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Math 117: Calculus & Functions II

Contents

0 Calculus Review from Math 116

Chapter 0 provides a brief summary of the essential content from Math 116: Calculus and Functions I. Specifically, we summarize the concepts of limits, continuity, and differentiation as well as some of the main theorems associated with these topics. Much of this material is provided without proof; see your Math 116 notes or textbook for further explanation. Our goal in this chapter is to remind you of the foundational skills that you have already learned as we will be advancing the theory by introducing trigonometric, exponential, and logarithmic functions in the chapters that follow.

0.1 Limits

0.1.1 Defining Limits – Intuitively and Formally

The idea of a limit is arguably the foundation of Calculus as we know it. By using this building block, we are able to define other concepts, including continuity, differentiability, and integrability – all of which have major applications in analytic geometry, the physical sciences, and engineering. Intuitively, if the values of the function $f(x)$ can be made as close to the number L as we like by taking values of x sufficiently close to (but not equal to) a, we say: "the limit of $f(x)$ as x approaches a is L" and we write:

$$\lim_{x\to a} f(x) = L$$

Formally, we define $\lim_{x\to a} f(x) = L$ as follows:

Definition 0.1.1 (Two-Sided Limit) *We say that* $\lim\limits_{x\to a} f(x) = L$ *if for all* $\epsilon > 0$*, there is a number* $\delta > 0$ *such that whenever* x *is in the interval* $(a-\delta, a) \cup (a, a+\delta)$ *it follows that* $f(x)$ *is in the interval* $(L-\epsilon, L+\epsilon)$*.*

Notice that it is required for $f(x)$ to get close to L when considering x's that are on *either* side of $f(x)$; hence, we call this a *two-sided limit.* There are instances when we are interested in the behavior of a function on only one side of $x = a$. This gives us the concept of a *one-sided limit.*

Definition 0.1.2 (One-Sided Limit) *We define the limit from the left and the limit from the right as follows:*

- **Left-Sided Limit:** *We say that* $\lim_{x \to a^-} f(x) = L$ *if for all* $\epsilon > 0$*, there is a number* $\delta > 0$ *such that whenever* x *is in the interval* $(a - \delta, a)$ *it follows that* $f(x)$ *is in the interval* $(L - \epsilon, L + \epsilon)$*.*

- **Right-Sided Limit:** *We say that* $\lim_{x \to a^+} f(x) = L$ *if for all* $\epsilon > 0$*, there is a number* $\delta > 0$ *such that whenever* x *is in the interval* $(a, a + \delta)$ *it follows that* $f(x)$ *is in the interval* $(L - \epsilon, L + \epsilon)$*.*

Notice that for the limit from the left, we are only concerned about what $f(x)$ approaches when considering values of x that are close to $x = a$ but a little smaller than $x = a$. Similarly, for the limit from the right, we are only concerned about what $f(x)$ approaches when considering values of x that are close to $x = a$ but a little larger than $x = a$. One key fact to remember is summarized in the following theorem:

Theorem 0.1.1 $\lim_{x \to a} f(x) = L$ *if and only if* $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

There are instances in which the values of a function $f(x)$ will continue to grow without bound for x near $x = a$. In these cases, we say $\lim_{x \to a} f(x) = +\infty$. Again, we can formalize this as follows:

Definition 0.1.3 *We say that* $\lim_{x \to a} f(x) = +\infty$ *if for all* $M > 0$*, there is a number* $\delta > 0$ *such that whenever* x *is in the interval* $(a - \delta, a) \cup (a, a + \delta)$ *it follows that* $f(x) > M$*.*

This means that whatever bound M you set, it will be possible to keep $f(x)$ above that bound by staying close (within δ) of $x = a$. Analogously, we may define limits which tend to $-\infty$. In situations where the values of $f(x)$ grow without bound near $x = a$, we say that the graph of $f(x)$ has a *vertical asymptote* of $x = a$.

Definition 0.1.4 (Vertical Asymptote) *We say that the graph of $f(x)$ has a vertical asymptote of $x = a$ if at least one of the following is true:*

- $\lim_{x \to a^-} f(x) = -\infty$
- $\lim_{x \to a^-} f(x) = +\infty$
- $\lim_{x \to a^+} f(x) = -\infty$
- $\lim_{x \to a^+} f(x) = +\infty$

Finally, we are often interested in the end behavior of a function. That is, we want to know how the function behaves as x tends to either ∞ or $-\infty$. Intuitively, we say $\lim_{x \to \infty} f(x) = L$ if the values of $f(x)$ can be made as close to the number L as we like by taking values of x that are large enough. Here is the formal definition:

Definition 0.1.5 *We say that $\lim_{x \to \infty} f(x) = L$ if for all $\epsilon > 0$, there is a number $N > 0$ such that whenever $x > N$ it follows that $f(x)$ is in the interval $(f(x) - L, f(x) + L)$.*

By analogy, one can define $\lim_{x \to -\infty} f(x) = L$.

Definition 0.1.6 (Horizontal Asymptote) *We say that the graph of $f(x)$ has a horizontal asymptote of $y = L$ if at least one of the following is true:*

- $\lim_{x \to -\infty} f(x) = L$
- $\lim_{x \to +\infty} f(x) = L$

0.1.2 Computing Limits

From the formal definitions provided in the preceding section, one can prove the following theorems about limits:

Theorem 0.1.2 (Properties of Limits) *Suppose* $\lim_{x\to a} f(x) = L_1$ *and* $\lim_{x\to a} g(x) = L_2$. *Then,*

- $\lim_{x\to a} [f(x) \pm g(x)] = L_1 \pm L_2$
- $\lim_{x\to a} [f(x)g(x)] = L_1 L_2$
- $\lim_{x\to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{L_1}{L_2}$ *provided* $L_2 \neq 0$.

It would also be useful to remember the following "building block" limit facts:

Theorem 0.1.3 *Let* a *and* k *be real numbers. Then:*

- $\lim_{x\to a} k = k$
- $\lim_{x\to a} x = a$
- $\lim_{x\to 0^-} \dfrac{1}{x} = -\infty$
- $\lim_{x\to 0^+} \dfrac{1}{x} = +\infty$

Many of the limit problems that you have already dealt with involve rational functions. Recall that a rational function has the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. These types of problems can be broken up in to 3 cases:

1. If $q(a) \neq 0$, then $\lim_{x\to a} f(x) = f(a)$
2. If $q(a) = 0$ but $p(a) \neq 0$, then either $\lim_{x\to a} f(x) = +\infty$, $\lim_{x\to a} f(x) = -\infty$, or the limit does not exist.
3. If both $q(a) = 0$ and $p(a) = 0$, then the limit is of the indeterminate form $\frac{0}{0}$. Since $p(x)$ and $q(x)$ are both polynomials with value 0 at $x = a$, there will be a common factor. Try to exploit this fact to replace the function with one that agrees everywhere (except potentially at $x = a$) so that you may try to determine the value of the limit.

Finally, here are some useful facts to remember about limits at infinity:

Theorem 0.1.4 *Let k be a real number and n be a positive real number. Then:*

- $\lim_{x \to \pm\infty} k = k$
- $\lim_{x \to +\infty} x^n = +\infty$
- $\lim_{x \to -\infty} x^n = \begin{cases} +\infty & n \text{ even} \\ -\infty & n \text{ odd} \end{cases}$
- $\lim_{x \to \pm\infty} \left(c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0\right) = \lim_{x \to \pm\infty} c_n x^n$
- $\lim_{x \to \pm\infty} \dfrac{1}{x^n} = 0$

Many of the limit problems that you have already dealt with involve rational functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. Using the limit properties in Theorem 0.1.2, will always give rise to the indeterminate form $\frac{\infty}{\infty}$. Typically, we divide both the numerator and denominator by the highest powered x in the denominator to rewrite the function in a "better form." After doing so, we appeal to the last fact in Theorem 0.1.4.

0.2 Continuity

Definition 0.2.1 *A function f is continuous at $x = a$ if all three of the following conditions are satisfied:*

1. *$f(a)$ is defined,*
2. *$\lim_{x \to a} f(x)$ exists,*
3. *$\lim_{x \to a} f(x) = f(a)$.*

If $f(x)$ is not continuous at $x = a$, we say that f has a *discontinuity* at $x = a$. We often classify common types of discontinuities as follows:

- If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $f(x)$ has a **jump discontinuity** at $x = a$.
- If $\lim_{x \to a^-} f(x) = +\infty$, $\lim_{x \to a^+} f(x) = +\infty$, $\lim_{x \to a^-} f(x) = -\infty$, or $\lim_{x \to a^+} f(x) = -\infty$, then $f(x)$ has an **infinite (essential) discontinuity** at $x = a$.
- If $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$, then f has a **removable discontinuity** at $x = a$.

If a function $f(x)$ is continuous for all x in the interval, we say that f is continuous on the interval. Using Definition 0.2.1, one can prove the following theorem:

Theorem 0.2.1 *All of the elementary functions are continuous on their respective domains.*

Theorem 0.2.2 *Suppose $f(x)$ and $g(x)$ are continuous at $x = a$. Then:*

- *$f \pm g$ is continuous at $x = a$.*
- *fg is continuous at $x = a$.*
- *$\frac{f}{g}$ is continuous at $x = a$ if $g(a) \neq 0$ and has a discontinuity at $x = a$ if $g(a) = 0$.*

0.3 Differentiation

0.3.1 The Derivative

Definition 0.3.1 (The Derivative) *The* **derivative of a function f with respect to x** *is defined by the following limit:*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

provided that this limit exists. If $f'(x_0)$ exists, we say that f is **differentiable at x_0**. *If f is differentiable at each point in its domain, we say that f is* **differentiable**.

If $y = f(x)$, we often use the following notations to denote the derivative:

Newton Notation	**Leibnitz Notation**
y'	$\frac{dy}{dx}$
$f'(x)$	$\frac{d}{dx}\left[f(x)\right]$

Physically, one may interpret $f'(x_0)$ as the instantaneous rate of change of $f(x)$ at x_0. Graphically, one may interpret $f'(x_0)$ as the slope of the line which is tangent to the graph of $f(x)$ at the point $(x_0, f(x_0))$.

Since the derivative of f is a function in its own right, it too may have a derivative. We can calculate the nth order derivative by differentiating n times:

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = \frac{d^2}{dx^2}[f(x)]$$

$$y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = \frac{d^3}{dx^3}[f(x)]$$
$$y^{(4)} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d^4y}{dx^4} = f^{(4)}(x) = \frac{d^4}{dx^4}[f(x)]$$

etc. Notice that it becomes cumbersome to put lots of primes for higher ordered derivatives. Hence, we often signify the order of the derivative by using an exponent in parentheses.

0.3.2 Techniques of Differentiation

The following theorem is a summary of the differentiation rules that you have learned through Math 116:

Theorem 0.3.1 (Derivative Rules) *Suppose $f(x)$ and $g(x)$ are differentiable functions, n is a real number, and c is a real number.*

- $\frac{d}{dx}[c] = 0$
- *Power Rule:* $\frac{d}{dx}[x^n] = nx^{n-1}$
- *Constant Multiple Rule:* $\frac{d}{dx}[cf(x)] = cf'(x)$
- *Sum/Difference Rule:* $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
- *Product Rule:* $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$
- *Quotient Rule:* $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$
- *Chain Rule:* $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

0.3.3 Theorems About Differentiation

The following is a summary of several important definitions and theorems on differential calculus and its applications.

Theorem 0.3.2 *If $f(x)$ is differentiable at x_0, then $f(x)$ is continuous at x_0.*

Definition 0.3.2 (Local/Relative Extrema) *A local extremum is either a local maximum or a local minimum.*

- *A function* $f(x)$ *will have a* **local/relative maximum** *at* x_0 *if* $f(x_0) \geq f(x)$ *for all* x *in a neighborhood surrounding* x_0*.*
- *A function* $f(x)$ *will have a* **local/relative minimum** *at* x_0 *if* $f(x_0) \leq f(x)$ *for all* x *in a neighborhood surrounding* x_0*.*

Definition 0.3.3 (Critical Point) *A critical point of* f *is a point in the domain of* f *at which either* $f'(x) = 0$ *or* $f'(x)$ *does not exist. These are the candidates for local (relative) extrema.*

Theorem 0.3.3 *Suppose that* $f(x)$ *is continuous on* $[a, b]$ *and differentiable on* (a, b)*. Then:*

- *If* $f'(x) > 0$ *for every value of* x *in* (a, b)*, then* f *is increasing on* (a, b)*.*
- *If* $f'(x) < 0$ *for every value of* x *in* (a, b)*, then* f *is decreasing on* (a, b)*.*
- *If* $f'(x) = 0$ *for every value of* x *in* (a, b)*, then* f *is constant on* (a, b)*.*

Theorem 0.3.4 *Suppose that* $f(x)$ *is twice differentiable on an interval* I*.*

- *If* $f''(x) > 0$ *for every value of* x *in* I*, then* f *is concave up on* I*.*
- *If* $f''(x) < 0$ *for every value of* x *in* I*, then* f *is concave down on* I*.*

Theorem 0.3.5 (First Derivative Test for Local Extrema) *Suppose that $f(x)$ is continuous at a critical point x_0.*

- *If $f'(x) > 0$ for every value of x on an open interval extending left from x_0 and $f'(x) < 0$ for every value of x on an open interval extending right from x_0, then $f(x)$ has a local maximum at x_0.*
- *If $f'(x) < 0$ for every value of x on an open interval extending left from x_0 and $f'(x) > 0$ for every value of x on an open interval extending right from x_0, then $f(x)$ has a local minimum at x_0.*
- *If the sign of $f'(x)$ remains unchanged on an open interval around x_0, then there is neither a local maximum nor a local minimum of $f(x)$ at x_0.*

Theorem 0.3.6 (Second Derivative Test for Local Extrema) *Suppose $f(x)$ is twice differentiable at a critical point x_0.*

- *If $f''(x_0) > 0$, then $f(x)$ has a local minimum at x_0.*
- *If $f''(x_0) < 0$, then $f(x)$ has a local maximum at x_0.*
- *If $f''(x_0) = 0$, then the test is inconclusive; $f(x)$ may or may not have a local extremum at x_0. Try another test.*

Definition 0.3.4 (Absolute/Global Extrema) *An absolute extremum is either an absolute maximum or an absolute minimum.*

- *A function $f(x)$ has an absolute maximum at x_0 in a domain if $f(x_0) \geq f(x)$ for all x in the domain.*
- *A function $f(x)$ has an absolute minimum at x_0 in a domain if $f(x_0) \leq f(x)$ for all x in the domain.*

Theorem 0.3.7 (Extrema Value Theorem) *Suppose $f(x)$ is continuous on the closed and bounded interval $[a, b]$, then, $f(x)$ must achieve both an absolute maximum and an absolute minimum.*

Theorem 0.3.8 (Rolle's Theorem) *Suppose $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). Furthermore, suppose $f(a) = f(b)$. Then, there must be at least one point x_0 in the interval (a, b) at which $f'(x_0) = 0$.*

Theorem 0.3.9 (Mean Value Theorem) *Suppose $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). Then, there must be at least one point x_0 in the interval (a, b) at which $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.*

1 Numerically Approximating Solutions to Equations

In Math 116, we reviewed methods for solving different types of equations. For example, when you are faced with a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, you learned how to solve for the solutions using algebraic methods including factoring, completing the square, or the quadratic formula. However, there are many instances in which finding the exact solution algebraically is impractical. In these situations, you may rely on numerical methods to approximate the solutions of a given equation. This chapter will discuss two numerical methods. In chapter 1.1, we discuss the **Bisection Method** and in chapter 1.2, we discuss **Newton's Method**.

1.1 Bisection Method (Interval Halving Method)

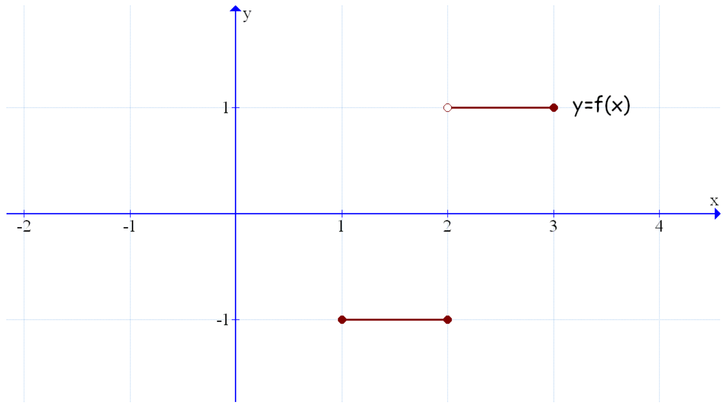
The Bisection Method is an iterative process which repeatedly applies the Intermediate Value Theorem to approximate a solution to the equation $f(x) = 0$. Before we demonstrate the method, let us first recall the statement of the key theorem on which this method is built.

Theorem 1.1.1 (Intermediate Value Theorem) *Suppose $f(x)$ is continuous on the interval $[a, b]$ with $f(a)f(b) < 0$. Then, there must be at least one $x = c$ in the interval (a, b) at which $f(c) = 0$.*

Notice that there are two assumptions which must be satisfied in order to apply the theorem. Firstly, the function must be continuous on $[a, b]$. For example, consider the following function

$$f(x) = \begin{cases} -1 & \text{if } 1 \leq x \leq 2 \\ 1 & \text{if } 2 < x \leq 3 \end{cases}$$

Notice that the $f(x)$ is not continuous at $x = 2$ because $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$. So, even though $f(1)f(3) < 0$, the conclusion of the intermediate value theorem does not necessarily have to hold because $f(x)$ has a discontinuity within this interval. As you can see from the following graph, the function jumps over the x-axis. And, in fact, there are no values of x in the interval $[1, 3]$ for which $f(x) = 0$. Thus, the continuity assumption must be satisfied in order to apply the theorem.



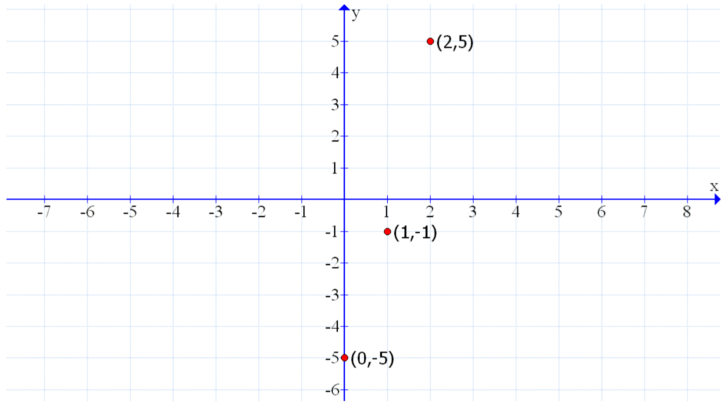
Secondly, the values of $f(x)$ at the endpoints must have different signs; that is, either $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$ must be satisfied. If not, there is no reason the function would be required to cross the x-axis. Try to sketch a continuous function which demonstrates why this second assumption is also essential to apply the theorem.

The following example demonstrates how you can apply the intermediate value theorem to justify the existence of a solution to $f(x) = 0$ within an interval (a, b).

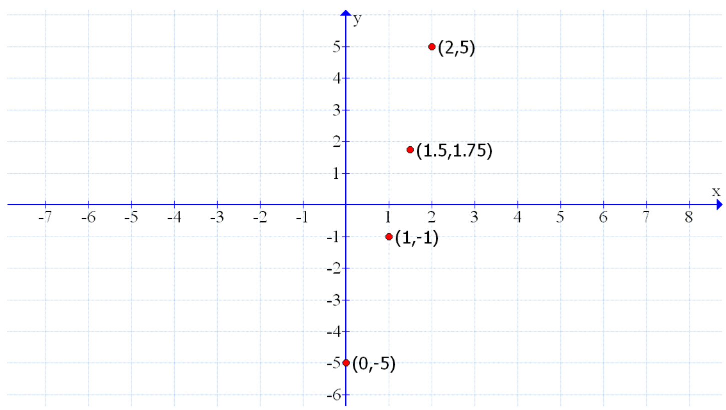
Example 1.1.1 *Use the intermediate value theorem to show that there is a solution to* $x^2 + 3x - 5 = 0$ *on the interval* $(0, 2)$.

Solution: *Let* $f(x) = x^2 + 3x - 5$. *We want to show that there is a solution to* $f(x) = 0$ *on the interval* $(0, 2)$. *First, we notice that* $f(x)$ *is a polynomial and is therefore continuous everywhere. Secondly, we notice that* $f(0) = -5 < 0$ *and* $f(2) = 5 > 0$. *Thus, the intermediate value theorem applies and there is at least one solution to* $f(x) = 0$ *in the interval* $(0, 2)$. □

Now that we know there is a solution to the given equation within the interval $(0, 2)$, let us try to narrow down the interval over multiple iterations to approximate the value of this solution. One way to do this is to keep halving the interval. Specifically, notice that the midpoint of the interval is $m_0 = 1$ and $f(1) = -1 < 0$. So, after this iteration, here is a graphical representation of what we have:



Notice that, according to the intermediate value theorem, a solution must now be in the interval $(1, 2)$ because $f(x)$ is continuous on $[1, 2]$, $f(1) < 0$, and $f(2) > 0$. For the next iteration, let us bisect this new interval; the new midpoint is $m_1 = 1.5$ and $f(1.5) = 1.75 > 0$. So, graphically, we have:



And, according to the intermediate value theorem, the solution that we are seeking is now in the interval $(1, 1.5)$. We can continue this way by repeatedly bisecting the intervals from the previous iterations. Eventually, if we were to continue indefinitely, this sequence of approximations should converge to the exact solution of $x = \frac{-3+\sqrt{29}}{2} \approx 1.19$.

Unfortunately, in practice, we cannot continue indefinitely. We would most likely program a computer to carry out the following algorithm for us. However, in order to do that, we need to implement a stopping procedure. Most common stopping procedures involve a preselected tolerance $\epsilon > 0$. An example of a stopping procedure is now described. If at some instant, the interval were (a, b), then the distance from the midpoint, m, to either of the endpoints would be half of the interval length: $\frac{b-a}{2}$. So, if we wanted to approximate our solution within some tolerance ϵ, we would need the radius of the interval to satisfy $r = \frac{b-a}{2} < \epsilon$. Using this as our stopping condition, we formalize the method below:

Algorithm 1.1.1 (Bisection Method) *Suppose $f(x)$ is a continuous function on $[a_0, b_0]$ satisfying $f(a_0)f(b_0) < 0$, where $a_0 < b_0$. Let ϵ be the tolerance that you are willing to accept. Repeat the following steps.*

1. *For each integer $k \geq 0$ Calculate $m_k = \dfrac{a_k + b_k}{2}$, the midpoint of the interval at the beginning of the kth iteration.*

2. *Calculate $f(m_k)$.*

3. *If $f(m_k) = 0$ or $r_k = \dfrac{b_k - a_k}{2} < \epsilon$, stop. The m_k is an approximate solution to $f(x) = 0$ within the preset tolerance. Otherwise, continue to step 4.*

4. *Since $f(m_k) \neq 0$, then $f(m_k)$ must have the same sign as $f(a_k)$ or $f(b_k)$.*

   - *If $f(m_k)$ has the same sign as $f(a_k)$, then the root must be in the interval (m_k, b_k). So, set $a_{k+1} = m_k$, $b_{k+1} = b_k$ and go to step 1.*

   - *If $f(m_k)$ has the same sign as $f(b_k)$, then the root must be in the interval (a_k, m_k). So, set $a_{k+1} = a_k$, $b_{k+1} = m_k$ and go to step 1.*

Example 1.1.2 *Apply the Bisection Method Algorithm described above with a tolerance of $\epsilon = 0.1$ to approximate a solution to $x^3 + 3x - 1 = 0$ in the interval $(0, 1)$.*

Solution:

Notice that $f(x) = x^3 + 3x - 1$ is continuous on the interval $[0, 1]$, $f(0) = -1 < 0$, and $f(1) = 3 > 0$. Thus, the algorithm is applicable; we will organize the output of the algorithm in the following table:

k	a_k	b_k	m_k	$f(m_k)$	$r_k = \frac{b_k - a_k}{2}$
0	0	1	0.5	0.625	0.5
1	0	0.5	0.25	-0.234375	0.25
2	0.25	0.5	0.375	0.177734375	0.125
3	0.25	0.375	0.3125	-0.031982422	0.0625

At this point, the radius of the interval $\frac{b_3 - a_3}{2} = 0.0625 < 0.1$. So, the algorithms stops because we are within the appropriate tolerance. It follows that the solution that we were seeking to the equation $x^3 + 3x - 1 = 0$ in the interval $(0, 1)$ is approximately $x = 0.3125$ with an accuracy of 0.0625. A more accurate solution (computed by a calculator) is 0.3221853540. □

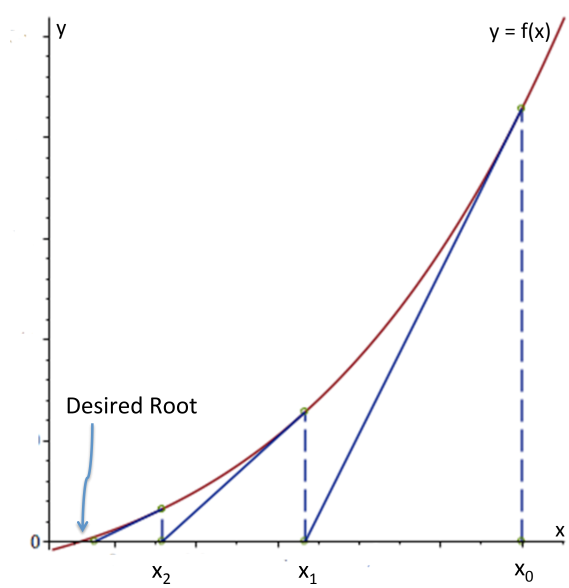
We conclude this section by mentioning that other stopping procedures are also possible. Here are a few other options:

- You can set a maximum number of iterations. For example, perform at most 10 iterations of the bisection method.
- You can pick a tolerance $\epsilon > 0$. At each step, check whether $|f(m_k)| \leq \epsilon$. If, indeed, $|f(m_k)| < \epsilon$, then at this point, $f(x)$ is rather small (close to 0); it follows that m_k is a reasonable approximation of the desired solution to $f(x) = 0$. Else, if $|f(m_k)| > \epsilon$, perform another iteration.
- You can pick a tolerance $\epsilon > 0$. At each iteration ($k \geq 1$), compute $|m_k - m_{k-1}|$. If $|m_k - m_{k-1}| < \epsilon$, then the two midpoint approximations are close to one another; stop the algorithm and choose m_k to be the estimated solution to $f(x) = 0$. Else, if $|m_k - m_{k-1}| > \epsilon$, perform another iteration.

You will be asked to implement these different stopping algorithms in the exercises.

1.2 Newton's Method

Newton's Method is an iterative process which is used to approximate solutions to equations of the form $f(x) = 0$. Graphically, we are trying to find a point where the graph of $y = f(x)$ crosses the x-axis. We begin the method by picking a point x_0. Then, each iteration will find the root of an appropriate linearization (tangent line). The goal is to move step-by-step towards a point where the graph of f crosses that x-axis, as demonstrated in the diagram below. Notice that because we are using tangent lines, we require that $f\prime(x)$ be differentiable.



Now that we have a general understanding about how this method works, let us make it more precise. As described in the introduction, we begin with a guess x_0 of the root that we are trying to find. The tangent line to the graph of $f(x)$ at the point $(x_0, f(x_0))$ is $y - f(x_0) = f'(x_0)(x - x_0)$. We can calculate the x-intercept of this tangent line by letting $y = 0$ and solving for the corresponding value of x.

$$\begin{aligned} 0 - f(x_0) &= f'(x_0)(x - x_0) \\ -\frac{f(x_0)}{f'(x_0)} &= x - x_0 && \text{Provided} f'(x_0) \neq 0 \\ x &= x_0 - \frac{f(x_0)}{f'(x_0)} \end{aligned}$$

Call this value x_1; that is, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. If $f(x_1) = 0$, we stop because we have found a root. Else, we should continue implementing the algorithm. Specifically, we can find the equation of the tangent line to $f(x)$ at $(x_1, f(x_1))$ and then solve for its x-intercept, which we will call x_2. Notice that $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, provided that $f'(x_1) \neq 0$. Again, if $f(x_2) = 0$, we stop as we have found the desired root. Else, if $f(x_2) \neq 0$, we perform another iteration. In fact, in each iteration, our next approximation is given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, provided $f'(x_n) \neq 0$.

Before demonstrating Newton's Method with an example, it is worthwhile to mentioning what would happen if $f'(x_n) = 0$ in the nth iteration. Aside from our formula not making sense (because of division by 0), if we have $f'(x_n) = 0$ and $f(x_n) \neq 0$, then Newton's Method will fail because, at the point $(x_n, f(x_n))$ the graph would have a horizontal tangent line which will not intersect the x-axis. As a result, the method cannot progress.

As with the Bisection Method discussed in the previous section, Newton's Method is typically implemented by a computer. So, one has to introduce a stopping procedure. Similar stopping procedures from the Bisection Method still apply:

- You can set a maximum number of iterations. For example, perform at most 10 iterations of Newton's Method.

- You can pick a tolerance $\epsilon > 0$. At each step, check whether $|f(x_k)| \leq \epsilon$. If, indeed, $|f(x_k)| < \epsilon$, then at this point, $f(x)$ is rather small (close to 0); it follows that x_k is a reasonable approximation of the desired solution to $f(x) = 0$. Else, if $|f(x_k)| > \epsilon$, perform another iteration.

- You can pick a tolerance $\epsilon > 0$. At each iteration $(k \geq 1)$, compute $|x_k - x_{k-1}|$. If $|m_k - m_{k-1}| < \epsilon$, then the two approximations are close to one another; stop the algorithm and choose x_k to be the estimated solution to $f(x) = 0$. Else, if $|x_k - x_{k-1}| > \epsilon$, perform another iteration.

You will be asked to implement these different stopping algorithms in the exercises. The following describes the general flow of the algorithm with one of the suggested stopping procedures.

Algorithm 1.2.1 (Newton's Method) *Suppose $f(x)$ is a differentiable function. Let ϵ be an appropriate tolerance. Repeat the following steps.*

1. *Initialize the algorithm by choosing a point x_0. (You may want to use the intermediate value theorem to choose a point in an interval where a solution is guaranteed to exist.)*

2. *For each $k \geq 0$, compute $f(x_k)$ and $f'(x_k)$.*

    - *If $f(x_k) = 0$ or $|f(x_k)| < \epsilon$, stop because you have found a reasonable estimate of the solution. Else, continue to the next bullet.*

    - *If $f'(x_k) \neq 0$, continue to step 3. Else, if $f'(x_k) = 0$, stop. The algorithm will fail. Perhaps reinitialize by choosing a different x_0.*

3. *For each integer $k \geq 0$ Calculate $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$ and return to step 2.*

Example 1.2.1 *Perform three iterations of Newton's Method to solve $x^3+3x-1 = 0$. Begin with $x_0 = 1$*

Solution:

Let $f(x) = x^3+3x-1$. We are trying to solve $f(x) = 0$ (or is sufficiently close to 0) and begin with $x_0 = 1$ as our initial guess. Clearly $f(1) = 3 \neq 0$, so, this beginning guess is not the solution we seek. Hence, we will apply an iteration of Newton's Method to get a (hopefully) better approximation of the solution. To do so, we need $f(1)$ and $f'(1)$. By the power rule, we know that $f'(x) = 3x^2 + 3$; so, $f'(1) = 6$.

$$
\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
x_1 &= 1 - \frac{3}{6} \\
x_1 &= \frac{1}{2}
\end{aligned}
$$

Notice that $f\left(\frac{1}{2}\right) = \frac{5}{8} \neq 0$; so, we apply another iteration of Newton's Method. Since $f\left(\frac{1}{2}\right) = \frac{5}{8}$ and $f'\left(\frac{1}{2}\right) = \frac{15}{4}$, we get our next approximation of the desired root:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$x_2 = \frac{1}{2} - \frac{\frac{5}{8}}{\frac{15}{4}}$$
$$x_2 = \frac{1}{3}$$

Notice that $f\left(\frac{1}{3}\right) = \frac{1}{27} \neq 0$; so, we apply another iteration of Newton's Method. Since $f\left(\frac{1}{3}\right) = \frac{1}{27}$ and $f'\left(\frac{1}{3}\right) = \frac{10}{3}$, we get our next approximation of the desired root:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$x_3 = \frac{1}{3} - \frac{\frac{1}{27}}{\frac{10}{3}}$$
$$x_3 = \frac{29}{90}$$

So, after three iterations of Newton's Method, we have an approximate solution of $x_3 = \frac{29}{90} \approx 0.3222222222$ to $f(x) = 0$. The exact solution that we were seeking is $x = \frac{(4+4\sqrt{5})^{2/3}-4}{2(4+4\sqrt{5})^{1/3}} \approx 0.3221853546$.

It may be useful to organize our results of Newton's Method as in the following table:

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	3	6	$\frac{1}{2}$	$\frac{1}{2}$
1	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{15}{4}$	$\frac{1}{6}$	$\frac{1}{3}$
2	$\frac{1}{3}$	$\frac{1}{27}$	$\frac{10}{3}$	$\frac{1}{90}$	$\frac{29}{90}$
3	$\frac{29}{90}$				

□

Example 1.2.2 *Estimate $\sqrt{2}$ using two iterations of Newton's Method. Initialize with $x_0 = 2$.*

Solution:

Our goal is to find a value of x which satisfies $x^2 = 2$. That is, we want to find a value of x which satisfies $x^2 - 2 = 0$. Let $f(x) = x^2 - 2$. We will apply two iterations of Newton's Method initialized with $x_0 = 2$. Notice that $f'(x) = 2x$ and consider the following table:

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
0	2	2	4	$\frac{1}{2}$	$\frac{3}{2}$
1	$\frac{3}{2}$	$\frac{1}{4}$	*3*	$\frac{1}{12}$	$\frac{17}{12}$
2	$\frac{17}{12}$				

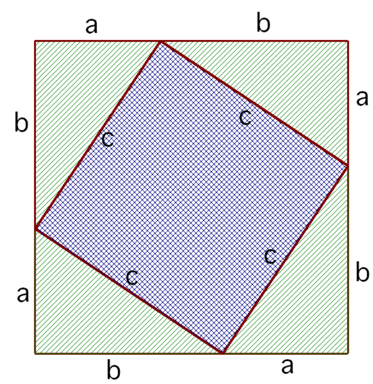
So, after two iterations of Newton's Method with $x_0 = 2$*, our estimate of* $\sqrt{2}$ *is* $\frac{17}{12} \approx 1.416666667$*. Using a calculator,* $\sqrt{2} \approx 1.414213562$*. Notice the accuracy that we have achieved with only two iterations.* $\square$

2 Geometry

2.1 Pythagorean Theorem

Theorem 2.1.1 (Pythagorean Theorem) *Suppose a right triangle has legs of length a and b and a hypotenuse with length c. Then, $a^2 + b^2 = c^2$.*

To prove this theorem, consider the diagram below. The green triangles are all identical; they have legs of length a and b and a hypotenuse of length c. Notice that in doing this, the blue region in the center is a square with side length c. (You should justify why this must be a square.)



We will calculate the area of the region in two different ways:

- Firstly, notice that this is a square with side length $a+b$. So, the area is Area $= (a+b)^2 = a^2+2ab+b^2$
- Secondly, we can compute the area by adding together the areas of the 4 green triangles and the 1 blue square. Thus, the area is Area $= 4\left(\frac{1}{2}ab\right) + c^2 = 2ab + c^2$.

Since both of these formulas represent the area of the shaded picture, we realize that they must be equal to each other. As a result, it follows that: $a^2 + 2ab + b^2 = 2ab + c^2$. That is, $a^2 + b^2 = c^2$. Hence, we have shown that if a right triangle has legs with lengths a and b and a hypotenuse with length c, then $a^2 + b^2 = c^2$.

Remark: The converse of this theorem also holds; that is, if $a^2 + b^2 = c^2$, then there must be a right triangle which has legs with lengths a and b and a hypotenuse of length c. This is often proven using the Law of Cosines which will be discussed later in the course.

Example 2.1.1 *Consider a right triangle which has legs with lengths 3 and 4. What is the length of the hypotenuse?*

Solution: *Since we have a right triangle, we can apply the Pythagorean Theorem,* $a^2 + b^2 = c^2$*, where* c *is the length of the hypotenuse. Since the legs have lengths 3 and 4, respectively, we can set* $a = 3$ *and* $b = 4$.

$$
\begin{aligned}
3^2 + 4^2 &= c^2 \\
25 &= c^2 \\
c &= \pm 5
\end{aligned}
$$

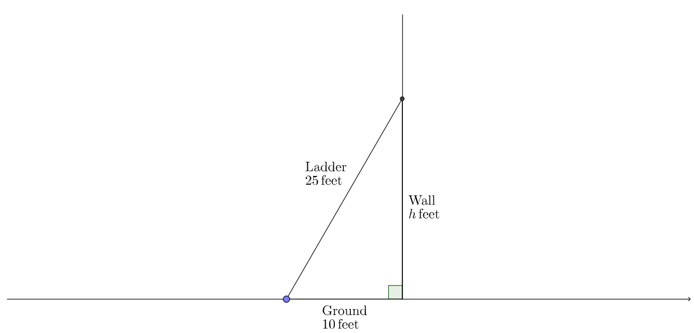
But, the hypotenuse has positive length, so the only reasonable answer is $c = 5$*. Notice that (3,4,5) is often called a Pythagorean Triple.* □

Example 2.1.2 *A ladder of length 25 feet is leaning against a vertical wall. The latter is initially 10 feet from the wall; but, it is being pulled away from the wall at a constant rate of 2 feet per second. This causes the top of the ladder to slide down the wall.*

(a) How high above the ground is the top of the ladder initially?

Solution:

The following diagram represents the initial position of the ladder before any time has passed.



We begin by applying the Pythagorean Theorem:

$$
\begin{aligned}
10^2 + h^2 &= 25^2 \\
100 + h^2 &= 625 \\
h^2 &= 525 \\
h &= \pm\sqrt{525}
\end{aligned}
$$

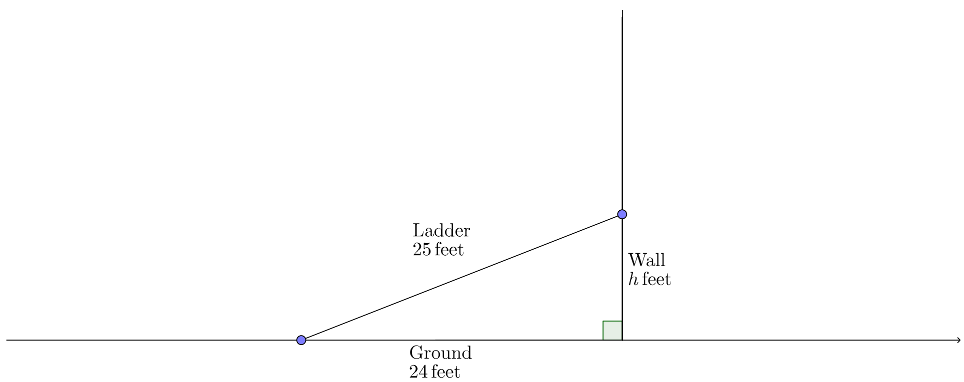
$$h = \pm 5\sqrt{21}$$

But, since h is a positive, we realize that the height above the ground before any time elapses is $h = 5\sqrt{21} \approx 22.91$ *feet.*

(b) *How high above the ground is the top of the ladder after 7 seconds have elapsed?*

Solution:

We know that the base of the ladder is being pulled away from the wall at a constant rate of 2 feet per second. So, after 7 seconds have elapsed, the base of the ladder has moved 14 feet from its original position. That is, at that instant, the base of the ladder is 24 feet away from the wall, as summarized in the diagram below.



As before, we begin by applying the Pythagorean Theorem:

$$24^2 + h^2 = 25^2$$

$$576 + h^2 = 625$$

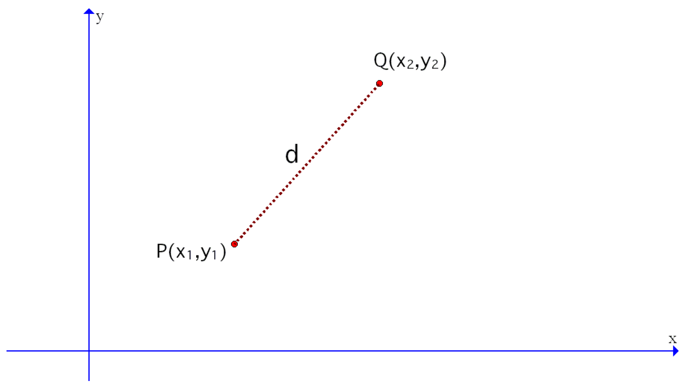
$$h^2 = 49$$

$$h = \pm 7$$

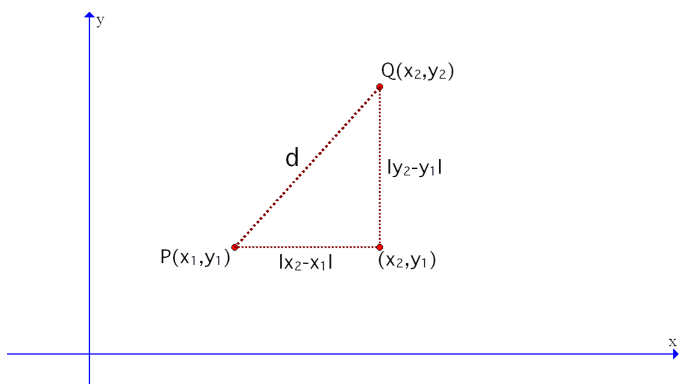
But, since h is a positive, we realize that the height above the ground after 7 seconds have elapsed is $h = 7$ *feet. As we expected, this is smaller than the height from part (a). Also, notice that (7,24,25) is another Pythagorean Triple.* □

2.2 Distance Formula

Now that we have the Pythagorean Theorem, we can derive the distance formula for points in the plane. Suppose that you have two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane. Our task is to calculate the distance, d, between these two points.



