Infinite Series

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Calculate the partial sums of a series.
- Recognize geometric and telescoping series, determine whether they converge, and if so, determine the sum of the series (i.e. what they converge to).
- Compute the sum of a finite number of terms from a geometric series.

PRACTICE PROBLEMS:

For problems 1 – 8, calculate the first four partial sums for each series.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2}$

$$\boxed{s_1 = \frac{1}{2}, s_2 = \frac{1}{2} + \frac{1}{2} = 1, s_3 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}, s_4 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2}$$

2. $\displaystyle\sum_{k=1}^{\infty} k$

$$\boxed{s_1 = 1, s_2 = 1 + 2 = 3, s_3 = 1 + 2 + 3 = 6, s_4 = 1 + 2 + 3 + 4 = 10}$$

3. $\displaystyle\sum_{k=1}^{\infty} (-1)^k$

$$\boxed{s_1 = -1, s_2 = -1 + 1 = 0, s_3 = -1 + 1 - 1 = -1, s_4 = -1 + 1 - 1 + 1 = 0}$$

4. $\displaystyle\sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j$

$$\boxed{s_0 = 1, s_1 = 1 + \frac{1}{2} = \frac{3}{2}, s_2 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}, s_3 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}}$$

5. $\displaystyle\sum_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{j+1}\right)$

$$
\boxed{\begin{aligned}
s_1 &= 1-\frac{1}{2}=\frac{1}{2}\\
s_2 &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)=1-\frac{1}{3}=\frac{2}{3}\\
s_3 &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)=1-\frac{1}{4}=\frac{3}{4}\\
s_4 &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)=1-\frac{1}{5}=\frac{4}{5}
\end{aligned}}
$$

6. $\displaystyle\sum_{j=0}^{\infty}(7^j-7^{j+1})$

$$
\boxed{\begin{aligned}
s_0 &= 1-7\\
s_1 &= (1-7)+(7-7^2)=1-7^2\\
s_2 &= (1-7)+(7-7^2)+(7^2-7^3)=1-7^3\\
s_3 &= (1-7)+(7-7^2)+(7^2-7^3)+(7^3-7^4)=1-7^4
\end{aligned}}
$$

7. $\displaystyle\sum_{\ell=3}^{\infty}\frac{3^{\ell+1}}{4^{\ell}}$

$$
\boxed{\begin{aligned}
s_3 &= \frac{3^4}{4^3}\\
s_4 &= \frac{3^4}{4^3}+\frac{3^4}{4^3}\left(\frac{3}{4}\right)\\
s_5 &= \frac{3^4}{4^3}+\frac{3^4}{4^3}\left(\frac{3}{4}\right)+\frac{3^4}{4^3}\left(\frac{3}{4}\right)^2\\
s_6 &= \frac{3^4}{4^3}+\frac{3^4}{4^3}\left(\frac{3}{4}\right)+\frac{3^4}{4^3}\left(\frac{3}{4}\right)^2+\frac{3^4}{4^3}\left(\frac{3}{4}\right)^3
\end{aligned}}
$$

8. $\displaystyle\sum_{\ell=1}^{\infty}\frac{5^{\ell}}{3^{\ell}}.$

$$
\boxed{\begin{aligned}
s_1 &= \frac{5}{3}\\
s_2 &= \frac{5}{3}+\left(\frac{5}{3}\right)^2\\
s_3 &= \frac{5}{3}+\left(\frac{5}{3}\right)^2+\left(\frac{5}{3}\right)^3\\
s_4 &= \frac{5}{3}+\left(\frac{5}{3}\right)^2+\left(\frac{5}{3}\right)^3+\left(\frac{5}{3}\right)^4
\end{aligned}}
$$

9. For numbers 1, 5, and 6 above, find a general formula for the n^{th} partial sum, s_n, for

each series. Use this to determine whether these series converge, and if so, determine the sum of the series.

Problem 1: $s_n = \frac{1}{2}n$, and so $\lim_{n\to+\infty} s_n = +\infty$. Thus, the series diverges.

Problem 5: $s_n = 1 - \frac{1}{n+1}$, and so $\lim_{n\to+\infty} s_n = 1$. Thus, the sum of the series is 1.

Problem 6: $s_n = 1 - 7^{(n+1)}$, and so $\lim_{n\to+\infty} s_n = -\infty$. Thus, the series diverges.

10. For numbers 3, 4, 7, and 8 above, determine whether these series converge, and if so, determine the sum of the series.

Problem 3: Geometric series with $a = -1$ and $r = -1$.
Since $|r| = 1$ the series diverges.
Alternatively, the sequence of partial sums oscillates bewteen -1 and 0 and thus diverges; hence, the series diverges.

Problem 4: Geometric series with $a = 1$ and $r = \frac{1}{2}$.

Since $|r| < 1$ the series converges to $\frac{1}{1-\frac{1}{2}} = 2$.

Problem 7: Geometric series with $a = \frac{3^4}{4^3}$ and $r = \frac{3}{4}$.

