Power Series

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Use sigma notation to write the Maclaurin series for a function $f(x)$.
- Use sigma notation to write the Taylor series for a function $f(x)$ about a specified $x = x_0$.
- Find the interval of convergence and the radius of convergence of a power series.
- Find the domain of a function that is expressed as a power series.

PRACTICE PROBLEMS:

For problems 1 & 2, use sigma notation to write the Macluarin series for the given function.

1. $f(x) = \ln(1 + x)$

   $\boxed{\sum_{k=1}^{\infty} (-1)^{k+1} \dfrac{x^k}{k}\text{ ; Compare this to Polynomial Approximations of Functions \#7.}}$

2. $f(x) = x \cos x$

   $\boxed{\sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+1}}{(2k)!}}$

For problems 3 & 4, use sigma notation to write the Taylor series for the given function about $x = x_0$.

3. $f(x) = e^{2x};\ x_0 = \ln 3$

   $\boxed{\sum_{k=0}^{\infty} \dfrac{2^k(9)}{k!}(x - \ln 3)^k\text{ ; Compare this to Polynomial Approximations of Functions \#9.}}$

4. $f(x) = \sin x;\ x_0 = \dfrac{\pi}{2}$

   $\boxed{\sum_{k=0}^{\infty} (-1)^k \dfrac{\left(x - \frac{\pi}{2}\right)^{2k}}{(2k)!}}$

For problems 5 – 13, find the interval of convergence and the radius of convergence R for the power series.

5. $x + x^2 + x^3 + x^4 + \ldots$

   $\boxed{(-1, 1); R = 1. \text{ Note that this is the power series from } \underline{\text{Infinite Series}} \text{ \#24.}}$

6. $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \ldots$

   $\boxed{(-1, 1]; R = 1. \text{ This is the Maclaurin series for } f(x) = \ln(1+x). \text{ See problem \#1.}}$

7. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$

   $\boxed{(-\infty, +\infty); R = +\infty. \text{ This is the Maclaurin series for } f(x) = e^x.}$

8. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$

   $\boxed{[-1, 1]; R = 1.}$

9. $\displaystyle\sum_{k=0}^{\infty} \frac{(-5)^k x^k}{\sqrt{k+10}}$

   $\boxed{\left(-\frac{1}{5}, \frac{1}{5}\right]; R = \frac{1}{5}; \text{ Detailed Solution: Here}}$

10. $\displaystyle\sum_{k=0}^{\infty} \left[(2k)!\ (2x+1)^k\right]$

    $\boxed{\left[-\frac{1}{2}, -\frac{1}{2}\right], \text{ or just } \left\{-\frac{1}{2}\right\}; R = 0. \text{ In other words, the series converges only when } x = -\frac{1}{2}.}$

11. $\displaystyle\sum_{k=0}^{\infty} \left[\left(\frac{2}{7}\right)^k (x+4)^{k+1}\right]$

    $\boxed{\left(-\frac{15}{2}, -\frac{1}{2}\right); R = \frac{7}{2}; \text{ Detailed Solution: Here}}$

12. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

    $\boxed{(-\infty, +\infty); R = +\infty. \text{ This is the Maclaurin series for } f(x) = \sin x.}$

13. $\displaystyle\sum_{k=2}^{\infty} \frac{(x-3)^k}{k \ln k}$

$\boxed{[2,4); R=1}$

For problems 14 – 16, a function is represented as a power series. Find the domain of the function.

14. $f(x) = \displaystyle\sum_{k=0}^{\infty} \left[(-1)^{k+1}\,(x-2)^k\right]$

The domain of $f(x)$ is $1 < x < 3$. This is the Taylor series for $f(x) = \dfrac{1}{1-x}$ about $x = 2$. See Polynomial Approximations of Functions #8.

15. $f(x) = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

The domain of $f(x)$ is all real numbers. This is the Maclaurin series for $f(x) = \cos x$.

16. $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{e^{(k^2)} x^k}{k!}$

The domain of $f(x)$ is only $x = 0$.; Detailed Solution: Here