Cutoff Characterization of Matching Mechanisms

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Cutoffs and Large Matching Markets

- Matching mechanisms in school choice:
  - School choice: Many-to-one matching
  - Deferred Acceptance (DA) and Top Trading Cycles (TTC)

- Cutoff characterizations:
  - **Azevedo & Leshno (2016)**: Cutoff characterization of DA
  - **Leshno & Lo (2018):** Cutoff characterization of TTC
  - Cutoffs can be used to define propensity scores

- Two simplifications:
  - A large matching market – many students, each college has many seats
  - Cutoffs via non-combinatorial equations

Cutoff Characterizations

- **Recap**
- Model
- Deferred Acceptance
- Top Trading Cycles

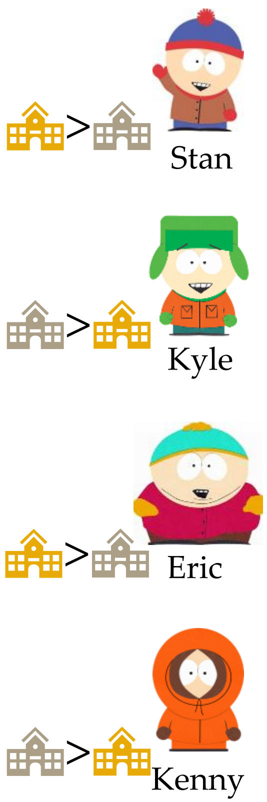
School Choice

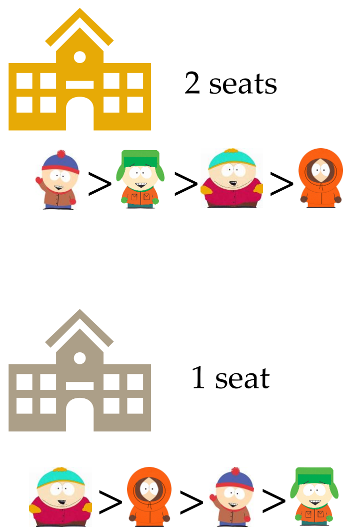
- Finite set of students Θ
  - Student θ has preferences $\succ^{\theta}$ over schools
- Finite set of schools C
  - School c can admit q_c students
  - $\succ^{c}$ a strict ranking over students, responsive preferences
- Assignment $\mu: \Theta \to C \cup \phi$ of students to schools
  - Feasibility constraint: $\mu^{-1}(c) \leq q_c$

- Abdulkadiroğlu & Sönmez 2013:
  - School choice is a mechanism design problem
  - Candidate mechanisms:
    Deferred Acceptance (DA) and Top Trading Cycles (TTC)

Deferred Acceptance

[Gale & Shapley ’62]





Student-Proposing Deferred Acceptance

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while some student can still
propose:
    all students propose to
    favorite school that has
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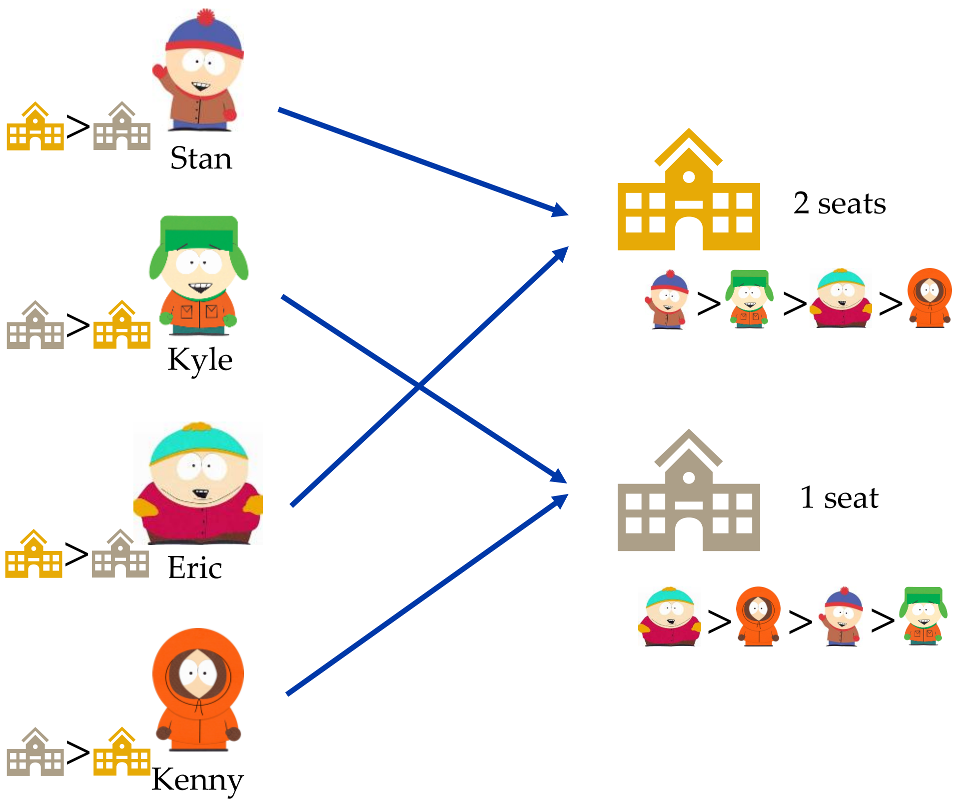
Deferred Acceptance

[Gale & Shapley '62]



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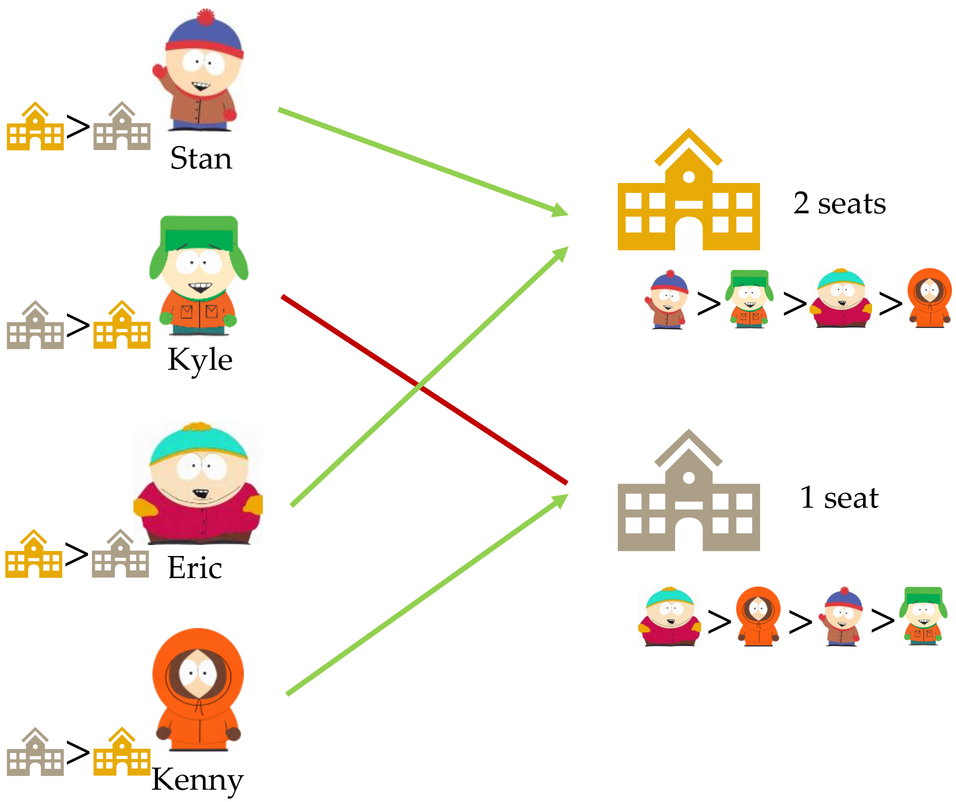


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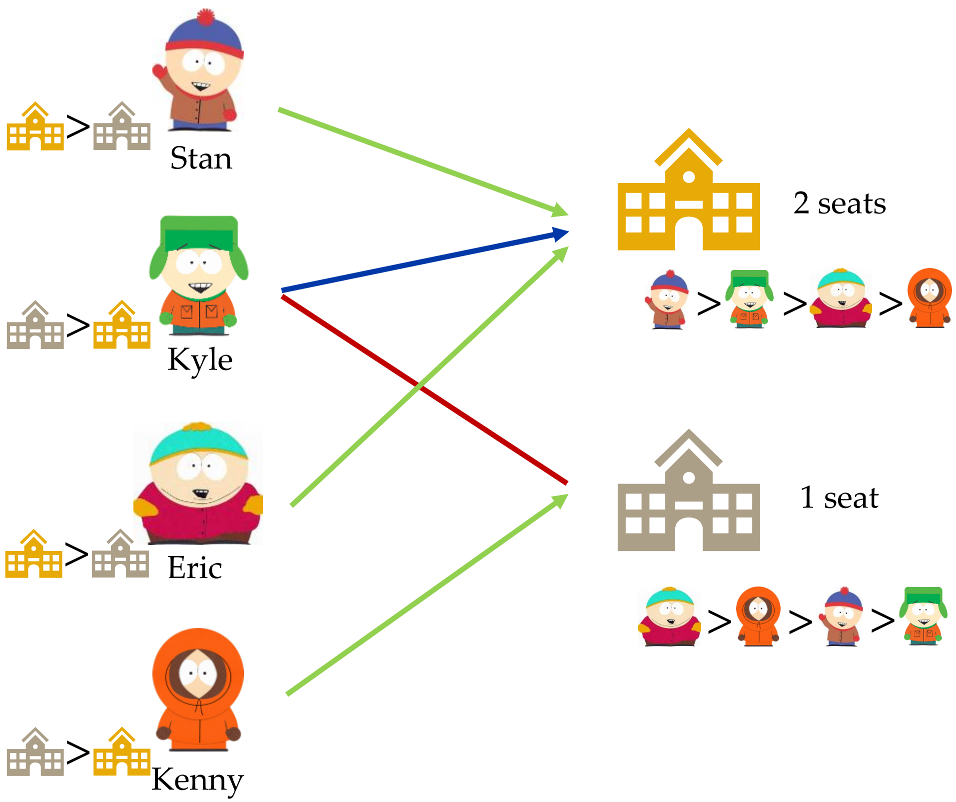


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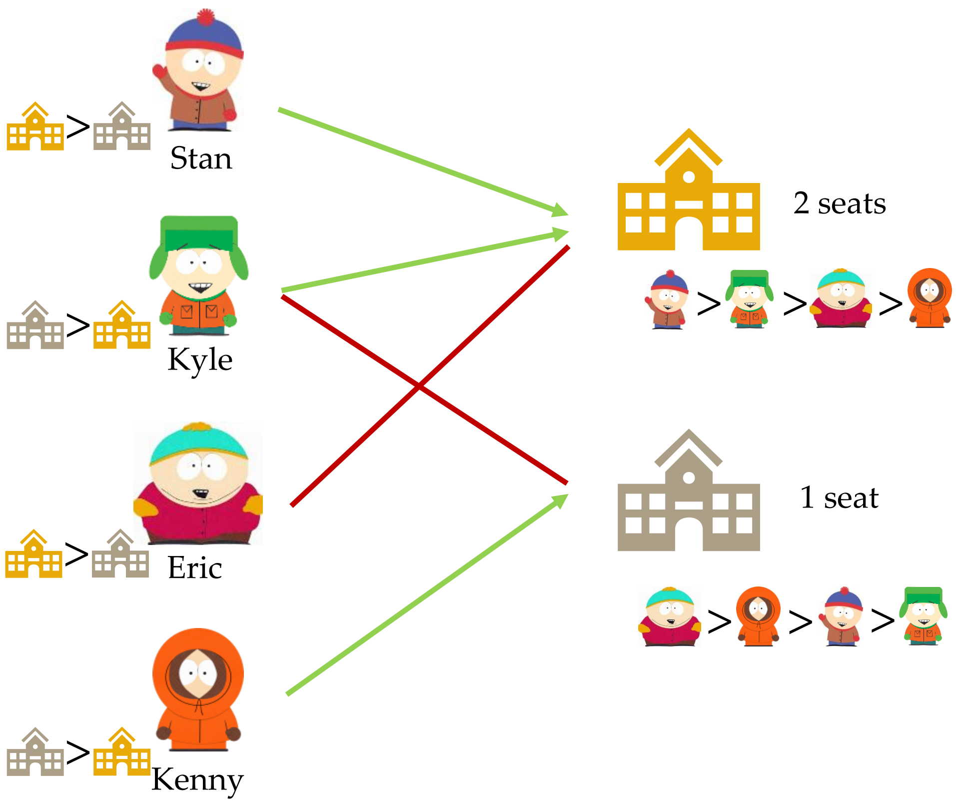


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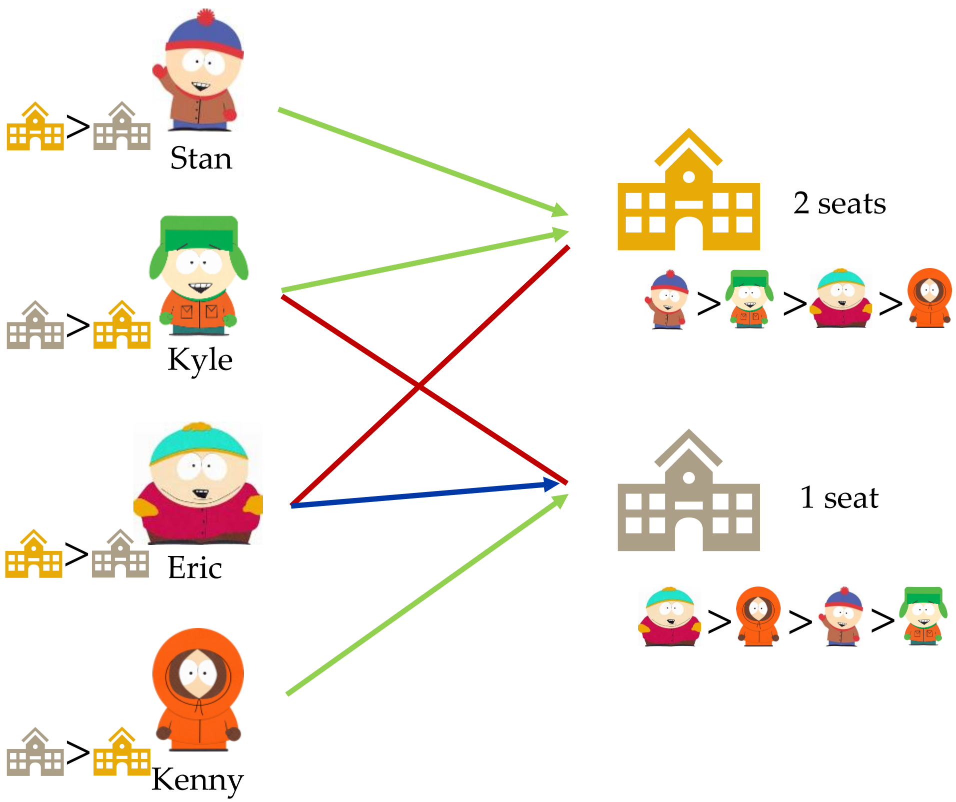


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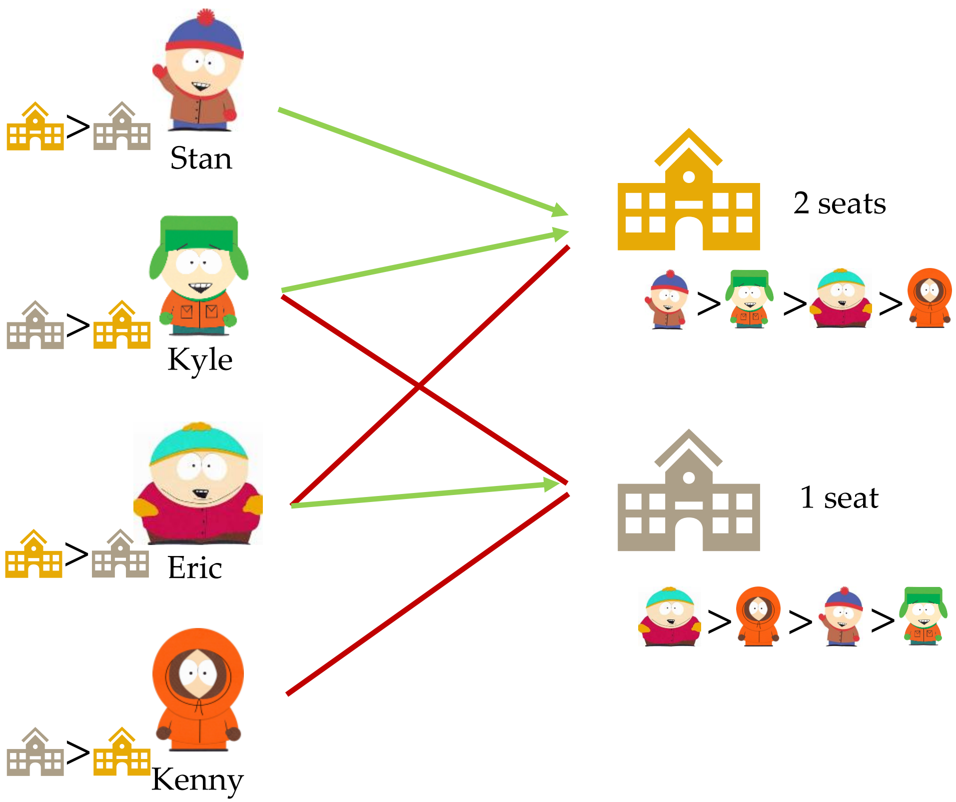


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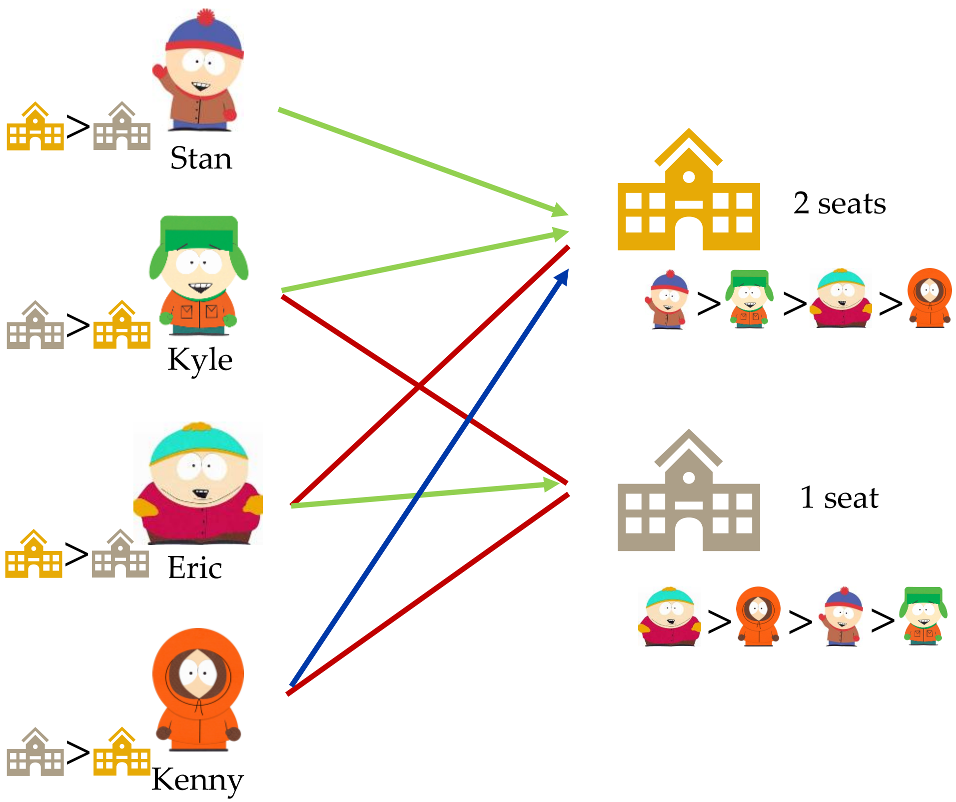


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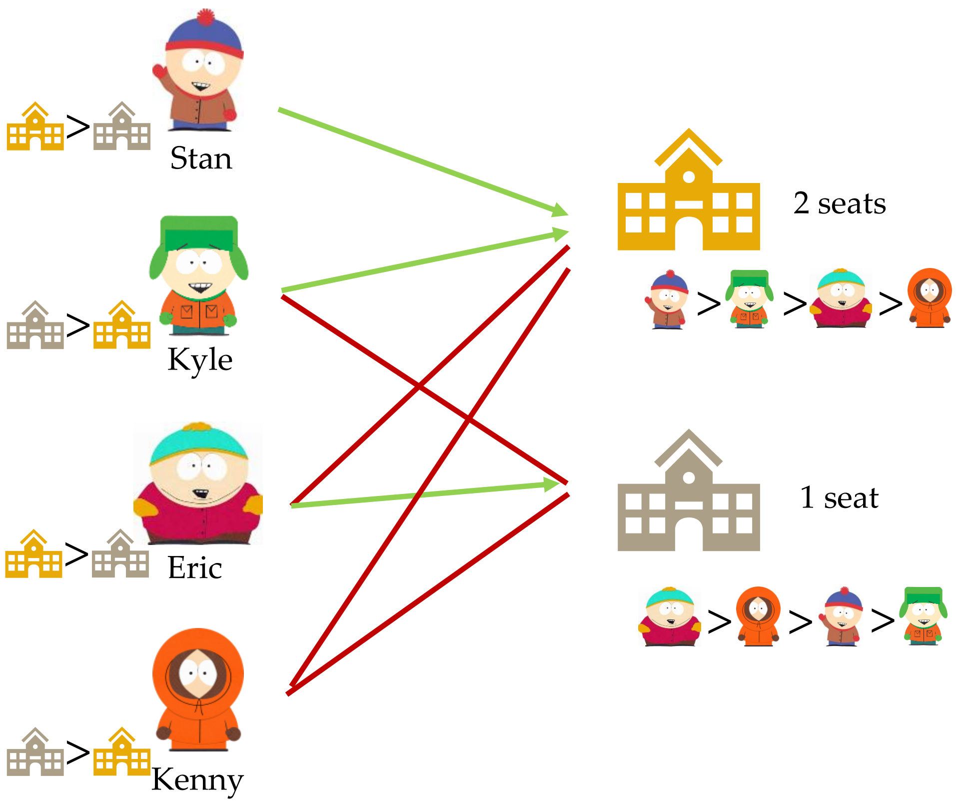


