American Economic Journal: Microeconomics 2017, 9(1): 88–122
https://doi.org/10.1257/mic.20150183

Dynamic Assignment of Objects to Queuing Agents[†]

By Francis Bloch and David Cantala*

We analyze the dynamic assignment of objects to agents organized in a constant size waiting list. Applications include the assignment of social housing and organs for transplants. We analyze the optimal design of probabilistic queuing disciplines, punishment schemes, and information release. With private values, all agents prefer first-come first-served to the lottery, but waste is lower at the lottery. With common values, all agents prefer first-come first-served to any other mechanism, and waste is minimized at the lottery. Punishment schemes accelerate turnover in the queue and information release increases the value of agents at the top of the waiting list. (*JEL* C78, D44, D82)

This paper analyzes the dynamic assignment of objects to agents organized in a constant size waiting list. Objects arrive over time, and each time a new object becomes available it is offered to agents according to a fixed sequence. Each agent in the sequence decides whether to accept the object or not. If the object is rejected by all agents in the waiting list, it is wasted. If one agent accepts the object, a new agent enters the queue and all agents following the agent who picked the object move up one rank in the waiting list. Agents have a common additive waiting cost. Our objective is to study how different probabilistic queuing disciplines, different punishment schemes after a rejection, and different policies of information release affect the behavior of agents in the waiting list, their expected welfare, the turnover in the queue, and the amount of waste.

The situation we study arises whenever there exists a huge imbalance between demand and supply for an object and monetary transfers cannot be used to match the two sides of the market. Examples include the assignment of social housing, of deceased donor organs for transplant, or of spots in daycare. In all these examples, objects are heterogeneous—apartments that become available have different sizes and locations, organs are harvested on deceased donors of different ages and health conditions, daycares have different staffs and amenities. Agents have preferences over the heterogeneous objects which are assumed to be uncorrelated over time. We

* Bloch: Paris School of Economics, Université Paris 1, 106-112 Boulevard de l'Hopital, 75647 Paris CEDEX 13, France (e-mail: francis.bloch@univ-paris1.fr); Cantala: El Colegio de Mexico, Col. Pedregal de Sta. Teresa C.P. 10740, Mexico, D.F. Mexico (e-mail: dcantala@colmex.mx). We thank two anonymous referees, Atila Abdulkadiroglu, Jean-François Arenes, Anna Bogomolnaia, Estelle Cantillon, Olivier Compte, Lars Ehlers, Ed Glaeser, Moshe Haviv, Sergiu Hart, Guy Laroque, Jacob Leshno, Hervé Moulin, Ütku Unver, and seminar participants at different institutions for their comments on the paper. This work was supported by the Agence Nationale de la Recherche under contract ANR-13-BSHS1-0010.

[†] Go to https://doi.org/10.1257/mic.20150183 to visit the article page for additional materials and author disclosure statement(s) or to comment in the online discussion forum.

consider both the private values case where agents have idiosyncratic preferences over the different objects (as in the case of public housing) and the common values case where all agents share the same preference over the objects (as in the case of organ transplants).

Because objects are heterogeneous, when an agent receives an offer, he faces an optimal stopping problem. Should he accept the current object or wait to receive a better object in the future? The answer to this question depends on the characteristics of the assignment system, like the queuing discipline, punishment scheme after a rejection, or information released to the agents which affect the continuation value. The features of the assignment system also influence the total number of assignments. If the queuing discipline gives an advantage to agents according to their rank in the waiting list, agents at the top of the waiting list have a higher continuation value and hence are more likely to be selective and to reject current proposals. This may result in a sequence of rejections, leading to waste for objects with a short lifetime, like organs and vacancies, and delays for durable objects like apartments.[1] The design of optimal assignment schemes must thus balance the values of agents in the waiting list with the inefficiencies resulting from the waste of objects.

Existing assignment mechanisms fall into three categories: first-come first served (FCFS) mechanisms with different priority groups, scoring rules, and lotteries. FCFS mechanisms with priority groups divide applicants into different priority groups and rank each applicant within a priority group according to the date of entry in the waiting list. When an object becomes available, it is first assigned to a priority group, either through absolute priorities—priority groups are ranked and the object is always proposed to priority groups in the same order—or through a rotating or quota scheme—different priority groups are offered the object according to a fixed rotating sequence or the objects are proportionally divided across priority groups. Within each priority group, objects are offered to applicants in sequence according to their order in the waiting list. In the United States, the 1964 Civil Rights Act requires public housing authorities to assign units to applicants according to a Tenant Selection and Assignment Plan (TSAP), which functions as a FCFS mechanism with priority groups.[2] FCFS mechanisms with priority groups are also used to assign hearts and intestines for transplant.[3] Scoring rules assign points to applicants and establish priorities among applicants according to their number of points. Scoring mechanisms are used to assign council housing in England and Wales and, since the fall of 2014, in Paris.[4] In both cases, waiting time is used as a tie-breaking rule to distinguish

[1] See the article in the *New York Times* dated September 20, 2012 for a discussion of waste in the assignment of kidneys for transplants: http://www.nytimes.com/2012/09/20/health/transplant-experts-blame-allocation-system-for-discarding-kidneys.html.

[2] The definition of priority groups and specific assignment of units across priority groups differ across public housing authorities. For example, in the TSAP of the New York Housing Authority, apartments rotate across five priority groups, while the Chicago Housing Authority assigns apartments across priority groups using absolute priorities. The definitions of priority groups are similar but not identical in the two TSAPs. General guidelines for TSAPs of the department of Housing and Urban Development are given in www.hud.gov/offices/pih/rhiip/phguidebooknew.pdf. Up-to-date descriptions of the TSAPs in New York and Chicago can be found at http://www.nyc.gov/html/nycha and http://www.thecha.org/.

[3] See the OPTN Policies 6.5.B and 7.3.A updated June 24, 2015 available at http://optn.transplant.hrsa.gov.

[4] See https://www.gov.uk/council-housing for council housing in England and Wales and http://www.paris.fr/services-et-infos-pratiques/demander-un-logement-social-37 for social housing in Paris.

between applicants with the same score. Scoring rules are also used to assign deceased donors' kidneys, livers, and lungs. For the allocation of kidneys, waiting time has a major weight in the computation of the score. In the case of liver and lung transplants, waiting time is only used to break ties among applicants with the same score.[5] Finally, lotteries are used in New York to allocate subsidized housing managed by private and nonprofit developers and listed by NYC Housing connect. Eligible applicants to affordable housing units are chosen by a uniform random draw.[6]

Overview of the Main Results.—The main result of the paper compares the value of all agents in the waiting list under FCFS and the lottery. With private values, we show that *all agents in the waiting list,* including the last one, prefer the FCFS scheme to the lottery. This surprising result rests on the following intuition. When the waiting list is of constant size n, the expected waiting time of any agent conditional on the fact that the object is picked by him or any agent with higher rank, is independent of the queuing discipline. An agent at rank i will on average wait i periods before obtaining the object. The expected value of the object depends both on the queueing discipline and on the strategies of the agents, as agents at different ranks may choose different strategies. However, in a pure strategy symmetric equilibrium in the lottery, all agents adopt the same strategy, and the expected value of the object is independent of the rank of the agents. The equilibrium value of an agent in the lottery, who on average waits n periods, is thus equal to the equilibrium value of the *last* agent under FCFS. As equilibrium values are increasing in rank under FCFS, all agents prefer FCFS to the lottery. As equilibrium values are positively correlated to agent's selectivity, FCFS also results in higher expected waste than the lottery.

With common values, we obtain a stronger result and show that *all agents have a higher equilibrium value under FCFS than under any other probabilistic queuing discipline.* This result is explained as follows. When the strategies of the agents are fixed, any increase in the probability that the object is proposed to agents with higher rank reduces the expected waiting time and increases the rate at which agents move up in the waiting list and obtain objects of higher quality. Both effects increase the equilibrium values of all agents. Furthermore, with common values case, the equilibrium behavior of agents is monotonic in the probability that the object is assigned to agents with higher rank.[7] Hence, agents are most selective in the FCFS scheme and least selective in the lottery. The values of all agents are thus highest under FCFS and lowest under the lottery, and waste is minimized under the lottery.

While these results are obtained in a model with binary values and identical agents, we show that the results extend to a model with continuous values and heterogeneous agents. In a two-agent model with continuous values, we show that the main result of the model holds in a stronger form. The FCFS rule dominates all other

[5] See the OPTN Policies 8.5.D, 9.6.D and 10.4.A updates June 24, 2015 available at http://optn.transplant.hrsa.gov.

[6] See https://a806-housingconnect.nyc.gov.

[7] This monotonicity fails with private values case because agents' continuation values after a rejection are not equal to their equilibrium values. After a rejection, an agent may still move up in the waiting list if an agent ordered after him in the sequence, but before him in the waiting list, draws a high value for the object. When the probability that the object is offered to an agent with higher rank goes up, agents are less likely to move up in the waiting list after a rejection, and the continuation value after a rejection may go down.

probabilistic queuing disciplines, both with private and common values. However, the technique of proof does not extend to longer waiting lists, and we do not know whether this comparison still holds for arbitrary waiting list sizes. We consider a two-agent model with heterogeneous waiting costs, and show that the optimal queuing discipline is lexicographic: it first proposes the object to the agent with the highest waiting cost, and breaks ties among agents with identical waiting costs by proposing the object to the agent with highest rank in the waiting list.[8]

We use the theoretical model to discuss other features of assignment rules. Punishment schemes are used in the assignment of social housing to prevent continuous rejections.[9] The guidelines of the department of Housing and Urban Development indicate that applicants who reject apartments without good cause should be taken off the waiting list. In Paris, an applicant who refuses an offer is kept out of the assignment process for six months and regains his rank on the waiting list after this waiting period. We observe that any punishment scheme reduces the value of agents in the queue, making them less selective. An alternative to the sequential offer mechanism is to ask agents to apply and commit ex ante to accepting objects after they observe their value. Housing Connect in New York, most councils in England, and the city of Paris have adopted the latter system, implementing a website where agents apply for specific housing units. We prove a neutrality result: This alternative system results in the same equilibrium value and behavior for all agents in the waiting list. We also consider the effect of information release about an agent's rank in the sequence of offers and show that this information release increases the value of the top agent in the waiting list but results in ambiguous effects on the other agents.

Our analysis shows the existence of a tension between the values of agents in the list (the "insiders") and the turnover in the queue, which allows "outsiders" to join the waiting list. The FCFS maximizes the value of insiders, but as a consequence induces highly selective behavior. By making agents very selective, the FCFS mechanism also minimizes the probability that an object is allocated to an agent when another agent in the waiting list values it more. But this selectivity comes at a price, as it results in a high fraction of objects being wasted, slowing down the turnover in the queue, and resulting in a loss for agents who are not currently in the waiting list. In order to accelerate turnover in the queue, the mechanism must instead induce low selectivity. The lottery then stands out as the mechanism to be used, as it results in low values for the insiders, and a high rate of acceptance of the objects.

We would like to mention at the outset some limitations of the model. First and foremost, we consider constant size waiting lists and do not allow for stochastic entry and exit in the waiting list. By fixing the size of the waiting list, we focus on the dynamic trajectory of individuals induced by endogenous acceptance decisions rather than the dynamic of the queue induced by exogenous entry and exit. In applications, the number of agents to which any object can be offered is typically very low. Given their short lifetime, organs can only be offered to a small number of

[8] This lexicographic rule corresponds to FCFS with priority groups and is very close to the scoring rule recently adopted by the Paris social housing authority.

