Convergence of Taylor Series

SUGGESTED REFERENCE MATERIAL:

As you work through the problems listed below, you should reference your lecture notes and the relevant chapters in a textbook/online resource.

EXPECTED SKILLS:

- Know (i.e. memorize) the Remainder Estimation Theorem, and use it to find an upper bound on the error in approximating a function with its Taylor polynomial.
- Find the value(s) of x for which a Taylor series converges to a function $f(x)$.

PRACTICE PROBLEMS:

1. Find an upper bound for the remainder error if the 4th Maclaurin polynomial for $f(x) = \cos x$ is used to approximate $\cos 5°$.

   If $f(x) = \cos x$, then $|f^{(5)}(x)| \leq 1$ for all x, and so by the Remainder Estimation Theorem, $|R_4(\frac{\pi}{36})| \leq \frac{1}{5!}\left(\frac{\pi}{36} - 0\right)^5$.

   This can be verified with a calculator as follows:
   The 4th Maclaurin polynomial for $\cos x$ is $p_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$.
   Thus $\cos\left(\frac{\pi}{36}\right) \approx p_4\left(\frac{\pi}{36}\right) \approx 0.996194698705$
   Now a calculator tells us that $\cos\left(\frac{\pi}{36}\right) \approx 0.996194698092$.
   So $|R_4(\frac{\pi}{36})| = |\cos(\frac{\pi}{36}) - p_4(\frac{\pi}{36})| \approx 6 \times 10^{-10} < \frac{1}{5!}\left(\frac{\pi}{36}\right)^5 \approx 4.218 \times 10^{-8}$.

2. Find an upper bound for the remainder error if the 2nd Maclaurin polynomial for $f(x) = e^x$ is used to approximate $\sqrt{e}$?
   Note: You may assume that $\sqrt{e} < 2$ (this should be clear since $\sqrt{e} < \sqrt{3} < \sqrt{4} = 2$).

   Note that $f(x) = e^x$ is an increasing function. So for all x on the interval $[0, \frac{1}{2}]$ we have $|f^{(3)}(x)| = e^x \leq e^{\frac{1}{2}} < 2$, and so by the Remainder Estimation Theorem, $|R_2(\frac{1}{2})| \leq \frac{2}{3!}\left(\frac{1}{2} - 0\right)^3 = \frac{1}{24}$.

   This can be verified with a calculator as follows:
   The 2nd Maclaurin polynomial for e^x is $p_2(x) = 1 + x + \frac{1}{2}x^2$.
   Thus $\sqrt{e} \approx p_2(\frac{1}{2}) = 1.625$
   Now a calculator tells us that $\sqrt{e} \approx 1.648721271$.
   So $|R_2(\frac{1}{2})| = |\sqrt{e} - p_2(\frac{1}{2})| \approx 0.023721271 < \frac{1}{24} = 0.041\overline{6}$.

3. Find the smallest value of n that is needed so that the n-th Macluarin polynomial $p_n(x)$ approximates $\sqrt{e}$ to four decimal-place accuracy. In other words, find the smallest value of n so that the n-th remainder $|R_n(\frac{1}{2})| \leq 0.00005$.
   Note: You may assume that $\sqrt{e} < 2$ (this should be clear since $\sqrt{e} < \sqrt{3} < \sqrt{4} = 2$).

Note that $f(x) = e^x$ is an increasing function and $f^{(n)}(x) = e^x$ for all n. So for all x on the interval $[0, \frac{1}{2}]$ we have $|f^{(n+1)}(x)| = e^x \leq e^{\frac{1}{2}} < 2$, and so by the Remainder Estimation Theorem, $|R_n(\frac{1}{2})| \leq \frac{2}{(n+1)!}\left(\frac{1}{2} - 0\right)^{n+1} = \frac{1}{2^n(n+1)!}$.

So we want $\frac{1}{2^n(n+1)!} \leq 0.00005$, or $2^n(n+1)! \geq 20,000$.
For $n = 4 : 2^4(5!) = 16(120) < 20,000$.
For $n = 5 : 2^5(6!) = 32(720) > 20,000$.
So we should let $n = 5$, i.e. the 5-th Macluarin polynomial for e^x approximates $\sqrt{e}$ to four decimal-place accuracy.

