Cross Product

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference Chapter 11.4 of the recommended textbook (or the equivalent chapter in your alternative textbook/online resource) and your lecture notes.

EXPECTED SKILLS:

- Know how to compute the cross product of two vectors in $\mathbb{R}^3$.
- Be able to use a cross product to find a vector perpendicular to two given vectors.
- Know how to use a cross product to find areas of parallelograms and triangles.
- Be able to use a cross product together with a dot product to compute volumes of parallelepipeds.

PRACTICE PROBLEMS:

1. For each of the following, compute $\overrightarrow{u} \times \overrightarrow{v}$ and verify that it is orthogonal to both $\overrightarrow{u}$ and $\overrightarrow{v}$.

   (a) $\overrightarrow{u} = \langle 3, -4, 1\rangle$; $\overrightarrow{v} = \langle 2, -2, 3\rangle$

   $\boxed{\langle -10, -7, 2\rangle}$

   (b) $\overrightarrow{u} = \langle 2, -2, 6\rangle$; $\overrightarrow{v} = \langle -1, 2, -1\rangle$

   $\boxed{\langle -10, -4, 2\rangle}$

   (c) $\mathbf{u} = 2\mathbf{i} + 3\mathbf{k}$; $\mathbf{v} = \mathbf{i} - \mathbf{j}$

   $\boxed{\langle 3, 3, -2\rangle}$

2. (a) Using appropriate properties of the cross product (**Not Determinants**), compute $(\mathbf{i} - \mathbf{j}) \times (\mathbf{j} - \mathbf{i})$.

   $\boxed{\mathbf{0}\text{; Detailed Solution: Here}}$

   (b) Verify that your answer to part (a) is correct by using determinants.

   $\boxed{\mathbf{0}\text{; Detailed Solution: Here}}$

3. Compute two unit vectors which are normal to the plane which is determined by the points $A(1, 2, 3)$, $B(6, 4, 7)$, and $C(1, 5, 2)$.

   $$\boxed{\overrightarrow{u_{1,2}} = \pm\frac{1}{\sqrt{446}}\langle -14, 5, 15\rangle}$$

4. Compute the area of the triangle with vertices $A(1, 2, 3)$, $B(6, 4, 7)$, and $C(1, 5, 2)$.

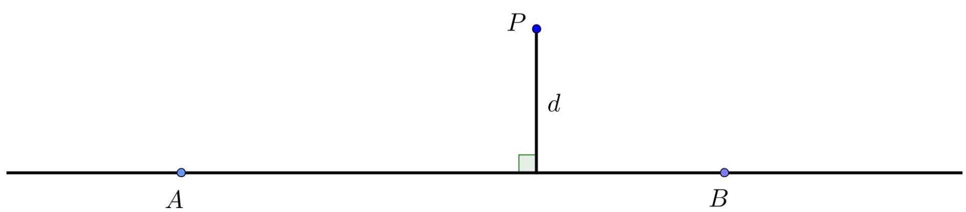
$\boxed{\frac{1}{2}\sqrt{446}}$

5. Compute $\|\mathbf{u} \times \mathbf{v}\|$ if $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 5$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $30°$.

$\boxed{5}$

6. The following questions deal with finding the distance from a point to a line:

(a) Given three points A, B, and P in 3-space as shown in the picture below, explain how you could use the cross product to calculate the distance, d, between the point P and the line which contains A and B.



(Hint: Consider the vectors **AP** and **AB**)

Let θ be the angle between **AP** and **AB**. Then:

$$\begin{aligned} d &= \|\mathbf{AP}\| \sin\theta \\ &= \frac{\|\mathbf{AP}\|\|\mathbf{AB}\| \sin\theta}{\|\mathbf{AB}\|} \\ &= \frac{\|\mathbf{AP} \times \mathbf{AB}\|}{\|\mathbf{AB}\|} \end{aligned}$$

(b) Use your method from part (a) to compute the distance from the point $P(5, 3, 0)$ to the line containing $A(1, 0, 1)$ and $B(2, 3, 1)$. Verify your answer with HW 11.3 #10(b).

$\boxed{d = \sqrt{\frac{91}{10}}}$

7. Consider the parallelepiped with adjacent edges $\vec{u} = \langle 1, 2, 3 \rangle$, $\vec{v} = \langle 3, 4, 0 \rangle$, and $\vec{w} = \langle -1, 3, -2 \rangle$.

(a) Compute the volume of the parallelepiped.

43; Detailed Solution: Here

(b) Determine the area of the face determined by $\vec{v}$ and $\vec{w}$.

$\sqrt{269}$; Detailed Solution: Here

(c) Compute the angle between $\overrightarrow{u}$ and the plane containing the face determined by $\overrightarrow{v}$ and $\overrightarrow{w}$.

$\boxed{\dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{43}{\sqrt{14}\sqrt{269}}\right)\text{; Detailed Solution: Here}}$

8. **Multiple Choice:** Suppose $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors in $\mathbb{R}^3$ and that $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\times\mathbf{v}\|$, which of the following is the angle between $\mathbf{u}$ and $\mathbf{v}$?

   (a) 0

   (b) $\dfrac{\pi}{6}$

   (c) $\dfrac{\pi}{4}$

   (d) $\dfrac{\pi}{3}$

   (e) $\dfrac{\pi}{2}$

   $\boxed{\text{c}}$

9. **True or False:** Mark each of the following as either true or false. If the statement is false, explain why or provide a counterexample.

   (a) The cross product of two vectors in $\mathbb{R}^3$ is anti-commutative; i.e., $\mathbf{v}\times\mathbf{w} = -(\mathbf{w}\times\mathbf{v})$.

   $\boxed{\text{True}}$

   (b) $\mathbf{i}\times\mathbf{k} = \mathbf{j}$.

   $\boxed{\text{False; } \mathbf{i}\times\mathbf{k} = -\mathbf{j}}$

   (c) For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{v}\times\mathbf{u}\|$.

   $\boxed{\text{True}}$

   (d) If $\mathbf{u}\times\mathbf{v} = \mathbf{0}$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

   $\boxed{\text{False; If } \mathbf{u} \text{ is parallel to } \mathbf{v}\text{, then } \mathbf{u}\times\mathbf{v} = \mathbf{0}.}$

   (e) If $\mathbf{u}\cdot\mathbf{v} = 0$ and $\mathbf{u}\times\mathbf{v} = \mathbf{0}$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

   $\boxed{\text{True}}$