Dot Product & Projections

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 11.3 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Know how to compute the dot product of two vectors.
- Be able to use the dot product to find the angle between two vectors; and, in particular, be able to determine if two vectors are orthogonal.
- Know how to compute the direction cosines of a vector.
- Be able to decompose vectors into orthogonal components. And, know how to compute the orthogonal projection of one vector onto another.

PRACTICE PROBLEMS:

1. For each of the following, compute $\overrightarrow{u} \cdot \overrightarrow{v}$ based on the given information.

   (a) $\overrightarrow{u} = \langle 3, -1 \rangle$; $\overrightarrow{v} = \langle 2, -5 \rangle$

   $\boxed{11}$

   (b) $\overrightarrow{u} = \langle 4, -5, 1 \rangle$; $\overrightarrow{v} = \langle 3, 6, -1 \rangle$

   $\boxed{-19}$

   (c) $\overrightarrow{u} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$; $\overrightarrow{v} = 9\mathbf{i} - 2\mathbf{j}$;

   $\boxed{15}$

   (d) $\|\overrightarrow{u}\| = 3$; $\|\overrightarrow{v}\| = 4$; the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$ is $\dfrac{\pi}{4}$

   $\boxed{6\sqrt{2}}$

2. Explain why the operation $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ does not make sense.

   $\boxed{\mathbf{u} \cdot \mathbf{v} \text{ is a scalar. We cannot take the dot product of a scalar with a vector.}}$

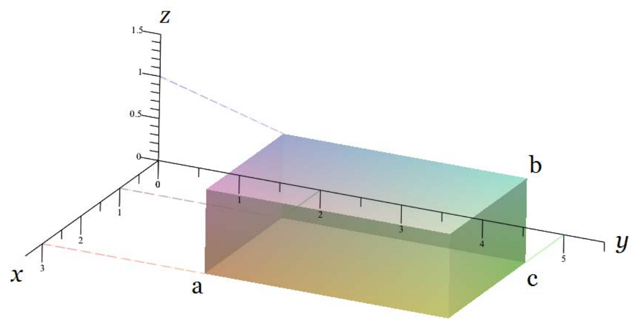
3. Determine whether the angle between $\overrightarrow{v} = \langle 1, 2, 3 \rangle$ and $\overrightarrow{w} = \langle -6, 4, -1 \rangle$ is acute, obtuse, or right. Explain.

   $\boxed{\text{Since } \overrightarrow{v} \cdot \overrightarrow{w} = -1 < 0 \text{, the angle between the two vectors is obtuse.}}$

4. Give an example of a vector which is orthogonal to both $\overrightarrow{v} = \langle 1, 1, 1 \rangle$ and $\overrightarrow{w} = \langle 2, 0, 4 \rangle$. (Hint: Set up a system of equations.)

   $\boxed{\text{Any scalar multiple of } \overrightarrow{n} = \langle -2, 1, 1 \rangle \text{ is orthogonal to both given vectors.}}$

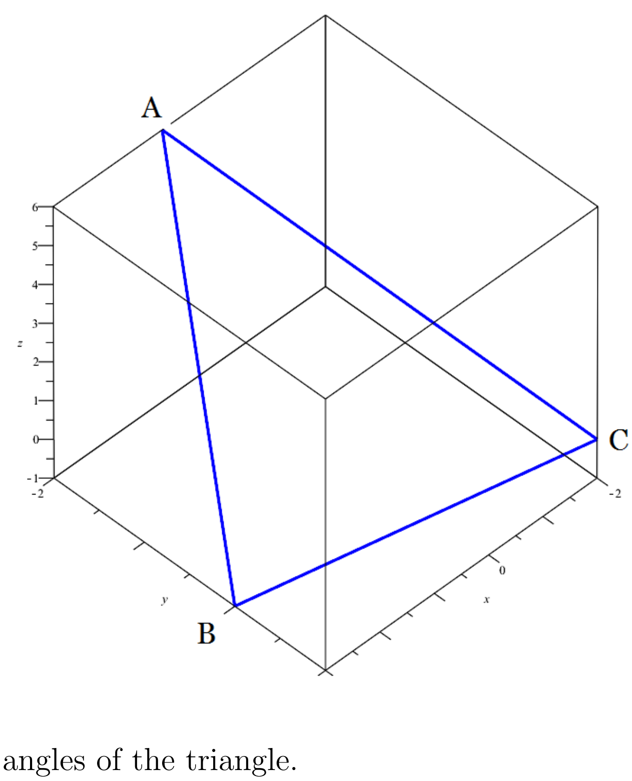
5. Consider the triangle with vertices a, b, and c.