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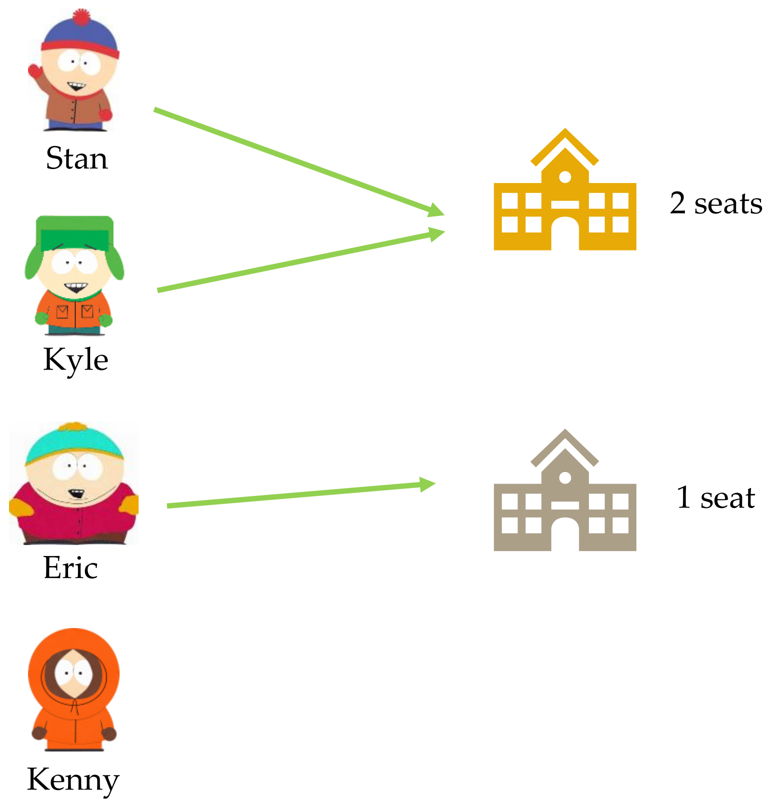


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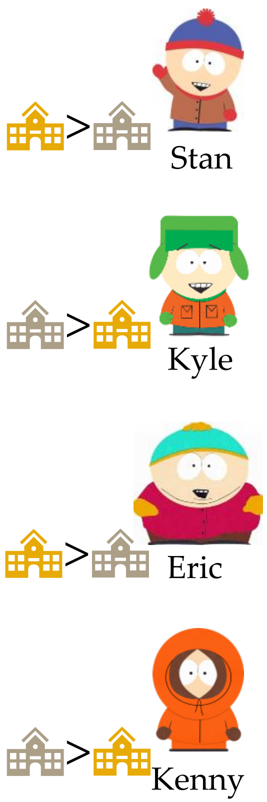
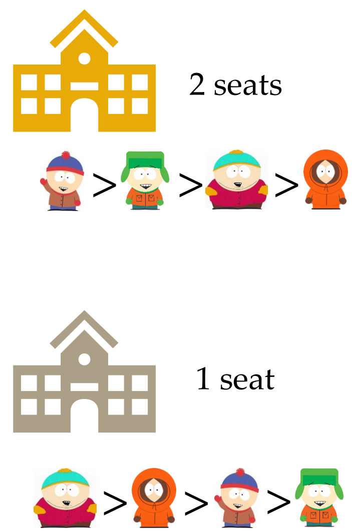
Top Trading Cycles

[Shapley & Scarf '74]



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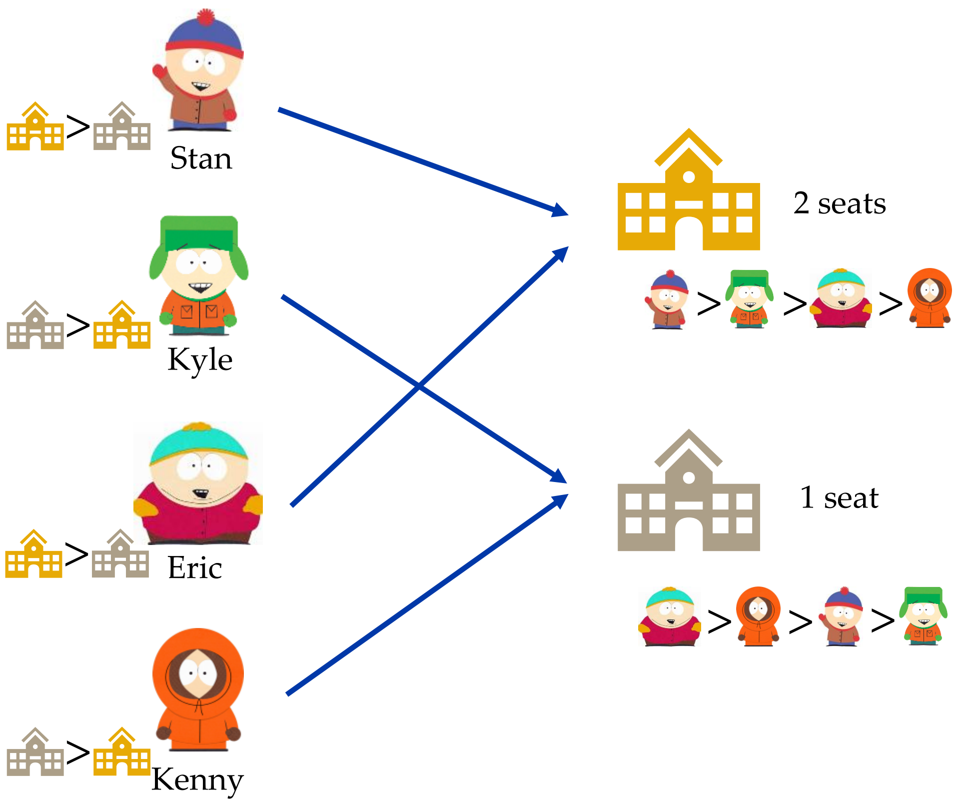


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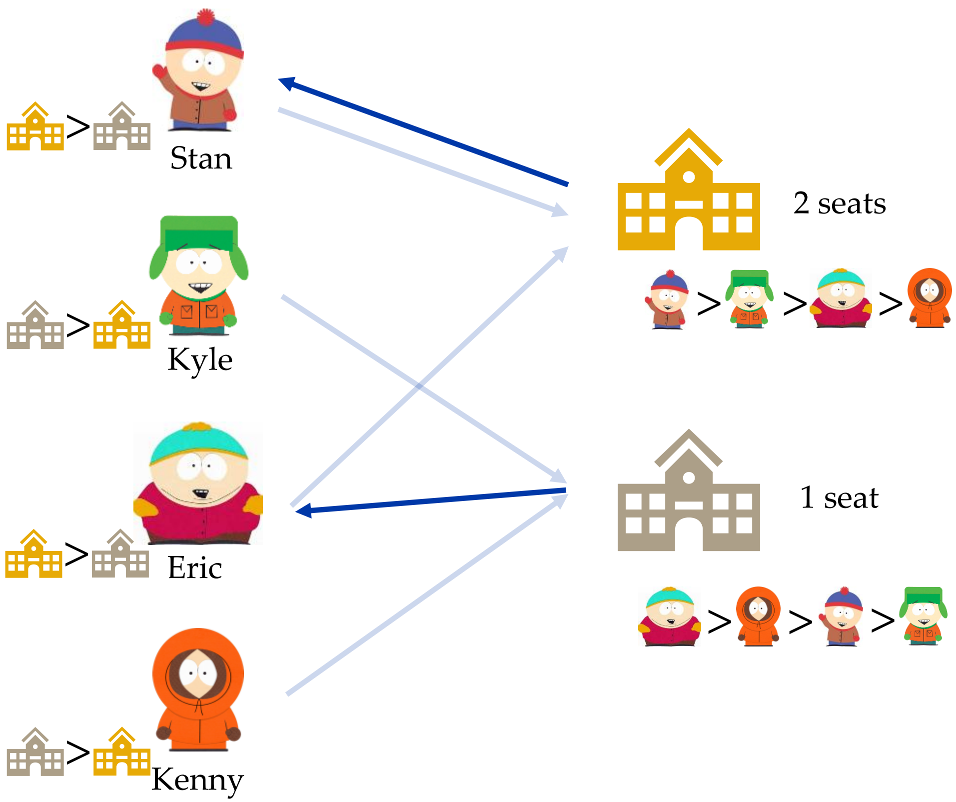


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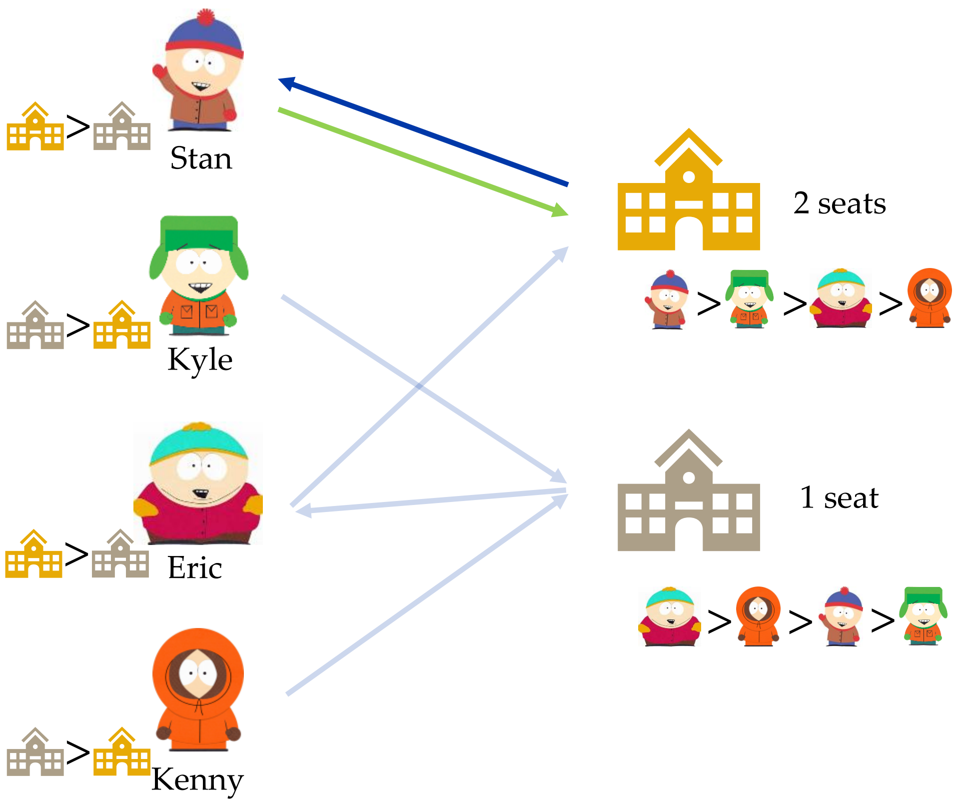


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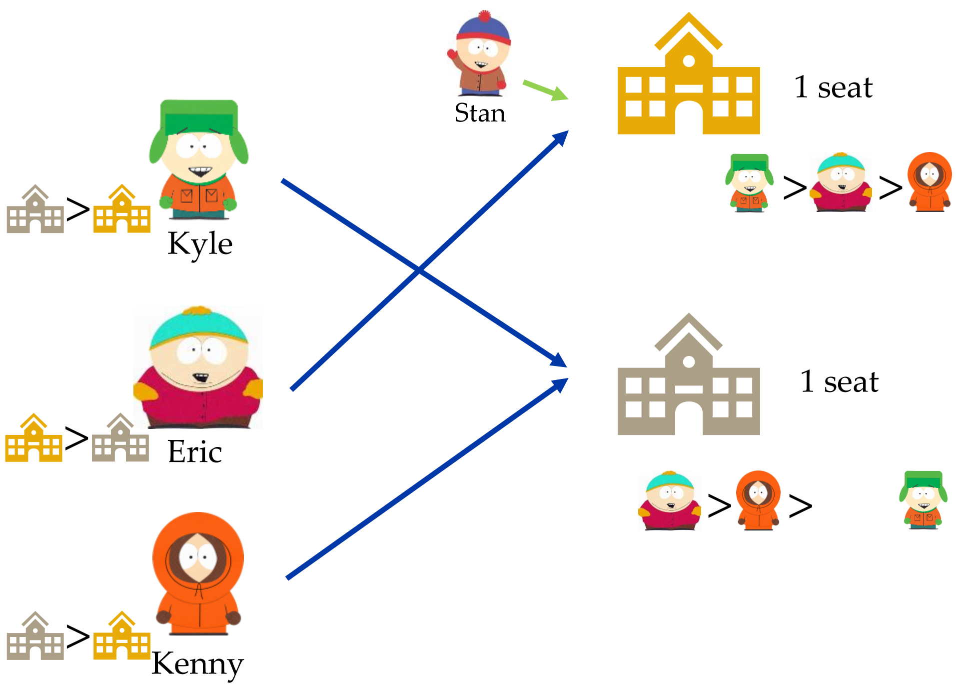


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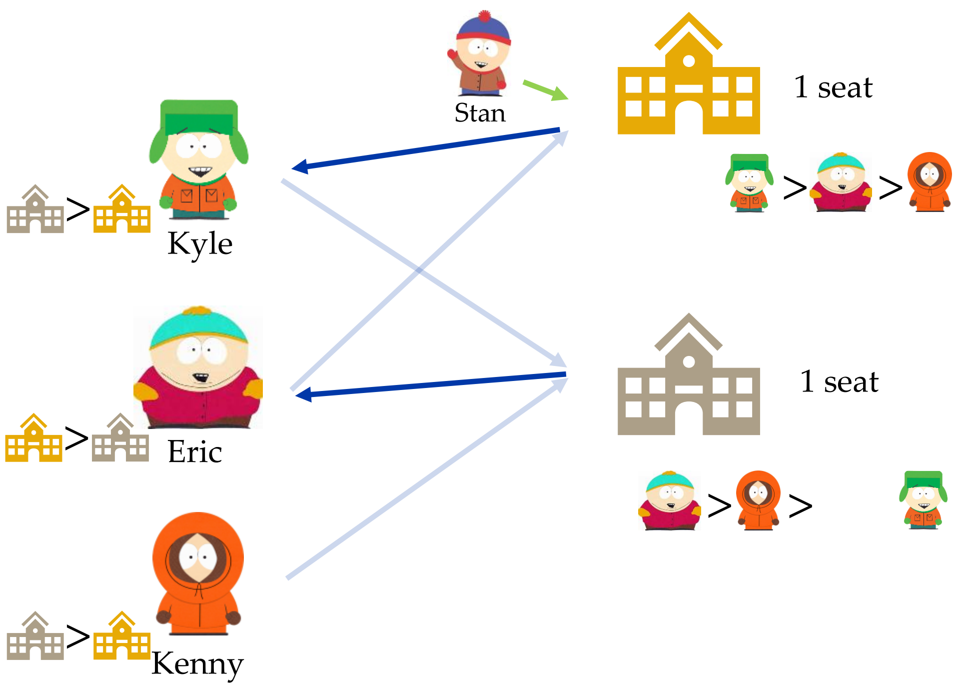


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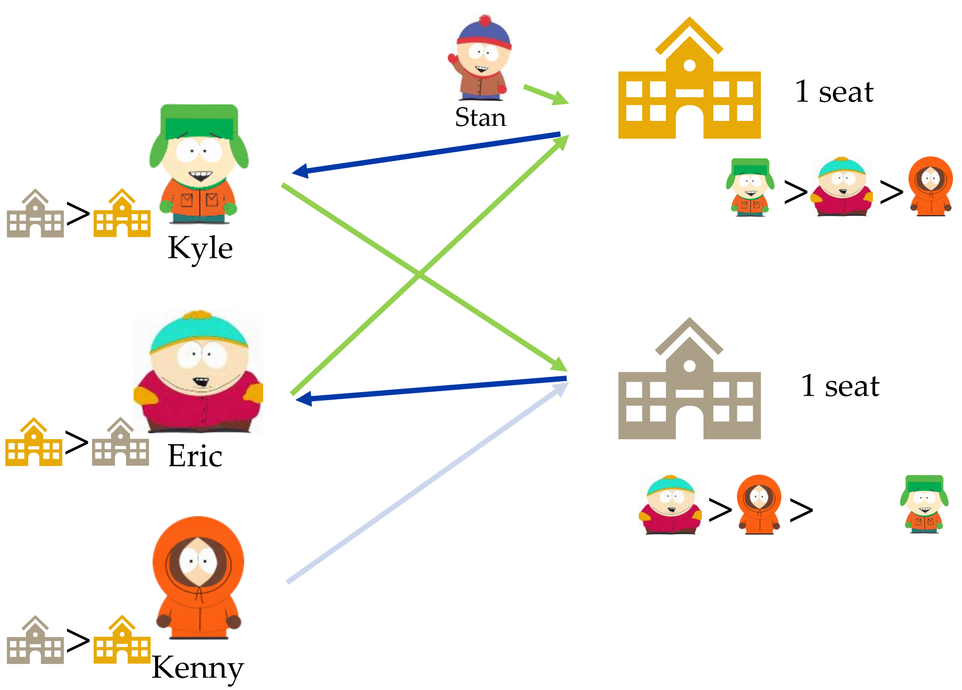


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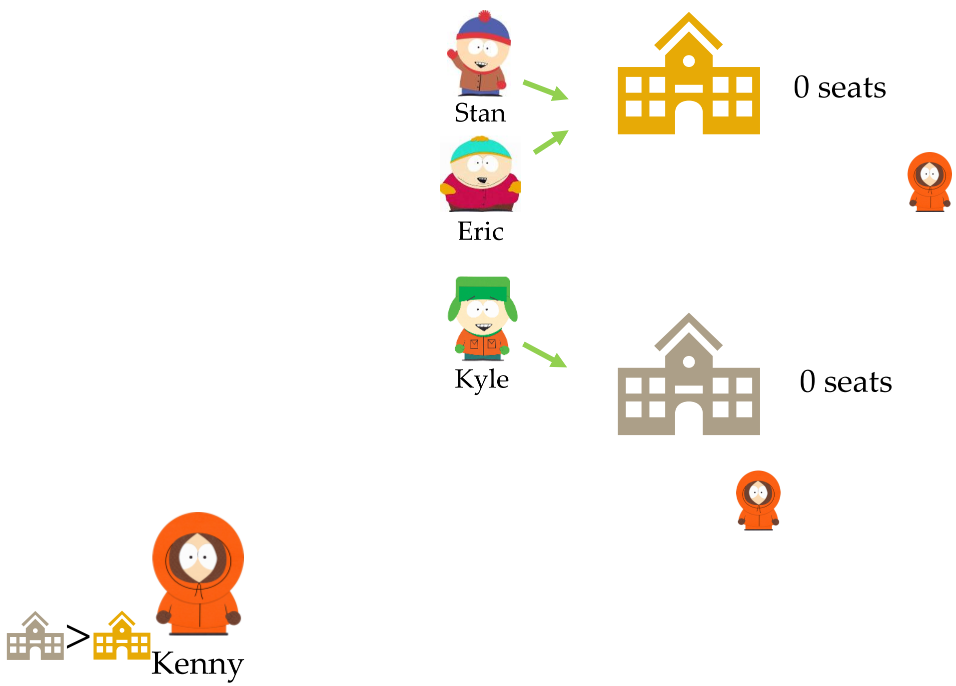

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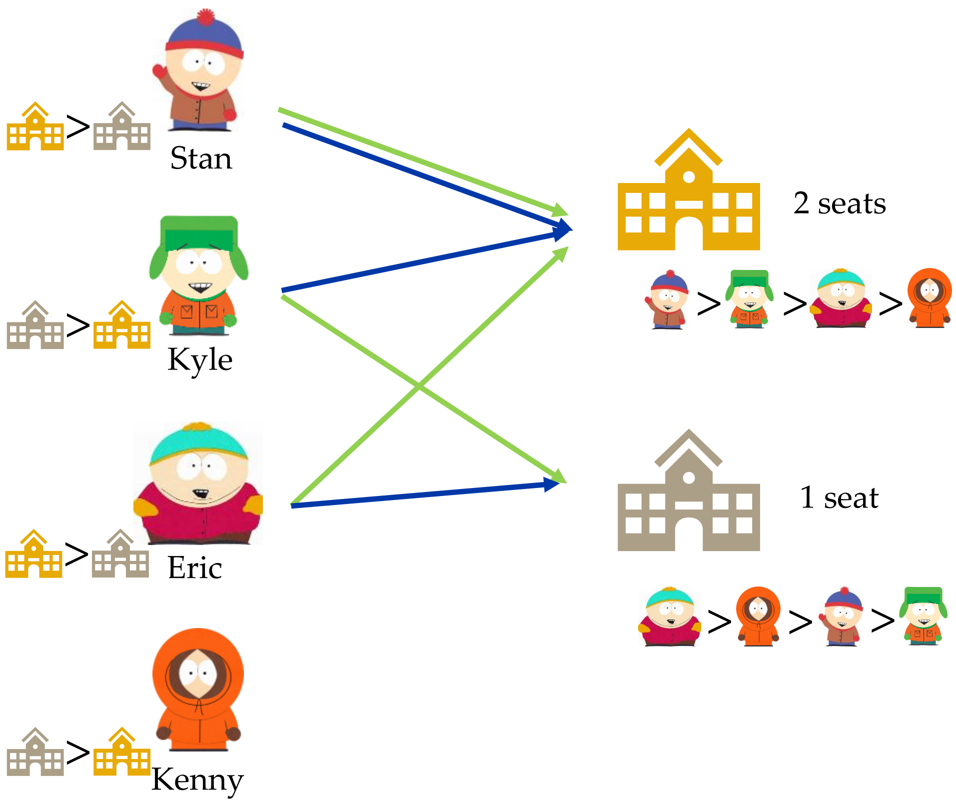
DA vs TTC



Desirable properties.

