Stanford University

Two-Sided Matching

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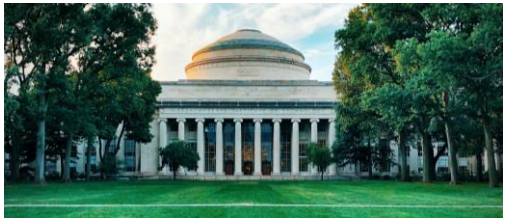
DSE Summer School, August 2022

Two-Sided Allocation



PhD Admissions

THE MATCH
NATIONAL RESIDENT MATCHING PROGRAM®

Medical Residency

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School Choice

Labor Markets

Refugee Resettlement

Judicial Clerkships

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Two-Sided Allocation

Q: How should we 'assign' students to Ph.D. programs? Discuss with your neighbor:

- How does the Ph.D. admissions process work in your program? What are the pros and cons of this process?

Two-Sided Allocation

- **Model**
- Deferred Acceptance
- Boston Mechanism
- Top Trading Cycles

Model

- Agents $a \in A = D \cup H$
  - Doctors $d \in D$
    - Preferences $\succ^d$ over hospitals,
      $i \succ^d j$ means d prefers hospital i to hospital j
  - Hospitals $h \in H$
    - Preferences $\succ^h$ over doctors,
      $a \succ^h b$ means h prefers doctor i to doctor j
    - Quotas q_h
  - ***Note:*** *Preferences are ordinal; agents cannot express intensity of preferences.*

- Matching $\mu: D \to H \cup \{\emptyset\}$
  - Many-to-one matching: $|\mu^{-1}(h)| \leq q_h$ for all $h \in H$
  - i.e. A matching is an allocation of doctors to hospitals
    (Some agents may be unmatched or not fully matched.)

Properties of Matchings

Def. A matching μ is **Pareto efficient** if no other matching μ' is weakly preferred by all agents and strictly preferred by some agent:

$\nexists \mu' \text{ s.t.:} \quad \mu'(a) \succcurlyeq^a \mu(a)\ \forall a \in A, \quad \mu'(a) \succ \mu(a) \text{ for some } a \in A$

No matching where everyone weakly improves, someone strictly improves

In some applications (e.g. school choice), we care about efficiency of one side of the market (e.g. students getting their preferred schools).

Def. A matching μ is **Pareto efficient** for agents in B if no other matching μ' is weakly preferred by all agents in B and strictly preferred by some agent in B:

$\nexists \mu' \text{ s.t.:} \quad \mu'(a) \succcurlyeq^a \mu(a)\ \forall a \in B, \quad \mu'(a) \succ \mu(a) \text{ for some } a \in B$

No matching where everyone in B weakly improves, someone in B strictly improves

Properties of Matchings

Def. A matching μ is in the **core** if there is no coalition of agents $B \subseteq A$ that can *profitably deviate* to a matching ν, i.e. $\nexists B \subseteq A$ such that

1. $\forall a \in B: \nu(a) \in B$.

   The agents in B swap assignments between themselves

2. $\nu(a) \succcurlyeq^a \mu(a) \;\; \forall a \in B$, and $\nu(a) \succ^a \mu(a)$ for some $a \in B$.

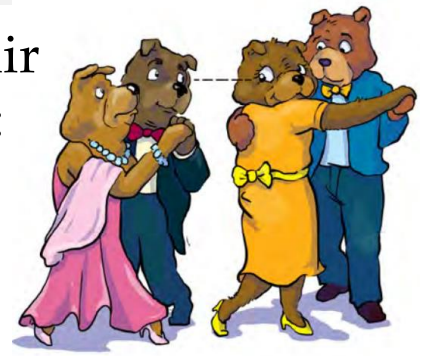
   Everyone in B weakly improves, someone in B strictly improves

Def. A matching μ is **stable** if there are no blocking pairs:

A **blocking pair** for matching μ is a doctor-hospital pair (d, h) that can *profitably deviate* to matching each other:

$$h \succ^d \mu(d) \quad \text{and} \quad d \succ^h d' \text{ for some } d' \in \mu(h)$$

Both the doctor and hospital prefer each other to their match in μ.



Source: Karlin & Peres, "Game Theory Alive"

Q. What is the relationship between stable matchings and the core? *A matching is stable iff it is in the core*

Mechanisms and Strategies

Def. A **mechanism** is an algorithm with *strategic input*.

$$M: \{\succ^a\}_{a \in A} \to \{\mu\}$$

Doctors and hospitals report their preferences $\succ^d, \succ^h$

Mechanism M is **strategyproof** if truth-telling is a dominant strategy:

$$\forall \succ, a \in A, \hat{\succ}^a: M(\succ)(a) \succcurlyeq^a M(\hat{\succ}^a, \succ^{-a})$$

Reporting true preferences $\succ^a$ gives the most preferred assignment no matter what preferences $\succ^{-a}$ everyone else reports

Mechanism M is **group strategyproof** if:

$$\forall \succ, B \subseteq A, \hat{\succ}_B: \text{Either } M(\succ)(a) \succ^a M(\hat{\succ}^B, \succ^{-B}) \text{ for some } a \in B$$
$$\text{Or } M(\succ)(a) \succcurlyeq^a M(\hat{\succ}^B, \succ^{-B}) \text{ for all } a \in B$$

Reporting true preferences $\succ^B$ gives B a Pareto undominated assignment no matter what preferences $\succ^{-B}$ everyone else reports

Mechanisms for Two-Sided Matching

- Boston Mechanism (Immediate Acceptance)
- Deferred Acceptance
- Top Trading Cycles
- And many others… *Efficiency-adjusted DA, DA with circuit-breaker…*

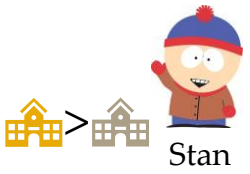
Mechanism	Stable	Pareto efficient	Strategy proof	Practice	Inference
Deferred Acceptance *(Stable matchings)*					
Boston Mechanism *(Immediate acceptance)*					
Top Trading Cycles *(Top agents trade)*					

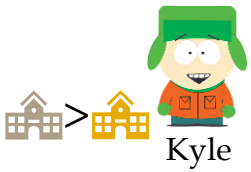
Two-Sided Allocation

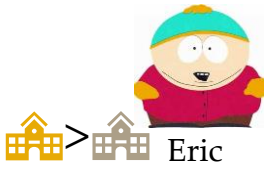
- Model
- **Deferred Acceptance**
- Boston Mechanism
- Top Trading Cycles

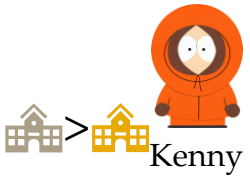
Deferred Acceptance

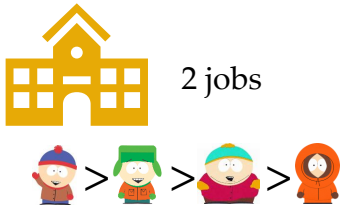
[Gale & Shapley 1962]

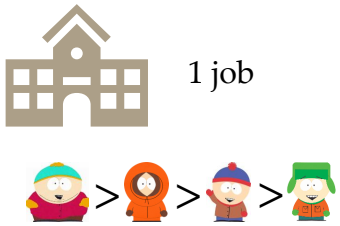










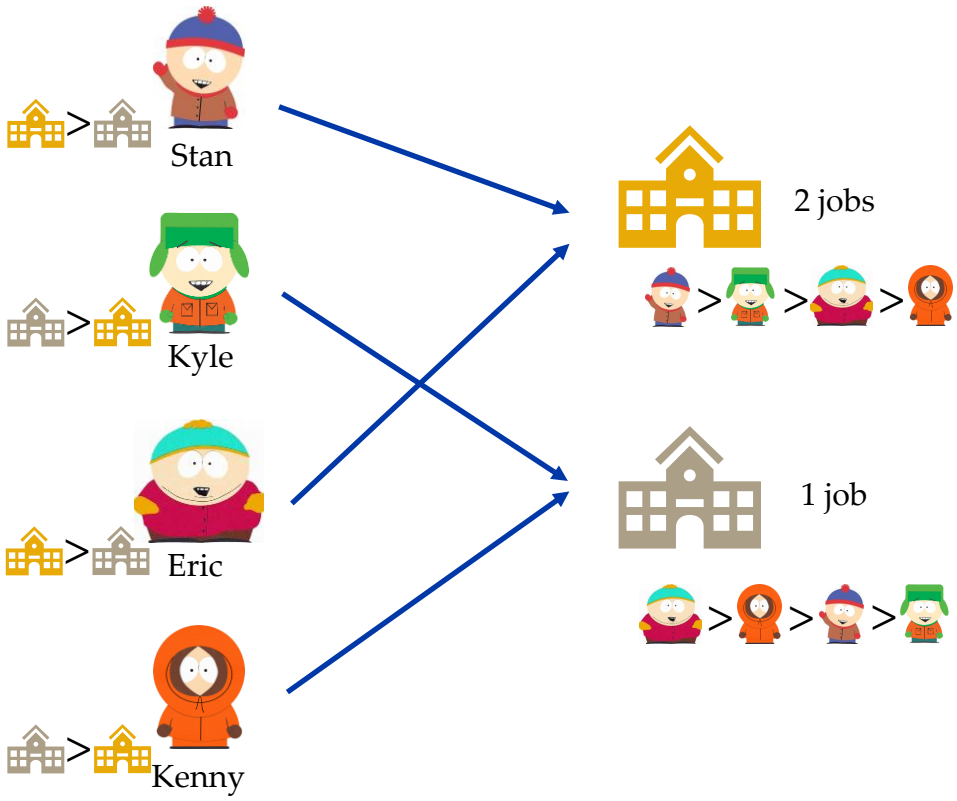


Doctor-Proposing Deferred Acceptance

```
while some doctor can still propose:
    all doctors propose to favorite hospital that has not rejected them  before
    all hospitals tentatively accept top doctors (up to capacity) who have proposed to them and reject the rest
each doctor assigned to favorite hospital that has still tentatively accepted them
```

Deferred Acceptance

[Gale & Shapley 1962]