[9] Punishment schemes are not used in the assignment of organs for transplants.

patients. Housing units are offered to a small number of applicants—the New York TSAP, for example, specifies the length of the queue as a function of the number of available units and the recent history of acceptances and rejections. Hence, we believe that the assumption of a constant size waiting list reflects real assignment systems and is a reasonable approximation. In the last section of the paper, we discuss a variant of the model where the arrival of new agents in the waiting list is stochastic. A state of the system is then described by two elements: the rank of an agent and the size of the waiting list. Because the queuing discipline affects the transition probabilities between queues of different sizes, the expected waiting time of an agent is no longer independent of the queueing discipline. The last agent has a longer expected waiting time under FCFS than under the lottery when the current queue is long (and the queue in the future is likely to be shorter) and a shorter expected waiting time when the current queue is short (and the queue in the future is likely to be longer). Hence, the domination of the FCFS over the lottery holds whenever the current queue is short, but may fail when the current queue is long. Second, we assume that agents' values are uncorrelated across time. The absence of intertemporal correlation in values seems a good approximation in the case of organ transplants, but is probably less compelling in the social housing application where agents have persistent geographic preferences. However, if we consider waiting lists restricted to specific projects or small geographic areas, the assumption of uncorrelated values becomes more plausible. Third, we restrict attention to probabilistic queuing disciplines that give an advantage to agents with higher seniority rank. We believe that in the applications we consider, giving an advantage to agents with lower rank in the waiting list would entail a high political cost and raise difficult ethical issues. In addition, as we show in a variant where we analyze the last come, first served (LCFS) queueing discipline, giving an advantage to agents with lower seniority rank may result in equilibria, where agents at the top of the waiting list are served with very low probability, raising the issue of voluntary participation in the waiting list. Finally, we focus attention on additive waiting costs. This assumption is made both for tractability and because we believe that in applications like social housing and organ transplant, agents experience flow utilities every period, before and after the assignment of the object. The additive waiting cost can be interpreted as the difference between the utility flow associated to the lowest quality object and the utility flow without the object.

Related Literature.—The assignment policies for social housing and the management of waiting lists for organs have long been the object of attention in the operations research literature and more recently been discussed in the economics literature on dynamic assignment mechanisms. Kaplan (1986) and (1987) studies tenant assignment policies as FCFS mechanisms with priority groups computing the expected probability of assignment and expected waiting time under different policies. In a similar vein, Zenios (1999) and Zenios, Chertow, and Wein (2000) model the kidney transplant waiting list as a FCFS mechanisms with priority groups with a random assignment of organs across policy groups, and computes expected waiting times and the expected fraction of agents receiving a kidney as a function of the random assignment policy.

Su and Zenios (2004); Su, Zenios, and Chertow (2004); and Su and Zenios (2005, 2006) explicitly introduce patient's choice in the queuing model for kidney transplants. (See also the book on strategic behavior in queues by Hassin and Haviv (2003) and in particular chapter 5.) Su and Zenios (2005) explicitly compute the optimal assignment policy for a fixed population of heterogeneous patients, while Su, Zenios, and Chertow (2004) simulate the assignments under different policies respecting the individual rationality of patients. Su and Zenios (2006) develop a mechanism design model to take into account the incentive constraints of patients who have private information about the value of kidneys. Closest to our analysis, Su and Zenios (2004) study the effect of the queueing discipline on the assignment of kidneys in a queueing model where agents can reject the offer. In their model, they show that the LCFS queuing discipline implements the socially optimal outcome and dominates the FCFS mechanism, which results in excessive waste. The optimality of the LCFS mechanism is due to the fact that when agents are homogeneous, the agent who enters the system internalizes perfectly the externalities he imposes on the other agents in the LCFS mechanism. (See also Hassin 1985). Su and Zenios (2004) point out, as we do, the existence of a tension between absolute priorities given to insiders and the negative externalities this behavior imposes on outsiders. Finally, Su and Zenios (2004) consider a family of queuing disciplines, which are convex combinations of LCFS and FCFS mechanisms and show that an increase in the weight on the FCFS rule results in more selective behavior. Our analysis differs from Su and Zenios (2004) in several key dimensions. First, we analyze both the private and common values cases, whereas they assume homogeneous agents. Second, we analyze the entire vector of equilibrium values, highlighting the dynamic trajectory of individual agents inside the queue, whereas they focus on a single measure of social welfare—the expected sum of values of all (homogeneous) agents in the waiting list. Third, we consider a constant size waiting list, whereas their analysis rests on the assumption that the waiting list (understood as the number of agents to whom the object may be offered) evolves over time.

Recent contributions in economics on the assignment of objects to queuing agents with a specific focus on social housing have been proposed by Leshno (2014), Schummer (2016), and Thakral (2015). Leshno (2014) studies dynamic allocation of objects to queueing agents when agent's preferences are unknown. He shows that, in the absence of transfers, a dynamic allocation mechanism where agents are placed in a priority buffer queue is incentive compatible and furthermore that it is optimal to choose uniformly among agents in the buffer queue. While his study is inspired by the same applications to social housing and organ transplant as ours, the two papers differ in many respects. First, in Leshno (2014), agent's preferences are perfectly correlated across time—some agents prefer to live in the north, others in the south, and the mechanism is designed to elicit this persistent type. In our model, there is no persistence of types and hence no information to be elicited from agents. Second, Leshno (2014) only considers two types of objects—there is no vertical quality differentiation—and assumes that there is always an agent who is assigned the object. Hence, there is no waste in his model, and the efficiency discussion is limited to one criterion: the probability of misallocation.

Schummer (2016) considers a model with homogeneous agents receiving heterogeneous objects over time as in Su and Zenios (2004). He considers a planner who can make arbitrary changes in agents' continuation values (by "influencing" them to accept objets of a given value) and analyzes the effect of influence over the values of agents in the waiting list. He shows that all agents are adversely affected by the intervention of the planner, echoing our result that the FCFS mechanism dominates any other mechanism for any agent in the queue. He also highlights, as we do, the tension between the value of agents in the waiting list and the waste of objects. Finally, Schummer (2016) extends the analysis by considering risk-averse agents who are harmed by uncertainty about the waiting time. He shows that an outside party whose objective is to minimize the variance of waiting times will typically prefer to exert influence so that all agents are treated symmetrically, a result which relates to our finding that the lottery minimizes asymmetries in the values of agents in the waiting list.

Thakral's (2015) analysis is motivated by the rules implemented by public housing authorities. He proposes a new assignment mechanism—the multiple waitlist procedure—by which agents can choose to reject an object and be placed on a waiting list for a different object. He shows that this mechanism satisfies strategy-proofness, efficiency, and the absence of justified envy. Thakral's (2015) model differs from ours in several dimensions: he assumes that preferences over objects are fixed and that uncertainty relates to the time at which any object becomes available rather than the values of objects. He does not explicitly consider waiting costs dynamically incurred by the agents, but measures the cost of waiting by the maximal number of periods an agent is willing to wait to get into his preferred building.

Finally, our work belongs more distantly to the emerging literature on dynamic matching models, enriching assignment mechanisms with dynamic considerations. Other papers in this literature include Abdulkadiroğlu and Loerscher (2007); Ünver (2010); Bloch and Houy (2012); Bloch and Cantala (2013); Akbarpour, Li, and Oveis Gharan (2014); Kennes, Monte, Tumennasan (2014); and Kurino (2014) for models without transfers; and Gerkshov and Moldovanu (2009a,b), and Dizdar, Gerkshov, and Moldovanu (2011) for models with transfers.

Contents of the Paper.—The remainder of the paper unfolds as follows. We present our model in Section I. In Section II, we characterize the equilibrium of the binary model with a two-agent waiting list, illustrating the main forces at work in our model. Section III extends the analysis to waiting lists of arbitrary sizes in the binary model. In Section IV, we analyze the continuous model with two-agent waiting lists. Section V briefly discusses robustness checks and extensions of the two-agent binary model. We conclude and give directions for future research in Section VI. All proofs of propositions in the main text are collected in the Appendix, and the online Appendix covers the variants of the baseline model that are discussed in Section V.

I. The Model

A. *Queues*, *Values*, *and Waiting Costs*

We consider a society with an infinite number of agents, organized in a waiting list of constant size n. We let $i = 1, 2, \ldots, n$ denote the rank of agents in the waiting list. Conforming to conventional usage, we say that agent i has *higher rank* than agent j in the waiting list if $i < j$. Time is discrete and at each period $t = 1, 2, \ldots$ a new object becomes available. Agents in the waiting list draw a value for the object, $\theta \in \mathbb{R}$. This value is observed privately by the agent, but not by the other agents nor by the planner. In the binary model, θ can only take on two values that we normalize without loss of generality to 0 and 1, $\theta \in \{0, 1\}$. In the continuous model, θ is taken from a continuous distribution F with support $\left[\underline{\theta}, \overline{\theta}\right]$.

We assume that each object is different, and that there is no persistence in agents' values. The values θ_t, $\theta_{t'}$ drawn by an agent for the objects available at periods t and $t' \neq t$ are uncorrelated. In the private values model at any period t, the values drawn by the different agents are independent. In the common values model, all agents draw the same value. Each time an agent waits in the queue, he incurs an additive cost $c > 0$. Assuming that the reservation utility of an agent outside the waiting list is sufficiently low, we can guarantee that voluntary participation constraints are satisfied, so that agents always have an incentive to enter and stay in the waiting list.

B. *Probabilistic Queuing Disciplines*

We restrict attention to one instrument that can be chosen by the designer of the allocation rule: the sequence in which agents will be proposed the object. A *probabilistic queuing discipline* assigns a probability distribution p over the finite set R of all $n!$ sequences of agents in the waiting list. We denote by $\rho : \mathbb{N} \to \mathbb{N}$ a typical sequence in R.

C. *Agents' Strategies and Values*

Using the probability distribution $p \in \mathcal{P}$, the planner picks a sequence ρ. If agent i is proposed an object of value θ and accepts it, he collects the value θ and leaves the waiting list. All other agents in the waiting list incur the cost c, a new agent enters the waiting list at position n, and all agents whose rank is higher than i move up one position in the waiting list. If no agent accepts the object, the object is wasted, all agents incur the waiting cost c, keep their rank in the waiting list, and no new agent is allowed to enter the waiting list.

In the benchmark model, we suppose that agents are not given any information about the sequence ρ chosen by the planner. A Markovian strategy for agent i specifies his acceptance rule for the object of value θ as a function of his rank in the waiting list.

In the binary model, agent i always accepts the object with value 1. Hence, the only choice of agent i is whether he accepts the object with value 0 or not. A Markovian strategy for agent i is then the probability $q(i) \in [0, 1]$ that he accepts the object with value 0. With this notation, we write the value of agent i (net of the waiting cost) as

$$\begin{aligned} V(i) = {} & \Pr[\text{object accepted by } j < i]\, V(i-1) \\ & + \Pr[\text{agent } i \text{ picks the object}]\Pr[\theta_i = 1] \\ & + (1 - \Pr[\text{object accepted by } j \leq i])\, V(i) - c. \end{aligned}$$

This expression distinguishes between three possible outcomes: either the object is picked by an agent with lower rank than i, and i moves up one position in the waiting list, or agent i picks the object and receives value 1 (with probability $\Pr[\theta_i = 1]$), or the object is not picked or picked by an agent with higher rank than i, and i remains in the same position in the waiting list.

In the continuous model, a Markovian strategy for agent i is a threshold value $\theta(i)$ such that agent i accepts any object of value $\theta \geq \theta(i)$ and rejects any object of value $\theta < \theta(i)$. The value of agent i is then given by

$$\begin{aligned} V(i) = & \Pr[\text{object accepted by } j < i] V(i-1) \\ & + \Pr[\text{agent } i \text{ picks the object}] \int_{\theta(i)}^{\overline{\theta}} \theta \, dF(\theta) \\ & + (1 - \Pr[\text{object accepted by } j \leq i] V(i) - c. \end{aligned}$$

Notice that there is a clear distinction between the optimal strategies of agents in the FCFS mechanism and in any other probabilistic queuing discipline. In the FCFS mechanism, player 1's problem is a classical optimal stopping problem that does not depend on the behavior of other agents in the waiting list. Given agent 1's threshold, agent 2's threshold can be computed as the solution of an optimal stopping problem, etc. In the FCFS mechanism, there is a unique vector of optimal strategies that can be computed as the solution of a recursive system of individual optimal stopping problems. By contrast, for any other probabilistic queuing discipline, the value of an agent depends on the strategies of other agents in the waiting list. Agents are playing a game against other agents in the waiting list, and the behavior of agents is characterized by a Markovian equilibrium of the game. Equilibrium is no longer guaranteed to be unique and cannot be computed as the solution to a recursive system.

