Area As A Limit & Sigma Notation

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 5.4 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Understand and know how to evaluate the summation (sigma) notation.
- Be able to use the summation operation's basic properties and formulas. (You do not need to memorize the "Useful Formulas" listed below; if they are needed, they will be provided to you).
- Know how to denote the approximate area under a curve and over an interval as a sum, and be able to find the exact area using a limit of the approximation.
- Be able to find the net signed area between the graph of a function and the x-axis on an interval using a limit.

USEFUL FORMULAS

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

PRACTICE PROBLEMS:

For problems 1-5, evaluate.

1. $\displaystyle\sum_{k=1}^{4} k^3$

   $\boxed{100}$

2. $\displaystyle\sum_{j=2}^{6} (j^3 - 1)$

   $\boxed{435}$

3. $\displaystyle\sum_{i=-1}^{3} 2i$

   $\boxed{10}$

4. $\displaystyle\sum_{k=0}^{5}(-1)^k$

$\boxed{0}$

5. $\displaystyle\sum_{k=1}^{5}\sin\left(\frac{\pi}{2}k\right)$

$\boxed{1}$

For problems 6-8, use the summation formulas at the top of page 1 to evaluate the given sum.

6. $\displaystyle\sum_{k=1}^{100}(3k-5)$

$\boxed{14{,}650}$

7. $\displaystyle\sum_{k=1}^{25}[k(k-1)(k+1)]$

$\boxed{105{,}300}$

8. $\displaystyle\sum_{k=3}^{120}(k+7)$

(CAUTION: In problem 8, the lower index is not 1; so, the summation formulas at the top of page 1 do not immediately apply!)

8,083; Video Solution: https://www.youtube.com/watch?v=Cq08CHq0wlY

For problems 9-12, write the given expression in sigma notation. Do not evaluate the sum. (For each, there are many different ways to write the expression in sigma notation; the answer key illustrates one such way for each.))

9. $4(1)+4(2)+4(3)+4(4)+\cdots+4(20)$

$\boxed{\displaystyle\sum_{k=1}^{20}4k}$

10. $3-6+9-12+\cdots-36$

$\boxed{\displaystyle\sum_{k=1}^{12}3(-1)^{k+1}k}$

11. $1+3+5+7+\cdots+21$

$$\boxed{\sum_{k=0}^{10}(2k+1)}$$

12. $2+4+8+16+\cdots+256$

$$\boxed{\sum_{k=1}^{8}2^k}$$

For problems 13-15, express the given summation in closed form.

13. $\displaystyle\sum_{j=1}^{n}\frac{j}{n}$

$$\boxed{\frac{n+1}{2}}$$

14. $\displaystyle\sum_{k=1}^{n-1}\frac{3k^3}{n}$

$$\boxed{\frac{3n(n-1)^2}{4}}$$

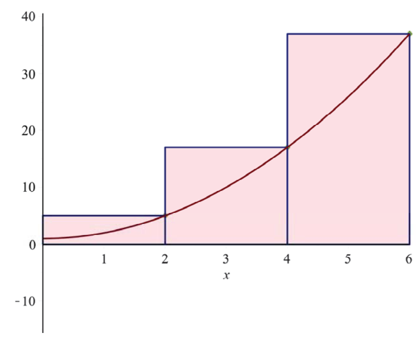
15. $\displaystyle\sum_{k=0}^{n}\left(\frac{1}{n}-\frac{k^2}{n}\right)$

(CAUTION: In problem 15, the lower limit is not 1; so the summation formulas at the top of page 1 do not immediately apply!)

$$\boxed{1-\frac{(n+1)(2n+1)}{6}+\frac{1}{n}}$$

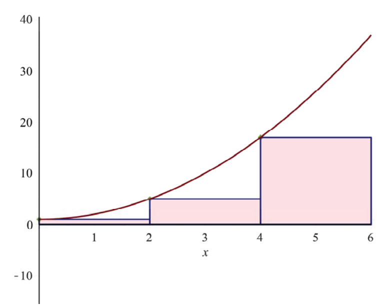
16. Consider $f(x)=x^2+1$.

(a) Estimate the area under the graph of $f(x)$ on the interval $[0,6]$ using 3 rectangles of equal width and right endpoints, as in the diagram below. Is your estimate an overestimate or an underestimate?



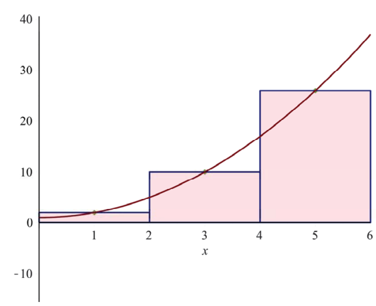
$\boxed{A \approx 118\text{; It is an overestimate.}}$

(b) Estimate the area under the graph of $f(x)$ on the interval $[0, 6]$ using 3 rectangles of equal width and left endpoints, as in the diagram below. Is your estimate an overestimate or an underestimate?