To calculate this distance, we form a right triangle as shown below:



Notice that this right triangle has legs with lengths $|x_2-x_1|$ and $|y_2-y_1|$. So, by Pythagorean Theorem, it follows that $|x_2-x_1|^2+|y_2-y_1|^2 = d^2$. Since the distance is a positive number, we can write the distance as $d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$. Furthermore, since $|x_2 - x_1|^2 = (x_2 - x_1)^2$ and $|y_2 - y_1|^2 = (y_2 - y_1)^2$, we can write the distance formula as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Theorem 2.2.1 (Distance Formula) *The distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 2.2.1 *Show that $A(1,2)$, $B(3,5)$, and $(6,3)$ are vertices of a right triangle.*

Solution:

We can solve the this problem by showing that the lengths of the three sides satisfy the Pythagorean Theorem. The lengths of the three sides are:

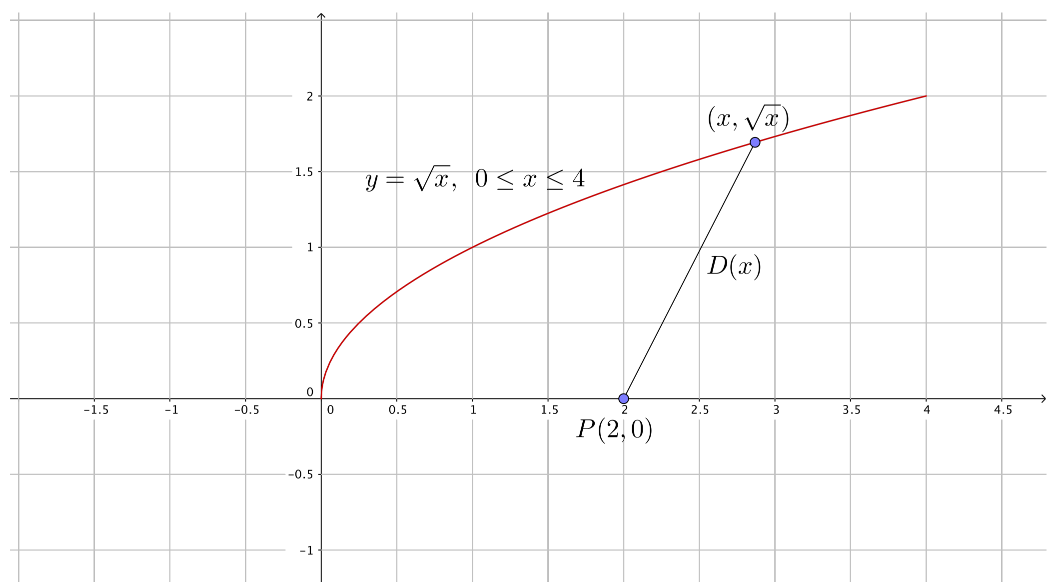
- $d_{AB} = \sqrt{(3-1)^2 + (5-2)^2} = \sqrt{13}$
- $d_{BC} = \sqrt{(6-3)^2 + (3-5)^2} = \sqrt{13}$
- $d_{AC} = \sqrt{(6-1)^2 + (3-2)^2} = \sqrt{26}$

Since $(d_{AB})^2 + (d_{BC})^2 = (d_{AC})^2$, it follows that the vertices of form a right triangle. In fact, the right angle is at vertex B. □

Example 2.2.2 *Find all points on the curve $y = \sqrt{x}$, with $0 \le x \le 4$, which are (a) closest to and (b) farthest from the point $P(2,0)$.*

Solution:

An arbitrary point on the curve has coordinates $(x, \sqrt{x})$. The following diagram shows the distance $D(x)$ from this arbitrary point to the point $P(2,0)$.



By applying the distance formula, we see that

$$\begin{aligned} D(x) &= \sqrt{(x-2)^2 + (\sqrt{x}-0)^2} \\ &= \sqrt{(x-2)^2 + x} \end{aligned}$$

Once can prove that $D(x)$ and $f(x) = (x-2)^2 + x$ have the same critical points. (This is part of your in-class worksheet). Hence, we will find the critical points of $f(x)$. Notice that $f'(x) = 2(x-2)+1 \implies f'(x) = 2x-3$. Since $f'(x)$ always exists on the given interval, the only critical points will be the values of x where $f'(x) = 0$:

$$\begin{aligned} f'(x) = 0 &\iff 2x-3=0 \\ &\iff x = \frac{3}{2} \end{aligned}$$

Since $D(x)$ is continuous on the closed and bounded interval $[0,4]$, there will be both a minimum distance and a maximum distance by the extreme value theorem. We evaluate $D(x)$ at each endpoint and at each critical point to identify where the distance is minimized and where it is maximized:

x	$D(x)$
0	2
$\frac{3}{2}$	$\frac{1}{2}\sqrt{7}$
4	$2\sqrt{2}$

Thus, the minimum distance is $\frac{1}{2}\sqrt{7}$ which occurs when $x = \frac{3}{2}$. The maximum distance will be $2\sqrt{2}$ when $x = 4$. That is, the point on the curve that is closest to $P(2,0)$ is $\left(\frac{3}{2}, \sqrt{\frac{3}{2}}\right)$ and the point on the curve at which farthest from to $P(2,0)$ is $(4,2)$. $\square$

2.3 Circles

A **circle** is the set of all points in the xy-plane whose distance from a central point is fixed. More specifically, suppose that the center of the circle is (h,k) and that all points on the circle have a distance of r to this central point. In this case, any (x,y) on the circle must satisfy the following equation:

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

That is,

$$(x-h)^2 + (y-k)^2 = r^2$$

Definition 2.3.1 (Circle) *A circle with a center (h,k) and a radius $r > 0$ has equation*

$$(x-h)^2 + (y-k)^2 = r^2$$

Example 2.3.1 *Find the center and radius of the circle with equation* $x^2 + 4x + y^2 + 10y - 13 = 0$*.*

Solution:

To determine the center and radius of this circle, we will complete the square twice (once for the x *terms and once for the* y*-terms).*

$$x^2 + 4x + y^2 + 10y - 13 = 0$$

$$(x^2 + 4x + \_\_\_) + (y^2 + 10y + \_\_\_) = 13$$

$$(x^2 + 4x + \mathbf{4}) + (y^2 + 10y + \mathbf{25}) = 13 + \mathbf{4} + \mathbf{25}$$

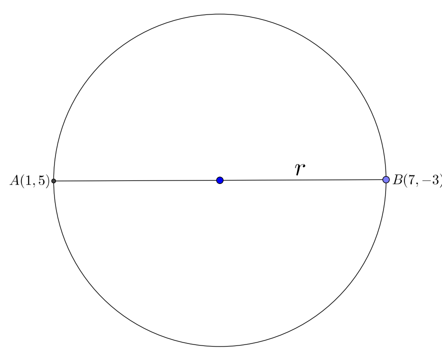
$$(x + 2)^2 + (y + 5)^2 = 42$$

Thus, the center is $(-2, -5)$ *and the radius is* $\sqrt{42}$*.* □

Example 2.3.2 *Suppose that* $A(1, 5)$ *and* $B(7, -3)$ *are endpoints of a diameter of the circle. What is an equation of this circle?*

Solution:

To find an equation of the circle, we can begin by finding the center and the radius. It may help to draw a representative picture of the given information. The picture does not necessarily have to be to scale; we will just label it appropriately to come up with a strategy.



The center will be the midpoint of $A(1, 5)$ *and* $B(7, -3)$*. Recall that the midpoint of two points in the plane* (x_1, y_1) *and* (x_2, y_2) *is* $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$*. So, the center of the circle is* $\left(\frac{1+7}{2}, \frac{5+(-3)}{2}\right) = (4, 1)$*. The radius will be half the distance from point* A *to point* B*. Since* $d_{AB} = \sqrt{(7-1)^2 + (-3-5)^2} = 10$*, it follows that the radius is* $r = 5$*. Hence, an equation of the circle which satisfies the given information is* $(x - 4)^2 + (y - 1)^2 = 25$*.*

To check that our result is reasonable, you should verify that points A and B satisfy our equation. Equivalently, you could verify that the distance from the center to A and the distance from the center to B are both 5. □

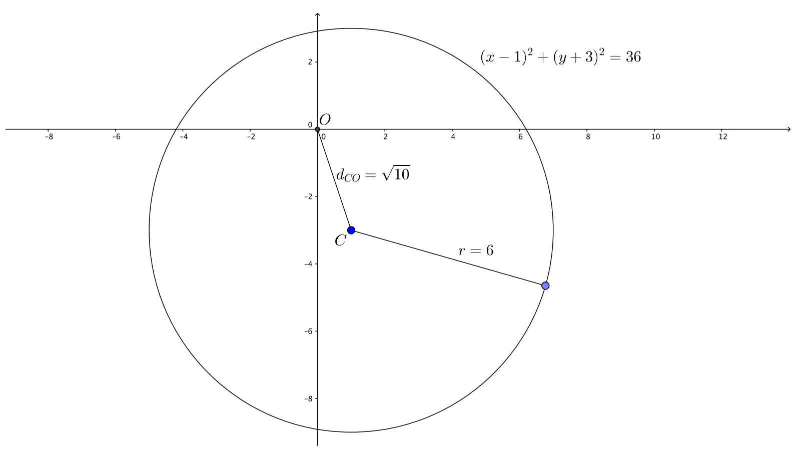
Example 2.3.3 *Does the origin lie inside, outside, or on the circle with equation $(x-1)^2+(y+3)^2 = 36$?*

Solution:

If the origin were on the circle, it would satisfy the given equation. However, when we substitute in $(x, y) = (0, 0)$, we get $10 = 36$ which is false. Thus, it follows that the origin must be either inside or outside the given circle. If the origin were inside the circle, then the distance from the center to the origin would have to be less than the radius of $r = 6$. Else, if the origin were outside of the circle, then the distance from the center to the origin would be greater than the radius of $r = 6$. So, to solve this problem, we calculate the distance from $C(1, -3)$ to $O(0, 0)$:

$$d_{CO} = \sqrt{(1 - 0^2 + (-3 - 0)^2} = \sqrt{10}$$

Since $\sqrt{10} < 6$, we know that the origin is inside the given circle.



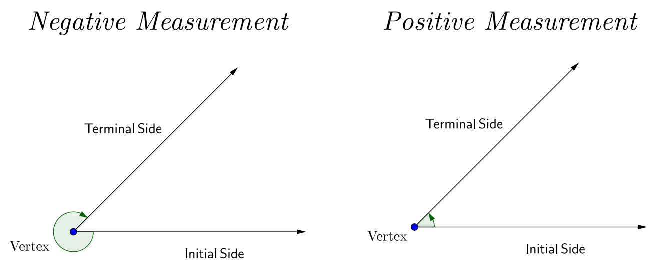
□

3 Trigonometry

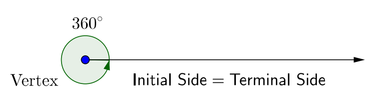
3.1 Angles

3.1.1 Basic Definitions & Degree Measurement

Definition 3.1.1 (Angle) *An* **angle** *consists of two rays in a plane with a common endpoint, as shown below. This common endpoint is called the* **vertex**. *If the angle is taken as a counterclockwise rotation from the initial side to the terminal side, we consider this to have positive measurement whereas if the angle is taken as a clockwise rotation from the initial side to the terminal side, we consider this angle to have negative measurement.*



In this course, we will use two different measuring systems. The first is **Degrees** and the second is **Radians**. We begin by describing degree measurement. When measuring in degrees, we associate a complete counterclockwise revolution to be $360°$.

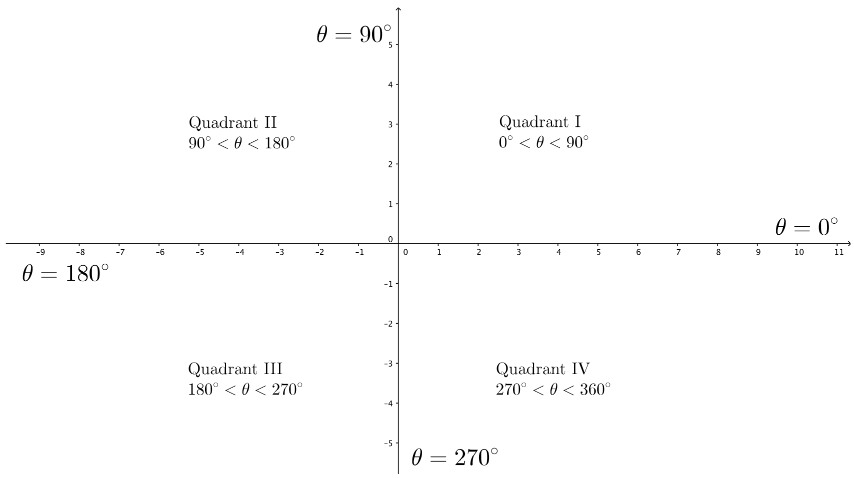


So $1°$ represents $\frac{1}{360}$ of a revolution. Typically, we position the initial ray along the positive x-axis. Angles which follow this convention are said to be in **standard position**. We can now identify angle measures for each part of the coordinate axes.

- The positive x-axis corresponds to $0°$.
- The positive y-axis is $\frac{1}{4}$ of a full counterclockwise rotation. Thus, it corresponds to $\frac{90}{360}$ of a rotation, which is $90°$.
- The negative x-axis is $\frac{1}{2}$ of a full counterclockwise rotation. Thus, it corresponds to $\frac{180}{360}$ of a rotation, which is $180°$.

- The negative y-axis is $\frac{3}{4}$ of a full counterclockwise rotation. Thus, it corresponds to $\frac{270}{360}$ of a rotation, which is $270°$.

This is summarized in the following diagram.

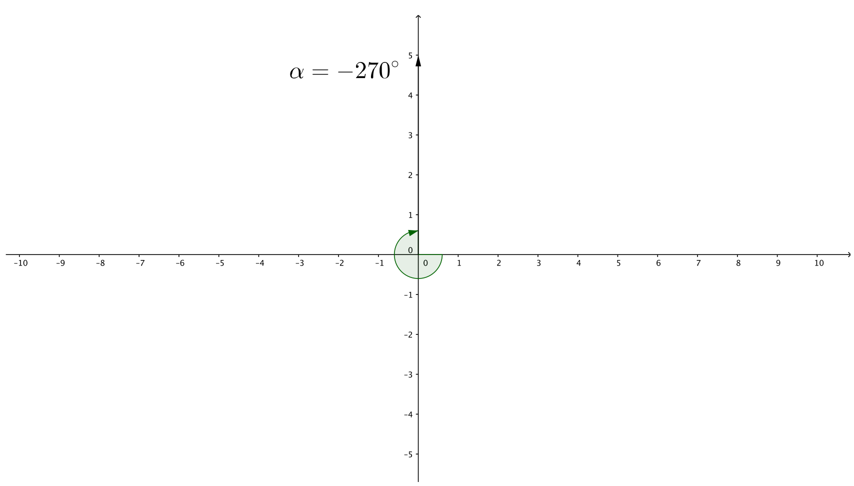


Notice that, based on our current discussion, the positive x-axis can be represented as $0°$ or $360°$. These are examples of **Coterminal Angles** since the terminal rays for each of these angles coincide. The following example demonstrates other coterminal angles.

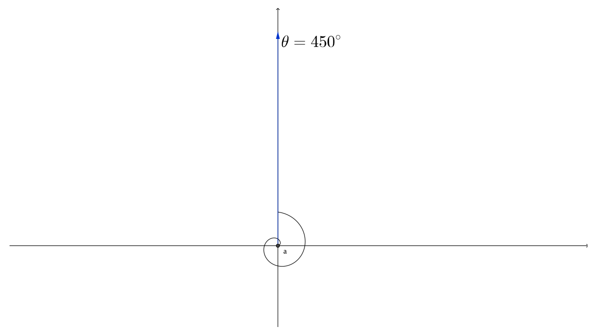
Example 3.1.1 *Sketch* $\alpha = -270°$, $\beta = 450°$, *and* $\gamma = 810°$.

Solution:

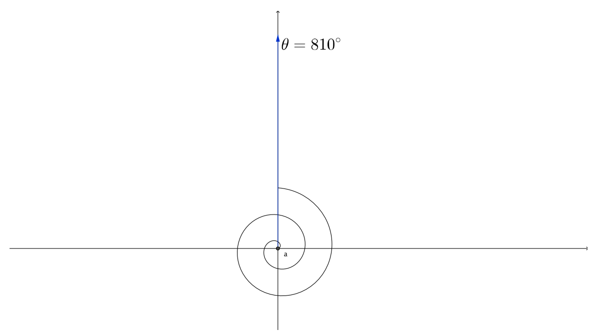
The angle $\alpha = -270°$ *means to rotate 270 degrees clockwise, as in the following diagram.*



The angle of $\beta = 450°$ means to rotate $\frac{450}{360} = 1.25$ times. So, the terminal ray will also lie on the positive y-axis, as shown below.



The angle of $\gamma = 810°$ means to rotate $\frac{810}{360} = 2.25$ times. So, again the terminal ray will lie on the positive y-axis, as shown below.



$\square$

Notice the following:

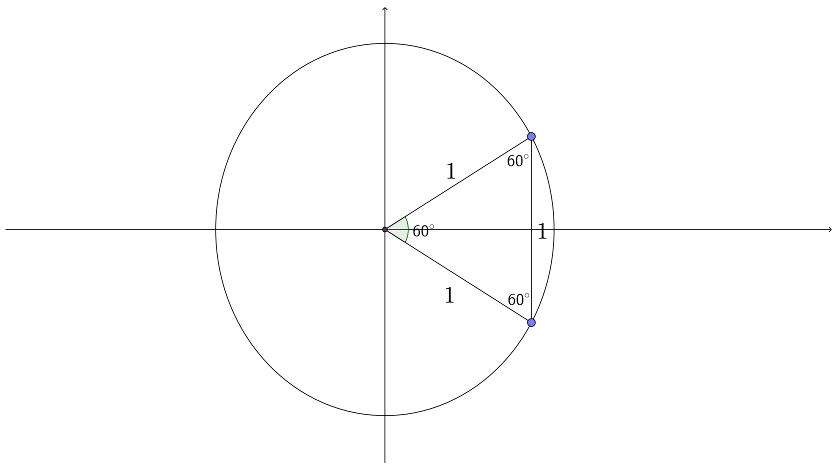
- $-270° = 90° - 360° = 90° + (-1) \cdot 360°$
- $450° = 90° + 360°$
- $810° = 90° + 720° = 90° + 2 \cdot 360°$

Based on the previous example, we arrive at the following definition:

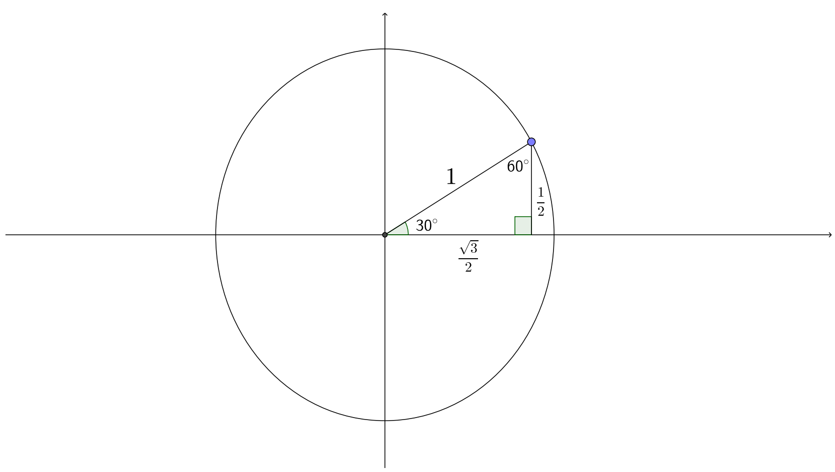
Definition 3.1.2 (Coterminal Angles) Coterminal Angles *are angles whose measures differ by a multiple of* $360°$. *The terminal rays will coincide.*

3.1.2 Common Angles & The Unit Circle

There are some angles which will come up frequently in our calculations. One triangle which will occur frequently is the $30^\circ - 60^\circ - 90^\circ$ triangle. To determine the relative side lengths, consider the equilateral triangle shown below:

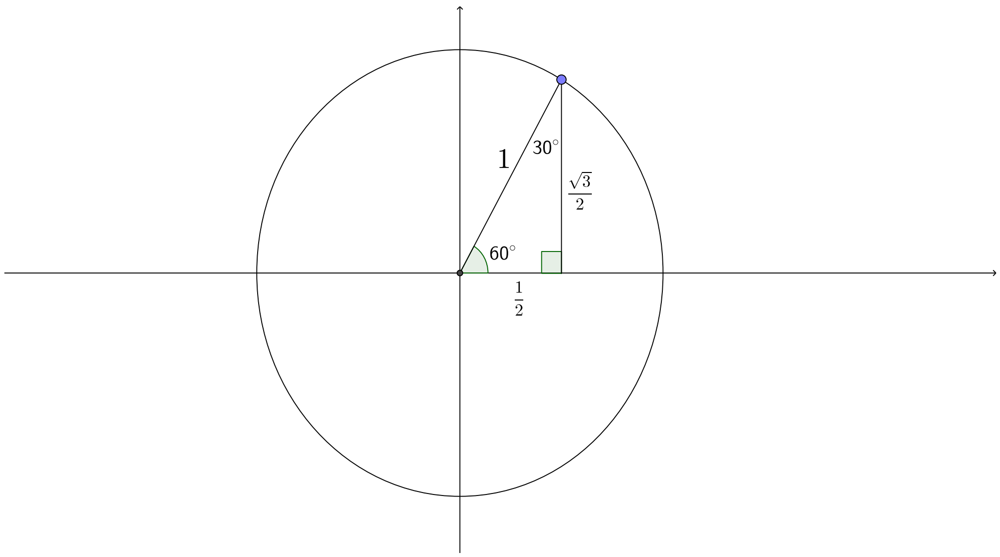


Notice that the positive x-axis cuts through the center of the triangle, dividing it into two equal right triangles. The upper triangle has angles of 30°, 60°, and 90°, as shown below:

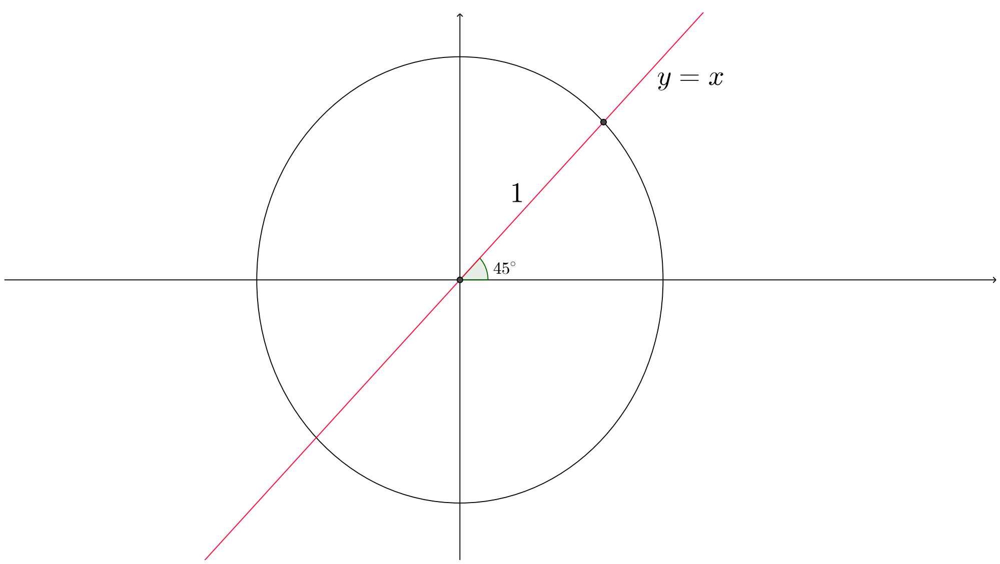


Notice that the vertical side has length $\frac{1}{2}$ since it is half the length of the original vertical line from the previous figure. Using the Pythagorean Theorem, you can show that the horizontal side has length $\frac{\sqrt{3}}{2}$.

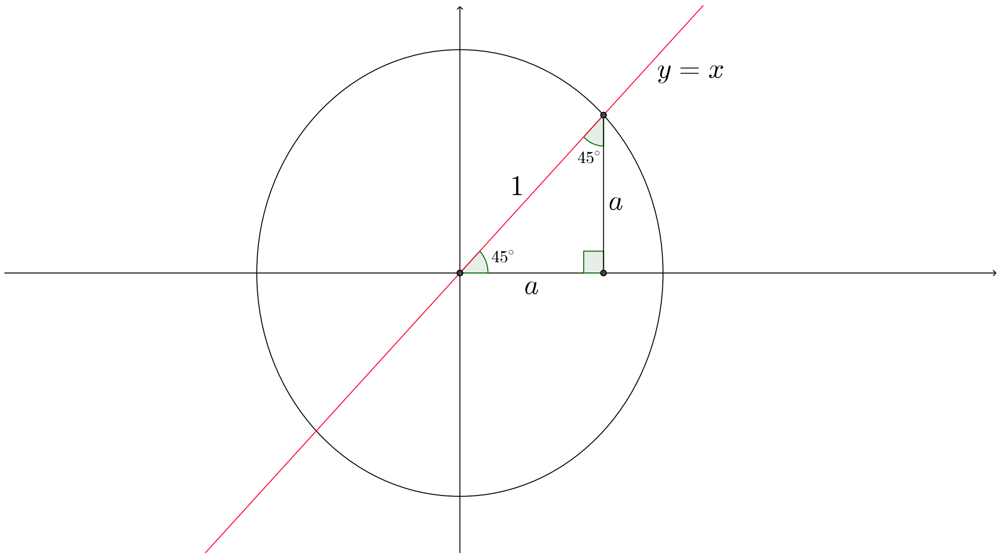
Now that we know the relative side lengths for a $30^\circ - 60^\circ - 90^\circ$ triangle, notice that if we take a 60° angle counterclockwise from the positive x-axis, we would have the following triangle:



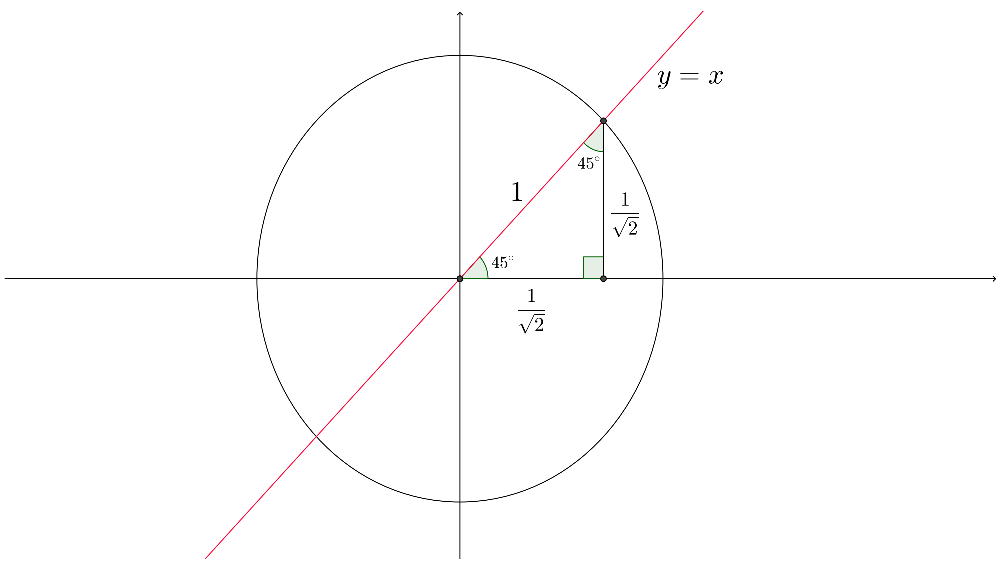
Another useful triangle to remember is the $45° - 45° - 90°$ triangle. Consider the line $y = x$, as shown below. Notice that the angle between the x-axis and the the $y = x$ is $45°$ since this line bisects the first quadrant.



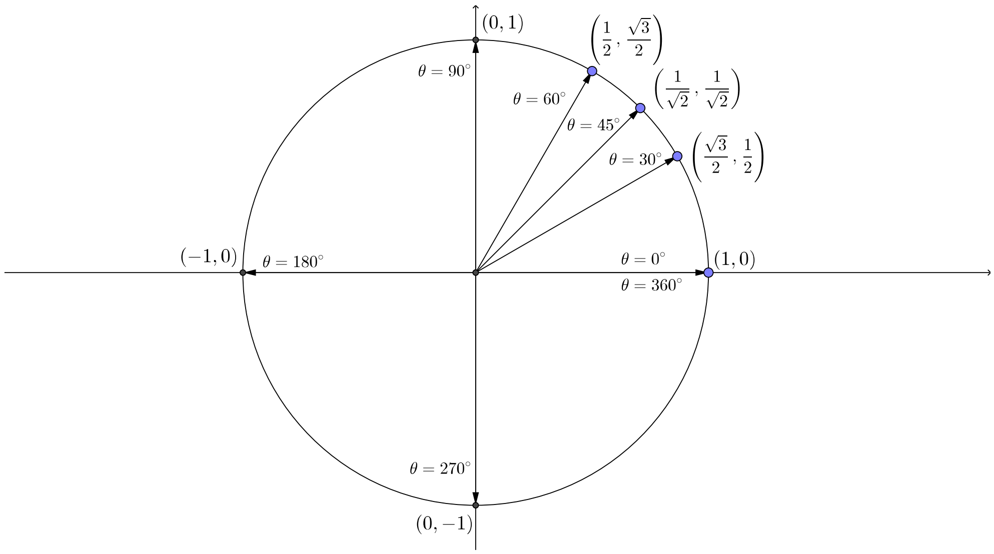
If we drop a perpendicular from the point at which this line intersects the unit circle, we form a right triangle, as shown below. Notice that the lengths of the two legs are equal since this triangle is formed from the line $y = x$. For the moment, call this length a.



By the Pythagorean Theorem, we have $a^2 + a^2 = 1$, which gives us $a = \frac{1}{\sqrt{2}}$. As a result, the relative lengths of a $45^\circ - 45^\circ - 90^\circ$ are as shown in the following diagram:



Using these triangles, we can label some important points on the unit circle. Specifically, we will label the ordered pairs for angles of 30°, 45°, 60°, and the quadrantal angles.

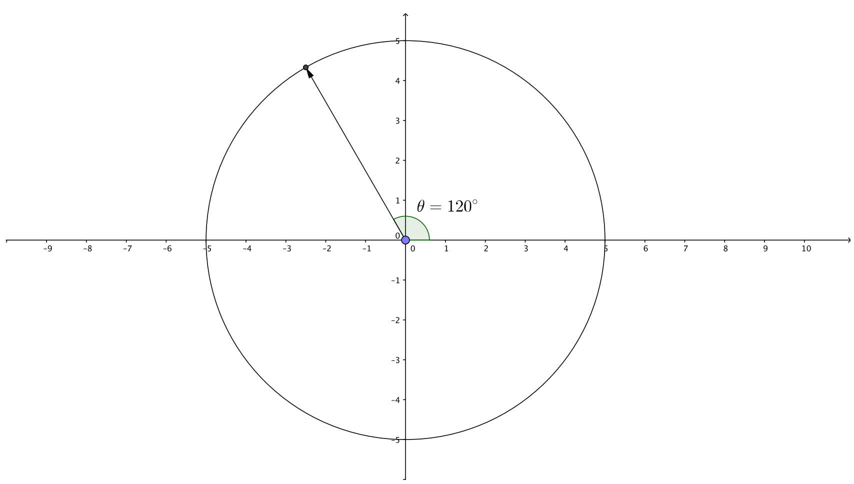


Now, let us label some points on the unit circle in the remaining quadrants. Consider the following example:

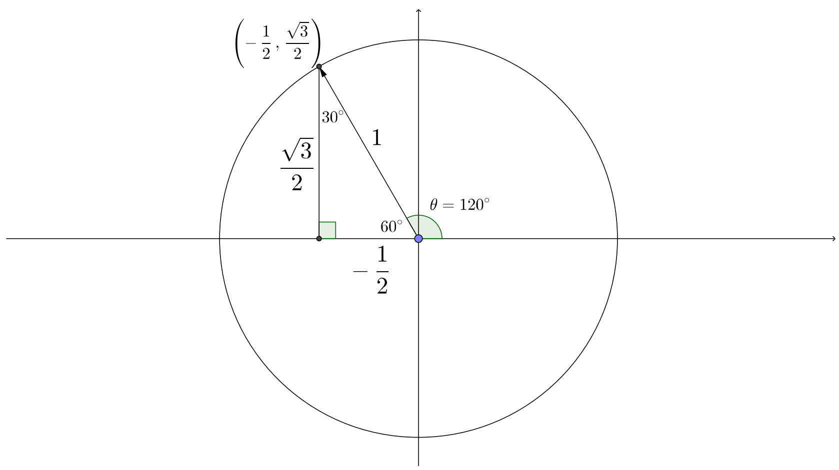
Example 3.1.2 *Determine the coordinates of the point on the unit circle corresponding to an angle of* $\theta = 120^\circ$.

Solution:

The terminal ray for this angle is in the second quadrant, as in the diagram below.

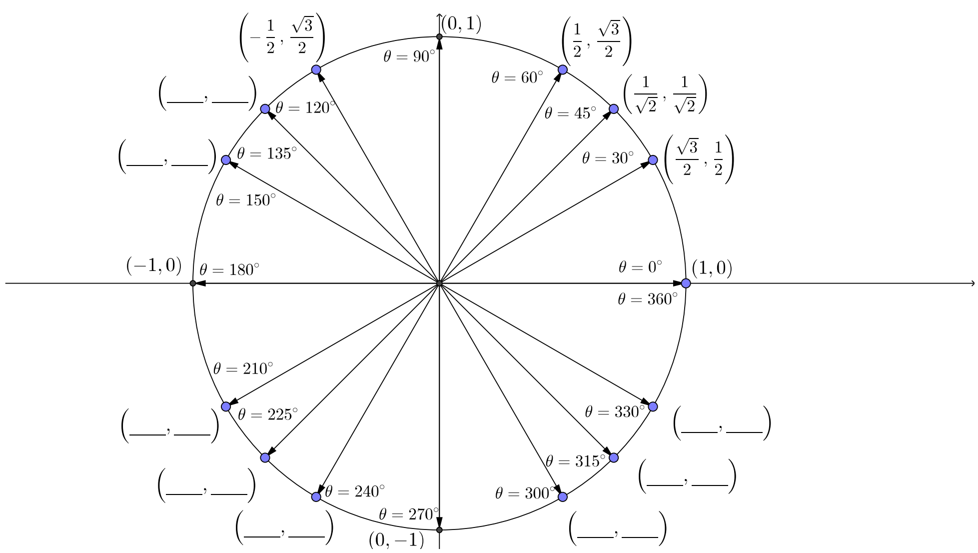


To determine the coordinates of this point, we drop a perpendicular and focus on the Reference Triangle that is formed. Notice that this reference triangle is a $30^\circ - 60^\circ - 90^\circ$ *triangle. As a result, we can label the lengths of the corresponding legs.*



Thus, it follows that the coordinates of this point are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. $\square$

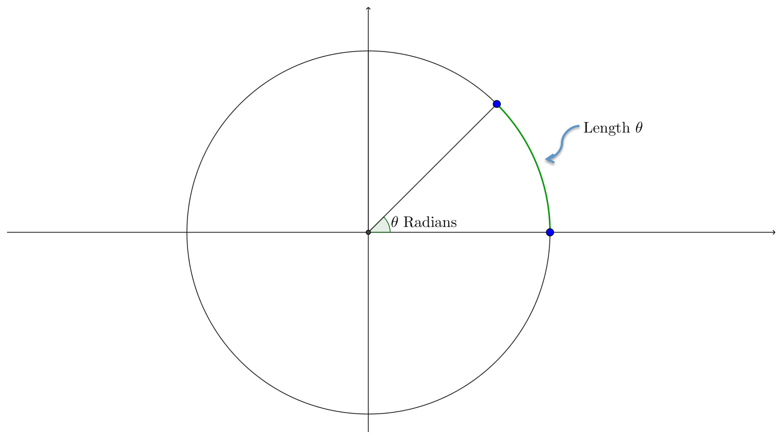
As an exercise, try to fill in the remaining points on the following copy of the unit circle:



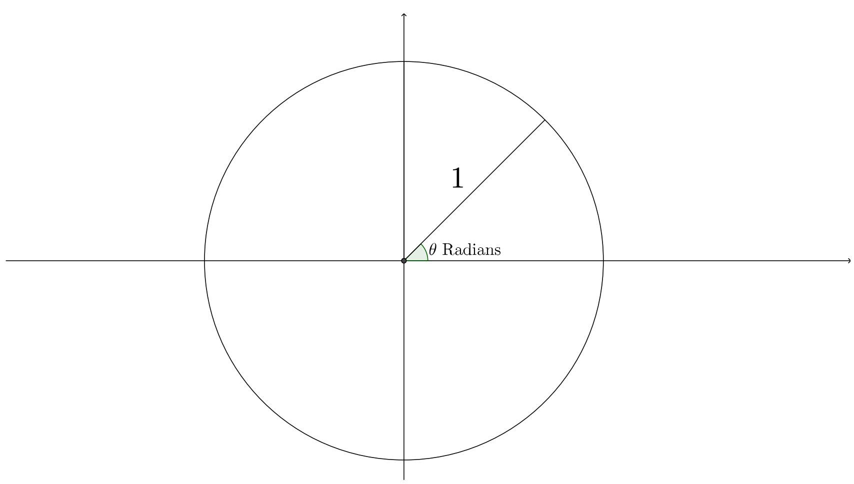
3.1.3 Radian Measurement

When measuring angles in degrees, we defined 360° to be the angle which makes a complete circle. Then, we took any other angle to be a fraction of this arbitrarily chosen convention. In doing so, we divided the circle into 360 pieces. However, this is not always the best system for measuring angles. In calculus, it is often better to use **radians** to measure angles as this unit of measurement will simplify many of our calculations with the trigonometric functions.

Definition 3.1.3 (Radians) *When measuring an angle in* **Radians**, *the measurement of the angle is equal to the length of the corresponding arc of the unit circle. As before, if the angle is a counterclockwise rotation, it will have positive measurement whereas if it is a clockwise rotation it will have negative measurement.*



It is now our goal to determine how to convert between our two measurement systems. Consider the unit circle, shown below.



The total length walked in one revolution of the circle is the circumference, 2π. So, we recognize that:

$$2\pi \text{ radians} = 360^\circ$$

Dividing both sides by 360° gives us a conversion factor:

$$\boxed{\frac{\pi \text{ radians}}{180^\circ} = 1}$$

Example 3.1.3 *Convert each of the following angles from degrees to radians.*

(a) $90°$

Solution:

We can use dimensional analysis to convert:

$$90° = 90° \cdot \frac{\pi \ radians}{180°} = \frac{\pi}{2} \ radians$$

Notice that our result make sense because $90°$ *corresponds to* $\frac{1}{4}$ *of a circle. So, if we multiply the circumference of the unit circle by* $\frac{1}{4}$*, we get* $\frac{1}{4}(2\pi) = \frac{\pi}{2}$.

(b) $180°$

Solution:

We can use dimensional analysis to convert:

$$180° = 180° \cdot \frac{\pi \ radians}{180°} = \pi \ radians$$

Notice that our result make sense because $180°$ *corresponds to* $\frac{1}{2}$ *of a circle. So, if we multiply the circumference of the unit circle by* $\frac{1}{2}$*, we get* $\frac{1}{2}(2\pi) = \pi$.

(c) $45°$

Solution:

We can use dimensional analysis to convert:

$$45° = 45° \cdot \frac{\pi \ radians}{180°} = \frac{\pi}{4} \ radians$$

Try to explain why this makes sense in terms of lengths.

□

In fact, we can convert all of our "common" angles. As an exercise, verify the results of the following table:

Degrees:	0	30	45	60	90	180	270	360
Radians:	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	π

Equivalently, we can convert from radian measurement to degree measurement, as in the following example.