D. *Efficiency Criteria*

We consider a society with a varying population—agents enter and leave the waiting list over time—so that there is no obvious efficiency criterion we can apply to rank assignment mechanisms. Instead, we define three different criteria that capture different facets of the problem. We first consider the *vector of values of agents in the waiting list*, $\mathbf{V} = (V(1), V(2), \ldots, V(n))$. This criterion captures the welfare of insiders—agents who are currently active in the waiting list. We compare two vectors of values using Pareto dominance. This is a strong criterion—stronger than the utilitarian criterion used by Su and Zenios (2004)—it turns out that this criterion can be applied to compare assignment mechanisms in our model. (We note that this is also the criterion used in Schummer 2016.) Our second criterion focuses on the static efficiency of the assignment mechanism. Given that monetary transfers are not allowed and that the assignment mechanism uses a priority rule that is unrelated to the values of objects, the assignment mechanism may result in a (static) misallocation. The object may be picked by an agent i, whereas there exists another agent j in the waiting list who would have accepted the object and such that $\theta_j > \theta_i$. We measure this misallocation loss by the expected probability μ that the object is given to an agent i when there exists another agent j who accepts the object and for whom $\theta_j > \theta_i$. Our third criterion is the probability that the object is rejected by all agents in the waiting list. When an object is rejected by all agents, it is wasted and no new agent is allowed to enter the queue. The expected waste, measured by the probability ν that any object is rejected by all agents, captures the speed at which the queue is served, and the welfare of outsiders—agents who have not yet entered the waiting list.

II. Two-Agent Waiting List with Binary Values

We start the analysis with the simple case of a two-agent waiting list with binary values. This simple case will help us introduce the main results of the paper in a setting where Markov equilibria can easily be computed and illustrated. When the waiting list only consists of two agents, a probabilistic queueing discipline is characterized by a single parameter p denoting the probability that the order $\rho_1 = 1, 2$ is chosen. We suppose that, for political and ethical reasons, the planner cannot give an advantage to agents with lower rank in the waiting list, so that $p \in \left[\frac{1}{2}, 1\right]$.

A. *Private Values*

We first consider the case of private values. Let π denote the (independent) probability that the high value is drawn by any of the two players. Given strategies $q(1)$ and $q(2)$, the values of the two players are computed as follows:

$$
\begin{aligned}
V(1) = {} & p[\pi + (1 - \pi - (1 - \pi)q(1))V(1)] \\
& + (1 - p)[(1 - (1 - \pi)(1 - q(2))(\pi + (1 - \pi)q(1)))V(1) \\
& + \pi(1 - \pi)(1 - q(2))] - c; \\
V(2) = {} & p\big[(\pi + (1 - \pi)q(1))V(1) + (1 - \pi)(1 - q(1))\pi \\
& \quad + (1 - \pi - (1 - \pi)q(1) - (1 - \pi)(1 - q(1))(\pi + (1 - \pi)q(2)))\big] \\
& + (1 - p)\big[\pi + (1 - \pi)(1 - q(2))(\pi + (1 - \pi)q(1))V(1) \\
& \qquad + (1 - \pi - (1 - \pi)(1 - q(2)) \\
& \qquad\quad - (1 - \pi)(1 - q(2))(\pi + (1 - \pi)q(1)))\big] - c,
\end{aligned}
$$

resulting in

$$
V(i) = \frac{A(i) - ic}{B(i)},
$$

with

$$
\begin{aligned}
A(1) &= \pi[p + (1-p)(1-q(2))(1-\pi)],\\
A(2) &= \pi[2 - \pi - (1-\pi)(pq(1) + (1-p)q(2))],\\
B(1) &= [\pi + (1-\pi)q(1)][p + (1-p)(1-q(2))(1-\pi)],\\
B(2) &= [\pi + (1-\pi)q(1)][p + (1-p)(1-\pi)(1-q(2))]\\
&\quad + [\pi + (1-\pi)q(2)][1 - p + p(1-\pi)(1-q(1))].
\end{aligned}
$$

Notice that $A(i)$ is the *expected value of the object picked by agent i* and $B(i)$ the *expected probability that the object is assigned to an agent whose rank is lower than or equal to i.* The probabilities of misallocation and waste are given by

$$
\begin{aligned}
\mu &= \pi(1-\pi)[pq(1) + (1-p)q(2)]\\
\nu &= (1-\pi)^2(1-q(1))(1-q(2)).
\end{aligned}
$$

There are four candidate pure strategy Markov equilibria in the game, corresponding to the four possible choices of $q(1)$ and $q(2)$ in $\{0, 1\}$. Player 1, upon rejection of the low-value object, receives a continuation value $V(1)$. Hence, player 1's optimal strategy is to choose $q(1) = 0$ if and only if

$$V(1) \geq 0.$$

Furthermore, player 2, when rejecting the low-value object, does not know which sequence ρ has been chosen, and computes his continuation value as

$$\frac{(1-p)(\pi + (1-\pi)\,q(1))\,V(1) + (1-\pi)(1-q(1))\,V(2)}{1 - p + p(1-\pi)(1-q(1))},$$

where $1 - p + p(1-\pi)(1-q(1))$ is the probability that the object is offered to agent 2, and $(1-p)(\pi + (1-\pi)\,q(1))$ the probability that agent 1 picks the object after agent 2's rejection. Hence, agent 2 chooses $q(2) = 0$ if and only if

$$(1-p)(\pi + (1-\pi)q(1))V(1) + (1-\pi)(1-q(1))V(2) \geq 0.$$

We notice that, in the two-agent model, we cannot generically obtain an equilibrium where agent 1 chooses $q(1) = 1$ and agent 2 chooses $q(2) = 0$. This equilibrium would imply that $V(1) \leq 0$ (because agent 1 accepts the low value object) and $V(1) \geq 0$ (because agent 2 knows that agent 1 accepts both objects and deduces that he will always move up one rank in the waiting list and obtain a continuation value $V(1)$ after a rejection). We thus focus attention on three pure strategy equilibria, labeled according to the number of selective agents with E^2 the equilibrium

Table 1—Markov Equilibria in the Two-Agent Model with Private Values

	$V(1)$	$V(2)$	μ	ν
E^2	$1-\frac{c}{\pi(1-\pi(1-p))}$	$1-\frac{2c}{\pi(2-\pi)}$	0	$(1-\pi)^2$
E^1	$1-\frac{c}{p\pi}$	$\pi(1+p(1-\pi))-2c$	$(1-p)\pi(1-\pi)$	0
E^0	$\pi-\frac{c}{p}$	$\pi-2c$	$\pi(1-\pi)$	0
E^m	0	0	$\pi(2-\pi)-2c$	$\frac{(c-p\pi)(c-(1-p)\pi)}{\pi^2 p(1-p)}$

where $q(1) = q(2) = 0$, E^1 the equilibrium where $q(1) = 0$, $q(2) = 1$, and E^0 the equilibrium where $q(1) = q(2) = 1$. For some parameter values, the game also admits a mixed strategy equilibrium denoted E^m, with $q(1) = \frac{\pi(1-p\pi)-c}{p\pi(1-\pi)}$ and $q(2) = \frac{\pi(1-\pi(1-p))-c}{(1-p)\pi(1-\pi)}$. The following table computes the equilibrium values of the two agents and the probabilities of misallocation and waste for the different equilibria.

We first use Table 1 to assess the effect of a change in p on the equilibrium values of the two agents within each equilibrium. The expected value of the object picked by agent 1 is independent of p, equal to 1 if $q(1) = 0$ and π if $q(1) = 1$, but the expected waiting time is decreasing in p, so that $V(1)$ is increasing in p. The probability that the object is assigned to any of the two agents in the queue is independent of p—equal to $\pi(2-\pi)$ under equilibrium E^2 and 1 for all other equilibria—so that the expected waiting time of agent 2 is independent of p. In equilibria E^2 and E^0, the expected value of the object picked by agent 2 is also independent of p, it is equal to 1 under equilibrium E^2 and π under equilibrium E^0, so that $V(2)$ is independent of p. In equilibrium E^1, an increase in p increases the rate at which the agent switches from accepting both objects to becoming selective, resulting in an increase in the expected value of the object picked by agent 2 and an increase in $V(2)$. In the mixed strategy equilibrium, the equilibrium values of both agents are equal to the value of the worst object and hence independent of p. We conclude that, in all equilibria, an increase in p weakly increases the equilibrium values of *both* agents.

We next compare the equilibrium values of the two agents across pure strategy equilibria for fixed parameters p, π, and c. For both agents, values are strictly increasing in the number of selective agents in equilibrium: for $\pi < 1$ and $p < 1$, $V^2(i) > V^1(i) > V^0(i)$. When an agent switches from accepting the lowest object to being selective, his value necessarily increases. But we also observe that the value of an agent increases when the other agent switches from accepting both objects to becoming selective. This is due to the fact that when the other agent becomes more selective, his equilibrium value increases (raising the expected value of the object for the least senior agent) and the probability that the object is offered to the agent goes up.

The probability of misallocation is negatively correlated to the equilibrium values of the agents. It is weakly decreasing in p and lower in equilibria where agents are more selective. By contrast, the probability of waste is positively correlated with the equilibrium values of the agents, and lower in equilibria where agents are less selective.

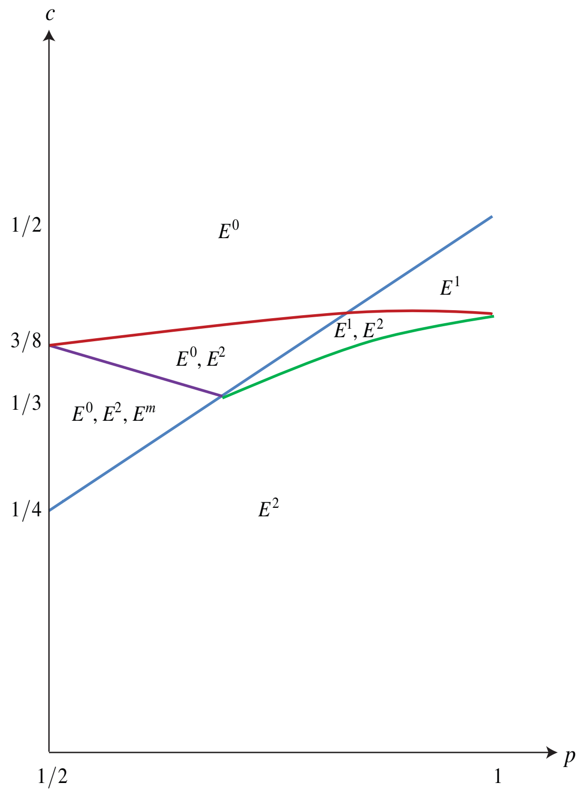

Figure 1. Equilibria in the Two-Agent Model with Private Values

We finally analyze conditions on the parameters under which each of the equilibria exists. Figure 1 illustrates the regions of parameters (p, c) under which the different equilibria exist for $\pi = \frac{1}{2}$.

We first note that there exist parameter regions where multiple equilibria exist. The multiplicity of equilibria stems from the fact that players' choices are strategic complements: if one player is selective, the probability that the other player is offered the object increases. This increases the equilibrium value of the other player, making him more selective. In particular, the two extreme equilibria E^0 and E^2 can be sustained when p is close to $\frac{1}{2}$. The mixed strategy equilibrium E^m only exists when both equilibria E^0 and E^2 exist, for intermediate regions of the cost and when p is close to $\frac{1}{2}$. The asymmetric pure strategy equilibrium only exists for extreme values of the probability p, when $p \geq \frac{2}{3}$.

Finally, we analyze the comparative statics effect of an increase in p on the existence of different equilibria. When p increases, both equilibrium values $V(1)$ and $V(2)$ increase. The continuation value after a rejection of agent 1 is equal to $V(1)$ and increasing in p, but the continuation value after a rejection of agent 2, $(1-p)(\pi + (1-\pi)q(1))V(1) + (1-\pi)(1-q(1))V(2)$, is not monotonic in p. When p increases, even though the values $V(1)$ and $V(2)$ increase, the probability that agent 2 moves up in the waiting list after a rejection goes down, inducing a possible non-monotonicity. For example, for values of the waiting cost

$c \in (0.375, 0.3818)$, there exists an equilibrium E^2 at $p = 0.9$ but not at $p = 1$, where the only equilibrium is an E^1 equilibrium. Intuitively, agent 2 may be more optimistic about his continuation value at $p = 0.9$ when he believes that the other agent is going to pick the object with positive probability than at $p = 1$ when he is sure that the other agent has already rejected the object.

