Convergence of Taylor Series

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Know (i.e. memorize) the Remainder Estimation Theorem, and use it to find an upper bound on the error in approximating a function with its Taylor polynomial.
- Find the value(s) of x for which a Taylor series converges to a function $f(x)$.

PRACTICE PROBLEMS:

1. Find an upper bound for the remainder error if the 4th Maclaurin polynomial for $f(x) = \cos x$ is used to approximate $\cos 5°$.

2. Find an upper bound for the remainder error if the 2nd Maclaurin polynomial for $f(x) = e^x$ is used to approximate $\sqrt{e}$?
Note: You may assume that $\sqrt{e} < 2$ (this should be clear since $\sqrt{e} < \sqrt{3} < \sqrt{4} = 2$).

3. Find the smallest value of n that is needed so that the n-th Macluarin polynomial $p_n(x)$ approximates $\sqrt{e}$ to four decimal-place accuracy. In other words, find the smallest value of n so that the n-th remainder $|R_n(\frac{1}{2})| \leq 0.00005$.
Note: You may assume that $\sqrt{e} < 2$ (this should be clear since $\sqrt{e} < \sqrt{3} < \sqrt{4} = 2$).

4. Find the smallest value of n so that the Taylor polynomial for $f(x) = \ln(x)$ about $x_0 = 1$ approximates $\ln(1.2)$ to three decimal-place accuracy.

5. The purpose of this problem is to show that the Maclaurin series for $f(x) = \cos x$ converges to $\cos x$ for all x.

   (a) Find the Maclaurin series for $f(x) = \cos x$.

   (b) Find the interval of convergence for this Maclaurin series.

   (c) Show that the n-th remainder goes to 0 as n goes to $+\infty$,
   i.e. show that $\lim\limits_{n\to+\infty} |R_n(x)| = 0$.

6. Show that the Maclaurin series for $f(x) = \frac{1}{1-x}$ converges to $f(x)$ for all x in its interval of convergence.

7. The pupose of this problem is to show that it is possible for a function $f(x)$ to have a Maclaurin series that converges for all x but does not always converge to $f(x)$.

   Consider the piecewise function $f(x) = \begin{cases} e^{(-1/x^2)}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

(a) Use the definition of the derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ to show that $f'(0) = 0$. Hint: Make the substitution $t = \frac{1}{h}$ and compute the one-sided limits as $h \to 0^+$ and $h \to 0^-$.

(b) Assuming that $f^{(n)}(0) = 0$ for $n \geq 2$, find the Macluarin series for $f(x)$ and the interval of convergence for the series.

(c) Find the values of x for which the Maclaurin series converges to $f(x)$.