- Strategyproof for students:
  Truthtelling is a dominant strategy
- Stable: No blocking pairs.
  i.e. no school and student who prefer each other to their match
- Pareto efficient for students:
  No other assignment is weakly preferred by all students

DA is strategyproof & stable.

TTC is strategyproof & Pareto efficient.

Theorem.
No mechanism is strategyproof, stable, and Pareto efficient.

DA and TTC via Cutoffs

Is there a non-combinatorial way of understanding DA and TTC?

- Azevedo, Eduardo M., and Jacob D. Leshno. "**A supply and demand framework for two-sided matching markets.**" *Journal of Political Economy* 124.5 (2016): 1235-1268.

  The Deferred Acceptance outcome can be characterized by one cutoff for every school. These cutoffs solve supply and demand equations.

- Leshno, Jacob D. and Irene Y. Lo. "**The cutoff structure of Top Trading Cycles in school choice.**"

  The Top Trading Cycles outcome can be characterized by one cutoff for every **pair** of schools. These cutoffs solve trade balance equations.

Cutoff Characterizations

- Recap
- **Large Market Model**
- Deferred Acceptance
- Top Trading Cycles

Large Matching Markets

- A large matching market – many students, each college has many seats

- A simpler matching model
  - Continuum of students (Aumann 1964)
  - Supply and demand characterization of stable matching
  - Trade balance characterization of TTC

- Allows for
  - Simpler derivation of outcomes and comparative statics
  - Complex preferences and no transfers (like Gale-Shapely 1962)

School Choice Model

- Finite number of students $\theta = (\succ^\theta, r^\theta)$
  - Student θ has preferences $\succ^\theta$ over schools
  - $r_c^\theta \in [0,1]$ is the rank of student θ at school c (percentile in c's priority list)

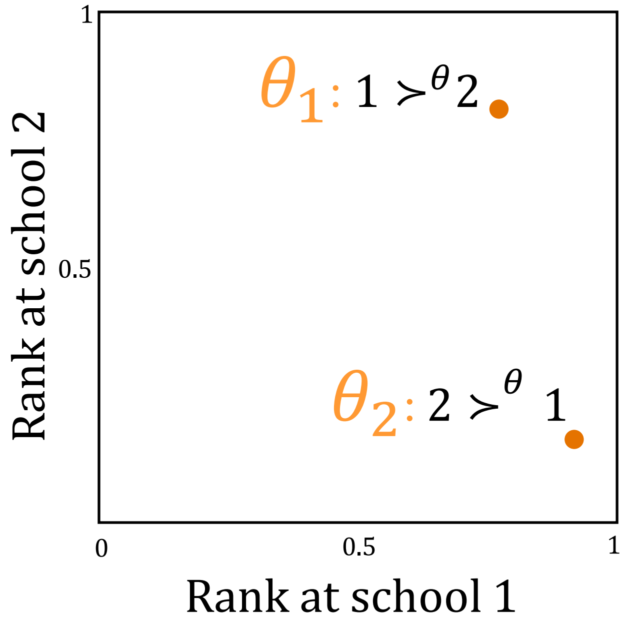
- Finite number of schools c
  - School c can admit q_c students
  - $\succ^c$ a strict ranking over students

School Choice Continuum Model

- **Continuous mass** of students $\theta = (\succ^\theta, r^\theta)$
  - Student θ has preferences $\succ^\theta$ over schools
  - $r_c^\theta \in [0,1]$ is the rank of student θ at school c (percentile in c's priority list)
  - **Distribution specified by measure η**

- Finite number of schools c
  - School c can admit q_c students

School Choice Visualization



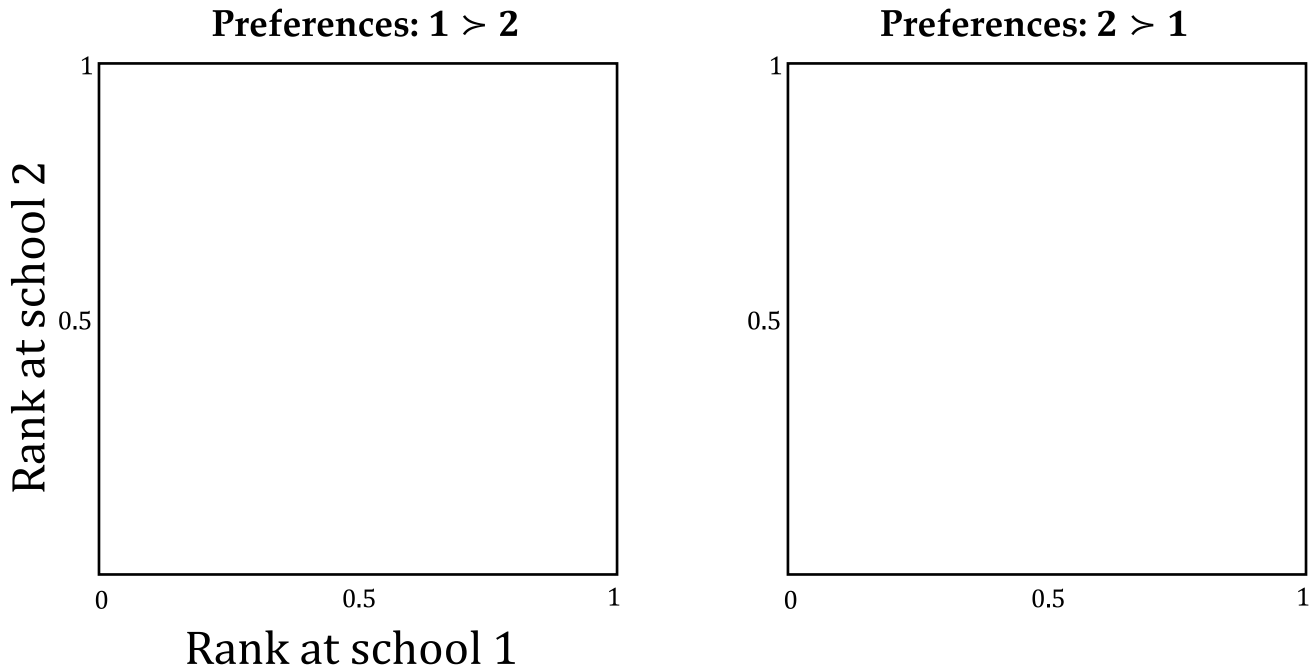
Student θ_1

- prefers 1 to 2
- highly ranked at 1
- highly ranked at 2

Student θ_2

- prefers 2 to 1
- highly ranked at 1
- poorly ranked at 2

School Choice Visualization



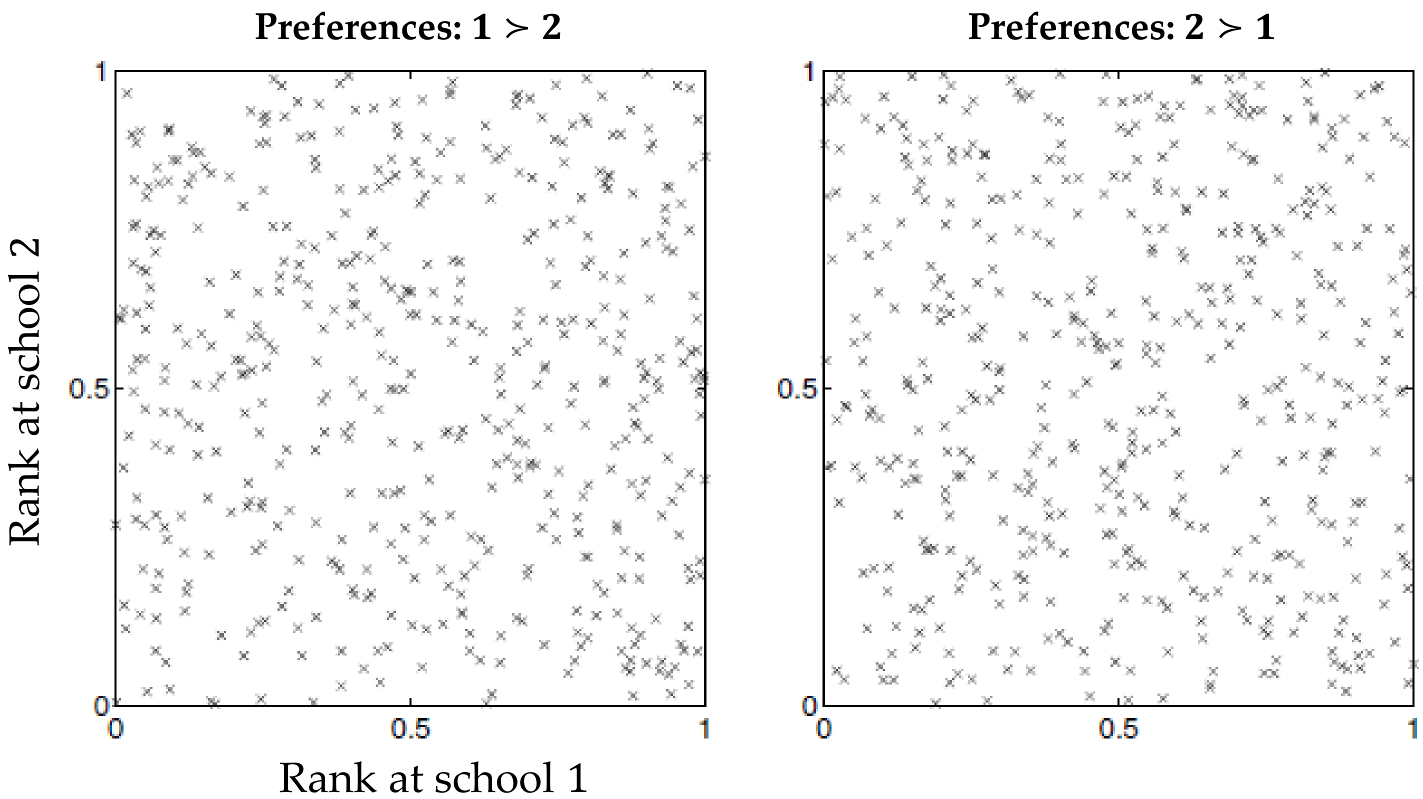
A Supply and Demand Framework for Two-Sided Matching Markets

Eduardo Azevedo & Jacob Leshno

Main Takeaways

The Deferred Acceptance outcome can be characterized by **cutoffs**.
These cutoffs solve **supply and demand equations**.

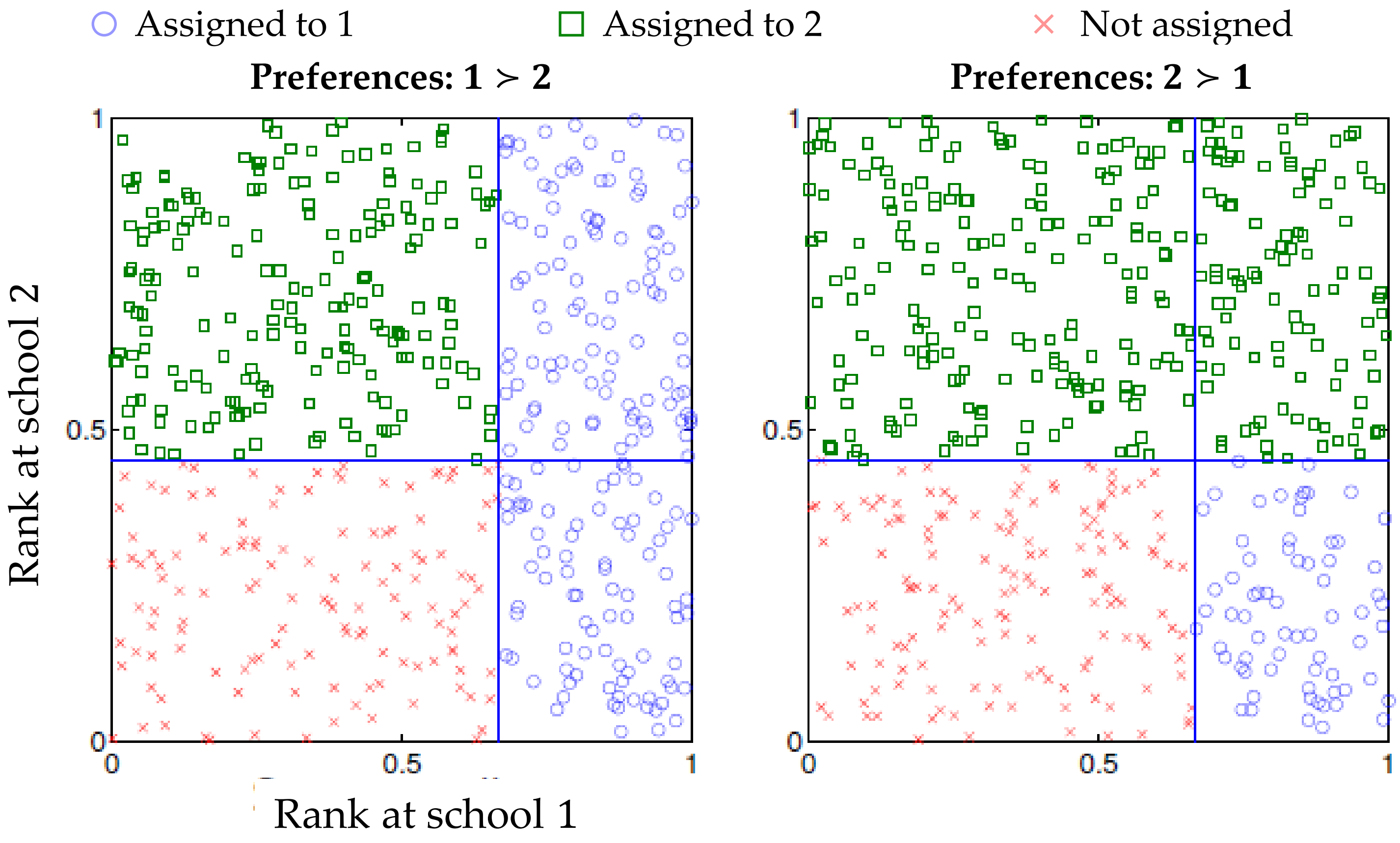
Example



1,000 students, 500 prefer 1, 500 prefer 2
$q_1 = 250, q_2 = 500$

Q. What is the outcome of DA on this economy?

Example



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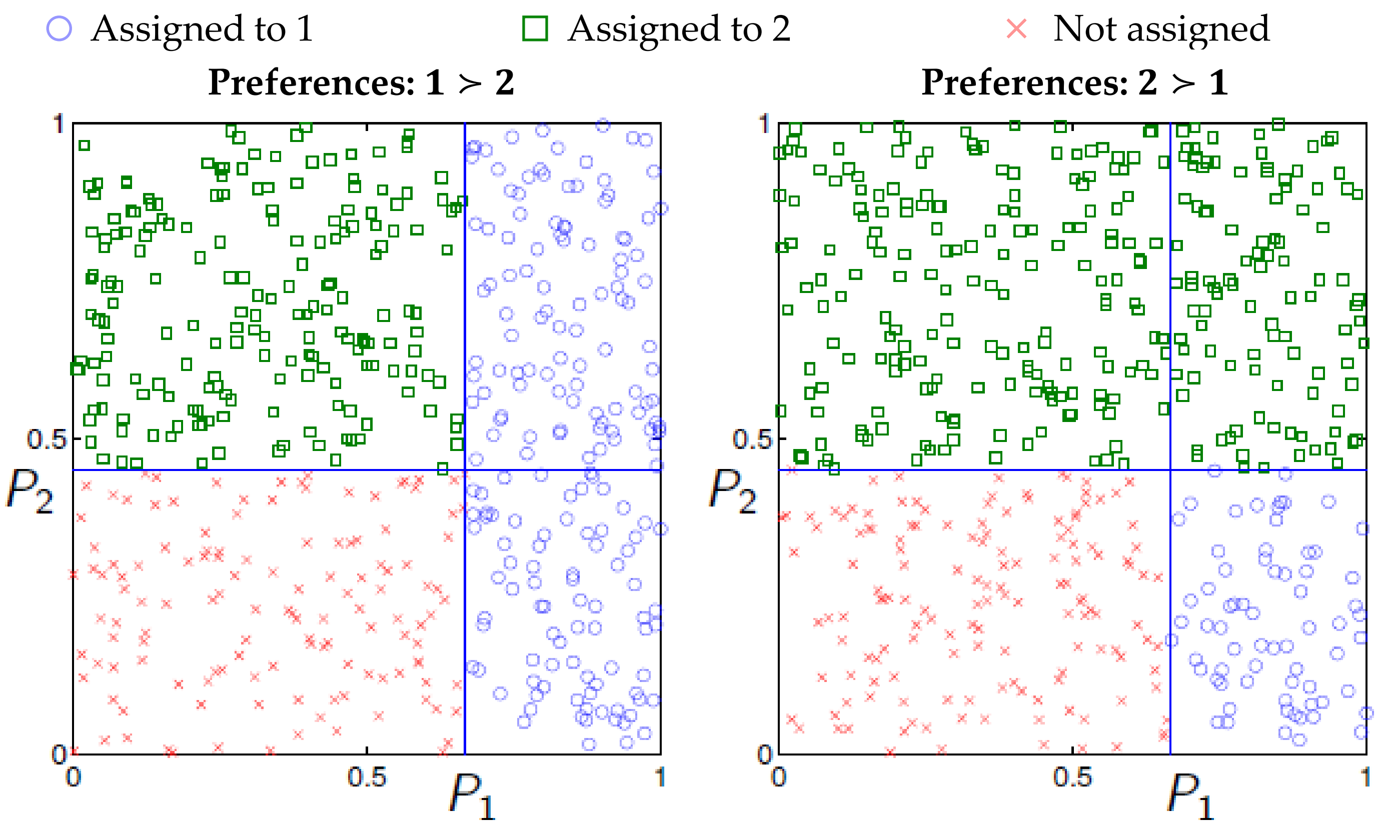
A Simpler description



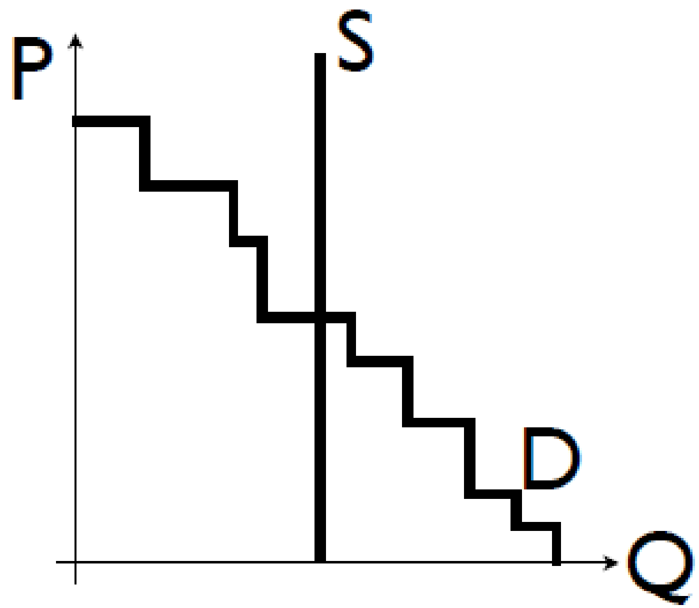
$$D(P) = S$$

Demand given cutoffs

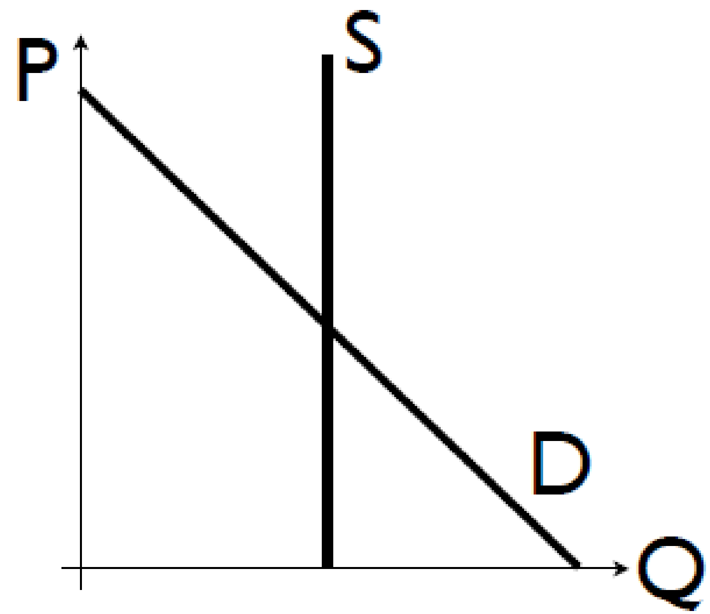
Supply of seats

Two Simplifications

1. Characterization of stable matching in terms of supply and demand



2. Continuum of agents



DA via Cutoffs (Azevedo Leshno)

- **Continuum Model**
- Cutoff Characterization
- Uniqueness and Convergence

Continuum Model

- Finite set of schools $c \in C = \{1, \dots, n\}$
  - School c can admit a mass q_c of students
- Measure η specifying a distribution of a continuous mass of students
  - A student $\theta \in \Theta$ is given by $\theta = \left(\succ^\theta, r^\theta\right)$
  - Student θ has preferences $\succ^\theta$ over schools
  - $r_c^\theta \in [0,1]$ is the student's rank at school c *(percentile in c's priority list)*

- An **economy** is $E = [\eta, q]$ where η is a distribution over student types and q is a vector of capacities
- **Assumption.** *(Strict preferences)* Schools' indifference curves have measure 0

$$\eta\left(\left\{\theta \in \Theta \mid r_c^\theta = x\right\}\right) = 0 \quad \forall x, c$$

Matchings

Def. A **matching** is a function

$$\mu: \Theta \to C \cup \{\phi\}$$

such that

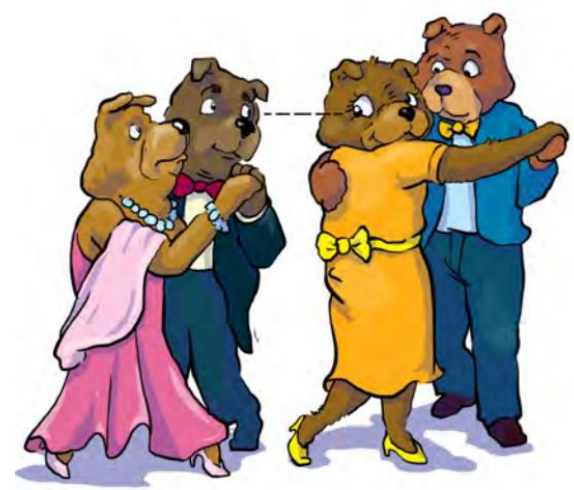
1. Each student is matched to a school or the empty set
2. *(Feasibility)* Each school is matched to a set of students $\mu^{-1}(c)$ such that $\eta\left(\mu^{-1}(c)\right) \leq q_c$
3. *(Right continuity)* For $\theta^k = \left(\succ, r^k\right), r^{k+1} \leq r^k$ then

$$\mu\left(\lim \theta^k\right) = \lim \mu\left(\theta^k\right).$$

Stable Matchings

Def. A matching μ is **stable** if there are no blocking pairs, i.e. a student-school pair (θ, c) such that

- Student θ prefers c over $\mu(\theta)$
- c either:
  I. did not fill its quota $\eta\big(\mu^{-1}(c)\big) < q_c$
  II. is matched to some less preferred student $\theta' \in \mu^{-1}(c)$ where $r_c^{\theta'} < r_c^{\theta}$

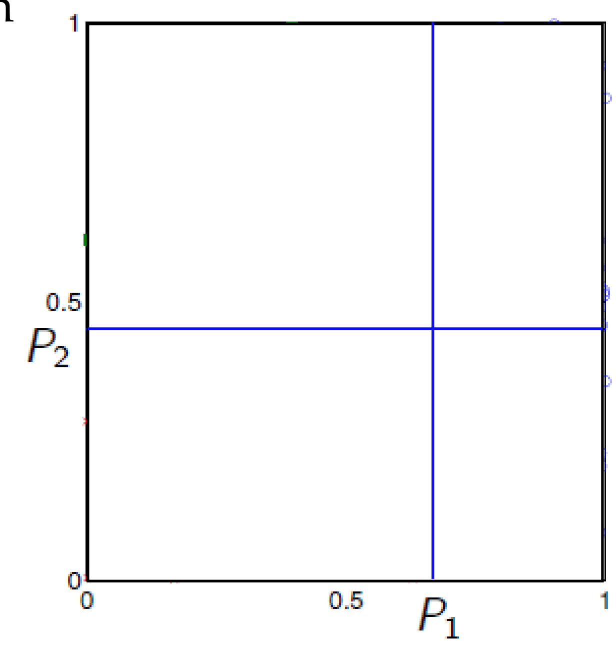


Source: Karlin & Peres, "Game Theory Alive"

Supply and Demand

- A vector of **cutoffs** $\boldsymbol{P}$ is a vector $P \in [0,1]^C$ specifying a minimal score (cutoff) P_c for each school
  How much does a school need to like a student for them to have that school as an ***option****?*

Preferences: 1 ≻ 2



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- The **demand** $\boldsymbol{D^\theta(P)}$ of student θ given cutoff P is her most preferred school where she meets the cutoff: $D^\theta(P) = \arg\max_{\succ^\theta}\{c \mid P_c \le r_c^\theta\}$
  What is student θ's favorite option?

