Dynamic Matching in Overloaded Waiting Lists

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Abstract

Commonly used waiting list mechanisms allocate items by offering agents a choice between items and associated expected waiting times. Because the expected waiting times fluctuate over time, different agents may face different waiting times, potentially resulting in misallocation of items. In a stylized model of overloaded waiting lists, we show that welfare is maximized by maximizing the value of assigned items and derive expressions for welfare loss from fluctuations. A simple randomized assignment policy can decrease expected wait fluctuations, thereby reducing misallocation and increasing welfare.

Keywords: Dynamic matching, waiting lists, queueing, misallocation, rationing.

1   Introduction

From nursery schools to nursing homes, waiting lists are a common tool for allocating scarce goods that arrive stochastically over time to agents that accumulate over time.[1] In such settings it is impossible to allocate all items at once, and waiting lists are used to dynamically allocate items over time. We focus on overloaded waiting lists, where items are scarce and many agents are waiting for any arriving item. Examples include waiting lists for public housing, organ transplant, nursing homes, and daycare centers.[2] Given the high demand, all items can be readily assigned. But when items are heterogeneous and agents have heterogeneous preferences,[3] to maximize welfare the assignment needs to efficiently match agents to items.

This paper introduces a new stylized model to capture distinctive features of waiting list allocation mechanisms. First, agents choose among items with associated expected wait times. Waiting times serve a similar role to that of monetary prices in directing agents' choices and rationing items. Second, the expected wait for an item is determined by queue. Waiting times cannot be arbitrarily set and randomly fluctuate over time as agents arrive and queue or as items arrive and are assigned to waiting agents. Different agents make their choice at different times and may face different menus of options.

In the model, agents join an overloaded waiting list. Two kinds of items stochastically arrive over time and are assigned to agents as they arrive. All agents incur the same waiting costs but differ in their preference over the two items. We say that agents are mismatched if they are assigned to their less-preferred item. Because waiting is costly, agents will choose to wait for their preferred item only if the required expected wait is

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[1]Waiting lists are common in practice. Some examples include publicly provided medical services (Lindsay and Feigenbaum, 1984; Martin and Smith, 1999), organs for transplant (Kessler and Roth, 2014), and public housing (Kaplan, 1984). The inspiration for this work came from problems arising from allocating senior citizens to publicly provided nursing homes in Quebec, Canada. Barzel (1974) describes how waiting time can be used to ration goods when monetary prices are fixed to be below the market price. Waiting lists are also used by for-profit firms; for example, many NFL teams hold waiting lists for season tickets (Forbes, 2007).

[2]The rate at which applicants join the waiting list often exceeds the rate at which items arrive, leading to long and growing waiting lists. For example, the Chicago Housing Authority runs a lottery to determine who can join its long waiting list, because the number of potential applicants is too large and the median applicant on the waiting list drops out without being assigned (Chicago Housing Authority, 2016). See also a similar assumption in Kaplan (1986). As of December 2016, more than 90,000 patients in the United States are waiting for a kidney transplant, with 22 people a day dying while waiting for transplant (UNOS, 2016). Private conversations with daycare centers and the administration of the public nursing home system in Quebec indicate that long waiting lists are common there as well.

[3]Patients differ in their preferences over organs for transplant based on immunocompatibility and proper organ size. Applicants to public housing, daycare centers, and nursing homes differ in their geographical preferences over locations.

sufficiently low.

As an illustration, consider two agents, a_1 and a_2, where a_1 prefers an A item and a_2 prefers a B item. Suppose that in period 1 a B item arrives, and in period t an A item arrives. There are two possible assignments: either both agents get their preferred item and a_1 waits, or both agents get their less-preferred item and a_2 waits. The former assignment maximizes welfare, but if t is large then a_1 prefers the inefficient[4] latter assignment because a_1 prefers to avoid the costly wait and instead have a_2 incur the wait.

In general, maximizing welfare is equivalent to minimizing the fraction of agents assigned to mismatched items. Because the waiting list is overloaded, every item that arrives reduces the waiting cost of one assigned agent while the other agents remain waiting. Total waiting costs are constant across all assignments and can only be transferred from one agent to another. Thus, the planner's goal is to maximize the value of assigned items.

Waiting costs serve a similar role to monetary prices in standard competitive equilibrium. If the fraction of agents that prefer A items is higher than the probability that an A item arrives, the mechanism can ration A items by requiring a sufficiently longer wait for A items. If it is feasible to assign all agents to their preferred items, the mechanism can induce agents to choose their preferred item by offering an approximately equal expected wait for both items. But in contrast to monetary prices, expected wait varies with the current state of the system; if the mechanism already promised future item arrivals to previously approached agents, the mechanism may be forced to offer a longer wait to the next agent.

We begin the analysis by considering a common waiting list mechanism: the waiting list with declines.[5] When an item arrives, this mechanism approaches an agent and offers a choice between taking the current item or declining the item and keeping their position. If the item is declined, the mechanism immediately approaches the next agent in line. The mechanism informs agents of their position, giving agents all available information about their expected wait.[6] We focus on the waiting list with declines because it is a common mechanism and because it can be analyzed in a model that abstracts away from

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[4]The latter assignment will be inefficient even if the two agents can eventually trade because the assignment is inefficient in the periods until trade takes place. In addition, the latter assignment is less efficient if there are transaction costs for trading.

[5]This mechanism is a simplification of common mechanisms. Variants of this mechanism or equivalent formulations of it are commonly used for organ allocation (UNOS, 2014) and allocation of spots in daycare centers. Thakral (2016) provides arguments in favor of this mechanism.

[6]This is a stylized modeling assumption. In many applications, agents are only given partial information about the current state of the system and their expected wait. See Section 4 and the comments below.

the agent arrival process.[7]

To analyze the waiting list with declines, it is useful to represent it as a buffer-queue mechanism. Under the buffer-queue representation, an agent who declines an item to wait for his preferred item is said to join a buffer-queue for the preferred item. Two buffer-queues, one for each kind of item, hold agents who declined a mismatched item and are waiting to be assigned their preferred item. The waiting list with declines is equivalent to a buffer-queue mechanism with the First-Come First-Served (FCFS) queueing policy.

The buffer-queue representation allows us to determine how often agents choose a mismatched item and calculate welfare. The system's dynamics are captured by a tractable Markov chain whose states are the number of agents in each buffer-queue. The number of agents in a buffer-queue can be thought of as the current imbalance between the demand from approached agents and the supply of arriving items. In states of higher imbalance (many agents accumulating in the buffer-queue), agents need to wait longer for their preferred item; under FCFS, the expected wait for an agent who joins the buffer-queue increases linearly with the number of agents already in the buffer-queue ahead of him.

