Partial Fraction Decomposition

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 7.5 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to recognize an improper rational function, and perform the necessary long division to turn it into a proper rational function.
- Know how to write down the partial fraction decomposition for a proper rational function, compute the unknown coefficients in the partial fractions, and integrate each partial fraction.

PRACTICE PROBLEMS:

For problems 1-3, write out the partial fraction decomposition. (Do not solve for the numerical values of the coefficients.)

1. $\dfrac{2x+3}{(x-2)(x-5)}$

2. $\dfrac{6}{x^2(x^2-9)}$

3. $\dfrac{5x^4-1}{x(x-2)(x^2+x+1)^2}$

For problems 4 & 5, use the given partial fraction decomposition to evaluate the integral.

4. $\displaystyle\int \frac{11x^2-28x+20}{(2x+1)(x-3)^2}\,dx$

   Hint: $\dfrac{11x^2-28x+20}{(2x+1)(x-3)^2} = \dfrac{3}{2x+1} + \dfrac{4}{x-3} + \dfrac{5}{(x-3)^2}$

5. $\displaystyle\int \frac{11x^2+7x-15}{(3x-4)(x^2+1)}\,dx$

   Hint: $\dfrac{11x^2+7x-15}{(3x-4)(x^2+1)} = \dfrac{5}{3x-4} + \dfrac{2x+5}{x^2+1}$

For problems 6-17, evaluate the given integral.

6. $\displaystyle\int \frac{4}{x^2-1}\,dx$

7. $\displaystyle\int \frac{4x-1}{x^2-5x+6}\,dx$

8. $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

9. $\displaystyle\int \frac{3x^2-4}{x+1}\,dx$

10. $\displaystyle\int \frac{4x-1}{2x^2-18x+36}\,dx$

11. $\displaystyle\int \frac{1}{x^2(x-1)^2}\,dx$

12. $\displaystyle\int \frac{2x+3}{(x-3)(x+1)^2}\,dx$

13. $\displaystyle\int \frac{x^4-4x^2+5}{x^3-4x}\,dx$

14. $\displaystyle\int \frac{x^5-3x^3+6}{x^3+x}dx$

15. $\displaystyle\int \frac{x^3-6x^2+3x-17}{x^2+3}\,dx$

16. $\displaystyle\int \frac{2x^2}{(x-1)^3}dx$

17. $\displaystyle\int \frac{4x^2-4x+2}{x^2-x}dx$

For problems 18-19, evaluate the given integral by making substitution that transforms the problem into integrating a rational function.

18. $\displaystyle\int \frac{\sin x}{\cos^2 x+6\cos x+5}\,dx$

19. $\displaystyle\int \frac{e^{5x}}{e^{4x}-1}\,dx$

20. By the end of this problem, you will know the antiderivatives of $\sec x$. Observe the following:

$$\begin{aligned}\int \sec x\,dx &= \int \frac{1}{\cos x}\,dx\\ &= \int \frac{\cos x}{\cos^2 x}\,dx\\ &= \int \frac{\cos x}{1-\sin^2 x}\,dx\end{aligned}$$

(a) Use the substitution $u = \sin x$ to convert the given integral to an integral of a rational function.

(b) Use partial fractions to evaluate your integral from part (a). Show that the antiderivatives have the form $\dfrac{1}{2}\ln\left|\dfrac{u+1}{u-1}\right| + C$

Notice that substituting back in for u yields:

$$\begin{aligned}\int \sec x\,dx &= \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + C\\ &= \frac{1}{2}\ln\left|\frac{(\sin x+1)(\sin x+1)}{(\sin x-1)(\sin x+1)}\right| + C\\ &= \frac{1}{2}\ln\left|\frac{(\sin x+1)^2}{\sin^2 x-1}\right| + C\\ &= \frac{1}{2}\ln\left|\frac{(\sin x+1)^2}{\cos^2 x}\right| + C\\ &= \ln\left|\frac{\sin x+1}{\cos x}\right| + C\\ &= \ln|\tan x+\sec x| + C\end{aligned}$$