Tangent Planes & Normal Lines

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 13.7 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Be able to compute an equation of the tangent plane at a point on the surface $z = f(x, y)$.
- Given an implicitly defined level surface $F(x, y, z) = k$, be able to compute an equation of the tangent plane at a point on the surface.
- Know how to compute the parametric equations (or vector equation) for the normal line to a surface at a specified point.
- Be able to use gradients to find tangent lines to the intersection curve of two surfaces. And, be able to find (acute) angles between tangent planes and other planes.

PRACTICE PROBLEMS:

For problems 1-4, find two unit vectors which are normal to the given surface S at the specified point P.

1. $S : 2x - y + z = -7$; $P(-1, 2, -3)$

$$\boxed{\overrightarrow{n_{1,2}} = \pm\left\langle \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle}$$

2. $S : x^2 - 3y + z^2 = 11$; $P(-1, -2, 2)$

$$\boxed{\overrightarrow{n_{1,2}} = \pm\left\langle -\frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right\rangle}$$

3. $S : z = y^4$; $P(3, -1, 1)$

$$\boxed{\overrightarrow{n_{1,2}} = \pm\left\langle 0, -\frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}} \right\rangle}$$

4. $S : z = 2 - x^2 \cos(xy)$; $P\left(-1, \frac{\pi}{2}, 2\right)$

$$\boxed{\overrightarrow{n_{1,2}} = \pm\frac{2}{\sqrt{\pi^2 + 8}}\left\langle -\frac{\pi}{2}, 1, -1 \right\rangle}$$

For problems 5-9, compute equations of the tangent plane and the normal line to the given surface at the indicated point.

5. $S : \ln(x+y+z) = 2;\ P(-1, e^2, 1)$

   $\boxed{x+y+z = e^2;\ \overrightarrow{\ell}(t) = \langle -1, e^2, 1\rangle + t\langle 1,1,1\rangle}$

6. $S : x^2+y^2+z^2 = 1;\ P\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

   $$\boxed{x+y+z = \sqrt{3};\ \overrightarrow{\ell}(t) = \left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right\rangle + t\langle 1,1,1\rangle}$$

7. $S : z = \arcsin\left(\frac{x}{y}\right);\ P\left(-1, -\sqrt{2}, \frac{\pi}{4}\right)$

   $$\boxed{-x + \frac{\sqrt{2}}{2}y - z = -\frac{\pi}{4};\ \overrightarrow{\ell}(t) = \left\langle -1, -\sqrt{2}, \frac{\pi}{4}\right\rangle + t\left\langle -1, \frac{\sqrt{2}}{2}, -1\right\rangle}$$

8. $S : x^2 - xy + z^2 = 9;\ P(2,2,3)$

   $\boxed{x - y + 3z = 9;\ \overrightarrow{\ell}(t) = \langle 2,2,3\rangle + t\langle 1,-1,3\rangle}$

9. $S : z = x\cos(x+y);\ P\left(\frac{\pi}{2}, \frac{\pi}{3}, -\frac{\sqrt{3}\pi}{4}\right)$

   $$(\pi + 2\sqrt{3})\left(x - \frac{\pi}{2}\right) + \pi\left(y - \frac{\pi}{3}\right) + 4\left(z + \frac{\sqrt{3}\pi}{4}\right) = 0$$

   $$\overrightarrow{\ell}(t) = \left\langle \frac{\pi}{2}, \frac{\pi}{3}, -\frac{\pi\sqrt{3}}{4}\right\rangle + t\left\langle \pi + 2\sqrt{3}, \pi, 4\right\rangle$$

   Detailed Solution: Here

10. Consider the surfaces $S_1 : x^2 + y^2 = 25$ and $S_2 : z = 2 - x$

    (a) Find an equation of the tangent line to the curve of intersection of S_1 and S_2 at the point $(3, 4, -1)$.

    $\boxed{\overrightarrow{\ell}(t) = \langle 3,4,-1\rangle + t\langle -4,3,4\rangle}$

    (b) Find the acute angle between the planes which are tangent to the surfaces S_1 and S_2 at the point $(3, 4, -1)$.

    $$\boxed{\pi - \cos^{-1}\left(\frac{-3}{5\sqrt{2}}\right)}$$

11. Consider the surfaces $S_1 : z = x^2 - y^2$ and $S_2 : y^2 + z^2 = 10$

   (a) Find an equation of the tangent line to the curve of intersection of S_1 and S_2 at the point $(2, 1, 3)$.

   $\boxed{\overrightarrow{\ell}(t) = \langle 2, 1, 3\rangle + t\langle 5, 12, -4\rangle}$; Detailed Solution: Here

   (b) Find the acute angle between the planes which are tangent to the surfaces S_1 and S_2 at the point $(2, 1, 3)$.