Use vectors to compute the angle between the diagonal which extends from vertex a to vertex b and the line segment which extends from vertex a to vertex c. (Verify your answer with HW 11.1 #3c.)

$$\boxed{\cos^{-1}\left(\frac{\sqrt{13}}{\sqrt{14}}\right)}$$

6. Consider the triangle, shown below,with vertices $A(1,-2,6)$, $B(3,0,-1)$, and $C(-2,1,0)$.



Compute all three angles of the triangle.

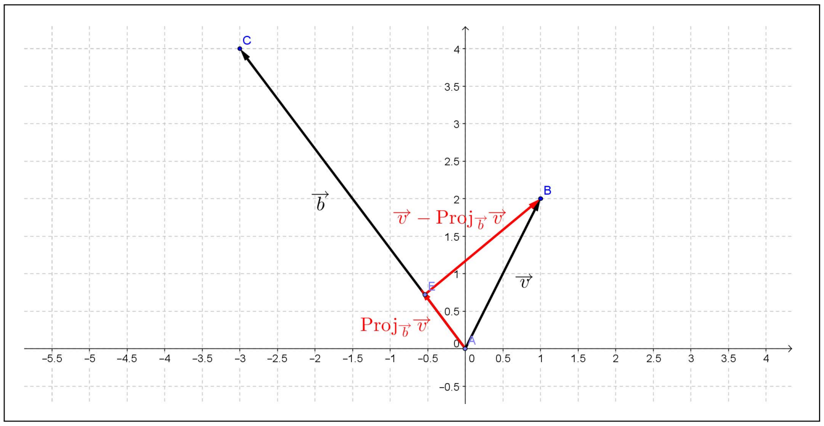
Angle A has a measure of $\cos^{-1}\left(\dfrac{42}{\sqrt{57}\sqrt{54}}\right)$ radians.
Angle B has a measure of $\cos^{-1}\left(\dfrac{15}{\sqrt{57}\sqrt{27}}\right)$ radians.
Angle C has a measure of $\cos^{-1}\left(\dfrac{12}{\sqrt{54}\sqrt{27}}\right)$ radians.
(NOTE: Once you find any two of the angles, you can use the fact that the sum of all of the angles must be π radians to compute the remaining angle.)

7. Let $\overrightarrow{v} = \langle 1, 2\rangle$ and $\overrightarrow{b} = \langle -3, 4\rangle$.

   (a) Find the vector component of $\overrightarrow{v}$ along $\overrightarrow{b}$ and the vector component of $\overrightarrow{v}$ orthogonal to $\overrightarrow{b}$.

   The vector component of $\overrightarrow{v}$ along $\overrightarrow{b}$ is $\text{Proj}_{\overrightarrow{b}}\overrightarrow{v} = \left\langle -\dfrac{3}{5}, \dfrac{4}{5}\right\rangle$ and the vector component of $\overrightarrow{v}$ orthogonal to $\overrightarrow{b}$ is $\overrightarrow{v} - \text{Proj}_{\overrightarrow{b}}\overrightarrow{v} = \left\langle \dfrac{8}{5}, \dfrac{6}{5}\right\rangle$.