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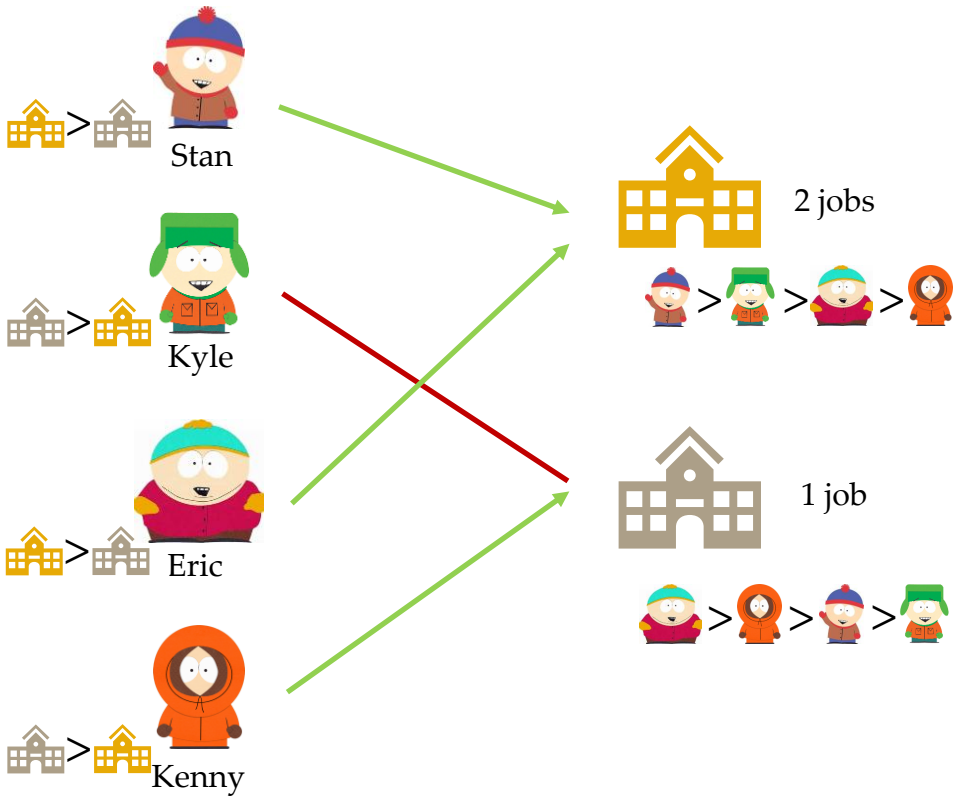

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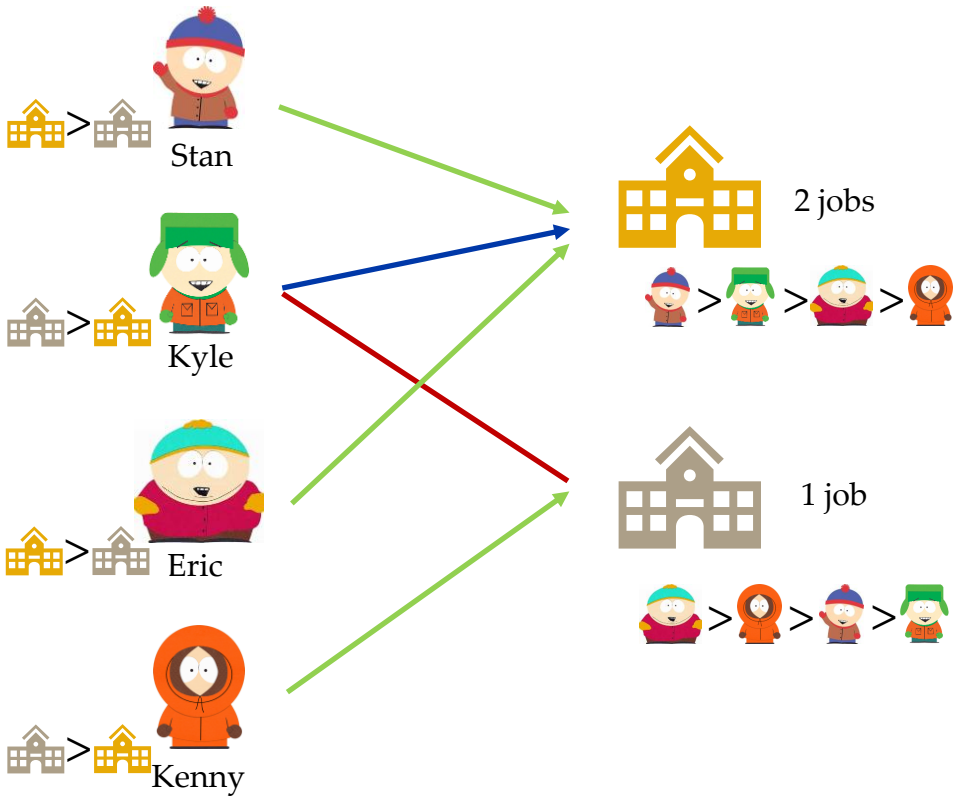

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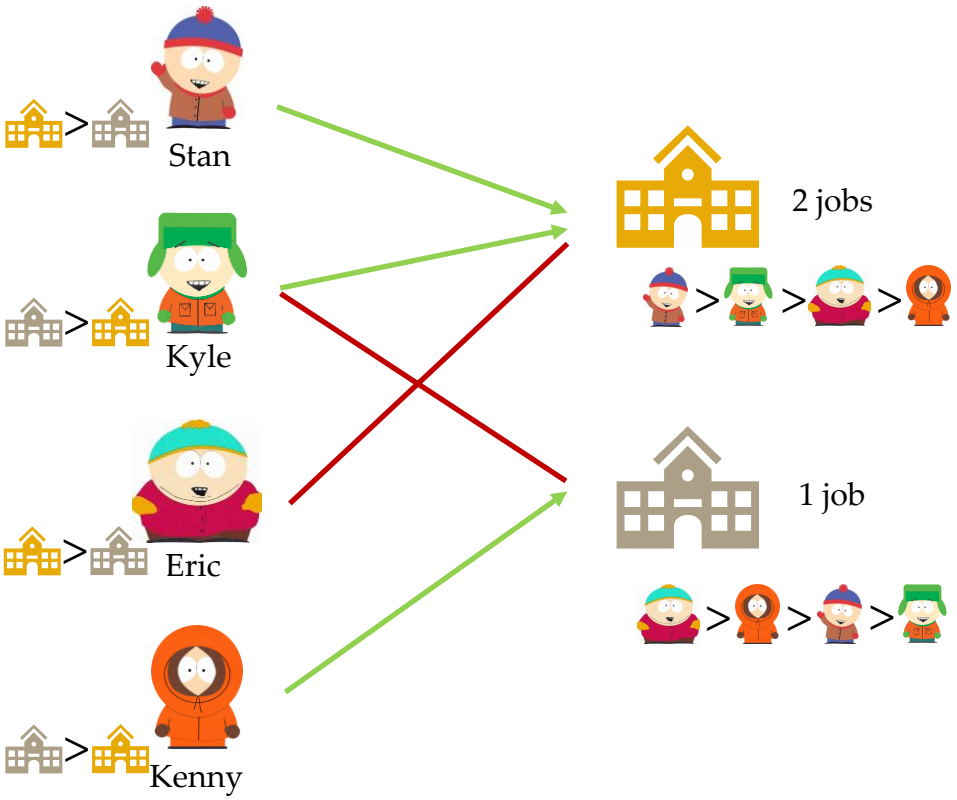

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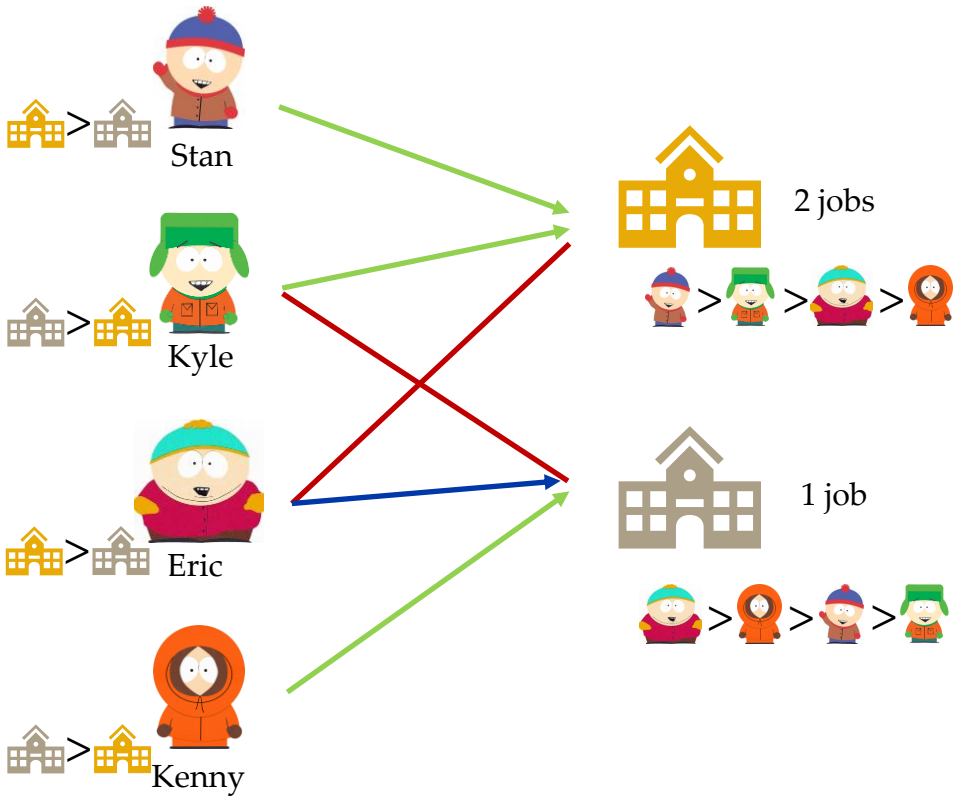

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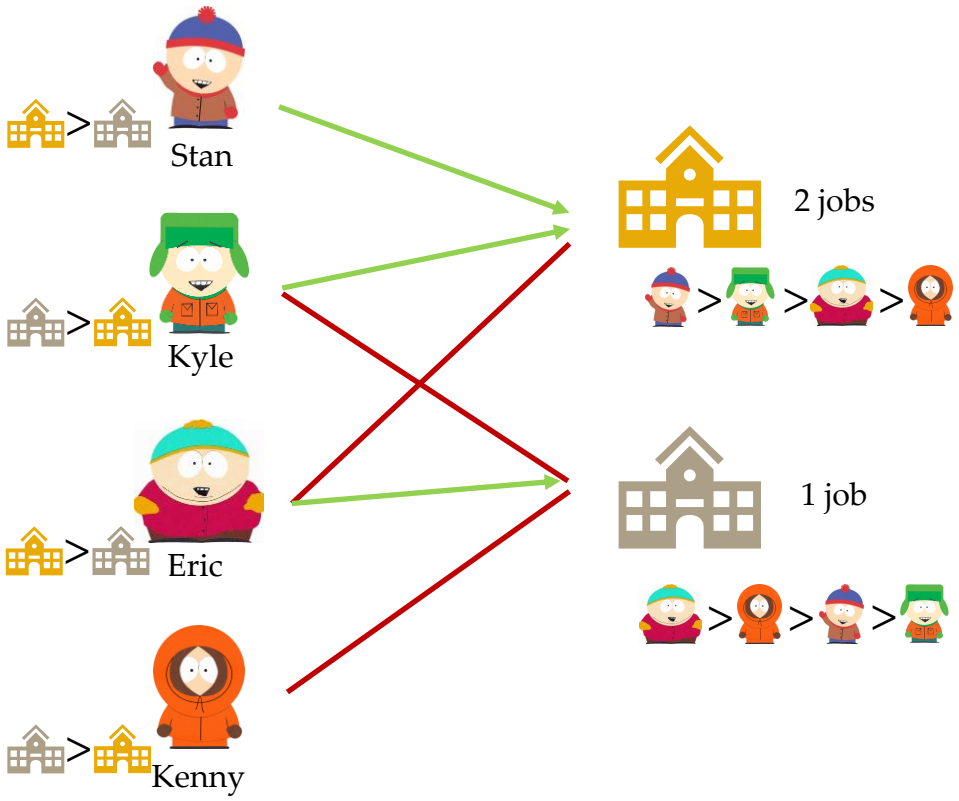

