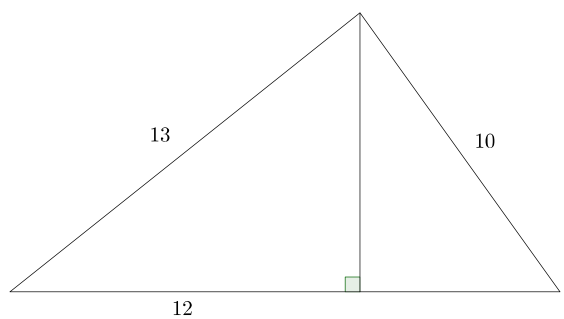
Chapters 2.1-2.3: Pythagorean Theorem, Distance Formula, & Circles

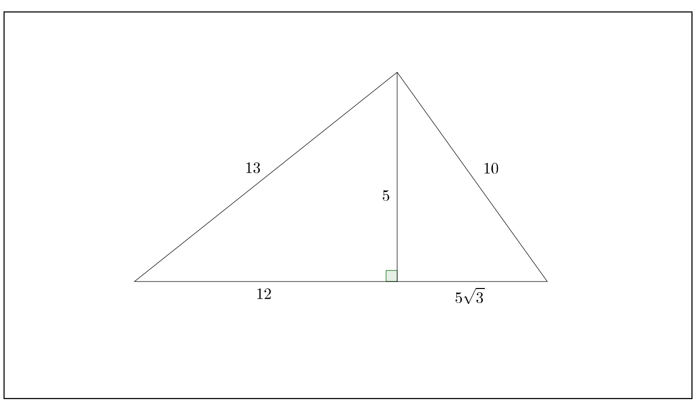
Expected Skills:

- Given the lengths of two sides of a right triangle, be able to use the Pythagorean Theorem to determine the length of the remaining side.
- Be able to calculate the distance between two points in the plane.
- Be able to write an equation of a circle which satisfies some given conditions. Also, be able to identify the center and radius of a circle.
- Be able to find the point on a curve which is closest to or farthest from a given point P.

Practice Problems:

1. For the triangle below, determine the lengths of the two unlabeled sides.





2. Televisions are advertised by the length of the screen's diagonal. A television has a rectangular screen with a height of 36 inches and a length of 64 inches. How should this television be advertised?

$\boxed{4\sqrt{337} \approx 73.43 \text{ inches}}$

3. A ladder of length 25 feet is leaning against a vertical wall. The ladder is initially 7 feet from the wall; but, it is being pushed towards the wall at a constant rate of 2 feet per second. This causes the top of the ladder to slide up the wall.

   (a) How high above the ground is the top of the ladder initially?

   $\boxed{24 \text{ feet}}$

   (b) How high above the ground is the top of the ladder after 1 second has elapsed?

   $\boxed{10\sqrt{6} \text{ feet}}$

4. Let T be the triangle with vertices $A(0,0)$, $B(8,0)$, and $C(4,4\sqrt{3})$.

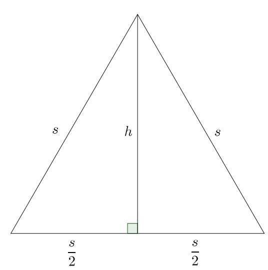
   (a) Show that T is an equilateral triangle.

   We will calculate the lengths of all three sides and verify that they are equal.

   - $d_{AB} = 8$
   - $d_{AC} = \sqrt{(4-0)^2 + (4\sqrt{3}-0)^2} = 8$
   - $d_{BC} = \sqrt{(4-8)^2 + (4\sqrt{3}-0)^2} = 8$

   (b) Show that the area of an equilateral triangle with sides of length s is $A = \dfrac{\sqrt{3}}{4}s^2$.

   Recall that the area of a triangle is $A = \dfrac{1}{2}bh$ and consider the triangle shown below.



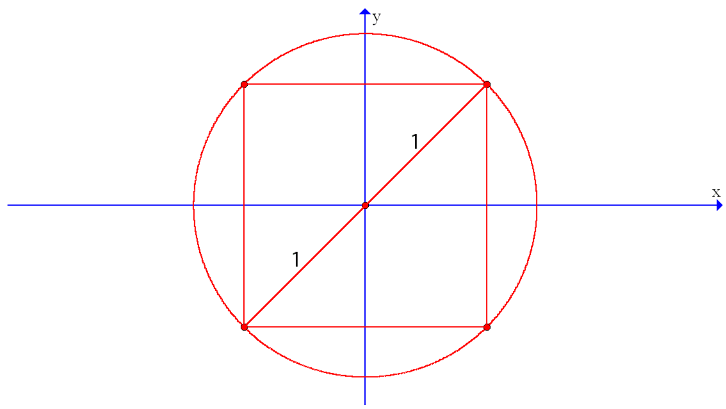
   The length of the base is s. Using the Pythagorean Theorem, you can determine that the height is $h = \dfrac{\sqrt{3}}{2}s$. Thus, the area is $A = \dfrac{1}{2}(s)\left(\dfrac{\sqrt{3}}{2}s\right) = \dfrac{\sqrt{3}}{4}s^2$.