Even if demand from agents and supply of items are balanced on average, the random arrivals of items and random preferences of approached agents cause the state to randomly fluctuate. In states of high imbalance, the mechanism offers a long expected wait that induces agents to prefer an immediate mismatched item. We calculate the misallocation rate by calculating the stationary distribution of the fluctuating state and the probability that imbalance becomes sufficiently high for the offered expected wait to be unacceptable. A similar approach can be applied to analyze other waiting list mechanisms as well.[8]

Better information design can reduce misallocation. If the mechanism can limit information about the current state, it can improve welfare by only giving agents a binary signal about their position: suggesting whether the agent should wait for the preferred item or take the immediate mismatch. Agents make their choice based on the average expected wait across states, allowing the mechanism to signal agents to wait and endure

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[7]Because the waiting list with declines approaches a new agent only when trying to assign the current item, it is likely to approach only a few agents at any given time. If the waiting list includes many agents, most agents will be waiting to be approached and the mechanism is unlikely to reach the end of the waiting list. In contrast, it is not possible to abstract way from the agent arrival process when analyzing other mechanisms such as the disjoint-queues mechanism (analyzed in Appendix B) that approaches agents as soon they enter the waiting list.

[8]Examples include the analysis of Caldentey et al. (2009) and Adan and Weiss (2012), and the subsequent work of Baccara et al. (2020). Appendix B analyzes the disjoint-queues mechanism which holds a separate queue for each item and asks agents to choose a single queue when they join the waiting list. The analysis uses a similar Markov chain that tracks imbalance between supply of items and demand from agents. Similarly, misallocation occurs when the imbalance is too large, inducing agents choose to join the shorter queue regardless of their preferred item.

longer wait in some states of high imbalance if on average the expected wait is acceptable because of the low wait in states of low imbalance. In other words, by hiding information, the mechanism can offer the same acceptable wait despite the fluctuating state. In practice many waiting list mechanism provide infrequently updated information to applicants, limiting their response to fluctuations.[9]

Without hiding information, the mechanism can reduce misallocation by using a randomized queueing policy.[10] The FCFS queueing policy is wasteful in that it offers a very short expected wait to agents approached in states of low imbalance, but an unacceptably long wait in states of high imbalance. By increasing the expected wait in low-imbalance states, a randomized queueing policy can decrease the expected wait in high-imbalance states. By having an expected wait that varies less with the state, the mechanism can offer an acceptable expected wait in more states, reducing misallocation due to random fluctuations.

A practical recommendation is the simple service-in-random order (SIRO) queueing policy. A SIRO buffer-queue mechanism has a simple description: agents who decline an item are allowed to join a priority pool for their preferred item, and agents in each priority pool have an equal probability of receiving an arriving item. We characterize the SIRO buffer-queue mechanism as the robustly optimal mechanism. This simple randomization does not fully equalize the expected wait across states, but it lessens the expected wait fluctuations and therefore reduces the misallocation probability and achieves higher welfare in equilibrium than FCFS. The analysis also characterizes a new optimal LIEW policy which uses randomization to offer agents the same expected wait regardless of their position. However, implementing a LIEW policy requires precise randomization and correct agent beliefs, making SIRO a more practical mechanism.

In summary, this paper offers two messages for the practical design of allocation through waiting lists. First, although many public-housing authorities have waiting list policies that discourage applicants from declining items, the analysis suggests agents should be encouraged to decline mismatched items. When the system is overloaded, an agent who declines a mismatched item allows the system to search further and assign

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[9]While such policies can reduce the misallocation that is the focus of the model, providing current information to applicants is important for reasons that are beyond the scope of our model. See the discussion in Section 4.

[10]Randomization is used in practice both implicitly and explicitly. For example, priority for liver transplants is determined by a lab test score (Wiesner et al. 2003). Because the score contains some random variation, an organ will be randomly assigned to one of the sickest patients. Arnosti and Shi (2017) and van Dijk (2019) analyze mechanisms that allocate housing via explicit lotteries. Verdier and Reeling (ming) study the allocation of hunting licenses through a mechanism that uses lotteries for tie-breaking.

the item to a matching agent. Furthermore, such an agent reduces the waiting costs of others by allowing them to be assigned before him. Second, equalizing the expected wait agents face when making their choice can improve welfare. This can be achieved by the SIRO buffer-queue mechanism or by partial information mechanisms. Both are practical mechanisms that offer agents more equal options at the time they make their choice, and thus reduce misallocation and improve welfare.

1.1 Related literature

A growing literature studies dynamic assignment markets. Public housing is a prominent example of assignment through waiting lists, studied by Kaplan in a series of papers (1984; 1986; 1987; 1988). Su and Zenios (2004; 2005; 2006) study the assignment of transplant organs through waiting lists and suggest mechanisms that induce agents to accept marginal kidneys to reduce wastage of organs. By contrast, our findings suggest that if the waiting list is long, patients should be induced to decline organs that can be better assigned to other agents. Bloch and Cantala (2017) analyze dynamic assignment to agents with idiosyncratic preferences and find an FCFS policy maximizes welfare. Schummer (2016) follows up on the current paper and derives conditions under which welfare improves when agents are induced to decline items. Thakral (2016) argues theoretically and empirically that waiting-list mechanisms should allow agents to decline items without penalty. Arnosti and Shi (2017) analyze trade-offs between efficiency and targeting in dynamic assignments.

A growing empirical literature evaluates the allocative efficiency of waiting-list mechanisms. Agarwal et al. (2019) empirically study the allocation of kidneys, and van Dijk (2019) and Waldinger (2018) study the allocation of public housing. Verdier and Reeling (Forthcoming) study the dynamic allocation of hunting licenses.

The subsequent work of Baccara et al. (2020) analyzes a dynamic two-sided matching market using a similar Markov chain to capture fluctuating imbalances. They find that agents would wait longer than is socially efficient, and that welfare can be improved by side payments or batching. Doval and Szentes (2018) analyze a dynamic two-sided matching market and characterize when agents will be more or less impatient than socially optimal. Doval (2015) develops a notion of stability in dynamic environments.[11]

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[11]See also Abdulkadiroglu and Loertscher (2007), Kurino (2014), and Kennes et al. (2014) for other dynamic considerations in matching markets. Fershtman and Pavan (2017) derive payments for optimal experimentation on a matching platform. Loertscher et al. (2016) analyze the thickness of a financial market. Coviello et al. (2014) study the dynamic allocation of time to projects. Gershkov and Moldovanu (2010) consider a sequential-assignment problem under incomplete information.

Ünver (2010), Akbarpour et al. (2020), Anderson et al. (2017), Ashlagi et al. (2019), and Das et al. (2015) explore the related issue of thickness of dynamic markets. This literature finds that a myopic policy can be optimal under some assignment feasibility constraints.

Our model is connected to but differs from standard queueing models. Starting with Naor (1969), a large literature considers waiting costs in strategic queueing settings (see Hassin and Haviv, 2003 for a survey). By contrast, this paper analyzes the matching between agents and items. From a technical perspective, our stochastic model is closer to the FCFS infinite bipartite matching problem studied by Caldentey et al. (2009). Our analysis relies heavily on their Markovian representation. Adan and Weiss (2012) and Adan et al. (2018) provide expression for calculating performance metrics, but conjecture that calculating welfare from these expressions is computationally hard.