$\boxed{A \approx 46\text{; It is an underestimate.}}$

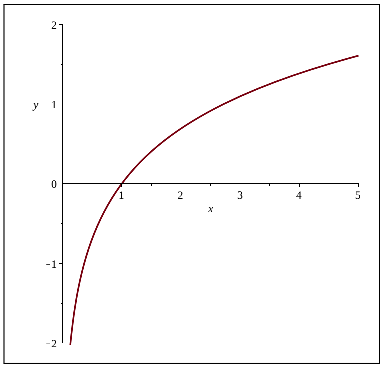
(c) Estimate the area under the graph of $f(x)$ on the interval $[0, 6]$ using 3 rectangles of equal width and midpoints, as in the diagram below. Is your estimate an overestimate or an underestimate?



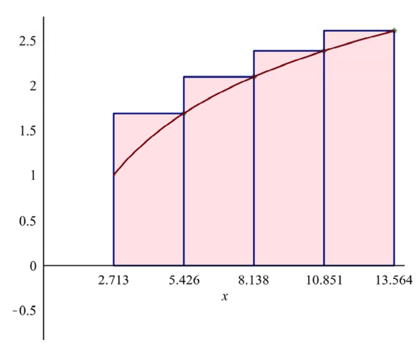
$A \approx 76$; By inspection, it is hard to judge whether this is an overestimate or an underestimate. In fact, in a future section, you will be able to show that the exact area is 78.

17. Let $f(x) = \ln x$.

(a) Sketch the graph of $f(x)$. Label all asymptotes and intercepts with the coordinate axes.



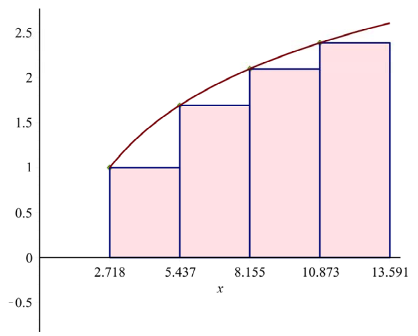
(b) Sketch the graph of $f(x)$ on the interval $[e, 5e]$. Divide the interval into 4 subintervals of equal width. On each subinterval, sketch a rectangle using the function value at the **right endpoint** as the height of the rectangle on that subinterval. Estimate the area between the graph of $f(x)$ and the x-axis on the interval $[e, 5e]$ using the 4 rectangles that you sketched. Is your estimate an overestimate or an underestimate?



$A \approx e\ln(2e) + e\ln(3e) + e\ln(4e) + e\ln(5e) \approx 23.89$

This is an overestimate.

(c) Sketch the graph of $f(x)$ on the interval $[e, 5e]$. Divide the interval into 4 subintervals of equal width. On each subinterval, sketch a rectangle using the function value at the **left endpoint** as the height of the rectangle on that subinterval. Estimate the area between the graph of $f(x)$ and the x-axis on the interval $[e, 5e]$ using the 4 rectangles that you sketched. Is your estimate an overestimate or an underestimate?



$A \approx e + e\ln(2e) + e\ln(3e) + e\ln(4e) \approx 19.51$
This is an underestimate.

18. Let $f(x) = x^2 + 1$. By the end of this problem, you will have computed the exact area under the graph of $f(x)$ on the interval $[1, 6]$.

(a) Find the Δx which is necessary to divide $[1, 6]$ into n subintervals of equal width.

$\boxed{\Delta x = \frac{5}{n}}$

(b) In each of the n subintervals of equal width, pick x_k^* to be the right endpoint. Fill in the following table:

Subinterval Number	Right Endpoint Number	Right Endpoint of Subinterval
$k = 1$	x_1^*	$\boxed{1 + \frac{5}{n}}$
$k = 2$	x_2^*	$\boxed{1 + \frac{5}{n}(2)}$
$k = 3$	x_3^*	$\boxed{1 + \frac{5}{n}(3)}$
$\vdots$	$\vdots$	$\vdots$
$k = n - 1$	x_{n-1}^*	$\boxed{1 + \frac{5}{n}(n-1)}$
$k = n$	x_n^*	$\boxed{1 + \frac{5}{n}(n) = 6}$

(c) **Fill in the blank:** A closed formula for the right endpoints found in the table above is $x_k^* =$ $\boxed{1 + \frac{5}{n}(k)}$, for $k = 1, 2, ..., n - 1, n$.

(d) Determine $f(x_k^*)$, the height of the k^{th} rectangle.

$\boxed{\left(1+\frac{5}{n}k\right)^2+1}$

(e) The right endpoint approximation of the area under the graph of $f(x)$ on the interval $[1,6]$ using n rectangles of equal width is:

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_{n-1}^*)\Delta x + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Using the appropriate formulas from the top of page 1, express the right endpoint approximation in closed form.

$$\boxed{\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[\left(1+\frac{5}{n}k\right)^2+1\right]\frac{5}{n} = 10 + \frac{25(n+1)}{n} + \frac{125(n+1)(2n+1)}{6n^2}}$$

(f) Repeating over finer and finer partitions is equivalent to the number of subintervals, n, approaching infinity. Using this information, compute the exact area under the graph of $f(x) = x^2+1$ on the interval $[1,6]$.

