Multivariable Chain Rule

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 13.5 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to compute partial derivatives with the various versions of the multivariate chain rule.
- Be able to compare your answer with the direct method of computing the partial derivatives.

PRACTICE PROBLEMS:

1. Find $\frac{dz}{dt}$ by using the Chain Rule. Check your answer by expressing z as a function of t and then differentiating.

   (a) $z = 2x - y,\ x = \sin t,\ y = 3t$

   (b) $z = x \sin y,\ x = e^t,\ y = \pi t$

   (c) $z = xy + y^2,\ x = t^2,\ y = t + 1$

   (d) $z = \ln\left(\frac{x^2}{y}\right),\ x = e^t,\ y = t^2$

2. Suppose $w = x^2 + y^2 + 2z^2,\ x = t + 1,\ y = \cos t,\ z = \sin t$. Find $\frac{dw}{dt}$ using the Chain Rule. Check your answer by expressing w as a function of t and then differentiating.

3. Suppose f is a differentiable function of x & y, and define $g(u, v) = f(3u - v, u^2 + v)$. Use the table of values shown below to calculate $\left.\frac{\partial g}{\partial u}\right|_{(u,v)=(2,-1)}$ and $\left.\frac{\partial g}{\partial v}\right|_{(u,v)=(2,-1)}$.

(x, y)	f	g	f_x	f_y
$(2, -1)$	6	-7	1	9
$(7, 3)$	4	2	-3	5

   Hint: Decompose $f(3u - v, u^2 + v)$ into $f(x, y)$ where $x = 3u - v$ and $y = u^2 + v$.

4. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ by using the appropriate Chain Rule.

   (a) $w = xy\sin\left(z^2\right),\ x = s - t,\ y = s^2,\ z = t^2$

   (b) $w = xy + yz,\ x = s + t,\ y = st,\ z = s - 2t$

5. Suppose that $J = f(x, y, z, w)$, where $x = x(r, s, t)$, $y = y(r, t)$, $z = z(r, s)$ and $w = w(s, t)$. Use the Chain Rule to find $\frac{\partial J}{\partial r}$, $\frac{\partial J}{\partial s}$, and $\frac{\partial J}{\partial t}$.

6. Suppose $g = f(u - v, v - w, w - u)$. Show that $\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v} + \frac{\partial g}{\partial w} = 0$.

7. Suppose $u = u(x, y)$, $v = v(x, y)$, $x = r\cos\theta$, and $y = r\sin\theta$.

   (a) Calculate $\dfrac{\partial u}{\partial r}$, $\dfrac{\partial u}{\partial \theta}$, $\dfrac{\partial v}{\partial r}$, and $\dfrac{\partial v}{\partial \theta}$

   (b) Suppose that $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

   Use this along with part (a) to derive the polar form of the Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$