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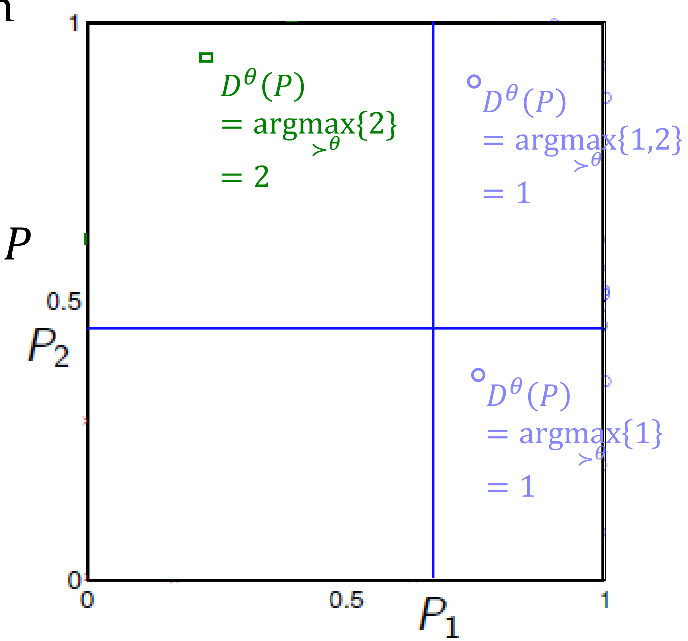

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- **Aggregate demand $D(P)$** is the mass of students demanding each school
$$D_c(P) = \eta(\{\theta \mid D^\theta(P) = c\})$$

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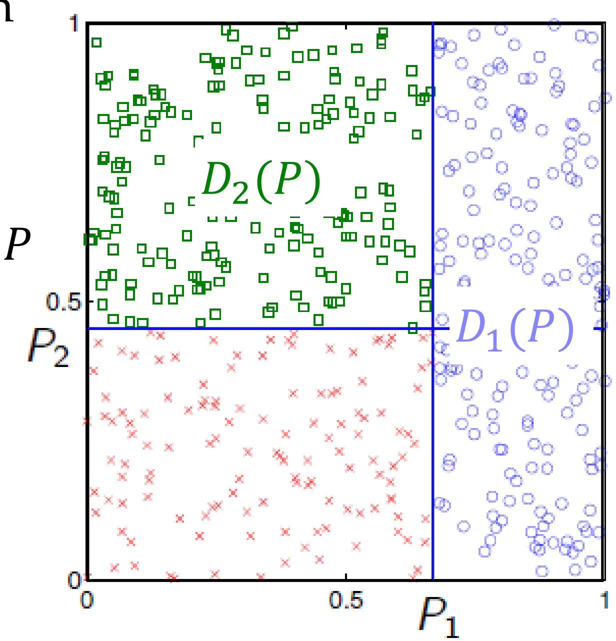

Market Clearing Cutoffs

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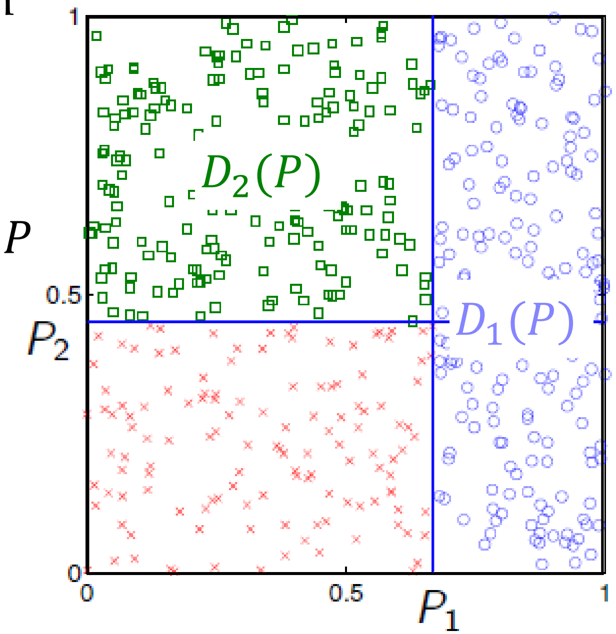


Def. P^* is a vector of **market clearing cutoffs** if
$D_c(P^*) \le q_c$ for all c with equality if $P_c^* > 0$

DA via Cutoffs (Azevedo Leshno)

- Continuum Model
- **Cutoff Characterization**
- Uniqueness and Convergence

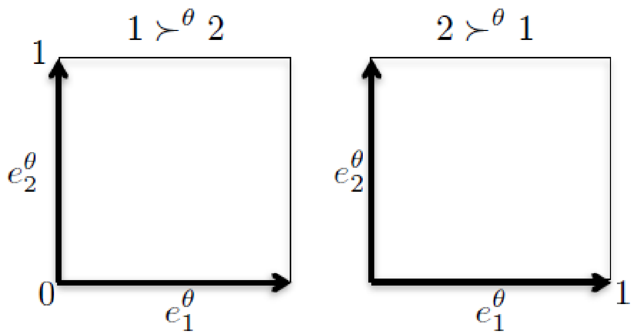
Matching and Cutoffs

- **$\mathcal{P}$: Matchings → Cutoffs:**
  Given a matching μ let $P = \mathcal{P}\mu$ be the scores of the marginal accepted students
  $$P_c = \inf_{\theta \in \mu(c)} e_c^\theta$$
- **$\mathcal{M}$: Cutoffs → Matchings:**
  Given cutoffs P let $\mu = \mathcal{M}P$ be the match resulting from the demand under P
  $$\mu(\theta) = D^\theta(P)$$

Lemma. $\mathcal{M}$ and $\mathcal{P}$ take stable matchings into market clearing cutoffs, and are inverses of each other

i.e. Stable matchings can be characterized using market-clearing cutoffs!

Intuition



$\frac{1}{2}$ students prefer 1, $\frac{1}{2}$ prefer 2

$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

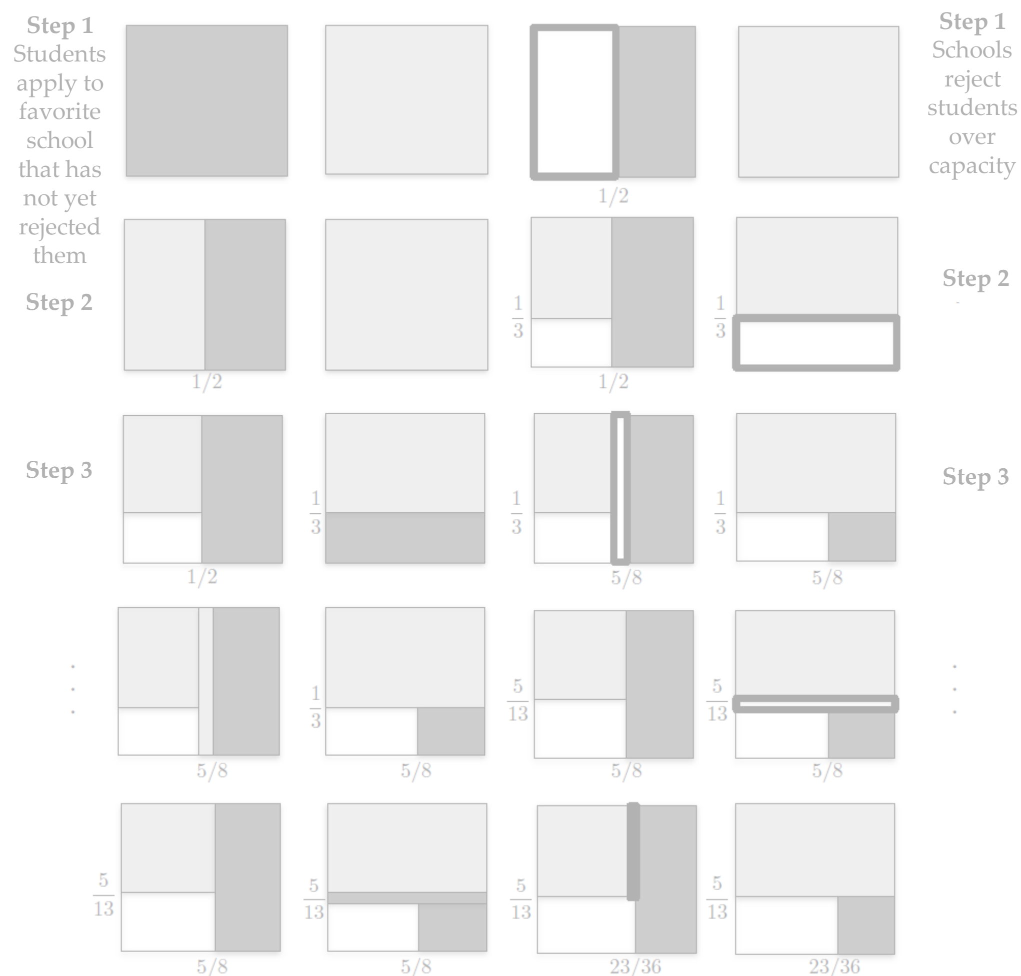
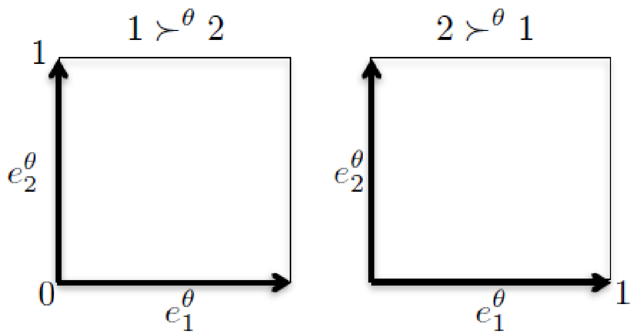


FIGURE 1. The Gale and Shapley algorithm

Intuition



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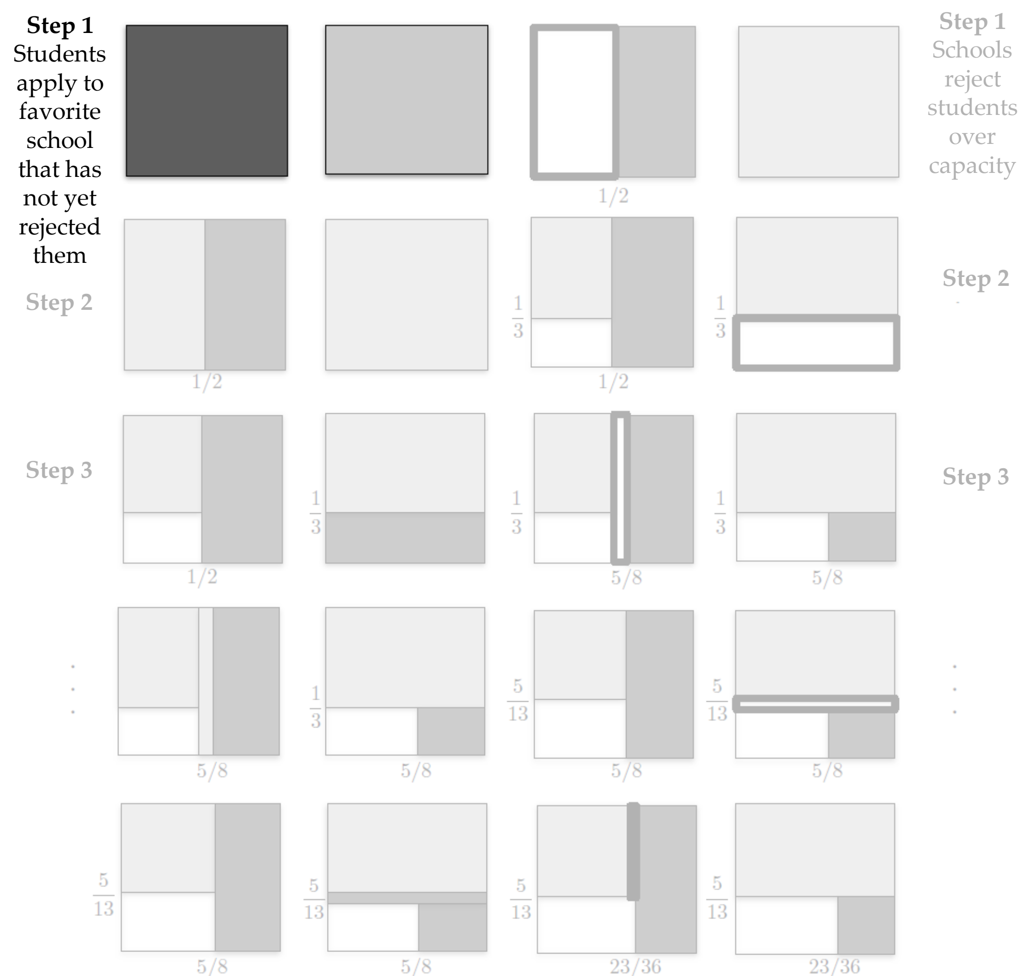

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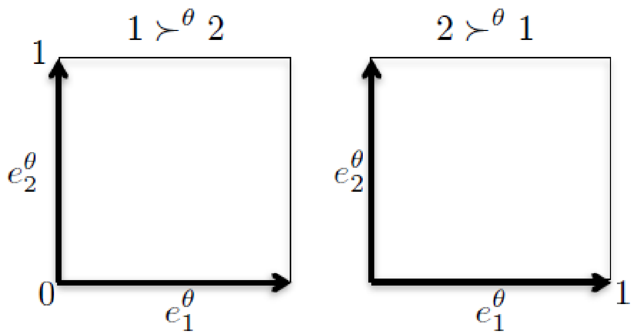

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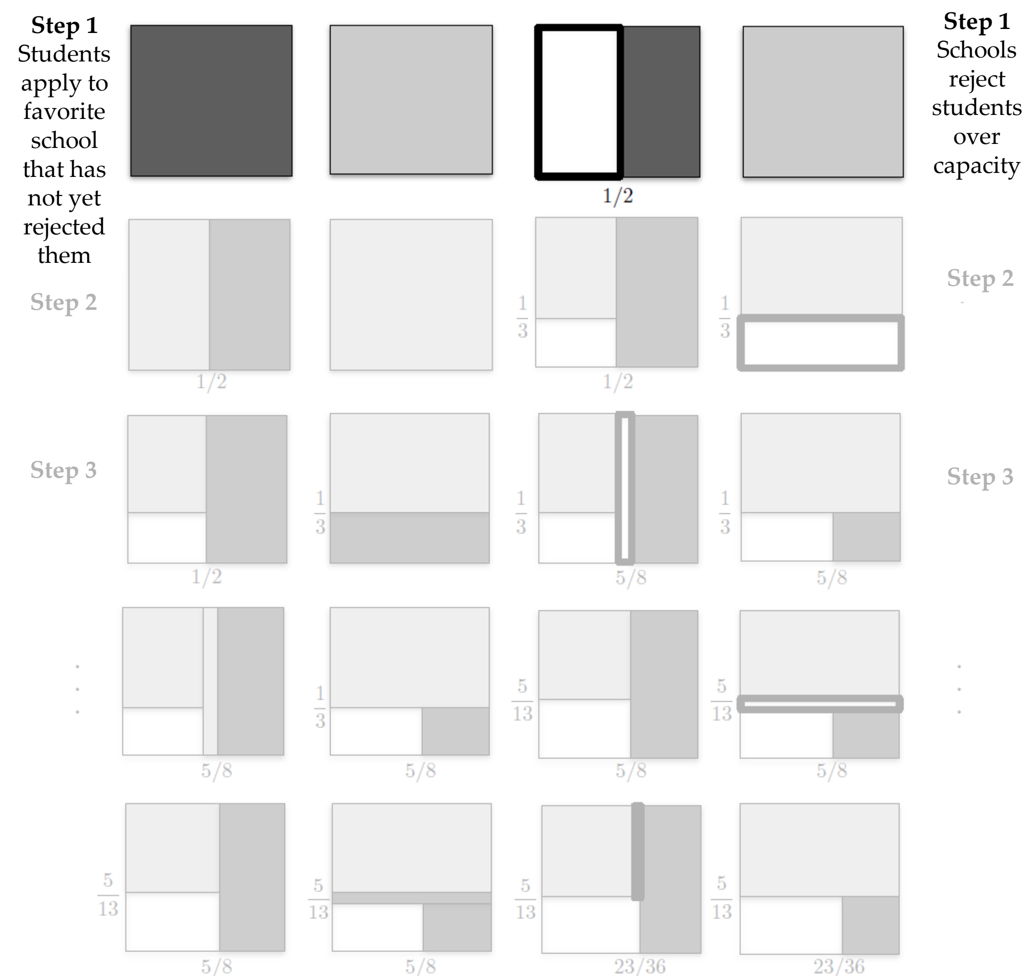

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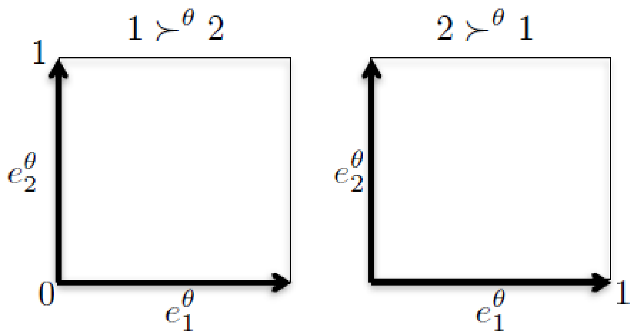

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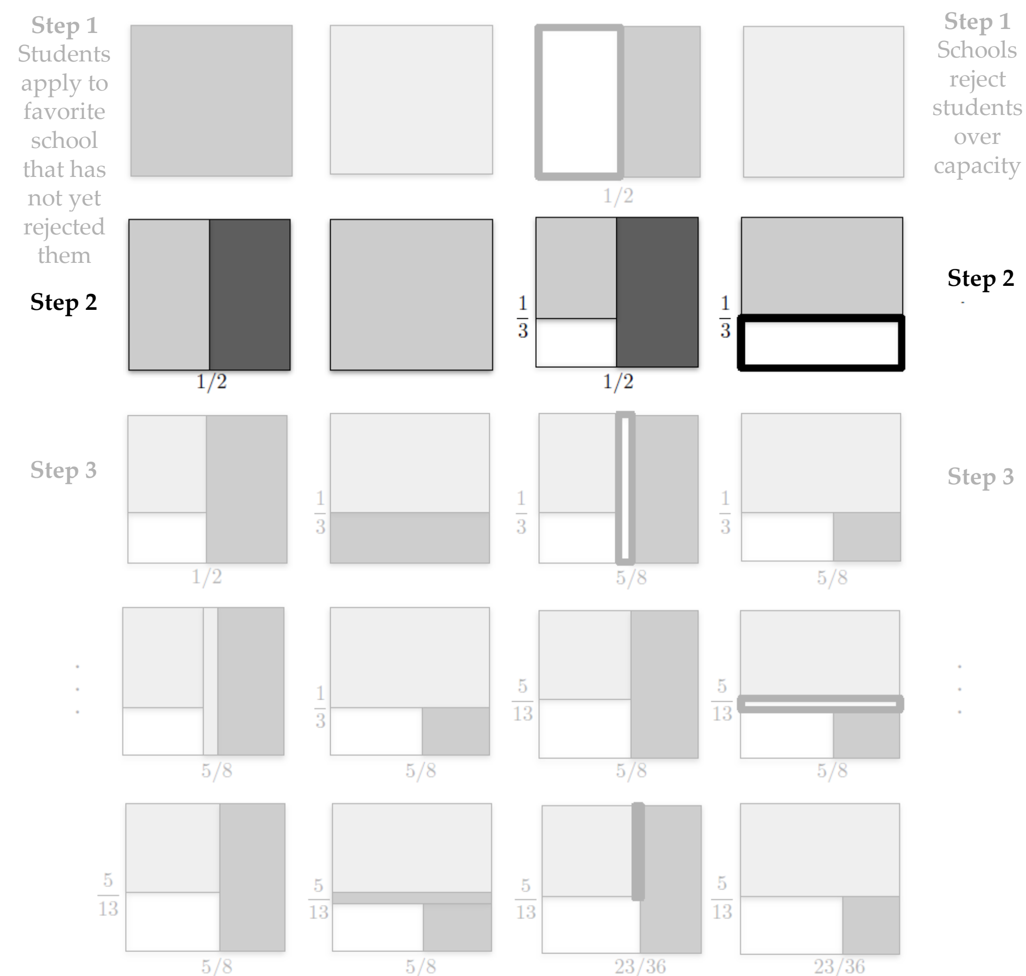