Example 3.1.4 *Convert each of the following angles from radian measurement to degree measurement.*

(a) $\dfrac{\pi}{5}$ *radians*

Solution:

Using dimensional analysis, we get:

$$\frac{\pi}{5}\,\textit{radians} = \frac{\pi}{5}\,\textit{radians} \cdot \frac{180^\circ}{\pi\,\textit{radians}} = 36^\circ$$

(b) $\dfrac{5\pi}{2}$ *radians*

Solution:

Using dimensional analysis, we get:

$$\frac{5\pi}{2}\,\textit{radians} = \frac{5\pi}{2}\,\textit{radians} \cdot \frac{180^\circ}{\pi\,\textit{radians}} = 450^\circ$$

Notice that since $\frac{5\pi}{2} > 2\pi$*, this angle involves more than one full revolution around the circle. In fact, the angle of* $\theta = \frac{5\pi}{2}$ *is coterminal with the angle* $\frac{\pi}{2}$ *since these angles differ by a multiple of* 2π*.*

(c) 10 *radians*

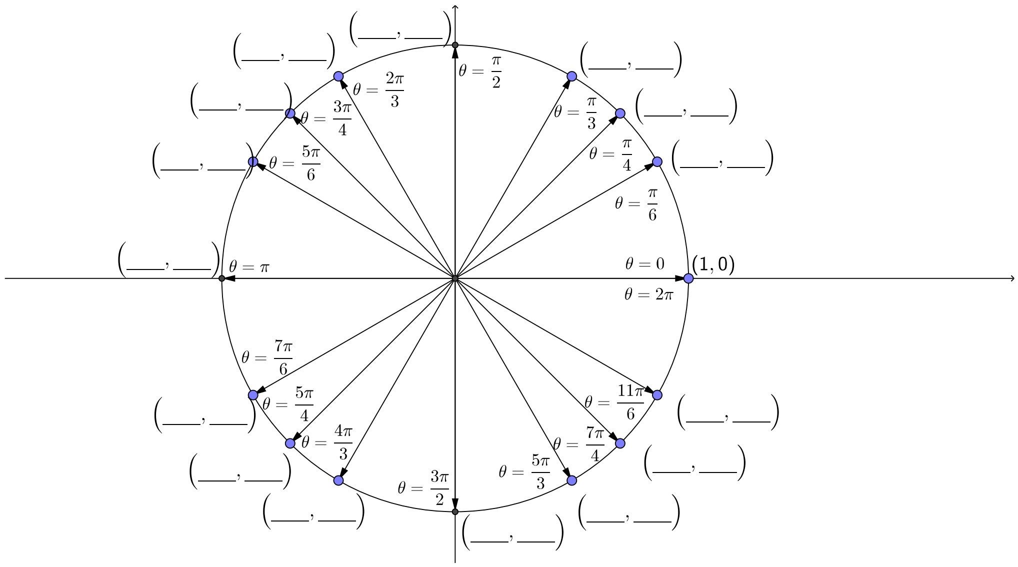
Solution: *Using dimensional analysis, we get:*

$$10\,\textit{radians} = 10\,\textit{radians} \cdot \frac{180^\circ}{\pi\,\textit{radians}} = \frac{1800^\circ}{\pi}$$

Using a calculator, this is approximately 572.96°*. Hence the terminal ray of this angle is in quadrant III.*

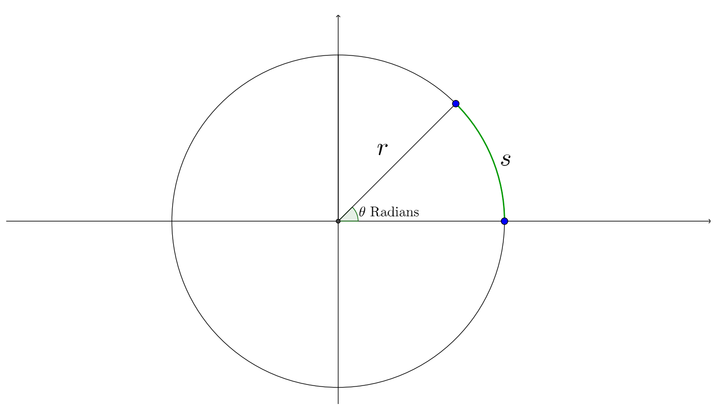
□

Before continuing to the next last topic of this section, take a moment to tie everything together. Earlier we discussed certain "common" angles (measured in degrees) and their coordinates on the unit circle. The following diagram represents those same angles (measured in radians). Label all of the coordinates of the corresponding points on the unit circle. It may be helpful to draw appropriate reference triangles so that you can use what you know about $30^\circ - 60^\circ - 90^\circ$ and $45^\circ - 45^\circ - 90^\circ$ triangles.



3.1.4 Length of a Circular Arc & Area of a Sector

We begin by determining the length of a circular arc. The following diagram shows an arc of the circle with radius r. The length of the arc is s and the angle is θ radians.



To determine how to calculate the length s, we will set up a proportion. Specifically, the ratio of the length s to the circumference must equal the ratio of the angle θ to 2π, the angle of 1 full revolution. This gives us:

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$$

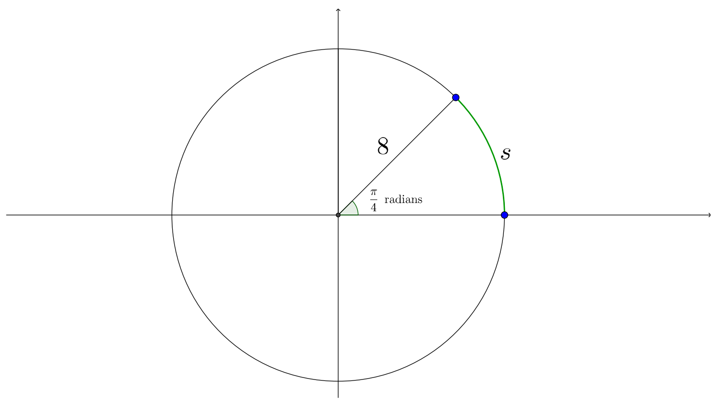
Solving for s gives us the desired formula:

$$\boxed{s = r\theta}$$

If θ is measured in degrees, then we can first convert θ to radians by multiplying by a conversion factor. Doing so will give us $\frac{\pi\theta}{180^\circ}$ radians. And, the resulting arc length formula will be:

$$\boxed{s = \frac{r\theta\pi}{180}}$$

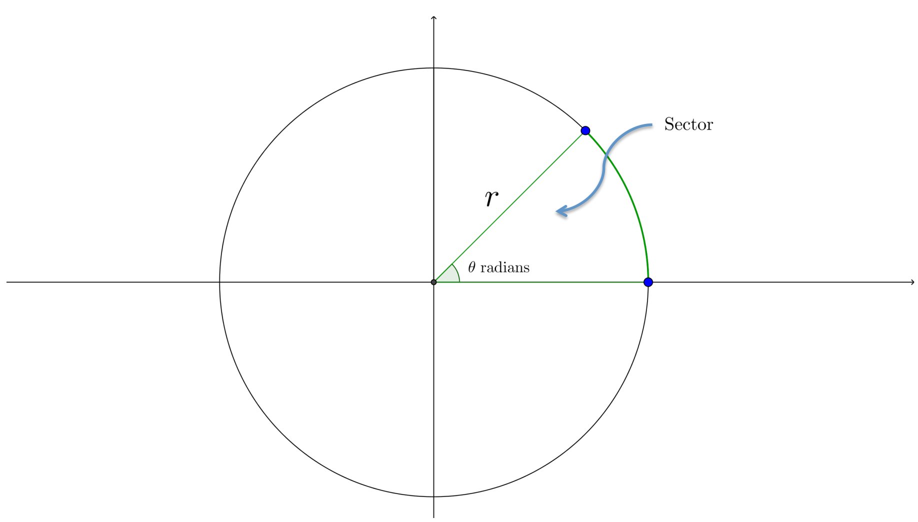
Example 3.1.5 *Compute the length of the circular arc shown below*



Solution:

We know that, in general, when θ is measured in radians, arc length is: $s = r\theta$. Since $r = 5$ and $\theta = \frac{\pi}{4}$ radians, it follows that $s = (8)\left(\frac{\pi}{4}\right) = 2\pi$. Notice that this makes sense geometrically because the angle of $\frac{\pi}{4}$ radians corresponds to $\frac{1}{8}$ of the circumference. If we multiply the circumference of 16π by $\frac{1}{8}$, we again get 2π as the length. $\square$

To finish off this section, we will discuss how to calculate the area of the sector of a circle of radius r with an angle of θ (measured in radians), shown below.



Notice that the area of this sector corresponds to a fraction of the total area of the circle. So, we will take the total area of πr^2 and we will multiply it by the fraction of the circle which this sector occupies. Since this section is $\frac{\theta}{2\pi}$ of the circle, it turns out that the area is

$$A = \left(\frac{\theta}{2\pi}\right)(\pi r^2)$$

Simplifying this gives us the formula for the area of a sector of radius r, when the angle θ is measured in radians:

$$\boxed{A = \frac{1}{2}r^2\theta}$$

If θ is measured in degrees, then we can first convert θ to radians by multiplying by a conversion factor. Doing so will give us $\frac{\pi\theta}{180°}$ radians. And, the resulting arc length formula will be:

$$\boxed{A = \frac{\pi r^2 \theta\pi}{360}}$$

Example 3.1.6 *Suppose you purchase a pizza with a diameter of 16 inches for 6 equally hungry people to share. Each person will eat a single slice of pizza in the shape of a sector. What is the area of each slice?*

Solution:

Naturally, we already know the answer. Since the diameter is 16 inches, the radius is 8 inches. As a result the area of the entire area is $\pi(8)^2 = 64\pi$. *So, the area of each sector will be* $A = \frac{64\pi}{6} = \frac{32}{3}\pi$ *square units.*

Equivalently, we can use our formula for the area of a sector. Since each slice will be equal, each sector will have an angle of $\frac{2\pi}{6} = \frac{\pi}{3}$ *radians. So, the area of each sector is* $A = \frac{1}{2}(8)^2\left(\frac{\pi}{3}\right) = \frac{32\pi}{3}$ *square units.*

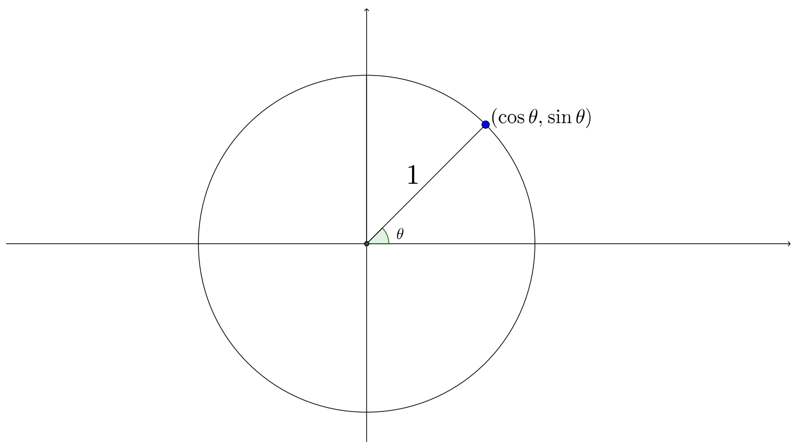
□

As you have already seen with the formulas for length of a circular arc and area of a sector, using radians provides us with a simpler formula. Especially when dealing with calculus problems, we will typically assume that θ is measured in radians, unless otherwise stated.

3.2 Definition of Trigonometric Functions

In this section, we define the trigonometric functions. For reasons that will soon be clear, these are sometimes called "Circular" Functions." We begin by defining sine and cosine. Then, we can define the remaining trigonometric functions in terms of these.

Definition 3.2.1 (Cosine & Sine) *Consider the unit circle shown below.*



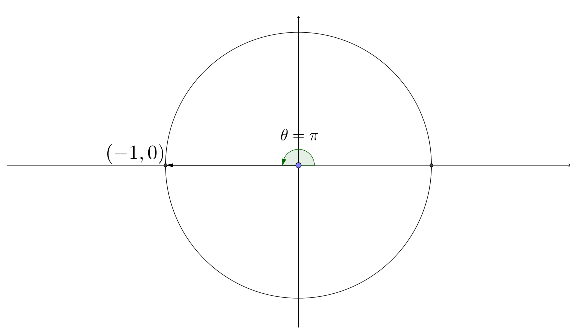
We define the **cosine of the angle** θ*, denoted* $\cos\theta$*, to be the first coordinate of the point at which the terminal ray of the angle* θ *in standard position intersects the unit circle. Similarly, we define the* **sine of the angle** θ*, denoted* $\sin\theta$*, to be the second coordinate of the point at which the terminal ray of the angle* θ *in standard position intersects the unit circle.*

Example 3.2.1 *By appealing to the appropriate definition, evaluate each of the following:*

(a) $\cos\pi$ *and* $\sin\pi$

Solution:

If we go to the terminal ray corresponding to an angle of $\theta = \pi$ *on the unit circle, we have:*

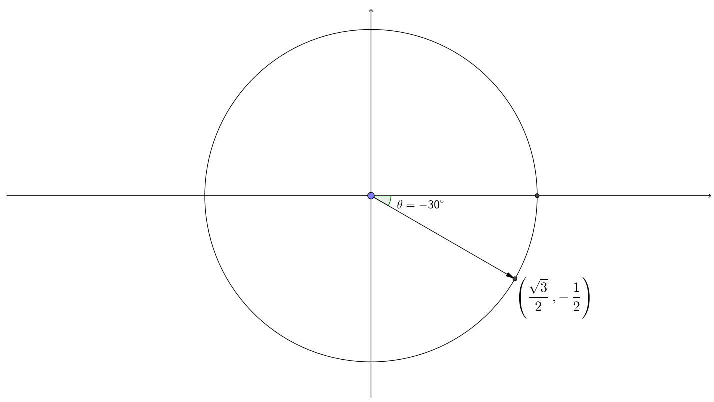


Hence, since the x*-coordinate of this point is* -1*, it follows that* $\cos\pi = -1$*. Similarly, since the* y*-coordinate of this point is* 0*, it follows that* $\sin\pi = 0$*.*

(b) $\cos(-30°)$ *and* $\sin(-30°)$

Solution:

The angle of $-30°$ means to rotate $30°$ clockwise from the positive x-axis. As a result, the terminal ray of this angle is in quadrant IV. It follows that the x-coordinate of this point on the unit circle will be positive and the y-coordinate of this point will be negative. Hence $\cos(-30°) > 0$ *and* $\sin(-30°) < 0$.



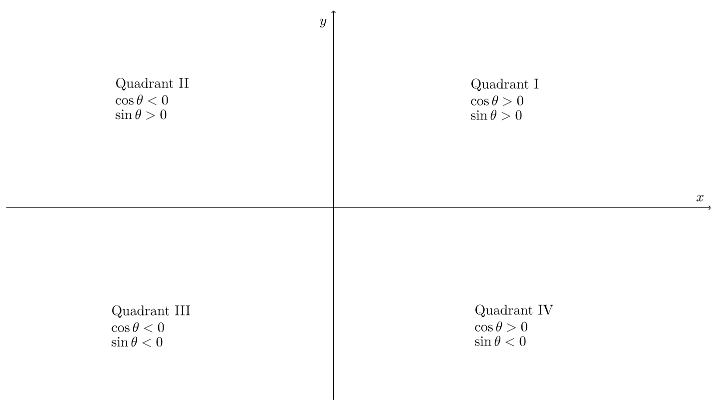
To evaluate the requested quantities, you could recognize that $\theta = -30°$ is coterminal with $330°$. Then, you can take the respective coordinates of that point on the unit circle. Or, you can draw a reference triangle and use the known ratios of a $30° - 60° - 90°$ triangle which we discussed earlier in the text. Either way, you would get $\cos(-30°) = \frac{\sqrt{3}}{2}$ *and* $\sin(-30°) = -\frac{1}{2}$.

□

Now that we have definitions for sine and cosine, we recognize that these are both functions. For any given angle, there is only one point associated with it. Hence, there is only one output for each input. Notice that these functions are always defined; positive inputs correspond to counterclockwise rotation around the origin from the positive x-axis. Inputs bigger that 2π radians correspond to more than one revolution. Negative inputs are interpreted as clockwise rotations with similar conventions. Hence, $f(\theta) = \sin\theta$ and $f(\theta) = \cos\theta$ both have domain $(-\infty, \infty)$. Also, notice that the x-coordinate of any point on the unit circle can be any real number from -1 to 1. So, the range of $f(\theta) = \cos\theta$ is $[-1, 1]$. Similarly, the y-coordinate of any point on the unit circle can be any real number from -1 to 1, So, the range of $f(\theta) = \sin\theta$ is also $[-1, 1]$. To summarize:

	$\sin\theta$	$\cos\theta$
Domain:	$(-\infty,\infty)$	$(-\infty,\infty)$
Range	$[-1,1]$	$[-1,1]$

In addition, since $\sin\theta$ is the y-coordinate of a point on the unit circle, it follows that $\sin\theta > 0$ for θ in the interval $(0,\pi)$ and $\sin\theta < 0$ for θ in the interval $(\pi, 2\pi)$. That is, $\sin\theta$ is positive for angles in the upper half plane and $\sin\theta$ is negative for angles in the lower half plane. Similarly, $\cos\theta$ is positive for angles in the right half plane and is negative for angles in the left half plane. These results are summarized in the diagram that follows.



Another useful result follows from the fact that $(\cos\theta, \sin\theta)$ is a point on the unit circle. As such, these coordinates must satisfy $x^2 + y^2 = 1$. This gives us the first of our Pythagorean Identities, namely:

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

Consequences of this identity will follow in the section on Trigonometric Identities.

Example 3.2.2 *Suppose* $\cos\theta = \frac{1}{4}$ *and* $\frac{3\pi}{2} < \theta < 2\pi$. *Compute the value of* $\sin\theta$.

Solution:

We will use the identity that $\cos^2\theta + \sin^2\theta = 1$ *to solve this problem.*

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1 \\
\left(\frac{1}{4}\right)^2 + \sin^2\theta &= 1 && \textit{Since } \cos\theta = \frac{1}{4} \textit{ is given.} \\
\frac{1}{16} + \sin^2\theta &= 1 \\
\sin^2\theta &= \frac{15}{16}
\end{aligned}$$

$$\sin\theta = \pm\frac{\sqrt{15}}{4}$$

But, because of the restriction on θ, we know that the angle is in quadrant IV. Since $\sin\theta < 0$ for angles whose terminal ray are in quadrant 4, it follows that $\sin\theta = -\frac{\sqrt{15}}{4}$. $\square$

One thing that will be important later in the course is that these functions are not one-to-one. For instance, $\cos 0 = \cos 2\pi = 1$. Similarly, the following example shows that the graphs of sine and cosine have infinitely many x-intercepts. This will be a problem that we will have to overcome in order to carefully define their inverse functions.

Example 3.2.3

(a) *Find all angles θ for which $\sin\theta = 0$.*

Solution:

Within the interval $[0, 2\pi)$, there are two points on the unit circle which have a y-coordinate of 0. These points correspond to $\theta = 0$ and $\theta = \pi$. But, keep in mind that we could go around the circle multiple times and end up at these same points after rotating through larger angles. For example, in the interval $[2\pi, 4\pi)$, the solutions to $\sin\theta = 0$ are $\theta = 2\pi$ and $\theta = 3\pi$. In the interval $[4\pi, 6\pi)$, the solutions to $\sin\theta = 0$ are theta $= 4\pi$ and $\theta = 5\pi$. We should mention that negative integer values of k are also permissible. Continuing in this way, we see that all solutions to the equation $\sin\theta = 0$ are integer multiples of π. That is, all solutions to $\sin\theta = 0$ are of the form $\theta = \pi k$ for some integer k.

(b) *Find all angles θ for which $\cos\theta = 0$.*

Solution:

Within the interval $[0, 2\pi)$, there are two points on the unit circle which have a x-coordinate of 0. These points correspond to $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. But, keep in mind that we could go around the circle multiple times and end up at these same points after rotating through larger angles. For example, in the interval $[2\pi, 4\pi)$, the solutions to $\cos\theta = 0$ are $\theta = \frac{5\pi}{2}$ and $\theta = \frac{7\pi}{2}$. In the interval $[4\pi, 6\pi)$, the solutions to $\cos\theta = 0$ are theta $= \frac{9\pi}{2}$ and $\theta = \frac{11\pi}{2}$. We should mention that negative integer values of k are also permissible. Continuing in this way, we see that all solutions to the equation $\cos\theta = 0$ are the odd integer multiples of $\frac{\pi}{2}$. That is, all solutions to $\cos\theta = 0$ are of the form $\theta = \frac{\pi}{2}(2k+1)$ for some integer k. $\square$

Now that we have defined $\sin\theta$ and $\cos\theta$, we can define the remaining trigonometric functions through algebraic methods. Consider the following definition:

Definition 3.2.2 (Tangent, Secant, Cosecant, Cotangent) *We define the remaining trigonometric functions as follows:*

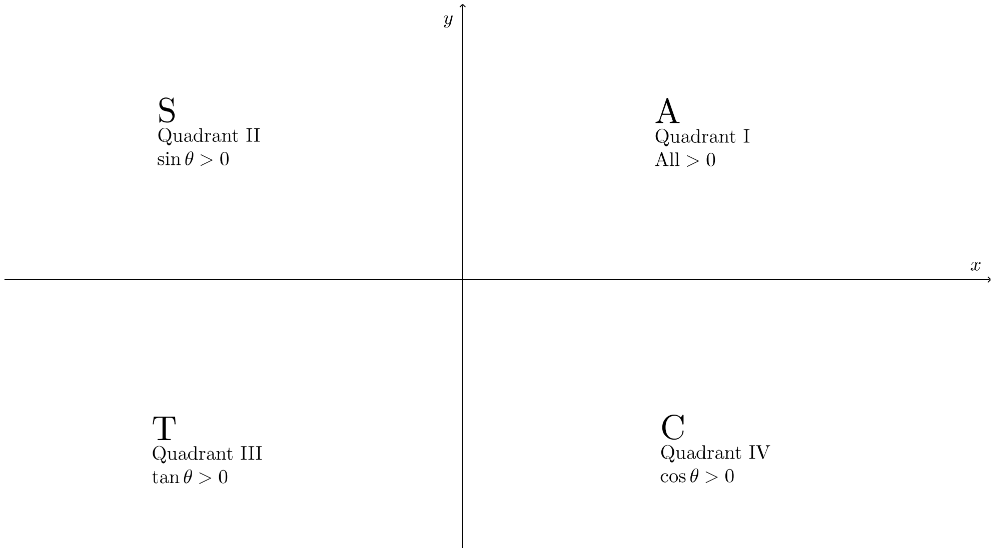
- *The* **tangent of an angle** θ *is* $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, *provided* $\cos\theta \neq 0$. *You can interpret this as the slope of the terminal ray corresponding to an angle* θ.
- *The* **secant of an angle** θ *is* $\sec\theta = \dfrac{1}{\cos\theta}$, *provided* $\cos\theta \neq 0$.
- *The* **cosecant of an angle** θ *is* $\csc\theta = \dfrac{1}{\sin\theta}$, *provided* $\sin\theta \neq 0$.
- *The* **cotangent of an angle** θ *is* $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, *provided* $\sin\theta \neq 0$.

For each of these trigonometric functions, we now summarize the domain in the table that follows, based on the results from example 3.2.3. Their ranges are also summarized in the following table; but, determining the respective ranges of these functions requires a little bit of additional thought. For instance, suppose you wanted to determine the range of $f(\theta) = \tan\theta$. By definition, 3.2.2, $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Since $\cos\theta$ and $\sin\theta$ are the x and y-coordinates, respectively, of the point on the unit circle corresponding to an angle of θ, we can interpret $f(\theta) = \tan\theta$ as the slope of the line containing the origin and this point. By sketching the terminal rays for angles in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, you should recognize that for any slope in $(-\infty, \infty)$ there is a corresponding terminal ray. Hence, the range of $\tan\theta$ is (∞, ∞).

Similarly, we give you an intuitive reason explaining why the range of $\sec\theta$ is as described in the following table. Recall from definition 3.2.2 that $f(\theta) = \sec\theta = \frac{1}{\cos\theta}$. Since $\cos 0 = 1$, it follows that $\sec 0 = 1$ as well. So, 1 is in the range of $\sec\theta$. For any value of θ in the interval $\left(0, \frac{\pi}{2}\right)$, we know that $\cos\theta$ is positive. Thus, $\sec\theta$ will also be positive. Furthermore, in this interval $0 < \cos\theta < 1$; so, when we take reciprocals of these values to obtain $\sec\theta$, we will get real numbers larger than 1. In fact, as θ gets closer to $\frac{\pi}{2}$, $\cos\theta$ will get closer to 0 and the resulting values of its reciprocal, $\sec\theta$, will get larger. Through a similar argument, you can see that for θ in the interval $\left(\frac{\pi}{2}, \pi\right]$ you can argue that the range of $\sec\theta$ will be $(-\infty, -1]$. Repeating this argument for the remaining two quadrants will give you the same results. As a result, the range of $f(\theta) = \sec\theta$ is $(-\infty, -1] \cup [1, \infty)$.

	$\tan\theta$	$\sec\theta$	$\csc\theta$	$\cot\theta$
Domain	$\theta \neq \frac{\pi}{2}(2k+1)$	$\theta \neq \frac{\pi}{2}(2k+1)$	$\theta \neq \pi k$	$\theta \neq \pi k$
Range	$(-\infty, \infty)$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, \infty)$

Also, using our knowledge of when $\sin\theta$ and $\cos\theta$ are positive or negative, we obtain the following more general result:



Students often learn the pneumonic "**A**ll **S**tudents **T**ake **C**alculus." In quadrant 1, all trig function are positive. In quadrant II, $\sin\theta$ and its reciprocal $\csc\theta$ are positive. In quadrant III, $\tan\theta$ and its reciprocal $\cot\theta$ are positive. Finally, in quadrant IV, $\cos\theta$ and its reciprocal $\sec\theta$ are positive.

Example 3.2.4 *Suppose* $\sin\theta = \frac{3}{5}$ *and* $\frac{\pi}{2} < \theta < \pi$. *Determine the values of the remaining trigonometric functions.*

Solution:

Because of the angle restrictions on θ*, we know that the terminal ray of the angle is in quadrant II. As a result, only* $\sin\theta$ *and* $\csc\theta$ *will be positive. The remaining trigonometric functions will be negative. We begin by calculating the value of* $\cos\theta$*; then, we can determine the remaining functions by appealing to the definitions.*

- *To calculate the value of* $\cos\theta$*, we will use the identity* $\cos^2\theta + \sin^2\theta = 1$*:*

$$\cos^2\theta + \sin^2\theta = 1$$

$$
\begin{aligned}
\cos^2\theta + \left(\frac{3}{5}\right)^2 &= 1 \\
\cos^2\theta + \frac{9}{25} &= 1 \\
\cos^2\theta &= \frac{16}{25} \\
\cos\theta &= \pm\frac{4}{5}
\end{aligned}
$$

But, as discussed, $cos\theta < 0$ in quadrant II. As a result, $\cos\theta = -\frac{4}{5}$.

- *Next, we calculate the value of* $\sec\theta$*:*

$$
\begin{aligned}
\sec\theta &= \frac{1}{\cos\theta} && \textit{By definition.} \\
&= \frac{1}{-\frac{4}{5}} && \textit{By the previous bullet.} \\
&= -\frac{5}{4}
\end{aligned}
$$

So, $\sec\theta = -\frac{5}{4}$.

- *Next, we calculate the value of* $\csc\theta$*:*

$$
\begin{aligned}
\csc\theta &= \frac{1}{\sin\theta} && \textit{By definition.} \\
&= \frac{1}{\frac{3}{5}} && \textit{By the given information} \\
&= \frac{5}{3}
\end{aligned}
$$

So, $\csc\theta = \frac{5}{3}$.

- *Next, we calculate the value of* $\tan\theta$*:*

$$
\begin{aligned}
\tan\theta &= \frac{\sin\theta}{\cos\theta} && \textit{By definition.} \\
&= \frac{\frac{3}{5}}{-\frac{4}{5}} \\
&= -\frac{3}{4}
\end{aligned}
$$

So, $\tan\theta = -\frac{3}{4}$.

- *Finally, we calculate the value of* $\cot\theta$*:*

$$
\begin{aligned}
\cot\theta &= \frac{\cos\theta}{\sin\theta} && \textit{By definition.} \\
&= \frac{-\frac{4}{5}}{\frac{3}{5}}
\end{aligned}
$$

$$= -\frac{4}{3}$$

So, $\cot\theta = -\frac{4}{3}$.

$\square$

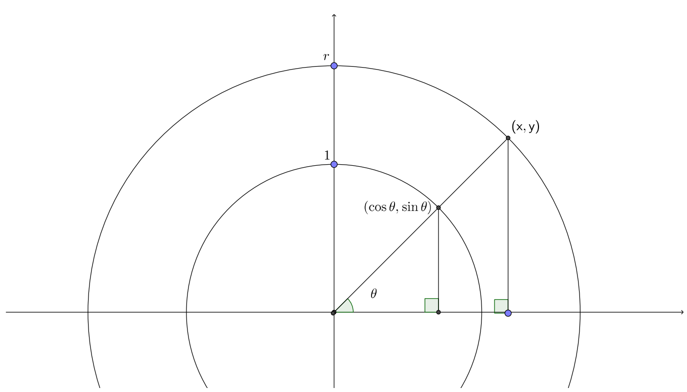
We conclude this section by making one final observation relating $tan\theta$ and $\cot\theta$. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$, we arrive at the following two results:

1. $\cot\theta = \frac{1}{\tan\theta}$, provided $\tan\theta \neq 0$

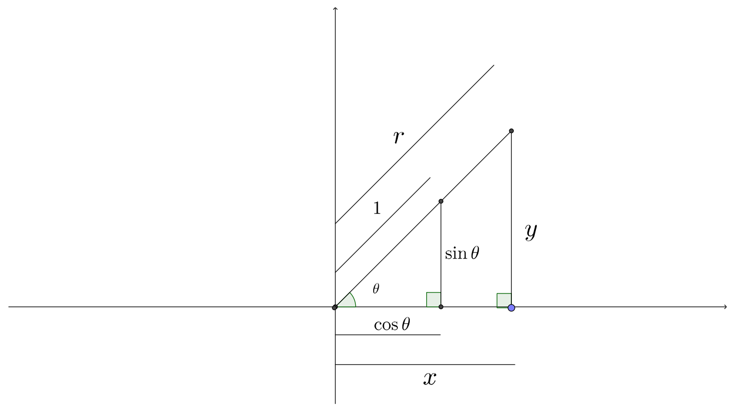
2. $\tan\theta = \frac{1}{\cot\theta}$, provided $\cot\theta \neq 0$

3.3 Right Triangle Trigonometry

Consider two concentric circles $x^2 + y^2 = 1$ and $x^2 + y^2 = r^2$, with $r > 0$. On both of these circles, mark the point which corresponds to an angle of θ. On the unit circle, this point has coordinates $(\cos\theta, \sin\theta)$ and on the circle of radius r, let us call the coordinates (x, y), as in the diagram below.



Notice that after dropping two vertical lines, we get the following similar triangles:



By similar triangles, we know that the ratio or corresponding sides must be equal. Thus

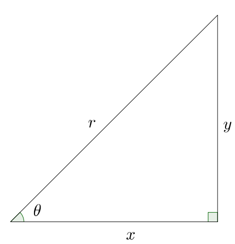
$$\frac{\cos\theta}{1} = \frac{x}{r}$$

That is, for a right triangle with sides of length x and y and a hypotenuse of length $r > 0$, it follows that $\cos\theta = \frac{x}{r}$. And, by appealing to similar triangles again, we get

$$\frac{\sin\theta}{1} = \frac{y}{r}$$

That is, $\sin\theta = \frac{y}{r}$. These results are summarized in the following box along with corresponding values for the remaining trigonometric functions.

Theorem 3.3.1 (Right Triangle Trigonometry) *Consider the right triangle shown below which has legs with lengths x and y and a hypotenuse of length r.*



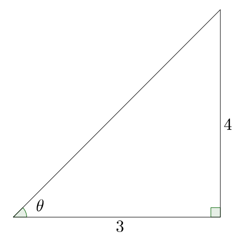
Then,

$$\sin\theta = \frac{y}{r} = \frac{\textit{opposite}}{\textit{hypotenuse}} \qquad \cos\theta = \frac{x}{r} = \frac{\textit{adjacent}}{\textit{hypotenuse}} \qquad \tan\theta = \frac{y}{x} = \frac{\textit{opposite}}{\textit{adjacent}}$$

$$\csc\theta = \frac{r}{y}\frac{\textit{hypotenuse}}{\textit{opposite}} \qquad \sec\theta = \frac{r}{x} = \frac{\textit{hypotenuse}}{\textit{adjacent}} \qquad \cot\theta = \frac{x}{y} = \frac{\textit{adjacent}}{\textit{opposite}}$$

Students often learn the pneumonic device "SOH-CAH-TOA." The SOH means that **S**ine is the **O**pposite side divided by the **H**ypotenuse. And the remaining are similar.

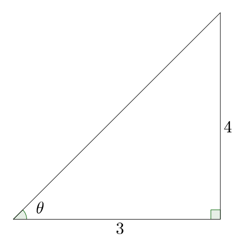
Example 3.3.1 *Consider the right triangle shown below:*



Determine the values of the 6 trigonometric functions when they are evaluated at θ.

Solution:

By appealing to the Pythagorean Theorem, we recognize that the hypotenuse has length 5*.*



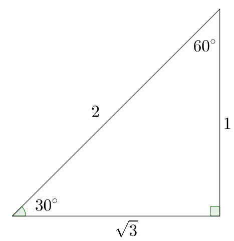
Hence, we get the following:

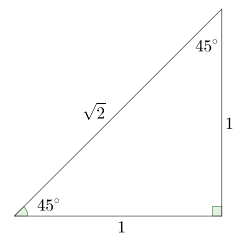
- $\sin\theta = \frac{4}{5}$
- $\cos\theta = \frac{3}{5}$
- $\tan\theta = \frac{4}{3}$
- $\sec\theta = \frac{5}{3}$
- $\csc\theta = \frac{5}{4}$
- $\cot\theta = \frac{3}{4}$

Recall that once you have the values of $\sin\theta$ *and* $\cos\theta$*, you can just appeal to the original definitions from chapter 3.2.* $\square$

Before we continue, recall that we have to useful triangles which often come up in our calculations:

$30° - 60° - 90°$ $\qquad$ $45° - 45° - 90°$



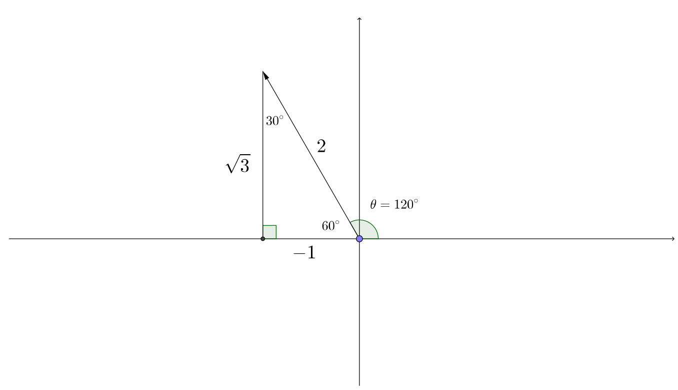


Notice that to avoid fractions, we scaled all sides of the $30° - 60° - 90°$ triangle by 2. We can always label the side opposite a $30°$ angle by 1 as long as we label the side opposite the $60°$ angle by $\sqrt{3}$ and the hypotenuse by 2. Similarly, we scaled and all sides of the $45° - 45° - 90°$ triangle by $\sqrt{2}$ to get the relative side lengths listed in the diagram.

Example 3.3.2 *Evaluate all 6 trigonometric functions at $\theta = 120°$.*

Solution:

We could go back to the unit circle to get the values of $\sin\theta$ *and* $\cos\theta$*. Then, we could calculate the remaining trigonometric functions using the original definitions. Instead, here we demonstrate how to use right triangles to solve this problem. First, we draw the given angle.*



Notice that the reference angle is $60°$ because this reference angle combined with $120°$ must equal $180°$. (They are supplementary angles.) As a result, we have a $30° - 60° - 90°$ triangle. We label all of the sides appropriately and can now compute the values of the trigonometric functions:

- $\sin\theta = \dfrac{\sqrt{3}}{2}$
- $\cos\theta = -\dfrac{1}{2}$

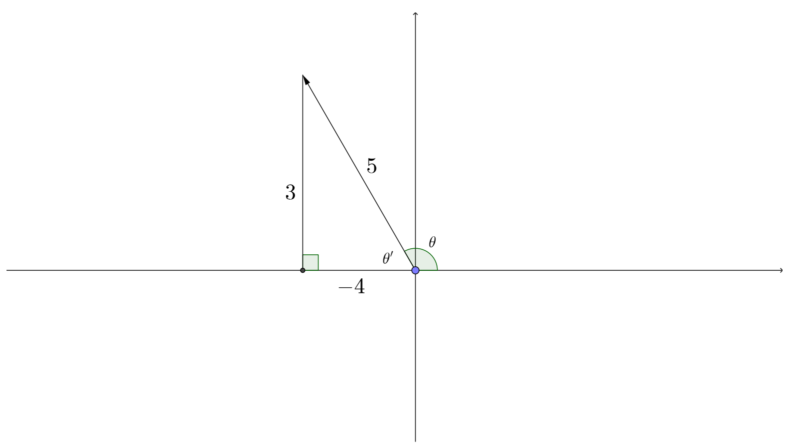
- $\tan\theta = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$
- $\sec\theta = \dfrac{2}{-1} = -2$
- $\csc\theta = \dfrac{2}{\sqrt{3}}$
- $\cot\theta = \dfrac{-1}{\sqrt{3}}$

$\square$

Example 3.3.3 *If* $\sin\theta = \frac{3}{5}$ *and* $\cos\theta < 0$, *evaluate* $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, *and* $\cos\theta$

Solution:

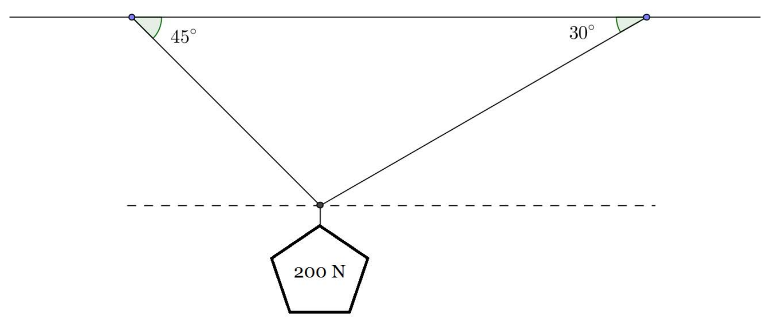
Since $\sin\theta > 0$ *and* $\cos\theta < 0$, *it follows that* θ *is an angle in the second quadrant. Once we draw this angle and the corresponding reference triangle, we can label the opposite side as 3 and the hypotenuse as 5 because of the given information. And, we can use the Pythagorean Theorem to label the remaining side of the right triangle.*



- $\cos\theta = -\dfrac{4}{5}$
- $\tan\theta = -\dfrac{3}{4}$
- $\sec\theta = -\dfrac{5}{4}$
- $\csc\theta = \dfrac{5}{3}$
- $\cot\theta = -\dfrac{4}{3}$

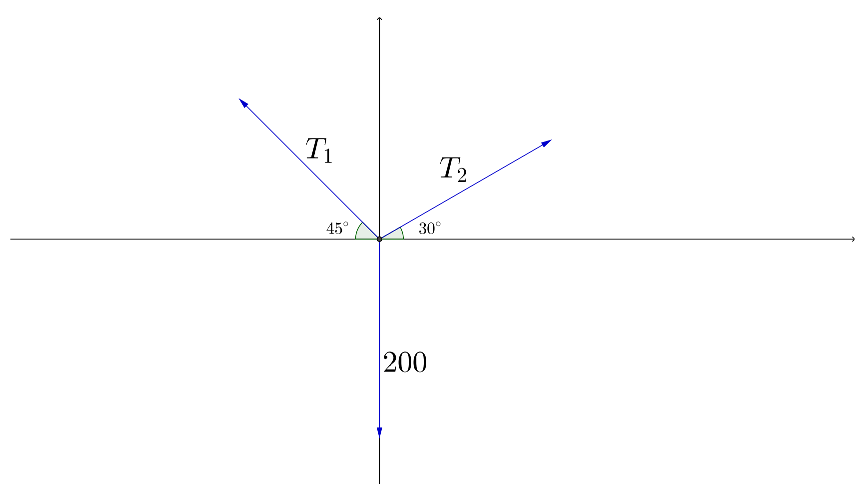
$\square$

Example 3.3.4 (Physics) *A weight of 200 Newtons (N) is being supported (in equilibrium) by two wires, as shown below. Find the tension in each wire.*

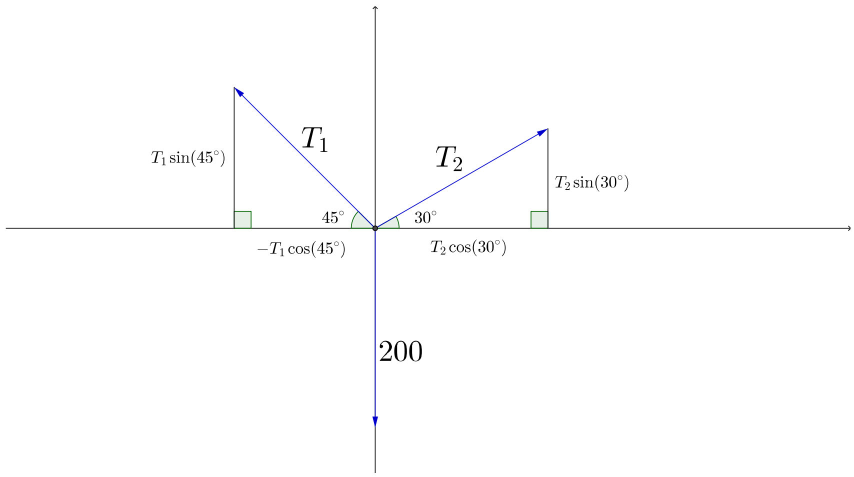


Solution:

Since the weight is being supported in equilibrium, we know that the sum of all of the horizontal forces is 0 as is the sum of all of the vertical forces. Let T_1 be the tension on the left wire and T_2 be the tension on the right wire. Notice that we can draw the following free body diagram. Using facts from geometry, we can label the angles as indicated.