Since $|r| < 1$ the series converges to $\frac{\frac{3^4}{4^3}}{1-\frac{3}{4}} = \frac{81}{16}$.

Problem 8: Geometric series with $a = \frac{5}{3}$ and $r = \frac{5}{3}$.

Since $|r| > 1$ the series diverges.

For problems 11 – 14, determine whether each series converges, and if so, determine the sum of the series.

11. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$

The series converges to $\frac{1}{3}$.

12. $2 + \frac{4}{7} + \frac{8}{49} + \frac{16}{343} + \ldots$

The series converges to $\frac{14}{5}$.

13. $2 + \frac{22}{10} + \frac{242}{100} + \frac{2662}{1000} + \ldots$

The series diverges.

14. $-3 - 1 - \frac{1}{3} - \frac{1}{9} - \ldots$

$\boxed{\text{The series converges to } -\frac{9}{2}.\text{; Detailed Solution: Here}}$

For problems 15 & 16, use a geometric series to write the repeating decimal as a fraction of integers.

15. 0.99999...

$$\boxed{0.99999... = 0.9 + 0.09 + 0.009 + \ldots = \sum_{k=0}^{\infty} 0.9\left(\frac{1}{10}\right)^k = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = 1}$$

16. 8.126262626...

$$\begin{aligned} 8.126262626... &= 8.1 + 0.026 + 0.00026 + 0.0000026 + \ldots \\ &= 8.1 + \sum_{k=0}^{\infty} 0.026\left(\frac{1}{100}\right)^k \\ &= 8.1 + \frac{\frac{26}{1000}}{1 - \frac{1}{100}} = \frac{81}{10} + \frac{26}{990} = \frac{8045}{990} = \frac{1609}{198} \end{aligned}$$

17. Calculate $\sum_{k=0}^{300} (-2)^k$.

$$\boxed{\frac{1 - (-2)^{301}}{3} = \frac{1 + 2^{301}}{3}}$$

18. Calculate $\sum_{j=1}^{13} 7^j$.

$\boxed{-\frac{7}{6}\left(1 - 7^{13}\right)\text{; Detailed Solution: Here}}$

19. Calculate $\sum_{\ell=2}^{73} \frac{1}{2^\ell}$.

$$\boxed{\frac{1}{2}\left(1 - \frac{1}{2^{72}}\right)}$$

20. An ordinary annuity is a sequence of equal payments made at the end of equal time periods, where the frequency of the payments is the same as the frequency of compounding.

(a) Suppose that 500 dollars is deposited at the end of each month into an account paying 3% interest compounded monthly.

i. How much is in the account at the end of 1 month?

$\boxed{\text{500 dollars.}}$

ii. How much is in the account at the end of 2 months?

$\boxed{500 + 500(1.03) \text{ dollars.}}$

iii. How much is in the account at the end of 3 months?

$\boxed{500 + 500(1.03) + 500(1.03)^2 \text{ dollars.}}$

iv. How much is in the account at the end of n months? Express your final answer in closed form, i.e. without sigma notation or "...".

$$\boxed{\sum_{k=0}^{n-1} 500(1.03)^k = \frac{500\left(1-(1.03)^n\right)}{1-1.03} \text{ dollars.}}$$

(b) Suppose that R dollars is deposited at the end of some fixed time period into an account paying an interest of i per period. How much is in the account at the end of n periods?

$$\frac{R\left(1-(1+i)^n\right)}{1-(1+i)} = \frac{R\left(1-(1+i)^n\right)}{-i} = R\left[\frac{(1+i)^n-1}{i}\right].$$

This sum is know as the future value of an ordinary annuity.

For problems 21 & 22, use partial fractions to determine the sum of the series.

21. $\displaystyle\sum_{k=0}^{\infty} \frac{10}{k^2+9k+20}$

$\boxed{\dfrac{5}{2}}$

22. $\displaystyle\sum_{k=0}^{\infty} \frac{4}{k^2+4k+3}$

3; Detailed Solution: Here

23. Consider the following formula:

$$\sum_{k=1}^{\infty}(x^k - x^{k+1}) = x.$$

For which values of x does the series on the left-hand side of the formula converge? For which values of x is the formula correct?

This is a telescoping series with an n^{th} partial sum of $s_n = x - x^{n+1}$.
Now $\lim_{n\to+\infty}(x - x^{n+1})$ equals x if $-1 < x < 1$ and equals 0 if $x = 1$. So the series converges if $-1 < x \leq 1$, but the formula is only correct if $-1 < x < 1$.

24. Consider the following formula::

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}.$$

For which values of x does the series on the left-hand side of the formula converge? For which values of x is the formula correct?

This is a geometric series with $a = x$ and $r = x$. Thus the series converges only if $|x| < 1$, i.e. $-1 < x < 1$. If this is true, then the sum of the series is $\frac{a}{1-r} = \frac{x}{1-x}$. So the formula is true if $-1 < x < 1$.