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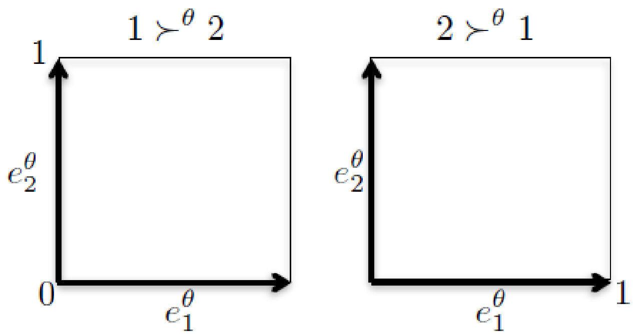

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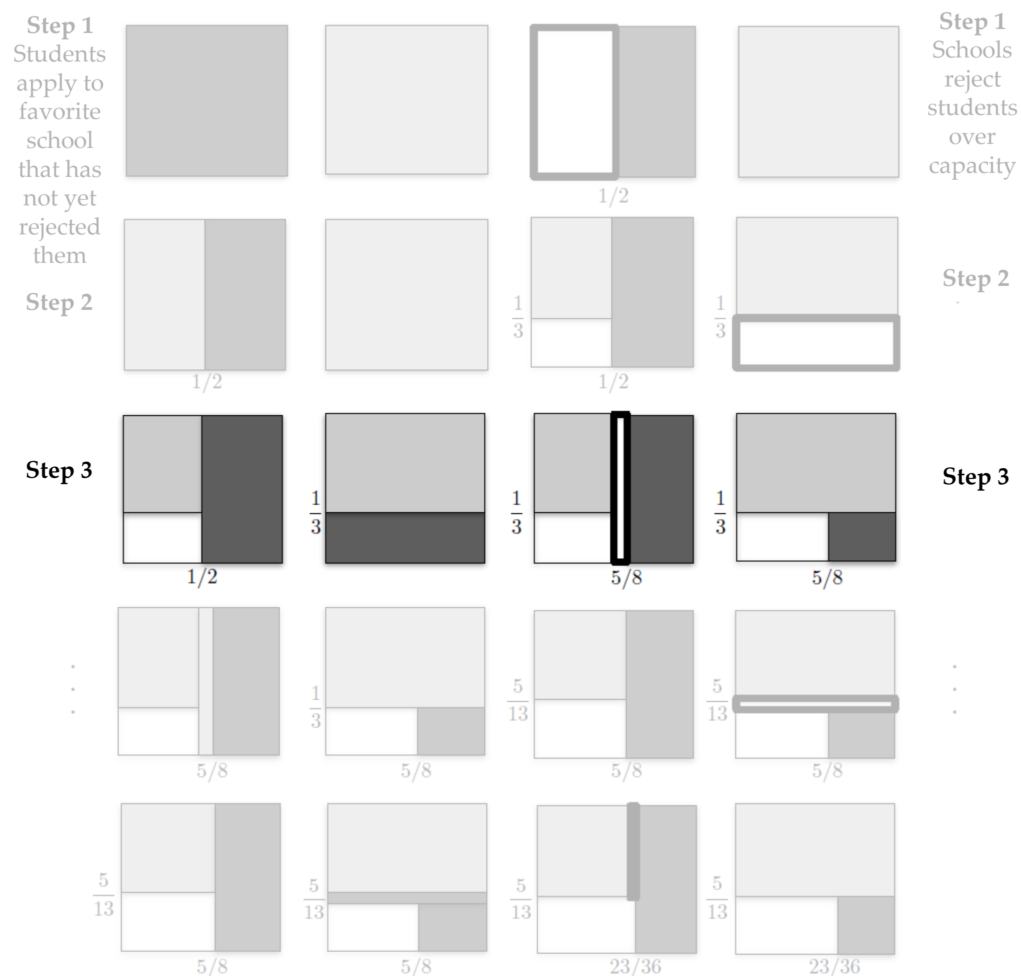

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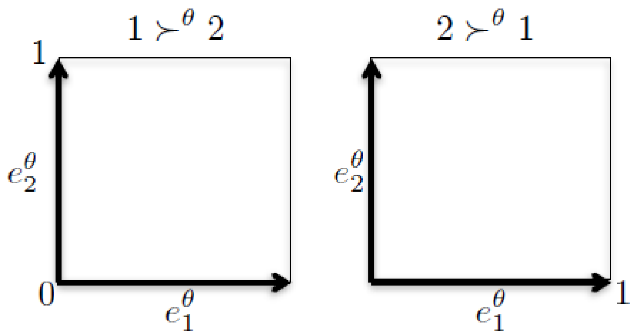

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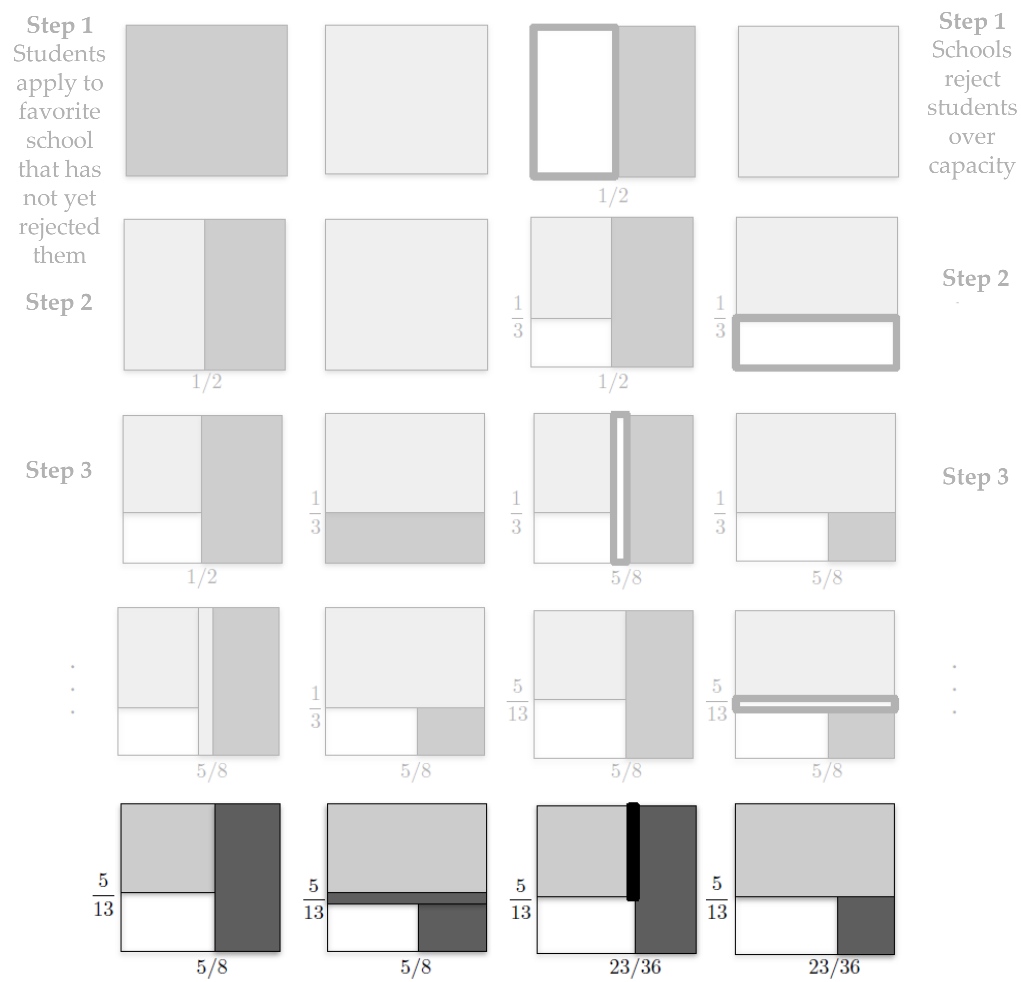
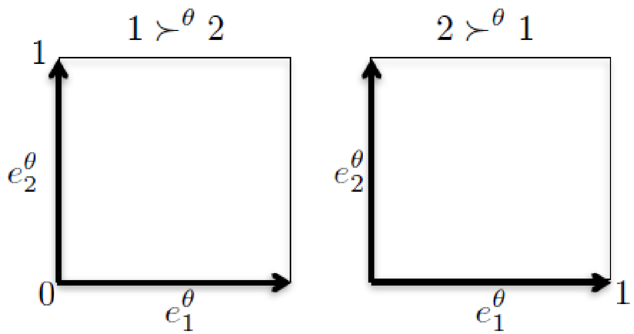


FIGURE 1. The Gale and Shapley algorithm

Intuition



$\frac{1}{2}$ students prefer 1, $\frac{1}{2}$ prefer 2

$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

Proof. Process converges to market-clearing cutoffs:

- Convergence: Monotonicity
- $D_c(P^*) \leq q_c$: True at every step
- If $P_c^* > 0$ then $D_c(P^*) = q_c$: School rejected some students, so school is full

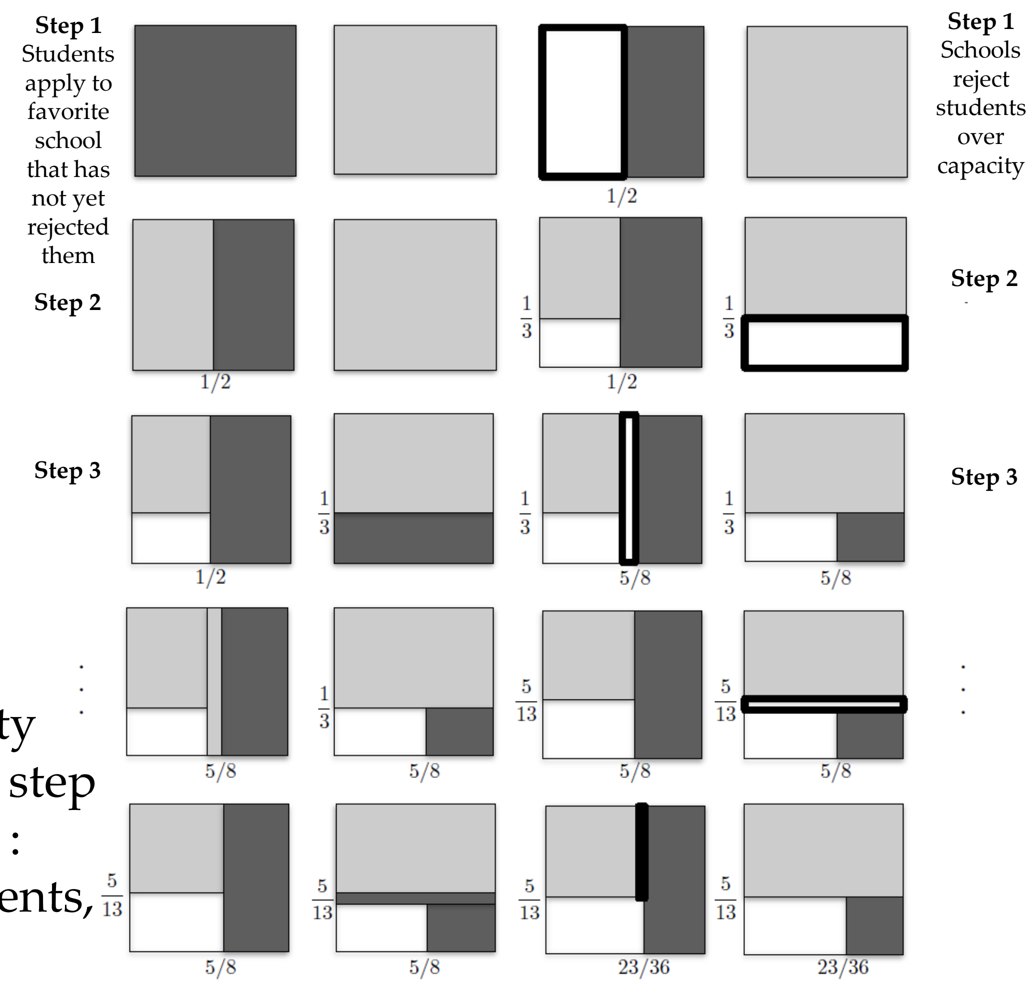


FIGURE 1. The Gale and Shapley algorithm

Standard Results Still Hold

- **Existence.** The deferred acceptance algorithm converges to a stable matching
- **Lattice.** The set of stable matchings is a lattice
  - The set of market clearing cutoffs is a lattice
- **Rural Hospitals Theorem (Roth 1986).**
  - The measure of students matched to each school is the same in all stable matchings
  - The set of students matched to each under-demanded school is the same in all stable matchings

Cutoffs and Inference

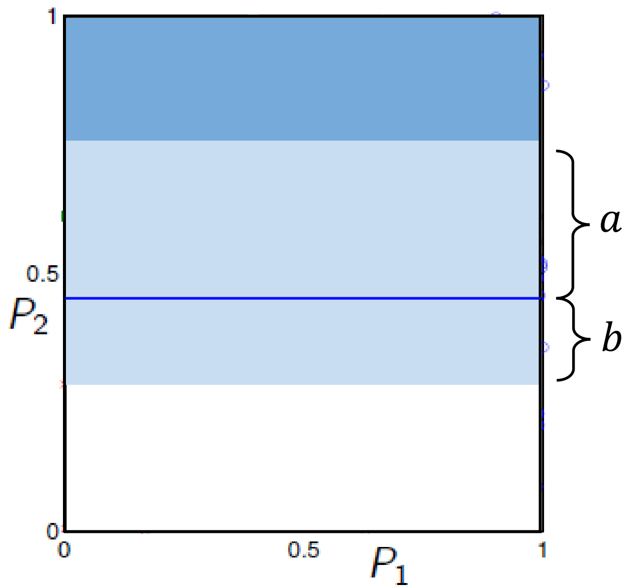
- In school choice, schools often have weak priorities
- Strict priorities are generated using random tiebreaking:
  - Each student gets a lottery number $U[0,1]$ at each school
  - **Single tie-breaking:** Same lottery used at every school
  - **Multiple tie-breaking:** New lottery drawn at every school
- Students are ordered first by priority, then by lottery number.
- **Main idea:** The lottery induces a quasi-experiment!
  (see, e.g., Abdulkadiroglu et al. 2017, "Research Design Meets Market Design" – i.e. Lecture 16 on Thursday 8/18)

Cutoffs and Inference

Example

- All students prefer $1 \succ 2$
- School 1 has strict priorities
- School 2 gives priority to siblings, then neighborhood, and breaks remaining ties randomly

Preferences: $1 \succ 2$



- Students in the neighborhood have the *option* to go to school 2 w.p. $\frac{a}{a+b}$

Cutoffs and Inference

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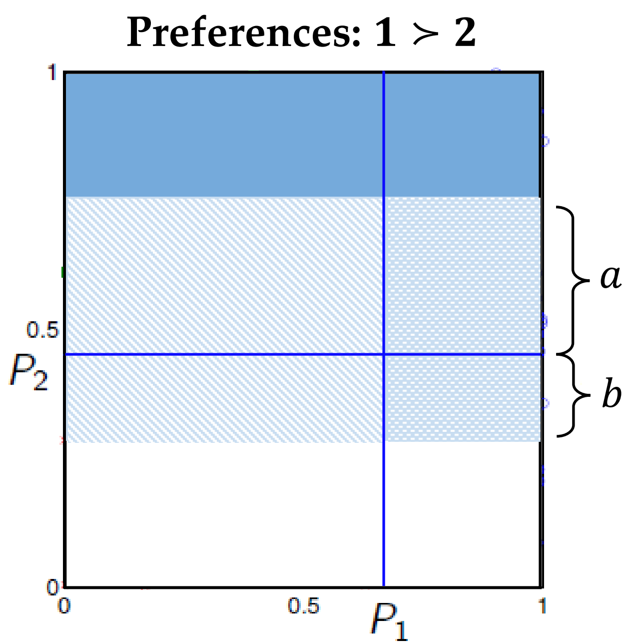


- Students in the neighborhood have the *option* to go to school 2 w.p. $\frac{a}{a+b}$
- Since students all prefer $1 \succ 2$, students in the neighborhood will *take the option* only if they don't also have school 1 as an option
- We can use the cutoffs to define separate propensity scores for students in &

DA via Cutoffs (Azevedo Leshno)

- Continuum Model
- Cutoff Characterization
- **Uniqueness and Convergence**

Uniqueness Theorem

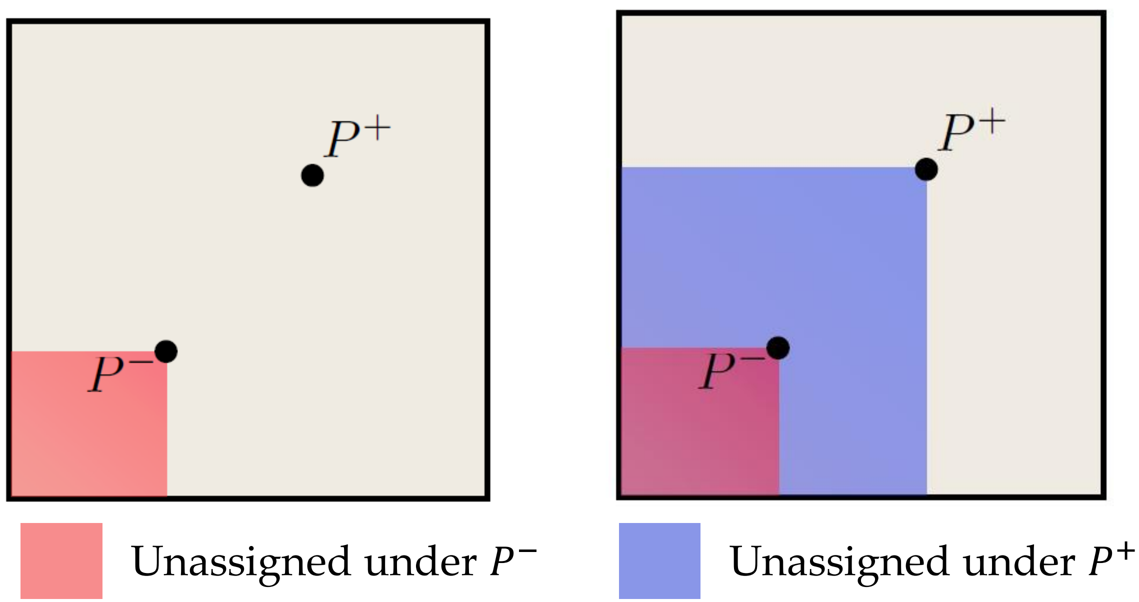
Theorem. Let $E = [\eta, q]$ be a continuum economy

I. If η has full support then there is a unique stable matching

II. If $D(\cdot)$ is continuously differentiable then for almost any q there is a unique stable matching.

Proof by picture:

I. Full support



Continuity and Convergence

Def. A **discrete economy** is $F = [\eta, q]$ where η is composed of a finite number of atoms.

Lemma. For a discrete economy $\mathcal{M}$ and $\mathcal{P}$ take stable matchings into market clearing cutoffs, and $\mathcal{MP}$ is the identity.

