Area As A Limit & Sigma Notation

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 5.4 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Understand and know how to evaluate the summation (sigma) notation.
- Be able to use the summation operation's basic properties and formulas. (You do not need to memorize the "Useful Formulas" listed below; if they are needed, they will be provided to you).
- Know how to denote the approximate area under a curve and over an interval as a sum, and be able to find the exact area using a limit of the approximation.
- Be able to find the net signed area between the graph of a function and the x-axis on an interval using a limit.

USEFUL FORMULAS

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

PRACTICE PROBLEMS:

For problems 1-5, evaluate.

1. $\displaystyle\sum_{k=1}^{4} k^3$

2. $\displaystyle\sum_{j=2}^{6} (j^3 - 1)$

3. $\displaystyle\sum_{i=-1}^{3} 2i$

4. $\displaystyle\sum_{k=0}^{5} (-1)^k$

5. $\displaystyle\sum_{k=1}^{5} \sin\left(\frac{\pi}{2}k\right)$

For problems 6-8, use the summation formulas at the top of page 1 to evaluate the given sum.

6. $\displaystyle\sum_{k=1}^{100} (3k-5)$

7. $\displaystyle\sum_{k=1}^{25} [k(k-1)(k+1)]$

8. $\displaystyle\sum_{k=3}^{120} (k+7)$

   (CAUTION: In problem 8, the lower index is not 1; so, the summation formulas at the top of page 1 do not immediately apply!)

For problems 9-12, write the given expression in sigma notation. Do not evaluate the sum. (For each, there are many different ways to write the expression in sigma notation; the answer key illustrates one such way for each.))

9. $4(1)+4(2)+4(3)+4(4)+\cdots+4(20)$

10. $3-6+9-12+\cdots-36$

11. $1+3+5+7+\cdots+21$

12. $2+4+8+16+\cdots+256$

For problems 13-15, express the given summation in closed form.

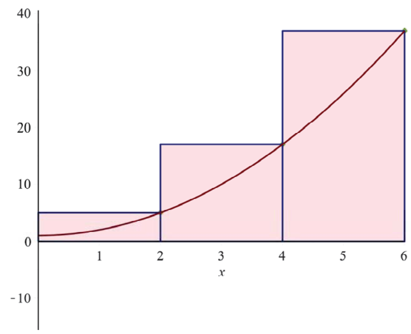
13. $\displaystyle\sum_{j=1}^{n} \frac{j}{n}$

14. $\displaystyle\sum_{k=1}^{n-1} \frac{3k^3}{n}$

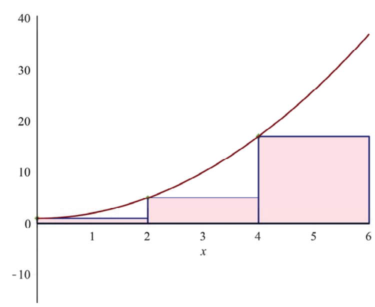
15. $\displaystyle\sum_{k=0}^{n} \left(\frac{1}{n}-\frac{k^2}{n}\right)$
   (CAUTION: In problem 15, the lower limit is not 1; so the summation formulas at the top of page 1 do not immediately apply!)

16. Consider $f(x) = x^2 + 1$.

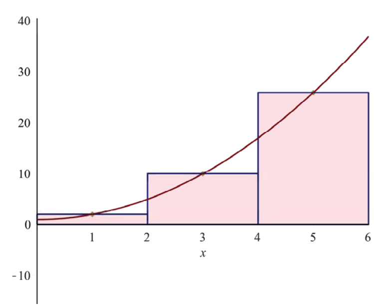
(a) Estimate the area under the graph of $f(x)$ on the interval $[0, 6]$ using 3 rectangles of equal width and right endpoints, as in the diagram below. Is your estimate an overestimate or an underestimate?



(b) Estimate the area under the graph of $f(x)$ on the interval $[0, 6]$ using 3 rectangles of equal width and left endpoints, as in the diagram below. Is your estimate an overestimate or an underestimate?



(c) Estimate the area under the graph of $f(x)$ on the interval $[0, 6]$ using 3 rectangles of equal width and midpoints, as in the diagram below. Is your estimate an overestimate or an underestimate?



17. Let $f(x) = \ln x$.

   (a) Sketch the graph of $f(x)$. Label all asymptotes and intercepts with the coordinate axes.

   (b) Sketch the graph of $f(x)$ on the interval $[e, 5e]$. Divide the interval into 4 subintervals of equal width. On each subinterval, sketch a rectangle using the function value at the **right endpoint** as the height of the rectangle on that subinterval. Estimate the area between the graph of $f(x)$ and the x-axis on the interval $[e, 5e]$ using the 4 rectangles that you sketched. Is your estimate an overestimate or an underestimate?

   (c) Sketch the graph of $f(x)$ on the interval $[e, 5e]$. Divide the interval into 4 subintervals of equal width. On each subinterval, sketch a rectangle using the function value at the **left endpoint** as the height of the rectangle on that subinterval. Estimate the area between the graph of $f(x)$ and the x-axis on the interval $[e, 5e]$ using the 4 rectangles that you sketched. Is your estimate an overestimate or an underestimate?