   (c) What is the area of T?

   $\boxed{16\sqrt{3}}$

5. What is the area of a square inscribed in a unit circle?

$\boxed{\text{The area is 2. Hint: consider the following diagram.}}$



6. Find an equation of the circle which has a center of $(3, 5)$ and which has a radius of 6.

$\boxed{(x-3)^2 + (y-5)^2 = 36}$

7. Suppose $A(1, 5)$ and $B(3, -2)$ are endpoints of a diameter of a circle. Find an equation of this circle.

$$\boxed{(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{53}{4}}$$

8. The set of points in the xy-plane which satisfy $x^2 - 2x + y^2 + 10y = -17$ forms a circle.

   (a) What are the center and radius of this circle?

   $\boxed{\text{Center } (1, -5), \text{ radius } r = 3}$

   (b) Does the origin lie inside or outside this circle? Explain.

   $\boxed{\text{The origin is outside of the circle.}}$

9. Consider the square which is centered at the origin and has sides of length 2 which are parallel to the coordinate axes.

   (a) Find an equation of the circle which is inscribed within this square.

   $\boxed{x^2 + y^2 = 1}$

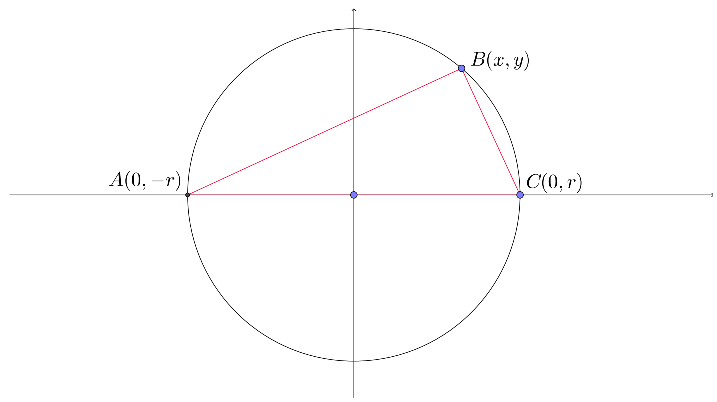
   (b) Find an equation of the circle which is circumscribed around this square.

   $\boxed{x^2 + y^2 = 2}$

10. Find equations of the two tangent circles of equal radii which have centers $C_1(-2,5)$ and $C_2(3,4)$, respectively.

$$\boxed{(x+2)^2+(y-5)^2=\frac{13}{2} \text{ and } (x-3)^2+(y-4)^2=\frac{13}{2}}$$

11. Consider the circle of radius r shown below:



(a) Calculate d_{AB}, d_{BC}, and d_{AC}.

Using the distance formula, we calculate the required lengths:

- $d_{AB} = \sqrt{(x-0)^2+(y+r)^2} = \sqrt{x^2+(y+r)^2}$
- $d_{BC} = \sqrt{(0-x)^2+(r-y)^2} = \sqrt{x^2+(r-y)^2}$
- $d_{AC} = 2r$

(b) Use your result from part (a) to argue that $\triangle ABC$ is a right triangle. (This is Thales' Theorem.)

We will show that $(d_{BC})^2+(d_{AB})^2=(d_{AC})^2$. At some point in our calculation, we will use the fact that the circle has equation $x^2+y^2=r^2$.

$$\begin{aligned}
(d_{BC})^2+(d_{AB})^2 &= \left(\sqrt{x^2+(r-y)^2}\right)^2+\left(\sqrt{x^2+(y+r)^2}\right)^2 \\
&= x^2+(r-y)^2+x^2+(y+r)^2 \\
&= x^2+r^2-2ry+y^2++x^2+y^2+2ry+r^2 \\
&= 2(x^2+y^2)+2r^2 \\
&= 2r^2+2r^2 \\
&= 4r^2 \\
&= (d_{AC})^2
\end{aligned}$$

Thus, since the Pythagorean Theorem is satisfied, $\triangle ABC$ is a right triangle.

12. Consider the curve $y = \sqrt{x}$ on $[0, 10]$ and the point $P(9.5, 0)$

   (a) Find the point on the curve which is closest to P.

   $\boxed{(9,3) \text{ which is a distance of } \frac{1}{2}\sqrt{37} \text{ from } P.}$

   (b) Find the point on the curve which is farthest from to P.

   $\boxed{(0,0) \text{ which is a distance of } \frac{19}{2} \text{ from } P}$

13. Consider the curve $y = x^2$ for $0 \leq x \leq 2$ and the point $P(3, 0)$.

   (a) Find the point on the curve which is closest to P.

   $\boxed{(1,1) \text{ which is a distance of } \sqrt{5} \text{ from } P.}$

   (b) Find the point on the curve which is farthest from to P.

   $\boxed{(2,4) \text{ which is a distance of } \sqrt{17} \text{ from } P}$