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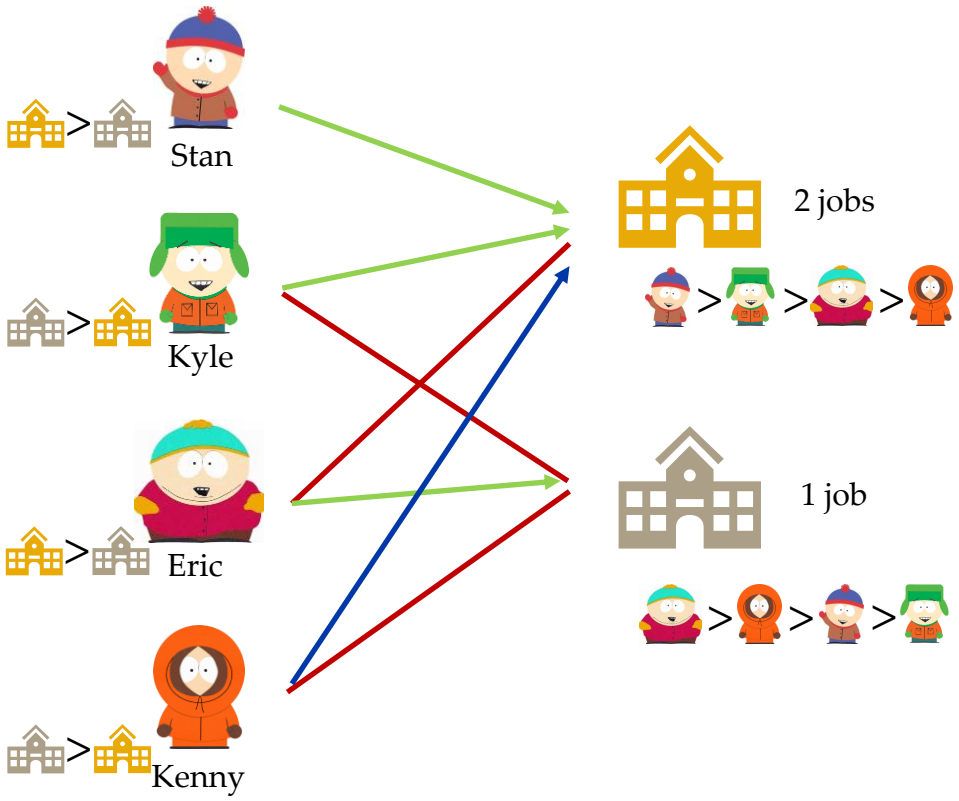

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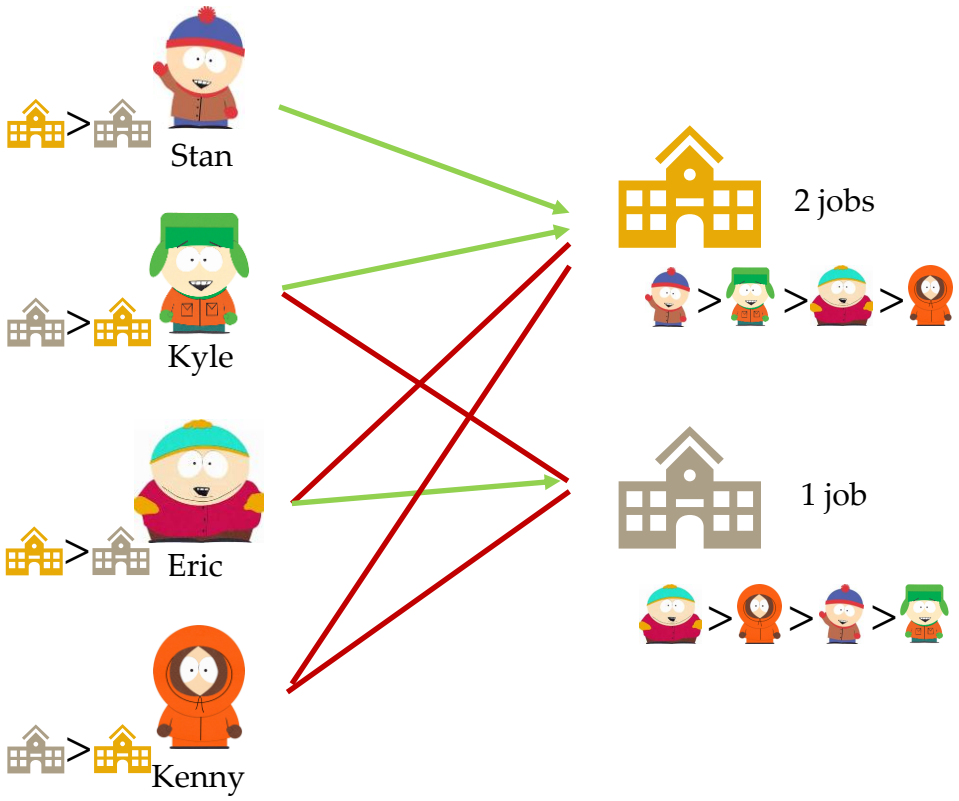

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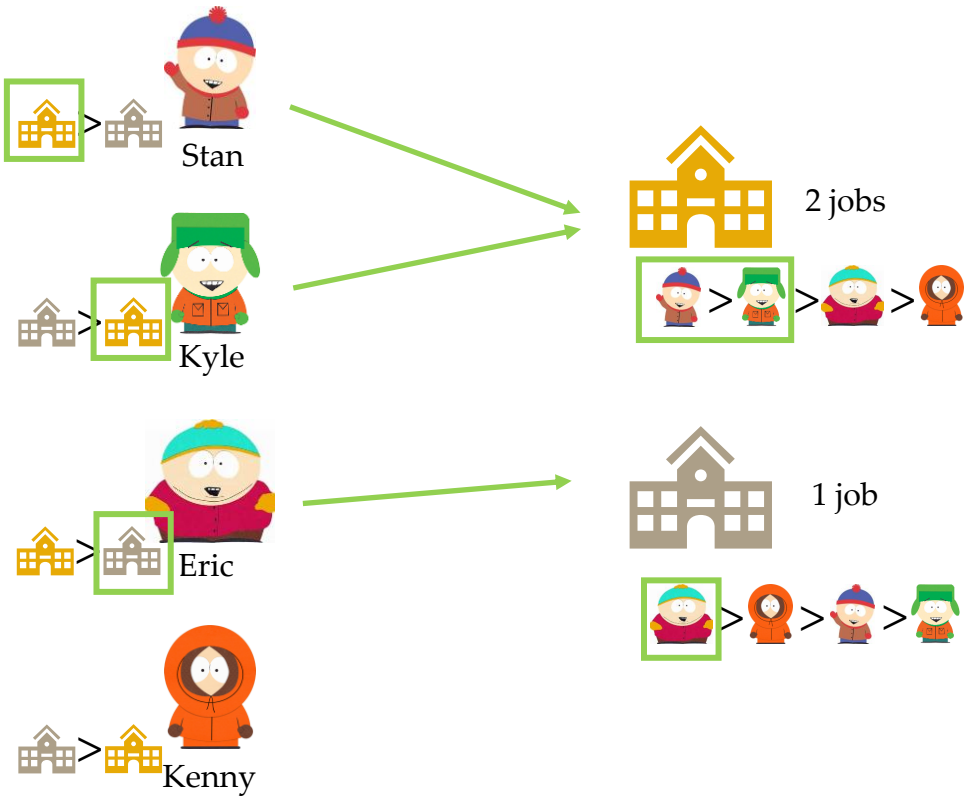

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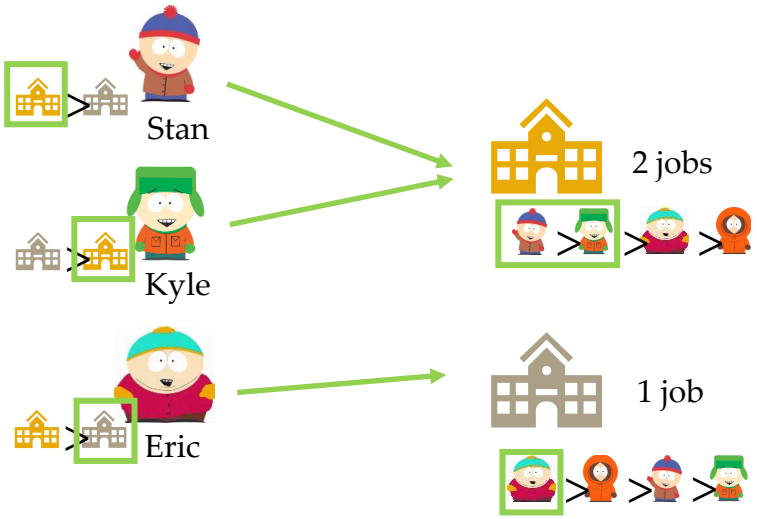
Deferred Acceptance (DA)

Q. Why might we like or dislike DA?

- Stability?
- Efficiency?
- Strategyproofness?
- Inference?



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```

Stability of DA ✔

Theorem. Doctor-proposing deferred acceptance terminates in a stable matching.

Proof.

- **Terminates** in $|D||H|$ rounds with a matching, since doctors never propose to the same hospital twice.
- **Stable when terminates.**
  Suppose not, i.e. there is a blocking pair (d, h) when terminates.
  - Since $h \succ^d \mu(d)$ and doctor d was not assigned to hospital h, this means (1) doctor d proposed to h, and (2) at some step hospital h rejected d in favor of q_h doctors it preferred.
  - But at each step hospital h only improves the set of doctors it is matched to, so hospital h prefers all doctors in $\mu(h)$ to d, contradiction.

□

Stability of DA ✔

Theorem. Doctor-proposing deferred acceptance terminates in the **doctor-optimal** stable matching.

Similarly, **hospital-proposing** deferred acceptance terminates in the **hospital-optimal** stable matching.

Definition *(Informal):*
The **doctor-optimal** (hospital-optimal) stable matching is the unique stable matching preferred by all doctors (hospitals) to all their stable matches

Formally: Theorem (Conway in Knuth 1976; Blair 1988)
The set of stable matchings is a non-empty **lattice.**

Define $\mu \vee \nu$ by assigning each doctor d to their preferred match out of $\mu(d)$ and $\nu(d)$
Similarly define $\mu \wedge \nu$ by assigning doctors to their less preferred match out of $\mu(d)$ and $\nu(d)$

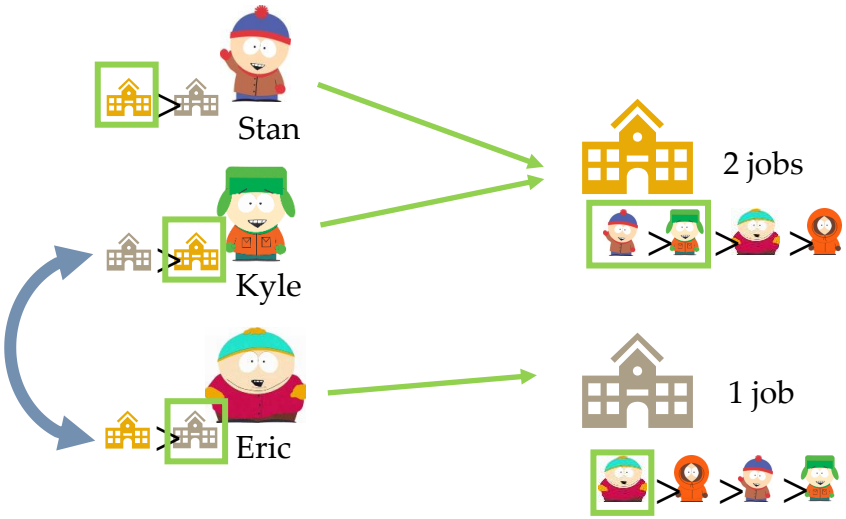
A **lattice** is a partially ordered set where every pair has:
a *greatest lower bound* (meet ∧)
and a *least upper bound* (join ∨).

