Multivariable Chain Rule

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 13.5 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to compute partial derivatives with the various versions of the multivariate chain rule.
- Be able to compare your answer with the direct method of computing the partial derivatives.

PRACTICE PROBLEMS:

1. Find $\frac{dz}{dt}$ by using the Chain Rule. Check your answer by expressing z as a function of t and then differentiating.

   (a) $z = 2x - y, \ x = \sin t, \ y = 3t$

   $\boxed{\frac{dz}{dt} = 2\cos t - 3}$

   (b) $z = x \sin y, \ x = e^t, \ y = \pi t$

   $\boxed{\frac{dz}{dt} = e^t \sin(\pi t) + \pi e^t \cos(\pi t)}$

   (c) $z = xy + y^2, \ x = t^2, \ y = t + 1$

   $\boxed{\frac{dz}{dt} = 3t^2 + 4t + 2}$

   (d) $z = \ln\left(\frac{x^2}{y}\right), \ x = e^t, \ y = t^2$

   $\boxed{\frac{dz}{dt} = 2 - \frac{2}{t}}$

2. Suppose $w = x^2 + y^2 + 2z^2, \ x = t + 1, \ y = \cos t, \ z = \sin t$. Find $\frac{dw}{dt}$ using the Chain Rule. Check your answer by expressing w as a function of t and then differentiating.

   $\boxed{\frac{dw}{dt} = 2t + 2 + 2\sin t \cos t;}$; Detailed Solution: Here

3. Suppose f is a differentiable function of x & y, and define $g(u,v) = f(3u - v, u^2 + v)$. Use the table of values shown below to calculate $\left.\dfrac{\partial g}{\partial u}\right|_{(u,v)=(2,-1)}$ and $\left.\dfrac{\partial g}{\partial v}\right|_{(u,v)=(2,-1)}$.

(x,y)	f	g	f_x	f_y
$(2,-1)$	6	-7	1	9
$(7,3)$	4	2	-3	5

Hint: Decompose $f(3u - v, u^2 + v)$ into $f(x,y)$ where $x = 3u - v$ and $y = u^2 + v$.

$\boxed{g_u(2,-1) = 11;\ g_v(2,-1) = 8}$

4. Find $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$ by using the appropriate Chain Rule.

   (a) $w = xy\sin\left(z^2\right),\ x = s - t,\ y = s^2,\ z = t^2$

   $$\boxed{\frac{\partial w}{\partial s} = s^2\sin\left(t^4\right) + 2s(s-t)\sin\left(t^4\right);\ \frac{\partial w}{\partial t} = -s^2\sin\left(t^4\right) + 4s^2t^3(s-t)\cos\left(t^4\right)}$$

   (b) $w = xy + yz,\ x = s + t,\ y = st,\ z = s - 2t$

   $$\boxed{\frac{\partial w}{\partial s} = 4st - t^2;\ \frac{\partial w}{\partial t} = 2s^2 - 2st}$$

5. Suppose that $J = f(x,y,z,w)$, where $x = x(r,s,t)$, $y = y(r,t)$, $z = z(r,s)$ and $w = w(s,t)$. Use the Chain Rule to find $\dfrac{\partial J}{\partial r}$, $\dfrac{\partial J}{\partial s}$, and $\dfrac{\partial J}{\partial t}$.

$$\boxed{\begin{aligned}\frac{\partial J}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r};\\ \frac{\partial J}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial s};\\ \frac{\partial J}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial t}\end{aligned}}$$

6. Suppose $g = f(u - v, v - w, w - u)$. Show that $\dfrac{\partial g}{\partial u} + \dfrac{\partial g}{\partial v} + \dfrac{\partial g}{\partial w} = 0$.

We can express g as $g = f(x, y, z)$, where $x = u - v$, $y = v - w$, and $z = w - u$. Then, we compute the partial derivatives of g with respect to u, v, and w.

$$\begin{aligned}
\frac{\partial g}{\partial u} &= f_x(u - v, v - w, w - u) - f_z(u - v, v - w, w - u) \\
\frac{\partial g}{\partial v} &= -f_x(u - v, v - w, w - u) + f_y(u - v, v - w, w - u) \\
\frac{\partial g}{\partial w} &= -f_y(u - v, v - w, w - u) + f_z(u - v, v - w, w - u)
\end{aligned}$$

Summing these three partial derivatives yields 0.

7. Suppose $u = u(x, y)$, $v = v(x, y)$, $x = r\cos\theta$, and $y = r\sin\theta$.

(a) Calculate $\dfrac{\partial u}{\partial r}$, $\dfrac{\partial u}{\partial \theta}$, $\dfrac{\partial v}{\partial r}$, and $\dfrac{\partial v}{\partial \theta}$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta; \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial u}{\partial x}\sin\theta + r\frac{\partial u}{\partial y}\cos\theta;$$
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta; \quad \frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial x}\sin\theta + r\frac{\partial v}{\partial y}\cos\theta$$

(b) Suppose that $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann Equations:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

Use this along with part (a) to derive the polar form of the Cauchy-Riemann Equations:

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\
\frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r}
\end{aligned}$$

From part (a), we know that $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$. We apply the Cauchy Riemann Equations to $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. Thus,

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
&= \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta \\
&= \frac{1}{r}\left(r\frac{\partial v}{\partial y}\cos\theta - r\frac{\partial v}{\partial x}\sin\theta\right) \\
&= \frac{1}{r}\frac{\partial v}{\partial \theta}
\end{aligned}$$

A similar argument yields the second Cauchy Riemann Equation in polar coordinates.