In order to determine the horizontal and vertical forces, we will decompose T_1 and T_2 using the right triangles shown below.



Thus, the sum of the horizontal forces is $-T_1 \cos(45°) + T_2 \cos(30°)$ *and the sum of the vertical forces is* $T_1 \sin(45°) + T_2 \sin(30°) - 200$*. For the system to be in equilibrium, we need to solve the following system of equations:*

$$\begin{cases} -T_1 \cos(45°) + T_2 \cos(30°) = 0 \\ T_1 \sin(45°) + T_2 \sin(30°) - 200 = 0 \end{cases}$$

Equivalently, we have

$$\begin{cases} -\left(\frac{1}{\sqrt{2}}\right) T_1 + \left(\frac{\sqrt{3}}{2}\right) T_2 = 0 \\ \left(\frac{1}{\sqrt{2}}\right) T_1 + \left(\frac{1}{2}\right) T_2 = 200 \end{cases}$$

Adding the equations together gives $\left(\frac{1+\sqrt{3}}{2}\right) T_2 = 200$*. As a result, the tension in the second wire is* $T_2 = \frac{400}{1+\sqrt{3}}$ *Newtons. By back substitution, it can be shown that* $T_1 = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{400}{1+\sqrt{3}}$ *Newtons.* $\square$

3.4 Common Trigonometric Identities

In this section, we will derive some frequently used trigonometric identities.

3.4.1 Pythagorean Identities

Recall that $(\cos\theta, \sin\theta)$ are the coordinates of the point on the unit circle which makes an angle of θ, counterclockwise, with the positive x-axis. As a result, these coordinates must satisfy the equation of the unit circle, $x^2 + y^2 = 1$. This gives us the first identity:

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

From this identity, we can derive two other Pythagorean Identities. If we divide both sides of the equation by $\cos^2\theta$, we get:

$$\begin{aligned}
\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\
1 + \frac{\sin^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\
1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 &= \left(\frac{1}{\cos\theta}\right)^2 \\
1 + \tan^2\theta = \sec^2\theta
\end{aligned}$$

That is, our second Pythagorean Identity is:

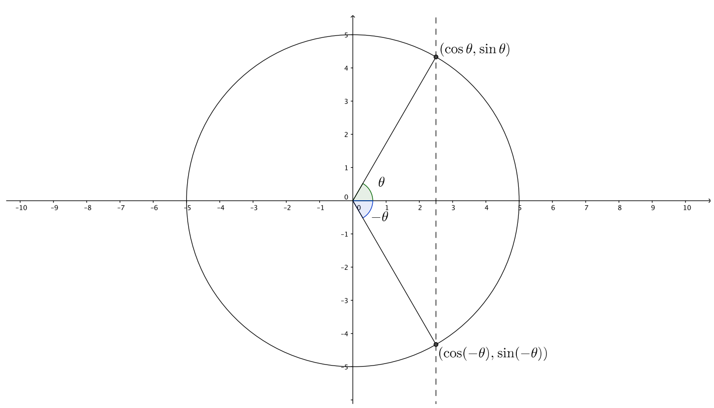
$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

If we divide both sides of $\cos^2\theta + \sin^2\theta = 1$ by $\sin^2\theta$ we get:

$$\boxed{\cot^2\theta + 1 = \csc^2\theta}$$

3.4.2 Identities With Negative Angles

Recall that $(\cos\theta, \sin\theta)$ are the coordinates of the point on the unit circle which makes an angle of θ, counterclockwise, with the positive x-axis. Similarly, $(\cos(-\theta), \sin(-\theta))$ are the coordinates of the point on the unit circle which makes an angle of θ, clockwise, with the positive x-axis. Consider the diagram below:



Notice that both of these points are on the same vertical line. As a result, the x-coordinates must be equal. That is,

$$\boxed{\cos(-\theta) = \cos\theta}$$

And, since the two points are reflections over the x-axis, it follows that

$$\boxed{\sin(-\theta) = -\sin\theta}$$

Once we have these trigonometric identities, we can derive identities for the remaining trigonometric functions. In the following example, we will show that $\tan(-\theta) = -\tan\theta$.

Example 3.4.1 *Show that* $\tan(-\theta) = -\tan\theta$.

Solution:

$$\begin{aligned}
\tan(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)} && \textit{By definition of } \tan\theta \\
&= -\frac{\sin\theta}{\cos\theta} && \textit{Because } \sin(-\theta) = -\sin\theta \textit{ and } \cos(-\theta) = \cos\theta \\
&= -\tan\theta && \textit{By definition of } \tan\theta
\end{aligned}$$

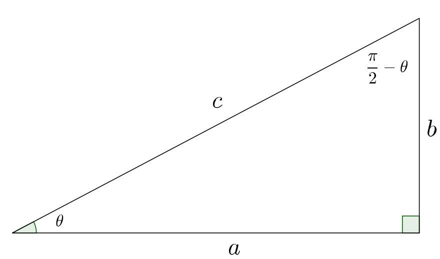
□

As a result, it follows that $f(\theta) = \cos\theta$ is an even function whereas $f(\theta) = \sin\theta$ and $f(\theta) = \tan\theta$ are odd functions. This will be useful when we graph the trigonometric functions as the graphs of even/odd functions contain certain symmetries.

3.4.3 Complementary Angle Identities

Definition 3.4.1 *Two angles* α *and* β *are called* **Complementary** *is* $\alpha + \beta = 90°$.

Suppose $0 < \theta < \frac{\pi}{2}$ and consider the following right triangle. Notice that we labeled the complementary angle as $\frac{\pi}{2} - \theta$.



Notice the following:

$$\sin\theta = \tfrac{b}{c} \qquad \sin\left(\tfrac{\pi}{2}-\theta\right) = \tfrac{a}{c}$$

$$\cos\theta = \tfrac{a}{c} \qquad \cos\left(\tfrac{\pi}{2}-\theta\right) = \tfrac{b}{c}$$

$$\tan\theta = \tfrac{b}{a} \qquad \cot\left(\tfrac{\pi}{2}-\theta\right) = \tfrac{a}{b}$$

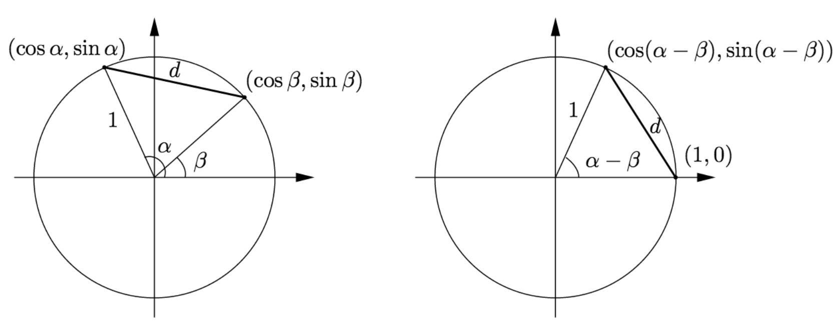
Thus, we arrive at some useful complementary angle identities:

- $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$
- $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$
- $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

Although we have only proven these for $0 < \theta < \frac{\pi}{2}$, it is possible to justify these identities for all other angles. We leave that as an exercise to the reader. Also, we will see the significance of these identities when we discuss the graphs of the trigonometric functions.

3.4.4 Sum/Difference Identities

In this section, we will derive an identity for $\cos(\alpha - \beta)$. To do so, consider the following diagram. The length d is the same in both triangles. So, we will calculate d in two different ways and then set them equal to each other.



From the triangle on the left, we will calculate d using the distance formula:

$$d = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$$

Similarly, from the triangle on the right, we will calculate d using the distance formula:

$$d = \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2} = \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta))^2}$$

Thus, equating these gives us:

$$
\begin{aligned}
\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2} &= \sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta))^2} \\
(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2 &= (\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta))^2 \\
\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta &= \cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta) \\
2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta &= 2-2\cos(\alpha-\beta) \\
\cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos(\alpha-\beta)
\end{aligned}
$$

And, we have derived the following identity:

$$\boxed{\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

Example 3.4.2 *Using the previous identity, derive a comparable identity for* $\cos(\alpha+\beta)$.

Solution:

$$
\begin{aligned}
\cos(\alpha+\beta) &= \cos(\alpha-(-\beta)) \\
&= \cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta) \\
&= \cos\alpha\cos\beta-\sin\alpha\sin\beta && \textit{Because } \cos(-\theta)=\cos\theta \textit{ and } \sin(-\theta)=-\sin\theta
\end{aligned}
$$

Hence, we have just derived the following trigonometric identity:

$$\boxed{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

Example 3.4.3 *Using the sum/difference identities for cosine, derive the formula* $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$.

Solution:

$$
\begin{aligned}
\sin(\alpha+\beta) &= \cos\left[\frac{\pi}{2}-(\alpha+\beta)\right] && \textit{Because } \cos\theta=\sin\left(\frac{\pi}{2}-\theta\right) \\
&= \cos\left[\left(\frac{\pi}{2}-\alpha\right)-\beta\right] \\
&= \cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta+\sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta && \textit{By the difference formula for cosine.} \\
&= \sin\alpha\cos\beta+\cos\alpha\sin\beta && \textit{Because } \sin\theta=\cos\left(\frac{\pi}{2}-\theta\right) \textit{ and } \cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)
\end{aligned}
$$

Hence, we have just derived the following trigonometric identity:

$$\boxed{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$

Example 3.4.4 *Derive the difference formula* $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$*.*

Solution:

$$\begin{aligned}
\sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) \\
&= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) && \textit{Because of the formula for } \sin(\alpha+\beta) \\
&= \sin\alpha\cos\beta - \cos\alpha\sin\beta && \textit{Because } \cos(-\theta) = \cos\theta \textit{ and } \sin(-\theta) = -\sin\theta
\end{aligned}$$

Hence, we have just derived the following trigonometric identity:

$$\boxed{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

Example 3.4.5 *Evaluate* $\sin(75^\circ)$

Solution:

$$\begin{aligned}
\sin(75^\circ) &= \sin(30^\circ + 45^\circ) \\
&= \sin(30^\circ)\cos(45^\circ) + \cos(30^\circ)\sin(45^\circ) && \textit{By the sum formula for sine} \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\
&= \frac{1+\sqrt{3}}{2\sqrt{2}} \\
&= \frac{1+\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} && \textit{If you choose to rationalize your answer} \\
&= \frac{\sqrt{2}+\sqrt{6}}{4}
\end{aligned}$$

□

3.4.5 Double Angle Formulas

Using the sum formulas for sine and cosine, we can now derive the double angle formulas.

Example 3.4.6 *Derive the double angle formula* $\cos(2\theta) = \cos^2\theta - \sin^2\theta$*.*

Solution:

$$\begin{aligned}
\cos(2\theta) &= \cos(\theta + \theta) \\
&= \cos\theta\cos\theta - \sin\theta\sin\theta && \textit{By for the formula for } \cos(\alpha+\beta) \textit{ with } \alpha = \beta = \theta \\
&= \cos^2\theta - \sin^2\theta
\end{aligned}$$

Hence, we have just derived the following trigonometric identity:

$$\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta}$$

Example 3.4.7 *Derive the double angle formula* $\sin(2\theta) = 2\sin\theta\cos\theta$.

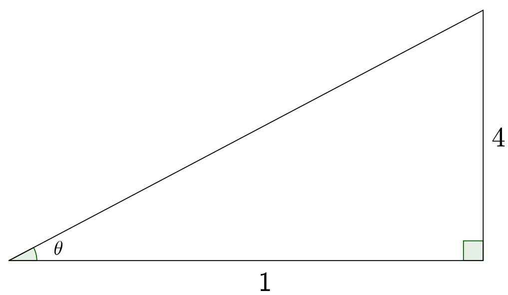
Solution:

$$\begin{aligned}
\sin(2\theta) &= \sin(\theta+\theta) \\
&= \sin\theta\cos\theta + \cos\theta\sin\theta && \textit{By for the formula for } \sin(\alpha+\beta) \textit{ with } \alpha=\beta=\theta \\
&= 2\sin\theta\cos\theta
\end{aligned}$$

Hence, we have just derived the following trigonometric identity:

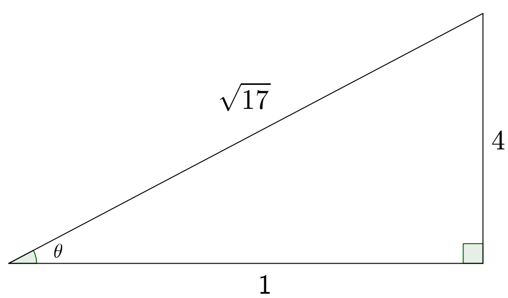
$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$

Example 3.4.8 *Evaluate* $\sin(2\theta)$, $\cos(2\theta)$, *and* $\tan(2\theta)$ *based on the following triangle:*



Solution:

By the Pythagorean Theorem, the hypotenuse has length $\sqrt{17}$. *The following triangle summarizes the given information:*



Based on this triangle, we know that $\sin\theta = \frac{4}{\sqrt{17}}$ *and* $\cos\theta = \frac{1}{\sqrt{17}}$. *Using these values, we can now answer the question.*

- *We begin by computing* $\sin(2\theta)$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \textit{By the double angle formula for sine.}$$

$$
\begin{aligned}
&= 2\left(\frac{4}{\sqrt{17}}\right)\left(\frac{1}{\sqrt{17}}\right) \\
&= \frac{8}{17}
\end{aligned}
$$

- *Next, we compute* $\cos(2\theta)$.

$$
\begin{aligned}
\cos(2\theta) &= \cos^2\theta - \sin^2\theta && \textit{By the double angle formula for cosine.} \\
&= \left(\frac{1}{\sqrt{17}}\right)^2 - \left(\frac{4}{\sqrt{17}}\right)^2 \\
&= \frac{1}{17} - \frac{16}{17} \\
&= -\frac{15}{17}
\end{aligned}
$$

- *Finally we compute* $\tan(2\theta)$

$$
\begin{aligned}
\tan(2\theta) &= \frac{\sin(2\theta)}{\cos(2\theta)} && \textit{By definition of tangent.} \\
&= \frac{\frac{8}{17}}{-\frac{15}{17}} \\
&= -\frac{8}{15}
\end{aligned}
$$

$\square$

3.4.6 Power Reducing Formulas

Recall the following two trigonometric identities:

- $\cos^2\theta + \sin^2\theta = 1 \implies \cos^2\theta = 1 - \sin^2\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

We will begin with the double angle formula for cosine and will derive a power reducing formula for sine. This will be useful in Integral Calculus (Math 122).

$$
\begin{aligned}
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\
\cos(2\theta) &= 1 - \sin^2\theta - \sin^2\theta && \text{Because } \cos^2\theta = 1 - \sin^2\theta \\
\cos(2\theta) &= 1 - 2\sin^2\theta \\
\sin^2\theta &= \frac{1 - \cos(2\theta)}{2}
\end{aligned}
$$

Hence, we have just derived the following trigonometric identity:

$$\boxed{\sin^2\theta = \frac{1-\cos(2\theta)}{2}}$$

In a similar way, you can derive

$$\boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}}$$

Example 3.4.9 *Rewrite $\cos^4\theta$ as an equivalent expression which does not have any trigonometric functions with powers greater than 1.*

Solution:

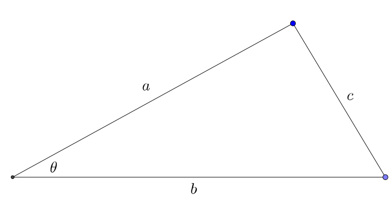
$$\begin{aligned}
\cos^4\theta &= (\cos^2\theta)^2 \\
&= \left(\frac{1+\cos(2\theta)}{2}\right)^2 && \textit{By the power reducing formula for cosine.} \\
&= \frac{1}{4}\left(1+2\cos(2\theta)+\cos^2(2\theta)\right) \\
&= \frac{1}{4}\left(1+2\cos(2\theta)+\frac{1+\cos(4\theta)}{2}\right) && \textit{By the power reducing formula for cosine.} \\
&= \frac{1}{4}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}(1+\cos(4\theta)) \\
&= \frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)
\end{aligned}$$

□

3.4.7 Law of Cosines

We conclude this section by discussing a generalization of the Pythagorean Theorem: the Law of Cosines.

Theorem 3.4.1 (Law of Cosines) *Consider a triangle with sides of length a, b, and c. Suppose that the side of length c is opposite the angle θ, as in the diagram below.*

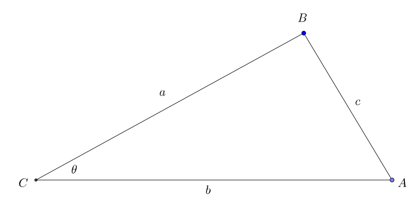


Then,

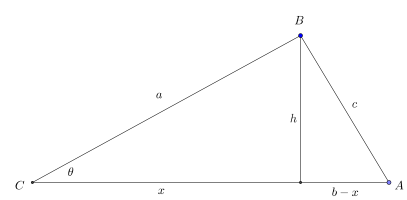
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

To prove this theorem, we consider two cases. First, we consider when θ is acute. Then, we discuss when θ is obtuse. Of course, if $\theta = \frac{\pi}{2}$, then we arrive at the Pythagorean Theorem since $\cos\left(\frac{\pi}{2}\right) = 0$.

- **Case 1:** Suppose θ is acute as in the following diagram.



We will drop a perpendicular line segment from vertex B to segment CA.



By applying the Pythagorean Theorem to the triangle on the left, we have:

$$x^2 + h^2 = a^2 \tag{1}$$

Furthermore, since $\cos\theta = \frac{x}{a}$, we have:

$$x = a\cos\theta \tag{2}$$

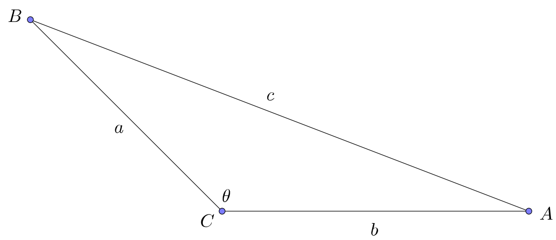
Similarly, we apply the Pythagorean Theorem to the triangle on the right to get

$$(b-x)^2 + h^2 = c^2 \tag{3}$$

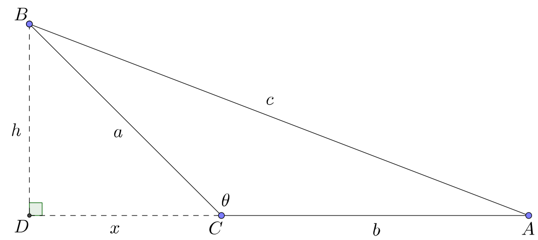
Combining these three results will give us the desired result.

$$\begin{aligned}
c^2 &= (b-x)^2 + h^2 && \text{By result (3).} \\
&= b^2 - 2bx + x^2 + h^2 \\
&= b^2 - 2ba\cos\theta + a^2 && \text{By results (1) and (2).} \\
&= a^2 + b^2 - 2ab\cos\theta
\end{aligned}$$

- **Case 2:** Suppose θ is obtuse as in the following diagram.



As before, we drop a perpendicular line segment from vertex B to the line containing A and C.



Applying the Pythagorean Theorem to triangle BDC, we get:

$$x^2 + h^2 = a^2 \tag{4}$$

Next, we introduce $\cos\theta$:

$$\begin{aligned} \cos(\pi - \theta) &= \frac{x}{a} && \text{Using the smaller right triangle.} \\ \cos\pi\cos\theta + \sin\pi\sin\theta &= \frac{x}{a} && \text{By the formula for } \cos(\alpha - \beta). \\ -\cos\theta &= \frac{x}{a} && \text{Because } \cos\pi = -1 \text{ and } \sin\pi = 0. \end{aligned}$$

Hence, we obtain:

$$x = -a\cos\theta \tag{5}$$

Finally, we apply the Pythagorean Theorem to the larger right triangle, $\triangle BDA$, to obtain:

$$(x+b)^2 + h^2 = c^2 \tag{6}$$

Combining these three results will give us the desired result.

$$\begin{aligned} c^2 &= (x+b)^2 + h^2 && \text{By (6)} \\ &= x^2 + 2bx + b^2 + h^2 \end{aligned}$$

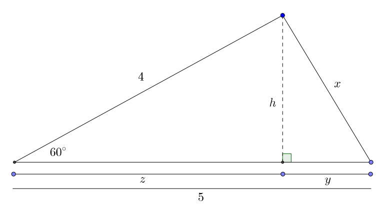
$$= (x^2 + h^2) + b^2 + 2bx$$

$$= a^2 + b^2 + 2b(-a\cos\theta) \qquad \text{By (4) and (5)}$$

$$= a^2 + b^2 - 2ab\cos\theta$$

We have proven the desired result. $\square$

Example 3.4.10 *Consider the triangle below:*



(a) *Determine the lengths x, y, z, and h.*

Solution:

We can begin by calculating length x by applying the Law of Cosines:

$$x^2 = 5^2 + 4^2 - 2(5)(4)\cos 60^\circ$$
$$= 25 + 16 - 40\left(\frac{1}{2}\right)$$
$$= 21$$

Thus, $x = \sqrt{21}$. Next, we calculate z. Since $\cos 60^\circ = \frac{z}{4}$, it follows that $z = 4\cos 60^\circ = 2$. And, since $y + z = 5$, we now know that $y = 3$. Finally, we apply the Pythagorean Theorem to the triangle on the right to obtain $(3)^2 + (h)^2 = (\sqrt{21})^2$. As a result, we know that $h = \sqrt{12}$.

(b) *Calculate the area of the triangle.*

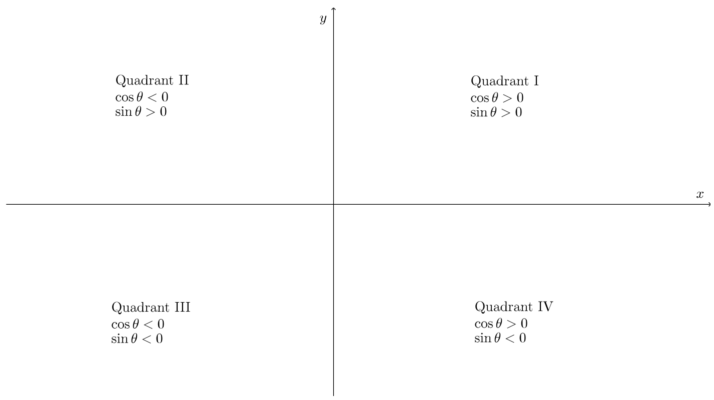
Solution:

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(5)(\sqrt{12})$$
$$= \frac{5\sqrt{12}}{2}$$

$\square$

3.5 Trigonometric Equations

In this section, we will solve trigonometric equations. Recall the following diagram from chapter 3.2:



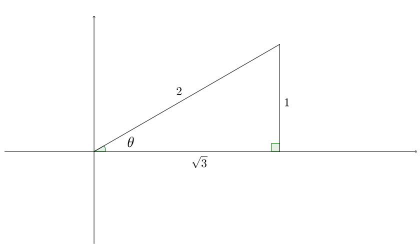
This can be useful when solving trigonometric equations. Consider the following examples.

Example 3.5.1 *Find all values of θ in the interval $[0, 2\pi)$ for which $2\sin\theta - 1 = 0$.*

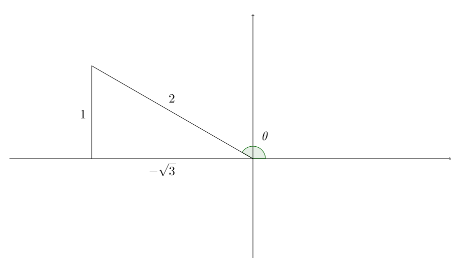
Solution:

To solve this equation, we isolate the $\sin\theta$ term to get $\sin\theta = \frac{1}{2}$. Since the value of $\sin\theta$ is positive, we know that θ must be either in quadrant I or quadrant II. So, we can draw appropriate reference triangles in these quadrants. And, since we know that $\sin\theta = \frac{1}{2}$, we can label side opposite θ as 1 and the hypotenuse as 2. As a result the adjacent side must be length $\sqrt{3}$.

Quadrant I



Quadrant II



Since these side lengths correspond to a 30-60-90 triangle, we recognize the reference angle as $\theta = \frac{\pi}{6}$. Thus, the two answers are $\theta = \frac{\pi}{6}$ and $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Equivalently, if you do not want to use reference triangles, you can find the points on the unit circle (in quadrants I and II) for which the y-coordinate is $\frac{1}{2}$. These will correspond to angles of $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, as before. □

Example 3.5.2 *Find all values of θ for which $2\sin\theta - 1 = 0$.*

Solution:

Since the period of $\sin\theta$ is 2π, we can find all solutions in the interval $[0, 2\pi)$ and use these results to find all of the remaining solutions. From the previous example, we know that $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ are the only solutions in $[0, 2\pi)$. All other solutions will be coterminal. In particular, all solutions must be of the form $\theta = \frac{\pi}{6} + 2\pi k$ for some integer k or $\theta = \frac{5\pi}{6} + 2\pi k$ for some integer k. □

Example 3.5.3 *Find all solutions to $\sec\theta\csc\theta = 2\csc\theta$ in the interval $[0, 2\pi)$.*

Solution:

To solve this equation, it is recommended that we bring all terms to the same side of the equation and then factor out the greatest common factor.

$$\begin{aligned} \sec\theta\csc\theta &= 2\csc\theta \\ \sec\theta\csc\theta - 2\csc\theta &= 0 \\ \csc\theta(\sec\theta - 2) &= 0 \end{aligned}$$

Thus, either $\csc\theta = 0$ or $\sec\theta - 2 = 0$. We will solve these equations one at a time.

- *First, we solve $\csc\theta = 0$.*

$$\begin{aligned} \csc\theta &= 0 \\ \frac{1}{\sin\theta} &= 0 && \textit{By the definition of } \csc\theta \\ 1 &= 0 && \textit{After multiplying both sides of the equation by } \sin\theta \end{aligned}$$

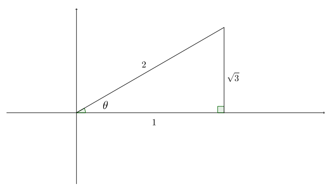
  Since the last equation is false, there are no values of θ for which $\csc\theta = 0$. We also knew this based on our discussion of the domain and range of $\csc\theta$.

- *Next, we solve $\sec\theta - 2 = 0$.*

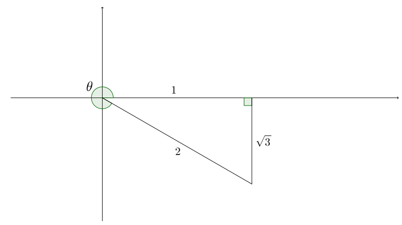
$$\begin{aligned} \sec\theta - 2 &= 0 \\ \sec\theta &= 2 \\ \frac{1}{\cos\theta} &= 2 && \textit{By the definition of } \sec\theta \\ \cos\theta &= \frac{1}{2} \end{aligned}$$

Since the value of $\cos\theta$ is positive, the angle must either be in quadrant I or IV. We draw the appropriate reference triangles, labeling the adjacent side as 1 and the hypotenuse as 2. As a result, the opposite side will have length $\sqrt{3}$.

Quadrant I



Quadrant IV



Since these side lengths correspond to a 30-60-90 triangle, we recognize the reference angle as $\theta = \frac{\pi}{3}$. Thus, the two answers are $\theta = \frac{\pi}{3}$ and $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

In the interval $[0, 2\pi)$, the only solutions to the original equation are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. □

Example 3.5.4 *Find all solutions in $[0, 2\pi)$ for which $2\sin^2\theta + \sin\theta - 1 = 0$.*

Solution:

This is a quadratic equation in $\sin\theta$ which happens to be factored. In particular, after factoring the left hand side, we get $(2\sin\theta - 1)(\sin\theta + 1) = 0$. Thus, we need θ such that either $2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$. As usual, we solve these separately.

- *First, we solve $2\sin\theta - 1 = 0$. The results, from example 3.5.1 are $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.*
- *Next, we solve $\sin\theta + 1 = 0$. Isolating the $\sin\theta$ term gives us $\sin\theta = -1$. It turns out that θ is a quadrantal angle. Specifically, if we find all points on the unit circle which have a y-coordinate of -1, we realize that the one point which satisfies this corresponds to $\theta = \frac{3\pi}{2}$.*

As a result, there are three solutions to the original equation in the interval $[0, 2\pi)$. Specifically, the solutions are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{3\pi}{2}$. □

Example 3.5.5 *Find all values of θ for which $\sin\theta = \cos\theta$*

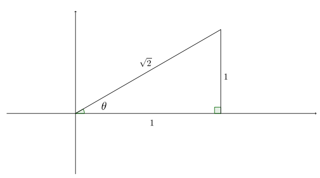
Solution:

Notice that if $\cos\theta = 0$, then $\sin\theta \neq 0$. Thus, we can divide both sides by $\cos\theta$ without losing any solutions.

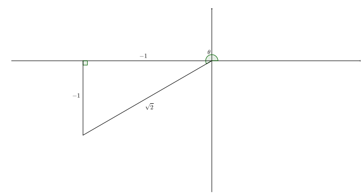
$$\sin\theta = \cos\theta$$
$$\frac{\sin\theta}{\cos\theta} = 1$$
$$\tan\theta = 1$$

Since the value of $\tan\theta$ *is positive, it follows that* θ *is in either quadrant I or quadrant III. We draw the corresponding reference triangles, labeling the opposite side as 1 and the adjacent side as 1 (since* $\tan\theta = 1$*). The hypotenuse will be* $\sqrt{2}$ *by Pythagorean Theorem.*

Quadrant I



Quadrant III



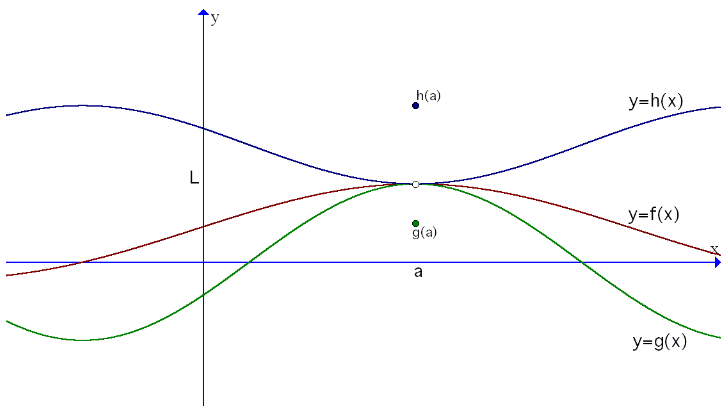
Since these side lengths correspond to a 45-45-90 triangle, we recognize the reference angle as $\theta = \frac{\pi}{4}$*. Thus, the two answers in the interval* $[0, 2\pi)$ *are* $\theta = \frac{\pi}{4}$ *and* $\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$*. Of course, to get all other coterminal angles, we add integer multiples of* 2π*. Thus, all of the solutions to the equation are* $\theta = \frac{\pi}{4} + 2\pi k$ *and* $\theta = \frac{5\pi}{4} + 2\pi k$ *for any integer* k*. A more compact way of expressing this solution set is* $\frac{\pi}{4} + \pi k$ *for any integer* k*.* $\square$

3.6 Limits & Continuity of Trigonometric Functions

3.6.1 The Squeeze Theorem

Although we have learned many algebraic techniques for determining limits, there are still some for which we will need more advanced tools. One particularly useful tool is called the **Squeeze Theorem**.

Theorem 3.6.1 (Squeeze Theorem) *Suppose $g(x) \le f(x) \le h(x)$ on an interval I containing $x = a$ and that $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, as shown in the diagram below. Then, $\lim_{x\to a} f(x) = L$.*

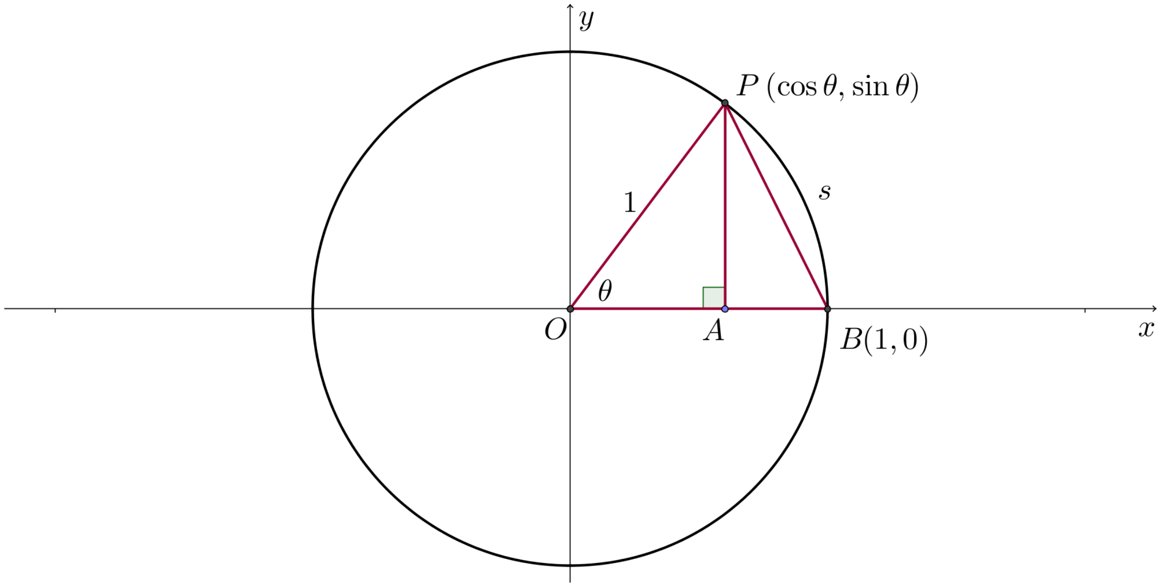


Example 3.6.1 *Evaluate* $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$.

Solution: *To evaluate this limit, we can appeal to the squeeze theorem. Notice that $-1 \le \sin x \le 1$ for all x. Now suppose $x > 0$; we can divide all terms of this inequality by x, preserving the order, to get $-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$. Since $\displaystyle\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to\infty} \frac{-1}{x} = 0$, it follows that $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x} = 0$.* □

3.6.2 Continuity of Trigonometric Functions

We begin this section by discussing continuity of the trigonometric functions. Specifically, we will first show that $f(\theta) = \sin\theta$ is continuous when $\theta = 0$. In particular, by appealing to the squeeze theorem, we will show that $\lim_{\theta\to 0} \sin\theta = \sin 0 = 0$. Consider the unit circle, shown below:



Suppose that $\theta > 0$ and that θ is near 0 (i.e., θ is a small angle in the first quadrant). Then, $0 < |AP| < |BP| < s$, where s is the length of the arc from point B to point P. Notice two facts:

- $|AP| = \sin\theta$
- Since $r = 1$, we have that $s = r\theta = \theta$

Thus, the inequality above becomes: $0 < \sin\theta < \theta$. Finally, using this inequality, $\lim_{\theta\to 0^+} 0 = 0$, and $\lim_{\theta\to 0^+} \theta = 0$, the squeeze theorem gives us that $\lim_{\theta\to 0^+} \sin\theta = 0$.

Similarly, if $\theta < 0$ and near 0 (i.e., a small clockwise angle in the fourth quadrant), we get the inequality that $\theta < \sin\theta < 0$. Using this inequality and the limits $\lim_{\theta\to 0^-} 0 = 0$, and $\lim_{\theta\to 0^-} \theta = 0$, the squeeze theorem gives us that $\lim_{\theta\to 0^-} \sin\theta = 0$.

So, the result follows that $\lim_{\theta\to 0} \sin\theta = 0$; and, the sine function is continuous for $\theta = 0$. Our next goal is to prove that $f(\theta) = \sin\theta)$ is continuous for all θ. In order to complete this justification, we need one additional fact:

$$\lim_{\theta\to 0} \cos\theta = 1$$

Here is the proof of this claim:

Suppose θ is near 0. Then $\sin^2\theta + \cos^2\theta = 1 \implies \cos\theta = \sqrt{1-\sin^2\theta}$. Then:

$$\begin{aligned}
\lim_{\theta\to 0}\cos\theta &= \lim_{\theta\to 0}\sqrt{1-\sin^2\theta} \\
&= \sqrt{1-\left(\lim_{\theta\to 0}\sin\theta\right)^2} \\
&= \sqrt{1-0^2} \\
&= 1
\end{aligned}$$

Now that we have all of the tools necessary to justify that $f(\theta) = \sin\theta$ is continuous for all θ, let us justify it. Specifically, we need to show that $\lim\limits_{\theta\to a}\sin\theta = \sin a$. To do so, let $h = \theta - a$. Then,

$$\begin{aligned}
\lim_{\theta\to a}\sin\theta &= \lim_{h\to 0}\sin(a+h) \\
&= \lim_{h\to 0}(\sin a\cos h + \cos a\sin h) && \text{Because of the addition identity} \\
&= \lim_{h\to 0}(\sin a)\lim_{h\to 0}(\cos h) + \lim_{h\to 0}(\cos a)\lim_{h\to 0}(\sin h) \\
&= (\sin a)(1) + (\cos a)(0) && \text{By the previous two calculations} \\
&= \sin a
\end{aligned}$$

Thus, we have shown that for all real numbers a, $\lim\limits_{\theta\to a}\sin\theta = \sin a$, which proves the following theorem:

Theorem 3.6.2 (Continuity of $\sin x$) *$f(x) = \sin x$ is continuous everywhere.*

Now that we know that $f(x) = \sin x$ is everywhere continuous, it immediately follows that the remaining trigonometric functions are continuous on their domains because compositions and quotients of continuous functions are continuous. Consider the following examples:

Example 3.6.2 *Argue that $f(x) = \cos x$ is everywhere continuous.*

Solution:

We know that $f(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right)$. Since $x + \frac{pi}{2}$ is everywhere continuous and $\sin x$ is everywhere continuous, it follows that $f(x) = \cos x$ is everywhere continuous because compositions of continuous functions are continuous. $\square$

Example 3.6.3 *Argue that $f(x) = \tan x$ is continuous on its domain.*

Solution:

We know that $f(x) = \tan x = \frac{\sin x}{\cos x}$. Since $\sin x$ and $\cos x$ are everywhere continuous, it follows that $f(x) = \tan x$ will be continuous everywhere except when $\cos x = 0$. That is, $f(x) = \tan x$ is continuous on its domain. □

The ideas in the previous two examples lead us to our next theorem. You should justify the continuity of the remaining trigonometric functions.

Theorem 3.6.3 (Continuity of Trigonometric Functions) *All of the trigonometric functions are continuous on their respective domains.*

3.6.3 Limits with Trigonometric Functions

Now that we know that the trigonometric functions are continuous on their respective domains, evaluating many limits involving trigonometric functions have become very simple. Consider the following examples:

Example 3.6.4 *Evaluate* $\lim\limits_{x \to \pi} \sin x$

Solution:

Since $\sin x$ is continuous at $x = \pi$, it follows that $\lim\limits_{x \to \pi} \sin x = \sin \pi = 0$. □

Example 3.6.5 *Evaluate* $\lim\limits_{x \to \pi/4} \tan x$

Solution:

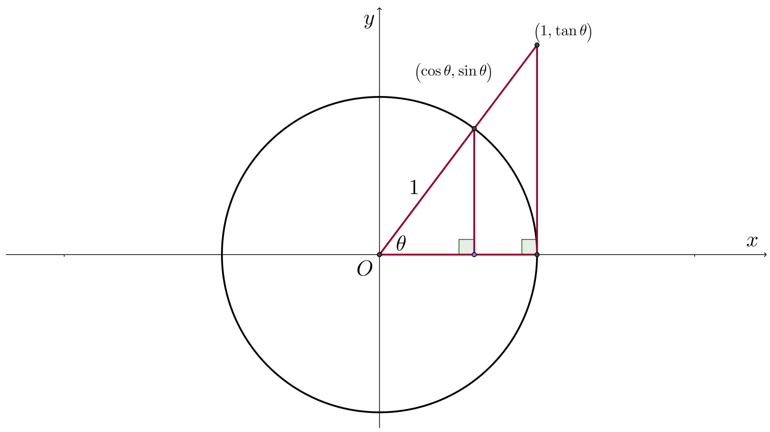
Since $\tan x$ is continuous at $x = \frac{\pi}{4}$, it follows that $\lim\limits_{x \to \pi/4} \tan x = \tan \dfrac{\pi}{4} = 1$. □

However, other limits will be more complicated. A useful limit to remember is discussed in the following example. The result of this example will be very helpful in some of our later calculations.

