Monotone Sequences

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Understand what it means for a sequence to be increasing, decreasing, strictly increasing, strictly decreasing, eventually increasing, or eventually decreasing.
- Use an approriate test for monotonicity to determine if a sequence is increasing or decreasing.
- Show that a sequence must converge to a limit by showing that it is montone and appropriately bounded.

PRACTICE PROBLEMS:

1. Give an example of a convergent sequence that is not a monotone sequence.

   One possibility is $\left\{(-1)^n \dfrac{1}{n}\right\}_{n=1}^{+\infty} = -1, \dfrac{1}{2}, -\dfrac{1}{3}, \dfrac{1}{4}, \ldots$, which converges to 0 but is not monotonic.

2. Give an example of a sequence that is bounded from above and bounded from below but is not convergent.

   One possibility is $\{(-1)^n\}_{n=1}^{+\infty} = -1, 1, -1, 1, -1, 1...$, which is bounded from above by 1 (or any number greater than 1) and is bounded below by -1 (or any number less than -1). However, the sequence diverges since its terms oscillate between 1 and -1.

For problems 3 and 4, determine if the sequence is increasing or decreasing by calculating $a_{n+1} - a_n$.

3. $\left\{\dfrac{1}{4^n}\right\}_{n=1}^{+\infty}$

   The sequence is (strictly) decreasing.

4. $\left\{\dfrac{2n-3}{3n-2}\right\}_{n=1}^{+\infty}$

   The sequence is (strictly) increasing.

For problems 5 amd 6, determine if the sequence is increasing or decreasing by calculating $\dfrac{a_{n+1}}{a_n}$.

5. $\left\{ \dfrac{1}{4^n} \right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) decreasing.}}$

6. $\left\{ \dfrac{e^n - e^{-n}}{e^n + e^{-n}} \right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) increasing.; Detailed Solution: Here}}$

For problems 7 and 8, determine if the sequence is increasing or decreasing by calculating the derivative a'_n.

7. $\left\{ \dfrac{1}{4^n} \right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) decreasing.}}$

8. $\left\{ \dfrac{\ln(2n)}{\ln(6n)} \right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) increasing.}}$

For problems 9 – 17, use an appropriate test for monotonicity to determine if the sequence increases, decreases, eventually increases, or eventually decreases.

9. $\left\{ \dfrac{3n}{2n+1} \right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) increasing.}}$

10. $\left\{ n - \dfrac{1}{n} \right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) increasing.}}$

11. $\left\{ \dfrac{n^2}{n!} \right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is eventually (strictly) decreasing.}}$

12. $\left\{ \dfrac{2n+1}{(2n)!} \right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) decreasing.; Detailed Solution: Here}}$

13. $\left\{\dfrac{e^{\sqrt{n}}}{n}\right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is eventually (strictly) increasing.}}$

14. $\left\{e^n \pi^{-n}\right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) decreasing.}}$

15. $\left\{\dfrac{3^{(n^2)}}{(1000)^n}\right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is eventually (strictly) increasing.}}$

16. $\left\{\dfrac{n!}{n^n}\right\}_{n=1}^{+\infty}$

$\boxed{\text{The sequence is (strictly) decreasing.}}$

17. $\left\{n^3 e^{-n}\right\}_{n=1}^{+\infty}$

The sequence is eventually (strictly) decreasing.; Detailed Solution: Here

18. In the previous set of assigned problems it was shown that **if** the sequence

$$\sqrt{30}, \sqrt{30+\sqrt{30}}, \sqrt{30+\sqrt{30+\sqrt{30}}}, \ldots$$

converged to a limit, that limit was 6. Now we will show that the sequence is bounded above and increasing; thus, it must converge.

(a) Define the sequence recursively.

$\boxed{a_1 = \sqrt{30}, a_{n+1} = \sqrt{30 + a_n} \text{ for integers } n \geq 1.}$

(b) Show that the sequence has an upper bound of 6.

$a_1 = \sqrt{30} < \sqrt{36} = 6$, so $a_1 < 6$.
$a_2 = \sqrt{30 + a_1} < \sqrt{30 + 6} = 6$, so $a_2 < 6$.
$a_3 = \sqrt{30 + a_2} < \sqrt{30 + 6} = 6$, so $a_3 < 6$.
This continues indefinitely, so $a_n < 6$ for all integers $n \geq 1$, i.e. the sequence is bounded from above by 6. (It is also bounded from below by 0).

(c) Show that the sequence is increasing by computing $a_{n+1}^2 - a_n^2$.

$a_{n+1}^2 - a_n^2 = 30 + a_n - a_n^2 = (5 + a_n)(6 - a_n)$.
Now from part (b) $0 < a_n < 6$, so $5 + a_n > 0$ and $6 - a_n > 0$, so $a_{n+1}^2 - a_n^2 > 0$.
Also, $a_{n+1}^2 - a_n^2 = (a_{n+1} - a_n)(a_{n+1} + a_n)$, so $(a_{n+1} - a_n)(a_{n+1} + a_n) > 0$.
Since every term in the sequence is positive, we now have $(a_{n+1} - a_n) > 0$, or $a_{n+1} > a_n$, i.e. the sequence is (strictly) increasing.