Convergence Tests: Divergence, Integral, and p-Series Tests

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Recognize series that cannot converge by applying the Divergence Test.
- Use the Integral Test on appropriate series (all terms positive, corresponding function is decreasing and continuous) to make a conclusion about the convergence of the series.
- Recognize a p-series and use the value of p to make a conclusion about the convergence of the series.
- Use the algebraic properties of series.

PRACTICE PROBLEMS:

For problems 1 – 9, apply the Divergence Test. What, if any, conlcusions can you draw about the series?

1. $\displaystyle\sum_{k=1}^{\infty}(-1)^k$

$\displaystyle\lim_{k\to\infty}(-1)^k$ DNE and thus is not 0, so by the Divergence Test the series diverges.

Also, recall that this series is a geometric series with ratio $r = -1$, which confirms that it must diverge.

2. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\,\frac{1}{k}$

$\displaystyle\lim_{k\to\infty}(-1)^k\,\frac{1}{k} = 0$, so the Divergence Test is inconclusive.

3. $\displaystyle\sum_{k=3}^{\infty}\frac{\ln k}{k}$

$\displaystyle\lim_{k\to\infty}\frac{\ln k}{k} = 0$, so the Divergence Test is inconclusive.

4. $\displaystyle\sum_{k=1}^{\infty} \frac{\ln 6k}{\ln 2k}$

$\boxed{\displaystyle\lim_{k\to\infty} \frac{\ln 6k}{\ln 2k} = 1 \neq 0 \text{ [see } \underline{\text{Sequences}} \text{ problem \#26], so by the Divergence Test the series diverges.}}$

5. $\displaystyle\sum_{k=1}^{\infty} ke^{-k}$

$\displaystyle\lim_{k\to\infty} ke^{-k} = 0$ [see <u>Limits at Infinity Review</u> problem #6], so the Divergence Test is inconclusive.

6. $\displaystyle\sum_{k=1}^{\infty} \frac{e^k - e^{-k}}{e^k + e^{-k}}$

$\displaystyle\lim_{k\to\infty} \frac{e^k - e^{-k}}{e^k + e^{-k}} = 1 \neq 0$ [see <u>Sequences</u> problem #21],

so by the Divergence Test the series diverges.

7. $\displaystyle\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^k$

$\displaystyle\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e \neq 0$ [see <u>Sequences</u> problem #34],

so by the Divergence Test the series diverges.

8. $\displaystyle\sum_{k=1}^{\infty} (\sqrt{k^2 + 8k - 5} - k)$

$\displaystyle\lim_{k\to\infty} (\sqrt{k^2 + 8k - 5} - k) = 4 \neq 0$ [see <u>Sequences</u> problem #28],

so by the Divergence Test the series diverges.

9. $\displaystyle\sum_{k=2}^{\infty} (\sqrt{k^2 + 3} - \sqrt{k^2 - 4})$

$\displaystyle\lim_{k\to\infty} (\sqrt{k^2 + 3} - \sqrt{k^2 - 4}) = 0$, so the Divergence Test is inconclusive.; Detailed Solution: Here

For problems 10 – 20, determine if the series converges or diverges by applying the Divergence Test, Integral Test, or noting that the series is a p-series. Explicitly state what test you are using. If you use the Integral Test, you must first verify that the test is applicable. If the series is a p-series, state the value of p.

10. $\displaystyle\sum_{k=3}^{\infty} \frac{\ln k}{k}$

$\boxed{\text{The series diverges by the Integral Test.}}$

11. $\displaystyle\sum_{k=1}^{\infty} ke^{-k}$

$\boxed{\text{The series converges by the Integral Test.}}$

12. $\displaystyle\sum_{k=1}^{\infty} \left(\arctan\left(\frac{1}{k}\right) - \arctan(k)\right)$

$\boxed{\text{The series diverges by the Divergence Test. [see \underline{Sequences} problem \#33.]}}$

13. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k+15}}$

$\boxed{\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k+15}} = \sum_{k=16}^{\infty} \frac{1}{\sqrt[4]{k}}, \text{ which is a } p\text{-series with } p = \frac{1}{4} < 1, \text{ so the series diverges.}}$

14. $\displaystyle\sum_{k=1}^{\infty} \pi^k e^{-k}$

The series diverges by the Divergence Test. Also, observe that this is a geometric series with ratio $r = \frac{\pi}{e} > 1$, which confirms that the series diverges.

15. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{4k^2}$

$\boxed{\text{The series is a constant multiple of a } p\text{-series with } p = 2 > 1, \text{ so the series converges.}}$

16. $\displaystyle\sum_{k=2}^{\infty} \frac{k^2}{4k^2+9}$

$\boxed{\text{The series diverges by the Divergence Test.}}$

17. $\displaystyle\sum_{k=2}^{\infty} \frac{k}{4k^2+9}$

$\boxed{\text{The series diverges by the Integral Test.}}$

18. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{4k^2+9}$

$\boxed{\text{The series converges by the Integral Test.; Detailed Solution: Here}}$

19. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{4k^2-9}$

$\boxed{\text{The series converges by the Integral Test.; Detailed Solution: Here}}$

20. $\displaystyle\sum_{k=10}^{\infty} 15k^{-0.999}$

$\boxed{\text{The series is a constant multiple of a } p\text{-series with } p = 0.999 < 1 \text{, so the series diverges.}}$

For problems 21 & 22, use algebraic properties of series to find the sum of the series.

21. $\displaystyle\sum_{k=1}^{\infty} \left[\frac{1}{6^k} - \left(\frac{1}{k} - \frac{1}{k+1}\right)\right]$

$\boxed{-\dfrac{4}{5}}$

22. $\dfrac{1}{2} + 2 - \dfrac{1}{4} + \dfrac{4}{7} + \dfrac{1}{8} + \dfrac{8}{49} - \dfrac{1}{16} + \dfrac{16}{343} + \dots$

[Hint: See Infinite Series problems #11 & #12.]

$\boxed{\dfrac{47}{15}}$