   $\pi - \cos^{-1}\left(\dfrac{-10}{\sqrt{21}\sqrt{40}}\right)$; Detailed Solution: Here

12. Find all points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 72$ where the tangent plane is parallel to the plane $4x + 4y + 12z = 3$.

   $(4, 2, 4)$ and $(-4, -2, -4)$

13. Find all points on the hyperboloid of 1 sheet $x^2 + y^2 - z^2 = 9$ where the normal line is parallel to the line which contains points $A(1, 2, 3)$ and $B(7, 6, 5)$.

   $\left(\dfrac{3\sqrt{3}}{2}, \sqrt{3}, -\dfrac{\sqrt{3}}{2}\right)$ and $\left(-\dfrac{3\sqrt{3}}{2}, -\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$; Detailed Solution: Here

14. Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point. Show that the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z^2 = x^2 + y^2$ are orthogonal at all points of intersection. (HINT: Assume that the surfaces intersect at the arbitary point (x_0, y_0, z_0).)

   Suppose that $S_1 : x^2 + y^2 + z^2 = 1$ and $S_2 : z^2 = x^2 + y^2$ intersect at (x_0, y_0, z_0). We will find a normal vector to each surface at the point P_0. To do this, let $F(x, y, z) = x^2 + y^2 + z^2$ and $G(x, y, z) = x^2 + y^2 - z^2$. Notice that S_1 is the level surface $F(x, y, z) = 1$ and S_2 is the level surface $G(x, y, z) = 0$. So, $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, 2z_0\rangle$ and $\nabla G(x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0\rangle$ are normal to S_1 and S_2, respectively, at the point P_0. And, as a result, these vectors are parallel to the normal lines to S_1 and S_2 at P_0.

   Showing that the surfaces are orthogonal is equivalent to showing that $\nabla F(x_0, y_0, z_0) \perp \nabla G(x_0, y_0, z_0)$; i.e, $\nabla F(x_0, y_0, z_0) \cdot \nabla G(x_0, y_0, z_0) = 0$.

   (Continued on next page)

$$\begin{aligned}\nabla F(x_0, y_0, z_0) \cdot \nabla G(x_0, y_0, z_0) &= \langle 2x_0, 2y_0, 2z_0 \rangle \cdot \langle 2x_0, 2y_0, -2z_0 \rangle \\ &= 4x_0^2 + 4y_0^2 - 4z_0^2 \\ &= 4(x_0^2 + y_0^2 - z_0^2)\end{aligned}$$

Since (x_0, y_0, z_0) is a point of intersection of S_1 and S_2, it must satisfy both equations. In particular, since it satisfies the equation for S_2, we have $x_0^2 + y_0^2 = z_0^2$. Using this fact, we see that

$$\begin{aligned}\nabla F(x_0, y_0, z_0) \cdot \nabla G(x_0, y_0, z_0) &= 4(x_0^2 + y_0^2 - z_0^2) \\ &= 4(z_0^2 - z_0^2) \\ &= 0\end{aligned}$$

As a result the surfaces are orthogonal to one another at the point of intersection, (x_0, y_0, z_0).

15. Show that every plane which is tangent to the cone $z^2 = x^2 + y^2$ must pass through the origin. (HINT: Compute the equation of the plane which is tangent to the surface at the point $P_0(x_0, y_0, z_0)$ and see what happens.)

Let $F(x, y, z) = x^2 + y^2 - z^2$. The given surface is the level surface $F(x, y, z) = 0$; so, $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0 \rangle$ is normal to the given surface at the point (x_0, y_0, z_0). Thus, an equation of the plane which is tangent to the given surface at the point (x_0, y_0, z_0) is $2x_0(x - x_0) + 2y_0(y - y_0) - 2z_0(z - z_0) = 0$; i.e., $x_0x + y_0y + z_0z - x_0^2 - y_0^2 + z_0^2 = 0$.

Now, since (x_0, y_0, z_0) is the point of tangency, it must also be a point on the surface. Thus, $x_0^2 + y_0^2 = z_0^2 \Rightarrow -x_0^2 - y_0^2 = -z_0^2$. Using this fact, the equation of the tangent plane can be written as:

$$\begin{aligned}x_0x + y_0y + z_0z - x_0^2 - y_0^2 + z_0^2 &= 0 \\ x_0x + y_0y + z_0z - z_0^2 + z_0^2 &= 0 \\ x_0x + y_0y + z_0z &= 0\end{aligned}$$

And, $(0, 0, 0)$ satisfies this equation. Thus, since (x_0, y_0, z_0) was an arbitrary point on the surface and its tangent plane passes through the origin, we have that all tangent planes to the surface must pass through the origin.