. Consider the following definitions:

   - **Definition:** A function $f(x)$ has a removable discontinuity at $x = a$ if $\lim\limits_{x \to a} f(x)$ exists but $f(x)$ is not continuous at $x = a$. This could be because $f(a)$ is undefined or because $\lim\limits_{x \to a} f(x) \neq f(a)$.
   - **Definition:** A function $f(x)$ has a jump discontinuity at $x = a$ if $\lim\limits_{x \to a^-} f(x)$ exists and $\lim\limits_{x \to a^+} f(x)$ exists, but $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

   For each of the follwing, determine the value(s) of x where the given function has a point of discontinuity. Classify each discontinuity as a removable discontinuity, a jump discontinuity, or neither.

   (a) $f(x) = \dfrac{x^2 - 4}{x - 2}$

   $\boxed{f(x) \text{ has a removable discontinuity when } x = 2}$

   (b) $f(x) = \dfrac{x - 1}{x - 4}$

   $\boxed{f(x) \text{ has a discontinuity when } x = 4\text{; it is neither a removable discontinuity nor a jump discontinuity.}}$

   (c) $f(x) = \begin{cases} x^2 - 3x + 4, & \text{if } x \leq 1 \\ x^4 - 4x^3 - 2x^2 + 6, & \text{if } x > 1 \end{cases}$

   $\boxed{f(x) \text{ has jump discontinuity when } x = 1}$

   (d) $f(x) = \dfrac{x - 1}{x^2 - 4x + 3}$

   $\boxed{f(x) \text{ has a removable discontinuity when } x = 1. \; f(x) \text{ has another discontinuity when } x = 3\text{; it is neither a removable discontinuity nor a jump discontinuity.}}$

20. **Multiple Choice:** Consider the function:

$$f(x) = \begin{cases} x^2 & \text{if } x < -2 \\ 4 & \text{if } -2 < x \leq 1 \\ 6 - x & \text{if } x > 1 \end{cases}$$

Which of the following statements is true about $f(x)$?

(a) $f(x)$ is continuous everywhere.

(b) If $f(-2)$ were defined to be 4, then $f(x)$ would be continuous everywhere.

(c) The only discontinuity of $f(x)$ occurs when $x = -2$.

(d) The only discontinuity of $f(x)$ occurs when $x = 1$.

(e) The only discontinuities of $f(x)$ occur when $x = -2$ and $x = 1$.

$\boxed{\text{e}}$

21. Show that the equation $x^3 - x^2 + 3x - 1 = 1$ has at least one solution in $(0, 1)$.

Let $f(x) = x^3 - x^2 + 3x - 2$. It suffices to show that there exists a c in $(0, 1)$ such that $f(c) = 0$. Since $f(x)$ is a polynomial, it is continuous everywhere on $(-\infty, \infty)$. Specifically, it is continuous on $[0, 1]$. Since $f(0) = -2 < 0$ and $f(1) = 1 > 0$, the Intermediate Value Theorem states that there exists some $cin(0, 1)$, $f(c) = 0$. The result follows.

22. Show that $f(x) = x^3 - 9x + 5$ has at least one x-intercept in $(1, 10)$.

We need to show that there exists at least one solution to $f(x) = 0$. Since $f(x)$ is a polynomial, it is continuous on $[1, 10]$. Notice that $f(1) = -3 < 0$ and $f(10) = 915 > 0$. Thus, the Intermediate Value Theorem states that there must be a c in $(1, 10)$ with $f(c) = 0$.

23. Use the intermediate value theorem to show that $x^3 - 2x^2 - 2x + 1 = 0$ has at least **TWO** solutions in $[0, 5]$.

We will apply the IVT twice – first on $[0, 1]$ and then on $[1, 5]$. Let $f(x) = x^3 - 2x^2 - 2x + 1$. Since $f(x)$ is a polynomial, it is continuous on $(-\infty, \infty)$. As a result, it is continuous on $[0, 1]$ and $[1, 5]$. Notice that $f(0) = 1 > 0$ and $f(1) = -2 < 0$. So, the IVT implies that there exists a c in $(0, 1)$ such that $f(c) = 0$. Similarly, notice that $f(1) = -2 < 0$ and $f(5) = 66 > 0$. So, the IVT implies that there exists a d in $(1, 5)$ such that $f(d) = 0$.