The current paper is related to the literature on dynamic mechanism design (see Bergemann and Said, 2011 for a survey), but differs from it in that we do not allow transfers. While expected waiting times serve as prices, the mechanism can offer only expected waiting times that can be feasibly generated by the stochastic dynamics.[12] Section 6 uses ideas from the literature on robust mechanism design (Bergemann and Morris, 2005).

Finally, our results demonstrate how fluctuations adversely affect the efficiency of resource allocation. De Vany (1976), Carlton (1977), and Carlton (1978) study how demand fluctuations affect firms and market behavior. Asker et al. (2014) provide empirical evidence that fluctuations cause misallocation and lower productivity.

1.2 Organization of the paper

Section 2 introduces the model and shows that in an overloaded waiting list, welfare is maximized by maximizing the value of assigned items. Section 3 analyzes the waiting list with declines mechanism and calculates the welfare loss from fluctuations. Section 4 shows that information design can be used to reduce welfare loss. Section 5 gives the technical intuition for the results by providing a buffer-queue representation for the waiting list with declines mechanism. Section 6 leverages the technical results to the design queueing policies that help control expected wait fluctuations and reduce welfare loss. It includes characterizations of the optimal policy and the practical SIRO policy. Section 7 uses simulation to assess concerns of realized envy and presents heuristics that can mitigate

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[12]The dynamic mechanisms we derive have features similar to Levin (2003). In both problems, the mechanism has a finite stock of value (the value of future arrivals) that is used for generating good incentives for agents.

such concerns. Section 8 concludes.

Appendix A discusses non-linear waiting costs. Appendix B analyzes the disjoint-queues mechanism and finds similar welfare loss from random fluctuation. Appendix C contains omitted proofs. Appendix C.1 provides the details regarding the Markov chain used in our analysis. Online Appendix D presents a more general class of mechanisms that use more general history-dependent prioritization rules, and shows that the optimal buffer-queue mechanism is optimal among this larger class.

2   Model of Dynamic Matching in Waiting Lists

Each period $t \geq 1$ begins with arrival of an item x_t and ends when the item is assigned to an agent. The item is of kind A with probability p_A and of kind B with probability $p_B = 1 - p_A$, independently across periods.[13]

All agents are infinitely lived, risk neutral, and incur a common linear waiting cost $c > 0$ per period until they are assigned. We assume that registering for the waiting list does not change the agent's waiting costs.[14] Thus, an agent's outside option, opting out of the waiting list, is equivalent to never getting assigned and entails a utility of $-\infty$.

Agents are of two types: agents of type α prefer A items and agents of type β prefer B items. We refer to the agent's non-preferred item as a *mismatched item*. Given an item, we refer to agents of the type that prefers the item as *matching*, and other agents are *mismatched*. Agent types are private information.

Once assigned, agents stop paying the waiting cost and receive a value $v > 0$ if they are assigned their preferred item, or a value of 0 if they are assigned their mismatched item. That is, the utility of an α agent who is assigned after waiting w periods is $v - c \cdot w$ if he is assigned an A item, or $-c \cdot w$ if assigned a B item. We assume agents break indifferences in favor of their preferred item. Because of the reduction in waiting costs, agents prefer receiving a mismatched item to never being assigned.

An *assignment* is $\mu : \{t \geq 1\} \to \mathcal{I}$, where $\mathcal{I}$ is the set of agents and $\mu(t) \in \mathcal{I}$ is the agent assigned the item x_t. We say the item arriving in period t is *misallocated* if $\mu(t)$ is a mismatched agent for the item x_t. Assignments are final, and assigned agents leave the

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[13]The model and analysis would remain essentially unchanged if item arrivals follow a Poisson processes with rates λ_A, λ_B, as we can normalize time so that $\lambda_A + \lambda_B = 1$ and set $p_A = \lambda_A$. Kaplan (1986) argues the arrival process of public housing apartments should be modeled as a Poisson process.

[14]For example, public housing applicants incur a waiting cost from having to pay higher private market rent. Applicants pay this waiting cost until receiving subsidized public housing regardless of whether or not they registered to the waiting list.

system.[15]

The mechanism dynamically assigns items as they arrive. At the beginning of each period, the mechanism learns which item arrived and may have information about the preferences of agents approached in previous periods. We say that an agent is unapproached if the agent enrolled in the waiting list, but had no other interaction with the mechanism. The mechanism may assign the item given its current information or sequentially approach new unapproached agents from the waiting list to learn their preferences. We assume all such unapproached agents appear interchangeable to the mechanism.[16] The probability that an unapproached agent is of type α is p_α, independently across agents. We denote $p_\beta = 1 - p_\alpha$. We say the system is *balanced* if $p_A = p_\alpha$, and in that case, denote $p = p_A = p_\alpha$.

The goal of the social planner is to allocate the limited supply of items to maximize total utility. Each assigned item makes two contributions to welfare: (i) agent's value of the item, and (ii) reduction in waiting costs. The following lemma shows that any assignment reduces total waiting costs by the same amount.[17] Intuitively, a public housing apartment generates a reduction of waiting costs equal to one month's rent reduction for each month the apartment is assigned. Any public housing assignment that immediately assigns all apartments as they arrive generates the same reduction in total rent paid by applicants. Since waiting costs are potentially unbounded due to the infinite time horizon, the lemma compares assignments up to an arbitrary finite time horizon.

Lemma 1. *The difference between the total utility under assignments μ and μ' up to period T depends only on the number of misallocations under μ and μ' up to period T.*

Therefore, the social planner can ignore waiting costs when comparing different assignments. Welfare is determined by the number of misallocations, as each misallocated item generates a value of 0 instead of v. Therefore, the social planner's objective is to minimize welfare loss from misallocation.

Definition 1. Given an assignment μ, let $\xi_t \in \{0, 1\}$ be an indicator equal to 1 if the item x_t is misallocated under μ. The long-run *misallocation rate* $\xi = \xi(\mu)$ is given by

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[15]In some applications, misallocated agents may eventually be able to trade their items. Such considerations are left out of the analysis in this paper, but can be partially captured by setting the utility of a mismatched agent to be $v' - c \cdot w$, where $0 \leq v' < v$ is the value of getting a mismatched item and trading it later (even if trade is possible, receiving a mismatched item is strictly worse than receiving the preferred item, because the agent spends some time assigned to the mismatched item before trading).

[16]In particular, the time at which an agent enrolled in the waiting list is not correlated with the agent's preferences.

[17]The lemma can be generalized to allow a common discount rate, as in Appendix A. In the presence of discounting, we define social welfare as the sum of agents' utilities evaluated at time 0.

$\xi = \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \xi_t$. We define *welfare loss from misallocation* (WFL) to be

$$\text{WFL} = v \cdot (\xi - |p_A - p_\alpha|).$$

If $p_\alpha \neq p_A$, one of the items is overdemanded and $|p_A - p_\alpha|$ of agents must be assigned to their mismatched item.[18] A misallocation rate approximately equal to $|p_A - p_\alpha|$ can be obtained if agents are patient (Corollary 1) or if the mechanism has full information (Lemma 2). Thus, WFL can be interpreted as the additional loss due to the dynamic allocation problem. As an illustration, we consider two simple mechanisms.