$\boxed{\frac{230}{3}}$

19. For each of the following, use sigma notation and the appropriate summation formulas to evaluate the net signed area between the graph of $f(x)$ and the x-axis on the given interval. Let x_k^* be the **right endpoint** of the k^{th} subinterval (where all subintervals have equal width).

(a) $f(x) = x - 3$ on $[1,5]$

0; Detailed Solution: Here

(b) $f(x) = \frac{x^2}{3}$ on $[2,5]$

13; Detailed Solution: Here

(c) $f(x) = x^3 - 1$ on $[0,2]$

$\boxed{2}$

20. For each of the following, use sigma notation and the appropriate summation formulas to evaluate the net signed area between the graph of $f(x)$ and the x-axis on the given interval. Let x_k^* be the **left endpoint** of the k^{th} subinterval (where all subintervals have equal width).

(a) $f(x) = x - 3$ on $[1,5]$

$\boxed{0}$

(b) $f(x) = \dfrac{x^2}{3}$ on $[2, 5]$

$\boxed{13}$

(c) $f(x) = x^3 - 1$ on $[0, 2]$

$\boxed{2}$

21. For each of the following, use sigma notation and the appropriate summation formulas to evaluate the net signed area between the graph of $f(x)$ and the x-axis on the given interval. Let x_k^* be the **midpoint** of the k^{th} subinterval (where all subintervals have equal width).

(a) $f(x) = x - 3$ on $[1, 5]$

$\boxed{0}$

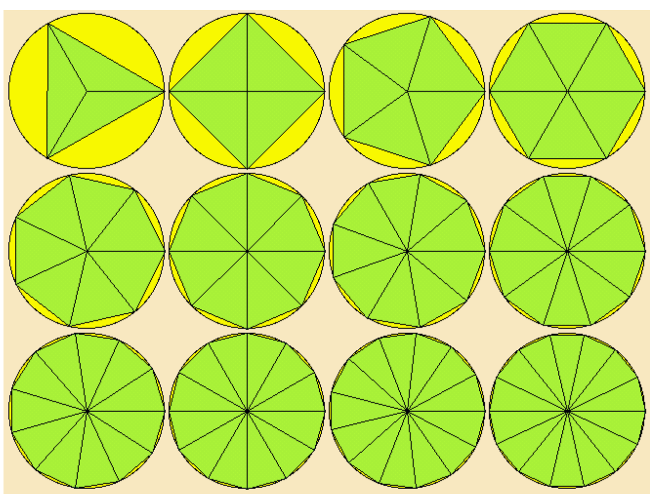
(b) $f(x) = \dfrac{x^2}{3}$ on $[2, 5]$

$\boxed{13}$

22. Use sigma notation and the appropriate summation formulas to formulate an expression which represents the net signed area between the graph of $f(x) = \cos x$ and the x-axis on the interval $[-\pi, \pi]$. Let x_k^* be the **right endpoint** of the k^{th} subinterval (where all subintervals have equal width). DO NOT EVALUATE YOUR EXPRESSION.

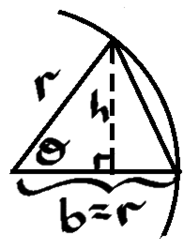
$$\boxed{\sum_{k=1}^{n} \cos\left(-\pi + \frac{2\pi}{n}k\right)\frac{2\pi}{n}}$$

23. A Regular Polygon is a polygon that is equiangular (all angles are equal in measure) and equilateral (all sides have the same length). The diagram below shows several regular polygons inscribed within a circle of radius r.



(a) Let A_n be the area of a regular n-sided polygon inscribed within a circle of radius r. Divide the polygon into n congruent triangles each with a central angle of $\frac{2\pi}{n}$ radians, as shown in the diagram above for several different values of n. Show that $A_n = \frac{1}{2}r^2 \sin\left(\frac{2\pi}{n}\right)n$.

We begin by examining one of the n triangles, pictured below.



The base of the triangle has a length of r. And, the height of the triangle is $r\sin\theta$, where θ is the central angle, $\frac{2\pi}{n}$. Thus, the area of one triangle is:

$$A = \frac{1}{2}(r)\left(r\sin\left(\frac{2\pi}{n}\right)\right) = \frac{1}{2}r^2\sin\left(\frac{2\pi}{n}\right)$$

But, the polygon is composed of n such triangles. So, the area of a regular n-sided polygon inscribed in the circle of radius r is:

$$A_n = \frac{1}{2}r^2\sin\left(\frac{2\pi}{n}\right)n$$

(b) What can you conclude about the area of the n-sided polygon as the number of sides of the polygon, n, approaches infinity? In other words, compute $\lim_{n\to\infty} A_n$.

$\boxed{\lim_{n\to\infty} A_n = \pi r^2}$