   (b) Sketch $\overrightarrow{v}$, $\overrightarrow{b}$, and the vector components that you found in part (a).



8. Express $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ as the sum of a vector parallel to $\mathbf{b} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and a vector perpendicular to $\mathbf{b}$

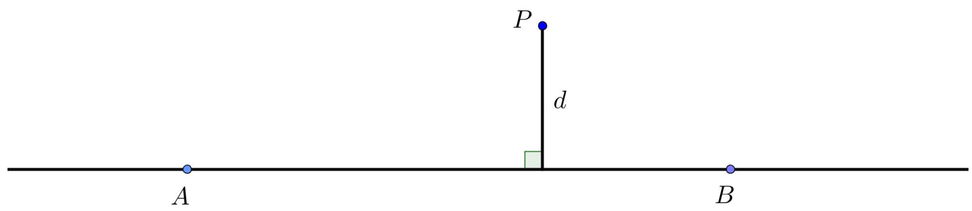
   $\mathbf{v} = \left\langle -\dfrac{2}{7}, \dfrac{4}{7}, -\dfrac{1}{7}\right\rangle + \left\langle \dfrac{9}{7}, \dfrac{10}{7}, \dfrac{22}{7}\right\rangle$; Detailed Solution: Here

9. Suppose that $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. Under what condition will $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$? Explain.

   The result follows if $\mathbf{u}$ is orthogonal to $\mathbf{v}$; Detailed Solution: Here

10. The following questions deal with finding the distance from a point to a line:

   (a) Given three points A, B, and P in 2-space or 3-space as shown in the picture below, describe two different ways that you could use the dot product to calculate the distance, d, between the point P and the line which contains A and B.



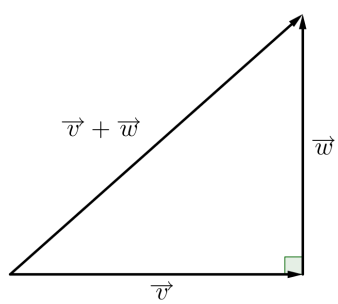
OPTION 1:
One can compute $\left\|\overrightarrow{AP}\right\|$ and $\left\|\text{Proj}_{\overrightarrow{AB}}\overrightarrow{AP}\right\|$. Then, use Pythagorean Theorem to calculate d.

OPTION 2:
One can compute $\text{Proj}_{\overrightarrow{AB}}\overrightarrow{AP}$. Then, $d = \left\|\overrightarrow{AP} - \text{Proj}_{\overrightarrow{AB}}\overrightarrow{AP}\right\|$.

   (b) Use one of your methods from part (a) to compute the distance from the point $P(5, 3, 0)$ to the line containing $A(1, 0, 1)$ and $B(2, 3, 1)$.

$d = \sqrt{\dfrac{91}{10}}$; Video Solution: Here

11. Consider the triangle shown below which is formed by vectors $\mathbf{v}$ and $\mathbf{w}$.



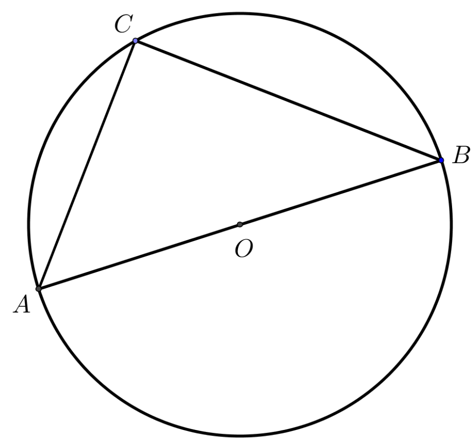
Prove Pythagorean's Theorem $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$. (Hint: Use properties of the dot product to expand $\|\mathbf{v} + \mathbf{w}\|^2$.)