Sequential assignment without choice To highlight the importance of facilitating agent choice, consider *the sequential assignment* mechanism. Each period, the sequential assignment mechanism assigns the arriving item to an (arbitrary) unapproached agent without offering that agent a choice.[19] This mechanism induces a misallocation rate equal to $\xi^{\text{SA}} = p_A p_\beta + p_B p_\alpha$. If the system is balanced, the misallocation rate simplifies to $\xi^{\text{SA}} = 2p(1 - p)$ and WFL is equal to $v \cdot 2p(1 - p)$.

Full information mechanism To clarify the mechanism design challenge, consider a simple mechanism that has full information of agent preferences. When an item arrives, the *full information mechanism* searches the entire waiting list for a matching agent and assigns the item to a matching agent if there is one. The full information mechanism will mismatch agents only if there is no matching agent in the waiting list.

Lemma 2. *Suppose the waiting list includes M agents at any point in time. The full information mechanism achieves a misallocation rate ξ^{FI} such that $\lim_{M \to \infty} \xi^{FI} = |p_A - p_\alpha|$.*

We say that the waiting list is *overloaded* for a given mechanism and agent arrival process if the waiting list always holds a new unapproached agent when the mechanism seeks one. A mechanism generates the same WFL under any arrival process that results in an overloaded waiting list. Under many mechanisms (including the waiting list with declines), agents are approached as items arrive, and there are only a few approached agents at the head of the waiting list. Specifically, for the mechanisms considered in this paper, the waiting list is overloaded if the waiting list always contains at least M agents,

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[18]Formally, if there is a constant that upper-bounds the number of agents in the waiting list at any time, $|p_A - p_\alpha|$ is the minimal expected misallocation rate among all allocations.

[19]Agents who decline the item are removed from the waiting list. If the waiting list is sufficiently long, even if an agent can decline an item and reenter the waiting list, the agent will not benefit from doing so.

where M is moderately small for plausible parameter values.[20] By assuming the waiting list is overloaded for the mechanisms considered, the analysis can abstract away from further details of the agent arrival process.[21]

3   The Waiting List with Declines Mechanism

The waiting list with declines mechanism holds a single ordered waiting list. Arriving items are offered to the first agent on the waiting list. When an agent is offered the current item, the agent can decline the item and keep his position in the waiting list. A declined item is immediately offered to the following agent. Note that the mechanism approaches a new agent only when it is trying to assign an arriving item that has been declined by all approached agents that are still in the system.

Agents know their position in the waiting list, which we denote by k. This assumption implies that agents can fully observe and react to the fluctuating state of the system. Section 4 explores partially informed agents.

Consider an α type agent who is offered a B item when he is in position k (the treatment of β agents who are offered an A item is symmetric). The agent faces a choice between taking B immediately or declining it and waiting for an A item. Let w_k denote the expected wait for an A item for an agent in position k. The α agent receives 0 utility from taking the B item immediately and $v - c \cdot w_k$ from waiting for an A item.[22] Thus, the α agent prefers to wait for the preferred A item if the expected wait w_k is below $\bar{w} = v/c$.

The expected wait w_k serves a similar role to prices in guiding the allocation of items. An expected wait $w_k \leq \bar{w}$ induces α agents to wait for an A item. An expected wait $w_k > \bar{w}$ rations A items by inducing α agent to take the immediate B item. Similarly to monetary transfers in standard competitive equilibrium models, waiting costs can only be transferred between agents (as the total waiting costs are constant across assignments). In particular, if $p_\alpha = p_A$, it is socially inefficient for an α agent to take a B item, as the waiting costs w_k are transferred to other agents.[23]

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[20]We formally show this in Section 5 by bounding the number of approached agents in the system under these mechanisms.

[21]In contrast, the disjoint-queues mechanism analyzed in Appendix B approaches each agent as they arrive, and there are no unapproached agents in the waiting list. Therefore, the analysis of the disjoint-queues mechanism cannot abstract away from the specifics of the agent arrival process.

[22]Note that an agent who declined a B once will prefer to decline all subsequent offers of B items to wait for an A, because past costs are sunk and the expected wait for an A can only decrease. Therefore, it is immaterial whether an agent who declined a B item will be offered B items again.

[23]If $p_\alpha > p_A$, the assignment must ration A items and assign $p_\alpha - p_A$ of α agents to a B, but it is socially inefficient to have more than a fraction $p_\alpha - p_A$ of α agents take a B item.

If the planner could choose the expected wait offered to agents, the planner could implement the optimal assignment by offering an expected wait above $\bar{w}$ only when items need to be rationed. But the planner cannot directly choose the expected wait offered to agents. If a B item is offered to the agent in position k, it has been declined by agents in positions $1, \ldots, k-1$, who are also waiting for an A item. Because these k agents are assigned in a First-Come First-Served priority order (FCFS), the expected wait for the agent in position k is the expected number of periods until k copies of A arrive, which is $w_k = k/p_A$. Different α agents face different expected waits for A depending on their position k when offered an item. If k is sufficiently large, an α agent prefers to take the immediate B item. Agent behavior is summarized in the following lemma.

Lemma 3. *The waiting list with declines has a unique equilibrium, under which an α agent in position k declines a B to wait for an A item iff[24] $k \leq K^A = \lfloor p_A \bar{w} \rfloor$. Likewise, β agents wait for B items iff $k \leq K^B = \lfloor p_B \bar{w} \rfloor$.*

The waiting list with declines incurs misallocation and WFL because agents are offered a randomly fluctuating expected wait. Depending on the randomly evolving state of the system, some agents will be offered a higher expected wait for A while others may be offered a higher expected wait for B. As the mechanism accumulates α agents who declined a B, it approaches agents with higher k who are offered a higher expected wait w_k. Even if $p_\alpha = p_A$, the state of of the system randomly evolves over time, and the expected wait offered to agents randomly fluctuates. When the mechanism randomly accumulates more than K^A agents waiting for an A item, it offers an expected wait that exceeds $\bar{w}$, inducing α agents to choose mismatched items.

Theorem 1. *The waiting list with declines has a unique equilibrium, under which welfare loss when $p_\alpha \neq p_A$ is given by*

$$
\begin{aligned}
WFL^{FCFS} &= v \left(\xi^{FCFS} - |p_\alpha - p_A| \right) \\
&= v \cdot 2 |p_\alpha - p_A| \left(\left(\left(p_\alpha/p_A \right)^{K^A+1} \left(p_\beta/p_B \right)^{-\left(K^B+1 \right)} \right)^{sgn(p_\alpha - p_A)} - 1 \right)^{-1},
\end{aligned}
$$

with $K^A = \lfloor p_A \bar{w} \rfloor$ and $K^B = \lfloor p_B \bar{w} \rfloor$. When $p_\alpha = p_A = p$, welfare loss simplifies to

$$
WFL^{FCFS} = v \frac{2p(1-p)}{(1-p)K^A + pK^B + 1}.
$$

The proof of Theorem 1 calculates WFL by solving for the stationary distribution in closed form. Section 5 provides the intuition and technical analysis.