B. *Common Values*

We now suppose that the values of the two agents are perfectly correlated. We compute the equilibrium values as

$$V(i) = \frac{A(i) - ic}{B(i)},$$

with

$$A(1) = p\pi,$$

$$A(2) = \pi,$$

$$B(1) = p[\pi + q(1)(1-\pi)] + (1-p)(1-\pi)q(1)(1-q(2)),$$

$$B(2) = 1 - (1-\pi)(1-q(1))(1-q(2)).$$

In the model with common values, the expected value of the object for the first agent is equal to $p\pi$ (the first agent only gets the high value object when he is the first to pick), whereas the expected value of the object to the second agent is equal to π (either the first agent picks the object, and then the second agent will pick the high value object next period with probability πp or the second agent immediately picks the high value object with probability $\pi(1-p)$). The object can never be misallocated ($\mu = 0$), and the probability of waste is

$$\nu = (1-\pi)(1-q(1))(1-q(2)).$$

In the model with common values, the first agent chooses $q(1) = 0$ if and only if

$$V(1) \geq 0.$$

The second agent computes his continuation value taking into account the fact that the object is of low quality. Hence, his continuation value is

$$\frac{(1-\pi)[(1-p)q(1)V(1) + (1-q(1))V(2)]}{(1-\pi)(1-pq(1))},$$

and he will choose $q(1) = 0$ if and only if

$$(1-p)q(1)V(1) + (1-q(1))V(2) \geq 0.$$

Table 2—Markov Equilibria in the Two-Agent Model with Common Values

	$V(1)$	$V(2)$	ν
E^2	$1-\frac{c}{p\pi}$	$1-\frac{2c}{\pi}$	$(1-\pi)$
E^1	$1-\frac{c}{p\pi}$	$\pi-2c$	0
E^0	$\pi-\frac{c}{p}$	$\pi-2c$	0

As in the case of private values, there cannot be an equilibrium with $q(1) = 1$ and $q(2) = 0$ as this implies $V(1) = 0$, which is generically not satisfied. Furthermore, we argue that generically the game with common values does not admit a mixed strategy equilibrium. In a mixed strategy equilibrium, the expected value of the most senior agent is necessarily equal to zero—implying that $p\pi = c$, a condition which cannot be satisfied generically.

Table 2 displays the equilibrium values and probability of waste in the three candidate pure strategy equilibria.

As in the private values case, the equilibrium value of agent 1 is always increasing in p. An increase in p does not affect the expected value of the object, equal to 1 if $q(1) = 0$ and π if $q(1) = 1$, and always reduces the expected waiting cost. The equilibrium value of agent 2 is independent of p. The expected equilibrium payoff of agent 2 is either equal to 1 $\left(\text{in equilibrium } E^2\right)$ or equal to π (in all other equilibria where the agent accepts a low quality object), and the probability that the object is allocated to any of the two agents is independent of the queuing discipline.

Comparing the equilibrium values across equilibria, we observe that the equilibrium payoff of agent 1 satisfies $V^2(1) = V^1(1) > V^0(1)$. For agent 2, $V^2(2) > V^1(2) = V^0(2)$. Hence, an increase in the number of selective agents increases the equilibrium values of both agents. The probability of waste is highest at the equilibrium where both agents are selective.

Finally the conditions on parameters for which equilibrium exists are easy to derive. When agent 2 is offered the object, he knows that agent 1 will never accept it, and hence computes his continuation payoff as $V(2)$. Hence, equilibrium E^2 only exists when $c \leq \frac{\pi}{2}$, equilibrium E^1 exists when $\frac{\pi}{2} \leq c \leq p\pi$, and equilibrium E^0 exists when $p\pi \leq c$. Figure 2 illustrates these different regions.

III. General Waiting List with Binary Values

We now analyze the general model when the constant size of the waiting list is an arbitrary number n. We restrict attention to probability queuing disciplines which satisfy the following assumption.

ASSUMPTION 1: *For any two agents $i < j$, and any sequences ρ, ρ', such that $\rho(k) = \rho'(k)$ for all $k \neq i$: j, $\rho(i) = \rho'(j) < \rho(j) = \rho'(i)$, $p(\rho) \geq p(\rho')$.*

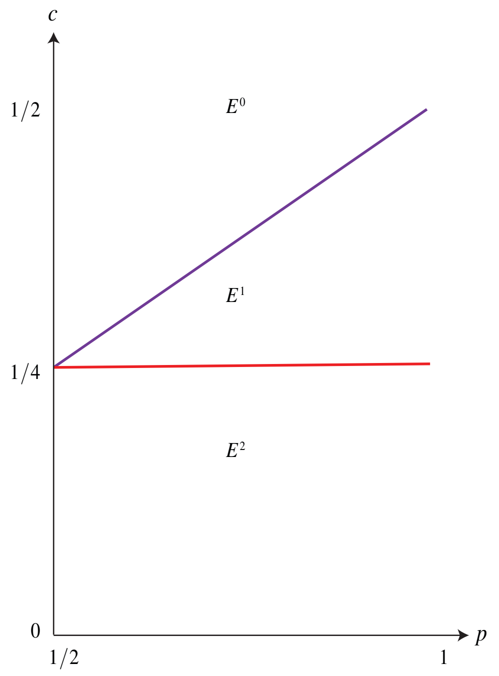

Figure 2. Equilibria in the Two-Agent Model with Common Values

Assumption 1 states that, whenever two agents i and j are permuted in the sequences ρ and ρ', then the sequence in which the agent with the highest rank is chosen first is picked with a probability at least as large as the sequence in which he is chosen last.[10] Assumption 1 excludes probabilistic queuing disciplines that favor less senior agents in the waiting list. The set of probabilistic queueing disciplines satisfying Assumption 1 admits two extreme elements: the FCFS queuing discipline $\hat{p}$ concentrated on the single sequence $1, 2, \ldots, n$ and the lottery $\tilde{p}$, where all sequences are chosen with equal probability $\frac{1}{n!}$. Formally, $\hat{p}$ and $\tilde{p}$ are the maximal and minimal elements of the lattice of all probabilistic queueing disciplines satisfying Assumption 1 equipped with the partial order $\succeq$ defined by $p \succeq p'$ if and only if, for any two agents $i < j$, and any sequences ρ, ρ', such that $\rho(k) = \rho'(k)$ for all $k \neq i, j$, $\rho(i) = \rho'(j) < \rho(j) = \rho'(i)$, $p(\rho) - p(\rho') \geq p'(\rho) - p'(\rho')$.

A. *Private Values*

For any mechanism p and vector of strategies $\mathbf{q}$, we compute the vector $\omega = (\omega(1), \ldots, \omega(n))$ which collects the expected probabilities that agent i is given the opportunity to pick the object, where the expectation is taken over the realization of the values θ and of the order ρ. We also denote by $\gamma = (\gamma(1), \ldots, \gamma(n))$

[10] In the special case where there are only two agents in the waiting list as in the previous section, this assumption reduces to $p \geq \frac{1}{2}$.

the vector collecting the expected probabilities that agent i picks the object. The expected probabilities $\omega(i)$ and $\gamma(i)$ are related by the equality

$$\gamma(i) = (\pi + (1-\pi)q(i))\omega(i).$$

We compute the values of the agents as

$$V(1) = \frac{\pi}{\pi + (1-\pi)q(1)} - \frac{c}{\gamma(1)},$$

$$V(i) = \frac{\sum_{t=1}^{i-1} \gamma(t)V(i-1) + \omega(i)\pi - c}{\sum_{i=1}^{t} \gamma(t)} \quad \text{for all} \quad i > 1.$$

Solving this recursive system,

$$V(i) = \frac{\sum_{t=1}^{i} \omega(t)\pi - ic}{\sum_{t=1}^{i} \gamma(t)}.$$

As in the two-agent case, the value of agent i, conditional on the fact that the object is allocated to an agent $k \leq i$, can be decomposed into two terms: the expected value of the object picked by i throughout his trajectory in the waiting list: $\sum_{t=1}^{i} \omega(t)\pi$, and the expected waiting cost ic, which is independent of the players' strategies and of the probabilistic queuing discipline.

We consider a family of equilibria indexed by the rank k of the last agent who rejects the low-value object. In a k-selective equilibrium, $q(i) = 0$ for all $i \leq k$ and $q(i) = 1$ for $i > k$. We first study the properties of k-selective equilibria and obtain the following Ppoposition, which generalizes the observations made in the two-agent case.

PROPOSITION 1: *Under private values, if Assumption* 1 *is satisfied, in any k-equilibrium, if $\rho \succeq \rho'$, the equilibrium values of all agents are higher under the probabilistic discipline ρ than under the probabilistic discipline ρ'. In addition, if for a fixed probabilistic discipline ρ the game admits two selective equilibria with k and $k+1$ selective agents, the equilibrium values of all agents are higher in the $k+1$-selective equilibrium than in the k-selective equilibrium.*

Proposition 1 establishes two facts. For fixed equilibrium strategies, the equilibrium values of all agents are higher when the probability that agents with higher rank choose first increases. Given a fixed probabilistic queuing discipline, if the game admits two equilibria, all agents obtain higher values in the equilibrium where the number of selective agents is higher. These two facts rely on the following intuitions.

In a k-selective equilibrium, we distinguish between agents with rank higher or equal to the threshold k (the "selective" agents) and agents with rank lower than the threshold (the "nonselective agents"). The expected value of the object for selective agents is constant, equal to 1, whereas the expected waiting cost, $\frac{c}{\sum_{t=1}^{i} \gamma(t)}$, is lower if the queuing discipline puts more weight on sequences where agents with higher rank choose first. For the nonselective agents, the expected value of the object is a convex combination of 1 (obtained when the agent is selective) and π (which is obtained when the agent is nonselective). Increasing the probability with which agents with higher rank are offered the object increases the speed at which agents move up in the waiting list, thereby increasing the expected value of the object. Hence, for nonselective agents, the two effects on expected value of the object and expected waiting time are compounded to result in an increase in equilibrium value.

The effect of a switch from an equilibrium with k selective agents to an equilibrium with $k+1$ agents is more difficult to assess. Agent $k+1$ switches behavior in the two equilibria. The switch affects his equilibrium value only when he is proposed the object. After the switch, he obtains a continuation value which is positive (because their equilibrium behavior now specifies that he rejects the low value object), whereas he obtained a value of 0 before the switch. For all other agents, we need to consider how the switch affects the probabilities that they are offered the object and the value they obtain. Consider an agent i. For any sequence ρ, where i chooses before agent $k+1$, the switch has no effect. If agent i is selective and the sequence lets agent $k+1$ choose before i, the switch will result in i obtaining a value 1 rather than $V(i)$ when the value of agent $k+1$ is low. As $1 > V(i)$, the switch increases the value of agent i. If agent i is nonselective, two cases need to be distinguished. If agent i is the first nonselective agent to choose in the sequence, the switch will result in i obtaining a value of π rather than $V(i-1)$ when the value of agent $k+1$ is low. For this sequence, the switch induces a gain of $\pi - V(i-1)$. If there is another agent j with lower rank than i who chooses in the sequence, the switch will result in i obtaining a value of $V(i)$ rather than $V(i-1)$ when the value of agent $k+1$ is low. For this sequence, the switch will induce a loss of $V(i-1) - V(i)$. A direct computation shows that $\pi - V(i-1) > V(i-1) - V(i)$. Furthermore, by Assumption 1, sequences where i chooses after another agent ranked after him are less likely. Hence, the total effect of the switch on the equilibrium value of nonselective agents is positive.

Proposition 1 ranks equilibrium values for fixed strategies and compares equilibrium values across equilibria. As in the two-agent model, the proposition is not sufficient to conclude that FCFS dominates other probabilistic queuing disciplines as the parameter configurations for which equilibria exist are not monotonic in the probability that agents with higher rank are offered the object. We restrict attention to the comparison between the two extreme probabilistic queueing disciplines: FCFS and the lottery.

Under FCFS, an equilibrium is characterized by the threshold rank k of the last agent who rejects the low-value object, and equilibrium values are given by

$$\hat{V}(i) = 1 - \frac{ic}{1-(1-\pi)^i} \quad \text{for} \quad i \leq k,$$

$$\hat{V}(i) = \pi - ic \quad \text{for} \quad i > k.$$

The threshold rank k is defined as the unique integer, such that

$$\frac{1-(1-\pi)^{k+1}}{k+1} \leq c \leq \frac{1-(1-\pi)^{k}}{k}.$$[11]

We can partition the positive real line into intervals $A^k = \left[\frac{1-(1-\pi)^{k+1}}{k+1}, \frac{1-(1-\pi)^{k}}{k}\right]$, such that the unique equilibrium of the game under the FCFS mechanism is a k equilibrium when $c \in A^k$.