Example 3.6.6 *Compute* $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta}$.

Solution:

This limit is of the indeterminate form $\frac{0}{0}$. We cannot use the same algebraic techniques from Math 116 to evaluate this limit. Instead, we will appeal to the squeeze theorem. Consider the diagram below.



Notice that the coordinates $(1, \tan\theta)$ *were found using similar triangles. From the diagram, we have the following inequality:*

$$\textit{Area of Smaller Triangle} \leq \textit{Area of Sector} \leq \textit{Area of Bigger Triangle}$$

Here are the areas of the three shapes:

- *From point* $P(\cos\theta, \sin\theta)$*, we know that the base of the smaller triangle is* $\cos\theta$ *and the height of the smaller triangle is* $\sin\theta$*. This, the area of the smaller triangle is* $\frac{1}{2}\cos\theta\sin\theta$*.*

- *Using similar triangles, you can show that the coordinates of point* Q *are* $Q(1, \tan\theta)$*. Thus, the larger triangle has a base of length 1 and a height of* $\tan\theta$*. As a result, the area of the larger triangle is* $\frac{1}{2}\tan\theta$*.*

- *Finally, the area of a sector of radius* r *with an angle of* θ *is* $\frac{1}{2}r\theta$*. Thus, the area of the sector in the diagram above is* $\frac{1}{2}\theta$*.*

Thus,

$$\frac{1}{2}\sin\theta\cos\theta \leq \frac{1}{2}\theta \leq \frac{1}{2}\tan\theta$$

For now, let us assume θ *is a small angle close to 0 radians in the first quadrant. Then we can divide the entire inequality by* $\sin\theta > 0$ *without changing the ordering. In particular, we arrive at the following inequality:*

$$\cos\theta \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$

Since $\lim_{\theta\to 0^+}\cos\theta = \lim_{\theta\to 0^+}\frac{1}{\cos\theta} = 1$*, it follows that* $\lim_{\theta\to 0^+}\frac{\theta}{\sin\theta} = 1$*. You can repeat the process when* θ *is a small angle close to 0 in the fourth quadrant to show that* $\lim_{\theta\to 0^-}\frac{\theta}{\sin\theta} = 1$*. Hence, we have proven that* $\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1$*. And, an immediate consequence of our calculation is that* $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$*.* □

> **Theorem 3.6.4** *If* θ *is measured in radians, then*
>
> $$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

In the previous theorem, it is important for θ to be measured in radians. The reason has to do with the area of the sector that showed up in the proof. If θ is not measured in radians, then the area of the sector would not be θ; instead, it would have been $\frac{\theta\pi}{360}$ and we would have ended up with a different result.

Example 3.6.7 *Evaluate* $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$

Solution:

This limit is of the indeterminate form of $\frac{0}{0}$*. To evaluate this limit, we will use the previous theorem.*

$$\begin{aligned}
\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} \\
&= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)} \\
&= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)} && \textit{Because } \sin^2\theta+\cos^2\theta = 1 \\
&= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta} \\
&= (1)\left(\frac{0}{2}\right) \\
&= 0
\end{aligned}$$

□

This result is going to be important for us in the next section of this chapter. So, let's call it a theorem:

> **Theorem 3.6.5** *If* θ *is measured in radians, then*
>
> $$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$$

We will conclude this section with some more examples:

Example 3.6.8 *Evaluate* $\lim_{x \to 0} \frac{\sin 3x}{x}$

Solution:

This limit is of the indeterminate form $\frac{0}{0}$. *To evaluate this limit, we will use Theorem 3.8.2. First, we multiply the numerator and denominator by 3.*

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot 3$$

Let $t = 3x$. *As* $x \to 0$, *it follows that* $t \to 0$. *So, we making this substitution, we get:*

$$\begin{aligned} \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot 3 &= \lim_{t \to 0} \frac{\sin t}{t} \cdot 3 \\ &= (1)(3) \\ &= 3 \end{aligned}$$

□

Example 3.6.9 *Evaluate* $\lim_{x \to 7} \frac{\sin(x-7)}{x^2 - 49}$

Solution:

This limit is of the indeterminate form $\frac{0}{0}$. *To evaluate this limit, we will again use Theorem 3.8.2. First, notice that* $x^2 - 49 = (x+7)(x-7)$. *Thus,*

$$\lim_{x \to 7} \frac{\sin(x-7)}{(x-7)(x+7)} = \lim_{x \to 7} \frac{\sin(x-7)}{(x-7)(x+7)}$$

Let $t = x - 7$. *As* $x \to 7$, *it follows that* $t \to 0$. *So, we making this substitution, we get:*

$$\begin{aligned} \lim_{x \to 7} \frac{\sin(x-7)}{(x-7)(x+7)} &= \lim_{t \to 0} \frac{\sin t}{t(t+14)} && \textit{Because } t = x - 7 \implies x = t + 7 \\ &= \lim_{t \to 0} \frac{\sin t}{t} \cdot \frac{1}{t+14} \\ &= (1)\left(\frac{1}{14}\right) \\ &= \frac{1}{14} \end{aligned}$$

□

Example 3.6.10 *Find the non-zero value(s) of k which make the* $f(x) = \begin{cases} \frac{\tan kx}{x} & \text{if } x \neq 0 \\ x+3 & \text{if } x = 0 \end{cases}$ *continuous at* $x = 0$.

Solution:

We need to find k such that $\lim_{x \to 0} f(x) = f(0)$.

- $f(0) = 3$
- $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{\tan kx}{x}$ *is of the indeterminate form* $\frac{0}{0}$. *Again, theorem 3.8.2 will help us unravel the value of this limit.*

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan kx}{x} &= \lim_{x\to 0} \frac{\frac{\sin kx}{\cos kx}}{x} && \text{Because } \tan\theta = \frac{\sin\theta}{\cos\theta} \\
&= \lim_{x\to 0} \frac{\sin kx}{x\cos kx} \\
&= \lim_{x\to 0} \frac{\sin kx}{x} \cdot \cos kx && \text{Indeterminate form of } \frac{0}{0} \\
&= \lim_{x\to 0} \frac{\sin kx}{kx} \cdot k\cos kx && \text{Because we multiplied and divided by } k \neq 0 \\
&= (1)(k) && \text{By theorem 3.8.2} \\
&= k
\end{aligned}$$

Hence, for $f(x)$ *to be continuous when* $x = 0$, *we need* $k = 3$. □

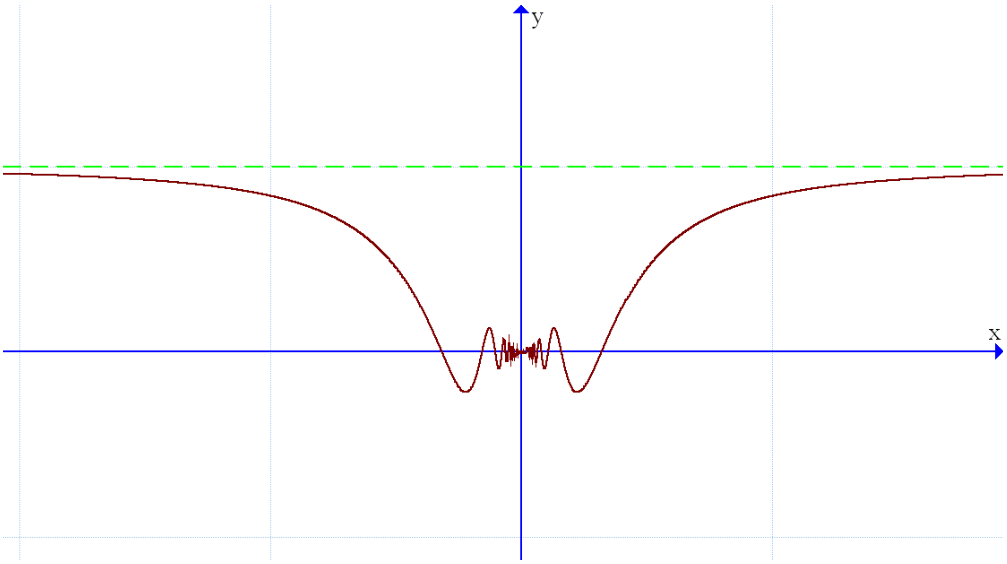
Example 3.6.11 *Evaluate* $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Solution:

This limit is of the indeterminate form $0 \cdot \infty$, *which will be discussed at the end of the course. Typically, with these types of indeterminate forms, we try to rewrite the function in such a way that the limit becomes either of the* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ *indeterminate form type. To do this, we bring a term to the denominator. In this case we will bring the x term to the denominator.*

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

Now, we have the indeterminate form of $\frac{0}{0}$. *Let us make a substitution. Specifically, let* $t = \frac{1}{x}$. *Then, since* $\lim_{x\to\infty} \dfrac{1}{x} = 0$, *we know that as* $x \to \infty$, *it follows that* $t \to 0$. *So, we can change variables to get* $\lim_{t\to 0} \dfrac{\sin t}{t} = 1$. *This result is supported by the following graph:*



$\square$

Example 3.6.12 *Compute* $\lim\limits_{x\to 0^+} \csc x$

Solution:

To begin, recognize that $\csc x = \frac{1}{\sin x}$*. So* $\lim\limits_{x\to 0^+} \csc x = \lim\limits_{x\to 0^+} \dfrac{1}{\sin x}$*. As* $x \to 0^+$*, the numerator approaches 1 and the denominator approaches 0. Hence, the possible values for this one-sided limit are* $+\infty$ *or* $-\infty$*. Because* $1 > 0$ *and* $\sin x > 0$ *as* $x \to 0^+$*, it follows that* $\lim\limits_{x\to 0^+} \csc x = +\infty$*.* $\square$

3.7 Derivatives of Trigonometric Functions

In this section, we will come up with derivative formulas for the six trigonometric functions. We will start with $f(x) = \sin x$.

Theorem 3.7.1 (Derivative of $\sin x$**)** *If* x *is measured in radians, then*

$$\frac{d}{dx}(\sin x) = \cos x$$

To justify this theorem, we will appeal to the definition.

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{Because of the trigonometric identity for } \sin(A+B) \\
&= \lim_{h\to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right)
\end{aligned}$$

$$= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]$$

$$= (\sin x)(0) + (\cos x)(1) \qquad \text{Because } \lim_{h \to 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

$$= \cos x$$

Now that we have proven that $\frac{d}{dx}(\sin x) = \cos x$, we can use various trigonometric identities to calculate the derivatives of the remaining trigonometric functions. The results are summarized in the following theorem.

Theorem 3.7.2 (Derivative of Trigonometric Functions) *If x is measured in radians, then*

1. $\frac{d}{dx}(\cos x) = -\sin x$
2. $\frac{d}{dx}(\tan x) = \sec^2 x$
3. $\frac{d}{dx}(\cot x) = -\csc^2 x$
4. $\frac{d}{dx}(\sec x) = \sec x \tan x$
5. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Selected Proofs:

1. To calculate the derivative of $\cos x$, we can use the trigonometric identity $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ as follows:

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] && \text{By the complementary angle identity} \\
&= \cos\left(\frac{\pi}{2} - x\right) \frac{d}{dx}\left(\frac{\pi}{2} - x\right) && \text{By the chain rule} \\
&= \cos\left(\frac{\pi}{2} - x\right)(-1) \\
&= (\sin x)(-1) && \text{By the complementary angle identity} \\
&= -\sin x
\end{aligned}$$

2. To calculate the derivative of $\tan x$, we can use the trigonometric identity $\tan x = \frac{\sin x}{\cos x}$ as follows:

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{(\cos x)\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} && \text{By the quotient rule}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} && \\
&= \frac{1}{\cos^2 x} && \text{Because } \sin^2 x + \cos^2 x = 1 \\
&= \sec^2 x && \text{Because } \frac{1}{\cos x} = \sec x
\end{aligned}
$$

4. To calculate the derivative of $\sec x$, we can use the trigonometric identity $\sec x = \frac{1}{\cos x}$ as follows:

$$
\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) && \\
&= \frac{d}{dx}\left[(\cos x)^{-1}\right] && \\
&= -1(\cos x)^{-2}\frac{d}{dx}(\cos x) && \text{By the chain rule} \\
&= -1(\cos x)^{-2}(-\sin x) && \\
&= \frac{\sin x}{\cos^2 x} && \\
&= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} && \\
&= \sec x \tan x &&
\end{aligned}
$$

You should try to justify the remaining derivative formulas.

Example 3.7.1 *Suppose $f(x) = \sin(2x)$. Calculate $f'(x)$.*

Solution:

There are several ways for us to do this.

- *Since we have a composition of functions, we can use the chain rule.*

$$
\begin{aligned}
f'(x) &= \cos(2x)\frac{d}{dx}(2x) && \textit{By the chain rule.} \\
&= 2\cos(2x) &&
\end{aligned}
$$

- *Alternatively, we can use the double angle identity first.*

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}[\sin(2x)] && \\
&= \frac{d}{dx}[2\sin x \cos x] && \textit{By the double angle formula for sine.} \\
&= 2\left[(\sin x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(\sin x)\right] && \textit{By the product rule.} \\
&= 2(-\sin^2 x + \cos^2 x) && \\
&= 2\cos(2x) && \textit{By the double angle formula for cosine.}
\end{aligned}
$$

$\square$

Example 3.7.2 *Find an equation of the line which is tangent to the graph of* $f(x) = \cot^3 x$ *at the point where* $x = \frac{\pi}{4}$*.*

Solution:

The point of tangency is $\left(\frac{\pi}{4}, f\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, 1\right)$*. And, by the chain rule we know that* $f'(x) = -3\cot^2 x \csc^2 x$*. Thus, the slope of the tangent line is* $f'\left(\frac{\pi}{4}\right) = -3(1)^2\left(\sqrt{2}\right)^2 = -6$*. So, an equation of the line which is tangent to the curve at the given point is* $y - 1 = -6\left(x - \frac{\pi}{4}\right)$*.* $\square$

Example 3.7.3 *Find all absolute maxima and absolute minima for* $f(x) = 2\sin x - x$ *on the interval* $[0, \pi]$*.*

Solution: *Since* $f(x) = 2\sin x - x$ *is continuous on the closed and bounded interval* $[0, \pi]$*,* $f(x)$ *must achieve both an absolute maximum and an absolute minimum by the Extreme Value Theorem. We must check the endpoints (*$x = 0$ *and* $x = \pi$*) and any critical points in the interval* $(0, \pi)$*. To calculate the critical points, we will find all* x *in* (0π) *for which* $f'(x) = 0$*:*

$$\begin{aligned} f'(x) &= 0 \\ 2\cos x - 1 &= 0 \\ \cos x &= \frac{1}{2} \end{aligned}$$

The only value of x *in* $(0, \pi)$ *which satisfies this condition is* $x = \frac{\pi}{3}$*. The following table shows the* y*-values at each of the endpoints and the critical point.*

x	$f(x) = 2\sin x - x$
0	0
$\frac{\pi}{3}$	$\sqrt{3} - \frac{\pi}{3}$
π	$-\pi$

From this data, we see that the absolute maximum is $\sqrt{3} - \frac{\pi}{3}$ *which occurs when* $x = \frac{\pi}{3}$*. The absolute minimum value is* $-\pi$ *which occurs when* $x = \pi$*.* $\square$

3.8 Graphs of Trigonometric Functions

In this section, we will graph each of the trigonometric functions. As usual, we begin by graphing $f(x) = \sin x$ and then we will progress to the remaining trigonometric functions. It may be a good idea to remember some of these graphs going forward.

Before graphing $f(x) = \sin x$, we will show that the period is 2π; that is, we will show that $\sin(x+2\pi) = \sin x$:

$$\begin{aligned} \sin(x+2\pi) &= \sin x \cos 2\pi + \cos x \sin 2\pi && \text{By the formula for } \sin(\alpha+\beta) \\ &= (\sin x)(1) + (\cos x)(0) \\ &= \sin x \end{aligned}$$

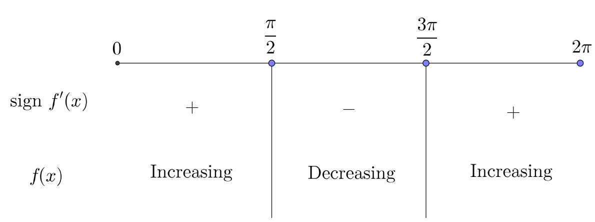
Now that we know that the period of $f(x) = \sin x$ is 2π, we will graph the function on the interval $[0, 2\pi]$, recognizing that the graph will repeat on the interval $[2\pi, 4\pi]$, and so on.

Example 3.8.1 *Graph $f(x) = \sin x$ on the interval $[0, 2\pi]$.*

Solution: *From example 3.2.3, we know that the x-intercepts of $f(x) = \sin x$ in the interval $[0, 2\pi]$ are $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$. Next, we will calculate the intervals on which the graph of $f(x)$ is increasing and those on which $f(x)$ is decreasing. To do so, we calculate the critical points:*

$$\begin{aligned} f'(x) &= 0 \\ \cos x &= 0 \\ x &= \frac{\pi}{2}, \frac{3\pi}{2} \end{aligned}$$

Next, we test the sign of $f'(x)$ using the following sign diagram:

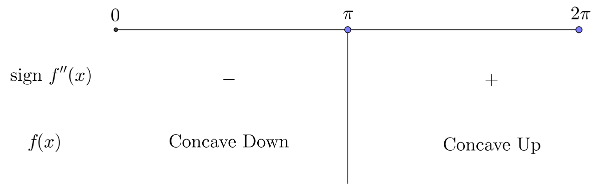


So, $f(x)$ is increasing on the intervals $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ and is decreasing on the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Notice that $f(x)$ will have a local maximum at $\left(\frac{\pi}{2}, f\left(\frac{\pi}{2}\right)\right) = \left(\frac{\pi}{2}, 1\right)$ and a local min at $\left(\frac{3\pi}{2}, f\left(\frac{3\pi}{2}\right)\right) = \left(\frac{3\pi}{2}, -1\right)$.

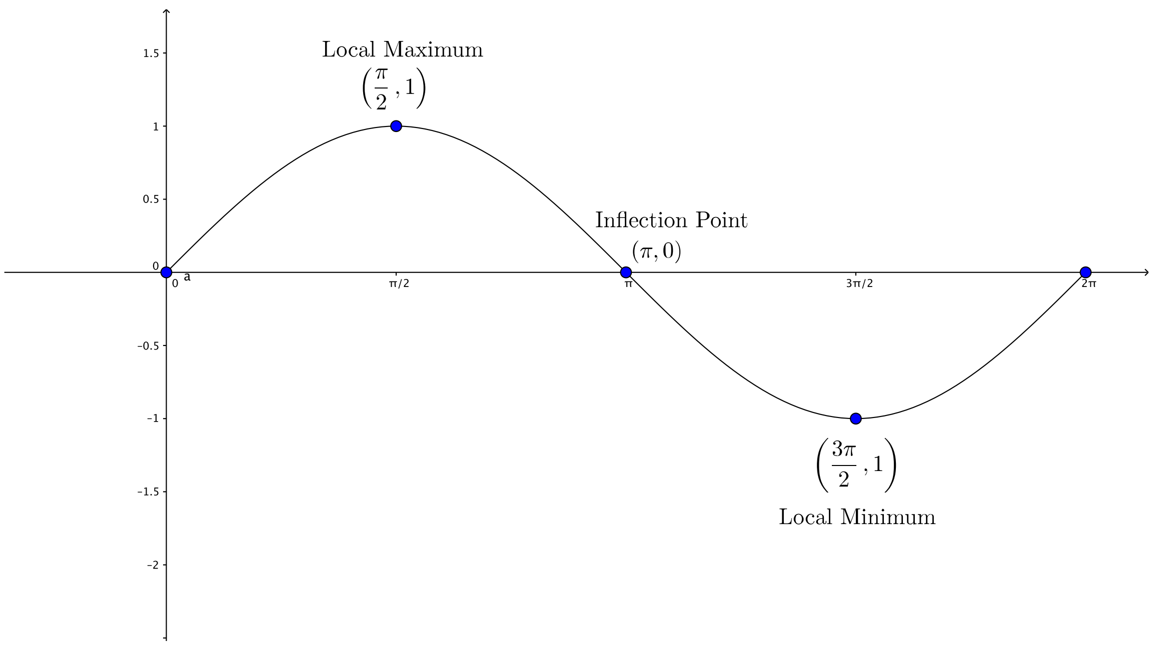
Next, we will calculate the intervals on which the graph of $f(x)$ is concave up and those on which $f(x)$ is concave down. To do so, we identify the potential inflection points in the interval $(0, 2\pi)$ as follows:

$$\begin{aligned} f''(x) &= 0 \\ -\sin x &= 0 \\ x &= \pi \end{aligned}$$

Again, we will test around this candidate on the following sign chart:

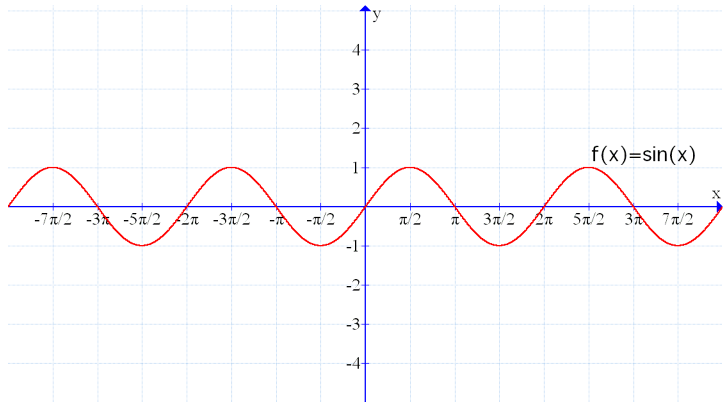


As a result, $f(x)$ *will have an inflection point at* $(\pi, f(\pi)) = (\pi, 0)$*. Using these results, we can graph* $f(x) = \sin x$*.*



$\square$

Now that we know the graph of $f(x) = \sin x$ on the interval $[0, 2\pi]$, we can use the periodicity to sketch the graph of $f(x) = \sin x$ on $(-\infty, \infty)$:



An immediate consequence of our work in developing the graph of $y = \sin x$ is the graph of $y = \cos x$.

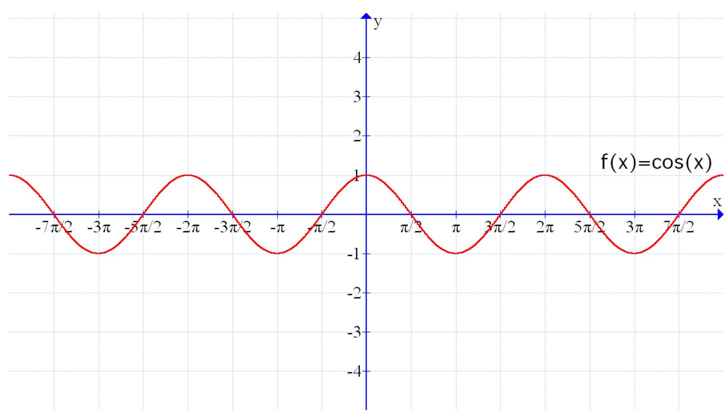
Example 3.8.2 *Graph $y = \cos x$ on the interval $(-\infty, \infty)$.*

Solution:

First, we will use trigonometric identities to relate the graphs of $y = \sin x$ and $y = \cos x$.

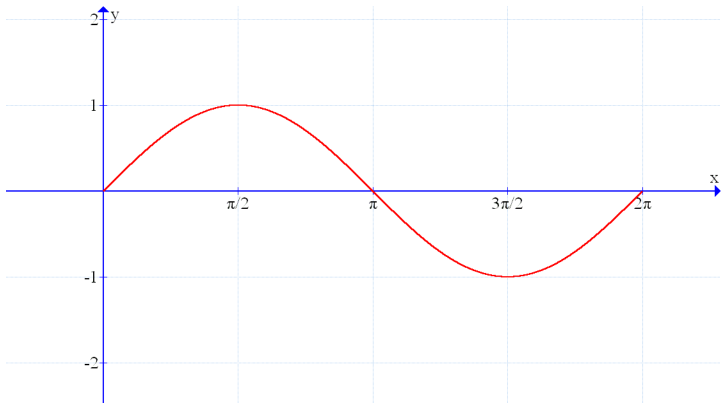
$$\begin{aligned} \cos x &= \sin\left(\frac{\pi}{2} - x\right) && \textit{By the complementary angle identity.} \\ &= \sin\left[-\left(x - \frac{\pi}{2}\right)\right] \\ &= -\sin\left(x - \frac{\pi}{2}\right) && \textit{Because } \sin(-x) = -\sin x. \end{aligned}$$

So, the graph of $y = \cos x$ can be obtained by reflecting the graph of $y = \sin x$ over the x-axis and then shifting the result to the right $\frac{\pi}{2}$ units. The graph of $f(x) = \cos x$ is as follows:

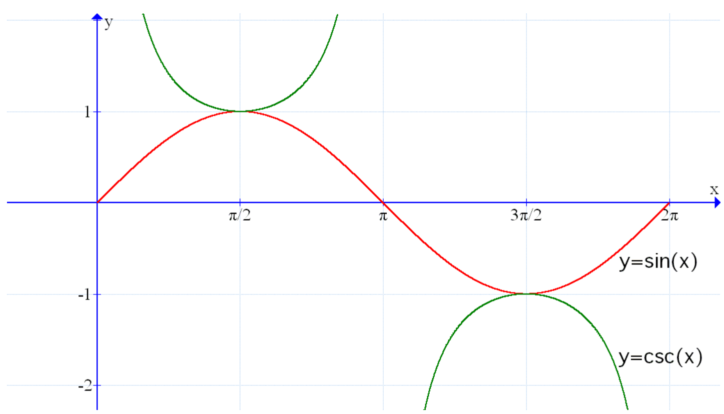


As an exercise, label the local extrema and the inflection points in the interval $[0, 2\pi]$. □

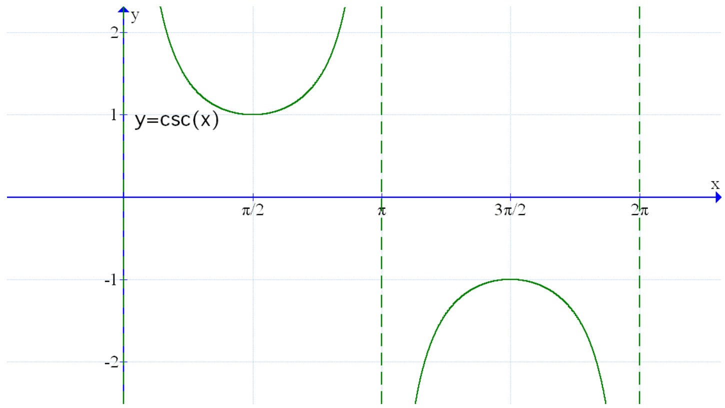
Also, once we know the graph of $y = \sin x$, we can take the reciprocal of all of the y-coordinates to get the graph of $y = \csc x$. Remember that $\csc x$ is not defined whenever $\sin x = 0$. It should be clear soon that the graph of $y = \csc x$ will have vertical asymptotes at these points. To begin, let us graph $y = \sin x$ on the interval $[0, 2\pi]$:



Notice that for x in the interval $(0, \pi)$, the values of $\sin x$ are in $(0, 1]$. When we take the reciprocals of these y-values in $(0, 1]$, we obtain corresponding y-values in $[1, \infty)$. Similarly, for x in the interval $(\pi, 2\pi)$, the y-values of $\sin x$ are in $[-1, 0)$. Taking reciprocals of these y-values will result in corresponding y-values in $(-\infty, 1]$. The following plot shows these reciprocal y-values ($\csc x$) along with the original y-values ($\sin x$).

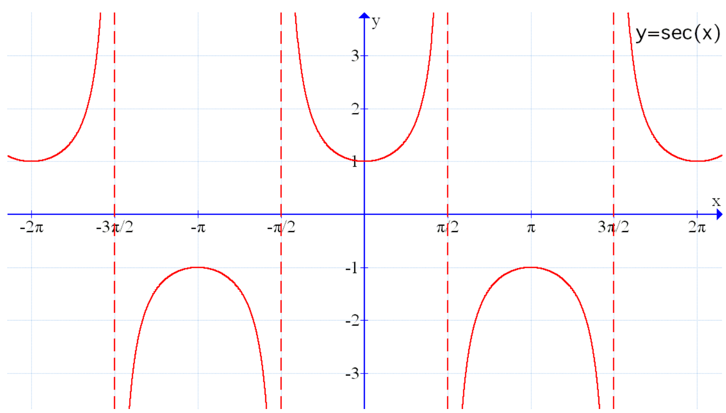


Finally, notice that for x near π, the values of $\sin x$ are very small (close to 0). So, their reciprocals are going to be very large in absolute value. Mathematically, we can show that $\lim_{x \to \pi^-} \csc x = \infty$ and $\lim_{x \to \pi^+} \csc x = -\infty$. As a result, the graph of $y = \csc x$ will have a vertical asymptote of $x = \pi$. Similarly, the graph of $\csc x$ will have vertical asymptotes at all other x where $\sin x = 0$. The following plot is the graph of $\csc x$ on the interval $[0, 2\pi]$. As usual, we can use the periodic nature of the function to sketch the graph on other intervals.



You should try to verify this result using the curve sketching techniques that you learned in Math 116. Specifically, use information from the first and second derivatives to identify intervals on which the graph is increasing, decreasing, concave up, or concave down. Furthermore, you should be able to identify all local extrema and inflection points (if any such points exist).

By performing a similar analysis, you can verify that the graph of $y = \sec x$ is as follows:



Finally, we will use our curve sketching techniques to develop the graph of $y = \tan x$. Then, we will take the reciprocal of all of the y-coordinates to get the graph of $y = \cot x$.

Example 3.8.3 *Show that the period of $f(x) = \tan x$ is π.*

Solution:

To justify the claim, we must show that $\tan(x+\pi) = \tan x$ *for all* x *in the domain.*

$$\begin{aligned}\tan(x+\pi) &= \frac{\sin(x+\pi)}{\cos(x+\pi)} && \textit{By definition of tangent.}\\ &= \frac{\sin x\cos\pi + \cos x\sin\pi}{\cos x\cos\pi - \sin x\sin\pi} && \textit{By the sum formulas for cosine and sine.}\end{aligned}$$

$$= \frac{-\sin x}{-\cos x}$$

$$= \tan x$$

Hence, the period of $\tan x$ *is* π. $\square$

We can sketch the graph of $\tan x$ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and then use the periodic nature of tangent to graph it on other intervals.

Example 3.8.4 *Sketch the graph of* $f(x) = \tan x$ *on the interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution:

We begin by calculating the x*-intercept(s) in the interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *by setting* $y = 0$:

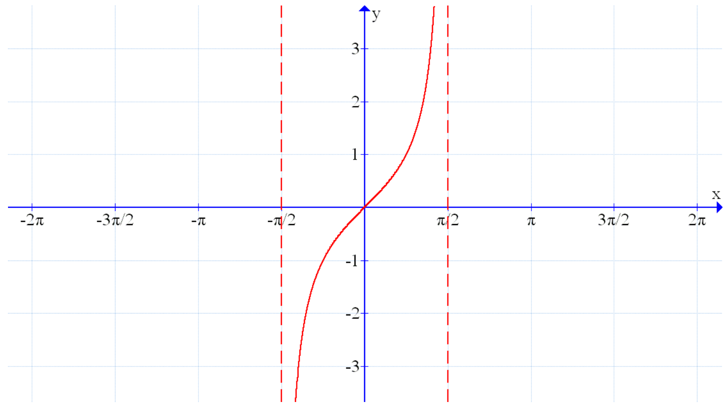
$$\begin{aligned} \tan x &= 0 \\ \frac{\sin x}{\cos x} &= 0 \\ \sin x &= 0 \\ x &= 0 \end{aligned}$$

So the only x*-intercept in this interval is* $(0, 0)$. *Next, we will compute* $\lim_{x \to \frac{\pi}{2}^-} \tan x$. *Thinking about* $\tan x$ *as the slope of the segment ray on the unit circle corresponding to* x*, we can deduce that this limit is* $+\infty$. *Similarly,* $\lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty$. *So, the graph of* $f(x) = \tan x$ *will have vertical asymptotes of* $x = \frac{\pi}{2}$ *and* $x = -\frac{\pi}{2}$ *on the given interval.*

Finally, we will use information from the first and second derivatives to analyze the graphs. Notice that $f'(x) = \sec^2 x$. *Since* $f'(x) > 0$ *for all* x *in* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, *it follows that* $f(x)$ *is always increasing on this interval and does not have any local extrema. Finally, we determine the concavity. The second derivative is* $f''(x) = 2\sec^2 x \tan x$. *Thus, we identify the possible inflection points as follows:*

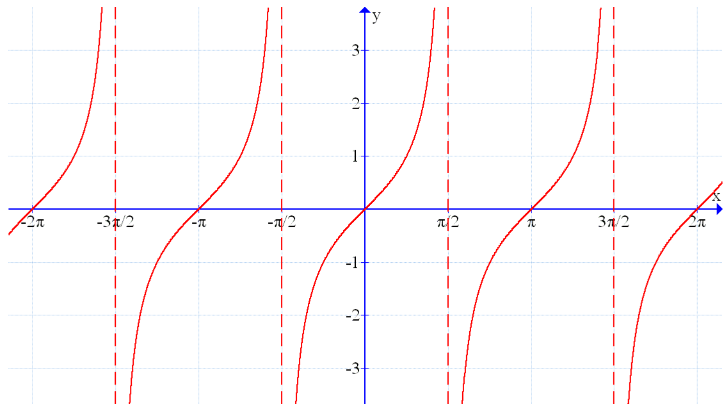
$$\begin{aligned} f''(x) &= 0 \\ 2\sec^2 x \tan x &= 0 \\ \tan x &= 0 && \textit{Since } \sec x \neq 0 \\ x &= 0 \end{aligned}$$

One can verify that $f''(x) < 0$ *on the interval* $\left(-\frac{\pi}{2}, 0\right)$ *and* $f''(x) > 0$ *on the interval* $\left(0, \frac{\pi}{2}\right)$. *So,* $f(x)$ *is concave down on* $\left(-\frac{\pi}{2}, 0\right)$, *is concave up on* $\left(0, \frac{\pi}{2}\right)$, *and has an inflection point at* $(0, 0)$. *By putting all of this information together will obtain the following graph of* $f(x) = \tan x$ *on the interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

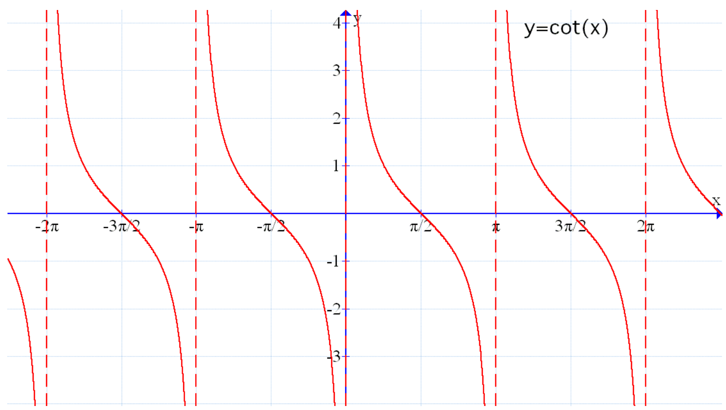


□

The graph of $f(x) = \tan x$ on the interval $(-\infty, \infty)$ is as follows:



As an exercise, you should verify that the graph of $f(x) = \cot x$ is as shown below.

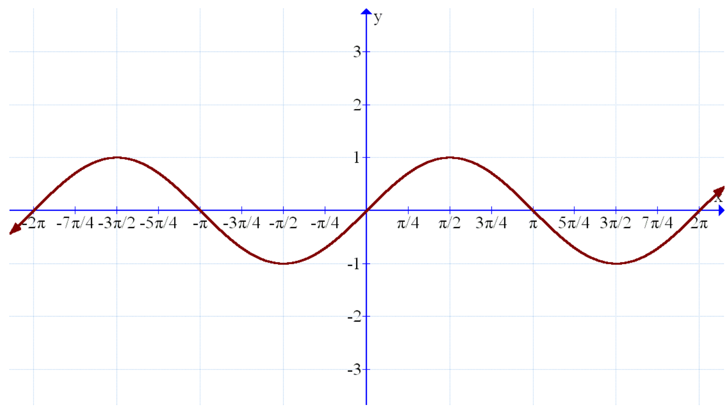


3.9 Inverse Trigonometric Functions

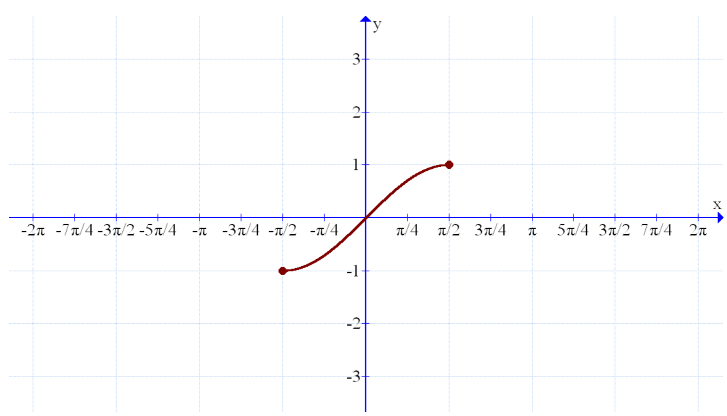
In this section, we will define the inverse sine, inverse cosine, and inverse tangent functions. And, after providing definitions of these functions, we will discuss some calculus concepts involving these functions.

3.9.1 Definitions of the Inverse Trigonometric Functions

Consider $f(x) = \sin x$ on $(-\infty, \infty)$, as sketched below.



$f(x) = \sin x$ does not have an inverse function on the given domain as its graph fails the horizontal line test. For example $f\left(\frac{\pi}{2}\right) = f\left(\frac{5\pi}{2}\right) = 1$. If we were to invert this function, it is unclear where to send 1. Does the inverse function return 1 to $\theta = \frac{\pi}{2}$? Does it return 1 to $\theta = \frac{\pi}{2}$? Or, does it return 1 to any of the other angles whose sine is 1? To get around this issue, we will restrict the domain carefully to have a one-to-one function. Specifically, it is the convention to restrict $f(x) = \sin x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to make it invertible.



On this interval, the range is the same as the unrestricted sine function; but, each y-value is the result of only one x-value. So, we are now able to define the inverse sine function $f^{-1}(x) = \sin^{-1}(x) = \arcsin x$.

The following table summarizes the domain and range of the inverse sine function.

	Domain	Range
Restricted Sine	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$
$\sin^{-1}(x) = \arcsin x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Conceptually, $\sin^{-1}(x)$ is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x.

Example 3.9.1 *Evaluate each of the following.*

1. $\sin^{-1}(1)$

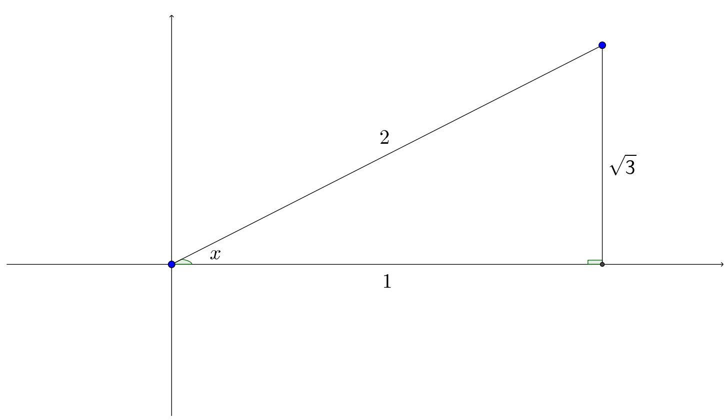
   Solution:

   $\sin^{-1}(1)$ *is the angle in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *whose sine is 1. This angle is* $x = \frac{\pi}{2}$.

2. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

   Solution:

   $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ *is the angle in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *whose sine is* $\frac{\sqrt{3}}{2}$. *The terminal ray of this angle must be in the first quadrant; we can represent this with a triangle where the "opposite side" is labeled* $\sqrt{3}$ *and the hypotenuse is labeled* 2, *as in the diagram below.*

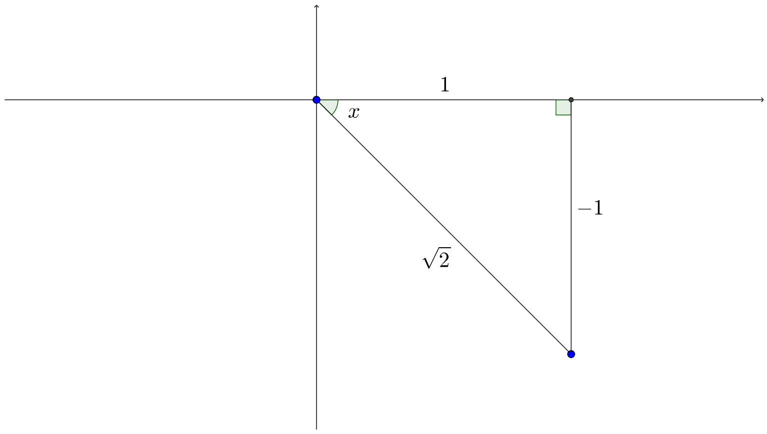


   Since this is a $30 - 60 - 90$ *triangle, it follows that* $x = \frac{\pi}{3}$.

3. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

   Solution:

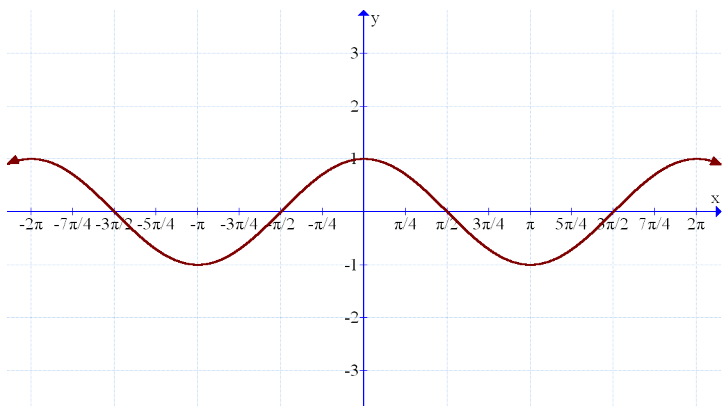
   $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ *is the angle in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *whose sine is* $-\frac{1}{\sqrt{2}}$. *The terminal ray of this angle must be in the fourth quadrant; we can represent this with a triangle where the "opposite side" is labeled* -1 *and the hypotenuse is labeled* $\sqrt{2}$, *as in the diagram below.*



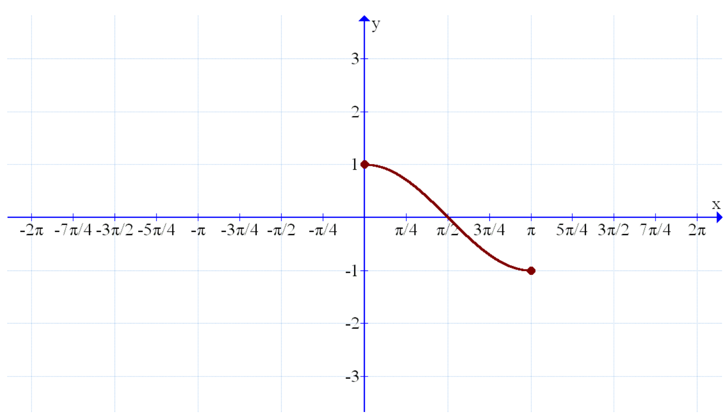
Since this is a $45-45-90$ *triangle, it follows that* $x = -\frac{\pi}{4}$.

□

In a similar way, we can define the inverse cosine function, $\cos^{-1}(x) = \arccos x$. As before, $f(x) = \cos x$ does not have an inverse function on the domain $(-\infty, \infty)$ as its graph fails to be one-to-one.



This time, the convention is to restrict $f(x) = \cos x$ to the interval $[0, \pi]$, as in the diagram below.

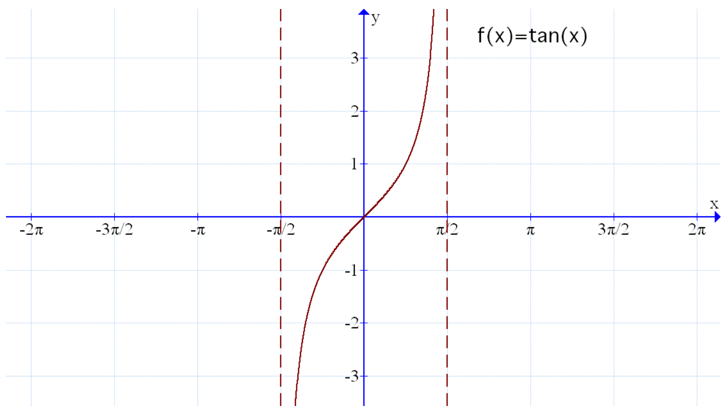


On this interval, the range is the same as the unrestricted cosine function; but, each y-value is the result of only one x-value. So, we are now able to define the inverse cosine function $f^{-1}(x) = \cos^{-1}(x) = \arccos x$. The following table summarizes the domain and range of the inverse sine function.

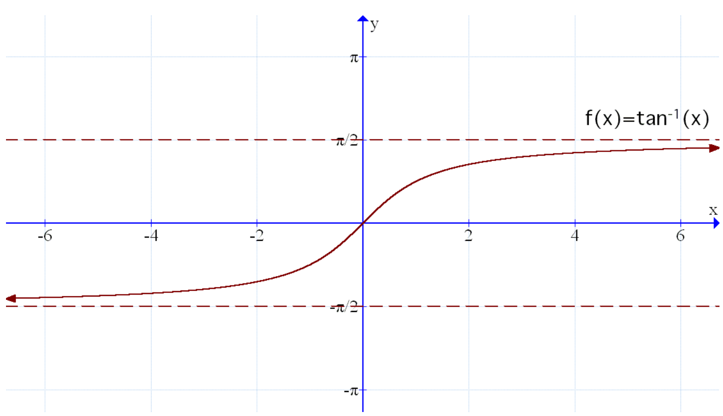
	Domain	Range
Restricted Cosine	$[0, \pi]$	$[-1, 1]$
$\cos^{-1}(x) = \arccos x$	$[-1, 1]$	$[0, \pi]$

Conceptually, $\cos^{-1}(x)$ is the angle in $[0, \pi]$ whose cosine is x.

Finally, we define the inverse tangent function. As usual, we must restrict the domain of $f(x) = \tan x$ in order to properly define its inverse function. The convention is to restrict the domain of $f(x) = \tan x$ to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, as shown in the following figure. On this domain, the tangent function is one-to-one.



In fact, we can graph the inverse tangent function by reflecting this graph over the line $y = x$. Notice that the vertical asymptotes of $f(x) = \tan x$ will become horizontal asymptotes for its inverse function.



Conceptually, $\tan^{-1}(x)$ is the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

The following table summarizes the domain and range of the inverse tangent function.

	Domain	Range
Restricted Tangent	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(-\infty, \infty)$
$\tan^{-1}(x) = \arctan x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example 3.9.2 *Evaluate each of the following.*

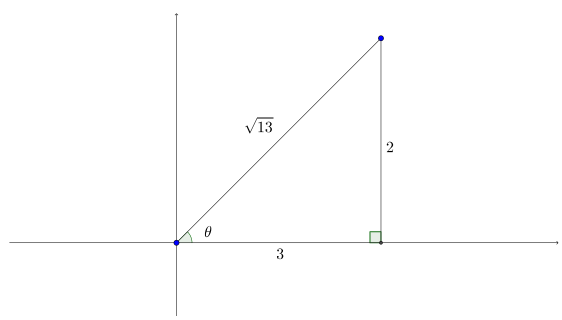
1. $\arctan 1$

   Solution:

   $\arctan 1$ *is the angle in* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *whose tangent is 1. This angle is* $x = \frac{\pi}{4}$.

2. $\sin\left(2\tan^{-1}\left(\frac{2}{3}\right)\right)$ **Solution:**

   First, recognize that $\tan^{-1}\left(\frac{2}{3}\right)$ *is the angle* θ *in the interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *whose tangent is* $\frac{2}{3}$. *Since this is not one of our standard angles, it may be helpful to represent this with an appropriate triangle. Since tangent is positive, the terminal ray of this angle is in the first quadrant.*



   Next, we will apply the double angle formula for sine:

$$\begin{aligned}
\sin\left(2\tan^{-1}\left(\frac{2}{3}\right)\right) &= 2\sin\left(\tan^{-1}\left(\frac{2}{3}\right)\right)\cos\left(\tan^{-1}\left(\frac{2}{3}\right)\right) && \textit{By the double angle formula} \\
&= 2\left(\frac{2}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right) && \textit{Because } \sin\theta = \frac{2}{\sqrt{13}} \textit{ and } \cos\theta = \frac{3}{\sqrt{13}} \\
&= \frac{12}{13}
\end{aligned}$$

$\square$

Example 3.9.3 *Find all solutions to the equation* $(1 - 2\cos\theta)(3 - 4\cos\theta) = 0$ *in the interval* $[0, 2\pi]$.

Solution:

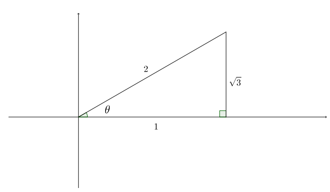
To solve this equation, we recognize that either $1 - 2\cos\theta = 0$ *or* $3 - 4\cos\theta = 0$. *So, we will solve these equations one at a time.*

- *First, we will solve* $1 - 2\cos\theta = 0$*:*

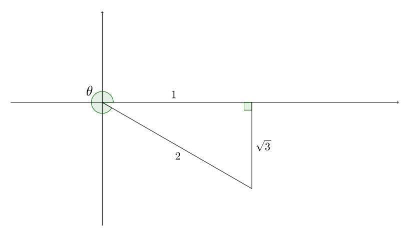
$$1 - 2\cos\theta = 0 \iff \cos\theta = \frac{1}{2}$$

Since $\cos x > 0$, *we know that the terminal side of angle* x, *when measured from the positive* x*-axis, must lie in either quadrant I or IV. We sketch the appropriate triangles below, labeling all sides using the Pythagorean Theorem.*

Quadrant I



Quadrant IV



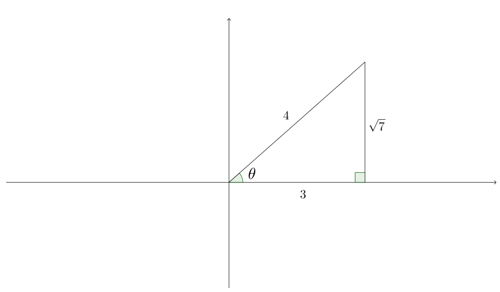
Since these side lengths correspond to a 30-60-90 triangle, we recognize the reference angle as $\theta = \frac{\pi}{3}$. *Thus, the two answers are* $\theta = \frac{\pi}{3}$ *and* $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

- *Next, we will solve* $3 - 4\cos\theta = 0$*:*

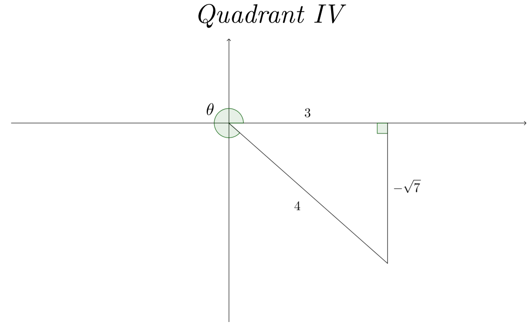
$$3 - 4\cos\theta = 0 \iff \cos\theta = \frac{3}{4}$$

There are definitely solutions to this equation since $\frac{3}{4}$ *is in the range of* $f(\theta) = \cos\theta$*.Since* $\cos x > 0$, *we know that the terminal side of angle* x, *when measured from the positive* x*-axis, must lie in either quadrant I or IV. We sketch the appropriate triangles below, labeling all sides using the Pythagorean Theorem.*

Quadrant I



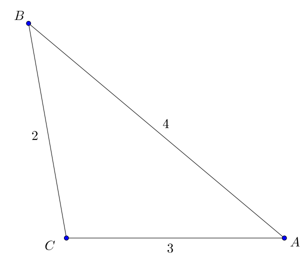
Quadrant IV



Unfortunately, this time are not familiar with the exact value of this angle. Instead, we will express the answers in terms of inverse trigonometric functions. Specifically, since the inverse cosine has a range of $[0, \pi]$, $\theta = \arccos\left(\frac{3}{4}\right)$ *corresponds to the angle in quadrant I. The angle* $\theta = 2\pi - \arccos\left(\frac{3}{4}\right)$ *corresponds to the ray in quadrant IV.*

In summary, the four solutions in the interval $[0, 2\pi]$ *are* $\frac{\pi}{3}$, $\frac{5\pi}{3}$, $\arccos\left(\frac{3}{4}\right)$, *and* $2\pi - \arccos\left(\frac{3}{4}\right)$ □

Example 3.9.4 *Find the angles at vertices A, B, and C for the triangle shown below.*



Solution:

Let $a = 2$, $b = 3$ *and* $c = 4$. *Let A, B, and C be the angles at the corresponding vertices. We will find angle C by using the law of cosines:*

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$16 = 4 + 9 - 2(2)(3)\cos C$$
$$-\frac{1}{4} = \cos C$$
$$C = \cos^{-1}\left(-\frac{1}{4}\right)$$

Notice that since the range of $f(x) = \cos^{-1} x$ *is* $[0, \pi]$, *the angle that is returned will always be in the upper half plane (Quadrants I or II) when drawn in standard position. And, since we are taking the inverse cosine of a negative number, the angle which is returned will be in quadrant II. As a result,* $\cos^{-1}\left(-\frac{1}{4}\right)$ *is an obtuse angle. Next, we can use the law of cosines to calculate angle B:*

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$9 = 4 + 16 - 2(2)(4)\cos B$$
$$\frac{11}{16} = \cos B$$
$$B = \cos^{-1}\left(\frac{11}{16}\right)$$

Finally, we could use the law of cosines a third time to find angle A. However, since all of the angles must add up to π, we can obtain angle A as follows:

$$A = \pi - \cos^{-1}\left(-\frac{1}{4}\right) - \cos^{-1}\left(\frac{11}{16}\right)$$

$\square$

3.9.2 Limits & Continuity with the Inverse Trigonometric Functions

Now that we have defined some of the inverse trigonometric functions, let us discuss move to calculus with them. It turns out that the inverse trigonometric functions are continuous on their respective domains. We will not prove this here; but, we will use this result to calculate some limits.

Example 3.9.5 *Compute the following limits:*

(a) $\lim_{x\to 1/2} \sin^{-1}(x)$

Solution:

$$\begin{aligned} \lim_{x\to 1/2} \sin^{-1}(x) &= \sin^{-1}\left(\frac{1}{2}\right) && \textit{Because } \sin^{-1}(x) \textit{ is continuous at } x = \frac{1}{2} \\ &= \frac{\pi}{6} && \textit{Because } \frac{\pi}{6} \textit{ is the angle in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \textit{ whose sine is } \frac{1}{2} \end{aligned}$$

(b) $\lim_{x\to\infty} \cos^{-1}\left(\frac{x}{1-2x}\right)$

Solution:

It may be useful to make a substitution. Let $t = \dfrac{x}{1-2x}$. Notice that

$$\lim_{x\to\infty} t = \lim_{x\to\infty} \frac{x}{1-2x} = -\frac{1}{2}$$

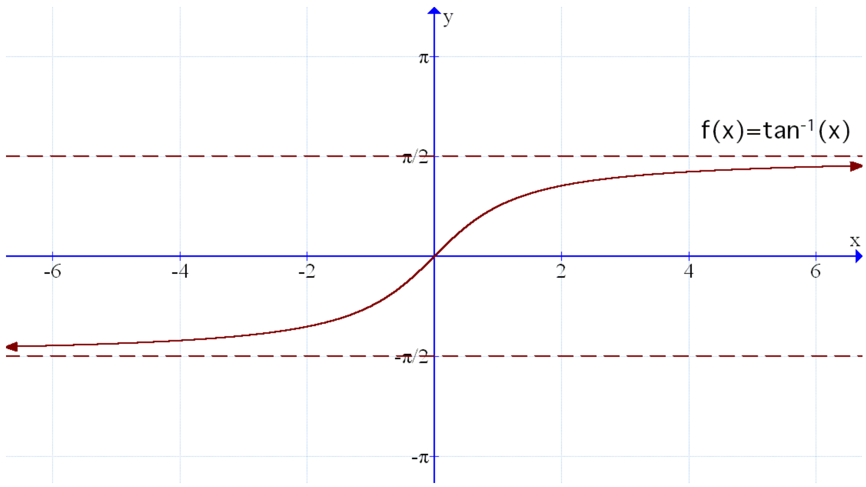
Thus, as $x \to \infty$, we have $t \to -\dfrac{1}{2}$. Thus, we make the change of variables as follows:

$$\begin{aligned} \lim_{x\to\infty} \cos^{-1}\left(\frac{x}{1-2x}\right) &= \lim_{t\to -1/2} \cos^{-1} t \\ &= \cos^{-1}\left(-\frac{1}{2}\right) && \textit{Because } \cos^{-1} x \textit{ is continuous at } x = -\frac{1}{2} \\ &= \frac{2\pi}{3} && \textit{Because } \frac{2\pi}{3} \textit{ is the angle in } [0,\pi] \textit{ whose cosine is } -\frac{1}{2} \end{aligned}$$

(c) $\lim_{x\to\infty} \tan^{-1} x$

Solution:

Recall from earlier that the graph of $y = \arctan x$ *is as follows:*



Thus, it appears that $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$.

(d) $\lim_{x\to 0^+} \tan^{-1}\left(\frac{1}{x}\right)$

Solution:

In this case, it is recommended to make a substitution. Let $t = \frac{1}{x}$. *Notice that*

$$\lim_{x\to 0^+}(t) = \lim_{x\to 0^+}\left(\frac{1}{x}\right) = +\infty$$

So, as $x \to 0^+$, *we have* $t \to +\infty$. *Thus, we make the change of variables as follows:*

$$\begin{aligned}\lim_{x\to 0^+} \tan^{-1}\left(\frac{1}{x}\right) &= \lim_{t\to+\infty} \tan^{-1}(t)\\ &= \frac{\pi}{2}\end{aligned}$$

□

3.9.3 Derivatives of the Inverse Trigonometric Functions

In this section, we will calculate the derivatives of the inverse sine, inverse cosine, and inverse tangent functions. Consider the following theorem:

Theorem 3.9.1 (Derivatives of the Inverse Trigonometric Functions) *Suppose x is in the domain of the given function, then:*

1. $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$
2. $\dfrac{d}{dx}\left(\cos^{-1}(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$
3. $\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{1+x^2}$

Proofs:

1. Suppose $y = \sin^{-1}(x)$. Recall that the domain is $-1 \le x \le 1$ and the range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. To calculate the derivative, we begin by recognizing that $y = \sin^{-1}(x) \iff \sin y = x$. We will use implicit differentiation:

$$\begin{aligned} \frac{d}{dx}(\sin y) &= \frac{d}{dx}(x) && \text{Take the derivative with respect to } x \text{ on both sides.} \\ \cos y \frac{dy}{dx} &= 1 && \text{Remember that } y \text{ is a function of } x. \\ \frac{dy}{dx} &= \frac{1}{\cos y} \end{aligned}$$

   To finish our calculation, we want to express the derivative solely as a function of the original variable x. Recognize the following:

$$\begin{aligned} \cos^2 y + \sin^2 y &= 1 \\ \cos^2 y + (x)^2 &= 1 && \text{Because } \sin y = x. \\ \cos^2 y &= 1 - x^2 \\ \cos y &= \pm\sqrt{1-x^2} \end{aligned}$$

   But, since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, it follows that the angle y is in the right half plane (quadrants I or IV). As a result, $\cos y > 0$ and we obtain $\cos y = \sqrt{1-x^2}$. Combining this with our earlier result gives us the derivative of the inverse sine function:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

   □

2. Suppose $y = \cos^{-1}(x)$. Recall that the domain is $-1 \le x \le 1$ and the range is $0 \le y \le \pi$. To calculate the derivative, we begin by recognizing that $y = cos-1(x) \iff \cos y = x$. We will use

implicit differentiation:

$$\begin{aligned}
\frac{d}{dx}(\cos y) &= \frac{d}{dx}(x) && \text{Take the derivative with respect to } x \text{ on both sides.}\\
-\sin y\frac{dy}{dx} &= 1 && \text{Remember that } y \text{ is a function of } x.\\
\frac{dy}{dx} &= \frac{-1}{\sin y}
\end{aligned}$$

To finish our calculation, we want to express the derivative solely as a function of the original variable x. Recognize the following:

$$\begin{aligned}
\cos^2 y + \sin^2 y &= 1\\
(x)^2 + \sin^2 y &= 1 && \text{Because } \cos y = x.\\
\sin^2 y &= 1 - x^2\\
\sin y &= \pm\sqrt{1-x^2}
\end{aligned}$$

But, since $0 \leq y \leq \pi$, it follows that the angle y is in the upper half plane (quadrants I or II). As a result, $\sin y > 0$ and we obtain $\sin y = \sqrt{1-x^2}$. Combining this with our earlier result gives us the derivative of the inverse sine function:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

□

3. Suppose $y = \tan^{-1}(x)$. Recall that the domain is $-\infty \leq x \leq \infty$ and the range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. To calculate the derivative, we begin by recognizing that $y = \tan -1(x) \iff \tan y = x$. We will use implicit differentiation:

$$\begin{aligned}
\frac{d}{dx}(\tan y) &= \frac{d}{dx}(x)\\
\sec^2 y\frac{dy}{dx} &= 1\\
\frac{dy}{dx} &= \frac{1}{\sec^2 y}
\end{aligned}$$

To finish our calculation, we want to express the derivative solely as a function of the original variable x. Recognize the following:

$$\begin{aligned}
1 + \tan^2 y &= \sec^2 y\\
1 + x^2 &= \sec^2 y && \text{Because } \tan y = x.
\end{aligned}$$

Combining this with our earlier result gives us the derivative of the inverse sine function:

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$\square$

As usual, we can use the product rule, quotient rule, and chain rule (where appropriate).

Example 3.9.6 *Find all values of x at which the tangent line to the graph of $f(x) = \tan^{-1}(2x)$ will be parallel to the line $y = x + 3$.*

Solution:

The slope of the given line is 1. So, any parallel line must also have slope 1. As a result, we must find all x at which $f'(x) = 1$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\tan^{-1}(2x)\right) \\ &= \frac{1}{1+(2x)^2}\frac{d}{dx}(2x) && \textit{By the chain rule.} \\ &= \frac{1}{1+4x^2}(2) \\ &= \frac{2}{1+4x^2} \end{aligned}$$

So, we can answer the question by solving the following equation:

$$\begin{aligned} f'(x) &= 1 \\ \frac{2}{1+4x^2} &= 1 \\ 2 &= 1 + 4x^2 && \textit{After multiplying both sides of the equation by } 1+4x^2. \\ x^2 &= \frac{1}{4} \\ x &= \pm\frac{1}{2} \end{aligned}$$

Thus, the tangent lines to the given function will be parallel to $y = x$ when $x = \pm\frac{1}{2}$. $\square$

3.9.4 (Optional) Derivatives of Inverse Functions

Suppose $f(x)$ and $f^{-1}(x)$ are inverse functions. Then, it follows from the definition that $f\left(f^{-1}(x)\right) = x$ for all x in the domain of $f^{-1}(x)$. Furthermore, suppose that these functions are differentiable. We differentiate both sides of the equation with respect to x:

$$\begin{aligned}
\frac{d}{dx}\left[f\left(f^{-1}(x)\right)\right] &= \frac{d}{dx}(x) && \\
f'\left(f^{-1}(x)\right)\frac{d}{dx}\left[f^{-1}(x)\right] &= 1 && \text{By the chain rule} \\
\frac{d}{dx}\left[f^{-1}(x)\right] &= \frac{1}{f'\left(f^{-1}(x)\right)} && \text{Provided } f'\left(f^{-1}(x)\right) \neq 0.
\end{aligned}$$

Hence, we arrive at the following theorem:

Theorem 3.9.2 (Inverse Function Theorem) *Suppose f and f^{-1} are inverse functions. Furthermore, suppose that these functions are continuously differentiable with $f'\left(f^{-1}(x)\right) \neq 0$. Then,*

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

We can use this theorem to calculate the derivative of the inverse trigonometric functions, as in the following example:

Example 3.9.7 *Use theorem 3.9.2 to calculate* $\frac{d}{dx}\left(\tan^{-1}(x)\right)$

Solution:

Suppose $f(x) = \tan x$ *and* $f^{-1}(x) = \tan^{-1}(x)$. *Then, since* $f'(x) = \sec^2(x)$, *it follows that:*

$$\begin{aligned}
f'\left(f^{-1}(x)\right) &= f'\left(\tan^{-1}(x)\right) && \\
&= \sec^2\left(\tan^{-1}(x)\right) && \\
&= 1 + \tan^2\left(\tan^{-1}(x)\right) && \textit{Because } 1 + \tan^2 x = \sec^2 x. \\
&= 1 + x^2 && \textit{Because } \tan\left(\tan^{-1}(x)\right) = x.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{d}{dx}\left(f^{-1}(x)\right) &= \frac{1}{f'\left(f^{-1}(x)\right)} \\
&= \frac{1}{1+x^2}
\end{aligned}$$

which is the desired result. □

And, you can use this result to verify the derivatives of the other inverse trigonometric functions which were discussed in section 3.9.3.

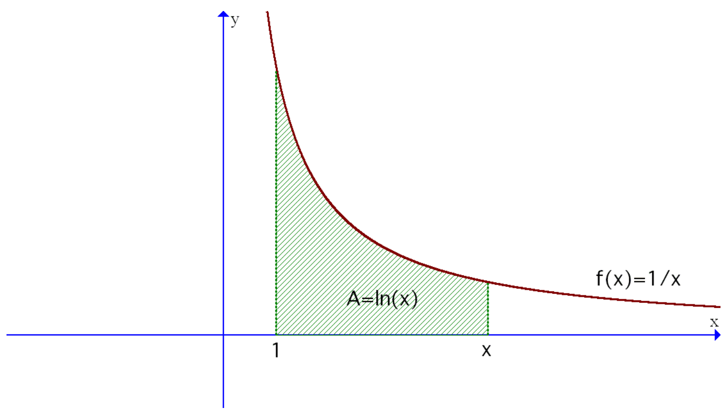
4 Exponential & Logarithmic Functions

4.1 Natural Logarithm & The Number e

4.1.1 Defining the Natural Logarithm

We begin this section by defining the natural logarithm function. Our definition may be different from other definitions that you have seen before; however, we hope that you can reconcile those differences by the end of this chapter. The definition listed below is a geometric definition of the natural logarithm function.

Definition 4.1.1 (Natural Logarithm) *The* **Natural Logarithm** *of a positive number x, written $\ln x$, is the area under the graph of $f(x) = \dfrac{1}{x}$ from 1 to x.*



Note: If x is in the interval $(1, \infty)$, then $\ln x > 0$. However, if x is in the interval $(0, 1)$, then $\ln x < 0$.

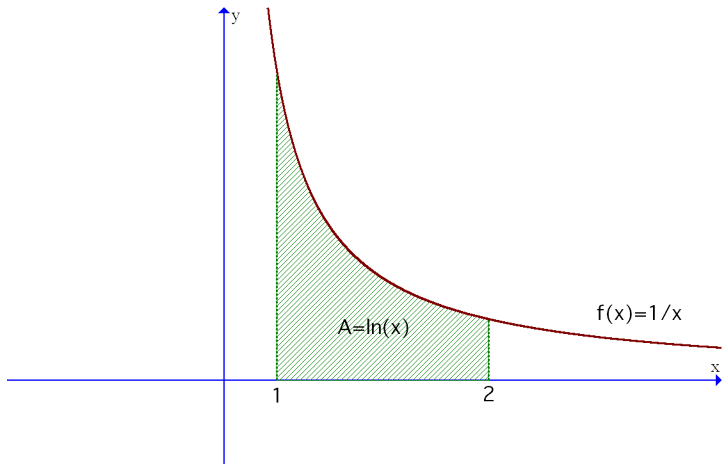
The domain of the $f(x) = \ln x$ is $(0, \infty)$. And, in the future, we will show that the range is $(-\infty, \infty)$. Before showing you some of the properties, let us practice evaluating this functions.

Example 4.1.1 *Evaluate* $\ln 1$

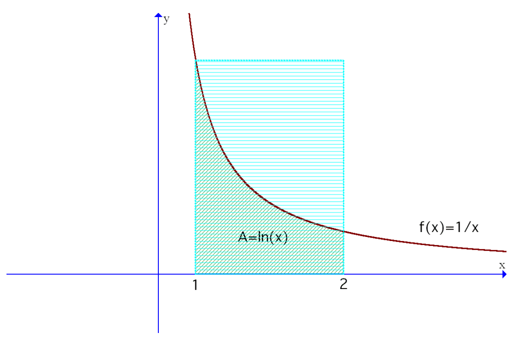
Solution: *By definition,* $\ln(1)$ *is the area under the graph of* $f(x) = \frac{1}{x}$ *from 1 to 1. This area is 0. Thus,* $\ln 1 = 0$. □

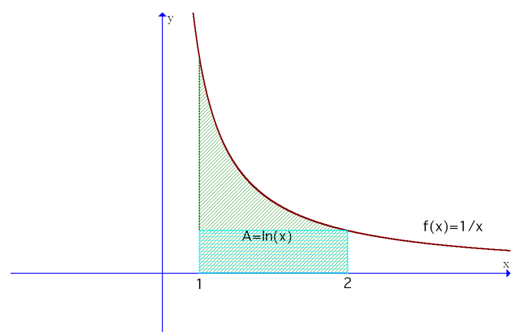
The following example shows how to approximate/bound the value of the natural logarithm function at a given point.

Example 4.1.2 *(a)* $\ln 2$ *is the area under* $\frac{1}{x}$ *from 1 to 2, as shown in the diagram below:*



To provide a some bounds on the value of $\ln 2$*, we will bound the green area between the area of two rectangles. Specifically, we will use the left and right endpoints of the interval* $[1, 2]$ *to build rectangles which are bigger and smaller, respectively, to the shaded area. Consider the following two graphs:*

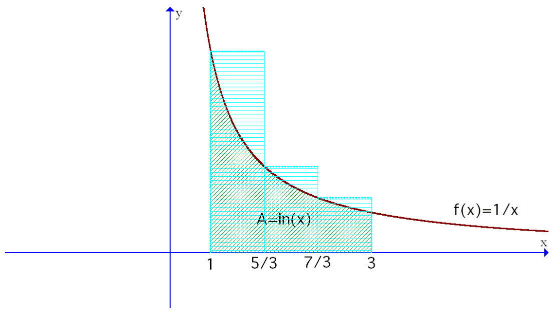


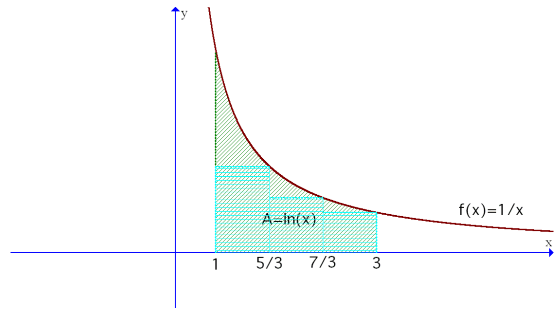


Notice that the height of $\frac{1}{x}$ *at the left endpoint* $x = 1$ *is 1. We take this as the representative height over the entire interval* $[1, 2]$*, which will give us an upper bound on the area represented by* $\ln 2$*. Specifically, if we calculate the area of the big blue rectangle in the left diagram, we get 1. So, we know that* $\ln 2 < 1$*. Similarly, the height of* $\frac{1}{x}$ *at the right endpoint of* $x = 2$ *is* $\frac{1}{2}$*. So, if we take this as the representative height on the entire interval* $[1, 2]$*, we clearly are underestimating the area represented by* $\ln 2$*, as depicted in the right picture above. The area of this blue rectangle is* $\frac{1}{2}$*. So,* $\ln 2 > \frac{1}{2}$*. Combining these results, we see that* $\frac{1}{2} < \ln 2 < 1$*.*

(b) Going through a similar analysis, we can provide upper and lower bounds for $\ln 3$*. Specifically, you can show* $\frac{2}{3} < \ln 3 < 2$

(c) *To get better bounds, you may use more than one rectangle. Let us use 3 rectangles of equal width to find a better bound on* $\ln 3$*. Since we are partitioning the interval* $[1, 3]$ *into 3 subintervals of equal width, each has to have a width of* $\frac{3-1}{3} = \frac{2}{3}$*, as demonstrated in the diagrams below:*





For the graph on the left, notice that we are using the left endpoints of the subintervals to calculate the heights of the rectangles. Specifically, the height of the first rectangle is $f(1) = 1$*. Similarly, the heights of the second and third rectangles in this diagram are* $f\left(\frac{5}{3}\right) = \frac{3}{5}$ *and* $f\left(\frac{7}{3}\right) = \frac{3}{7}$*. So, the total area enclosed by these three rectangles is:*

$$1 \cdot \frac{2}{3} + \frac{3}{5} \cdot \frac{2}{3} + \frac{3}{7} \cdot \frac{2}{3} = \frac{142}{105} \approx 1.352$$

For the graph on the right, notice that we are using the right endpoints of the subintervals to calculate the heights of the rectangles. Specifically, the height of the first rectangle is $f\left(\frac{5}{3}\right) = \frac{3}{5}$*. Similarly, the heights of the second and third rectangles in this diagram are* $f\left(\frac{7}{3}\right) = \frac{3}{7}$ *and* $f(3) = \frac{1}{3}$*. So, the total area enclosed by these three rectangles is:*

$$\frac{3}{5} \cdot \frac{2}{3} + \frac{3}{7} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{286}{315} \approx 0.9079$$

The calculation using left endpoints is clearly an overestimate of the area whereas the calculation using the right endpoint is clearly an underestimate of the area. This is clearly an under approximation of $\ln 3$*. As a result, we now know that* $0.9079 < \ln 3 < 1.352$*.*

(d) *If you were to repeat the process using 7 rectangles of equal width, you can show that the following is true:*

$$1.0093 < \ln 3 < 1.1998$$

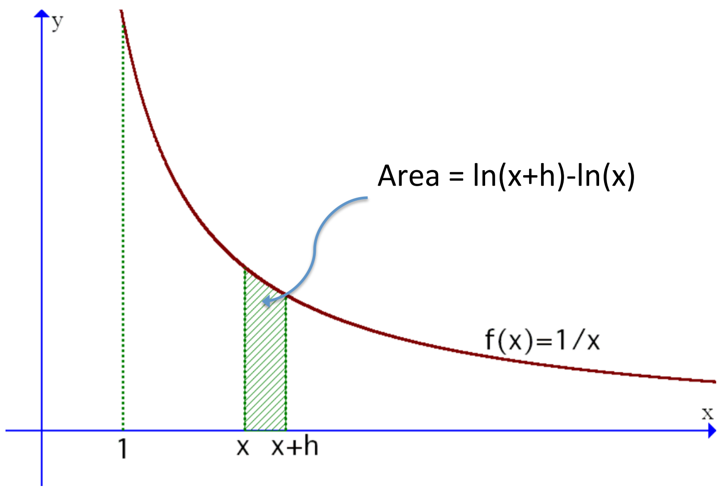
In the previous example, we showed how you can approximate and bound the value of $\ln 3$. If you wanted to get better bounds, you can repeat the process of using more rectangles as we did in parts (c) and (d). Eventually, you will realize that $\ln 3 \approx 1.0986$. You can use this same method to estimate $\ln c$ for any $c > 0$.

4.1.2 Derivative of The Natural Logarithm

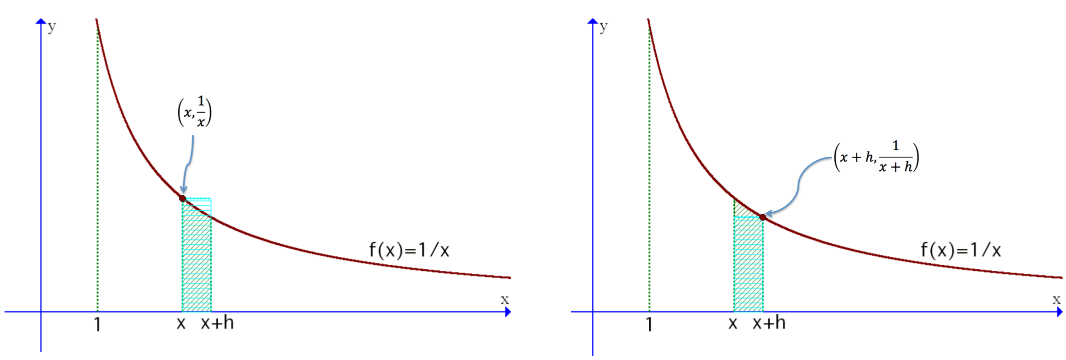
We begin this section by calculating the derivative of the Natural Log function, $f(x) = \ln x$. By definition, we know that, if the derivative exists, it will be given by the following limit:

$$\frac{d}{dx}(\ln x) = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

To evaluate this limit, we will appeal to the squeeze theorem. Recall that $\ln x$ was defined to be the area under the graph of $y = \frac{1}{x}$ from 1 to x. Similarly, $\ln(x+h)$ is the area under the graph of $y = \frac{1}{x}$ from 1 to $x + h$. So, $\ln(x+h) - \ln x$ this is the area under $y = \frac{1}{x}$ from x to $x + h$ as depicted in the diagram below (for $h > 0$).



We will bound this area using two rectangles. For the first rectangle, we will use the height at the left endpoint of the shaded region as the representative height. For the second rectangle, we will use the height at the right endpoint of the shaded region as the representative height.



From these images, we have the following inequality:

$$\text{Area of Smaller Rectangle} \le \ln(x+h) - \ln x \le \text{Area of Bigger Rectangle}$$
$$\left(\frac{1}{x}\right)(h) \le \ln(x+h) - \ln x \le \left(\frac{1}{x+h}\right)(h)$$

Thus, for $h > 0$, we have

$$\frac{1}{x} \le \frac{\ln(x+h) - \ln x}{h} \le \frac{1}{x+h}$$

Since $\lim_{h\to 0^+} \frac{1}{x} = \lim_{h\to 0^+} \frac{1}{x+h} = \frac{1}{x}$, it follows from the squeeze theorem that $\lim_{h\to 0^+} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x}$. A similar calculation shows that $\lim_{h\to 0^-} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x}$. This leads us to the following result:

Theorem 4.1.1 (Derivative of $\ln x$)

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Now that we have added the derivative of the natural logarithmic function to our vocabulary, let us calculate a few derivatives. Keep in mind that we should still appeal to product, quotient, and chain rules when appropriate.

Example 4.1.3 *(a) Suppose $y = \ln((2x+1))$. Determine the domain and then calculate $\frac{dy}{dx}$.*

Solution:

To calculate the domain, recall that we can only take the natural log of a positive number. Thus, we need to find the values of x which satisfy $2x+1 > 0$. Solving this inequality gives $x > -\frac{1}{2}$. So, the domain of this function is $\left(-\frac{1}{2}, \infty\right)$.

Now that we have determined the domain, we will calculate the derivative:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\ln(2x+1) \\ &= \frac{1}{2x+1} \cdot \frac{d}{dx}(2x+1) && \textit{By the chain rule.} \\ &= \frac{1}{2x+1}(2) \\ &= \frac{2}{2x+1} \end{aligned}$$

(b) Calculate an equation of the line which is tangent to the graph of $f(x) = x\ln x$ at $x = 1$.

Solution:

Notice that the point of tangency is $(1, f(1)) = (1, 0)$, since $\ln 1 = 0$. Next, we calculate the slope of *the tangent line at this point by calculating the derivative:*

$$\begin{aligned}
\frac{d}{dx}(x\ln x) &= (x)\frac{d}{dx}(\ln x) + \frac{d}{dx}(x)(\ln x) && \textit{By the product rule.}\\
&= (x)\left(\frac{1}{x}\right) + (1)(\ln x) && \textit{Because } \frac{d}{dx}(\ln x) = \frac{1}{x} \textit{ and } \frac{d}{dx}(x) = 1.\\
&= 1 + \ln x
\end{aligned}$$

Thus, the slope of the tangent line at the given point is $f'(1) = 1 + \ln 1 = 1$. And, the resulting equation is $y = x - 1$.

□

Theorem 4.1.2 *The function $f(x) = \ln x$ is continuous on its domain.*

Proof: Recall that the domain of $f(x) = \ln x$ is $(0, \infty)$. And, from Theorem 4.1.1, we know that $\frac{d}{dx}(\ln x) = \frac{1}{x}$. Since this derivative exists for all x in the domain, it follows that $f(x)$ is differentiable on its domain. Finally, recall from Math 116 that differentiability at a point implies continuity at that point. As a result, we can conclude that $f(x) = \ln x$ is continuous for all x in its domain. □

Next, we want to find a value of x for which $\ln x = 1$. First, let us argue that such an x actually exists. Notice that $f(x) = \ln x$ is continuous on its domain, by theorem 4.1.2. Furthermore, from parts (a) and (d) of the example 4.1.2, we know that $\ln 2 < 1$ and $\ln 3 > 1$. Hence, by the intermediate value theorem, there must be at least one $x = c$ in the interval $(2, 3)$ for which $f(x) = 1$. Furthermore, as we will see in the next section, $f(x) = \ln x$ is increasing on its domain. As a result, there can only be one such number. We will define this number to be e, Euler's constant.