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[24]We use the notation $\lfloor x \rfloor = \max \{ n \in \mathbb{N} \, | \, n \leq x \}$.
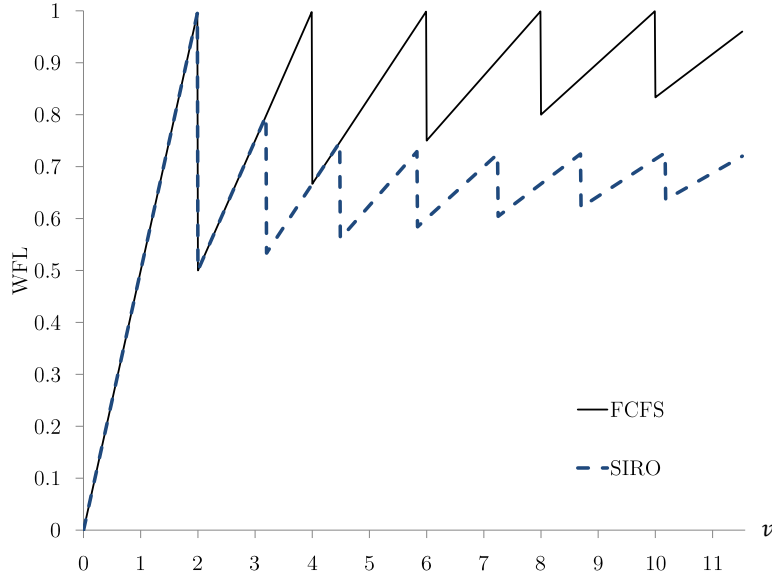
Figure 1: Equilibrium WFL for $p_\alpha = p_A = 1/2, c = 1$, and varying values of v under the waiting list with declines (labeled FCFS) and the SIRO buffer-queue mechanism.

The two following corollaries show how WFL varies with agent preferences. First, misallocation decreases as the cost of waiting c decreases. When agents are more patient $\bar{w} = v/c$ is larger, and K^A, K^B are larger. In other words, expected waiting times can fluctuate within a larger range without exceeding $\bar{w}$ and causing misallocation.

Corollary 1. *As the cost of waiting c decreases,* $\lim_{c \to 0} \xi^{FCFS} = |p_A - p_\alpha|$ *and* $\lim_{c \to 0} WFL^{FCFS} = 0$.

Second, when the system is balanced (i.e., $p_\alpha = p_A$), WFL can be substantial even if mismatched items are undesirable. As v increases, agents are willing to wait longer for their preferred item, reducing the misallocation rate. However, as v increases, each misallocation is a greater loss. Taken together, these two countervailing effects roughly cancel each other, $\xi^{FCFS} \approx 1/\bar{w}$ and WFL is approximately equal to $v \cdot 1/\bar{w} \approx c$.

Corollary 2. *If the system is balanced, we have that* $\lim_{v \to \infty} WFL^{FCFS} \to c$.
If $p_\alpha \neq p_A$, we have that $\lim_{v \to \infty} \xi^{FCFS} = |p_A - p_\alpha|$ *and* $\lim_{v \to \infty} WFL^{FCFS} \to 0$.

Figure 1 depicts the welfare loss under the waiting list with declines (labeled FCFS). When v is close to 0, the preferred item and the mismatched item are almost identical; little loss results from misallocation, and agents do not wait for their preferred item. As v increases, each misallocation becomes more costly, but agents are willing to wait for their

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preferred item at higher positions, reducing the misallocation rate. Discontinuity points correspond to values for which agents in some position are indifferent between waiting for their preferred item and taking an immediate mismatched item.

Figure 1 also provides a comparison with an alternative mechanism with lower welfare loss: the service-in-random-order buffer-queue (SIRO buffer-queue). Under the SIRO buffer-queue mechanism agents who decline an item join a priority pool for their preferred item, and agents in the pool have an equal probability of receiving an arriving item. By doing so, the expected wait offered to agents varies less with the agent's position, as shown in Figure 2. This allows the SIRO buffer-queue mechanism to offer more agents an expected wait that is below $\bar{w}$ and reduce misallocation and welfare loss.
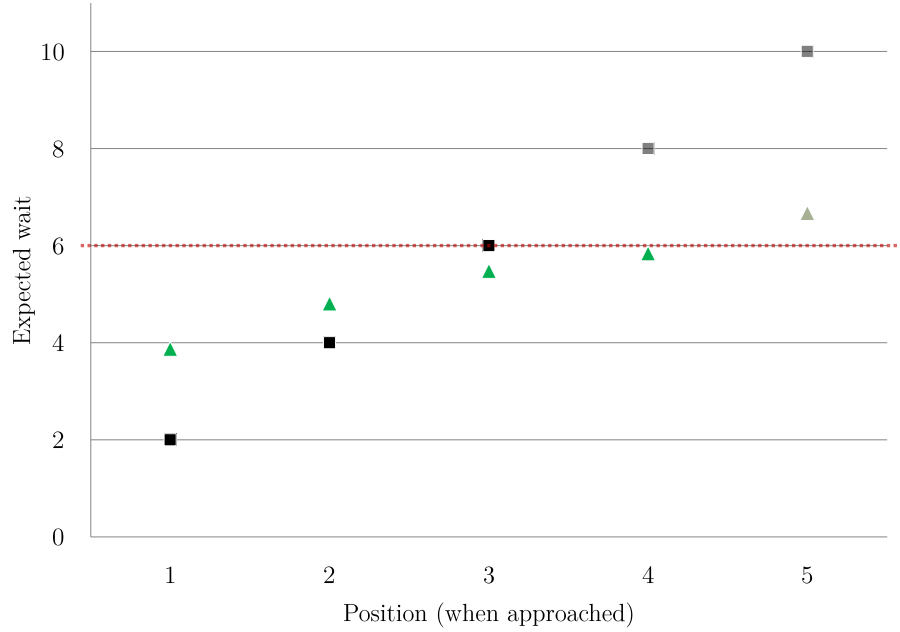
Figure 2: Equilibrium expected wait w_k for agent in position k under the waiting list with declines (black squares) and the SIRO buffer-queue mechanism (green triangles). Parameters used are $p_\alpha = p_A = 1/2, c = 1$ and $v = 6$. The dotted line indicates $\bar{w} = v/c = 6$. In equilibrium, agents wait for their preferred item in positions $k \leq 3$ under the waiting list with declines, or positions $k \leq 4$ under the SIRO buffer-queue. Markers for positions in which agents are not willing to wait for their preferred item are shaded.

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4 Information Design

Providing information to agents about their expected wait is useful for rationing overdemanded items (e.g., when $p_\alpha \neq p_A$), but the previous analysis shows that revealing the fluctuating expected wait leads to misallocations. By hiding the agent's position, the mechanism can control the agent's perceived expected wait and reduce welfare loss.