Given two stable matchings μ, ν: $\mu \vee \nu$ and $\mu \wedge \nu$ are also stable; and
All doctors prefer $\mu \vee \nu$, and all hospitals prefer $\mu \wedge \nu$.

Efficiency of DA

1. **Theorem (Conway in Knuth 1976; Blair 1988).**
   Doctor-proposing DA finds the doctor-optimal stable matching (which is hospital-pessimal).
2. **Theorem (Roth 1982; Kojima 2008).**
   The doctor-optimal stable matching μ^D is weakly Pareto efficient for doctors, i.e. there is no matching μ (stable or not) such that $\mu(d) \succ^d \mu^D(d)$ for all $d \in D$.

(Note: Not strongly Pareto efficient for doctors!)



Efficiency of DA ?

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2. **Theorem (Roth 1982; Kojima 2008).**
   The doctor-optimal stable matching μ^D is weakly Pareto efficient for doctors, i.e. there is no matching μ (stable or not) such that $\mu(d) \succ^d \mu^D(d)$ for all $d \in D$. *(Note: Not strongly Pareto efficient for doctors!)*
3. **Theorem (McVitie-Wilson 1970; Roth 1986).** ***"Rural Hospitals Theorem"***
   The set of matched doctors (hospitals) is the same in every stable matching.

Efficiency: ?

- Can do reasonably well for one side (doctor-optimal stable matching is weakly but not strongly PE for doctors; Theorems 1,2)
- This is at the expense of the other side (hospital-pessimal stable matching; Theorem 1).
- Cannot do better for 'rural hospitals' (Theorem 3).

Incentives under DA ?

Theorem (Dubins-Freedman 1981, Roth 1982). Doctor-proposing deferred acceptance is group-strategyproof for the doctors in one-to-one matching, and strategyproof for the doctors in many-to-one matching.

Theorem (Roth 1982). There is no strategyproof matching mechanism that always outputs a stable matching.

Example. Consider two doctors, two hospitals:

$$d_1 : h_1 \succ h_2 \qquad h_1 : d_2 \succ d_1$$
$$d_2 : h_2 \succ h_1 \qquad h_2 : d_1 \succ d_2$$

This has two stable matchings: doctor-optimal and hospital-optimal

Q. *What could be a profitable deviation for a hospital under doctor-proposing DA?*

A. *Truncating your preferences at your most preferred stable matching.*

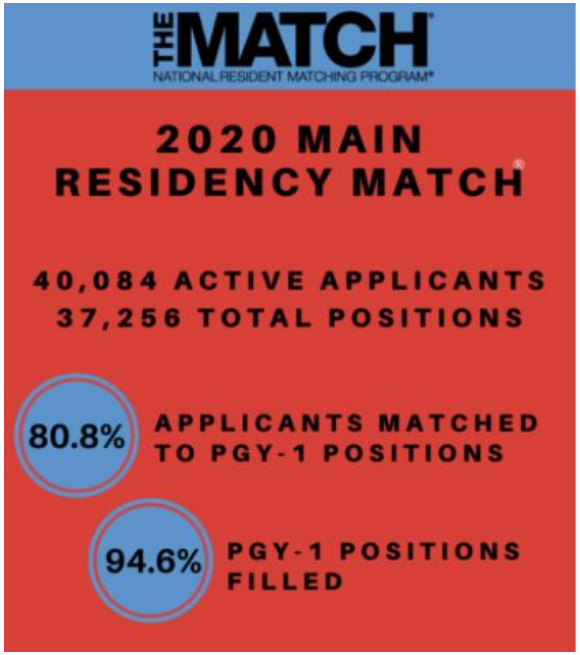
Deferred Acceptance in NRMP

March 1998.

National Residency Matching Program (NRMP) switched from unstable matching algorithm to Doctor-Proposing Deferred Acceptance

August 2022.

Doctor-Proposing Deferred Acceptance is still being used today!



Deferred Acceptance in NRMP

Stability helps with market unraveling (offers being made earlier and earlier)

Market	Stable	Still in use (halted unraveling)
NRMP	yes	yes (new design in '98)
Edinburgh ('69)	*yes*	*yes*
Cardiff	*yes*	*yes*
Birmingham	*no*	*no*
Edinburgh ('67)	*no*	*no*
Newcastle	*no*	*no*
Sheffield	no	no
Cambridge	no	yes
London Hospital	no	yes
Medical Specialties	yes	yes (~30 markets, **1 failure**)
Canadian Lawyers	yes	yes (Alberta, no BC, Ontario)
Dental Residencies	yes	yes (5) (no 2)
Osteopaths (< '94)	no	no
Osteopaths (≥ '94)	yes	yes
Pharmacists	yes	yes
Reform rabbis	yes (first used in '97-98)	yes
Clinical psych	yes (first used in '99)	yes

Source: Roth, Congressional Briefing 2010

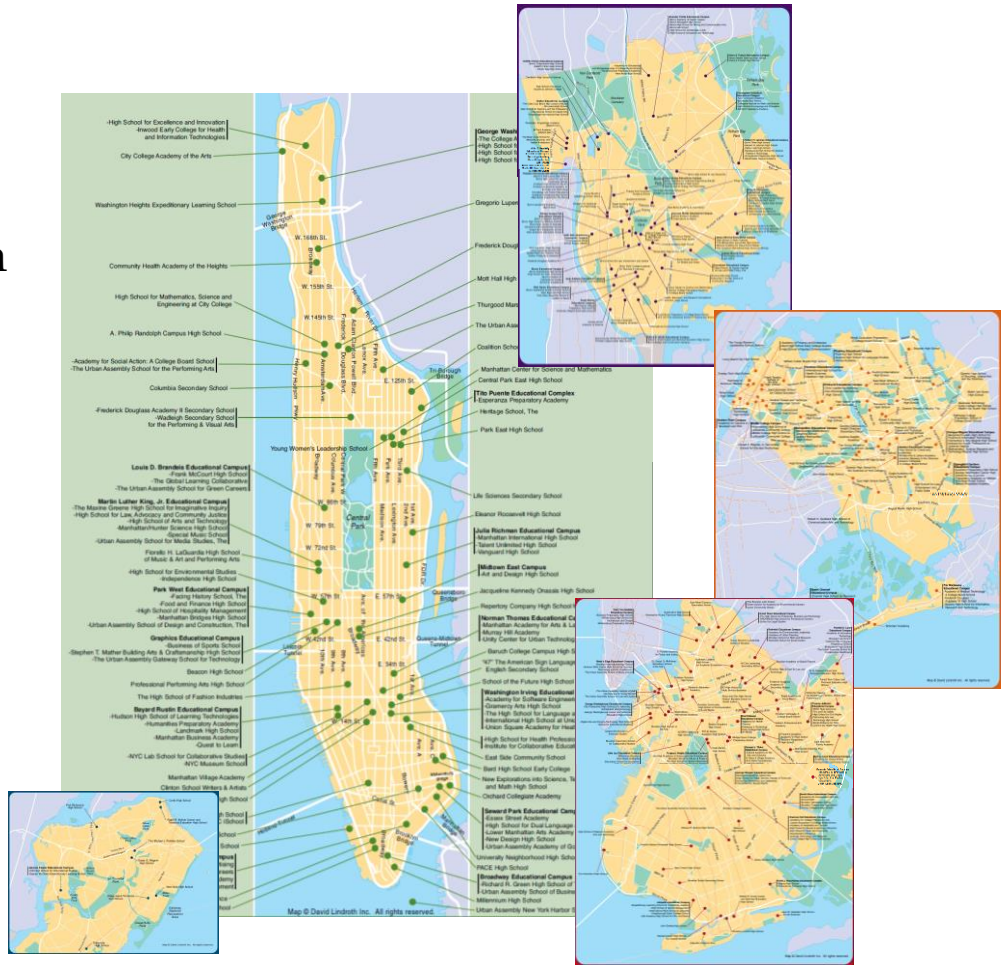
Deferred Acceptance in NYC

October 2003.

New York City Department of Education switched from unstable matching algorithm to Student-Proposing Deferred Acceptance

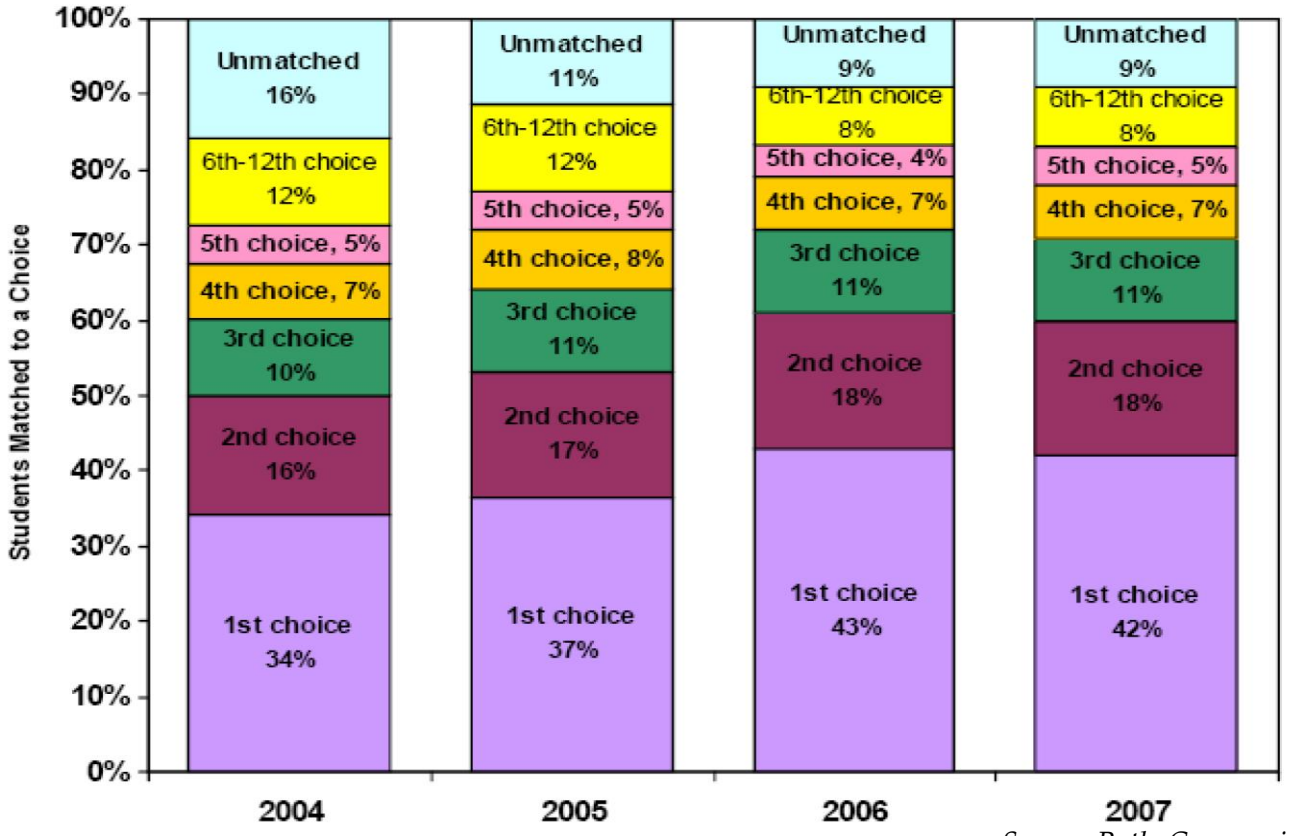
August 2022.