In the lottery, we consider symmetric equilibria where all agents adopt the same strategy. The only candidate equilibria are equilibria where all agents accept both objects (the 0 equilibrium), where all agents refuse the low value object (the n equilibrium), and the mixed strategy equilibrium. The values in the zero and n equilibria are given by

$$\tilde{V}^0 = \frac{\pi}{n} - c,$$

$$\tilde{V}^n = 1 - \frac{cn}{1-(1-\pi)^n}.$$

In a mixed strategy equilibrium, the probability q is the unique solution to

$$\tilde{V} = \frac{\pi}{n(1-(1-\pi)(1-q)} - \frac{c}{1-(1-\pi)^n(1-q)^n} = 0.$$

Notice that the 0 equilibrium exists if and only if $c \geq \frac{\pi}{n}$, whereas the n equilibrium exists if and only if $c \leq \frac{1-(1-\pi)^n}{n}$. The mixed strategy equilibrium exists whenever both the 0 and n equilibria exist, i.e., when $\frac{\pi}{n} \leq c \leq \frac{1-(1-\pi)^n}{n}$.

In both the 0 and n equilibria, the equilibrium value of the last agent under the FCFS rule is equal to the common equilibrium value of all agents in the lottery. Because under FCFS values are ordered according to the agent's rank, and more senior agents have a higher value, equilibrium values of all agents are higher under FCFS mechanism than under the lottery. In addition, as agents are more selective under FCFS than under the lottery, the misallocation loss is lower, and the expected waste higher under FCFS than under the lottery.

PROPOSITION 2: *With private values, the equilibrium values of all agents are higher under the FCFS rule $\hat{p}$ than under the lottery $\tilde{p}$. The misallocation loss is lower under $\hat{p}$ than $\tilde{p}$ and the expected waste is lower under $\tilde{p}$ than under $\hat{p}$.*

[11] To check that k is unique, notice that the function $f(k) = \frac{1-(1-\pi)^k}{k}$ is decreasing in k.

B. *Common Values*

We now turn to the case of common values. If the object is of high value, it will be picked by the first agent in the sequence. Hence, we define $\delta(i)$, a parameter that only depends on the probabilistic queuing discipline and not on the strategies, as the expected probability that agent i is first in the sequence ρ. If the object is of low value, the expected probability that the object is offered to i depends both on p and on the strategy profile $\mathbf{q}$, and we let $\omega^0(i)$ and $\gamma^0(i)$ denote the expected probabilities that the low value object is offered to i and picked by i with $\gamma^0(i) = q(i)\,\omega^0(i)$. The value of agent i is then given by

$$V(i) = \frac{\pi \sum_{t=1}^{i} \delta(t) - ic}{\sum_{t=1}^{i} \left[\pi\delta(t) + (1-\pi)\,\gamma^0(t)\right]}.$$

Our next proposition compares the values of agents, misallocation, and waste probabilities under the FCFS rule, the lottery, and any other probabilistic queuing discipline satisfying Assumption 1.

PROPOSITION 3: *With common values, the equilibrium values of all agents are higher under the FCFS rule $\hat{p}$ than under any probabilistic queuing discipline p satisfying Assumption* 1. *The expected waste is lower under the lottery $\tilde{p}$ than under any probabilistic queuing discipline satisfying Assumption* 1.

Proposition 3 establishes a stronger result with common values than with private values: the FCFS rule dominates any probabilistic queuing discipline for all the agents. The difference between the two situations arises in the computation of continuation values after a rejection. With common values, the agent knows that agents with higher rank are more selective and will not pick the low value object after a rejection. Hence, the continuation value after rejection of an agent is equal to his equilibrium value. As equilibrium values are higher when the probability that agents with higher rank choose first, agents become more selective when this probability increases. Comparing FCFS with any other probabilistic queuing discipline p, if an agent is selective (or nonselective) in equilibrium under the two queuing disciplines, his equilibrium value under FCFS must be higher. If the agent is selective in equilibrium under FCFS and nonselective under p, he must obtain a positive equilibrium value under FCFS and a negative equilibrium value under p, so that again his equilibrium value is higher under FCFS. As agents are most selective under $\hat{p}$ and least selective under $\tilde{p}$, we also conclude that the probability of waste is minimized at the lottery.

IV. The Continuous Model

In the continuous model, agents draw their values from a continuous, atomless distribution $F(\cdot)$ with density function $f(\cdot)$. Agents select the minimal quality of the object that they accept, a threshold value $\theta(i)$ in the support of the distribution.

We restrict attention to the case $n = 2$ and discuss whether our results can be generalized to arbitrary waiting list sizes.

A. *The Continuous Model with Private Values*

With private values, the draws of the agents are independent. The values of the two agents are computed as

$$\begin{aligned} V(1) &= p\Big[\int_{\theta(1)}^{\overline{\theta}} tf(t)\,dt + F(\theta(1))V(1)\Big] \\ &\quad + (1-p)\Big[F(\theta(2))\int_{\theta(1)}^{\overline{\theta}} tf(t)\,dt + (1 - F(\theta(2))) \\ &\qquad \times (1 - F(\theta(1)))V(1)\Big] - c \\ &= \frac{\int_{\theta(1)}^{\overline{\theta}} tf(t)\,dt}{1 - F(\theta(1))} - \frac{c}{[p + (1-p)F(\theta(2))](1 - F(\theta(2)))}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} V(2) &= p\Big[(1 - F(\theta(1)))V(1) + F(\theta(1))\Big[\int_{\theta(2)}^{\overline{\theta}} t(f)\,dt + F(\theta(2))V(2)\Big]\Big] \\ &\quad + (1-p)\Big[\int_{\theta(2)}^{\overline{\theta}} tf(t)\,dt + F(\theta(2)) \\ &\qquad \times [(1 - F(\theta(1)))V(1) + F(\theta(1))V(2)]\Big] - c \\ &= \frac{[pF(\theta(1) + (1-p)]\int_{\theta(2)}^{\overline{\theta}} tf(t)\,dt}{1 - F(\theta(1))F(\theta(2))} \\ &\quad + \frac{(1 - F(\theta(1)))[p + (1-p)F(\theta(2))]V(1) - c}{1 - F(\theta(1))F(\theta(2))}. \end{aligned} \tag{2}$$

Assuming that c is sufficiently small, the Markov equilibrium is interior and can easily be characterized. Let $V^*(1)$, $V^*(2)$, $\theta^*(1)$, and $\theta^*(2)$ denote the equilibrium values and thresholds of the two agents. For a fixed value $V^*(1)$, the optimal threshold $\theta^*(1)$ of agent 1 is given by

$$\theta^*(1) = V^*(1). \tag{3}$$

Using equation (1), we compute the equilibrium threshold as the solution to the equation

$$\frac{\int_{\theta^*(1)}^{\overline{\theta}} (t - \theta^*(1))f(t)\,dt}{1 - F(\theta^*(1))} - \frac{c}{[p + (1-p)F(\theta^*(2))](1 - F(\theta^*(1)))} = 0. \tag{4}$$

For fixed $V^*(1)$ and $V^*(2)$, agent 2 selects the optimal threshold $\theta^*(2)$ as

$$(5) \qquad \theta^*(2) = \frac{(1-p)(1-F(\theta^*(1)))}{pF(\theta^*(1)) + 1 - p} V^*(1) + \frac{F(\theta^*(1))}{pF(\theta^*(1)) + 1 - p} V^*(2).$$

As opposed to the top agent, the second agent does not choose a threshold value $\theta^*(2)$ equal to his equilibrium value $V^*(2)$ because he expects to move to the top of the waiting list with some probability after a rejection. Using equations (1) and (2), we compute the equilibrium threshold as the solution to the equation

$$(6) \qquad F(\theta^*(1)) \int_{\theta^*(2)}^{\overline{\theta}} (t - \theta^*(2)) f(t)\, dt + (1 - F(\theta^*(1)))(\theta^*(1) - \theta^*(2)) - \frac{cF(\theta^*(1))}{1 - p + pF(\theta^*(1))} = 0.$$

We now compare the equilibrium values and strategies under FCFS and other probabilistic queuing disciplines.

PROPOSITION 4: *Let $n = 2$. The equilibrium values of the two agents, $V^*(1)$ and $V^*(2)$, are strictly higher under FCFS than under any other probabilistic queuing discipline.*

Proposition 4 shows that the ranking of equilibrium values obtained in the binary model also holds in the continuous model when $n = 2$. The intuition parallels the intuition in the binary case. For fixed thresholds $\theta(1)$ and $\theta(2)$, the equilibrium values of both agents are increasing in p. As in the binary case, if the strategies of the two players are fixed, an increase in p reduces the expected waiting time of both agents and accelerates the time at which agent 2 moves to the top of the waiting list. We also prove that the equilibrium threshold of agent 1 (and hence his equilibrium value) is maximized under FCFS, as he then solves an unconstrained optimization problem. We also argue that the equilibrium value of agent 2 under FCFS is increasing in the threshold value chosen by the first agent for values of $\theta(1)$ smaller than the optimum $\theta^*(1)$. This result stems from two observations. First, an increase in $\theta(1)$ increases the value $V(1)$ whenever $\theta(1) \leq \theta^*(1)$. Second, an increase in $\theta(1)$ increases the probability that agent 2 gets to pick the object and obtain an expected payoff of $\frac{\int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta}{1 - F(\theta^*(2))}$ rather than move up in the waiting list and obtain a payoff $V(1)$. For $\theta(1) \leq \theta^*(1)$, we show that $V(1) < \frac{\int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta}{1 - F(\theta^*(2))}$ by using the first order condition (6). Proposition 4 now derives from a combination of the different observations. For a fixed probabilistic queuing discipline p, with thresholds $\theta^*(1)_p$ and $\theta^*(2)_p$, the equilibrium value of agent 2 is higher when $p = 1$ than at p. Because the equilibrium value is increasing in $\theta(1)$ and $\theta^*(1)_p < \theta^*(1)_1$, the equilibrium value of agent 2 must be higher under FCFS than under p. At the two extreme cases $p = 1$ and $p = \frac{1}{2}$, the equilibrium values of the agents are equal to their thresholds, so that the argument of Proposition 4 provides a comparison between the equilibrium

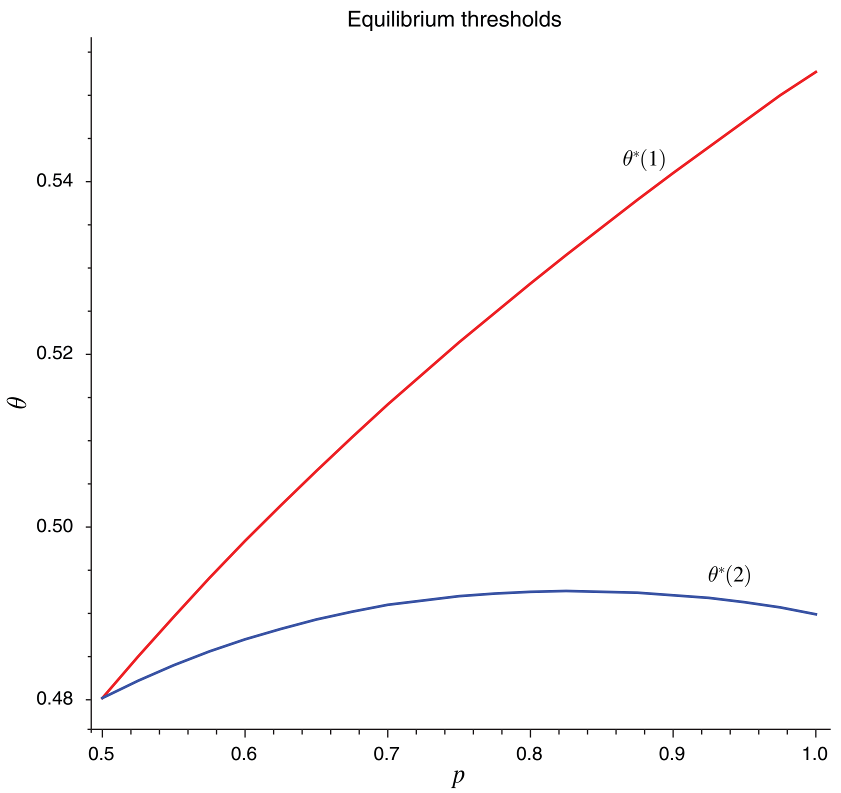

Figure 3. Equilibrium Thresholds in the Two-Agent Continuous Model

thresholds $\theta^*(1)$ and $\theta^*(2)$ under FCFS and under the lottery. The following example computes the equilibrium thresholds for general values of p.