Formally, consider a partial information mechanism that is identical to the waiting list with declines except: (i) agents who decline an item are not offered that item again,[25] and (ii) agents may be given partial information about their position. A partial information mechanism commits to information disclosure $\Upsilon : \{A, B\} \times S \to \Delta(\mathfrak{S})$ that discloses a signal given the state of the system and the current item kind. An α agent who is offered a B item and observes the disclosed signal $\mathfrak{s} \in \mathfrak{S}$ believes that the expected wait for A is[26] $w_\mathfrak{s} = \mathbb{E}[w_{\tilde{k}} \mid \mathfrak{s}]$, and will prefer to wait for an A if $w_\mathfrak{s} \leq \bar{w}$.

As an illustration, consider an α agent in the setting $p_\alpha = p_A = 1/2$, $\bar{w} = 6$ depicted in Figure 2. Under full disclosure, the α agent is willing to decline a B and wait for an A in positions $1, 2$ or 3. But an agent in position 4 prefers to take an immediate B as $w_4 = 4/p_A = 8 > \bar{w}$. Because the expected wait in position 2 is equal to $w_2 = 2/p_A = 4$ and is strictly below $\bar{w}$, an α agent is willing to wait for a B if he believes that his position is equally likely to be 2 or 4. That is, by hiding information the mechanism can induce an α agent in position 4 to wait for an A and avoid misallocation.

For general $p_\alpha = p_A$ and $\bar{w}$, a simple information disclosure allows the mechanism to minimize welfare loss.

Theorem 2. *Suppose that $p_\alpha = p_A$, that $2p_A\bar{w}, 2p_B\bar{w}$ are integers,[27] and assume that agents do not know their position k. Consider the information disclosure Υ^* under which agents offered an arriving B item are only informed whether $k \in \{1, \ldots, 2p_A\bar{w} - 1\}$ or $k > K_A^* = 2p_A\bar{w} - 1$ (and symmetrically for an arriving A item). Under information disclosure Υ^*, there is a unique equilibrium that yields the minimal welfare loss of any equilibrium*

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[25]Under full information, it is immaterial whether agents who previously declined a B item are offered a B again because the expected wait for the preferred item can only decrease over time. Che and Tercieux (2020) consider a partial information setting in which agents may learn over time and may change their decision with time.

[26]We assume that agents know the steady state distribution of the system and infer the distribution of their possible position $\tilde{k}$ given the signal $\mathfrak{s}$ to calculate their expected wait $w_\mathfrak{s} = \mathbb{E}[w_{\tilde{k}} \mid \mathfrak{s}]$.

[27]If $2p_A\bar{w}$ is not an integer, the information disclosure that minimizes welfare loss sends a randomly selected message to agent in position $\lfloor 2p_A\bar{w} \rfloor$. Let the message space be $\mathfrak{S} = \{$ "wait", "mismatch"$\}$. Agents in positions $k < \lfloor 2p_A\bar{w} \rfloor$ are sent the message "wait". Agents in positions $k > \lfloor 2p_A\bar{w} \rfloor$ are sent the message "mismatch". Agents in position $k = \lfloor 2p_A\bar{w} \rfloor$ are sent the message "wait" with probability q and the message "mismatch" with probability $1 - q$, where q is selected so that the expected wait conditional on receiving a "wait" message is equal to $\bar{w}$.

under any information disclosure policy. Welfare loss under information disclosure Υ^* is equal to

$$
\begin{aligned}
WFL^{FCFS:\Upsilon^*} &= v\frac{2p(1-p)}{(1-p)K_A^* + pK_B^* + 1} \\
&= c/2 \,.
\end{aligned}
$$

Under the information disclosure Υ^*, an α agent who is informed that his position is $k \geq \lfloor 2p_A\bar{w} \rfloor$ prefers to take the current B item, as the expected wait for A is above $\bar{w}$. Under the stationary distribution, an α agent who is informed that $k < \lfloor 2p_A\bar{w} \rfloor$ is equally likely to be in either of the positions $1, \ldots, \lfloor 2p_A\bar{w} \rfloor - 1$, and thus believes his expected wait is $\lfloor 2p_A\bar{w} \rfloor / 2p_A \leq \bar{w}$ and prefers to wait for an A. For an illustration, see Figure 5. A similar approach can be used when $p_\alpha \neq p_A$.

This partial information mechanism eliminates approximately half of the welfare loss of the waiting list with declines. By equalizing the expected wait offered to agents, the mechanism can offer an acceptable expected wait to more agents. Section 6.1 shows that the same welfare loss can be achieved with mechanisms that fully inform agents of their position. Instead of hiding information, the mechanism can use a queueing policy that offers all agents the same expected wait regardless of their position.

The partial information disclosure Υ^* does not eliminate all welfare loss. For example, in the setting depicted in Figure 2, an α agent in position 6 will take an immediate B item. Because the mechanism reveals some information by disclosing the currently available item, no partial information mechanism can achieve lower welfare loss. For example, if agents in position 6 are given a signal to wait for an A, the average expected wait of agents who receive the signal will be above $\bar{w} = 6$, and some α agents would have preferred to take an immediate B item.

A mechanism that hides all information, including which item is currently offered, can induce all agents to choose their preferred item if the system is balanced and agents hold correct steady-state beliefs. However, hiding all information may be problematic in practice for reasons that are beyond the scope of the model.[28] For example, the mechanism may want to provide information to agents to ensure they are not misinformed, or to reduce agents' incentive to collect information from other sources. In addition, the planner may want to provide agents with information to help manage expectations. Sim-

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[28]In addition, such a mechanism can accumulate an unbounded number of agents in the buffer-queue, and will not satisfy the overloaded waiting list assumption. Because the state follows an unbiased random walk, the expected wait for one item can grow unbounded. Moreover, the system can spend an arbitrary long time in states in which agents face a long wait for A items while B items are offered for immediate assignment, making it difficult to hide the difference between waiting times.

ilar concerns arise for partial information mechanisms as well. First, in many situations the planner will have imperfect control over the information available to agents. Under partial information disclosure agents will have an incentive to collect information about their position from external sources, such as online forums. Equity concerns may arise, as agents with access to better information will receive more favorable outcomes. Second, to implement Υ^* the planner needs to know the environment parameters, while the waiting list with declines is parameter free. Moreover, the planner will need to adjust the mechanism with any change in item arrival rates or agent preferences. Third, the behavior of agents depends on their beliefs about preferences of other agents and the stationary distribution of the system. For example, if the planner implements Υ^* for $p_\alpha = p_A = 1/2$, an α agent who believes $\hat{p}_\alpha = 3/4$ will refuse to wait for an A.

In practice, the mechanism should strike a balance between the need to provide agents with useful information and hiding the system's fluctuations. This can be achieved by disclosing historic expected waits that are updated sufficiently frequently to be relevant and sufficiently infrequently to avoid random fluctuations.

5   Analysis via a Buffer-Queue Representation

This section introduces the class of buffer-queue mechanisms that allows for a tractable representation and analysis of the waiting list with declines mechanism. This class of mechanisms is also useful for considering alternative mechanisms that use different queueing priorities to control expected wait fluctuations. Section 6 leverages the results in this section to derive improved mechanisms.