Definition 4.1.2 (Euler's Constant) *Euler's constant, e, is the solution to $\ln x = 1$. It can be shown that e is an irrational number whose value is approximated as follows:*

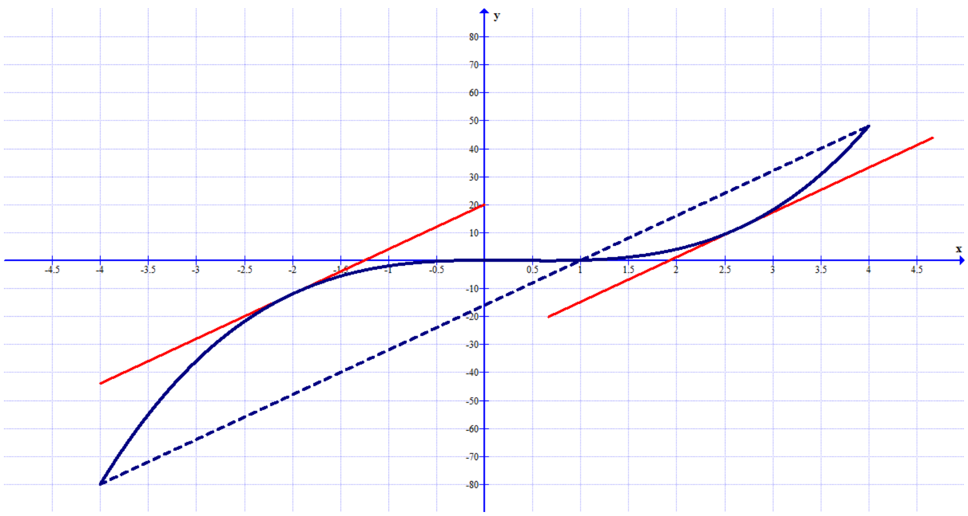
$$e \approx 2.718281828$$

4.1.3 Properties of the Natural Log Function

In order to derive certain useful properties of the natural log function, we will introduce some theorems which are consequences of the Mean Value Theorem:

Theorem 4.1.3 (Mean Value Theorem) *Suppose $f(x)$ is continuous on the closed interval $[a, b]$ and is differentiable on the open interval (a, b). Then, there must be at least one $x = c$ in the interval (a, b) at which $f'(c) = \frac{f(b)-f(a)}{b-a}$*

For a formal proof of this theorem, refer back to your notes from Math 116. Conceptually, this is saying that as long as the assumptions of the theorem are satisfied, then there will be at least one point in the interval at which the slope of the tangent line will be parallel to the slope of the secant line which passes through the endpoints of the interval, as demonstrated below.



Theorem 4.1.4 *Suppose $f'(x) = 0$ for all x in some interval I. Then, $f(x)$ is constant on the interval.*

Proof: Pick any two numbers a and b in the interval I, where $a < b$. Since, by assumption, we have $f(x)$ is differentiable for all x in I, we have the following:

- $f(x)$ is continuous on $[a, b]$.
- $f(x)$ is differentiable on (a, b).

Thus, by the Mean Value Theorem, there is a number c in the interval (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

But, $f'(x) = 0$ for all x in the interval I implies that $f'(c) = 0$. And, as a result, we have the following:

$$0 = f'(c)$$

$$0 = \frac{f(b) - f(a)}{b - a}$$

$$0 = f(b) - f(a)$$

$$f(b) = f(a)$$

Since we have chosen arbitrary points in the interval I and have shown that they have the same y-value, we have demonstrated that $f(x)$ is constant on the interval I. $\square$

Theorem 4.1.5 *Suppose $g'(x) = f'(x)$ for all x in an interval I. Then, for some constant c it follows that $g(x) = f(x) + c$ for all x in the interval I.*

Proof: Assume $g'(x) = f'(x)$ for all x in the interval I and define $h(x) = g(x) - f(x)$. Then, for all x in the interval I, it follows that $h'(x) = g'(x) - f'(x) = 0$. And, by the previous theorem, we know that $h(x) = c$, for some constant c. Thus, $g(x) - f(x) = c \implies g(x) = f(x) + c$. $\square$

We will now apply Theorem 4.1.5 to derive some useful properties of the natural logarithm function.

Theorem 4.1.6 (Properties of Natural Logarithm) *Suppose $M > 0$ Then:*

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln (Mx) = \ln M + \ln x$
4. $\ln \left(\frac{x}{M} \right) = \ln x - \ln M$

Selected Proofs:

3. Suppose M is a positive constant. Let $f(x) = \ln(Mx)$ and let $g(x) = \ln M + \ln x$. Notice the following:

   - By the chain rule, $f'(x) = \frac{d}{dx}(\ln(Mx)) = \frac{1}{Mx} \cdot M = \frac{1}{x}$
   - $g'(x) = \frac{d}{dx} (\ln M + \ln x) = 0 + \frac{1}{x} = \frac{1}{x}$

   Since $f'(x) = g'(x)$ for all x in $(0, \infty)$, it follows from Theorem 4.1.5 that $f(x) = g(x) + c$ for some constant c. That is, $\ln(Mx) = \ln M + \ln x + c$. Substitute in $x = 1$ to get:

$$\ln(M) = \ln M + \ln 1 + c$$

$$\ln(M) = \ln M + c \qquad \text{Because } \ln 1 = 0$$

$$c = 0$$

Thus, we conclude that $f(x) = g(x) + 0$ which gives us the desired result that $\ln(Mx) = \ln M + \ln x$.

$\square$

4. To prove (2), we can use the sum property of logs:

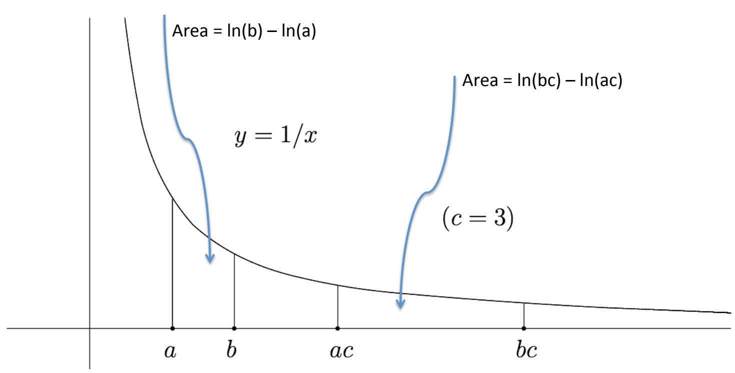
$$\ln\left(M \cdot \frac{x}{M}\right) = \ln(M) + \ln\left(\frac{x}{M}\right) \qquad \text{By the sum property of natural log}$$

$$\ln x = \ln M + \ln\left(\frac{x}{M}\right)$$

$$\ln\left(\frac{x}{M}\right) = \ln x - \ln M$$

$\square$

Interestingly, one can prove these statements geometrically as well. In 1647, Gregory of Saint-Vincent studied areas under the graph of $f(x) = \frac{1}{x}$ and noticed that the area on the interval $[a, b]$ is equal to the area on the interval $[ca, cb]$ for a positive number c, as shown below.



So, it follows that

$$\ln(bc) - \ln(ac) = \ln(b) - \ln(a)$$

Substituting in $a = 1$, $b = M$, $c = x$ gives us

$$\ln(Mx) - \ln(x) = \ln(M) - \ln(1)$$

and since $\ln(1) = 0$, we arrive at the summation property (property 3 in Theorem 4.1.6).

Theorem 4.1.7 (Exponent Property of Natural Logarithm) *For any real number r,*

$$\ln x^r = r \ln x$$

Proof: We will prove the statement for integer powers and rational powers. For now, we will not prove this fact for irrational powers.

1. Suppose N is an integer. we will show that $\ln x^N = N \ln x$.

   - **Case 1:** If $N = 0$, then $\ln x^N = \ln x^0 = \ln 1 = 0$ and $N \ln x = 0 \ln x = 0$. Thus, if $N = 0$, the conclusion holds.

   - **Case 2:** Suppose $N > 0$. Then, x^N is the product of N x's. That is, $x^N = \underbrace{x \cdot x \cdots x}_{N}$. We will apply the sum property N times:

   $$\ln x^N = \ln(\underbrace{x \cdot x \cdots x}_{N}) = \underbrace{\ln x + \ln x + \cdots + \ln x}_{N} = N \ln x$$

   Thus, the conclusion holds for positive integers N.

   - **Case 2:** Suppose $N < 0$. Then,

   $$\begin{aligned} \ln x^N &= \ln \left[\left(\frac{1}{x}\right)^{-N}\right] && \text{Because } \frac{1}{x} = x^{-1} \\ &= -N \ln \left(\frac{1}{x}\right) && \text{By case 2, since } -N > 0 \\ &= -N(\ln 1 - \ln x) && \text{By the difference property of the natural log function.} \\ &= -N(0 - \ln x) && \text{Because } \ln 1 = 0 \\ &= N \ln x \end{aligned}$$

   Thus, the conclusion holds for negative integers N.

   As a result of cases 1-3, we have justified the exponent property for all integer exponents. □

2. Next, we will justify the exponent rule for any rational exponent of the form $\frac{m}{n}$, where m and n are integers.

   - Consider the case where $m = 1$. We will show that $\ln x^{1/n} = \frac{1}{n} \ln x$:

   $$\begin{aligned} n \ln(x^{1/n}) &= \left[\left(x^{1/n}\right)^n\right] && \text{Because the exponent rule has been proven for integers.} \\ &= \ln x && \text{Because } (x^{1/n})^n = x \end{aligned}$$

Thus, dividing both sides by n gives $\ln(x^{1/n}) = \frac{1}{n}\ln x$, as desired.

- Finally, we will show that $\ln(x^{m/n}) = \frac{m}{n}\ln x$.

$$\begin{aligned}
\ln(x^{m/n}) &= \ln\left[(x^{1/n})^m\right] && \text{Because } (a^b)^c = a^{bc} \\
&= m\ln(x^{1/n}) && \text{Because the exponent rule has been proven for integers.} \\
&= \frac{m}{n}\ln x && \text{By the previous bullet point.}
\end{aligned}$$

Thus, we can conclude that for any rational number $\frac{m}{n}$, we have $\ln(x^{m/n}) = \frac{m}{n}\ln x$, as desired.

□

Example 4.1.4 *Use properties of logs to rewrite* $\ln\left(\frac{a^2b}{\sqrt{c}}\right)$ *as the sum/difference of simpler logarithmic expressions. Assume that a, b, and c are all positive.*

Solution:

$$\begin{aligned}
\ln\left(\frac{a^2b}{\sqrt{c}}\right) &= \ln(a^2b) - \ln(\sqrt{c}) && \textit{Because } \ln\frac{x}{M} = \ln x - \ln M. \\
&= \ln(a^2) + \ln b - \ln\sqrt{c} && \textit{Because } \ln Mx = \ln M + \ln x. \\
&= \ln(a^2) + \ln b - \ln c^{1/2} && \\
&= 2\ln a + \ln b - \frac{1}{2}\ln c && \textit{Because } \ln x^r = r\ln x
\end{aligned}$$

□

Example 4.1.5 (Logarithmic Differentiation) *Differentiate the following functions:*

(a) $y = \ln\left(\frac{e^x\sqrt{x^2+3}}{4x-7}\right)$

Solution: *We could differentiate by using a long combination of chain, product, and quotient rules. Instead, let us use properties of logs to simplify the function into a form which is easier to differentiate.*

$$\begin{aligned}
y &= \ln\left(\frac{e^x\sqrt{x^2+3}}{4x-7}\right) && \\
y &= \ln\left(e^x\sqrt{x^2+3}\right) - \ln(4x-7) && \textit{Because } \ln\left(\frac{M}{N}\right) = \ln M - \ln N \\
y &= \ln e^x + \ln\left(\sqrt{x^2+3}\right) - \ln(4x-7) && \textit{Because } \ln(MN) = \ln M + \ln N \\
y &= x\ln e + \frac{1}{2}\ln(x^2+3) - \ln(4x-7) && \textit{Because } \ln(M^p) = p\ln M \\
y &= x + \frac{1}{2}\ln(x^2+3) - \ln(4x-7) && \textit{Because } \ln e = 1
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{d}{dx}\left[\ln\left(\frac{e^x\sqrt{x^2+3}}{4x-7}\right)\right] &= \frac{d}{dx}\left(x+\frac{1}{2}\ln(x^2+3)-\ln(4x-7)\right) \\
&= 1+\frac{1}{2}\left(\frac{1}{x^2+3}\right)(2x)-\left(\frac{1}{4x-7}\right)(4) \\
&= 1+\frac{x}{x^2+3}-\frac{4}{4x-7}
\end{aligned}$$

(b) $y = \dfrac{e^x\sqrt{x^2+3}}{4x-7}$

Solution:

Unlike part (a) we cannot simplify the right hand side down to something which is easier to differentiate because we are not able to exploit properties of logs. So, one might decide to introduce logs into the problem by taking the natural log of both sides:

$$\ln y = \ln\left(\frac{e^x\sqrt{x^2+3}}{4x-7}\right)$$

Now that we have introduced logs into the problem, let us use properties of logs to simplify the right hand side as we did in part (a):

$$\ln y = x+\frac{1}{2}\ln(x^2+3)-\ln(4x-7)$$

Next, we can differentiate both sides with respect to x*. Keep in mind that we are now differentiating implicitly:*

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}\left(x+\frac{1}{2}\ln(x^2+3)-\ln(4x-7)\right) \\
\frac{1}{y}\frac{dy}{dx} &= 1+\frac{x}{x^2+3}-\frac{4}{4x-7} \\
\frac{dy}{dx} &= y\left(1+\frac{x}{x^2+3}-\frac{4}{4x-7}\right)
\end{aligned}$$

And, finally, since we started with a function of x*, it would be ideal if our derivative were a function of* x *alone. So, we can substitute back in for* y *to get:*

$$\frac{dy}{dx} = \left(\frac{e^x\sqrt{x^2+3}}{4x-7}\right)\left(1+\frac{x}{x^2+3}-\frac{4}{4x-7}\right)$$

The technique that we used in example 4.1.5b is called **Logarithmic Differentiation**. To summarize what we did:

1. Take the natural log of both sides.
2. Simplify the right hand side using rules of logs.
3. Differentiate both sides with respect to x. Remember that you are using implicit differentiation.
4. Solve for $\frac{dy}{dx}$.
5. Substitute back in for y so that your end result is only a function of x.

This is a useful technique to remember when you have many products and quotients in the original function because logs have properties that help us break these apart into simpler terms. It is also useful to use logarithmic differentiation when you have unknowns in the exponent. Consider the following example:

Example 4.1.6 *Calculate* $\dfrac{dy}{dx}$ *if* $y = x^x$

Solution:

Since we have a variable in the exponent, we may want to use properties of logs to pull it down. So, let us use logarithmic differentiation.

$$
\begin{aligned}
\ln y &= \ln x^x && \textit{Take the natural log of both sides} \\
\ln y &= x \ln x && \textit{Use properties of logs to simplify the right hand side} \\
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(x \ln x) && \textit{Differentiate both sides with respect to } x \\
\frac{1}{y} \cdot \frac{dy}{dx} &= x\left(\frac{1}{x}\right) + (1)(\ln x) && \textit{Notice the product rule} \\
\frac{dy}{dx} &= y\,(1 + \ln x) && \textit{Solve for } \frac{dy}{dx} \\
\frac{dy}{dx} &= x^x\,(1 + \ln x) && \textit{Substitute back in for } y
\end{aligned}
$$

Example 4.1.7 *In this example, we will derive additional properties of the natural log function which are consequences of the properties that we have already discovered. Then we will combine all of this information to sketch the graph of $f(x) = \ln x$.*

Show that $f(x) = \ln x$ is always increasing on its domain of $(0, \infty)$.

Solution:

Recall that $f'(x) = \frac{1}{x}$. Since $f'(x) > 0$ for all x in the domain of $f(x) = \ln x$, it follows that $f(x)$ is always increasing.

(b) *Show that the graph of $f(x) = \ln(x)$ is always concave down.*

Solution: *Notice that $f''(x) = -\frac{1}{x^2}$ Since $f''(x) < 0$ for all x in the domain of $f(x)$, it follows that the graph is always concave down.*

(c) *Show that the range of $f(x) = \ln x$ is $(-\infty, \infty)$.*

Solution:

- *Let $b > 1$. Consider the sequence $x = b^n$. As n approaches infinity, it follows that $x = b^n$ also approaches infinity. Then,*

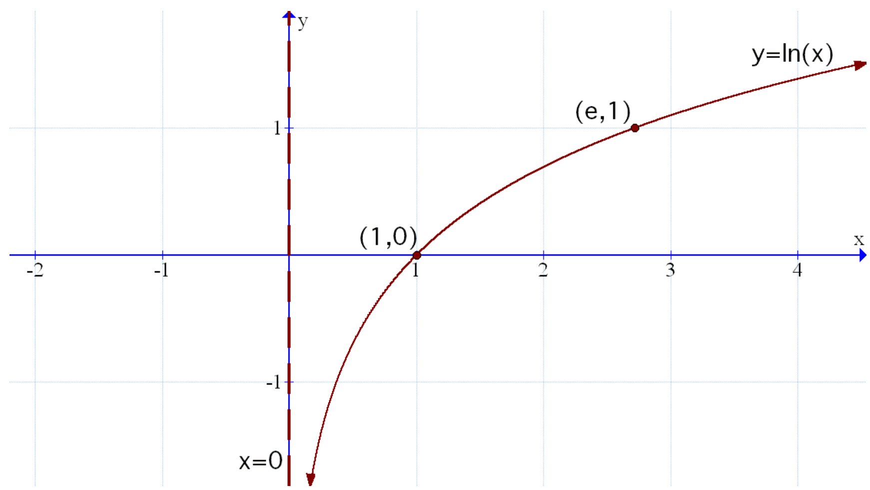
$$\begin{aligned}
\lim_{x\to\infty} \ln x &= \lim_{n\to\infty} \ln b^n \\
&= \lim_{n\to\infty} n \ln b && \textit{Because of the exponent rule of natural log.} \\
&= +\infty && \textit{Since } \ln b > 0 \textit{ by definition.}
\end{aligned}$$

- *Let $0 < c < 1$. Consider the sequence $x = c^n$. As n approaches infinity, it follows that $x = c^n$ approaches 0. Then,*

$$\begin{aligned}
\lim_{x\to 0^+} \ln x &= \lim_{n\to 0^+} \ln c^n \\
&= \lim_{n\to 0^+} n \ln c && \textit{Because of the exponent rule of natural log.} \\
&= -\infty && \textit{Since } \ln c < 0 \textit{ by definition.}
\end{aligned}$$

Since the function is continuous, it will take on all intermediate values as well. That is, the range of $\ln x$ is $(-\infty, \infty)$. Notice that in showing this result, we have also shown that $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to+\infty} \ln x = +\infty$. In addition, we now know that the graph of $f(x) = \ln x$ will have a vertical asymptote of $x = 0$.

(d) Use parts (a)-(c) and previous theorems/examples to graph $f(x) = \ln x$ sketch all intersections with the coordinate axes.



4.1.4 Exponential Function

Since the graph of $f(x) = \ln x$ is strictly increasing on its domain, it follows that the $f(x)$ is one-to-one. As a result, it is an invertible function. We define its inverse as follows:

Definition 4.1.3 (Exponential Function) *The exponential function, $f(x) = e^x$, is the inverse of the natural logarithm $f^{-1}(x) = \ln x$.*

Example 4.1.8 *Determine the domain and range of $f(x) = e^x$.*

Solution:

Since the exponential function is defined as the inverse of the natural logarithm function, it follows that:

- *The domain of the natural logarithm function will be the range of the exponential function.*
- *The range of the natural logarithm function will be the domain of the exponential function.*

The results are summarized in the table below:

	Domain	*Range*
$\ln x$	$(0, \infty)$	$(-\infty, \infty)$
e^x	$(-\infty, \infty)$	$(0, \infty)$

$\square$

Furthermore, recall that if f and f^{-1} are inverse functions, then:

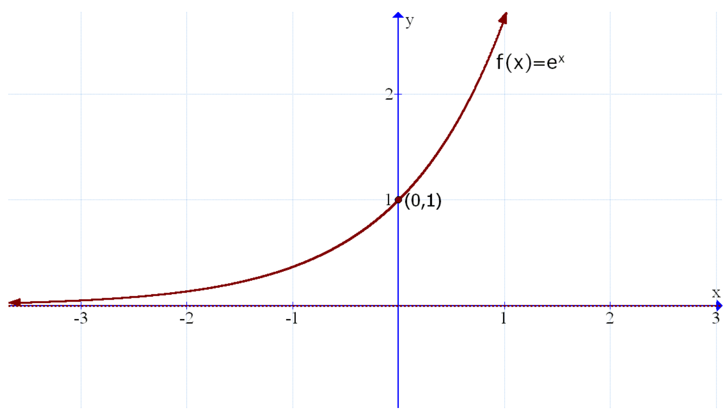
- $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}
- $f^{-1}(f(x)) = x$ for all x in the domain of f

As a result, we obtain the following property:

Theorem 4.1.8 *Because e^x and $\ln x$ are inverse functions:*

- $e^{\ln x} = x$ *for all x in the interval* $(0, \infty)$*.*
- $\ln(e^x) = x$ *for all x in the interval* $(-\infty, \infty)$*.*

In addition, if we reflect the graph of $y = \ln x$ over the line $y = x$, we can obtain the graph of $f(x) = e^x$, shown below:



As expected, the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$. Notice that there is a horizontal asymptote of $y = 0$. From this graph, we can intuitively figure out the end behavior limits:

Theorem 4.1.9 *The following are the end-behavior limits of $f(x) = e^x$:*

- $\lim_{x \to -\infty} e^x = 0$
- $\lim_{x \to +\infty} e^x = +\infty$

Example 4.1.9 *Solve the following equations:*

(a) $2^x = 7$

Solution:

To solve this equation, we will apply the natural log function so that we can use its exponent rule to bring down the exponent.

$$\begin{aligned} 2^x &= 7 \\ \ln(2^x) &= \ln(7) \\ x\ln 2 &= \ln 7 && \textit{Because } \ln x^r = r\ln x \\ x &= \frac{\ln 7}{\ln 2} \end{aligned}$$

(b) $\ln x - \ln(x+1) = 2$

Solution:

One way to begin is to combine the terms on the left side of the equation using rules of logs.

$$\begin{aligned} \ln x - \ln(x+1) &= 2 \\ \ln\left(\frac{x}{x+1}\right) &= 2 && \textit{Because } \ln\left(\frac{M}{N}\right) = \ln M - \ln N \\ e^{\ln\left(\frac{x}{x+1}\right)} &= e^2 \\ \frac{x}{x+2} &= e^2 && \textit{Because } e^{\ln x} = x \\ x &= e^2(x+1) \\ x - e^2 x &= e^2 \\ x(1-e^2) &= e^2 \\ x &= \frac{e^2}{1-e^2} \end{aligned}$$

When solving logarithmic equations, make sure that your solution satisfies domain requirements. In this particular problem, we need $x > 0$ *in order for* $\ln x$ *to be well defined. Since* $x = \frac{e^2}{1-e^2} < 0$ *the solution that we found is invalid and there are no solutions.* □

Example 4.1.10 *In a certain nutrient culture, a cell of the bacterium E. coli divides into two every 10 minutes. Let* $y = y(t)$ *be the number of cells that are present* t *minutes after 6 cells are placed in the culture. Assume that the growth of the bacteria obeys the exponential growth model* $y(t) = y_0 e^{kt}$*. Determine how many cells are present after 1 hour.*

Solution: *First, we will find the values of the parameters* y_0 *and* k*. Then, once we have the specific model, we can answer the question. Since* $y(0) = 6$*, we have*

$$y(0) = y_0 e^{k \cdot 0} = y_0 e^0 = y_0(1) = y_0$$

That is, we now know that $y(t) = 6e^{kt}$*. Next, we know that after 10 minutes, the initial population of cells will double. So,* $y(10) = 12$*. Substituting this in gives:*

$$\begin{aligned}
y(10) &= 6e^{10k} \\
12 &= 6e^{10k} && \text{\textit{Substitute in the point} } y(10) = 12 \\
2 &= e^{10k} && \text{\textit{Divide by the coefficient of 6.}} \\
\ln 2 &= \ln(e^{10k}) && \text{\textit{Take the natural log of both sides.}} \\
\ln 2 &= 10k && \text{\textit{Because} } \ln(e^x) = x \\
\frac{\ln 2}{10} &= k
\end{aligned}$$

Hence, we know that $y(t) = 6e^{\frac{t \ln 2}{10}}$*. We can simplify this as follows:*

$$\begin{aligned}
y(t) &= 6e^{\frac{t \ln 2}{10}} \\
&= 6e^{\frac{t}{10} \ln 2} \\
&= 6e^{\ln\left(2^{t/10}\right)} && \text{\textit{By the exponent rule for natural logarithm.}} \\
&= 6\left(2^{t/10}\right) && \text{\textit{Because} } e^{\ln x} = x
\end{aligned}$$

So, the simplified form is $y(t) = 6\left(2^{t/10}\right)$*, for* $t \geq 0$*. To determine the number of cells present after 1 hours, we must evaluate* $y(60)$*. Notice that we have to be careful of the units!* $y(60) = 6(2^{60/10}) = 6(2^6) = 384$ *bacteria cells.* □

Example 4.1.11 *Find all value(s) of* x *at which the graph of* $f(x) = x \ln x$ *has horizontal tangent lines.*

Solution: *To find the locations of the horizontal tangent lines, we need to find all value(s) of* x *at which* $f'(x) = 0$*. We differentiate* $f(x)$ *by appealing to the product rule:*

$$\begin{aligned}
f'(x) &= x \frac{d}{dx}(\ln x) + \frac{d}{dx}(x) \ln x \\
&= x\left(\frac{1}{x}\right) + (1)(\ln x) \\
&= 1 + \ln x
\end{aligned}$$

Thus,

$$\begin{aligned} f'(x) = 0 &\iff 1 + \ln x = 0 \\ &\iff \ln x = -1 && \textit{To solve this, we apply the exponential function} \\ &\iff e^{\ln x} = e^{-1} \\ &\iff x = e^{-1} \end{aligned}$$

$\square$

4.1.5 Differential Calculus of Exponential Functions

The goal of this section is to determine the derivatives of the exponential functions. Let us start with $y = e^x$. One way to calculate the derivative of this function is to use logarithmic differentiation which was discussed in the previous section. Since we know $y = e^x$, this is equivalent to $\ln y = x$. Now, let us differentiate both sides with respect to x:

$$\begin{aligned} \frac{d}{dx}(\ln y) &= \frac{d}{dx}(x) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y \\ \frac{dy}{dx} &= e^x \end{aligned}$$

Hence, we arrive at the following theorem:

Theorem 4.1.10 (Derivative of e^x)

$$\frac{d}{dx}(e^x) = e^x$$

Example 4.1.12 *Suppose* $y = 5e^{x \sin x}$ *Calculate* $\frac{dy}{dx}$.

Solution:

$$\begin{aligned} \frac{dy}{dx} &= 5e^{x \sin x} \frac{d}{dx}(x \sin x) && \textit{By the chain rule} \\ &= 5e^{x \sin x} \left(x \frac{d}{dx}(\sin x) + \frac{d}{dx}(x) \sin x \right) && \textit{By the product rule} \\ &= 5e^{x \sin x} \left(x \cos x + (1) \sin x \right) \\ &= 5(x \cos x + \sin x) e^{x \sin x} \end{aligned}$$

Example 4.1.13 *Find and classify all relative (local) extrema of* $f(x) = \frac{x}{e^x}$

Solution:

We begin by finding the critical points of $f(x)$*. To do this, we calculate* $f'(x)$*.*

$$\begin{aligned} f'(x) &= \frac{e^x \frac{d}{dx}(x) - x\frac{d}{dx}(e^x)}{(e^x)^2} && \textit{By the quotient rule} \\ &= \frac{e^x(1) - xe^x}{e^{2x}} \\ &= \frac{1-x}{e^x} \end{aligned}$$

Since this derivative always exists, the only critical points that we get are from solving $f'(x) = 0$*. That is, we get a critical point of* $x = 1$*. Next, we test this critical point using the second derivative test.*

$$\begin{aligned} f''(x) &= \frac{e^x \frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(e^x)}{(e^x)^2} && \textit{By the quotient rule} \\ &= \frac{e^x(1) - (x-1)e^x}{e^{2x}} \\ &= \frac{2-x}{e^x} \end{aligned}$$

Notice that $f''(1) = \frac{1}{e} > 0$*. Thus, there is a relative minimum when* $x = 1$*. Try to verify this result using the first derivative test.* □

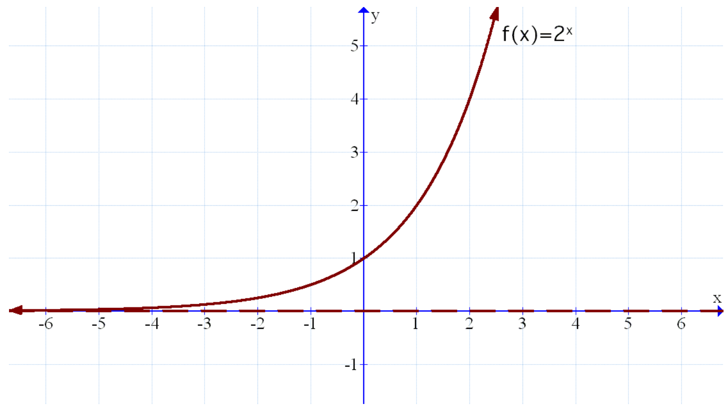
4.2 Other Exponential & Logarithmic Functions

4.2.1 Definition of Exponential Function of Base b

In the previous section, we defined $f(x) = e^x$ as the exponential function. The number e in this function is called the **base** of the exponential function. It is possible to generalize and discuss exponential functions of other bases.

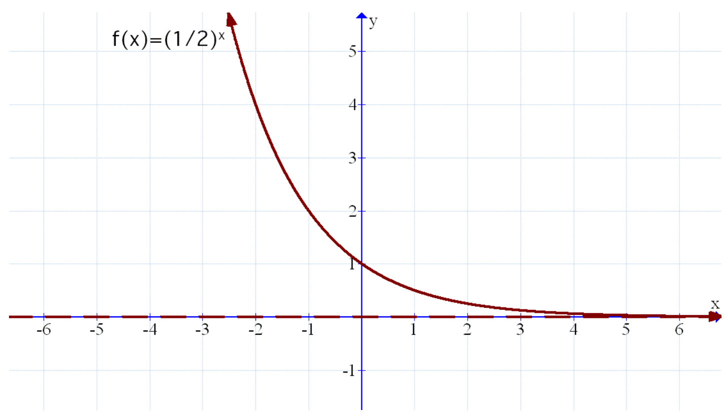
Definition 4.2.1 (Exponential Function of Base b) *Suppose* $b > 0$ *and* $b \neq 1$*. The exponential function with a base of* b *is* $f(x) = b^x$*.*

As with the exponential function that we have already discussed, the domain of the exponential function $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. If $b > 1$, the graph will resemble that of $y = e^x$ and can model exponential growth. The graph below shows $y = 2^x$.



Notice that, for $b > 1$, we have $\lim_{x\to\infty} b^x = +\infty$ and $\lim_{x\to-\infty} b^x = 0$. Also, notice the horizontal asymptote of $y = 0$.

Now, let us consider $y = \left(\frac{1}{2}\right)^x$. Using properties of exponents, we can rewrite the function as $y = 2^{-x}$. So, the graph of $y = \left(\frac{1}{2}\right)^x$ can be obtained by reflecting the graph of $y = 2^x$ over the $y - axis$. The graph shown below is that of $y = \left(\frac{1}{2}\right)^x$.



In general, if $0 < b < 1$, then the graph will be resemble that of $y = \left(\frac{1}{2}\right)^x$ and can model exponential decay.

4.2.2 Differential Calculus with Exponential Functions

Suppose $y = b^x$, where $b > 0, b \neq 1$. Will will differentiate with respect to x. Specifically, we will use logarithmic differentiation.

$$
\begin{aligned}
y &= b^x \\
\ln y &= \ln\left(b^x\right) \\
\ln y &= x \ln b && \text{Because } \ln M^p = p \ln M
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(x \ln b) \\
\frac{1}{y}\frac{dy}{dx} &= \ln b && \text{Because } \ln b \text{ is a constant.} \\
\frac{dy}{dx} &= y \ln b \\
\frac{dy}{dx} &= b^x \ln b && \text{Because } y = b^x.
\end{aligned}
$$

Thus, we arrive at the following theorem:

Theorem 4.2.1 (Derivative of b^x) *Suppose $b > 0, b \neq 1$. Then,*

$$\frac{d}{dx}(b^x) = b^x \ln b$$

As before, notice that if we take $b = e$, then Theorem 4.2.1 reduces to Theorem 4.1.10.

Example 4.2.1 *Suppose $y = x^2 \cdot 3^{\sin x}$. Calculate $\frac{dy}{dx}$.*

Solution:

$$
\begin{aligned}
\frac{dy}{dx} &= (x^2)\frac{d}{dx}\left(3^{\sin x}\right) + \frac{d}{dx}(x^2)\left(3^{\sin x}\right) && \textit{By the product rule.} \\
&= (x^2)\left(3^{\sin x}\ln 3\right)\frac{d}{dx}(\sin x) + (2x)\left(3^{\sin x}\right) && \textit{By the chain rule.} \\
&= x^2\left(3^{\sin x}\ln 3\right)\cos x + 2x\left(3^{\sin x}\right)
\end{aligned}
$$

□

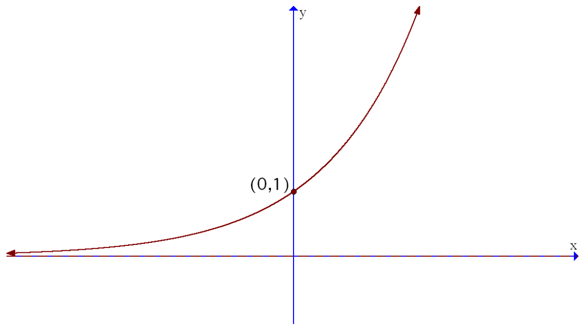
4.2.3 Definition of Logarithmic Function of Base b

At the beginning of this chapter, we defined the natural logarithm function ($y = \ln x$) and its inverse, the exponential function ($y = e^x$). In this section of the chapter, we will define other logarithm functions. As usual, these will be intimately related to the exponential functions.

Recall that we defined and graphed the exponential function with base b (satisfying $b > 0, b \neq 1$) as $f(x) = b^x$, Specifically, if $b > 1$ we had exponential growth whereas if $0 < b < 1$ we had exponential decay, as shown below.

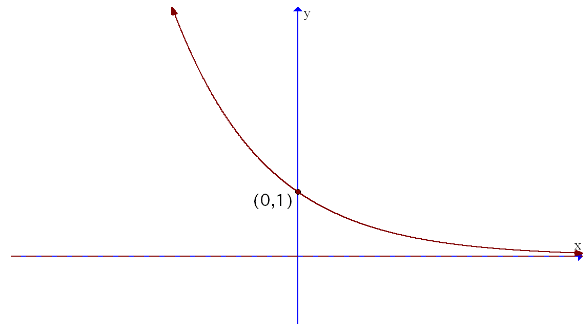
Exponential Growth

$f(x) = b^x$, $b > 1$



Exponential Decay

$f(x) = b^x$, $0 < b < 1$



As a result, the exponential function of base b, $f(x) = b^x$, is one-to-one and has an inverse function. The inverse will be a logarithmic function.

Definition 4.2.2 (Logarithm Base b) *Suppose $b > 0$ and $b \neq 1$. The logarithm base b of x, denoted $f(x) = \log_b(x)$, is the inverse of the exponential function $y = b^x$.*

If we take the base to be e, we return to the specific case of chapter 4.1. That is, we can think of $y = \ln x$ as the logarithm with a base of e. As a result, much of the following discussion on general logarithm functions will mimic that of chapter 4.1. The following table summarizes the domain and range of $y = \log_b(x)$ and $y = b^x$.

	Domain	Range
b^x	$(-\infty, \infty)$	$(0, \infty)$
$\log_b(x)$	$(0, \infty)$	$(-\infty, \infty)$

As usual, notice that we can only take the log of a positive quantity. Furthermore, the following theorem summarizes properties of logarithms. You should recognize these as generalizations from chapter 4.1.

Theorem 4.2.2 (Properties of Logarithmic Functions) *Suppose $b > 0, b \neq 1$. Then, the following properties hold:*

1. $\log_b(b^x) = x$ *for all x in the domain of b^x*

2. $b^{log_b(x)} = x$ *for all x in the domain of* $\log_b x$

3. $\log_b(1) = 0$

4. $\log_b(M) + \log_b(N) = \log_b(MN)$

5. $\log_b(M) - \log_b(N) = \log_b\left(\frac{M}{N}\right)$

6. $\log_b(M^p) = p\log_b(M)$

7. **Change of Base:** $\log_b x = \frac{\log_a x}{\log_a b}$ *where a is any real number satisfying $a > 0, a \neq 1$.*

Selected Proofs:

1. This property follows from the fact that $y = b^x$ and $y = \log_b(x)$ are inverse functions.

2. This property follows from the fact that $y = b^x$ and $y = \log_b(x)$ are inverse functions.

3. We know that $b^0 = 1$ So, the corresponding logarithmic statement is $\log_b(1) = 0$. Again, this follows from the fact that $y = b^x$ and $y = \log_b(x)$ are inverse functions.

4. Suppose $x = \log_b(M)$ and $y = \log_b(N)$. In order to use properties of exponential functions, we should convert these logarithmic statements to exponential statements. The equivalent exponential statements are $M = b^x$ and $N = b^y$, respectively. Consider the following:

$$MN = b^x b^y$$

$$MN = b^{x+y} \qquad \text{By rules of exponents.}$$

Taking Log base b of both sides of equation (2) gives:

$$\log_b(MN) = \log_b\left(b^{x+y}\right)$$

$$\log_b(MN) = x + y \qquad \text{Because of property 1}$$

$$\log_b(MN) = \log_b(M) + \log_n(N) \qquad \text{Because } x = \log_b M \text{ and } y = \log_b N$$

And, we have proven the desired result.

5. You can prove this by following the arguments in (3). You will need to use the fact that $\dfrac{b^x}{b^y} = b^{x-y}$.

6. Suppose $y = \log_b(M^p)$. The equivalent exponential statement is $b^y = M^p$. We take the p-th root of both sides of this equation, which gives $b^{y/p} = M$. Now, take log base b of both sides to arrive at $\log_b\left(b^{y/p}\right) = \log_b(M)$. Simplifying using property (1) gives us $\frac{y}{p} = \log_b(M)$. Multiplying by p gives us the desired result: $y = p\log_b(M)$.

7. Suppose $y = \log_b x$. This is equivalent to $b^y = x$. Taking log base a of both sides, we get $\log_a b^y = \log_a x$. Using property #6 to the left hand side, we have $y\log_a b = \log_a x$. After dividing by $\log_a b$ (which is non-zero since $b \neq 1$), we get $y = \frac{\log_a x}{\log_a b}$. In other words, we get the desired conclusion that $\log_b x = \frac{\log_a x}{\log_a b}$. $\square$

Example 4.2.2 *Suppose* $\log_b x = A$, $\log_b y = B$, *and* $\log_b z = C$. *Express* $\log_b\left(\frac{x^2}{\sqrt{y}b^z}\right)$ *completely in terms of* A, B, *and* C.

Solution:

$$\begin{aligned}
\log_b\left(\frac{x^2}{\sqrt{y}b^z}\right) &= \log_b(x^2) - \log_b\left(\sqrt{y}b^z\right) && \textit{Because } \log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N \\
&- \log_b(x^2) - (\log_b(\sqrt{y}) + \log_b(b^z)) && \textit{Because } \log_b(MN) = \log_b M + \log_b N \\
&= 2\log_b x - \frac{1}{2}\log_b y - z\log_b b && \textit{Because } \log_b M^p = p\log_b M \\
&= 2A - \frac{1}{2}B - z && \textit{Because of the given information and } \log_b b = 1 \\
&= 2A - \frac{1}{2}B - b^C && \textit{Because } \log_b z = C \iff b^C = z
\end{aligned}$$

$\square$

Example 4.2.3 *Solve* $2^x = 7$

Solution:

The inverse of 2^x *is* $log_2 x$. *So, we may solve for* x *by applying this inverse function to both sides of the equation. That is,*

$$\begin{aligned}
2^x &= 7 \\
\log_2(2^x) &= \log_2 7 \\
x &= \log_2 7
\end{aligned}$$

Equivalently, at the beginning of the problem, we could have taken the natural log of both sides:

$$\begin{aligned}
2^x &= 7 \\
\ln(2^x) &= \ln 7 \\
x \ln 2 &= \ln 7 && \textit{Because } \ln M^p = p \ln M \\
x &= \frac{\ln 7}{\ln 2}
\end{aligned}$$

The two solutions are equivalent. Specifically, according to the change of base formula, we know that $\log_2 7 = \frac{\ln 7}{\ln 2}$. □

Example 4.2.4 *Solve for x:* $\log_2(x+1) + \log_2(x+2) = 1$.

