Chapter 3.6: Limits & Continuity of Trig. Functions

Expected Skills:

- Know where the trigonometric functions are continuous and be able to evaluate basic trigonometric limits.
- Be able to use $\lim_{x \to 0} \frac{\sin x}{x} = 1$ or $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ to help find the limits of functions involving trigonometric expressions, when appropriate.
- Understand the squeeze theorem and be able to use it to compute certain limits.

Practice Problems:

Evaluate the following limits using the squeeze theorem.

1. Let $f(x)$ be a function which satisfies $5x - 6 \leq f(x) \leq x^2 + 3x - 5$ for all $x \geq 0$. Compute $\lim_{x \to 1} f(x)$.

   $\boxed{-1}$

2. $\lim_{x \to \infty} \frac{x + \cos x}{3x + 1}$

   Notice that $f(x) = \frac{x + \cos x}{3x + 1}$ can be bounded as follows:

   $$\frac{x-1}{3x+1} \leq \frac{x + \cos x}{3x + 1} \leq \frac{x+1}{3x+1}$$

   Since $\lim_{x \to \infty} \frac{x-1}{3x+1} = \lim_{x \to \infty} \frac{x+1}{3x+1} = \frac{1}{3}$, it follows that $\lim_{x \to \infty} \frac{x + \cos x}{3x + 1} = \frac{1}{3}$.

For problems 3-19, evaluate the given limit. If a limit does not exist, write DNE, $+\infty$, or $-\infty$ (whichever is most appropriate).

3. $\lim_{x \to \frac{\pi}{4}} \sin(2x)$

   $\boxed{1}$

4. $\lim_{\theta \to \pi} (\theta \cos \theta)$

   $\boxed{-\pi}$

5. $\lim_{x \to 0^+} \csc x$

   $\boxed{+\infty}$

6. $\lim_{x \to \frac{\pi}{2}^+} \tan x$

$\boxed{-\infty}$

7. $\lim_{x \to \frac{\pi}{2}^-} \tan x$

$\boxed{+\infty}$

8. $\lim_{x \to \frac{\pi}{4}} \sec x$

$\boxed{\sqrt{2}}$

9. $\lim_{x \to 0} \left(\frac{\sin x}{3x} \right)$

$\boxed{\frac{1}{3}}$

10. $\lim_{x \to 0} \left(\frac{\sin 3x}{3x} \right)$

$\boxed{1}$

11. $\lim_{x \to 0} \left(\frac{\sin x}{|x|} \right)$

$\boxed{\text{DNE}}$

12. $\lim_{x \to 0} \left(\frac{1 - \cos x}{4x} \right)$

$\boxed{0}$

13. $\lim_{x \to 0^-} \left(\frac{\cos x}{x} \right)$

$\boxed{-\infty}$

14. $\lim_{x \to 0} \left(\frac{\sin 2x}{x} \right)$

$\boxed{2}$

15. $\lim_{x \to 0} \left(\frac{\tan 2x}{x} \right)$

$\boxed{2}$

16. $\lim_{x\to 0}\left(\frac{1-3\cos x}{3x}\right)$

$\boxed{\text{DNE}}$

17. $\lim_{x\to 0}\left(\frac{3x^2}{1-\cos^2 x}\right)$

$\boxed{3}$

18. $\lim_{x\to 0}\left(\frac{\tan 5x}{\sin 9x}\right)$

$\boxed{\frac{5}{9}}$

For problems 19-20, evaluate the given limit by making an appropriate substitution (change of variables). If a limit does not exist, write DNE, $+\infty$, or $-\infty$ (whichever is most appropriate).

19. $\lim_{x\to 8}\frac{\sin(x-8)}{x^2-64}$

$\boxed{\frac{1}{16}}$

20. $\lim_{x\to\infty} x\sin\left(\frac{2}{x}\right)$

$\boxed{2}$

For problem 21-23, determine the value(s) of x where the given function is continuous.