A buffer-queue mechanism represents an α agent declining a B to wait for an A as having the α agent join a *buffer-queue* for A items. A buffer-queue mechanism maintains a buffer-queue for A and a buffer-queue for B. All approached agents who previously declined a B item are held in the A buffer-queue until assigned. Arrivals of A items are assigned to an agent in the A buffer-queue according to the queueing policy (for example, the waiting list with declines uses the FCFS queueing policy). If an A item arrives when the A buffer-queue is empty, the mechanism approaches new agents (skipping agents in the B buffer-queue, if there are any) and offers them a choice between taking the immediate A item or joining the B buffer-queue. Arrivals of B items are treated symmetrically.

The buffer-queue mechanism representation offers several benefits. First, this representation provides a natural state space that is used in the analysis to track the stochastic evolution of the system and calculate the stationary distribution. Second, this represen-

tation allows us to generate different schedules of expected wait by specifying a queueing policy φ that determines the relative priority among agents who declined items.

Last, the representation allows us to specify a direct mechanism, in which agents make their choice by reporting their type. The mechanism determines in each state (subject to incentive constraints) whether a mismatched agent is to join the buffer-queue to wait for the preferred item, or is to be assigned an immediate mismatched item. We consider a simple parameterization, under which mismatched α agents are to join the A buffer-queue for their preferred item if it holds less than K^A agents. We refer to K^A as the maximal buffer-queue size and say the A buffer-queue is full if it holds K^A agents. Similarly, the mechanism specifies K^B for the B buffer-queue.

Definition 2. A *buffer-queue mechanism* $\mathcal{M} = \left(K^A, \varphi^A, K^B, \varphi^B\right)$ is a dynamic mechanism parameterized by two buffer-queue policies: $\left(K^A, \varphi^A\right)$ for the buffer-queue holding agents waiting for A items, and $\left(K^B, \varphi^B\right)$ for the buffer-queue holding agents waiting for B items.

To simplify notation, we restrict attention to queueing policies that track and prioritize agents based on their position in the buffer-queue. Agents who join the buffer-queue take the first empty position, and move forward when an agent ahead of them is assigned. This class is sufficiently general to capture mechanism such as the waiting list with declines, and the SIRO buffer-queue.[29]

Definition 3. A *buffer-queue policy* (K, φ) is given by the maximal buffer-queue size $K \in \mathbb{N}$, and nonnegative assignment probabilities $\varphi = \{\varphi(k, i)\}_{1 \leq i \leq k \leq K}$ such that $\sum_{i=1}^{k} \varphi(k, i) = 1$ for all $1 \leq k \leq K$.

That is, if an item arrives when there are k agents in the A buffer-queue, it will be allocated to the agent in position i with probability $\varphi(k, i)$. This class includes common queueing policies. The First-Come First-Served (FCFS) queueing policy is equivalent to $\varphi^{\mathrm{FCFS}}(k, i) = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}$. The Last-Come First-Served (LCFS) queueing policy is equivalent to $\varphi^{\mathrm{LCFS}}(k, i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$.

---

[29]Definitions and analysis of an extended class of mechanisms with general queueing policies with general state spaces (and which are not restricted to using only position in the buffer-queue) can be found in Online Appendix D. Theorem 7 in that appendix shows that the optimal mechanism derived in Section 6 is also optimal among queueing policies with general state spaces.

5.1 Dynamics and Welfare

The evolution of a buffer-queue mechanism is a stochastic process due to the random arrival of items and agent types. The following analysis describes the dynamics, assuming all agents report their type truthfully, and calculates the implied misallocation rate.

Lemma 4. *The evolution of a buffer-queue mechanism $\mathcal{M} = \left(K^A, \varphi^A, K^B, \varphi^B\right)$ is a stochastic process that is generated by an ergodic Markov chain over the state space*

$$S = \{-K^B, \ldots, -1, 0, 1, 2, \ldots, K^A\},$$

where $k \geq 0$ corresponds to k agents of type α waiting in the A buffer-queue and $k \leq 0$ corresponds to $|k|$ agents of type β waiting in the B buffer-queue. At most one buffer-queue is non-empty at any given time. Each transition of the Markov chain corresponds to one period and one assigned item.
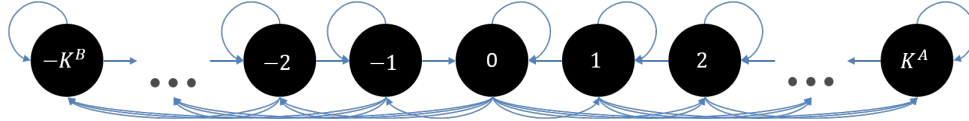

Figure 3: *Illustration of the Markov chain S. Transitions toward state 0 correspond to an assignment of the current item to an agent on the respective buffer-queue. Transitions away from 0 correspond to the mechanism offering the current item to (potentially multiple) new agents.*

The state of the system can be thought of as the imbalance between the supply of arriving items and the demand from approached agents. The maximal sizes of the buffer-queues give the range of imbalances the mechanism can sustain, and the mechanism is forced to misallocate items when the imbalance exceeds the range $\left\{-K^B, \ldots, K^A\right\}$. Within this range, imbalance randomly fluctuates.

Transition probabilities are calculated in Appendix C.1 from the arrival probabilities of items and choices of agents. To calculate the stationary distribution, we use the Markov chain $\hat{S}$, which includes all the original states of S as well as two additional sets of states S^A and S^B. This construction builds on the Markov chain introduced by Caldentey et al. (2009). A state $(k, B) \in S^B$ indicates k agents of type α are in the A buffer-queue, and a current B item is about to be offered to a new agent. Similarly, a state $(-k, A) \in S^A$ indicates k agents of type β are in the B buffer-queue, and a current A item is about to be offered. The original states are relabeled as $(k, \phi) \in S^\phi \cong S$. Each period starts and ends in a state in S^ϕ.

The Markov chain on $\hat{S}$ is depicted in Figure 4. Appendix C.1 contains the full analysis of the Markov chain and related proofs. It allows us to calculate the stationary
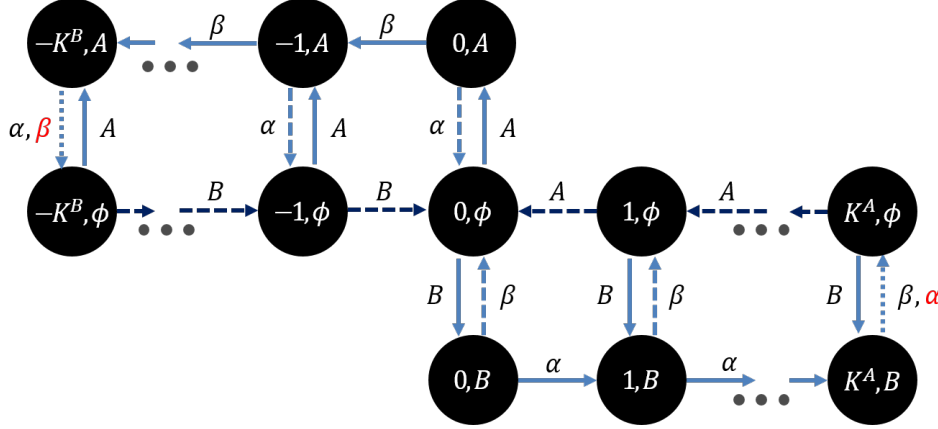
Figure 4: The Markov chain over state space $\hat{S}$. Arrows labeled A, B correspond to the arrival of that item. Arrows labeled α, β correspond to an offer to a new agent of that type, assuming agents are truthful. Dashed lines are transitions that assign the current item, and dotted lines indicate the current item is assigned to a potentially mismatched agent.

distribution over $\hat{S}$ and the misallocation rate.

