Chapter 5.1: L'Hôpital's Rule & Indeterminate Forms

Expected Skills:

- Know how to use L'Hôpital's Rule to help compute limits involving indeterminate forms of $\frac{0}{0}$ and $\frac{\infty}{\infty}$
- Be able to compute limits involving indeterminate forms $\infty - \infty$, $0 \cdot \infty$, 0^0, ∞^0, and 1^∞ by manipulating the limits into a form where L'Hôpital's Rule is applicable.

Practice Problems:

For problems 1-27, calculate the indicated limit. If a limit does not exist, write $+\infty$, $-\infty$, or DNE (whichever is most appropriate). Make sure that L'Hôpital's rule applies before using it. And, whenever you apply L'Hôpital's rule, indicate that you are doing so.

1. $\displaystyle\lim_{x\to 3} \frac{x^2+4x-21}{x^2-7x+12}$

   $\boxed{-10}$

2. $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{\ln(1+x)}$

   $\boxed{3}$

3. $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^2}$

   $\boxed{0}$

4. $\displaystyle\lim_{x\to 0} \frac{\sin(6x)}{\sin(3x)}$

   $\boxed{2}$

5. $\displaystyle\lim_{x\to 1^-} \frac{x-1}{x^2-2x+1}$

   $\boxed{-\infty}$

6. $\displaystyle\lim_{x\to \infty} \frac{e^{-x}}{x^{-2}}$

   $\boxed{0}$

7. $\displaystyle\lim_{x\to 0} \frac{\ln(\cos 3x)}{5x^2}$

$\boxed{-\dfrac{9}{10}}$

8. $\displaystyle\lim_{x\to 1} \frac{\tan^{-1} x - \frac{\pi}{4}}{x-1}$

$\boxed{\dfrac{1}{2}}$

9. $\displaystyle\lim_{x\to 0^+} \frac{8^{\sqrt{x}} - 1}{1 - 5^{\sqrt{x}}}$

$\boxed{-\dfrac{3\ln 2}{\ln 5}}$

10. $\displaystyle\lim_{x\to 0^+} \frac{5\sin x}{\sqrt{x}}$

$\boxed{0}$

11. $\displaystyle\lim_{x\to -\infty} \frac{x^3 + 4x - 5}{5x^2 - 5x - 89}$

$\boxed{-\infty}$

12. $\displaystyle\lim_{x\to 0} \frac{\sin^{-1}(2x)}{\tan^{-1}(3x)}$

$\boxed{\dfrac{2}{3}}$

13. $\displaystyle\lim_{x\to 1} \frac{\ln x^2}{x^2 - 9}$

$\boxed{0}$

14. $\displaystyle\lim_{x\to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\boxed{1}$

15. $\displaystyle\lim_{x\to \frac{\pi}{2}^+} \frac{\ln\left(x - \frac{\pi}{2}\right)}{\tan x}$

$\boxed{0}$

16. $\displaystyle\lim_{x \to 1^-} \frac{x-1}{\arccos x}$

$\boxed{0}$

17. $\displaystyle\lim_{x \to +\infty} \frac{e^{\sqrt{x}}}{x}$

$\boxed{+\infty}$

18. $\displaystyle\lim_{x \to +\infty} xe^{-6x}$

$\boxed{0}$

19. $\displaystyle\lim_{x \to +\infty} \frac{\sqrt{4+3x^2}}{2+2x}$

$\boxed{\dfrac{\sqrt{3}}{2}}$

20. $\displaystyle\lim_{x \to 0^+} x \csc 3x$

$\boxed{\dfrac{1}{3}}$

21. $\displaystyle\lim_{x \to +\infty} [\ln(x+2) - \ln(3x+5)]$

$\boxed{\ln\left(\dfrac{1}{3}\right)}$

22. $\displaystyle\lim_{x \to \infty} 3^x 7^{-x}$

$\boxed{0}$

23. $\displaystyle\lim_{x \to \infty} \left(\sqrt{x^2-x} - x\right)$

$\boxed{-\dfrac{1}{2}}$

24. $\displaystyle\lim_{x \to 0^+} \tan x \sec x$

$\boxed{0}$

25. $\displaystyle\lim_{x \to 0^+} x^{1/x}$

$\boxed{0}$

26. $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{5x}$

$\boxed{e^{10}}$

27. $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{-x}$

$\boxed{e}$

28. Which of the following are indeterminate forms?

$$\frac{0}{0} \quad \frac{0}{\infty} \quad \frac{\infty}{0} \quad \frac{\infty}{\infty}$$
$$\infty-\infty \quad \infty+\infty \quad 0\cdot\infty \quad \infty\cdot\infty$$
$$0^0 \quad \infty^0 \quad 0^\infty \quad 1^\infty \quad \infty^\infty \quad \infty^1$$

$\boxed{\frac{0}{0},\ \frac{\infty}{\infty},\ \infty-\infty,\ 0\cdot\infty,\ 0^0,\ \infty^0,\ 1^\infty}$

29. Calculate each of the following limits:

(a) $\lim_{x\to 0^+}(1+3^x)^{1/x}$

$\boxed{+\infty}$

(b) $\lim_{x\to 0^-}(1+3^x)^{1/x}$

$\boxed{0}$

(c) $\lim_{x\to +\infty}(1+3^x)^{1/x}$

$\boxed{3}$

(d) $\lim_{x\to -\infty}(1+3^x)^{1/x}$

$\boxed{1}$

30. Show that $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for any positive integer n.

$\lim_{x\to\infty} \frac{x^n}{e^x}$ is of the indeterminate form $\frac{\infty}{\infty}$, so, we may apply L'Hopital's Rule:

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x}$$

This new limit is also of the indeterminate form $\frac{\infty}{\infty}$, so, we may again apply L'Hopital's Rule:

$$\lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x}$$

In fact, we repeat the process until we end up with the following limit:

$$\lim_{x\to\infty} \frac{n(n-1)(n-2)\dots(2)(1)}{e^x}$$

which equals 0. Thus, $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$

31. Find the value(s) of of the constant k which make $f(x) = \begin{cases} \dfrac{\sin x - 1}{x - \frac{\pi}{2}} & \text{if } x \neq \dfrac{\pi}{2} \\ k & \text{if } x = \dfrac{\pi}{2} \end{cases}$ continuous at $x = \frac{\pi}{2}$.

$\boxed{k = 0}$

32. Find all values of k and m such that $\lim_{x\to 1} \frac{k + m\ln x}{x - 1} = 5$

$\boxed{k = 0 \text{ and } m = 5}$

33. **Multiple Choice:** What is $\lim_{x\to 1^+} \frac{x}{\ln(x)}$?

(a) 0
(b) 1
(c) e
(d) e^{-1}
(e) $+\infty$

$\boxed{\text{E}}$

34. **Multiple Choice:** What is $\lim_{x \to 0} \dfrac{e^x - 1}{\tan(x)}$?

(a) -1

(b) 0

(c) 1

(d) 2

(e) The limit does not exist.

$\boxed{\text{C}}$

35. **Multiple Choice:** If $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} g(x) = +\infty$ and $f'(x) = 1$ and $g'(x) = e^x$, what is $\lim_{x \to +\infty} \dfrac{f(x)}{g(x)}$?

(a) -1

(b) 0

(c) 1

(d) e

(e) The limit does not exist.

$\boxed{\text{B}}$