The Comparison, Limit Comparison, Ratio, & Root Tests

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Use the following tests to make a conclusion about the convergence of series with no negative terms:
  - Comparison Test
  - Limit Comparison Test
  - Ratio Test
  - Root Test

PRACTICE PROBLEMS:

For problems 1 & 2, apply the Comparison Test to determine if the series converges. Clearly state to which other series you are comparing.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k + 5}$

$\dfrac{1}{3^k+5} < \dfrac{1}{3^k}$ for $k \geq 1$.

Since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k}$ converges (geometric series, $r = \frac{1}{3}$), $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k+5}$ must converge.

2. $\displaystyle\sum_{k=3}^{\infty} \frac{1}{3(k-2)}$

$\dfrac{1}{3(k-2)} = \dfrac{1}{3k-6} > \dfrac{1}{3k}$ for $k \geq 3$.

Since $\displaystyle\sum_{k=3}^{\infty} \frac{1}{3k}$ diverges (p-series with $p = 1$), $\displaystyle\sum_{k=3}^{\infty} \frac{1}{3(k-2)}$ must diverge as well.

For problems 3 & 4, apply the Limit Comparison Test to determine if the series converges. Clearly state to which other series you are comparing.

3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3(k+2)}$

$\displaystyle\lim_{k\to\infty} \frac{\frac{1}{3(k+2)}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{3k+6} = \frac{1}{3}$, which is finite and nonzero.

So since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (Harmonic Series), $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3(k+2)}$ must diverge as well.

4. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{3^k - 5}$

$\displaystyle\lim_{k\to\infty} \frac{\frac{1}{3^k-5}}{\frac{1}{3^k}} = \lim_{k\to\infty} \frac{3^k}{3^k - 5} = 1$, which is finite and nonzero.

So since $\displaystyle\sum_{k=2}^{\infty} \frac{1}{3^k}$ converges (geometric series, $r = \frac{1}{3}$), $\displaystyle\sum_{k=2}^{\infty} \frac{1}{3^k - 5}$ must converge.

For problems 5 – 7, apply the Ratio Test to determine if the series converges. If the Ratio Test is inconclusive, apply a different test.

5. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$

The series converges by the Ratio Test.

6. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{3^k}$

The series converges by the Ratio Test. [Of course this just confirms what we already knew as this is a geometric series with $r = \frac{1}{3}$.]

7. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3(k+2)}$

The Ratio Test is inconclusive; however, as shown in #3 above, the series diverges by the Limit Comparison Test.

For problems 8 – 10, apply the Root Test to determine if the series converges. If the Root Test is inconclusive, apply a different test.

8. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{3^k}$

The series converges by the Root Test. [Of course this just confirms what we already knew as this is a geometric series with $r = \frac{1}{3}$.]

9. $\displaystyle\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^k$

The Root Test is inconclusive; however, as shown in the previous assigned problems, Convergence Tests #7, the series diverges by the Divergence Test.

10. $\displaystyle\sum_{k=1}^{\infty}\frac{k}{7^k}$

The series converges by the Root Test.; Detailed Solution: Here

For problems 11 – 22, apply the Comparison Test, Limit Comparison Test, Ratio Test, or Root Test to determine if the series converges. State which test you are using, and if you use a comparison test, state to which other series you are comparing to.

11. $\displaystyle\sum_{k=1001}^{\infty}\frac{1}{\sqrt[3]{k}-10}$

The series diverges by the Comparison Test. Compared to $\displaystyle\sum_{k=1001}^{\infty}\frac{1}{\sqrt[3]{k}}$.

12. $\displaystyle\sum_{k=1}^{\infty}\frac{4k^2+5k}{\sqrt{10+k^5}}$

The series diverges by the Limit Comparison Test. Compared to $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$.;
Detailed Solution: Here

13. $\displaystyle\sum_{k=0}^{\infty}\frac{2k+1}{(2k)!}$

The series converges by the Ratio Test.

14. $\displaystyle\sum_{k=1}^{\infty}\frac{k^2\cos^2 k}{2+k^5}$

The series converges by the Comparison Test. Compared to $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^3}$.

15. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{(5k)}}$

The series converges by the Root Test.

16. $\displaystyle\sum_{k=0}^{\infty} \frac{2+2^k}{3+3^k}$

$\boxed{\text{The series converges by the Limit Comparison Test. Compared to } \displaystyle\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k.}$

17. $\displaystyle\sum_{k=0}^{\infty} \frac{6^{(k+1)}}{k!}$

$\boxed{\text{The series converges by the Ratio Test.}}$

18. $\displaystyle\sum_{k=0}^{\infty} \frac{6^k+k}{k!+6}$ [Hint:Use the result from the previous problem.]

$\boxed{\text{The series converges by the Comparison Test. Compared to } \displaystyle\sum_{k=0}^{\infty} \frac{6^{(k+1)}}{k!}.\text{; Detailed Solution: Here}}$

19. $\displaystyle\sum_{k=2}^{\infty} \frac{k^3-2}{(k^2+1)^2}$

$\boxed{\text{The series diverges by the Limit Comparison Test. Compared to } \displaystyle\sum_{k=2}^{\infty} \frac{1}{k}.}$

20. $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan k}{k^{1.5}}$

$\boxed{\text{The series converges by the Comparison Test. Compared to } \displaystyle\sum_{k=1}^{\infty} \frac{\pi/2}{k^{1.5}}.}$

21. $\displaystyle\sum_{k=1}^{\infty} \frac{7k}{k^2+|\sin k|}$

$\boxed{\text{The series diverges by the Limit Comparison Test. Compared to } \displaystyle\sum_{k=1}^{\infty} \frac{1}{k}.}$

22. $\displaystyle\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2}$

$\boxed{\text{The series diverges by the Ratio Test.}}$