We expand $\|\mathbf{v}+\mathbf{w}\|^2$ using properties of the dot product:

$$\begin{aligned}
\|\mathbf{v}+\mathbf{w}\|^2 &= (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) \\
&= \mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}+\mathbf{w}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w} \\
&= \|\mathbf{v}\|^2 + 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2 \text{ (since } \mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v}) \\
&= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \text{ (since } \mathbf{v}\perp\mathbf{w} \Leftrightarrow \mathbf{v}\cdot\mathbf{w}=0)
\end{aligned}$$

Thus, $\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$, as promised.

12. Let A and B be endpoints of a diameter of a circle with a radius of r . And, suppose that C is any other point on the circle, as shown below.



Prove that triangle ABC is a right triangle. (Hint: Express each of $\overrightarrow{CA}$ and $\overrightarrow{CB}$ as the combination of $\overrightarrow{CO}$ and some other vector.)

Notice that $\mathbf{CA} = \mathbf{CO} + \mathbf{OA}$ and $\mathbf{CB} = \mathbf{CO} + \mathbf{OB}$. But, $\mathbf{OB} = -\mathbf{OA}$. Thus, $\mathbf{CB} = \mathbf{CO} - \mathbf{OA}$. We will show that $\mathbf{CA} \perp \mathbf{CB}$ by showing that $\mathbf{CA}\cdot\mathbf{CB} = 0$.

$$\begin{aligned}
\mathbf{CA}\cdot\mathbf{CB} &= (\mathbf{CO}+\mathbf{OA})\cdot(\mathbf{CO}-\mathbf{OA}) \\
&= \mathbf{CO}\cdot\mathbf{CO} - \mathbf{CO}\cdot\mathbf{OA} + \mathbf{OA}\cdot\mathbf{CO} - \mathbf{OA}\cdot\mathbf{OA} \\
&= \|\mathbf{CO}\|^2 - \|\mathbf{OA}\|^2 \\
&= r^2 - r^2 \\
&= 0
\end{aligned}$$

So, $\mathbf{CA} \perp \mathbf{CB}$ and the triangle is a right triangle.

13. Let $\overrightarrow{v}$ and $\overrightarrow{w}$ be vectors, either both in $\mathbb{R}^2$ or in $\mathbb{R}^3$. Prove the Cauchy-Schwarz Inequality: $|\overrightarrow{v} \cdot \overrightarrow{w}| \leq \|\overrightarrow{v}\|\|\overrightarrow{w}\|$.

Suppose that θ is the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$. Then:

$$\begin{aligned} |\overrightarrow{v} \cdot \overrightarrow{w}| &= |\|\overrightarrow{v}\|\|\overrightarrow{w}\| \cos\theta| \\ &= (\|\overrightarrow{v}\|)\,(\|\overrightarrow{w}\|)\,(|\cos\theta|) \\ &\leq (\|\overrightarrow{v}\|)\,(\|\overrightarrow{w}\|)\,(1) \\ &= \|\overrightarrow{v}\|\|\overrightarrow{w}\| \end{aligned}$$

Thus, $|\overrightarrow{v} \cdot \overrightarrow{w}| \leq \|\overrightarrow{v}\|\|\overrightarrow{w}\|$, as promised.

14. Let $\overrightarrow{v} = \langle 1, 2, 3 \rangle$.

(a) Compute the direction cosines of $\overrightarrow{v}$.

Let α, β, and γ be the angles between $\overrightarrow{v}$ and **i**, **j**, and **k**, respectively. Then: $\cos\alpha = \dfrac{1}{\sqrt{14}}$, $\cos\beta = \dfrac{2}{\sqrt{14}}$, and $\cos\gamma = \dfrac{3}{\sqrt{14}}$

(b) Verify that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, where α, β, and γ be the angles between $\overrightarrow{v}$ and **i**, **j**, and **k**, respectively.

Using the information from part (a), we obtain:

$$\begin{aligned} \cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2 \\ &= \frac{1}{14} + \frac{4}{14} + \frac{9}{14} \\ &= 1 \end{aligned}$$