21. $f(x)=\csc x$

$\boxed{f(x) \text{ is continuous for all } x \neq \pi k, \text{ where } k \text{ is any integer.}}$

22. $f(x)=\frac{1}{1-2\cos x}$ on $[0,2\pi]$

$\boxed{f(x) \text{ is continuous for all } x \text{ in } [0,2\pi] \text{ except for } x=\frac{\pi}{3} \text{ and } x=\frac{5\pi}{3}}$

23. $f(x)=\begin{cases}\cos x & \text{if } \quad x<\frac{\pi}{4}\\ \sin x & \text{if } \quad x\geq\frac{\pi}{4}\end{cases}$

$\boxed{f(x) \text{ is always continuous.}}$

24. Find all non-zero value(s) of k so that $f(x) = \begin{cases} \dfrac{3\sin(kx)}{x} & \text{if} \quad x > 0 \\ 6k^2 + 5x & \text{if} \quad x \leq 0 \end{cases}$ is continuous at $x = 0$.

$\boxed{k = \dfrac{1}{2}}$

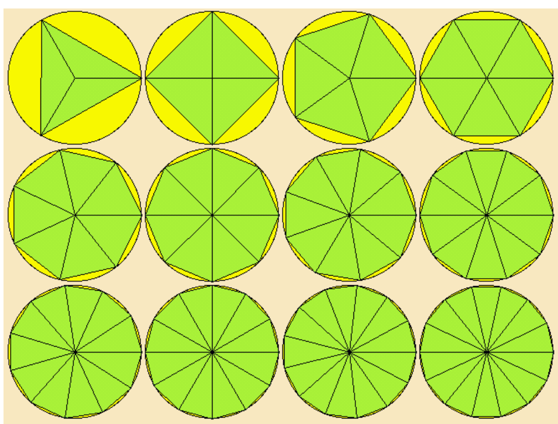
25. Use the Intermediate Value Theorem to prove that there is at least one solution to $\cos x = x^2$ in $(0, 1)$.

Let $f(x) = \cos(x) - x^2$. Since $f(x)$ is continuous on $(-\infty, \infty)$, it is also continuous on $[0, 1]$. Notice that $f(0) = 1 > 0$ and $f(1) = \cos(1) - 1 < 0$. Thus, the Intermediate Value Theorem states that there must be some c in $(0, 1)$ such that $f(c) = 0$. i.e., there must be at least one c in $(0, 1)$ such that $\cos(c) - c^2 = 0 \implies \cos(c) = c^2$, as desired.

26. Let x be a fixed real number. Compute $\lim\limits_{h \to 0} \dfrac{\sin(x+h) - \sin x}{h}$. (Hint: The identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$ will be useful.)

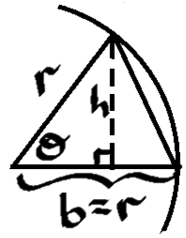
$\boxed{\cos x}$

27. A Regular Polygon is a polygon that is equiangular (all angles are equal in measure) and equilateral (all sides have the same length). The diagram below shows several regular polygons inscribed within a circle of radius r.



(a) Let A_n be the area of a regular n-sided polygon inscribed within a circle of radius r. Divide the polygon into n congruent triangles each with a central angle of $\dfrac{2\pi}{n}$ radians, as shown in the diagram above for several different values of n. Show that $A_n = \dfrac{1}{2} r^2 \sin\left(\dfrac{2\pi}{n}\right) n$.

We begin by examining one of the n triangles, pictured below.



The base of the triangle has a length of r. And, the height of the triangle is $r\sin\theta$, where θ is the central angle, $\frac{2\pi}{n}$. Thus, the area of one triangle is:

$$A = \frac{1}{2}(r)\left(r\sin\left(\frac{2\pi}{n}\right)\right) = \frac{1}{2}r^2\sin\left(\frac{2\pi}{n}\right)$$

But, the polygon is composed of n such triangles. So, the area of a regular n-sided polygon inscribed in the circle of radius r is:

$$A_n = \frac{1}{2}r^2\sin\left(\frac{2\pi}{n}\right)n$$

(b) What can you conclude about the area of the n-sided polygon as the number of sides of the polygon, n, approaches infinity? In other words, compute $\lim_{n\to\infty} A_n$.

$\boxed{\lim_{n\to\infty} A_n = \pi r^2}$