4. Find the smallest value of n so that the Taylor polynomial for $f(x) = \ln(x)$ about $x_0 = 1$ approximates $\ln(1.2)$ to three decimal-place accuracy.

$n = 3$.; Detailed Solution: Here

5. The purpose of this problem is to show that the Maclaurin series for $f(x) = \cos x$ converges to $\cos x$ for all x.

   (a) Find the Maclaurin series for $f(x) = \cos x$.

   $$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.}$$

   (b) Find the interval of convergence for this Maclaurin series.

   $(-\infty, +\infty)$. See Power Series #15.

   (c) Show that the n-th remainder goes to 0 as n goes to $+\infty$, i.e. show that $\lim_{n\to+\infty} |R_n(x)| = 0$.

   If $f(x) = \cos(x)$, then $|f^{(n+1)}(x)| \leq 1$ for all n and for all x.

   So by the Remainder Estimation Theorem, $0 \leq |R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$.

   Now $\lim_{n\to+\infty} 0 = \lim_{n\to+\infty} \frac{1}{(n+1)!}|x|^{n+1} = 0$.

   So by the Squeeze Theorem $\lim_{n\to+\infty} |R_n(x)| = 0$

6. Show that the Maclaurin series for $f(x) = \frac{1}{1-x}$ converges to $f(x)$ for all x in its interval of convergence.

The Maclaurin series for $f(x) = \frac{1}{1-x}$ is $1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{k=0}^{\infty} x^k$, which is a geometric series with $a = 1$ and $r = x$. Thus the series converges if, and only if, $-1 < x < 1$. For these values of x, the series converges to $\frac{a}{1-r} = \frac{1}{1-x} = f(x)$.

7. The pupose of this problem is to show that it is possible for a function $f(x)$ to have a Maclaurin series that converges for all x but does not always converge to $f(x)$.

Consider the piecewise function $f(x) = \begin{cases} e^{(-1/x^2)}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

(a) Use the definition of the derivative $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ to show that $f'(0) = 0$. Hint: Make the substitution $t = \frac{1}{h}$ and compute the one-sided limits as $h \to 0^+$ and $h \to 0^-$.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{(-1/h^2)}-0}{h}$$

Now let $t = \frac{1}{h}$ and examine the limit as $h \to 0^+$ and $h \to 0^-$.

So $\displaystyle\lim_{h\to 0^+} \frac{e^{(-1/h^2)}}{h} = \lim_{t\to+\infty} \frac{e^{-t^2}}{\frac{1}{t}} = \lim_{t\to+\infty} \frac{t}{e^{t^2}} = \lim_{t\to+\infty} \frac{1}{2te^{t^2}} = 0,$

and $\displaystyle\lim_{h\to 0^-} \frac{e^{(-1/h^2)}}{h} = \lim_{t\to-\infty} \frac{e^{-t^2}}{\frac{1}{t}} = \lim_{t\to-\infty} \frac{t}{e^{t^2}} = \lim_{t\to-\infty} \frac{1}{2te^{t^2}} = 0.$

Therefore $f'(0) = 0$.

(b) Assuming that $f^{(n)}(0) = 0$ for $n \geq 2$, find the Macluarin series for $f(x)$ and the interval of convergence for the series.

Since $f(0) = 0$ from the function definiton, $f'(0) = 0$ by part (a), and $f^{(n)}(0) = 0$ for $n \geq 2$ by assumption, the Macluarin series for $f(x)$ is

$$0 + 0x + \frac{0x^2}{2!} + \frac{0x^3}{3!} + \ldots = \sum_{k=0}^{\infty} 0 = 0.$$

Thus the series converges (to 0) for all x, i.e. the interval of convergance is $(-\infty, +\infty)$.

(c) Find the values of x for which the Maclaurin series converges to $f(x)$.

From part (b) we know that the Maclaurin series for $f(x)$ converges to 0 for all x. Now $f(0) = 0$, but if $x \neq 0$ then $f(x) = e^{(-1/x^2)} > 0$. Thus the Maclaurin series for $f(x)$ converges for all x but only converges to $f(x)$ for $x = 0$.