Student-Proposing Deferred Acceptance is still being used today!



Deferred Acceptance in NYC

Deferred Acceptance Created Efficiency Gains.



Schools stopped withholding capacity

Source: Roth, Congressional Briefing 2010

Two-Sided Allocation

Mechanism	Stable	Pareto efficient	Strategy proof	Practice	Inference
Deferred Acceptance *(Proposer-optimal stable matchings)*	✔	Doctor-opt ? weakly PE for doctors	Proposing ? side	✔	✔ *Stay tuned!*
Boston Mechanism *(Immediate acceptance)*					
Top Trading Cycles *(Top agents trade)*					

BREAK

In small groups:

- Share:
  - Name
  - Affiliation
  - Where you're from
- Find:
  - A fun fact that is shared by everyone in the group

Two-Sided Allocation

- **Model – One-Sided Incentives**
- Deferred Acceptance
- Boston Mechanism
- Top Trading Cycles

Model: One-Sided Incentives

- One-sided strategic agents
  - Students $s \in S$
    - Preferences $\succ^s$ over schools,
      $i \succ^s j$ means s prefers school i to school j
  - Schools/colleges $c \in C$ are not strategic
    - Priorities $\succ^c$ over students, quotas q_c
      $a \succ^c b$ means c prefers student a to student b

- Matching $\mu: S \to C \cup \{\emptyset\}$
  - Many-to-one matching: $|\mu^{-1}(h)| \leq q_i$ for all $h \in H$
  - i.e. A matching is an allocation of students to schools

Two-Sided Allocation

One-Sided Incentives

Mechanism	Stable	Pareto efficient	Strategy proof	Practice	Inference
Deferred Acceptance *(Proposer-optimal stable matchings)*	✓	Doctor-opt ✗ weakly PE for doctors	✓ Proposing side	✓	✓ *Stay tuned!*
Boston Mechanism *(Immediate acceptance)*					
Top Trading Cycles *(Top agents trade)*					

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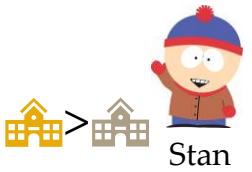
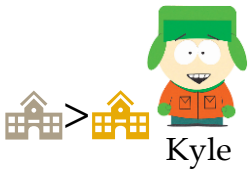
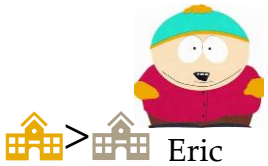
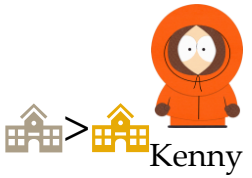
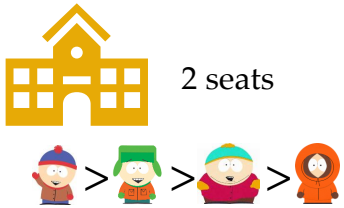
Boston Mechanism (Immediate Acceptance)











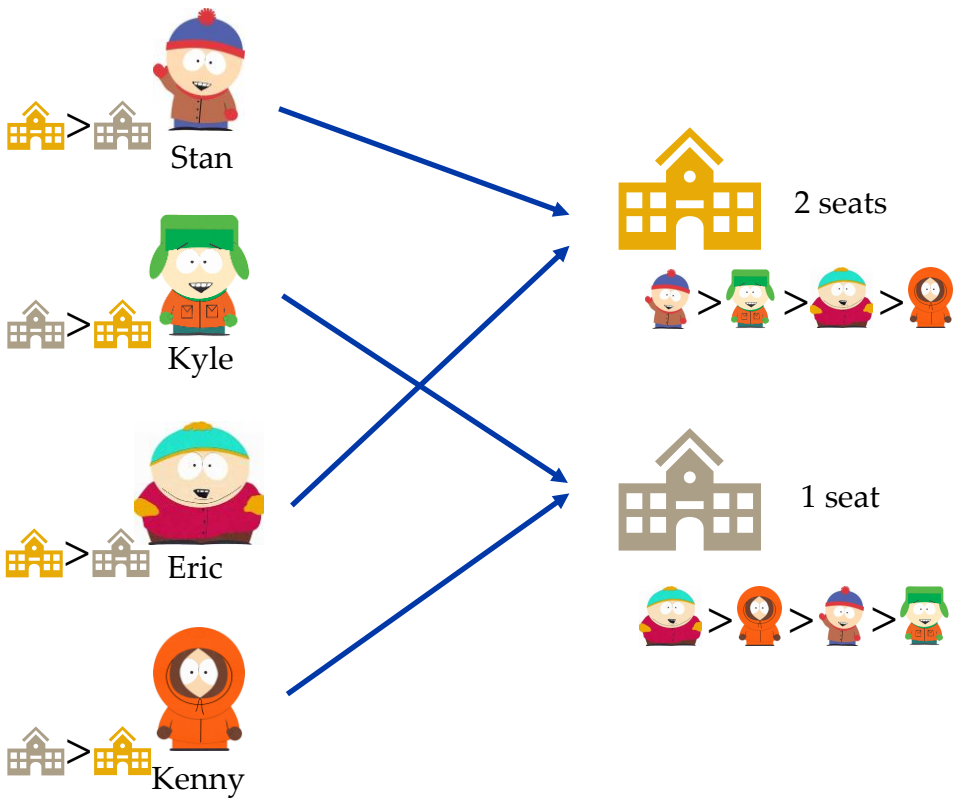
Boston Mechanism ("Immediate Acceptance")

while some student can still propose:

- all unassigned students propose to favorite school that has not rejected them
- all schools **immediately** accept top students (up to residual capacity) who have proposed to them this round and reject the rest

each student assigned to school that accepted them

Boston Mechanism (Immediate Acceptance)



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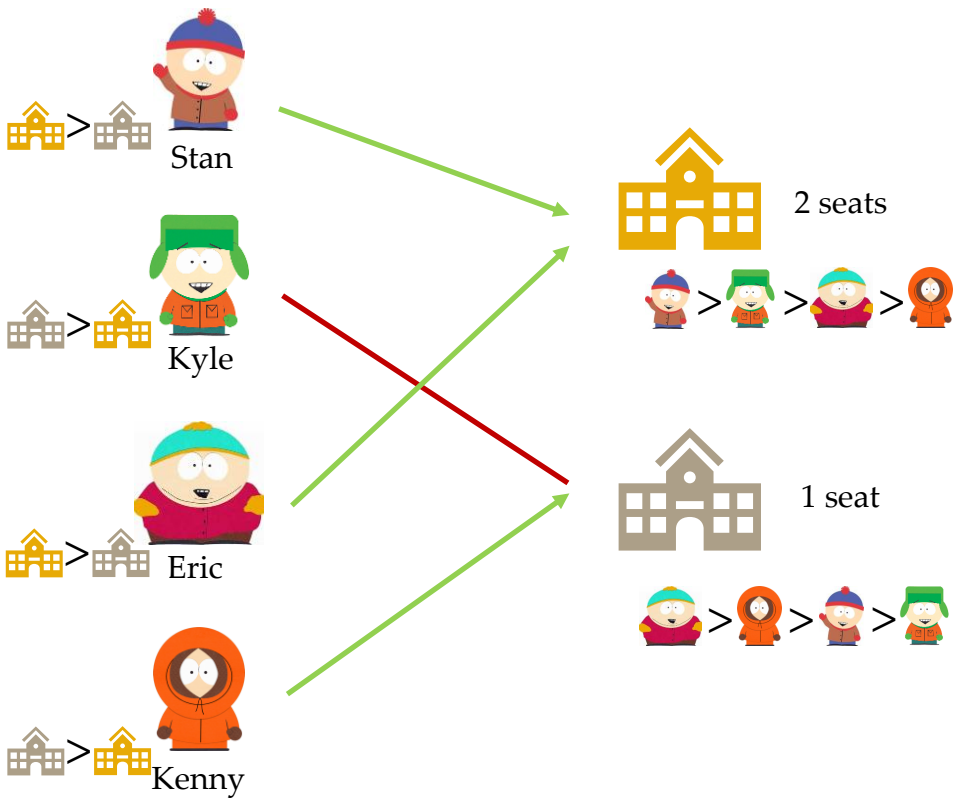

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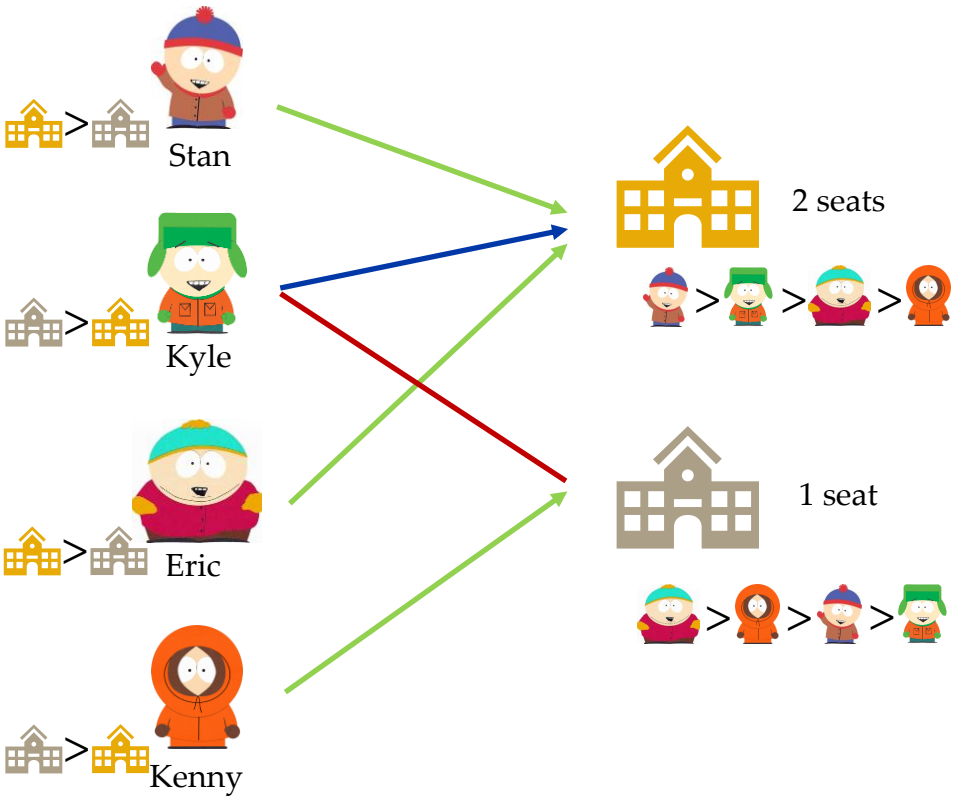