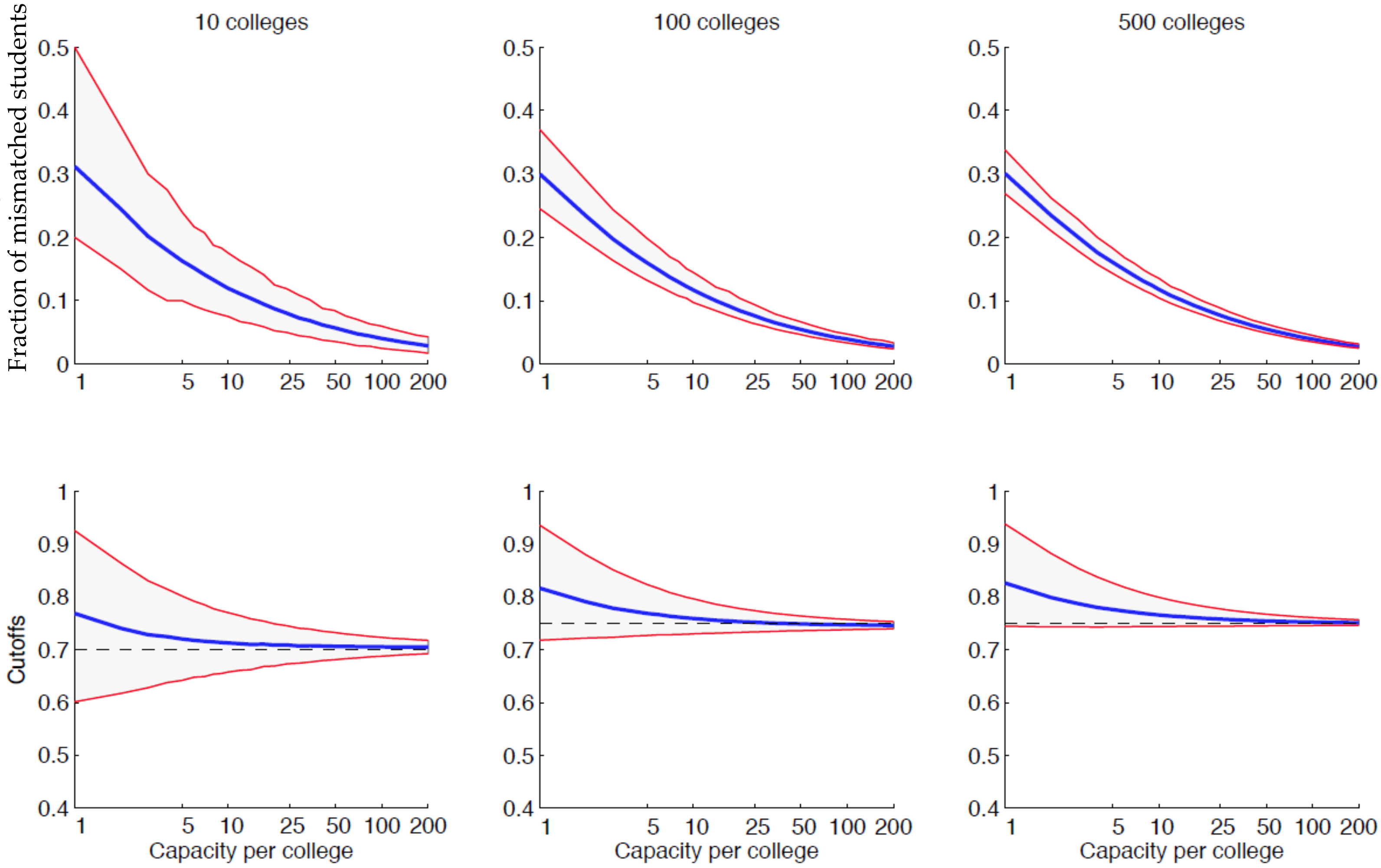
i.e. Stable matchings in discrete economies can also be characterized using market-clearing cutoffs:

1. $\mathcal{M}$: Find the marginal students assigned to each school c
2. $\mathcal{P}$: Let students more prioritized at c than marginal students have the *option* of attending c.

Theorem *(Informal)*. If discrete economies F^k converge to an economy E with a unique stable matching μ, then stable matchings and cutoffs converge to μ and $\mathcal{P}(\mu)$. Moreover, the stable matching correspondence is continuous at E

Implication: *Results for continuum economies should hold for large finite economies, and randomly sampled economies*

Simulations



Summary: DA via Cutoffs

- New model of matching allowing for complex preferences but simple derivation
- Cutoff characterization of DA outcome
  - Can be solved using supply/demand equations
  - Enables inference
- The continuum model approximates large markets

BREAK

In small groups:

- Share:
  - Name
  - Affiliation
  - Where you're from
- Find:
  - A fun fact for each person that is unique to them in the group

The Cutoff Structure of Top Trading Cycles in School Choice

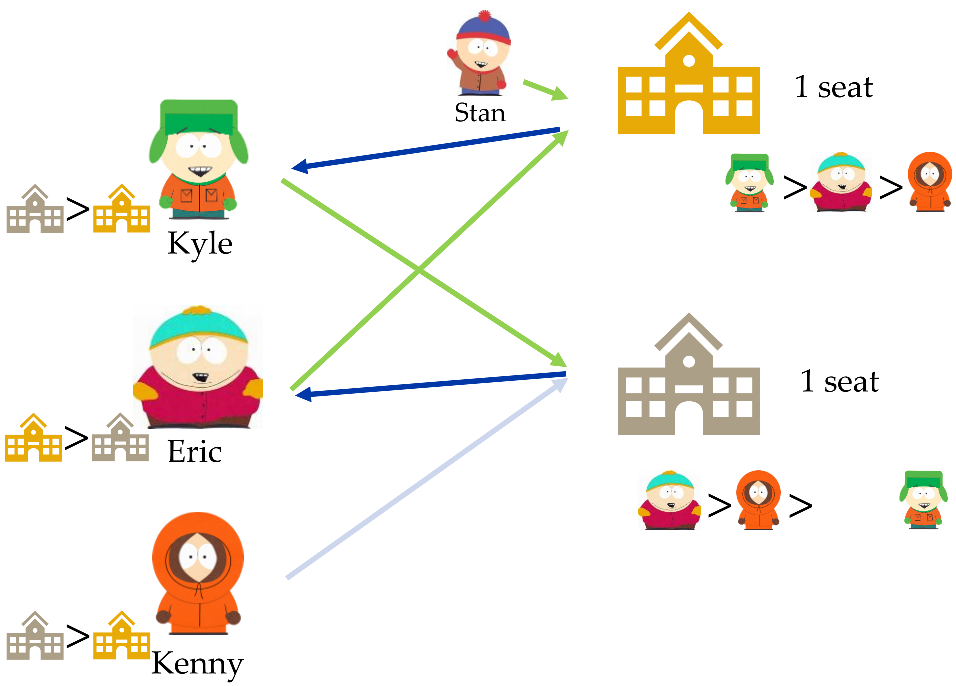
Jacob Leshno & Irene Lo

Main Takeaways

The Top Trading Cycles outcome can be characterized by **cutoffs**.
These cutoffs solve **trade balance equations**.

Top Trading Cycles

[Shapley & Scarf '74]

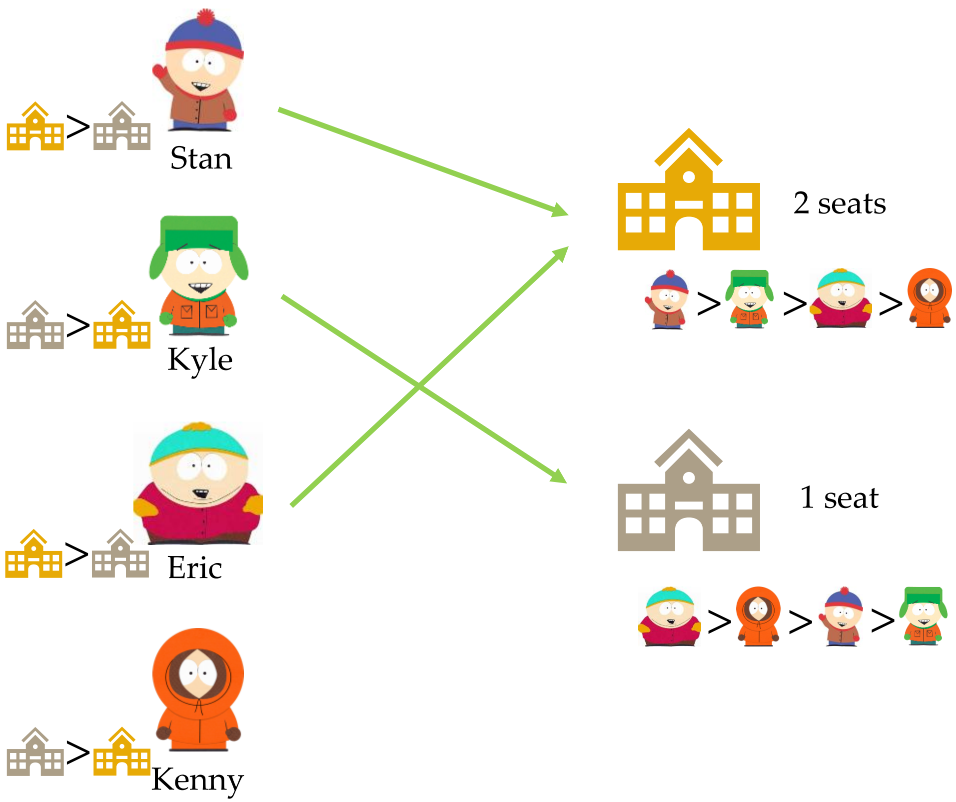



```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
    all remaining students
    point to favorite
    remaining school
    all remaining schools
    point to favorite
    remaining students
    select a cycle, assign
    students in cycle to
    school they point to
```

Top Trading Cycles

[Shapley & Scarf '74]



Top Trading Cycles

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    all remaining students
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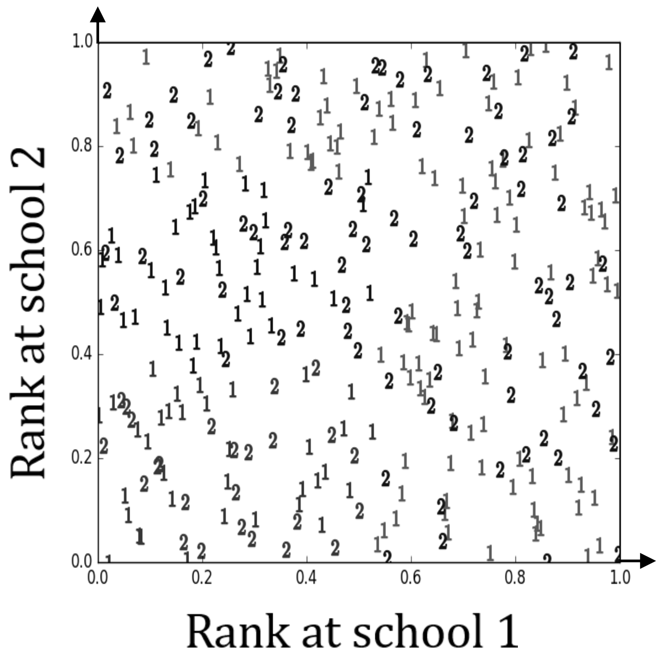
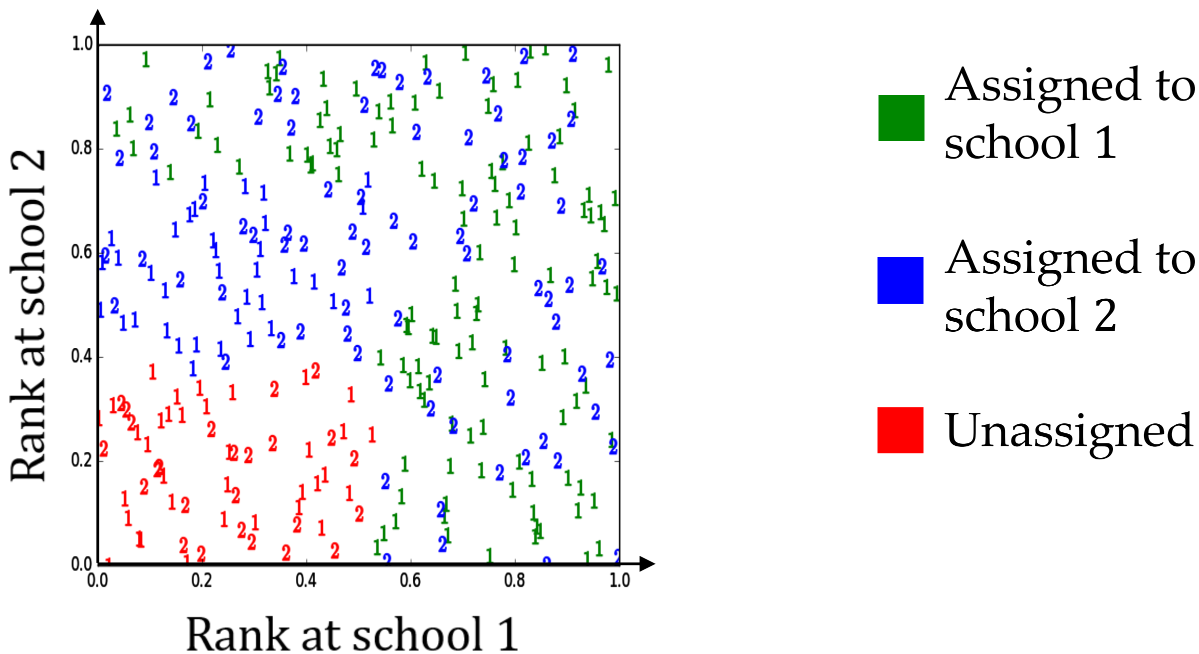
Example



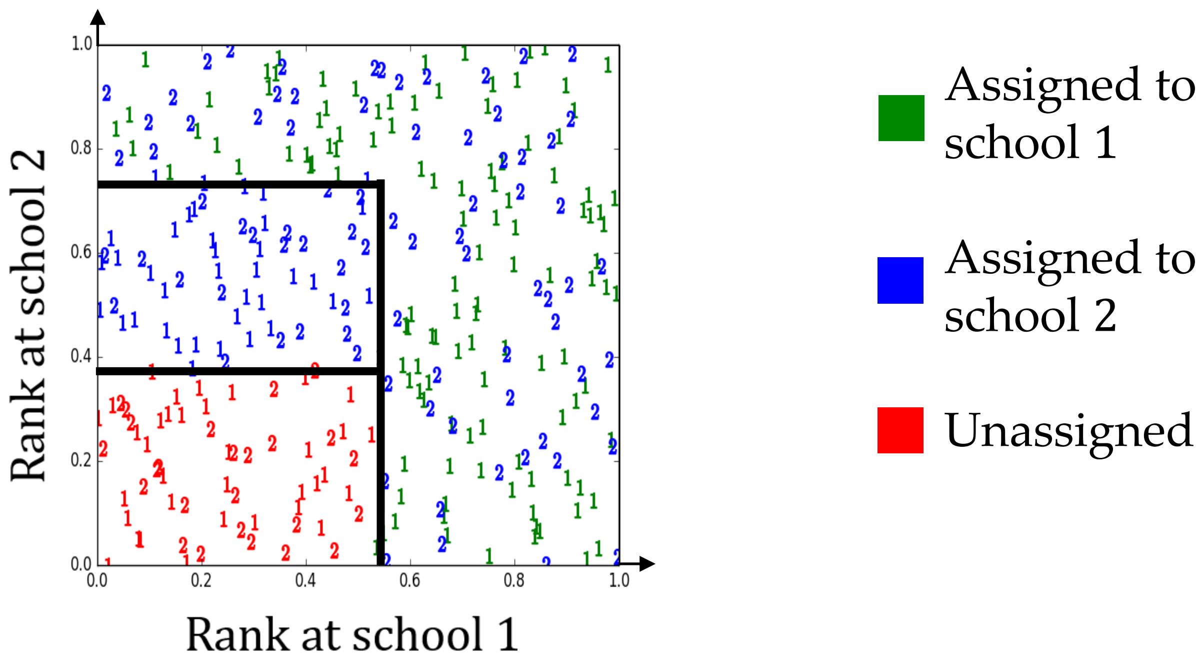
- 2/3 students prefer school 1
- Students marked with preferred school
- Ranks are uniformly i.i.d. across schools
- $q_1 = q_2$

Q. What is the outcome of TTC on this economy?

Example: TTC Assignment



Example: TTC Assignment



The TTC assignment can (like DA) also be described using cutoffs!

Q. How are TTC cutoffs different to DA cutoffs?

TTC via Cutoffs (Leshno Lo)

- **Cutoff Characterization**
- Continuum Model
- Uniqueness, Convergence, and Welfare

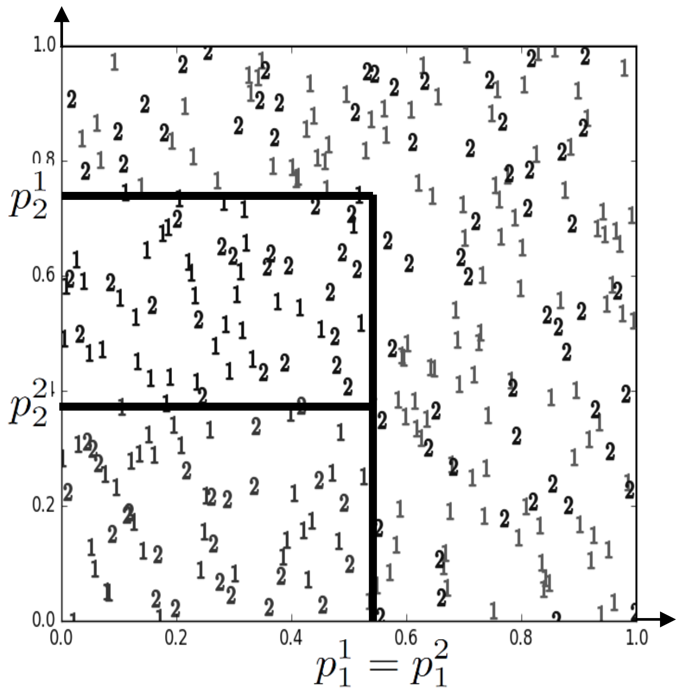
TTC Assignment via Cutoffs

Theorem.
The TTC assignment is given by cutoffs $\{p_b^c\}$ where:

- Each student θ has a budget set
$$B(p, \theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}$$
- Students assigned to their favorite school in their budget set
$$\mu(\theta) = \max_{\succ^\theta}(B(p, \theta))$$

Interpretation. p_b^c is the minimal priority at school b that allows trading a seat at school b for a seat at school c

Example: Assignment via Cutoffs



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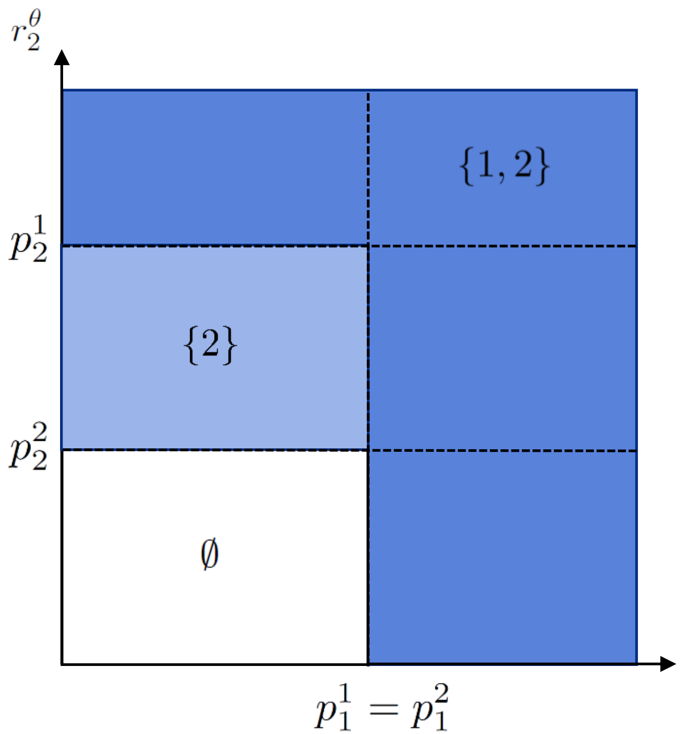
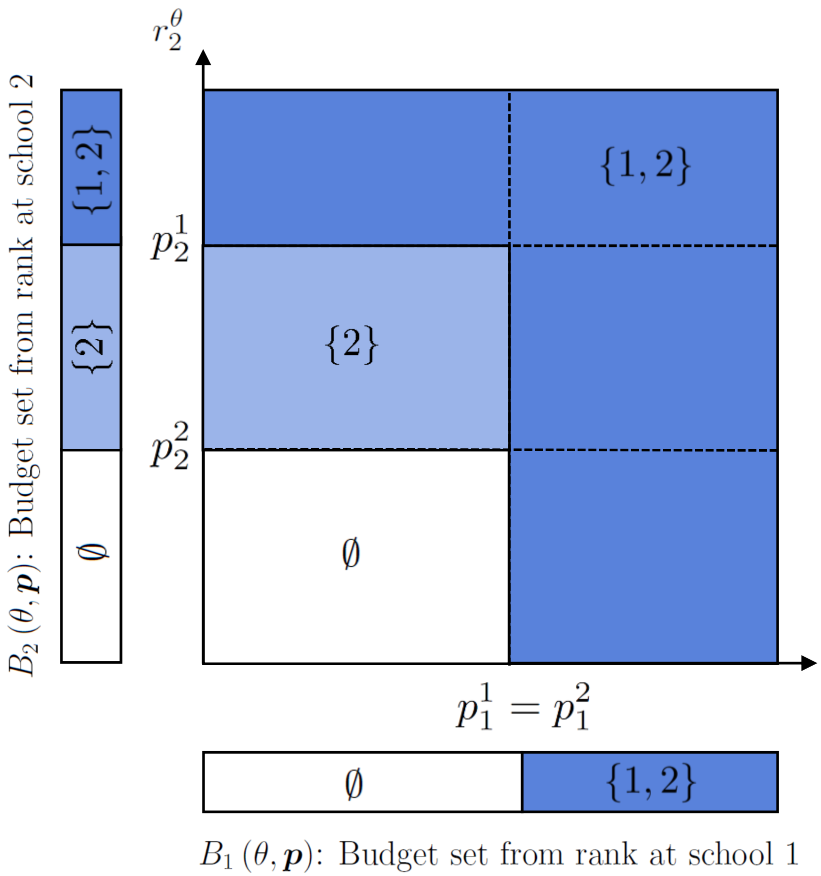
Example: Assignment via Cutoffs



$$B(p,\theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}$$

- Budget set {1,2}
- Budget set {2}

Example: Assignment via Cutoffs



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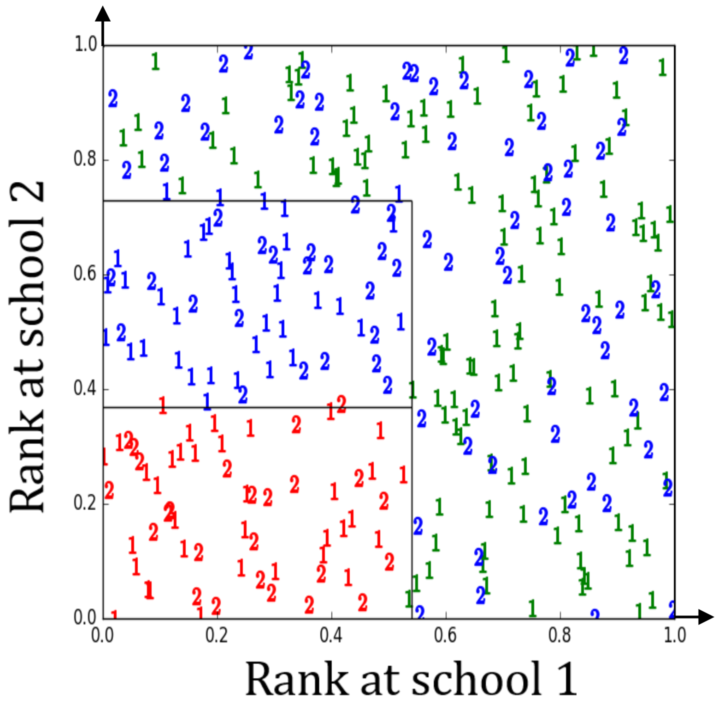


$$\mu(\theta) = \max_{\succ^\theta}(B(p,\theta))$$

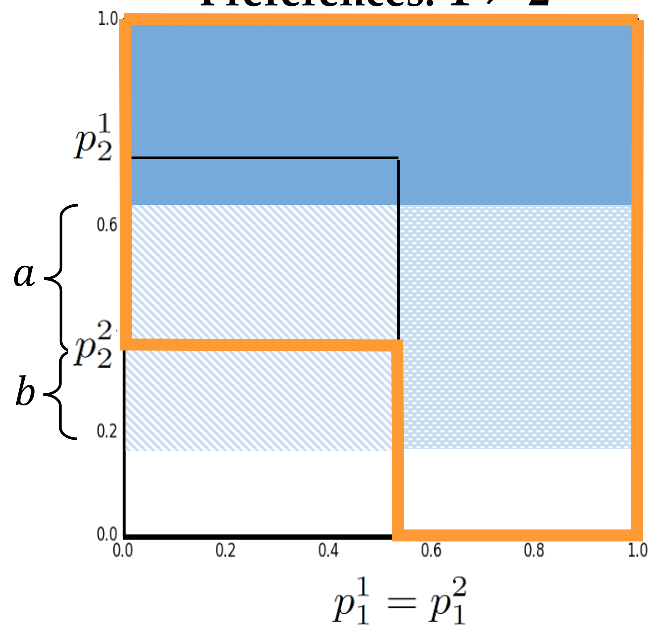
- Assigned to school 1
- Assigned to school 2
- Unassigned