18. Let $f(x) = x^2 + 1$. By the end of this problem, you will have computed the exact area under the graph of $f(x)$ on the interval $[1, 6]$.

   (a) Find the Δx which is necessary to divide $[1, 6]$ into n subintervals of equal width.

   (b) In each of the n subintervals of equal width, pick x_k^* to be the right endpoint. Fill in the following table:

Subinterval Number	Right Endpoint Number	Right Endpoint of Subinterval
$k = 1$	x_1^*	
$k = 2$	x_2^*	
$k = 3$	x_3^*	
$\vdots$	$\vdots$	$\vdots$
$k = n-1$	x_{n-1}^*	
$k = n$	x_n^*	

   (c) **Fill in the blank:** A closed formula for the right endpoints found in the table above is $x_k^* =$ ____________________, for $k = 1, 2, ..., n-1, n$.

   (d) Determine $f(x_k^*)$, the height of the k^{th} rectangle.

   (e) The right endpoint approximation of the area under the graph of $f(x)$ on the interval $[1, 6]$ using n rectangles of equal width is:

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_{n-1}^*)\Delta x + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Using the appropriate formulas from the top of page 1, express the right endpoint approximation in closed form.

(f) Repeating over finer and finer partitions is equivalent to the number of subintervals, n, approaching infinity. Using this information, compute the exact area under the graph of $f(x) = x^2 + 1$ on the interval $[1, 6]$.

19. For each of the following, use sigma notation and the appropriate summation formulas to evaluate the net signed area between the graph of $f(x)$ and the x-axis on the given interval. Let x_k^* be the **right endpoint** of the k^{th} subinterval (where all subintervals have equal width).

    (a) $f(x) = x - 3$ on $[1, 5]$

    (b) $f(x) = \dfrac{x^2}{3}$ on $[2, 5]$

    (c) $f(x) = x^3 - 1$ on $[0, 2]$

20. For each of the following, use sigma notation and the appropriate summation formulas to evaluate the net signed area between the graph of $f(x)$ and the x-axis on the given interval. Let x_k^* be the **left endpoint** of the k^{th} subinterval (where all subintervals have equal width).

    (a) $f(x) = x - 3$ on $[1, 5]$

    (b) $f(x) = \dfrac{x^2}{3}$ on $[2, 5]$

    (c) $f(x) = x^3 - 1$ on $[0, 2]$

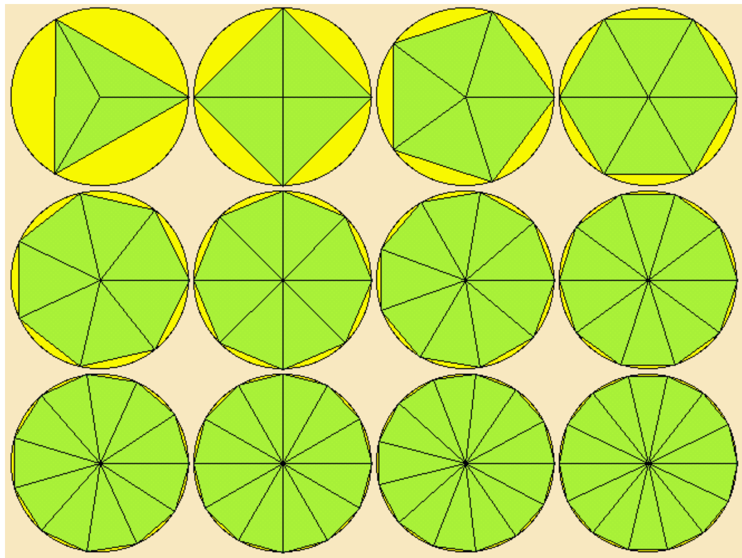
21. For each of the following, use sigma notation and the appropriate summation formulas to evaluate the net signed area between the graph of $f(x)$ and the x-axis on the given interval. Let x_k^* be the **midpoint** of the k^{th} subinterval (where all subintervals have equal width).

    (a) $f(x) = x - 3$ on $[1, 5]$

    (b) $f(x) = \dfrac{x^2}{3}$ on $[2, 5]$

22. Use sigma notation and the appropriate summation formulas to formulate an expression which represents the net signed area between the graph of $f(x) = \cos x$ and the x-axis on the interval $[-\pi, \pi]$. Let x_k^* be the **right endpoint** of the k^{th} subinterval (where all subintervals have equal width). DO NOT EVALUATE YOUR EXPRESSION.

23. A Regular Polygon is a polygon that is equiangular (all angles are equal in measure) and equilateral (all sides have the same length). The diagram below shows several regular polygons inscribed within a circle of radius r.



(a) Let A_n be the area of a regular n-sided polygon inscribed within a circle of radius r. Divide the polygon into n congruent triangles each with a central angle of $\dfrac{2\pi}{n}$ radians, as shown in the diagram above for several different values of n. Show that $A_n = \dfrac{1}{2}r^2 \sin\left(\dfrac{2\pi}{n}\right)n$.

(b) What can you conclude about the area of the n-sided polygon as the number of sides of the polygon, n, approaches infinity? In other words, compute $\lim\limits_{n\to\infty} A_n$.