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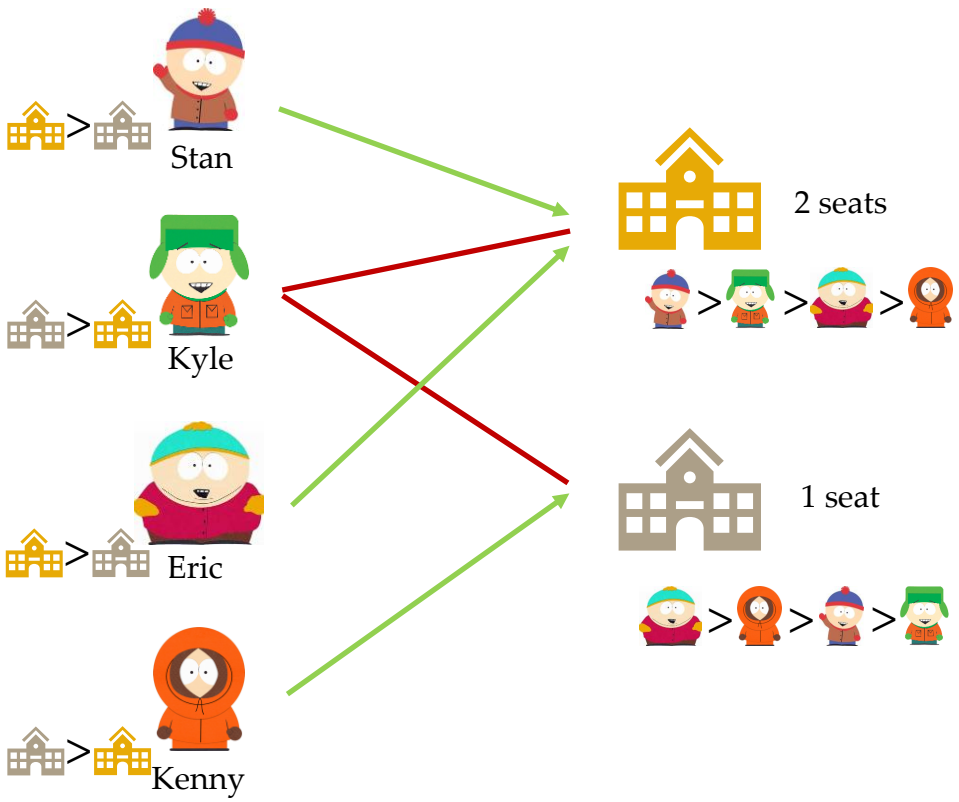

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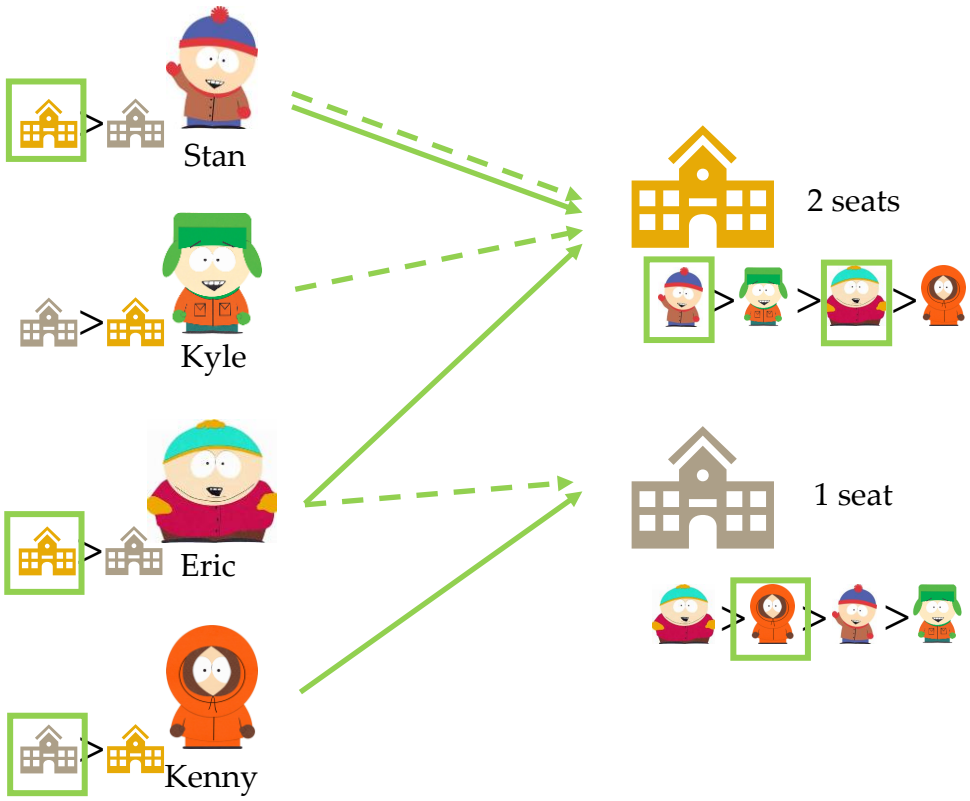

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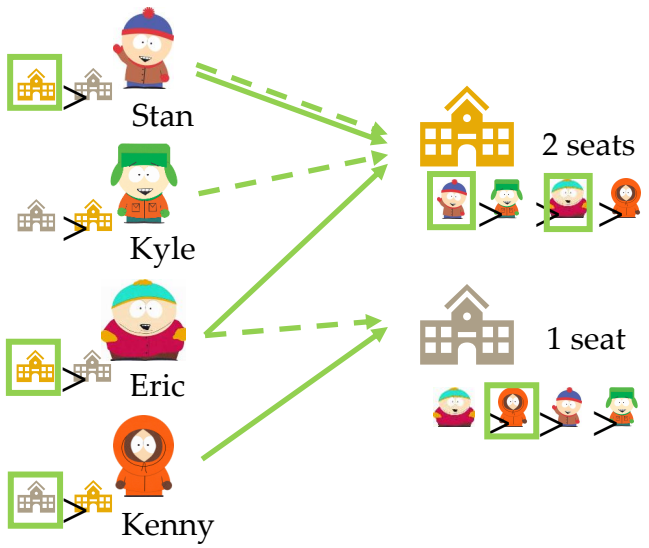
Boston mechanism assignment — — — DA assignment

Boston Mechanism (Immediate Acceptance)

Q. Why might we like or dislike the Boston Mechanism?

- Stability?
- Efficiency?
- Strategyproofness?
- Inference?



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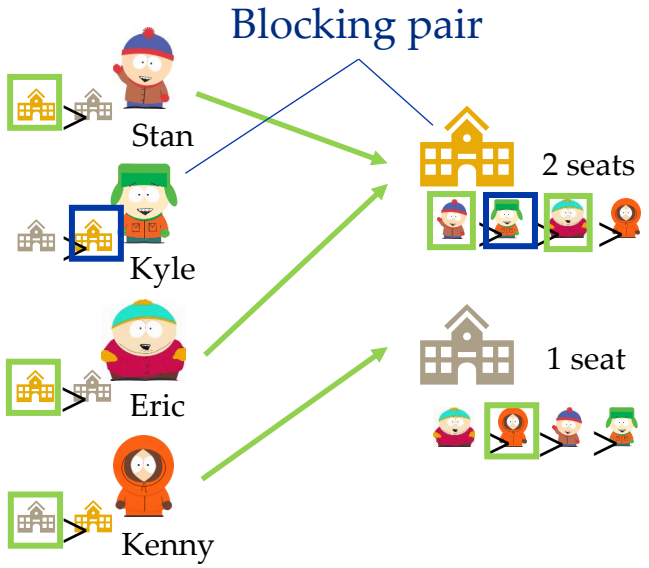
Boston Mechanism (Immediate Acceptance)

Q. Why might we like or dislike the Boston Mechanism?

- Stability? ✗
- Efficiency?
- Strategyproofness?
- Inference?



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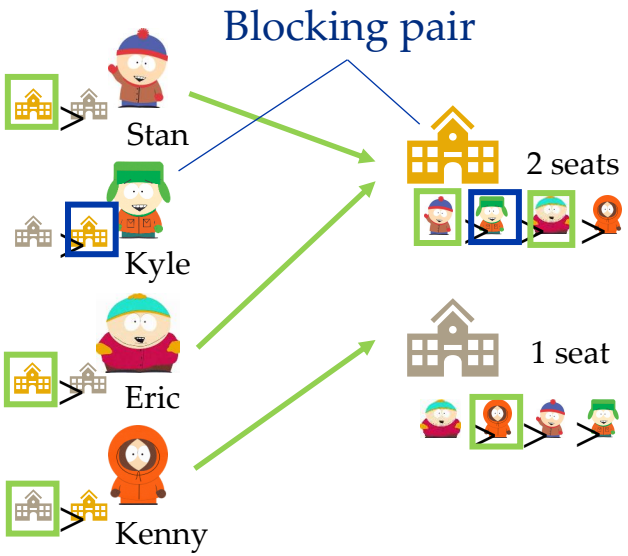
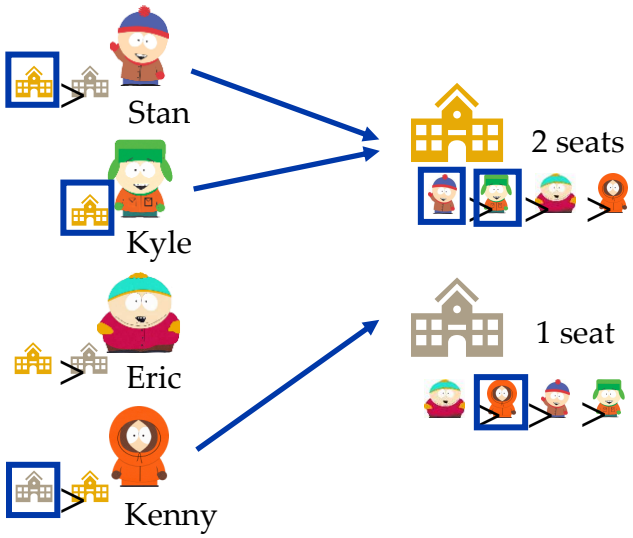
Incentives under the Boston Mechanism

Lemma. The Boston Mechanism is not strategyproof.



Suppose Kyle misreports

instead of



Theorem (Ergin and Sonmez 2006).
The set of Nash equilibrium outcomes of the Boston Mechanism is equal to the set of stable matchings under true preferences.

Boston Mechanism (Immediate Acceptance)

Q. Why might we like or dislike the Boston Mechanism?

- Stability ?
  - Not stable w.r.t. reported preferences
  - Stable in equilibrium
  - Do students report in equilibrium?
- Efficiency ?
  - Efficient w.r.t. reported preferences
  - In equilibrium, weakly less efficient for students than student-optimal stable matching
- Strategyproofness ✗
  - Not strategyproof (on either side)
- Inference ?