Cutoffs and Inference

Example

- All students prefer $1 \succ 2$
- School 1 has strict priorities
- School 2 gives priority to siblings, then neighborhood, and breaks remaining ties randomly

Preferences: $1 \succ 2$



- Students in $\square$ have the *option* to go to school 2
- Students in the neighborhood have the *option* to go to school 2

$$w.p. \begin{cases} \dfrac{a}{a+b} & if\ r^1 < p_1^2 \\ 1 & if\ r^1 \geq p_1^2 \end{cases}$$

- As with DA, can use the cutoffs to define propensity scores

TTC via Cutoffs (Leshno Lo)

- Cutoff Characterization
- **Continuum Model**
- Uniqueness, Convergence, and Welfare

Continuum Model

- Finite set of schools $c \in C = \{1, \dots, n\}$
  - School c can admit a mass q_c of students

- Measure η specifying a distribution of a continuous mass of students
  - A student $\theta \in \Theta$ is given by $\theta = \left(\succ^\theta, r^\theta\right)$
  - Student θ has preferences $\succ^\theta$ over schools
  - $r_c^\theta \in [0,1]$ is the student's rank at school c (percentile in c's priority list)

- TTC produces an allocation $\mu: \Theta \to C \cup \{\phi\}$

Continuum Model

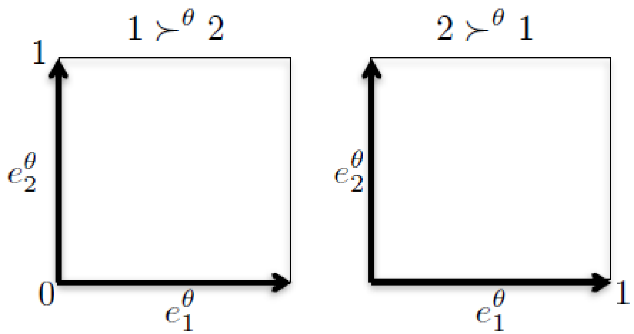
Assumption:

- (*Lipschitz density*) η has a density ν,
$$\eta(A) = \int_A \nu(\theta) d\theta \quad \forall A \subset \Theta,$$
where ν is Lipschitz continuous except on a finite grid, and bounded above

This implies:

- (*Strict preferences*) Colleges' indifference curves have measure 0, that is,
$$\eta\left(\left\{\theta \in \Theta : r_c^\theta = x\right\}\right) = 0 \quad \forall x, c$$

TTC in the Continuum



$\frac{1}{2}$ students prefer 1, $\frac{1}{2}$ prefer 2

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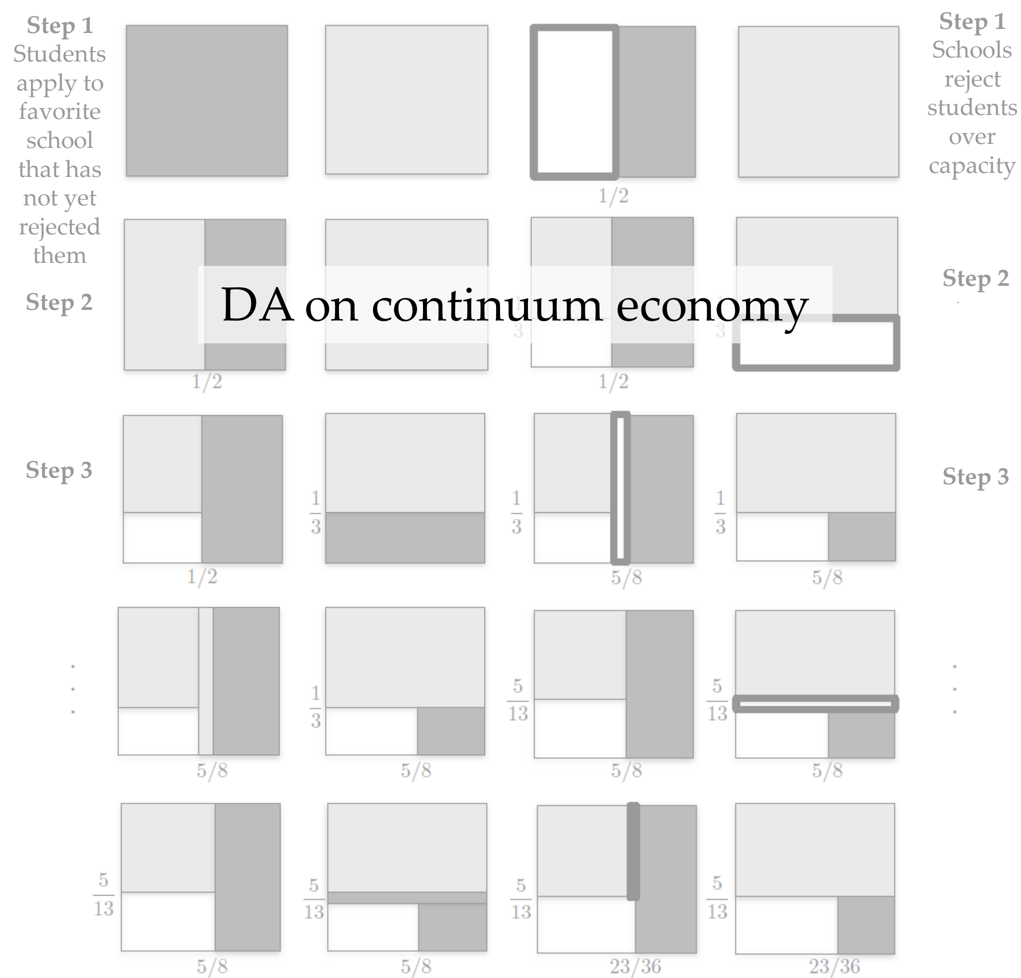
Q. How does TTC run on this continuum economy?



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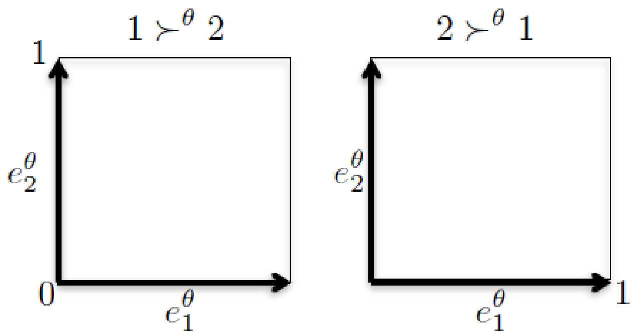

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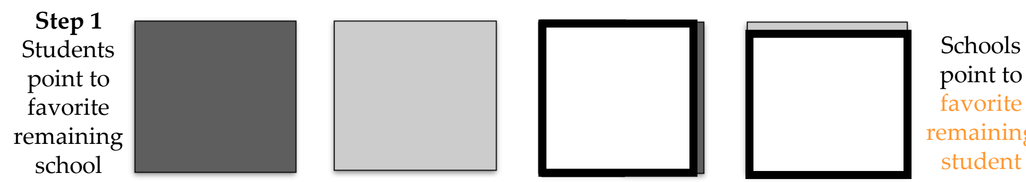
Q. How does TTC run on this continuum economy?



Problem. How to point to top student in a continuum?

Challenges

- Multiplicity in pointing
  - In continuum, colleges give priority (‘point’) to a **set** of students
  - In the discrete setting, when colleges ‘point’ to multiple students, outcome can depend on the cycle selection rule

- How to define the continuum TTC algorithm?
  - Continuous time instead of discrete steps
  - How do you select a continuum cycle?
  - How do you provide measure 0 capacity updates?

- How to track the progression of the algorithm?

Defining TTC in the Continuum Model

- 'Top' trading cycles → Measure zero cycles
  - Schools can only point to measure zero sets
  - Limit of infinitesimally small trading cycles

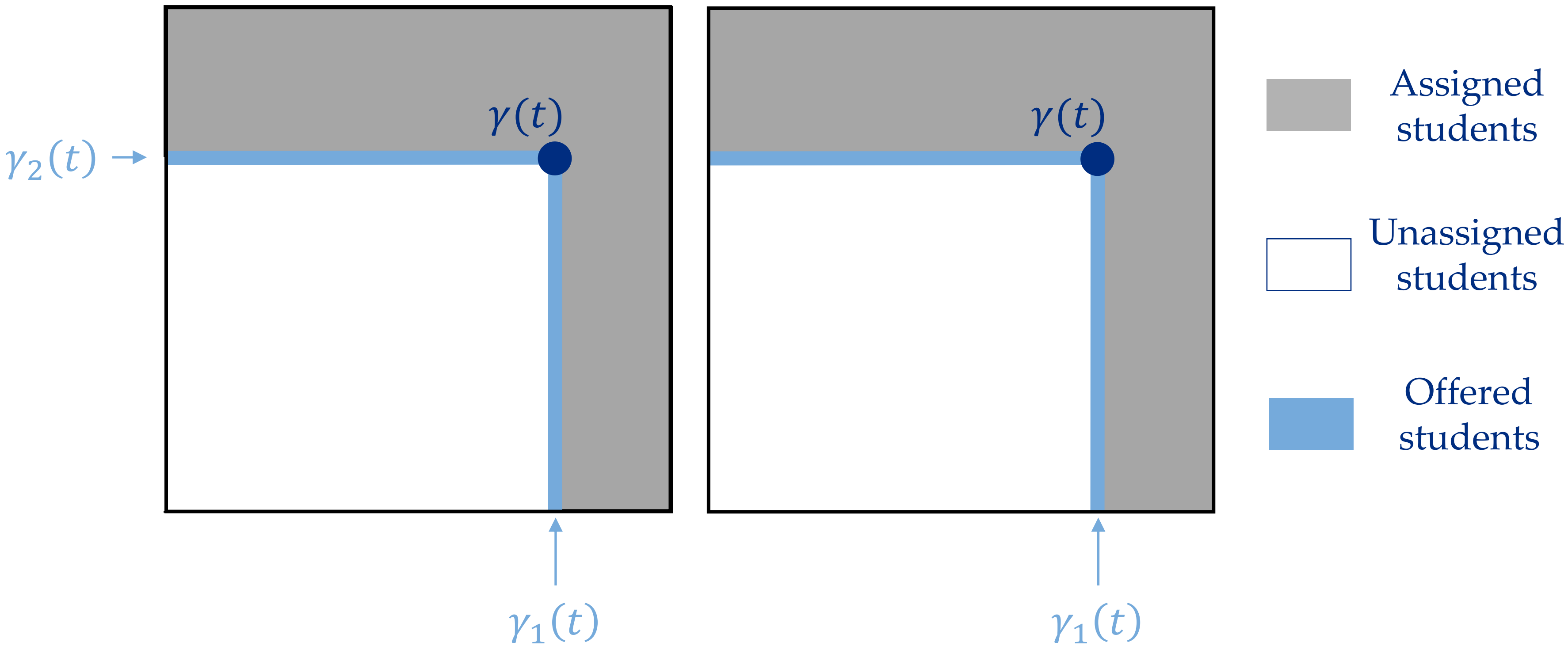
- Track progression of the algorithm using counter $\gamma_c(t)$

  $\gamma_c(t)$: Rank of students pointed to by school c at time t

- Aggregated cycles satisfy trade balance
  - The number of students offered a seat by a school is the same as the number of students assigned a seat at the school

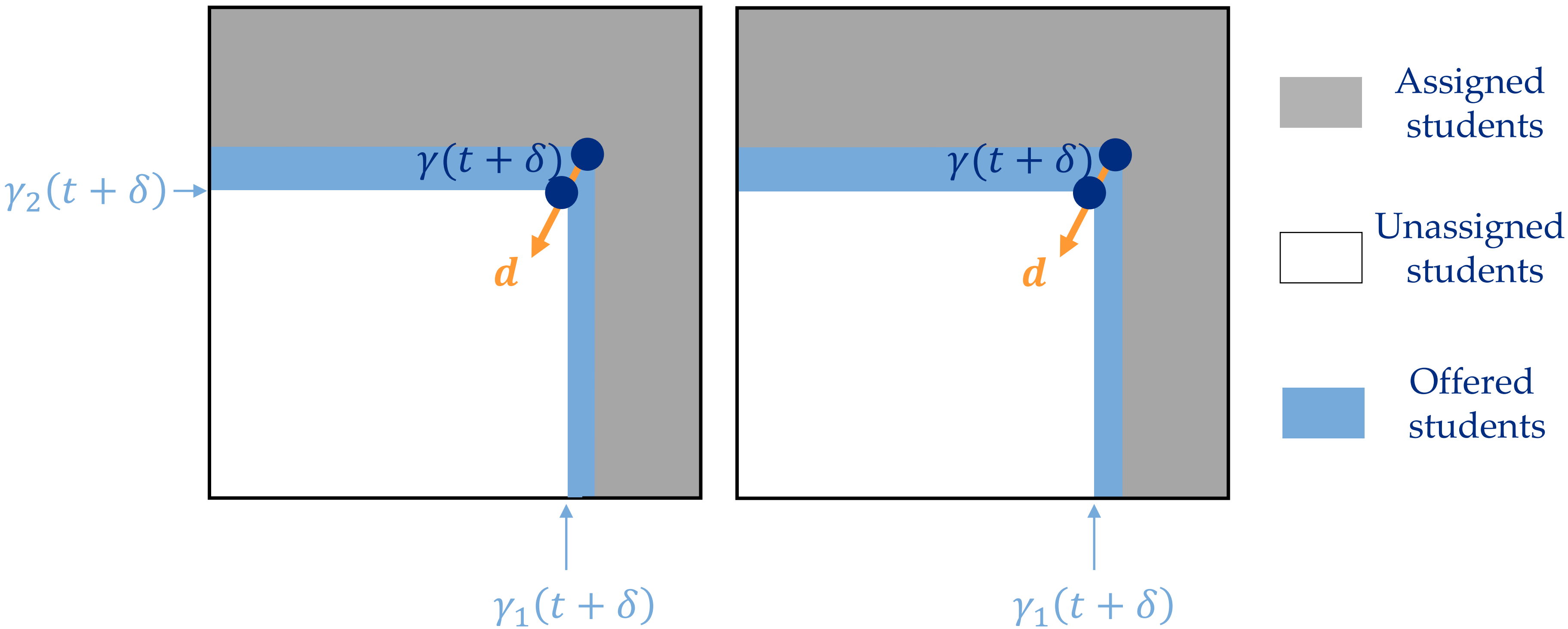
Trade Balance - Visualization

$\gamma_c(t)$: Rank of students pointed to by school c at time t



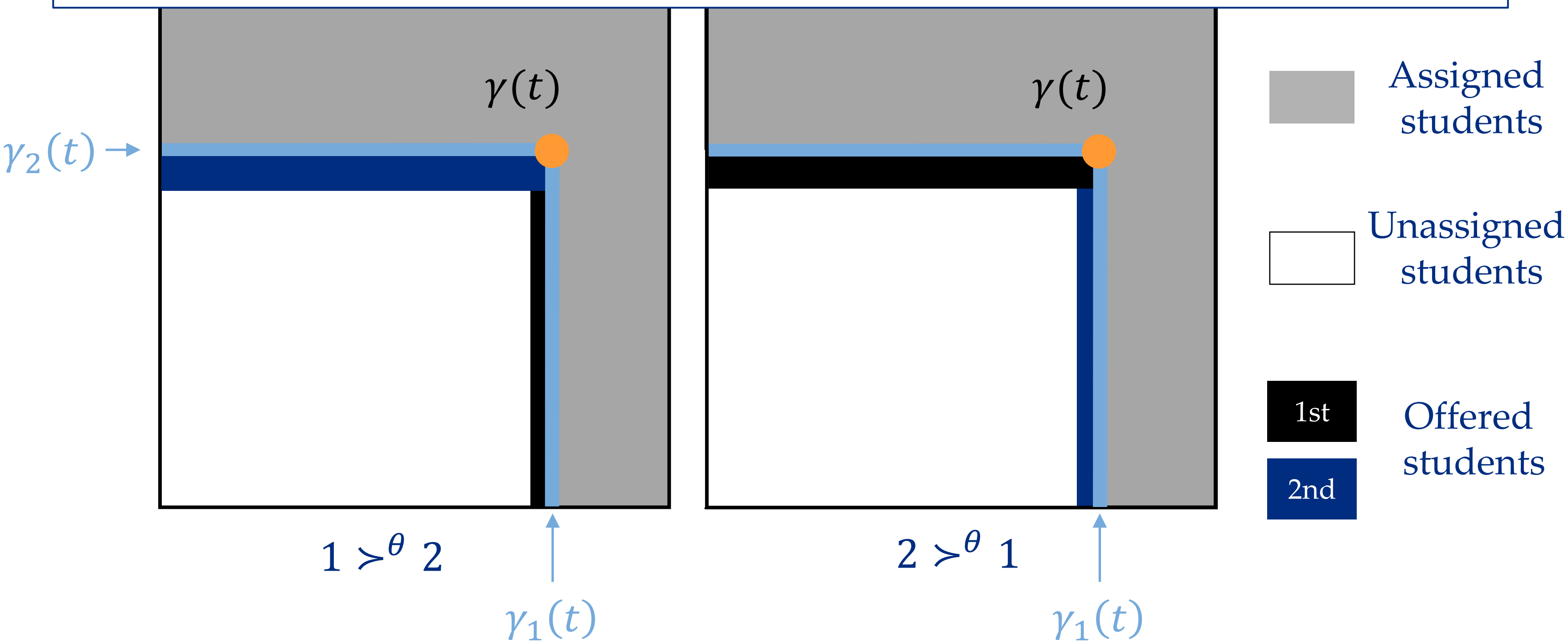
Trade Balance - Visualization

$\gamma_c(t+\delta)$: Rank of students pointed to by school c at time $t+\delta$



Trade Balance - Visualization

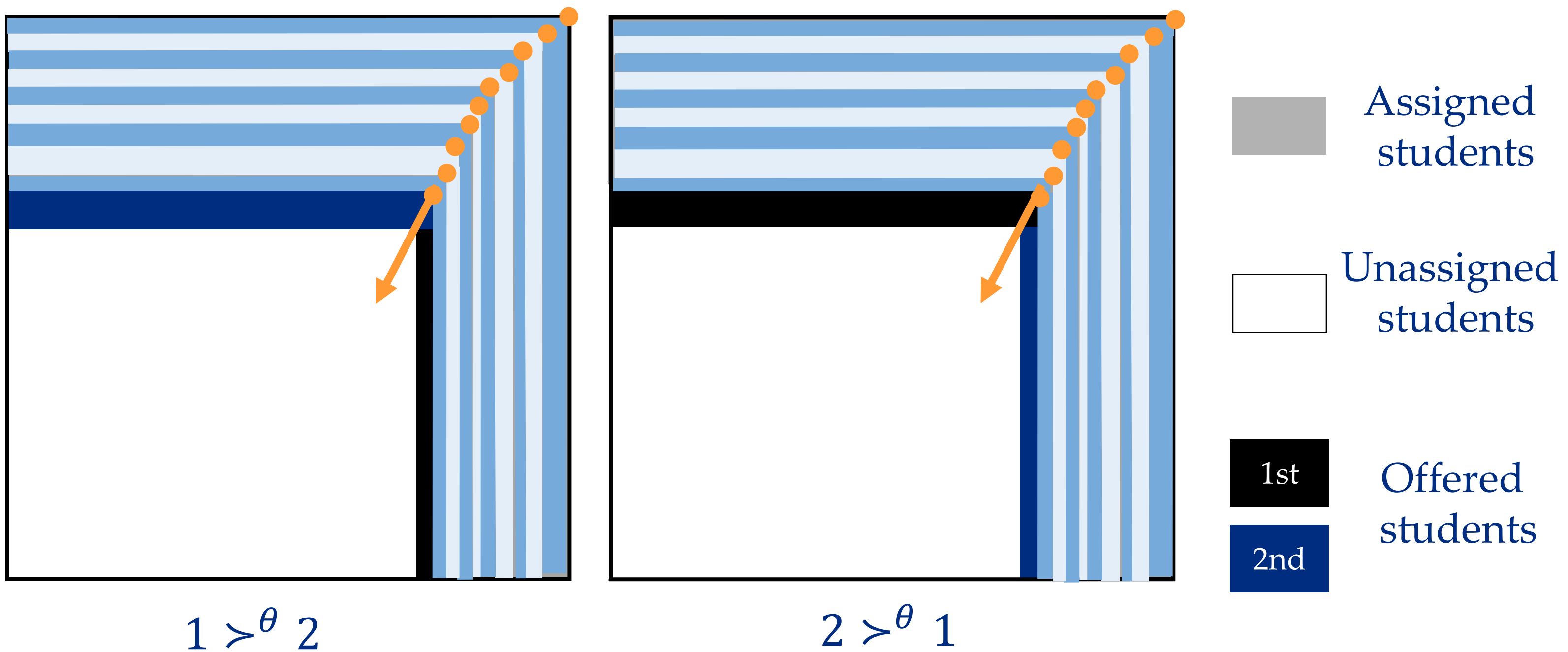
Trade balance from time t to $t + \delta$: *Offered a seat by 2* = *Assigned a seat at 2*



