Improper Integrals

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 7.8 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Given an improper integral, which either has an infinite interval of integration or an infinite discontinuity, be able to evaluate it using a limit.

- Know how to determine if such an integral converges (and if so, what it converges to) or diverges.

PRACTICE PROBLEMS:

For problems 1-13, evaluate each improper integral or show that it diverges.

1. $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt{x}}\, dx$

2. $\displaystyle\int_{-\infty}^{3} \frac{3x}{x^2 + 1}\, dx$

3. $\displaystyle\int_{1}^{\infty} e^{-x}\, dx$

4. $\displaystyle\int_{1}^{\infty} xe^{-3x^2}\, dx$

5. $\displaystyle\int_{0}^{4} \frac{1}{x^{2/3}}\, dx$

6. $\displaystyle\int_{4}^{\infty} \frac{1}{(x-2)^3}\, dx$

7. $\displaystyle\int_{2}^{6} \frac{1}{\sqrt{x-2}}\, dx$

8. $\displaystyle\int_{0}^{2} \frac{2}{\sqrt{4-x^2}}\, dx$

9. $\displaystyle\int_{0}^{\infty} \frac{1}{x^2 + 4x + 5}\, dx$ (Hint: Complete the square)

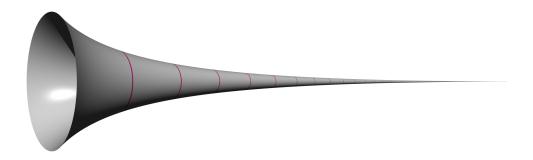10. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt[3]{\cos x}}\, dx$

11. $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\,\sec^2 x\, dx$

12. $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{(x-1)^2}}\, dx$

13. $\displaystyle\int_0^1 \frac{1}{x\ln x}\, dx$

14. Find the value of the constant k so that $\displaystyle\int_{-\infty}^{\infty} \frac{k}{1+x^2}\, dx = 1$.

15. Compute the exact area between the graph of $y = \dfrac{4}{x^2-1}$ and the x-axis for $x \geq 7$.

16. Consider Gabriel's Horn (shown below) which is formed by revolving the curve $y = \dfrac{1}{x}$ on $[1, \infty)$ around the x-axis.

Show that the volume within this horn is finite.

Note: It can be shown the surface area of this horn is infinite. Thus, it appears that the horn can be filled with a finite amount of paint; but, there is not enough to paint the inside of the surface for a coating of uniform thickness. This is called **The Paradox of Gabriel's Horn**.

17. Determine the values of the constant p for which the following integral will converge and those for which it will diverge.

$$\int_1^\infty \frac{1}{x^p}\, dx$$

(Hint: Consider two cases – when $p = 1$ and when $p \neq 1$.)

18. Determine the values of the constant p for which the following integral will converge and those for which it will diverge.

$$\int_0^1 \frac{1}{x^p}\, dx$$

(Hint: Consider two cases – when $p = 1$ and when $p \neq 1$.)

19. Consider the Gamma Function: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$, which is defined for $\alpha > 0$.

    (a) Compute $\Gamma(1)$.
    (b) Compute $\Gamma(2)$.
    (c) Compute $\Gamma(3)$.

20. If $f(x)$ is a continuous for $x \geq 0$, the **Laplace Transform** of $f(x)$ is given by:

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty f(x) e^{-sx}\, dx$$

    and the domain of $\mathcal{L}\{f(x)\}(s)$ is the set consisting of all numbers s for which the integral converges. Laplace Transforms are useful for solving differential equations

    (a) Compute the Laplace Transform of $f(x) = 1$ and state its domain.
    (b) Compute the Laplace Transform of $f(x) = e^x$ and state its domain.
    (c) Compute the Laplace Transform of $f(x) = x$ and state its domain.

21. **Definition:** In probability theory, the **probability density function (pdf)** of a random variable X is a function $f(x)$ from which we can compute the probability that X lies in the interval $[a, b]$ as follows:

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

    And, in order for $f(x)$ to be a valid pdf, it must satisfy the following:

    - $f(x) \geq 0$ for all values of x
    - $\int_{-\infty}^\infty f(x)\, dx = 1$

    (a) Verify that $f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$ is a valid probability density function.

    (b) Using the density function in part a, compute $P(0 \leq X \leq 1)$.

(c) **Definition:** The **cumulative distribution function (CDF)** for a continuous random variable X is defined as:

$$F(t) = P(X \leq t) = \int_{-\infty}^{t} f(x)\,dx$$

This CDF describes the accumulation of probability up to the real number t. Compute the CDF for the random variable X which has the density function from part a.