Solution:

Before beginning, notice that there are domain restrictions. Specifically, when we solve the equation, only values of x *satisfying* $x > -1$ *will be valid solutions as other values of* x *will cause us to take the log of non-positive numbers. Having realized this, one way to solve it to first use properties of logs to combine the left hand side to a single logarithm. Then, we can apply the inverse of* $\log_2 x$ *to solve for* x.

$$\begin{aligned}
\log_2(x+1) + \log_2(x+2) &= 1 \\
\log_2\left[(x+1)(x+2)\right] &= 1 && \textit{Because } \log_b M + \log_b N = \log_B MN \\
2^{\log_2[(x+1)(x+2)]} &= 2^1 \\
(x+1)(x+2) &= 2 && \textit{Because } 2^x \textit{ and } \log_2 x \textit{ are inverse functions.} \\
x^2 + 3x + 2 &= 2 \\
x^2 + 3x &= 0 \\
x(x+3) &= 0
\end{aligned}$$

As a result, we realize that either $x = 0$ *or* $x = -3$. *But,* $x = -3$ *is an invalid solution as it violated the domain restrictions from the original equation. As a result, the only solution is* $x = 0$. □

As of now, we have discussed in depth the **natural logarithm** (logarithm with a base of e), denoted $\ln x$. Another logarithm which frequently occurs is called the **common logarithm** (logarithm of base 10), denoted $\log x$. For instance, the following is an example from Chemistry.

Example 4.2.5 *In Chemistry, the pH of a substance is a scale from 1 to 14. A pH less than 7 is an acidic substance whereas a pH greater than 7 is basic. A pH closer to 1 is more acidic whereas a pH closer to 14 is more basic. A pH of 7 is considered neutral. The pH of a substance is defined as follows:*

$$pH = -\log\left[H^+\right]$$

where $\left[H^+\right]$ *is the concentration of Hydrogen ions present in the substance.*

(a) Determine the concentration of Hydrogen ions present in a substance with a pH of 1.

Solution:

A substance with a pH of 1 satisfies the following:

$$\begin{aligned} 1 &= -\log\left[H^+\right] \\ -1 &= \log\left[H^+\right] \\ 10^{-1} &= 10^{\log\left[H^+\right]} \\ \frac{1}{10} &= \left[H^+\right] && \textit{Because } 10^x \textit{ and } \log x \textit{ are inverse functions.} \end{aligned}$$

(b) Determine the concentration of Hydrogen ions present in a substance with a pH of 2.

Solution:

A substance with a pH of 2 satisfies the following:

$$\begin{aligned} 2 &= -\log\left[H^+\right] \\ -2 &= \log\left[H^+\right] \\ 10^{-2} &= 10^{\log\left[H^+\right]} \\ \frac{1}{100} &= \left[H^+\right] && \textit{Because } 10^x \textit{ and } \log x \textit{ are inverse functions.} \end{aligned}$$

Notice that a unit change in pH corresponds to factor of 10 change in the concentration of Hydrogen ions.

4.2.4 Differential Calculus with Logarithms of Base b

In this section we calculate the derivative of $y = \log_b x$ (where $b > 0, b \neq 1$). Rather than reverting to the definition, we can use the results that we have already obtained to calculate this derivative. By appealing to the change of base formula, we can convert the given log statement to an equal statement involving the natural logarithm.

$$\begin{aligned}
\frac{d}{dx}(\log_b x) &= \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) && \text{By Change of Base Formula} \\
&= \frac{1}{\ln b} \cdot \frac{d}{dx}(\ln x) && \text{Because } \frac{1}{\ln b} \text{ is a constant} \\
&= \frac{1}{\ln b} \cdot \frac{1}{x} && \text{Because } \frac{d}{dx}(\ln x) = \frac{1}{x} \\
&= \frac{1}{x \ln b}
\end{aligned}$$

This leads us to the following result. The reader is encouraged to think about what happens to this formula if $b = e$, Euler's constant.

Theorem 4.2.3 (Derivative of $\log_b x$) *Suppose $b > 0, b \neq 1$. Then,*

$$\frac{d}{dx}\left(\log_b x\right) = \frac{1}{x \ln b}$$

Example 4.2.6 *Find all values of x at which the tangent lines to the graph of $f(x) = \log_3(2x - 7)$ are parallel to $y = 4x + 5$.*

Solution:

First, notice that the domain of $f(x)$ is $\left(\frac{7}{2}, \infty\right)$. We need to find all x in the domain of $f(x)$ at which $f'x = 4$. By appealing to the chain rule, notice that $f'(x) = \frac{2}{(2x-7)\ln 3}$. Thus,

$$\begin{aligned}
f'(x) &= 4 \\
\frac{2}{(2x-7)\ln 3} &= 4 \\
2 &= 4(2x-7)\ln 3 \\
2 &= (8\ln 3)x - 28\ln 3 \\
(8\ln 3)x &= 2 + 28\ln 3 \\
x &= \frac{2 + 28\ln 3}{8\ln 3}
\end{aligned}$$

Using a calculator, we realize that $\frac{2+28\ln 3}{8\ln 3} \approx 3.73$, which is in the domain of $f(x)$. As a result, the tangent line to the graph of $f(x)$ will be parallel to the given line at this particular value of x. □

4.3 Hyperbolic Trigonometric Functions

A telephone wire which is supported by two poles will sag under its own weight and form the shape of a **catenary**, as shown below.

This shape can be modeled with **Hyperbolic Functions**, specifically hyperbolic cosine. Consider the following definitions:

Definition 4.3.1 (Hyperbolic Sine & Hyperbolic Cosine)

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

These are called hyperbolic functions because it can be shown that $x = \cosh t$ and $y = \sinh t$ satisfy the hyperbola $x^2 - y^2 = 1$. We verify this claim in the following example.

Example 4.3.1 *Show that $x = \cosh t$ and $y = \sinh t$ satisfy the equation $x^2 - y^2 = 1$.*

Solution:

$$\begin{aligned} x^2 - y^2 &= (\cosh t)^2 - (\sinh t)^2 \\ &= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 \\ &= \frac{1}{4}\left(e^{2t} + 2 + e^{-2t}\right) - \frac{1}{4}\left(e^{2t} - 2 + e^{-2t}\right) \\ &= \frac{1}{4}\left(e^{2t} + 2 + e^{-2t} - e^{2t} + 2 - e^{-2t}\right) \\ &= \frac{1}{4}(4) \\ &= 1 \end{aligned}$$

Notice that in the previous example, we derived an identity that the hyperbolic functions must satisfy.

Theorem 4.3.1 (Hyperbolic Identity) *The following identity holds for all x:*

$$\cosh^2 x - \sinh^2 x = 1$$

Example 4.3.2 *Find all x and y intercepts of $f(x) = \cosh x$*

Solution:

To calculate the x-intercepts, we set $y = 0$.

$$\begin{aligned} \cosh x &= 0 \\ \frac{e^t + e^{-t}}{2} &= 0 \\ e^t + e^{-t} &= 0 \end{aligned}$$

The final equation does not have any solutions. Thus, there are no values of x for which $\cosh x = 0$. And, as a result, there are no x-intercepts.

To calculate the y-intercept, we set $x = 0$.

$$\begin{aligned} f(0) &= \cosh 0 \\ &= \frac{e^0 + e^0}{2} \\ &= 1 \end{aligned}$$

Thus, the y-intercept is $(0, 1)$.

Example 4.3.3 *Compute $\lim\limits_{x\to\infty} \cosh x$ and $\lim\limits_{x\to-\infty} \cosh x$.*

Solution:

- *We begin by calculating the limit at $+\infty$:*

  $$\lim_{x\to\infty} \cosh x = \lim_{x\to\infty} \frac{e^x + e^{-x}}{2} = +\infty$$

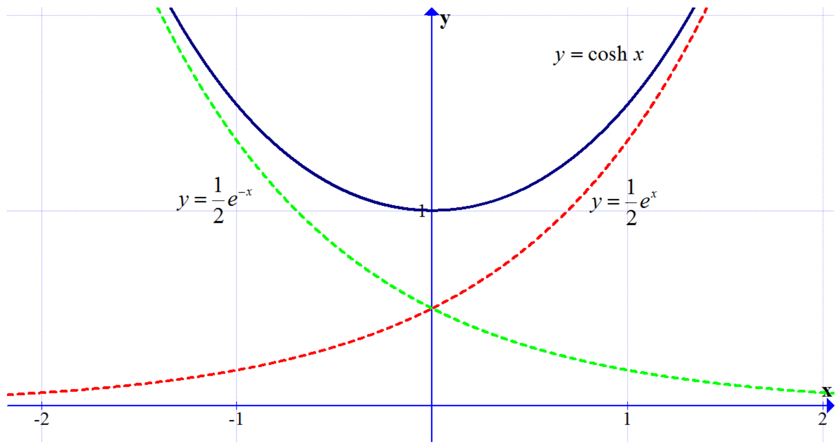
  The result follows because as x approaches $+\infty$, e^{-x} approaches 0 but e^x approaches $+\infty$. Notice that as x gets large, the e^{-x} term dies off to 0 and the function behaves roughly like $\frac{1}{2}e^x$. We say that $\cosh x \approx \frac{1}{2}e^x$ for large x; equivalently, $y = \frac{1}{2}e^x$ is a **curvilinear asymptote** *of $y = \cosh x$.*

- *Next, we calculate the limit at $-\infty$:*

  $$\lim_{x\to-\infty} \cosh x = \lim_{x\to-\infty} \frac{e^x + e^{-x}}{2} = +\infty$$

The result follows because as x approaches $-\infty$, e^{-x} approaches $+\infty$ but e^x approaches 0. Notice that as x approaches $-\infty$, the e^x term dies off to 0 and the function behaves roughly like $\frac{1}{2}e^{-x}$. We say that $\cosh x \approx \frac{1}{2}e^{-x}$ as x approaches $-\infty$; equivalently, $y = \frac{1}{2}e^{-x}$ is a **curvilinear asymptote** *of $y = \cosh x$.*

In fact, to demonstrate the results of the past few examples, here is a graph of $y = \cosh x$.



The dashed curves are the curvilinear asymptotes that were discussed in the previous example.

Theorem 4.3.2 (Derivatives of Hyperbolic Functions) *The derivatives of cosh$x$ and $\sinh x$ are:*

- $\frac{d}{dx}\cosh x = \sinh x$
- $\frac{d}{dx}\sinh x = \cosh x$

Proof: Here we justify that $\frac{d}{dx}\cosh x = \sinh x$. We leave the other formula as an exercise.

$$\begin{aligned} \frac{d}{dx}\cosh x &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) && \text{By definition of } \cosh x \\ &= \frac{1}{2}\left(e^x - e^{-x}\right) \\ &= \sinh x && \text{By definition of } \sinh h \end{aligned}$$

$\square$

Example 4.3.4 *Suppose* $y = \sinh(x^2)$*. Calculate* $\dfrac{d^2y}{dx^2}$.

Solution:

By the chain rule, we see that the first derivative is $\frac{dy}{dx} = 2x\cosh(x^2)$*. Now, we calculate the second derivative:*

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[2x\cosh(x^2)\right] \\
&= (2x)\frac{d}{dx}\left[\cosh(x^2)\right] + \frac{d}{dx}(2x)[\cosh(x^2)] && \textit{By the product rule.} \\
&= (2x)(\sinh(x^2))(2x) + 2\cosh(x^2) && \textit{By the chain and power rules.} \\
&= 4x^2\sinh(x^2) + 2\cosh(x^2)
\end{aligned}
$$

□

Example 4.3.5 *Find all values of* x *at which the tangent line to the graph of* $f(x) = \cosh x$ *are parallel to the graph of* $y = x$*.*

Solution:

Since the slope of the given line is 1, we need to find all values of x *at which* $f'(x) = 1$*. Since* $\frac{d}{dx}(\cosh x) = \sinh x$*, we solve the problem as follows:*

$$
\begin{aligned}
f'(x) &= 1 \\
\sinh x &= 1 \\
\frac{e^x - e^{-x}}{2} &= 1 && \textit{By definition of } \sinh x \\
e^x - e^{-x} &= 2 \\
e^{2x} - 1 &= 2e^x && \textit{Because we multiplied both sides of the equation by } e^x. \\
e^{2x} - 2e^x - 1 &= 0
\end{aligned}
$$

Notice that if $u = e^x$*, we arrive at the quadratic equation* $u^2 - 2u - 1 = 0$*. By appealing to the quadratic formula, we see that* $u = 1 \pm \sqrt{2}$*. So, all of the solutions of the original equation come from* $e^x = 1 + \sqrt{2}$ *or* $e^x = 1 - \sqrt{2}$*. Since* $1 - \sqrt{2} < 0$*, the second equation does not have any solutions. That is, the tangent lines to* $f(x) = \cosh x$ *will be parallel to* $y = x$ *only when* $x = \ln(1 + \sqrt{2})$*.* □

5 Applications & Extensions

5.1 L'Hôpital's Rule & Indeterminate Forms

5.1.1 Indeterminate form of $\frac{0}{0}$

In Math 116, we have discussed limits similar to $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$. In particular, we recognize that this limit is of the indeterminate form $\frac{0}{0}$ and performed algebraic manipulations to determine the value of this limit. Specifically, we calculated this limit as follows:

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2-9}{x-3} &= \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3} \\
&= \lim_{x\to 3}(x+3) \\
&= 3+3 \\
&= 6
\end{aligned}$$

Our argument relied on the fact that $y = \frac{x^2-9}{x-3}$ and $y = x+3$ agree everywhere except $x = 3$. So, for x sufficiently close to 3, the y-values of both of these functions will be equal.

In Math 117, we again encountered the indeterminate form of $\frac{0}{0}$ when we encountered $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$. We used a geometric argument and the squeeze theorem to realize that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$. (See chapter 3.6.3 for full details.)

In this section, we discuss a general rule which can be applied to evaluate limits of the indeterminate form $\frac{0}{0}$. This new rule, called **L'Hôpital's Rule** after Marquis de L'Hôpital. As a historical note, the rule is named after L'hôpital because it first appeared in his calculus textbook *Analyse des Infiniment Petits*. However, the rule was actually discovered by Swiss mathematician John (Johann) Bernoulli.

Theorem 5.1.1 (L'Hôpital's Rule (Special Case)) *Suppose f and g are differentiable functions and that $g'(x) \neq 0$ near $x = a$ (except possibly at $x = a$). Furthermore, suppose that $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$. Finally assume that f' and g' are continuous. Then,*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided this limit exists, is $+\infty$, or is $-\infty$.

Proof: Since f and g are differentiable functions on an interval containing $x = a$, it follows that f and g are continuous on this interval. In particular we have that f and g are continuous at $x = a$. So,

$$\lim_{x \to a} f(x) = f(a)$$

$$\lim_{x \to a} g(x) = g(a)$$

And, since we assumed that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, we have that $f(a) = 0$ and $g(a) = 0$. Thus,

$$\begin{aligned}
\lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f(x) - 0}{g(x) - 0} \\
&= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} \\
&= \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\
&= \frac{f'(a)}{g'(a)} && \text{Because } f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ by definition.} \\
&= \lim_{x \to a} \frac{f'(x)}{g'(x)} && \text{Since we assumed that } f' \text{ and } g' \text{ are continuous.}
\end{aligned}$$

□

In general, we do not need to assume that f' and g' are continuous. Theorem 5.1.2 is a more general statement of L'Hôpital's Rule for handling the indeterminate form $\frac{0}{0}$. To prove the more general case, we would need to appear to a generalization of the Mean Value Theorem, called the Cauchy Mean Value Theorem which is beyond the scope of this course.

Theorem 5.1.2 (L'Hôpital's Rule for Indeterminate Form $\frac{0}{0}$) *Suppose $f(x)$ and $g(x)$ are differentiable functions and that $g'(x) \neq 0$ near $x = a$ (except possibly at $x = a$). Furthermore, suppose that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Then,*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided this limit exists, is $+\infty$, or is $-\infty$.

Example 5.1.1 *Evaluate* $\lim_{x\to 3} \frac{x^2-9}{x-3}$.

Solution: *Since this limit is of the indeterminate form* $\frac{0}{0}$*, we can apply L'Hôpital's rule.*

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2-9}{x-3} &= \lim_{x\to 3} \frac{\frac{d}{dx}(x^2-9)}{\frac{d}{dx}(x-3)} \\
&= \lim_{x\to 3} \frac{2x}{1} \\
&= \lim_{x\to 3} 2x \\
&= 2(3) \\
&= 6
\end{aligned}$$

Thus, since this limit exists, we see that $\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$ *which matches the result that we obtained in the introduction to this section.* □

Example 5.1.2 *Use L'Hôpital's Rule to evaluate* $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution:

Since $\lim_{x\to 0} \sin x = 0$ *and* $\lim_{x\to 0} x = 0$*, we have the indeterminate form* $\frac{0}{0}$ *and may apply L'Hôpital's Rule.*

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} \\
&= \lim_{x\to 0} \frac{\cos x}{1} \\
&= \lim_{x\to 0} \cos x \\
&= \cos 0 \\
&= 1
\end{aligned}$$

Thus, since this limit exists, we see that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ *which matches the result that we obtained in chapter 3.6. Notice that even if we had known L'Hôpital's rule back in chapter 3.6, we still would not have been able to use it at that time since we did not know* $\frac{d}{dx}(\sin x)$ *at that time. In fact, this limit was needed for us to calculate the derivative of* $y = \sin x$ *by definition.* □

Example 5.1.3 *Compute* $\lim_{x\to 2} \frac{x^2-9}{x-3}$. **Solution:**

Notice that the given function is continuous at $x = 2$*. As a result, the requested limit will equal the function value. That is,*

$$\lim_{x\to 2} \frac{x^2-9}{x-3} = \frac{2^2-9}{2-3} = 5$$

If, instead, we had applied L'Hôpital's rule, we would have calculated as follows:

$$\lim_{x\to 2}\frac{x^2-9}{x-3}=\lim_{x\to 2}\frac{2x}{1}=2(2)=4$$

which is clearly incorrect. Many students try to blindly apply L'Hôpital's Rule without first checking that it is applicable. And, as a result, they often get incorrect answers. From this point on, we strongly encourage the reader to check that a particular limit of an appropriate indeterminate for prior to applying L'Hôpital's Rule. □

From here, we will deal with some more complicated limits. Some of these you may be able to evaluate using earlier techniques. For others, we would not have been able to figure out the value of the limit without this new tool.

Example 5.1.4 *Evaluate* $\lim\limits_{x\to 0}\dfrac{\tan 5x}{\sin 9x}$.

Solution:

Since $\lim\limits_{x\to 0}\tan 5x=\tan 0=0$ *and* $\lim\limits_{x\to 0}\sin 9x=\sin 0=0$, *we have the indeterminate form of* $\frac{0}{0}$ *and may apply L'Hôpital's Rule to evaluate the limit.*

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan 5x}{\sin 9x} &= \lim_{x\to 0}\frac{\frac{d}{dx}(\tan 5x)}{\frac{d}{dx}(\sin 5x)} && \textit{By L'Hôpital's Rule}\\
&= \lim_{x\to 0}\frac{5\sec^2(5x)}{9\cos(9x)}\\
&= \frac{5(\sec 0)^2}{9\cos 0}\\
&= \frac{5}{9}
\end{aligned}$$

□

Example 5.1.5 *Evaluate* $\lim\limits_{x\to 0}\dfrac{e^x-1}{x^3}$

Solution:

Since $\lim\limits_{x\to 0}(e^x-1)=0$ *and* $\lim\limits_{x\to 0}x^3=0$, *we have the indeterminate form* $\frac{0}{0}$ *and may apply L'Hôpital's Rule. Thus,*

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x-1}{x^3} &= \lim_{x\to 0}\frac{e^x}{3x^2} && \textit{By L'Hôpital's Rule}\\
&= \frac{1}{0^+}\\
&= +\infty
\end{aligned}$$

□

5.1.2 Indeterminate form of $\frac{\infty}{\infty}$

In Math 116, we were interested in computing $\lim_{x\to\infty} \dfrac{x^2-1}{3x^2+5}$ and $\lim_{x\to-\infty} \dfrac{x^2-1}{3x^2+5}$ to determine whether or not the graph of $f(x) = \frac{x^2-1}{3x^2+5}$ had any horizontal asymptotes. Unfortunately, in trying to calculate this limit we ended up with the confusing result of $\frac{\infty}{\infty}$. To determine the value of a limit in the indeterminate form of $\frac{\infty}{\infty}$, our technique was to divide by the most dominant term in the denominator in order to rewrite the function into a form from which we could immediately see the desired limit. For instance,

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2-1}{3x^2+5} &= \lim_{x\to\infty} \frac{x^2-1}{3x^2+5} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{3+\frac{5}{x^2}} \\
&= \frac{1-0}{3+0} && \text{Because } \lim_{x\to\infty} \frac{1}{x^2} = 0 \\
&= \frac{1}{3}
\end{aligned}$$

As a result, we were able to conclude that the function $f(x)$ had a horizontal asymptote of $y = \frac{1}{3}$.

It turns out that L'Hôpital's Rule can also be immediately applied to evaluate limits of the indeterminate form $\frac{\infty}{\infty}$, as described in the following theorem. The proof of the theorem is omitted.

Theorem 5.1.3 (L'Hôpital's Rule for Indeterminate Form $\frac{\infty}{\infty}$) *Suppose $f(x)$ and $g(x)$ are differentiable functions and that $g'(x) \neq 0$ near $x = a$ (except possibly at $x = a$). Furthermore, suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Then,*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided this limit exists, is $+\infty$, or is $-\infty$.

Example 5.1.6 *Evaluate* $\lim_{x\to\infty} \dfrac{x^2-1}{3x^2+5}$.

Solution:

This limit is of the indeterminate form $\frac{\infty}{\infty}$. So, we may directly apply L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2-1}{3x^2+5} &= \lim_{x\to\infty} \frac{\frac{d}{dx}(x^2-1)}{\frac{d}{dx}(3x^2+5)} && \textit{By L'Hôpital's Rule} \\
&= \lim_{x\to\infty} \frac{2x}{6x} \\
&= \lim_{x\to\infty} \frac{1}{3}
\end{aligned}$$

$$= \frac{1}{3}$$

Thus, we arrive at the same conclusion as earlier in this subsection. □

5.1.3 Indeterminate forms of $0 \cdot \infty$ and $\infty - \infty$

For limits of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we know that we may directly apply L'Hôpital's Rule. For other indeterminate forms, we may perform some manipulations to convert the given limit into a form in which L'Hôpital's Rule is applicable.

Example 5.1.7 *Evaluate* $\lim_{x\to\infty} x \sin\left(\frac{2}{x}\right)$.

Solution:

This limit is of the indeterminate form $\infty \cdot 0$. To solve this problem, we will bring the x term to the denominator. Doing so should reduce the problem to an indeterminate form of $\frac{0}{0}$ in which we can immediately apply L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to\infty} x \sin\left(\frac{2}{x}\right) &= \lim_{x\to\infty} \frac{\sin\left(\frac{2}{x}\right)}{x^{-1}} \\
&= \lim_{x\to\infty} \frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}} && \textit{This is of the indeterminate form } \frac{0}{0} \\
&= \lim_{x\to\infty} \frac{\cos\left(\frac{2}{x}\right)\cdot\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}} && \textit{By L'Hôpital's Rule} \\
&= \lim_{x\to\infty} 2\cos\left(\frac{2}{x}\right) \\
&= 2\cos 0 && \textit{Because } \lim_{x\to\infty}\frac{2}{x} = 0 \\
&= 2
\end{aligned}$$

□

Example 5.1.8 *Evaluate* $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$.

Solution: *This limit is of the indeterminate form $\infty - \infty$. So, we may not directly apply L'Hôpital's Rule. To convert the limit into a form in which we may apply L'Hôpital's Rule, we begin by combining the fractions.*

$$\begin{aligned}
\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) &= \lim_{x\to 0^+}\left(\frac{e^x - 1}{x(e^x-1)} - \frac{x}{x(e^x-1)}\right) \\
&= \lim_{x\to 0^+}\frac{e^x - 1 - x}{x(e^x-1)}
\end{aligned}$$

This limit is of the indeterminate form $\frac{0}{0}$; so, we may directly apply L'Hôpital's Rule.

$$= \lim_{x\to 0^+} \frac{e^x - 1}{xe^x + e^x - 1}$$

This new limit is of the indeterminate form $\frac{0}{0}$*; so, we may apply L'Hôpital's Rule again.*

$$= \lim_{x\to 0^+} \frac{e^x}{xe^x + e^x + e^x}$$
$$= \lim_{x\to 0^+} \frac{e^x}{xe^x + 2e^x}$$
$$= \frac{1}{0+2}$$
$$= \frac{1}{2}$$

Notice from this example that we may have to apply L'Hôpital's Rule multiple times. □

Example 5.1.9 *Evaluate* $\lim_{x\to\infty} \left(\sqrt{x^2+3x} - x\right)$.

Solution:

This limit is of the indeterminate form $\infty - \infty$*. It is our intention to combine these terms into a fraction so that we may attempt to apply L'Hôpital's Rule. Thus, we begin by multiplying and dividing by the conjugate.*

$$\lim_{x\to\infty} \left(\sqrt{x^2+3x} - x\right) = \lim_{x\to\infty} \left(\sqrt{x^2+3x} - x\right) \cdot \left(\frac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x}\right)$$
$$= \lim_{x\to\infty} \frac{x^2+3x-x^2}{\sqrt{x^2+3x}+x}$$
$$= \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+3x}+x} \tag{1}$$

This limit is of the indeterminate form $\frac{\infty}{\infty}$*. We apply L'Hôpital's Rule.*

$$= \lim_{x\to\infty} \frac{3}{\frac{2x+3}{2\sqrt{x^2+3x}}+1}$$
$$= \lim_{x\to\infty} \frac{6\sqrt{x^2+3x}}{2x+3+2\sqrt{x^2+3x}}$$

= *We have the indeterminate form of* $\frac{\infty}{\infty}$ *so, we apply L'Hôputal's Rule again.*

$$= \lim_{x\to\infty} \frac{\frac{6x+9}{\sqrt{x^2+3x}}}{2+\frac{2x+3}{\sqrt{x^2+3x}}}$$
$$= \lim_{x\to\infty} \frac{6x+9}{2\sqrt{x^2+3x}+2x+3}$$

Notice that as we keep applying L'Hôpital's Rule, the computation does not seem to be improving. In fact, L'Hôpital's Rule, although applicable, will not help with this problem. As a result, we should try earlier techniques for computing this limit. Specifically, after combining the two terms together into one fraction

(1), we can divide all terms in the numerator and denominator by x.

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+3x}-x\right) &= \lim_{x\to\infty}\frac{3x}{\sqrt{x^2+3x}+x} \\
&= \lim_{x\to\infty}\frac{3x}{\sqrt{x^2+3x}+x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty}\frac{3}{\frac{\sqrt{x^2+3x}}{x}+1} \\
&= \lim_{x\to\infty}\frac{3}{\frac{\sqrt{x^2+3x}}{\sqrt{x^2}}+1} \\
&= \lim_{x\to\infty}\frac{3}{\sqrt{1+\frac{3}{x}}+1} \\
&= \frac{3}{2} && \textit{Because } \lim_{x\to\infty}\frac{3}{x}=0
\end{aligned}$$

The point of this example is for you to be aware that there are instances where L'Hôpital's rule will not help you, even if it is applicable. In these cases, you should try to use earlier techniques to try to evaluate the given limit. □

5.1.4 Indeterminate forms of 1^∞, 0^0, and ∞^0

As with the indeterminate forms of $0\cdot\infty$ and $\infty-\infty$, we cannot immediately apply L'Hôpital's Rule to limits of the exponential indeterminate forms. For these exponential indeterminate forms, the strategy is to use logs to eventually reduce the function to a form where L'Hôpital's Rule is applicable. Consider the following examples.

Example 5.1.10 *Evaluate* $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution:

This limit is of the exponential indeterminate form 1^∞. Recalling that $e^{\ln x}=x$, we introduce logs into the limit as follows:

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x &= \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} \\
&= e^{\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x} && \textit{Because the exponential function is continuous.}
\end{aligned}$$

To continue, we will perform a side calculation in which we compute the limit of the exponent in the last line.

$$\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \qquad \textit{By properties of logs.}$$

This limit is now of the indeterminate form $\infty \cdot 0$. So, we will bring down the x term to the denominator.

$$\begin{aligned}
\lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right) &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} \\
&= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&\textit{This limit is of the indeterminate form } \frac{0}{0}\textit{. We apply L'Hôpital's Rule.} \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x \text{ø} \infty} \frac{1}{1+\frac{1}{x}} \\
&= 1
\end{aligned}$$

Thus, we see that $\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x = 1$*. And, as a result,*

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x &= e^{\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x} \\
&= e^1 \\
&= e
\end{aligned}$$

That is, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. □

Typically, for these exponential indeterminate forms, a standard procedure of evaluating $\lim_{x\to a} f(x)$ is:

1. Rewrite the limit using a composition of exponential and logarithmic functions:

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln[f(x)]}$$

2. Since the exponential function is continuous, bring the limit inside. That is, rewrite the limit as:

$$\lim_{x\to a} f(x) = e^{\lim_{x\to a} \ln[f(x)]}$$

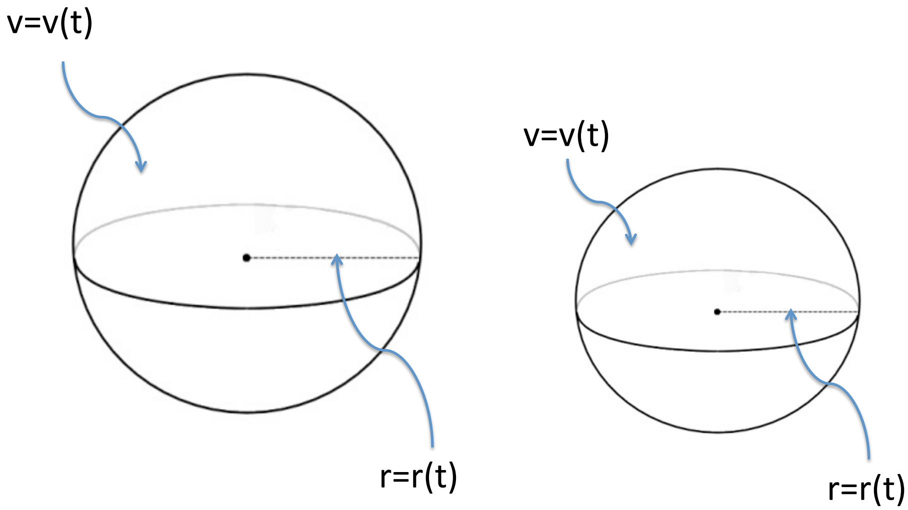
3. Perform a side calculation to compute the limit of the exponent. That is, compute $\lim_{x\to a} \ln[f(x)] = L$.

4. Since from step 3, the exponent has a limit of L, we conclude:

$$\lim_{x\to a} f(x) = e^{\lim_{x\to a} \ln[f(x)]} = e^L$$

5.2 Related Rates

In many real world situations, there are multiple quantities which vary with time. For instance, suppose you form a perfectly spherical snowball. Rather than throwing the snowball at an unexpected friend, you decide to bring the snowball into your apartment (which is significantly warmer than the outdoors). As a result, the snowball begins to melt. Assuming that the snowball remains perfectly spherical as it melts, the radius is decreasing as is the volume, as shown below.

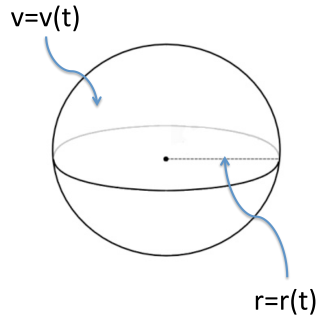


Our task is to relate these quantities so that if we know how fast one of them decreases we may determine the rate at which the other is decreasing. Let us formulate this problem more precisely in the following example and formulate a strategy for solving problems of this type.

Example 5.2.1 *A spherical snowball melts in such a way that its volume decreases at a constant rate of 3 cubic centimeters per minute. Find the rate at which the radius is changing at the instant when the diameter is* $\frac{1}{2}$ *centimeter?*

Solution:

To solve this problem, we begin by drawing the snowball at an arbitrary moment in time. We introduce variables for the quantities which are changing with time.



Now that we have introduced variables, let us summarize the given information in terms of these variables:

- **Given:** $\frac{dV}{dt} = -3$ *cubic feet per second*
- **Find:** $\frac{dr}{dt}$ *at the instant when* $r = \frac{1}{2}$ *cm*

Notice that $\frac{dV}{dt} < 0$ *since the volume is decreasing as time passes. Also, notice that* $\frac{dr}{dt}$ *will be negative since the radius is decreasing as time passes. Next, we relate the given quantities with each other. Specifically:*

- **Relationship:** $V = \frac{4}{3}\pi r^3$

This relationship hold at any instant in time. We will differentiate both sides with respect to time t*, keeping in mind that* V *and* r *are functions of* t*:*

- **Differentiate With Respect To Time:**

$$\begin{aligned} \frac{d}{dt}(V) &= \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \\ \frac{dV}{dt} &= \frac{4}{3}\pi(3r^2)\frac{dr}{dt} \\ \frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt} \end{aligned}$$

This is the general relationship between the rate of change of the radius and the rate of change of the volume. Having discovered this general relationship, we finally use the fact that at the instant when $r = \frac{1}{2}$ *cm we know* $\frac{dV}{dt} = -3$ *cubic feet per second to find the desired rate of change.*

- **Substitute In The Given Information:**

$$\begin{aligned} -3 &= 4\pi\left(\frac{1}{2}\right)^2\frac{dr}{dt} \\ \frac{dr}{dt} &= -\frac{3}{\pi} \text{ } \textit{cm per second} \end{aligned}$$

To summarize, at the instant when $r = \frac{1}{2}$ *cm, the radius is decreasing at a rate of* $\frac{3}{\pi}$ *cm per second.* $\square$

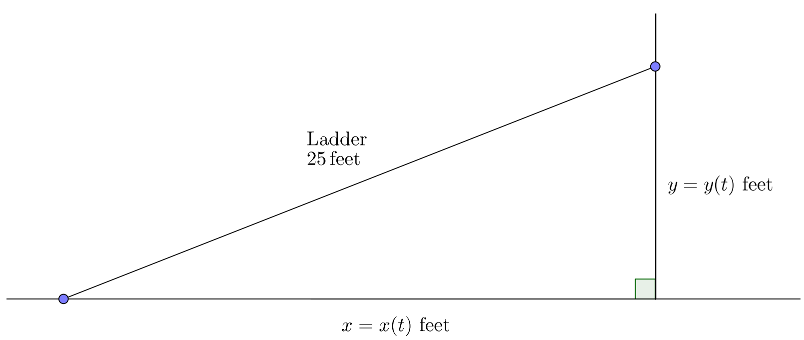
Strategy 5.2.1 (Related Rates) *Strategy for solving related rates problems:*

1. *Read the problem carefully.*
2. *Draw a diagram, if possible, representing the situation at an arbitrary time t.*
3. *Introduce notation, making sure to assign symbols to all quantities which are functions of time.*
4. *Express the given information and the rate that you are trying to find in terms of derivatives.*
5. *Write an equation which relates the various quantities of the problem.*
6. *Differentiate both sides with respect to t. (Keep in mind which quantities are functions of t and which quantities are remaining constant.)*
7. *Substitute all of the relevant information into the resulting equation and solve for the the unknown rate.*

Example 5.2.2 *Suppose a 25 feet ladder is leaning against a vertical wall. The base of the ladder is being pushed towards the wall at a constant rate of 2 feet per second. How fast is the top of the ladder climbing up the wall at the instant when the base is 7 feet from the wall. (Assume that the wall is perpendicular to the ground.)*

Solution:

- **Diagram at time t with variables introduced:**



- **Given Information:** $\frac{dx}{dt} = -2$ *feet per second. This rate is negative because the length of the corresponding side of the triangle is getting smaller as time passes.*

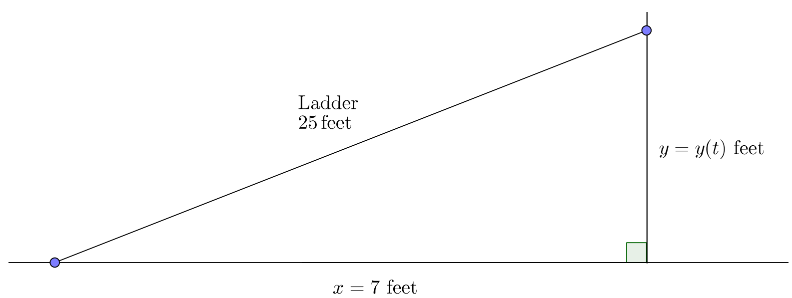
- **Find:** $\frac{dy}{dt}$ *at the instant when* $x = 7$ *feet. (Notice that* $\frac{dy}{dt} > 0$ *since this side of the triangle is getting longer as time passes.)*

- **Relate Variables:** *By the Pythagorean Theorem,* $x^2 + y^2 = 25^2$.

- **Differentiate with respect to t:** *Remember that x and y are functions of t.*

$$\begin{aligned}\frac{d}{dt}(x^2+y^2) &= \frac{d}{dt}(25^2)\\ 2x\frac{dx}{dt}+2y\frac{dy}{dt} &= 0\\ \frac{dy}{dt} &= \frac{-x\frac{dx}{dt}}{y} && (*)\end{aligned}$$

- *At the instant when $x = 7$ feet, the triangle is as follows:*



Applying the Pythagorean Theorem, we see that $y = 24$ feet. Thus, we may now solve the problem by substituting in all of the given information into equation (*):

$$\begin{aligned}\frac{dy}{dt} &= \frac{-(7)(-2)}{24}\\ &= \frac{7}{12}\end{aligned}$$

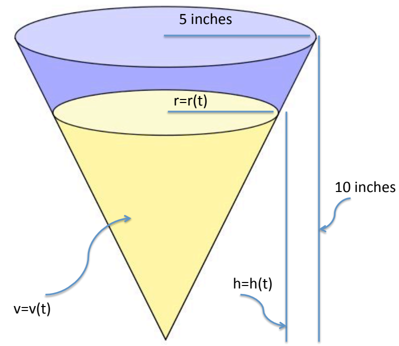
That is, at the instant when the base of the ladder is 7 feet away from the wall, the top of the ladder is climbing up the wall at a rate of $\frac{7}{12}$ feet per second.

□

Example 5.2.3 *Suppose liquid is being drained from an inverted right circular cone at a constant rate of 6 cubic inches per minute. The radius at the top of the cone is 5 inches and the height of the cone is 10 inches. Determine the rate of change of the height of the liquid at the instant when the volume is 18 cubic inches.*

Solution:

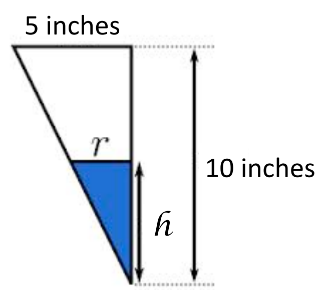
- **Diagram at time t with variables introduced:**



- **Given Information:** $\frac{dV}{dt} = -6$ *cubic inches per minute. This rate is negative because the volume of liquid in the cone is decreasing as time passes.*

- **Find:** $\frac{dh}{dt}$ *at the instant when* $V = 18$ *cubic inches. (Notice that* $\frac{dh}{dt} < 0$ *since this height of the cone-shaped liquid is decreasing as time passes.)*

- **Relate Variables:** $V = \frac{1}{3}\pi r^2 h$. *In the next step, we would differentiate both sides with respect to* t. *The right hand side would require the product rule since* r *and* h *are both functions of* t. *Alternatively, we can relate the height and the radius so that we may express the volume as a function of* h *alone. Specifically, we will use similar triangles.*



$$\frac{r}{5} = \frac{h}{10}$$
$$r = \frac{1}{2}h$$

Thus, using this, we have:

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$
$$V = \frac{\pi}{12}h^3$$

- **Differentiate with respect to t:** *Remember that V and h are functions of t.*

$$\begin{aligned} \frac{d}{dt}(V) &= \frac{d}{dt}\left(\frac{\pi}{12}h^3\right) \\ \frac{dV}{dt} &= \frac{\pi}{4}h^2\frac{dh}{dt} \qquad \textit{(*)} \end{aligned}$$

- *At the instant when $v = 18$ cubic centimeters, we know that $18 = \frac{\pi}{12}h^3$. Thus, we have $h = \frac{6}{\sqrt[3]{\pi}}$ inches. We may now solve the problem by substituting in all of the given information from the instant when $V = 18$ cubic inches into equation (*):*

$$\begin{aligned} -6 &= \frac{\pi}{4}\left(\frac{6}{\sqrt[3]{\pi}}\right)^2\frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{2}{3\sqrt[3]{\pi}} \end{aligned}$$

That is, at the instant when the volume is 18 cubic inches, the height of the liquid in the cone is decreasing at a rate of $\frac{2}{3\sqrt[3]{\pi}}$ inches per minute.

$\square$