Trade Balance - Visualization

$\gamma_c(t)$: Rank of students pointed to by school c at time t

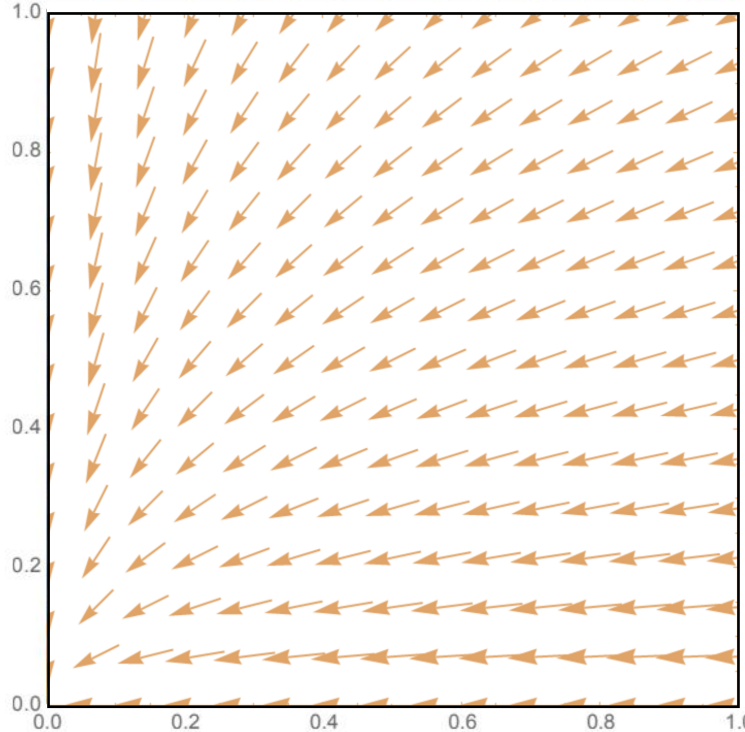
Trade balance: $\gamma_2'(t)(\textit{density of } 1 \succ 2) = \gamma_1'(t)(\textit{density of } 2 \succ 1)$



Trade Balance - Visualization

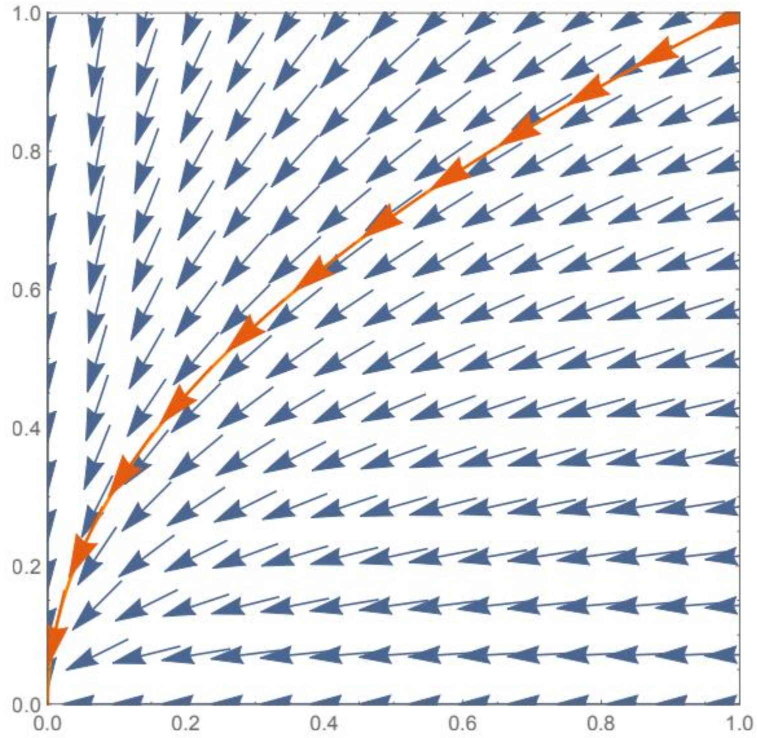
Trade balance: # Offered seat by 2 = # Assigned seat at 2

$\Rightarrow \gamma_2'(t)(\textit{density of } 1 \succ 2) = \gamma_1'(t)(\textit{density of } 2 \succ 1)$



TTC in the Continuum

The TTC path γ starts at $\gamma(0) = 1$ and follows $\gamma'(t) = d(\gamma(t))$, where d satisfies trade balance.



Trade Balance and Capacity Equations

Trade Balance Equation. For all times t, $\gamma'(t) = d(\gamma(t))$, where $\gamma_c(t)$ is the rank of students pointed to by school c at time t, and $d(x)$ balances the relative marginal densities.

- Necessary condition for aggregate trade
- Intuition: $\#\begin{Bmatrix}\text{Assigned at } c \\ \text{by time } t\end{Bmatrix} = \#\begin{Bmatrix}\text{Offered seat by } c \\ \text{by time } t\end{Bmatrix}$

Capacity Equation *(Informal)*.
Students stop being assigned to school c at

$$t^{(c)} = min\left\{t: \#\begin{Bmatrix}\text{Assigned at } c \\ \text{by time } t\end{Bmatrix} \geq q_c\right\}.$$

Calculating TTC Cutoffs

Theorem.

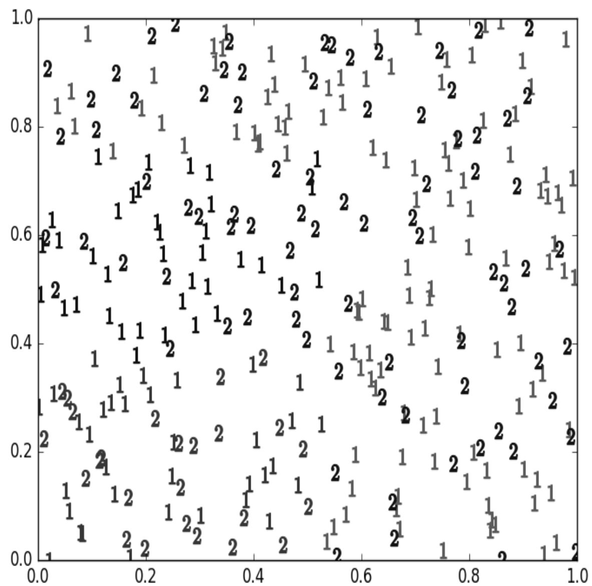
- (Existence) There exist a TTC path $\gamma(\cdot)$ and stopping times $\left\{t^{(c)}\right\}_{c\in C}$ that satisfy the trade balance and capacity equations.
- (Cutoffs) The n^2 TTC cutoffs $\{p_b^c\}$ are given by
$$p_b^c = \gamma_b\left(t^{(c)}\right).$$
- (Definition) The continuum TTC assignment is given by
$$\mu_{cTTC}(\theta) = \max_{\succ^\theta}\{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}.$$
- (Uniqueness) Any $\gamma(\cdot)$ and $\left\{t^{(c)}\right\}_{c\in C}$ that satisfy trade balance and capacity equations yield the same assignment μ_{cTTC}.

Calculating TTC Cutoffs

Recipe for calculating the TTC assignment

1. Compute 'trade balance' marginal densities $d(\cdot)$
2. Calculate the TTC path γ using trade balance equations
3. Calculate the TTC cutoffs p_b^c using capacity equations

Example



- Trade balance

$$d(x) = -\left[\frac{x_1}{x_1 + 2x_2} \quad \frac{2x_2}{x_1 + 2x_2}\right]$$

- TTC path

$$\gamma(t) = \left(t^{1/3}, t^{2/3}\right)$$

- TTC cutoffs

$$p^1 = \left((1 - 3q_1)^{1/3}, \left((1 - 3q_1)^{2/3}\right)\right)$$

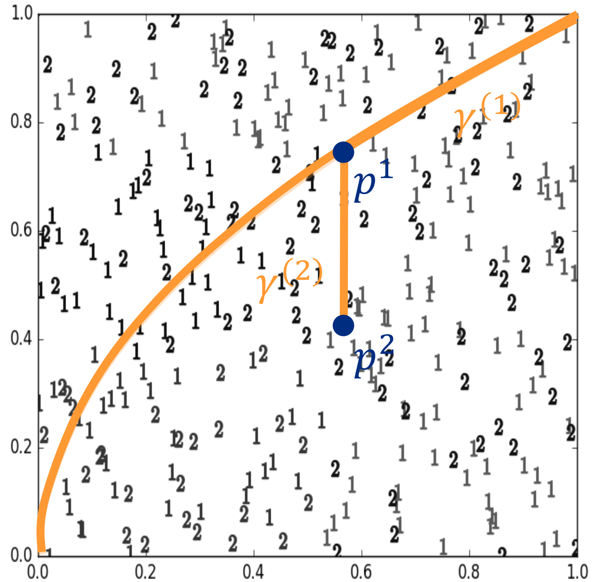
2/3 of students prefer school 1*, ranks are uniformly i.i.d. across schools,* $q_1 = q_2$

Calculating TTC Cutoffs

Recipe for calculating the TTC assignment

1. Compute 'trade balance' marginal densities $d(\cdot)$
2. Calculate the TTC path γ → *indicates the run of TTC*
3. Calculate the TTC cutoffs p_b^c → *when schools reach capacity*

Example



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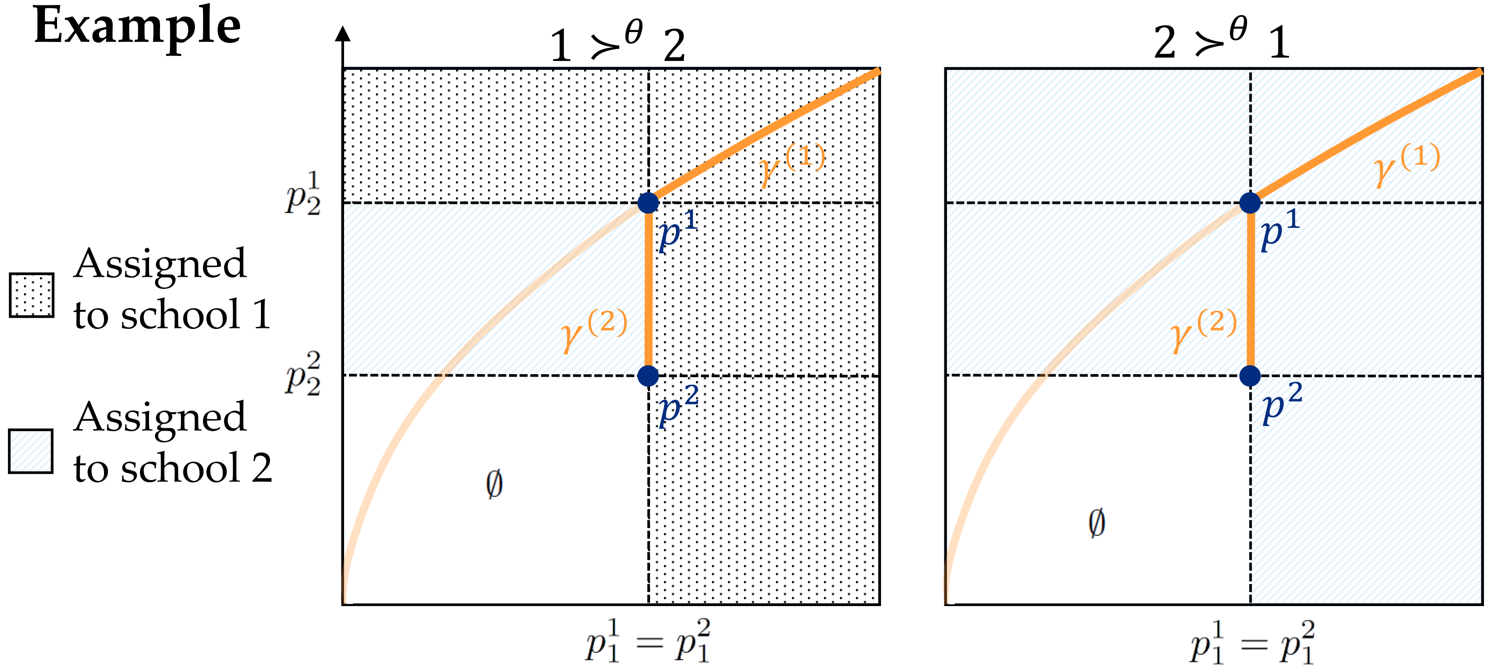

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TTC via Cutoffs (Leshno Lo)

- Cutoff Characterization
- Continuum Model
- **Uniqueness, Convergence, and Welfare**

Continuum TTC Generalizes Discrete TTC

- **Trade Balance Uniquely Determines the Allocation**
  - Differential equation and TTC path may not be unique, but all give the same allocation
- **Consistent with Discrete TTC**
  - Can naturally embed discrete TTC in the continuum model
  - The continuum embedding gives the same allocation as TTC in the discrete model
- **Convergence**
  - If two distributions of students have full support and total variation distance ε, then the TTC allocations differ on a set of students of measure $O(\varepsilon|C|^2)$.

Welfare via Cutoffs

- Cutoffs determine **budget set**
  - $B(p,\theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}$

- Budget sets determine assignment and **welfare**
  - **Assignment.** Favorite school in budget set
$$\mu(\theta) = \max_{\succ^\theta}(B(p,\theta)) = \operatorname*{argmax}_{c \in B(p,\theta)}(u^\theta(c))$$
  - **Welfare.** Highest utility school in budget set
$$W(\theta) = \max_{c \in B(p,\theta)}(u^\theta(c))$$

Comparing TTC & DA: Welfare

2 schools, $q_1 = q_2 = 0.4$, distance-based utilities:

- Student s's preferences given by:

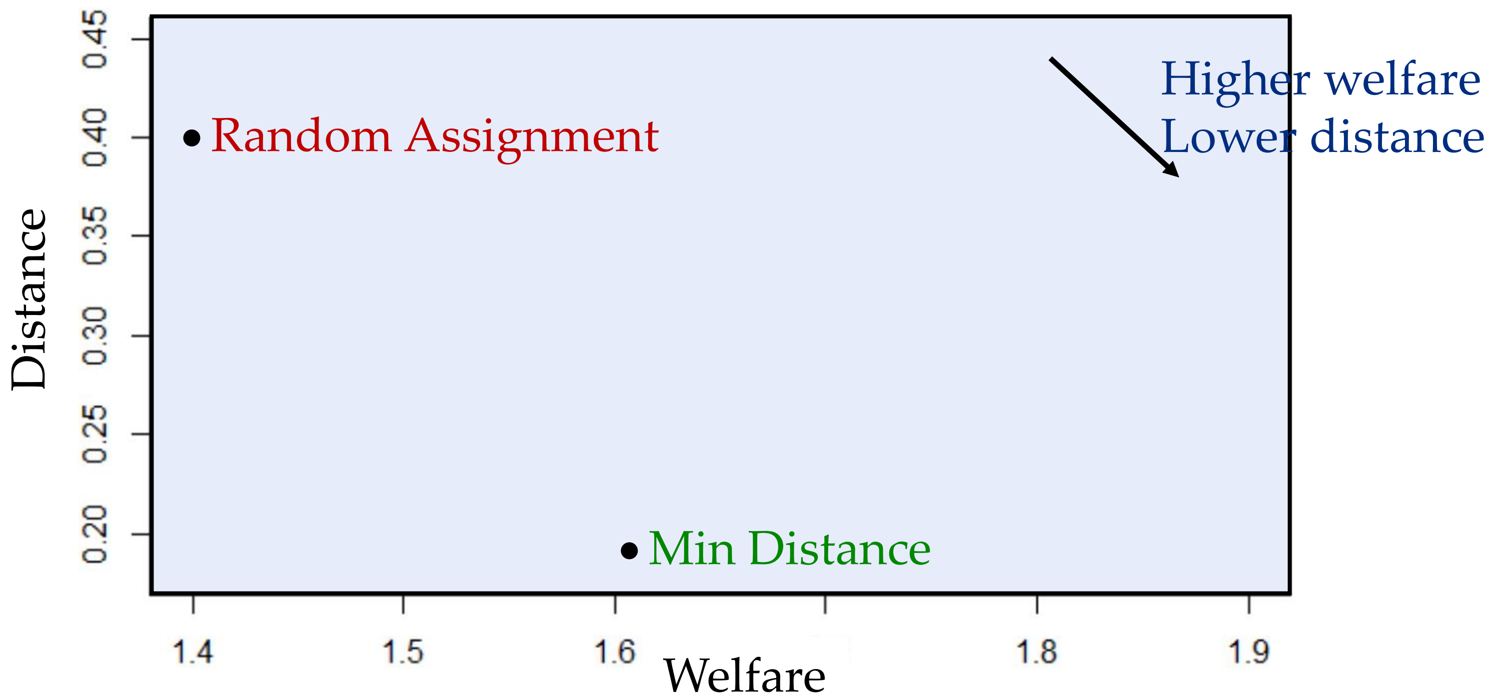
$$u_s(c) = 2 - \underbrace{d_{sc}}_{\text{distance}} + \underbrace{\varepsilon_{sc}}_{\text{idiosyncratic match value}}$$

  - $u_s(\phi) = 0$, $d_{sc} \sim U[0,1]$, $\varepsilon_{s1} \sim U[-1,2]$, $\varepsilon_{s2} \sim 0$.
  - 2/3 of students prefer school 1

- Welfare under TTC vs DA?
  - Uncorrelated priorities?
  - Distance-based priorities?

Comparing TTC & DA: Welfare

2 schools, $q_1 = q_2 = 0.4$, distance-based utilities:



Comparing TTC & DA: Welfare

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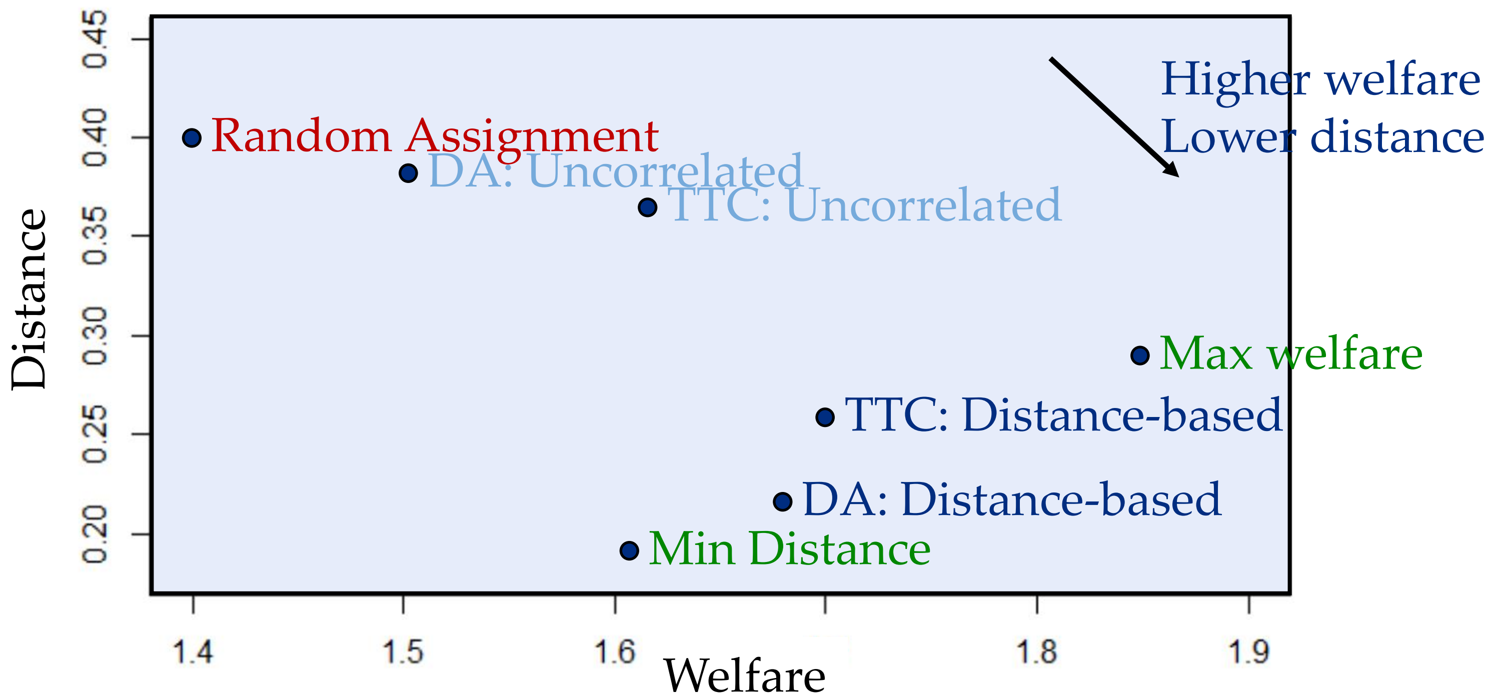

Priority design can have higher welfare effects than choice of mechanism

Summary: TTC via Cutoffs

- Cutoff description of TTC
  - n^2 admissions cutoffs
  - Enables inference

- Tractable framework for analyzing TTC
  - Trade balance equations
  - TTC cutoffs are a solution to a differential equation
  - Can give closed form expressions

- Quantifying Welfare
  - Priorities have larger welfare effects than DA vs TTC