Example 1: Suppose that the distribution of values is uniform over $[0, 1]$ and let $c = 0.1$. The interior equilibrium is characterized by the two conditions:

$$\theta^*(1) = \phi_1(\theta^*(2)) \equiv 1 - \sqrt{\frac{0.2}{p + (1-p)\theta^*(2)}},$$

$$\theta^*(2) = \phi_2(\theta^*(1)) \equiv 1 - \sqrt{1 - \theta^*(1)(-2\theta^*(1)) + 3 - \frac{0.2}{p\theta^*(1) + 1 - p}}.$$

Figure 3 displays the equilibrium threshold values $\theta^*(1)$ and $\theta^*(2)$ as a function of p. The threshold of agent 1 is increasing in p but, as in the binary model, the threshold of agent 2 is non-monotonic in p. This non-monotonicity of $\theta^*(2)$ with respect to p also explains why the argument of Proposition 4 does not extend beyond the case of two agents. The argument relies on the fact that the equilibrium threshold $\theta^*(1)$ is maximal at $p = 1$, and does not hold for $n = 3$ as $\theta^*(2)$ is not maximized at the FCFS rule. We now turn to the two other measures of efficiency.

The probability of misallocation measures the sum of the probability that the object is picked by the first agent when the second agent draws a higher value, and of the probability that the second agent picks the object when the first agent draws a higher value above $\theta^*(1)$. Hence, the probability of misallocation μ is given by

$$\begin{aligned}
\mu &= p \int_{\theta^*(1)}^{\overline{\theta}} \int_{\theta}^{\overline{\theta}} f(t)\, dt\, f(\theta)\, d\theta + (1-p) \int_{\theta^*(2)}^{\overline{\theta}} \int_{\max\{\theta, \theta^*(1)\}}^{\overline{\theta}} f(t)\, dt\, f(\theta)\, d\theta \\
&= p \int_{\theta^*(1)}^{\overline{\theta}} \int_{\theta}^{\overline{\theta}} f(t)\, dt\, f(\theta)\, d\theta + (1-p) \\
&\quad \times \left[\int_{\theta^*(2)}^{\theta^*(1)} \int_{\theta^*(1)}^{\overline{\theta}} f(t)\, dt\, f(\theta)\, d\theta + \int_{\theta^*(1)}^{\overline{\theta}} \int_{\theta}^{\overline{\theta}} f(t)\, dt\, f(\theta)\, d\theta\right] \\
&= \int_{\theta^*(1)}^{\overline{\theta}} (1 - F(\theta))\, d\theta + (1-p) \int_{\theta^*(2)}^{\theta^*(1)} (1 - F(\theta^*(1)))\, f(\theta)\, d\theta.
\end{aligned}$$

LEMMA 1: *When $n = 2$, the probability of misallocation is minimized at the FCFS rule.*

Finally, the probability of waste is given by

$$\nu = F(\theta^*(1))F(\theta^*(2)),$$

a strictly increasing function of $\theta^*(1)$ and $\theta^*(2)$. We observe that the probability of misallocation is always lower at the lottery than at the FCFS mechanism and conjecture—but could not prove—that the lottery minimizes waste.

B. *Common Values*

With common values, the values of the two agents are given by

$$\begin{aligned}
V(1) &= p\left[\int_{\theta(1)}^{\overline{\theta}} t f(t)\, dt + F(\theta(1))\right] + (1-p)V(1) - c, \\
V(2) &= p\left[\int_{\theta(2)}^{\theta(1)} t f(t)\, dt + (1 - F(\theta(1))V(1) + F(\theta(2))V(2))\right] \\
&\quad + (1-p) \int_{\theta(2)}^{\overline{\theta}} f(t)\, dt - c,
\end{aligned}$$

yielding

$$\begin{aligned}
V(1) &= \frac{\int_{\theta(1)}^{\overline{\theta}} t f(t)\, dt}{1 - F(\theta(1))} - \frac{c}{p(1 - F(\theta(1))}, \\
V(2) &= \frac{\int_{\theta(2)}^{\overline{\theta}} t f(t)\, dt}{1 - F(\theta(2))} - \frac{2c}{1 - F(\theta(2))}.
\end{aligned}$$

With common values, there is no interaction between the strategies of the two agents. The values of both agents only depend on their optimal thresholds: agent 1 obtains an object of expected value $\int_{\theta(1)}^{\overline{\theta}}$ and agent 2 an object of expected value $\int_{\theta(2)}^{\overline{\theta}}$, as he will either get an object of value $\int_{\theta(2)}^{\overline{\theta}}$ this period if he picks first, or an object of value $\int_{\theta(2)}^{\theta(1)}$ this period and of value $\int_{\theta(1)}^{\overline{\theta}}$ next period if the first agent picks first. The probability p only affects the equilibrium value of the top agent, and does not have any impact on the equilibrium value of the second agent. The optimal thresholds satisfy $W^*(1) = \theta^*(1)$ and $W^*(2) = \theta^*(2)$, and the equilibrium thresholds are solutions to the equations

$$p \int_{\theta^*(1)}^{\overline{\theta}} (t - \theta^*(1)) f(t)\, dt - c = 0, \tag{7}$$

$$\int_{\theta^*(2)}^{\overline{\theta}} (t - \theta^*(2)) f(t)\, dt - 2c = 0. \tag{8}$$

The value of agent 1 is increasing in p, the value of agent 2 is independent of p, and the probability of waste $F(\theta^*(2))$ is independent of p.

V. Variants

In this section, we discuss several variants of the two-agent model with binary values. Detailed computations can be found in the online Appendix.

Stochastic Queues.—We consider a model where the size of the waiting list varies. At each period t, a new agent enters the queue with probability s when the queue is of size 1. (The model corresponds to the baseline model when $s = 1$). In this variant, for a fixed strategy of the agents, the expected waiting time of an agent depends on the queuing discipline. If the queue is short (of size 1), it is expected to be longer in the future, so that the expected waiting time is smaller under FCFS rule than under the lottery. If the queue is long (of size 2), it is expected to be shorter in the future, so the expected waiting time is smaller under the lottery than under FCFS. We show that, in the two pure strategy equilibria that exist under both rules (the equilibrium where both agents are selective and where they both are non-selective), the equilibrium value of the last agent is higher under FCFS than under the lottery when the queue is of size 1, but higher under the lottery than under FCFS when the queue is of size 2. With stochastic queues, the computation of values must take into account the effect of the queuing discipline on the transition probabilities in the sizes of the queue, and the dominance of FCFS over lottery only holds when the current queue is short.

Heterogeneous Waiting Costs.—We allow for heterogeneous waiting costs. Assume that agents can either draw a low waiting cost or a high waiting cost that is observable. The state space is expanded to include the waiting costs of the two players. There are four possible states $(c(1), c(2))$ with $c(i) \in \{\underline{c}, \overline{c}\}$. The strategies of the players and the equilibrium values are now defined over the expanded state space. In addition, the probability of letting the first agent be offered the object first is a function of the observed waiting costs, $p(c(1), c(2))$. The equilibrium value of the first agent is always increasing in the probability that he is offered the object. However, contrary to the homogeneous case, the equilibrium value of the second agent is not increasing in the probability p when the second agent has a high waiting cost and the first agent a low waiting cost. In fact, we observe in an example that the optimal probabilistic queuing discipline for the second agent should follow a lexicographic specification: (i) first propose to the agent with a high waiting cost if the two agents have different waiting costs, and (ii) use the seniority rank to break ties between the agents if they have the same waiting cost, letting the more senior agent choose first.

Last Come First Serve.—We consider a probabilistic queuing discipline which gives an advantage to the agent with lower seniority. In the LCFS rule, agents are ordered in the sequence opposite to the seniority ranking. We first observe that an equilibrium where both agents accept both objects exists for all possible parameter configurations. In this equilibrium, agent 1 never obtains the object and agent 2 accepts the object with low value because after a rejection agent 1 accepts the object, so that agent 2 moves up one position in the waiting list and never gets any object afterward. In this equilibrium, the equilibrium value of agent 1 is lower than the equilibrium value of agent 2. As agent 1 is never served, he may have an incentive to leave the queue.

Assignment with Prior Application.—We change the timing of the agent's decisions. We suppose that agents apply for the object at time t after observing its value, and that the probabilistic priority mechanism assigns the object to some member of the pool of applicants. We obtain a neutrality result: the equilibria are identical in the sequential assignment model and in the assignment model with prior application. The equilibrium values of the two agents in the queues are equal under the two mechanisms. The equilibrium conditions are also identical. Agent 1 applies for the low-value object if and only if $V(1) \leq 0$ as in the sequential model. Agent 2 realizes that his application decision only matters if agent 1 does not pick the object first. His computation, conditional on the fact that agent 1 does not pick the object first, is identical under the sequential mechanism and the mechanism with prior application, and he applies for the low-value object if and only if his continuation value $(1-p)(\pi + (1-\pi)q(1))V(1) + (1-\pi)(1-q(1))V(2)$ is smaller than 0.

Information about the Sequence.—We suppose that agents are told their rank in the sequence ρ. In that case, agent 2 may behave differently when he learns that he is second to choose (and hence has a continuation value after rejection equal to $V(2)$) and when he learns that he is the first to choose (and has a continuation value after rejection equal to a convex combination of $V(1)$ and $V(2)$). Clearly, this additional information makes agent 2 more likely to accept the object when he is the second to choose and less likely to accept the object when he is the first to choose. Giving information about the order in the sequence thus makes it more likely that agent 2 rejects the object when he is first to choose and hence increases the probability that agent 1 is offered the object, thereby increasing the region of parameters where the selective equilibrium exists and increasing the equilibrium value of agent 1. On the other hand, providing information about the sequence has an ambiguous effect on agent 2's equilibrium behavior and value.

Eviction from the Waiting List.—We suppose that agents are evicted from the waiting list after a rejection with some positive probability. This eviction scheme reduces the continuation value after a rejection and gives an incentive for agents to accept the lower value object. The introduction of an eviction mechanism lowers the value of all agents currently in the waiting list but also lowers the probability of waste. Hence, eviction schemes are useful to accelerate the turnover in the waiting list, but at the expense of the agents currently in the queue.

VI. Conclusion

This paper analyzes the optimal assignment of objects which arrive sequentially to agents organized in a constant size waiting list. Applications include the assignment of social housing and organs for transplants. We analyze the optimal design of probabilistic queuing disciplines, punishment schemes, the optimal timing of applications, and information releases. With private values, we show that all agents prefer the FCFS mechanism to the lottery, but that waste is lower at the lottery. With common values, we show that all agents prefer the first-come first served mechanism to any other mechanism and that waste is minimized at the lottery. Punishment schemes accelerate turnover in the queue at the expense of agents in the waiting list and information release always increases the value of agents at the top of the waiting list.

Our analysis thus gives support to the use of waiting time as a primary criterion in the priority order in order to maximize the value of agents inside the queue, and of the use of lotteries in order to minimize waste. It also shows that punishment schemes like eviction from the queue harm agents currently inside the queue but accelerate turnover in the queue. There remain a number of aspects of dynamic allocation that deserve further study. The dynamics of the size of the waiting list and individual trajectories in a model with stochastic arrival of objects, entry and exit in the queue, still needs to be analyzed in detail. The design of mechanisms to elicit information about values and waiting costs also requires investigation. We plan to tackle these issues in future research.

Appendix: Proofs

PROOF OF PROPOSITION 1:

We first observe that, under Assumption 1, if $q(i) = q(j)$ and $i < j$, then $\omega(i) \geq \omega(j)$ and $\gamma(i) \geq \gamma(j)$. To see this, fix the strategies of all agents $k \neq i, j$ and consider two sequences ρ and ρ', where i and j are permuted and i precedes j in ρ. The expected probability that i is offered the object in ρ is equal to the expected probability that j is offered the object in ρ'. In addition, as $q(i) = q(j)$, the expected probability that i is offered the object in ρ' is equal to the expected probability that j

is offered the object in ρ. By Assumption 1, $p(\rho) \geq p(\rho')$. The result follows from taking expectations over all possible sequences of agents $k \neq i, j$. In addition, this reasoning shows that if $p \succeq p'$ and $i < j$, then $\omega(i) - \omega(j) \geq \omega'(i) - \omega'(j)$ and $\gamma(i) - \gamma(j) \geq \gamma'(i) - \gamma'(j)$.