Boston Mechanism ("Immediate Acceptance")

```
while some student can still propose:
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Two-Sided Allocation

One-Sided Incentives

Mechanism	Stable	Pareto efficient	Strategy proof	Practice	Inference
Deferred Acceptance *(Proposer-optimal stable matchings)*	✓	Doctor-opt ✗ weakly PE for doctors	Proposing ✓ side	✓	✓ *Stay tuned!*
Boston Mechanism *(Immediate acceptance)*	?	?	✗	?	?
Top Trading Cycles *(Top agents trade)*					

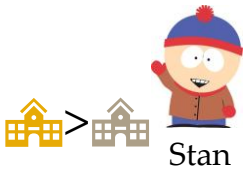
Depends on equilibrium reporting behavior

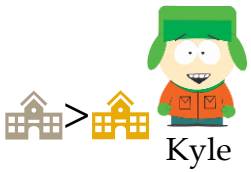
Two-Sided Allocation

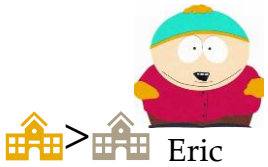
- Model – One-Sided Incentives
- Deferred Acceptance
- Boston Mechanism
- **Top Trading Cycles**

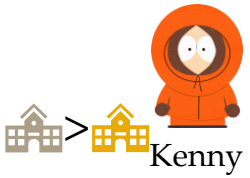
Top Trading Cycles

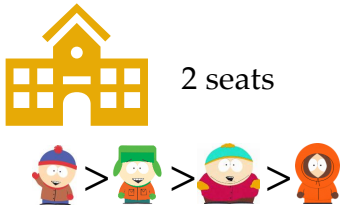
[Shapley & Scarf '74, Abdulkadiroglu & Sonmez '03]

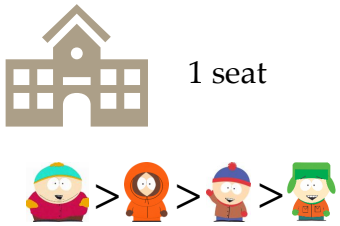












Top Trading Cycles

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while some students unassigned
or some schools unfilled:
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Top Trading Cycles

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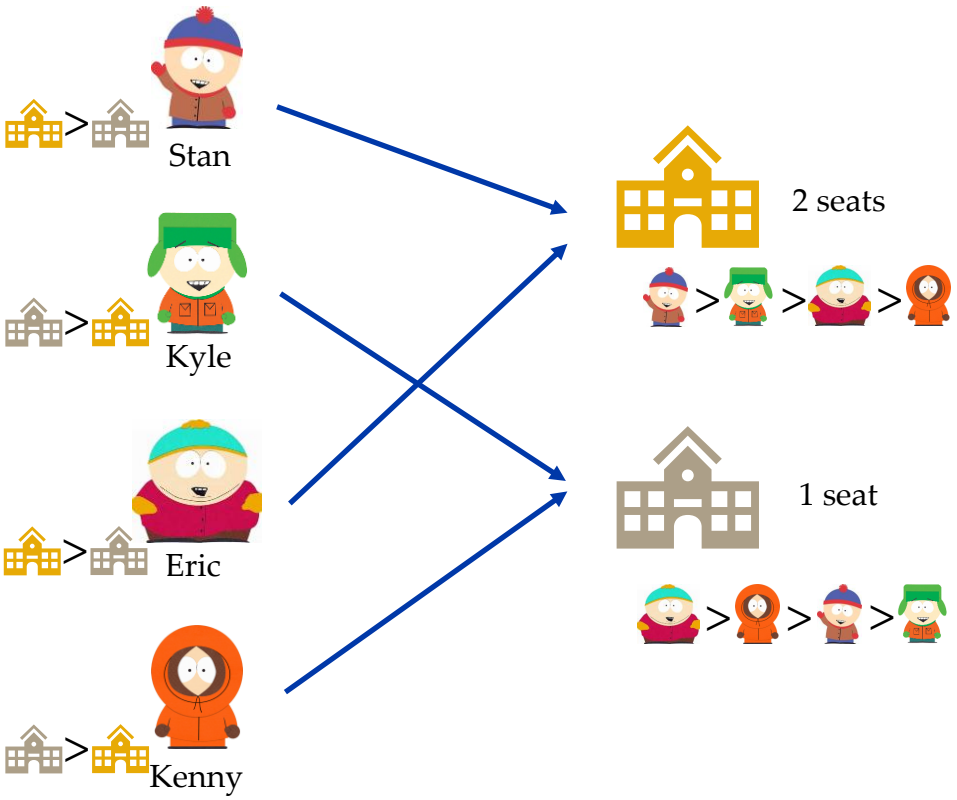

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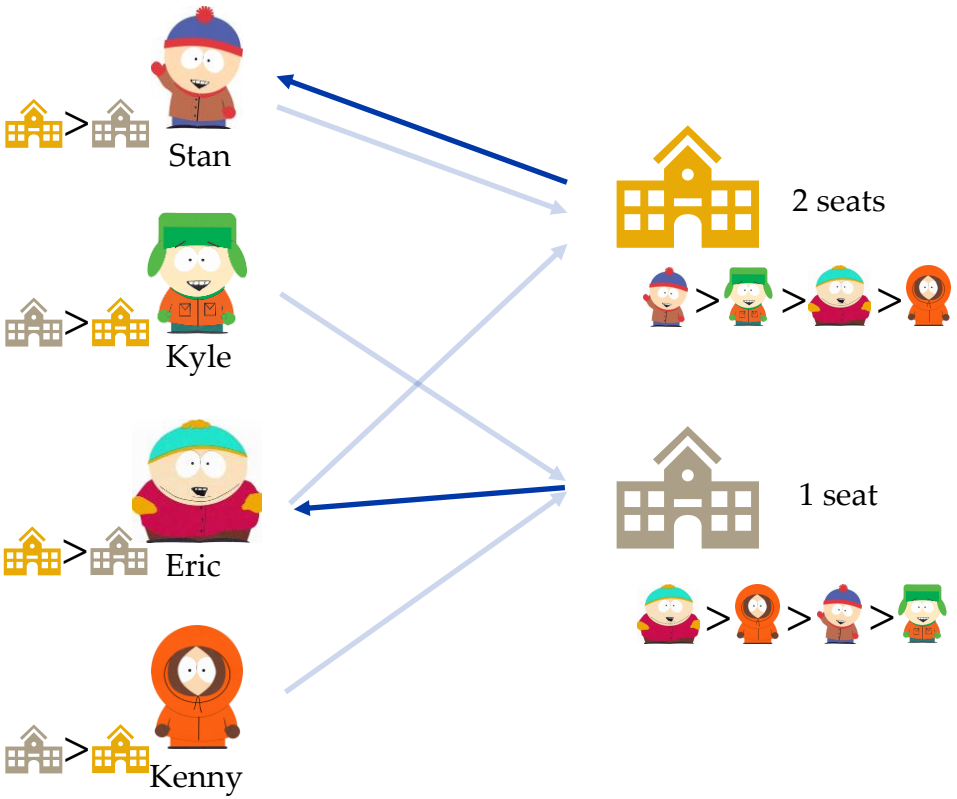

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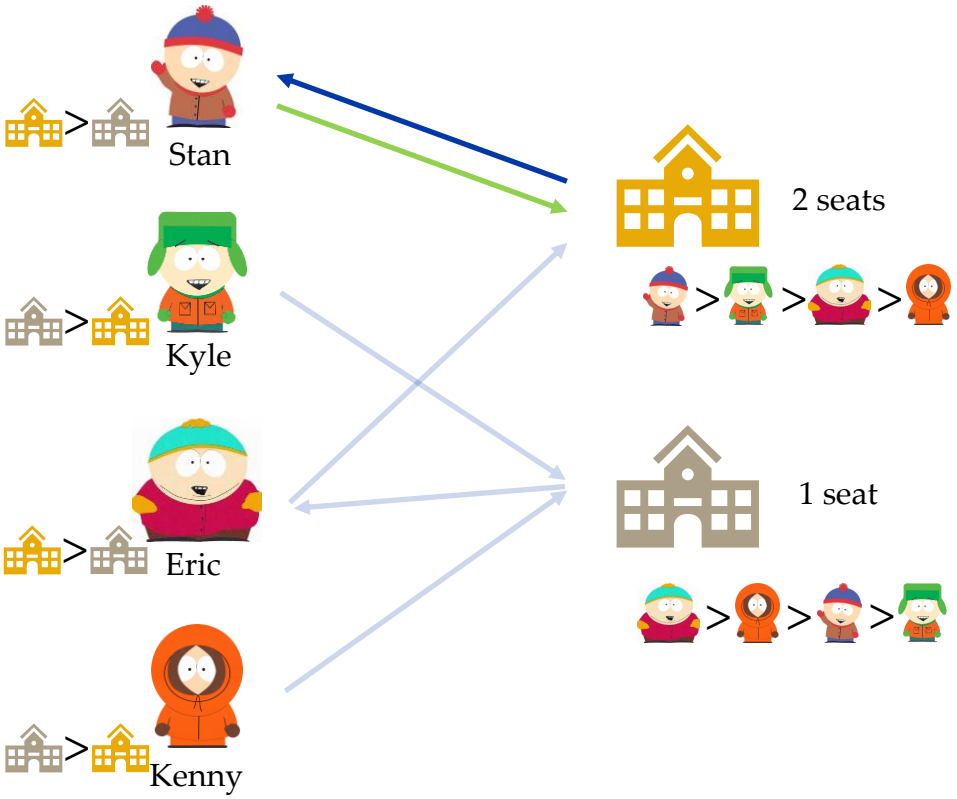

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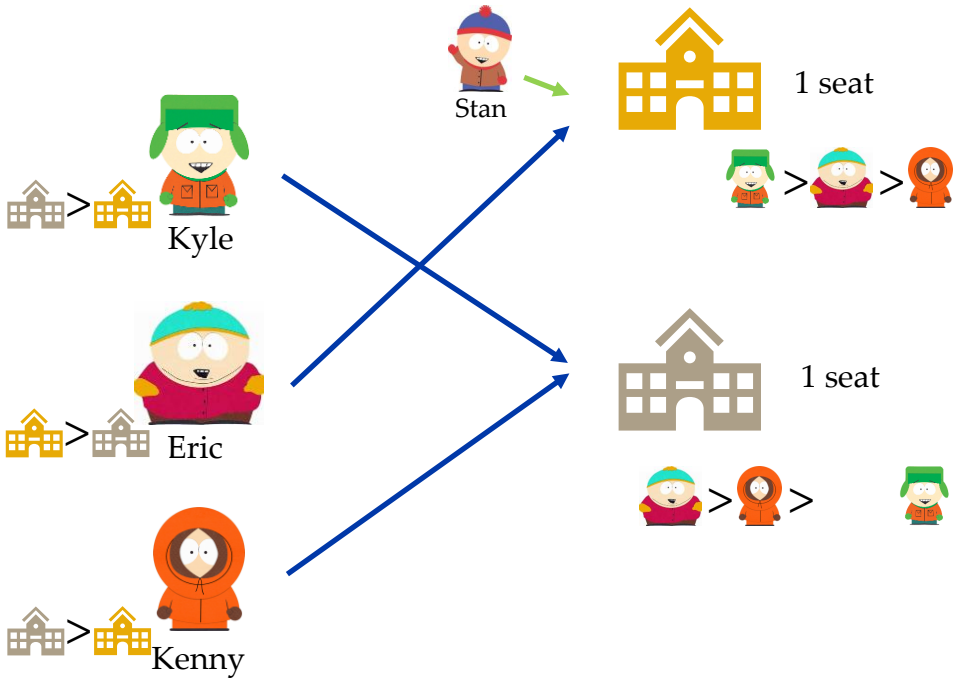

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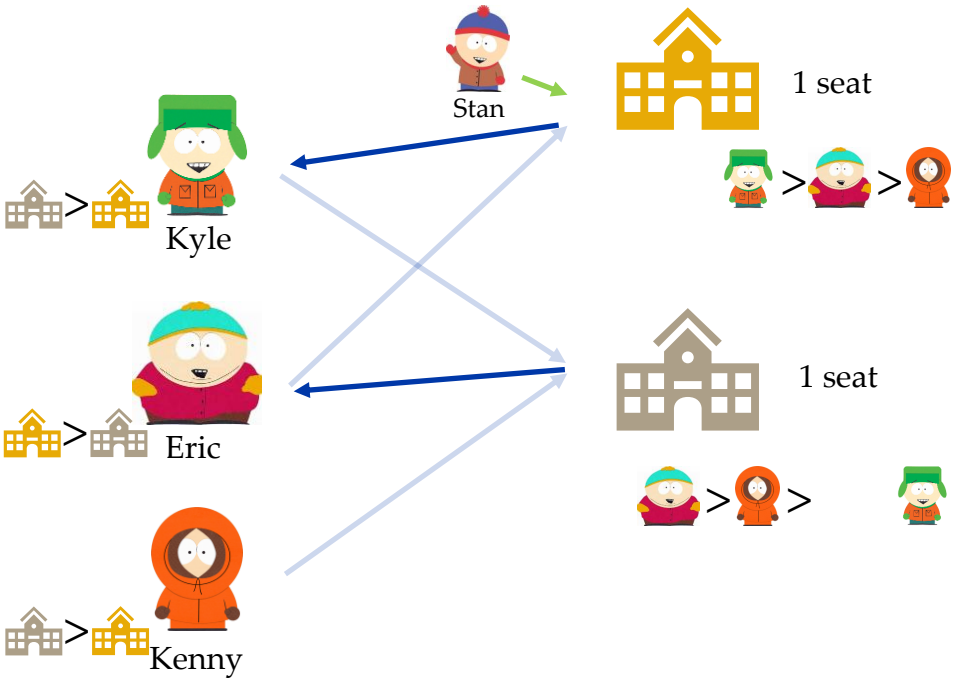

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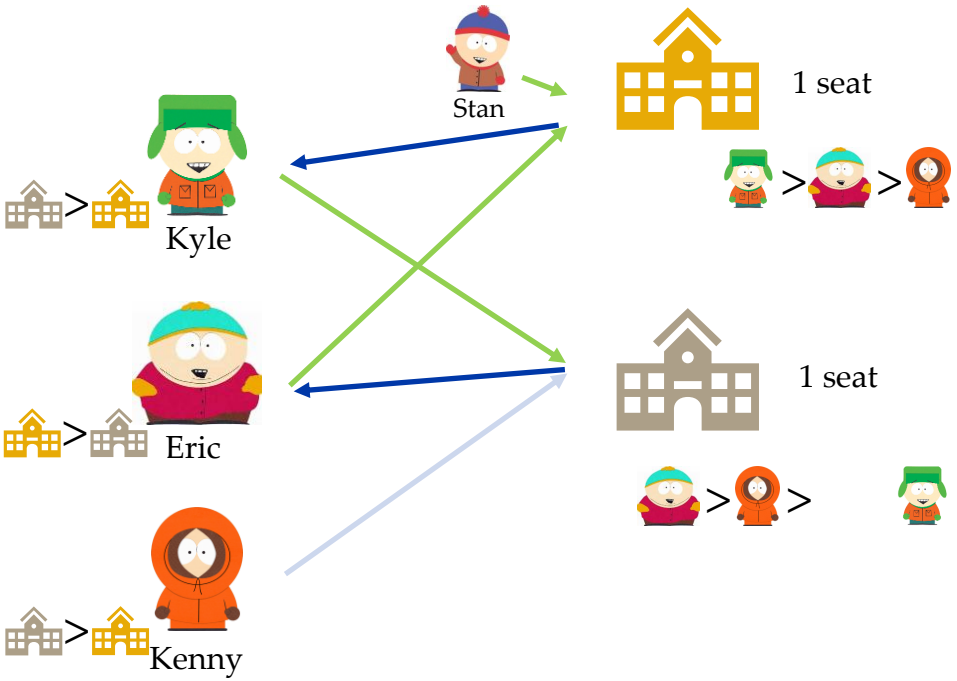

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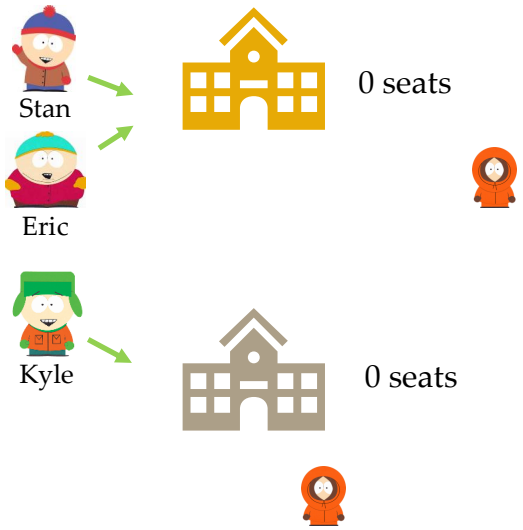

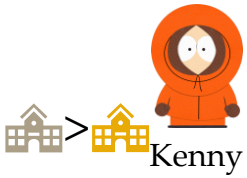

