Partial Fraction Decomposition

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 7.5 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to recognize an improper rational function, and perform the necessary long division to turn it into a proper rational function.
- Know how to write down the partial fraction decomposition for a proper rational function, compute the unknown coefficients in the partial fractions, and integrate each partial fraction.

PRACTICE PROBLEMS:

For problems 1-3, write out the partial fraction decomposition. (Do not solve for the numerical values of the coefficients.)

1. $\dfrac{2x+3}{(x-2)(x-5)}$

   $$\boxed{\frac{A}{x-2}+\frac{B}{x-5}}$$

2. $\dfrac{6}{x^2(x^2-9)}$

   $$\boxed{\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-3}+\frac{D}{x+3}}$$

3. $\dfrac{5x^4-1}{x(x-2)(x^2+x+1)^2}$

   $$\boxed{\frac{A}{x}+\frac{B}{x-2}+\frac{Cx+D}{x^2+x+1}+\frac{Ex+F}{(x^2+x+1)^2}}$$

For problems 4 & 5, use the given partial fraction decomposition to evaluate the integral.

4. $\displaystyle\int \frac{11x^2-28x+20}{(2x+1)(x-3)^2}\,dx$

Hint: $\dfrac{11x^2 - 28x + 20}{(2x+1)(x-3)^2} = \dfrac{3}{2x+1} + \dfrac{4}{x-3} + \dfrac{5}{(x-3)^2}$

$\boxed{\dfrac{3}{2}\ln|2x+1| + 4\ln|x-3| - \dfrac{5}{(x-3)} + C}$

5. $\displaystyle\int \frac{11x^2 + 7x - 15}{(3x-4)(x^2+1)}\,dx$

   Hint: $\dfrac{11x^2 + 7x - 15}{(3x-4)(x^2+1)} = \dfrac{5}{3x-4} + \dfrac{2x+5}{x^2+1}$

   $\boxed{\dfrac{5}{3}\ln|3x-4| + \ln(x^2+1) + 5\tan^{-1}x + C}$

For problems 6-17, evaluate the given integral.

6. $\displaystyle\int \frac{4}{x^2-1}\,dx$

   $\boxed{2\ln|x-1| - 2\ln|x+1| + C}$

7. $\displaystyle\int \frac{4x-1}{x^2-5x+6}\,dx$

   $\boxed{11\ln|x-3| - 7\ln|x-2| + C}$

8. $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

   $\boxed{x - \tan^{-1}x + C}$

9. $\displaystyle\int \frac{3x^2-4}{x+1}\,dx$

   $\boxed{\dfrac{3}{2}x^2 - 3x - \ln|x+1| + C}$

10. $\displaystyle\int \frac{4x-1}{2x^2-18x+36}\,dx$

    $\boxed{\dfrac{23}{6}\ln|x-6| - \dfrac{11}{6}\ln|x-3| + C}$

11. $\displaystyle\int \frac{1}{x^2(x-1)^2}\,dx$

    $\boxed{2\ln|x| - \dfrac{1}{x} - 2\ln|x-1| - \dfrac{1}{(x-1)} + C}$; Detailed Solution: Here

12. $\displaystyle\int \frac{2x+3}{(x-3)(x+1)^2}\,dx$

$\boxed{\dfrac{9}{16}\ln|x-3| - \dfrac{9}{16}\ln|x+1| + \dfrac{1}{4(x+1)} + C}$

13. $\displaystyle\int \frac{x^4-4x^2+5}{x^3-4x}\,dx$

$\boxed{\dfrac{x^2}{2} - \dfrac{5}{4}\ln|x| + \dfrac{5}{8}\ln|x+2| + \dfrac{5}{8}\ln|x-2| + C}$

14. $\displaystyle\int \frac{x^5-3x^3+6}{x^3+x}dx$

$\boxed{\dfrac{1}{3}x^3 - 4x + 6\ln|x| - 3\ln|x^2+1| + 4\arctan(x) + C\text{; Detailed Solution: Here}}$

15. $\displaystyle\int \frac{x^3-6x^2+3x-17}{x^2+3}\,dx$

$\boxed{\dfrac{1}{2}x^2 - 6x + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right) + C}$

16. $\displaystyle\int \frac{2x^2}{(x-1)^3}dx$

$\boxed{2\ln|x-1| - \dfrac{4}{x-1} - \dfrac{1}{(x-1)^2} + C}$

17. $\displaystyle\int \frac{4x^2-4x+2}{x^2-x}dx$

$\boxed{4x - 2\ln|x| + 2\ln|x-1| + C}$

For problems 18-19, evaluate the given integral by making substitution that transforms the problem into integrating a rational function.

18. $\displaystyle\int \frac{\sin x}{\cos^2 x + 6\cos x + 5}\,dx$

$\boxed{\dfrac{1}{4}\ln|\cos x + 5| - \dfrac{1}{4}\ln|\cos x + 1| + C}$

19. $\displaystyle\int \frac{e^{5x}}{e^{4x}-1}\,dx$

$\boxed{e^x + \dfrac{1}{4}\ln|e^x-1| - \dfrac{1}{4}\ln(e^x+1) - \dfrac{1}{2}\tan^{-1}(e^x) + C}$

20. By the end of this problem, you will know the antiderivatives of $\sec x$. Observe the following:

$$\begin{aligned}\int \sec x\,dx &= \int \frac{1}{\cos x}\,dx\\ &= \int \frac{\cos x}{\cos^2 x}\,dx\\ &= \int \frac{\cos x}{1-\sin^2 x}\,dx\end{aligned}$$

(a) Use the substitution $u = \sin x$ to convert the given integral to an integral of a rational function.

$\boxed{\int \frac{1}{1-u^2}\,du}$

(b) Use partial fractions to evaluate your integral from part (a). Show that the antiderivatives have the form $\frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + C$

$$\boxed{\begin{aligned}\int \frac{1}{1-u^2}\,du &= \int \frac{1}{2}\left(\frac{1}{u+1}\right) - \frac{1}{2}\left(\frac{1}{u-1}\right)\,du\\ &= \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1| + C\\ &= \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + C\end{aligned}}$$

Notice that substituting back in for u yields:

$$\begin{aligned}\int \sec x\,dx &= \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + C\\ &= \frac{1}{2}\ln\left|\frac{(\sin x+1)(\sin x+1)}{(\sin x-1)(\sin x+1)}\right| + C\\ &= \frac{1}{2}\ln\left|\frac{(\sin x+1)^2}{\sin^2 x-1}\right| + C\\ &= \frac{1}{2}\ln\left|\frac{(\sin x+1)^2}{\cos^2 x}\right| + C\\ &= \ln\left|\frac{\sin x+1}{\cos x}\right| + C\\ &= \ln|\tan x + \sec x| + C\end{aligned}$$