Consider two probabilistic queuing disciplines p, p', such that $p \succeq p'$. We will show that for any $i, \sum_{t=1}^{i} \gamma(t) \geq \sum_{t=1}^{i} \gamma'(t)$. The proof is by induction. Take $t = 1$. By the preceding step, $\gamma(1) - \gamma(j) \geq \gamma'(1) - \gamma'(j)$ for all $j \neq 1$. Adding up over all j, $(n-1)\gamma(1) - \sum_{j\neq 1} \gamma(j) \geq (n-1)\gamma'(1) - \sum_{j \neq 1} \gamma'(j)$. Using the fact that $\sum_j \gamma(j) = \sum_j \gamma'(j) = 1$, the result follows. Now suppose that $\sum_{t=1}^{i} \gamma(t) \geq \sum_{t=1}^{i} \gamma'(t)$. We again have $\sum_{t=1}^{i} \gamma(t) + \gamma(i+1) - \gamma(j) \geq \sum_{t=1}^{i} \gamma'(t) + \gamma'(i+1) - \gamma'(j)$ for all $j > i+1$ because of the inductive step and the assumption that $p \succeq p'$. Adding up over all $j > i+1$, $(n-i-1)\sum_{t=1}^{i+1} \gamma(t) - \sum_{t=i+2}^{n} \gamma(t) \geq (n-i-1) \sum_{t=1}^{i+1} \gamma'(t) - \sum_{t=i+2}^{n} \gamma'(t)$. Using the fact that $\sum_{t=1}^{i+1} \gamma(t) + \sum_{t=i+2}^{n} \gamma(t) = \sum_{t=1}^{i+1} \gamma'(t) + \sum_{t=i+2}^{n} \gamma'(t) = 1$, the result follows.

Consider a selective agent $i \leq k$. Then $V(i) = 1 - \frac{c}{\sum_{t=1}^{i} \gamma(t)}$ and the result follows immediately because $\sum_{t=1}^{i} \gamma(t) \geq \sum_{t=1}^{i} \gamma'(t)$. Consider next a nonselective agent $i > k$. Then $V(i) = \frac{\pi \sum_{t=1}^{i} \omega(t) - ic}{\sum_{t=1}^{i} \gamma(t)}$. Now $\sum_{t=1}^{i} \pi\omega(t) = \sum_{t=1}^{k} \gamma(t) + \pi \sum_{t=k}^{i} \gamma(t) = (1-\pi) \sum_{t=1}^{i} \gamma(t) + \pi \sum_{t=1}^{k} \gamma(t) \geq (1-\pi) \sum_{t=1}^{i} \gamma'(t) + \pi \sum_{t=1}^{k} \gamma'(t) = \sum_{t=1}^{i} \pi\omega'(t)$. Hence,

$$V(i) \geq \frac{\pi \sum_{t=1}^{i} \omega'(t) - ic}{\sum_{t=1}^{i} \gamma(t)}.$$

In addition, as $i > k, \pi \sum_{t=1}^{i} \omega'(t) - ic \leq 0$ and, hence,

$$\frac{\pi \sum_{t=1}^{i} \omega'(t) - ic}{\sum_{t=1}^{i} \gamma(t)} \geq \frac{\pi \sum_{t=1}^{i} \omega'(t) - ic}{\sum_{t=1}^{i} \gamma'(t)} = V'(i),$$

concluding the proof of the first statement.

For the second statement, consider two equilibria with k and $k+1$ selective agents. We index the values in the $k+1$ selective equilibrium with primes. Notice first that, along any realization of ρ where i precedes $k+1$, the probability that i is proposed, the object remains the same under the k and the $k+1$ equilibrium, but along any realization of ρ where $k+1$ precedes i, the probability that i is proposed the object weakly increases. Hence, denoting by $\omega(i)$ the expected probability that i is proposed the object in the k equilibrium, $\omega(i) \leq \omega'(i)$ for all $i \neq k+1$ and $\omega(k+1) = \omega'(k+1)$. Now consider first $i \leq k$, as

$$V'(i) = 1 - \frac{c}{\pi \sum_{t=1}^{i} \omega'(t)},$$

and $\omega'(t) \geq \omega(t)$ for all t, $V'(i) \geq V(i)$. Next, consider $i \geq k+1$. We will prove the stronger statement:

$$\left[1 - \sum_{t=i+1}^{n} \omega'(t)\right] V'(i) \geq \left[1 - \sum_{t=i+1}^{n} \omega(t)\right] V(i).$$

Notice that for $i = k+1$, this statement is equivalent to

$$\pi \sum_{t=1}^{k} \omega'(t) V'(k) + \pi\omega'(k+1) - c \geq \pi \sum_{t=1}^{k} \omega(t) V(k) + \pi\omega(k+1) - c.$$

Using the fact that $\omega'(t) \geq \omega(t)$ for all $t \neq k+1$, $\omega'(k+1) = \omega(k+1)$, and $V'(k) \geq V(k)$, the inequality follows. Suppose now that the statement is true for all $t < i$ and consider i; we compute

$$\left[1 - \sum_{t=i+1}^{n} \omega'(t)\right] V'(i) = \sum_{t=1}^{i-1} \gamma'(t) V'(i-1) + \pi\omega'(i) - c.$$

Now, as $i - 1 > k$, $\sum_{t=1}^{i-1} \gamma'(t) = 1 - \sum_{t=i}^{n} \omega'(t)$, and using the induction hypothesis and the fact that $\omega'(i) \geq \omega(i)$,

$$\begin{aligned}
\left[1 - \sum_{t=i+1}^{n} \omega'(t)\right] V'(i) &= \left[1 - \sum_{t=i}^{n} \omega'(t)\right] V'(i-1) + \pi\omega'(i) - c \\
&\geq \left[1 - \sum_{t=i}^{n} \omega(t)\right] V(i-1)\pi\omega(i) - c \\
&= \left[1 - \sum_{t=i+1}^{n} \omega(t)\right] V(i),
\end{aligned}$$

completing the proof of the proposition. ∎

PROOF OF PROPOSITION 2:

The proof that equilibrium values are higher under FCFS than under the lottery is given in the text. We remark that whenever the n equilibrium exists under the lottery it is the unique equilibrium under FCFS and the probability of misallocation is zero, and the probability of waste is $(1-\pi)^n$ under both rules. Whenever the n equilibrium does not exist in the lottery, the only equilibrium is a 0 equilibrium, resulting in the highest misallocation probability $\pi(1-\pi)$ and the lowest probability of waste 0. ∎

PROOF OF PROPOSITION 3:

The proof follows a series of steps.

Step 1: In any equilibrium, if $q(i) = 1$, then $q(j) = 1$ for all $j > i$.

Consider the lowest index i, such that $q(i) = 1$. For all $j < i$, $q(j) = 0$ and, hence, $V(j) \geq 0$ for all $j \leq i-1$ and $V(i) \leq 0$. This implies that $\pi\delta(i) - c \leq 0$. Suppose by contradiction that there exists $k > i$, such that $q(k) = 0$, and pick the lowest index k for which this is true. Because $q(k-1) = 1$, this implies that $V(k-1) \geq 0$. If $k = i+1$, we get an immediate contradiction for generic values of c. If $k > i+1$, we must have $V(k-2) \leq 0$ as $q(k-2) = q(k-1) = 1$ and hence $\pi\delta(k-1) - c \geq 0$.

Now observe that under Assumption 1, whenever $i < j$, $\delta(i) \geq \delta(j)$ with equality only under $\tilde{p}$. Hence, $\pi\delta(i) > \pi\delta(k-1)$, resulting in a contradiction.

Step 2: Consider a fixed equilibrium where $q(i) = \hat{q}(i) = 0$ for all $i \leq k$ and $q(i) = \hat{q}(i) = 1$ for all $i > k$. Then, for all agents i, $\hat{V}(i) \geq V(i)$.

By step 1, any equilibrium is parametrized by the index of the last agent k who refuses the low-value object. Equilibrium values are given by

$$V(i) = \begin{cases} 1 - \dfrac{c}{\pi \sum_{t=1}^{i} \delta(t)} & \text{if } i \leq k \\[2ex] \dfrac{\pi \sum_{t=1}^{i} \delta(t) - ic}{\pi \sum_{t=1}^{i} \delta(t) + (1-\pi) \sum_{t=k+1}^{i} \omega^0(t)} & \text{if } i \geq k+1 \end{cases}.$$

If $i \leq k$, $\sum_{t=1}^{i} \delta(t) \leq 1$ and 1 is always achieved for the FCFS rule, which is the only rule for which $\sum_{t=1}^{i} \delta(t) = 1$ for all i. If $i \geq k+1$, in equilibrium $V(i)$, $\hat{V}(i) \leq 0$. Hence, $\pi \leq ic$ and as $\pi \sum_{t=1}^{i} \delta(t) + (1-\pi) \sum_{t=k+1}^{i} \omega^0(t) \leq 1$, so that

$$V(i) \leq \frac{\pi - ic}{\pi \sum_{t=1}^{i} \delta(t) + (1-\pi) \sum_{t=k+1}^{i} \omega^0(t)}$$

$$\leq \pi - ic = \hat{V}(i).$$

Step 3: If, at equilibrium, $\hat{q}(i) = 1$ under FCFS, then $q(i) = 1$ for all other queuing disciplines p.

Suppose that $\hat{q}(k) = 1$ and k is the first agent who accepts the low-value object. Then $\pi - kc \leq 0$ and, hence, $\pi \sum_{t=1}^{k} \delta(t) - kc \leq \pi - kc \leq 0$ and $q(k) = 1$. By step 1, $q(i) = 1$ for all $i \geq k$.

Finally, consider the values $V(1), \ldots, V(n)$ at any rule p. Consider i, such that $q(i) = 0$. By step 3, $\hat{q}(i) = 0$, and by step 2, $\hat{V}(i) > V(i)$. Suppose next that $q(i) = 1$. If $\hat{q}(i) = 0$, then $\hat{V}(i) \geq 0 \geq V(i)$. Finally if $q(i) = \hat{q}(i) = 1$, observe that $\hat{V}(i) = \pi - ic \geq V(i)$ by step 2. ∎

PROOF OF PROPOSITION 4:

We start by proving two lemmas.

LEMMA A1: *For any* $\theta(1)$, $\theta(2)$, $V(1)$ *is strictly increasing in p. For any* $\theta(1) \geq \theta(2)$, $V(2)$ *is strictly increasing in p.*

PROOF OF LEMMA A1:

By a trivial computation, $V(1)$ is increasing in p. Now compute

$$\frac{\partial V(2)}{\partial p} = -(1 - F(\theta(1))) \int_{\theta(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta + (1 - F(\theta(2))) \int_{\theta(1)}^{\overline{\theta}} \theta f(\theta)\, d\theta.$$

As the function $\frac{\int_{\theta}^{\overline{\theta}} t f(t)\, dt}{1 - F(\theta)}$ is increasing in θ, whenever $\theta(1) \geq \theta(2), \frac{\partial V(2)}{\partial p} \geq 0$ with strict inequality when $\theta(1) > \theta(2)$. ∎

LEMMA A2: *At any equilibrium, the threshold of the second agent in the waiting list is smaller than the threshold of the first agent*, $\theta^*(2) \leq \theta^*(1)$.

PROOF OF LEMMA A2:

Notice that

$$\theta^*(1) \geq \theta^*(2) \Leftrightarrow V^*(1) \geq V^*(2).$$

Suppose that $\theta^*(1) < \theta^*(2)$ and $V^*(1) < V^*(2)$. Now,

$$V^*(2) = \frac{pF(\theta^*(1)) + 1 - p}{1 - F(\theta^*(1))F(\theta^*(2))} \left[\int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta - \frac{c}{pF(\theta^*(1)) + 1 - p}\right]$$

$$+ \frac{(p + (1-p)F(\theta^*(2)))(1 - F(\theta^*(1))}{1 - F(\theta^*(1))F(\theta^*(2))} V^*(1).$$

Hence, $V^*(2)$ is a convex combination of $V^*(1)$ and $\int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta - \frac{c}{pF(\theta^*(1)) + 1 - p}$, and if $V^*(1) < V^*(2)$, then

$$\int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta - \frac{c}{pF(\theta^*(1)) + 1 - p} > V^*(1) \tag{9}$$

$$= \frac{1}{1 - F(\theta^*(1))} \left[\int_{\theta^*(1)}^{\overline{\theta}} \theta f(\theta)\, d\theta - \frac{c}{p + (1-p)F(\theta^*(2))}\right]$$

$$> \int_{\theta^*(1)}^{\overline{\theta}} \theta f(\theta)\, d\theta - \frac{c}{p + (1-p)F(\theta^*(2))}.$$

But, as $\theta^*(1) < \theta^*(2)$, $\int_{\theta^*(1)}^{\overline{\theta}} \theta f(\theta)\, d\theta > \int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta$ and, because $p > \frac{1}{2}$, $p(1 - F(\theta^*(1))) > (1-p)(1 - F(\theta^*(2)))$, so that $p + (1-p)F(\theta^*(2)) > pF(\theta^*(1)) + 1 - p$, contradicting inequality (9). ∎

In the next step of the proof, we show that $\theta^*(1) = V^*(1)$ is maximized at $p = 1$. Consider the mapping $g(\theta(1), p) = V(1)$ fixing $\theta(2)$. By Lemma A1, $g(\theta(1), p)$ is increasing in p. Hence, for all θ and all $p < 1$, $g(\theta(1), 1) > g(\theta(1), p)$. Now, by the first order condition (3), the best response $\theta^*(1)$ is a fixed point of the mapping $g(\theta(1), p)$. By theorem 1 in Milgrom and Roberts (1994), because $g(\theta(1), p)$ is monotonically increasing in p, the lowest and highest fixed points of $g(\theta(1), p)$ are increasing in p. In addition, at $p = 1$, because the function $V(1)$ is strictly quasi-concave in $\theta(1)$, there exists a unique fixed point $\theta^*(1)$, which is independent of $\theta(2)$. Hence, the unique fixed point $\theta^*(1)$ at $p = 1$ is higher than any fixed point $\theta^*(1)$ evaluated at any $\theta(2)$, so that $V^*(1) = \theta^*(1)$ is maximized at $p = 1$.