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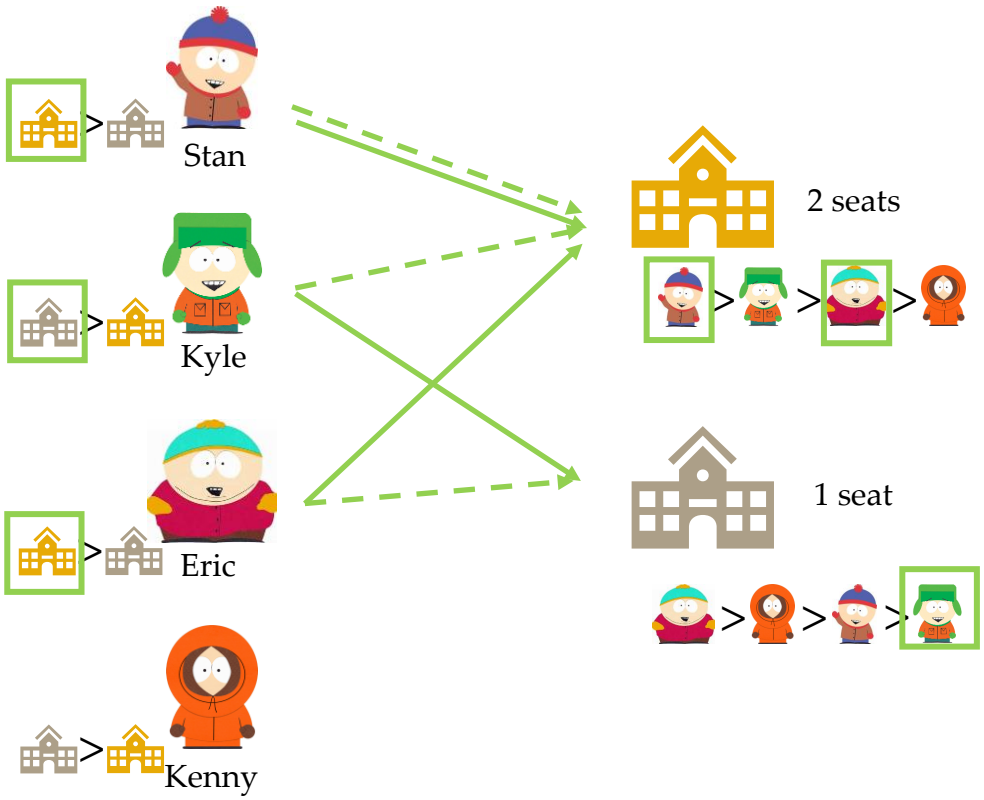

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TTC assignment

DA assignment

Top Trading Cycles

Q. Why might we like or dislike TTC?

- Stability?
- Efficiency?
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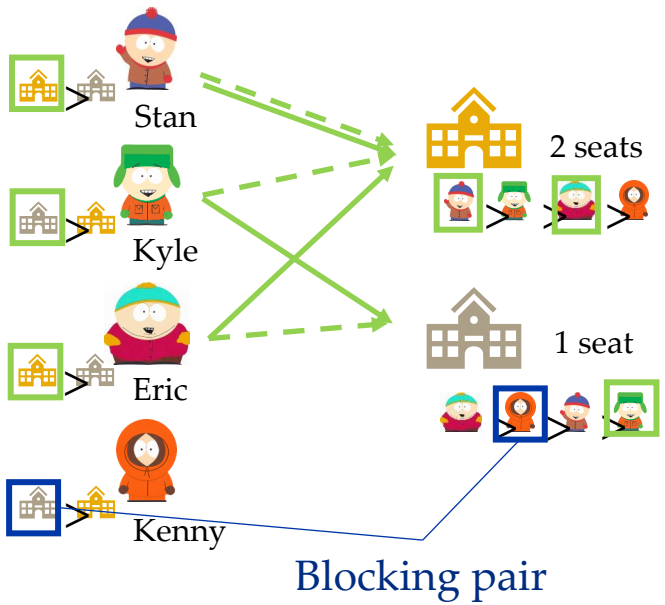

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Properties of TTC

Desirable properties.

- Strategyproof for students:
  Truthtelling is a dominant strategy
- Pareto efficient for students:
  No other assignment is weakly preferred by all students (strictly by some)
- Stable: No blocking pairs.
  i.e. No school and student who prefer each other to their match

DA is strategyproof for students and stable.

Theorem (Abdulkadiroglu and Sonmez 2003).

TTC is strategyproof and Pareto efficient for students.

Theorem.
No mechanism is stable and strategyproof and Pareto efficient for students.

Top Trading Cycles

[Shapley & Scarf '74, Abdulkadiroglu & Sonmez '03]

Q. Why might we like or dislike TTC?

- Stability ✗
- Efficiency ✓
  - TTC is **Pareto efficient** for students
- Strategyproofness ✓
  - TTC is **strategyproof** for students
- Works in practice ✗
  - We have records that (some version of) TTC was used in exactly two cities: New Orleans and San Francisco.
  - Both cities no longer use TTC.

Top Trading Cycles

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One-Sided Incentives

Mechanism	Stable	Pareto efficient	Strategy proof	Practice	Inference
Deferred Acceptance *(Proposer-optimal stable matchings)*	✓	Doctor-opt ✗ weakly PE for doctors	Proposing ✓ side	✓	✓ *Stay tuned!*
Boston Mechanism *(Immediate acceptance)*	? *Depends on equilibrium reporting behavior*	?	✗	?	?
Top Trading Cycles *(Top agents trade)*	✗	✓ One -sided	✓ One -sided	✗	✓ *Stay tuned!*