First-Order Linear Equations (Integrating Factors)

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Be able to solve first-order linear equations by using the appropriate integrating factors.
- Be able to set up and solve application problems using integrating factors.

PRACTICE PROBLEMS:

For problems 1-6, use an integrating factor to solve the given differential equation. Express your answer as an explicit function of x.

1. $\dfrac{dy}{dx} - 4y = e^{5x}$

   $\boxed{y = e^{5x} + Ce^{4x}}$

2. $\dfrac{dy}{dx} + 3x^2 y = x^2$

   $\boxed{\dfrac{1}{3} + Ce^{-x^3}}$

3. $y' = x - 2y$

   $\boxed{Ce^{-2x} + \dfrac{1}{2}x - \dfrac{1}{4}}$

4. $\dfrac{dy}{dx} - y = \sin\left(e^{-x}\right)$

   $\boxed{y = e^{x}\cos\left(e^{-x}\right) + Ce^{x}}$

5. $y' + \dfrac{y}{x\ln x} = x$, for $x > 1$

   $\boxed{y = \dfrac{1}{2}x^2 - \dfrac{x^2}{4\ln x} + \dfrac{C}{\ln x}}$

6. $\dfrac{dy}{dx} + y = \dfrac{1}{e^{2x} - 5e^{x} + 4}$

   $\boxed{y = \dfrac{1}{3}e^{-x}\ln\left|\dfrac{e^{x}-4}{e^{x}-1}\right| + Ce^{-x}}$; Detailed Solution: Here

7. Look at the First-Order Separable Equations practice problems 3 – 9 and determine which ODE's, if any, are first-order linear equations. If there are any, solve them using integrating factors.

   Problem #5 is a linear equation since it can be written as $y' - x^2 y = 0$. The solution is (of course) still $y = Ce^{x^3/3}$.

For problems 8-9, solve the initial value problem. Express your answer as an explicit function of x.

8. $\dfrac{dy}{dx} + \dfrac{1}{x}y = \dfrac{1}{x + x^3}$, for $x > 0$; $y(1) = 0$

   $\boxed{y = \dfrac{\arctan x}{x} - \dfrac{\pi}{4x}}$

9. $(\cos x)\dfrac{dy}{dx} + y \sin x = \sin x \cos x$, for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$; $y(0) = 5$

   $y = (\cos x) \ln(\sec x) + 5 \cos x$; Detailed Solution: Here

10. A tank intially contains 7 pounds of salt dissolved in 100 gallons of water. Then, salt water containing 3 pounds of salt per gallon enters the tank at a rate of 8 gallons per minute, and the mixed solution is drained from the tank at a rate of 8 gallons per minute. Let $y = y(t)$ be the amount of salt in the tank at time t.

    (a) Using this information, set up an initial value problem (IVP) whose solution is $y(t)$.

    $$\boxed{\begin{cases} \dfrac{dy}{dt} = 24 - \dfrac{2y}{25} \\ y(0) = 7 \end{cases}}$$

    (b) Using integrating factors, solve the IVP from part (a).

    $\boxed{y(t) = 300 - 293e^{-2t/25}}$

    (c) Using separation of variables, solve the IVP from part (a).

    $\boxed{y(t) = 300 - 293e^{-2t/25}}$

11. Suppose the saltwater solution in problem #10 is drained from the tank at a rate of 6 gallons per minute.

    (a) Set up an initial value problem (IVP) whose solution is $y(t)$. [Hint: The volume of saltwater is no longer a constant, but rather a function of t.]

$$\boxed{\begin{cases} \dfrac{dy}{dt} = 24 - \dfrac{3y}{50+t} \\ \\ y(0) = 7 \end{cases} \text{; Detailed Solution: Here}}$$

(b) Using integrating factors, solve the IVP from part (a).
[Note that unlike problem #10 the ODE is no longer separable.]

$$\boxed{y(t) = 6(50+t) - 293(50)^3(50+t)^{-3}}$$

(c) Suppose that the tank has a capacity of 200 gallons. How much salt is in the tank when it reaches the point of overflowing?

> The tank overflows at time $t = 50$ minutes.
> At that time the amount of salt is $y\,(50) = \dfrac{4507}{8} \approx 563.4$ pounds

12. Suppose that an object with mass m falls to the earth with a velocity $v = v(t)$ and is subjected to the force of gravity as well as air resistance (which is propotional to its velocity). Using Newton's Second Law it can be shown that

$$m\frac{dv}{dt} = -mg - kv$$

where g is the acceleration due to gravity and k is some positive constant of proportionality.

(a) Assuming that the object's initial velocity is v_0, set up an initial value problem (IVP) whose solution is $v(t)$.

$$\boxed{\begin{cases} \dfrac{dv}{dt} + \dfrac{k}{m}v = -g \\ \\ v(0) = v_0 \end{cases}}$$

(b) Solve the IVP from part (a).

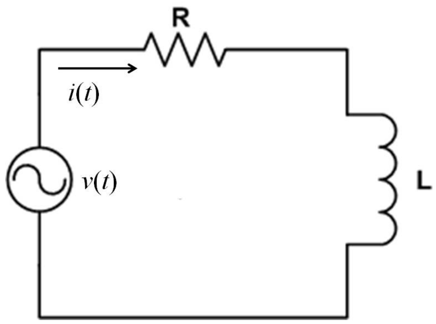
$$\boxed{v(t) = -\frac{gm}{k} + \left(v_0 + \frac{gm}{k}\right)e^{-kt/m}}$$

(c) Evaluate $\lim\limits_{t\to\infty} v(t)$.

> $\lim\limits_{t\to\infty} v(t) = -\dfrac{gm}{k}$. This is known as the terminal velocity of the object and occurs when the opposing forces of air resistance and gravity are equal, causing the object to experience no acceleration.

13. Consider the simple electrical circuit shown below. An electromotive force (e.g. a generator) produces a voltage of $V(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t. The circuit also contains a resistor with a constant resistance of R ohms (Ω)

and an inductor with a constant inductance of L henries (H). Such a circuit is called an $\underline{RL \text{ circuit}}$.



Using Ohm's Law and Kirchoff's Law it can be shown that

$$L\frac{dI}{dt} + RI = V(t)$$

Suppose that the RL circuit above has a resistance of 6 Ω and an inductance of 3 H. If a generator produces a variable voltage of $V(t) = 9\sin t$ and the initial current is $I(0) = 2$ A, find $I(t)$.

Hint: Recall to solve an integral of the form $\displaystyle\int e^x \sin x\, dx$, use integration by parts twice.

$$\boxed{I(t) = -\frac{3}{5}\cos t + \frac{6}{5}\sin t + \frac{13}{5}e^{-2t}}$$