Next suppose that $p = 1$. We define the mapping associating to each $\theta(1)$, the maximal payoff of the second agent

$$\hat{V}(2)(\theta(1)) = \max_{\theta(2)} \frac{\int_{\theta(1)}^{\overline{\theta}} tf(t)\, dt + F(\theta(1)) \int_{\theta(2)}^{\overline{\theta}} tf(t)\, dt - 2c}{1 - F(\theta(1))F(\theta(2))}.$$

Because, at $p = 1$, $V(1)$ is independent of $\theta(2)$, the value of $\theta(2)$ that maximizes the payoff is equal to the best response function $\theta^*(2)$ and

$$\hat{V}(2)(\theta(1)) = \frac{\int_{\theta(1)}^{\overline{\theta}} tf(t)\, dt + F(\theta(1)) \int_{\theta^*(2)}^{\overline{\theta}} tf(t)\, dt - 2c}{1 - F(\theta(1))F(\theta^*(2))}.$$

We now prove the following Lemma.

LEMMA A3: *For any* $\theta(1) < \theta^*(1)$, $\hat{V}(2)$ *is strictly increasing in* $\theta(1)$.

PROOF OF LEMMA A3:

We differentiate $\hat{V}(2)$ with respect to $\theta(1)$ to obtain, by the envelope theorem,

$$\frac{\partial \hat{V}(2)}{\partial \theta(1)} = \frac{f(\theta(1))}{(1 - F(\theta(1))(\theta^*(2)))^2} \left[-\theta(1) + \int_{\theta(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta\right][1 - F(\theta(1))F(\theta^*(2))]$$

$$+ F(\theta^*(2))\left[\int_{\theta(1)}^{\overline{\theta}} \theta f(\theta)\, d\theta + F(\theta(1)) \int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta - 2c\right].$$

Using the first order condition characterizing the optimal choice of $\theta(2)$,

$$\theta^*(2)(1 - F(\theta(1))F(\theta^*(2))) = \int_{\theta(1)}^{\overline{\theta}} \theta f(\theta)\, d\theta + F(\theta(1)) \int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta - 2c,$$

so that the sign of $\frac{\partial \hat{V}(2)}{\partial \theta(1)}$ is the same as the sign of

$$h(\theta(1)) \equiv -\theta(1) + \int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta + \theta^*(2)F(\theta^*(2)).$$

Next, notice that $h'(\theta(1)) = -1 + \frac{\partial \theta^*(2)}{\partial \theta(1)} F(\theta^*(2))$. Suppose that $h(\theta(1)) = 0$. Then, $\frac{\partial \theta^*(2)}{\partial \theta(1)} = 0$ and $h'(\theta(1)) = -1 < 0$. Hence, whenever $h(\theta(1)) = 0$, $h'(\theta(1)) < 0$. We now show that, at the optimal value $\theta^*(1)$, $h(\theta^*(1)) > 0$. As $h(\theta(1))$ is continuous and $h(0) > 0$, this implies that there cannot be $\theta(1) < \theta^*(1)$ for which $h(\theta(1)) = 0$. Hence, for any $\theta(1) \leq \theta^*(1)$, $h(\theta(1)) > 0$. Recall that $\theta^*(1)$ is characterized by

$$\theta^*(1) = \frac{\int_{\theta^*(1)}^{\overline{\theta}} \theta f(\theta)\, d\theta - c}{1 - F(\theta^*(1))}.$$

Now,

$$\begin{aligned}
&h(\theta^*(1))(1 - F(\theta^*(1))) \\
&\quad = -\int_{\theta^*(1)}^{\overline{\theta}} \theta f(\theta)\, d\theta + c + (1 - F(\theta^*(1))) \int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta \\
&\qquad + (1 - F(\theta^*(1)))F(\theta^*(2))\theta^*(2) \\
&\quad = -\left(\int_{\theta^*(1)}^{\overline{\theta}} \theta f(\theta)\, d\theta + F(\theta^*(1)) \int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta - 2c \right) + \int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta \\
&\qquad + (1 - F(\theta^*(1)))F(\theta^*(2))\theta^*(2) - c \\
&\quad = -\theta^*(2)(1 - F(\theta^*(1))F(\theta^*(2))) + \int_{\theta^*(2)}^{\overline{\theta}} \theta f(\theta)\, d\theta \\
&\qquad + (1 - F(\theta^*(1)))F(\theta^*(2))\theta^*(2) - c \\
&\quad = \int_{\theta^*(2)}^{\overline{\theta}} (\theta - \theta^*(2)) f(\theta)\, d\theta - c.
\end{aligned}$$

Next, observe that the mapping: $\int_{\theta}^{\overline{\theta}} (t - \theta) f(t)\, dt$ is *decreasing* in θ and that, at $\theta^*(1)$, $\int_{\theta^*(1)}^{\overline{\theta}} (t - \theta^*(1)) f(t)\, dt - c = 0$. Because, by Lemma A2, $\theta^*(1) \geq \theta^*(2)$,

$$h(\theta^*(1))(1 - F(\theta^*(1))) = \int_{\theta^*(2)}^{\overline{\theta}} (t - \theta^*(2)) f(t)\, dt - c \geq 0,$$

concluding the proof of the lemma. ∎

To show that the equilibrium value of agent 2 is maximized at $p = 1$, let $W_2(p, \theta(1), \theta(2))$ be the value of agent 2 as a function of p, $\theta(1)$, $\theta(2)$. Consider any $p < 1$ and pick equilibrium thresholds $\theta^*(1)_p$ and $\theta^*(2)_p$. By Lemma A2, $\theta^*(1)_p \geq \theta^*(2)_p$. By Lemma A1, W_2 evaluated at $(\theta^*(1)_p, \theta^*(2)_p)$ is increasing in p, so that

$$W_2(1, \theta^*(1)_p, \theta^*(2)_p) > W_2(p, \theta^*(1)_p, \theta^*(2)_p).$$

In addition, because agent 2 optimally chooses $\theta^*(2)$ as a best response to $\theta^*(1)_p$ when $p = 1$,

$$\hat{V}_2(\theta^*(1)_p) \geq W_2(1, \theta^*(1)_p, \theta^*(2)_p).$$

Finally, notice that we have shown that $\theta^*(1)_1 > \theta^*(1)_p$ and, by Lemma A3, that $\hat{V}(2)(\theta(1))$ is increasing in $\theta(1)$ for all $\theta(1) < \theta^*(1)_1$. Hence,

$$\hat{V}(2)(\theta^*(1)_1) > \hat{V}(2)(\theta^*(1)_p),$$

completing the proof of the proposition. ∎

PROOF OF LEMMA 1:

Consider the probability of misallocation μ and notice that, as $\theta^*(1)_1 > \theta^*(1)_p$ for all p,

$$\begin{aligned}
\mu(1) &= \int_{\theta^*(1)_1}^{\overline{\theta}} (1 - F(\theta))\, d\theta \\
&< \int_{\theta^*(1)_p}^{\overline{\theta}} (1 - F(\theta))\, d\theta \\
&< \int_{\theta^*(1)_p}^{\overline{\theta}} (1 - F(\theta))\, d\theta + (1-p) \int_{\theta^*(2)_p}^{\theta^*(1)_p} (1 - F(\theta^*(1)p)) f(\theta)\, d\theta \\
&= \mu(p). \blacksquare
\end{aligned}$$

REFERENCES

Abdulkadiroğlu, Atila, and Simon Loertscher. 2007. "Dynamic house allocations." http://www.simonloertscher.net/data/downloads/12120/dha_201107.pdf.

Akbarpour, Mohammad, Shengwu Li, and Shayan Oveis Gharan. 2014. "Dynamic Matching Market Design." http://economics.mit.edu/files/10375.

Bloch, Francis, and David Cantala. 2013. "Markovian assignment rules." *Social Choice and Welfare* 40 (1): 1–25.

Bloch, Francis, and Nicolas Houy. 2012. "Optimal assignment of durable objects to successive agents." *Economic Theory* 51 (1): 13–33.

Dizdar, Deniz, Alex Gerkshov, and Benny Moldovanu. 2011. "Revenue maximization in the dynamic knapsack problem." *Theoretical Economics* 6 (2): 157–84.

Gershkov, Alex, and Benny Moldovanu. 2009a. "Learning about the Future and Dynamic Efficiency." *American Economic Review* 99 (4): 1576–87.

Gershkov, Alex, and Benny Moldovanu. 2009b. "Dynamic Revenue Maximization with Heterogeneous Objects: A Mechanism Design Approach." *American Economic Journal: Microeconomics* 1 (2): 168–98.

Hassin, Refael. 1985. "On the Optimality of First Come Last Served Queues." *Econometrica* 53 (1): 201–02.

Hassin, Refael, and Moshe Haviv. 2003. *To Queue or Not to Queue: Equilibrium Behavior in Queueing Systems*. International Series in Operations Research and Management Science. Boston: Kluwer Academic Publishers.

Kaplan, Edward H. 1986. "Tenant Assignment Models." *Operations Research* 34 (6): 832–43.

Kaplan, Edward H. 1987. "Analyzing Tenant Assignment Policies." *Management Science* 33 (3): 395–408.

Kennes, John, Daniel Monte, and Norovsambuu Tumennasan. 2014. "The Day Care Assignment: A Dynamic Matching Problem." *American Economic Journal: Microeconomics* 6 (4): 362–406.

Kurino, Morimitsu. 2014. "House Allocation with Overlapping Agents." *American Economic Journal: Microeconomics* 6 (1): 258–89.

Leshno, Jacob D. 2014. "Dynamic matching in overloaded systems." Unpublished.

Milgrom, Paul, and John Roberts. 1994. "Comparing Equilibria." *American Economic Review* 84 (3): 441–59.

Schummer, James. 2016. "Influencing Waiting Lists." http://www.kellogg.northwestern.edu/faculty/schummer/ftp/research/waiting/Waiting.pdf.

Su, Xuanming, and Stefanos Zenios. 2004. "Patient Choice in Kidney Allocation: The Role of the Queueing Discipline." *Manufacturing and Service Operations Management* 6 (4): 280–301.

Su, Xuanming, and Stefanos A. Zenios. 2005. "Patient Choice in Kidney Allocation: A Stochastic Sequential Assignment Model." *Operations Research* 53 (3): 443–55.

Su, Xuanming, Stefanos A. Zenios, and Glenn M. Chertow. 2004. "Incorporating Recipient Choice in Kidney Transplantation." *Journal of the American Society of Nephrology* 15 (6): 1656–63.

Thakral, Neil. 2015. "Matching with Stochastic Arrival: Theory and Application to Public Housing." http://www.matching-in-practice.eu/wp-content/uploads/2016/02/Thakral.pdf.

Ünver, M. Utku. 2010. "Dynamic Kidney Exchange." *Review of Economic Studies* 77 (1): 372–414.

Zenios, Stefanos A. 1999. "Modeling the transplant waiting list: A queueing model with reneging." *Queuing Systems: Theory and Applications* 31 (3): 239–51.

Zenios, Stefanos A., Glenn M. Chertow, and Lawrence M. Wein. 2000. "Dynamic Allocation of Kidneys to Candidates on the Transplant Waiting List." *Operations Research* 48 (4): 549–69.