Theorem 3. *Let $\mathcal{M} = \left(K^A, \varphi^A, K^B, \varphi^B\right)$ be a buffer-queue mechanism. If $p_\alpha \neq p_A$, the misallocation rate under $\mathcal{M}$ when agents are truthful is equal to*

$$\xi = (p_A - p_\alpha) \frac{\left(p_\beta/p_B\right)^{K^B+1} + \left(p_\alpha/p_A\right)^{K^A+1}}{\left(p_\beta/p_B\right)^{K^B+1} - \left(p_\alpha/p_A\right)^{K^A+1}}.$$

If $p_\alpha = p_A = p$, the misallocation rate ξ is

$$\xi = \frac{2p(1-p)}{(1-p)K^A + pK^B + 1}.$$

Moreover, ξ is monotonically decreasing in K^A, K^B and

$$\lim_{K^A \to \infty} \xi = \lim_{K^B \to \infty} \xi = |p_A - p_\alpha|.$$

The misallocation rate for $p_\alpha = p_A = p$ has an intuitive interpretation. If an item arrives and the respective buffer-queue is full, the mechanism assigns the item to the next approached agent regardless of their type. The numerator $2p(1-p)$ captures the probability this assignment results in misallocation, and it is equal to misallocation rate in the sequential assignment without choice mechanism. If the respective buffer-queue is not full, misallocation is avoided by having a mismatched agent join the buffer-queue. When K^A, K^B are larger it is less likely that the buffer-queue is full, which is captured by the denominator $(1-p)K^A + pK^B + 1$ that is increasing in K^A, K^B.

Theorem 1 is a direct corollary of Theorem 3 and the following lemma.

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Lemma 5. *The waiting list with declines has a unique equilibrium which is equivalent to the buffer-queue mechanism* $\mathcal{M} = \left(K^A, \varphi^{FCFS}, K^B, \varphi^{FCFS}\right)$ *with* $K^A = \lfloor p_A \bar{w} \rfloor$ *and* $K^B = \lfloor p_B \bar{w} \rfloor$.

6 Controlling Expected Wait Fluctuations via Queueing Policy Design

The preceding analysis shows that maximizing welfare is equivalent to minimizing misallocation. To incentivize agents to avoid misallocation, the mechanism needs to offer an acceptable expected wait. Figure 1 shows that using a different queue design can allow the mechanism to control the expected wait offered to agents to maintain an acceptable expected wait under a wider range of fluctuations and reduce welfare loss. This section analyzes possible queue designs to derive an optimal design and give a characterization of the SIRO queueing policy as the robustly optimal policy.

Formally, given a mechanism $\mathcal{M} = \left(K^A, \varphi^A, K^B, \varphi^B\right)$, let w_k^A denote the implied expected wait for an agent who declines a B and joins position k in the A buffer-queue.[30] Let w_k^B be defined symmetrically. To simplify notation, we assume $\bar{w} \geq \max\{1/p_A, 1/p_B\}$ throughout this section, ruling out trivial parameters in which agents are unwilling to wait for the first arrival of their preferred item.

Lemma 6. *The expected waits* $\left\{w_k^A\right\}_{k=1}^{K^A}$ *depend only on* $\left(K^A, \varphi^A\right)$*, and* p_α, p_A *(and symmetrically for* w_k^B*).*

Therefore, the following is well defined.

Definition 4. A buffer-queue policy (K, φ) with expected waits $\{w_k\}_{k=1}^{K}$ is *incentive compatible (IC)* if $w_k \leq \bar{w}$ for all $k \leq K$.

A buffer-queue mechanism $\mathcal{M} = \left(K^A, \varphi^A, K^B, \varphi^B\right)$ is *IC* if both $\left(K^A, \varphi^A\right)$ and $\left(K^B, \varphi^B\right)$ are IC.

Under an IC mechanism, it is an equilibrium for all agents to report their type truthfully.[31] The following lemma uses Little's law to show that the average expected wait depends only on K and is independent of the queueing policy φ.

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[30]That is, w_k^A is the expected number of periods from when the agent joins the buffer-queue until he is assigned an A item, conditional on joining the A buffer-queue when it holds $k-1$ agents, and assuming the following agents truthfully report their type.

[31]Every agent makes at most a single choice. An α agent will always take a current A item to attain the maximal possible utility. Because immediate assignment to B is preferable to never being assigned, the mechanism can force the agent to take the current B item. If the current item is a B item and the

Lemma 7. *Let (K, φ) be a buffer-queue policy. Then, independently of φ, the average expected wait for an agent who joins the buffer-queue is*

$$W(K) = \mathbb{E}[w_{\tilde{k}}] = \begin{cases} \frac{K+1}{2p} & \text{if } p_\alpha = p_A = p \\ \frac{K}{p_A} + \frac{1}{p_A - p_\alpha} + \frac{1}{p_A} \frac{K}{(p_\alpha/p_A)^K - 1} & \text{if } p_\alpha \neq p_A. \end{cases}$$

Proof. By Little's Law (Little, 1961), if $W = \mathbb{E}[w_{\tilde{k}}]$ is the average time an agent spends in the buffer-queue, L is equal to the average number of agents in the buffer-queue, and λ is the arrival rate at which agents join/leave the buffer-queue, then we have that $L = \lambda W$. The expected number of agents that leave the buffer-queue in any given period is $\lambda = p_A$. If $p_\alpha = p_A = p$, the buffer-queue is equally likely to hold any number of agents k for $1 \leq k \leq K$ (by Lemma 11 in Appendix C.1), and the average number of agents in the buffer-queue is $L = (K + 1)/2$. Therefore, $\mathbb{E}[w_{\tilde{k}}] = W = L/\lambda = (K + 1)/2p$, which is independent of φ. The case $p_\alpha \neq p_A$ is proved similarly. □

Intuitively, Lemma 7 shows that a mechanism with higher K (that is, avoiding misallocation under states of greater imbalance) needs to offer some agents a longer expected wait. For a fixed K, the queueing policy can only redistribute the expected wait. The FCFS policy offers agents the minimal feasible expected wait at that state, which is low in low states but increases with the state. A different queue policy can distribute the expected wait more equally and offer agents an expected wait below $\bar{w}$ even under greater imbalance.

6.1 Optimal Queueing Policy

An immediate corollary of Lemma 7 is that there is no IC buffer-queue policy (K, φ) with $K > \kappa^*$ defined by $\kappa^*(\bar{w}, p_\alpha, p_A) = \sup \{K' \mid W(K') \leq \bar{w}\}$. To see that, observe that for any IC buffer-queue policy (K, φ) it must be that $\mathbb{E}[w_{\tilde{k}}] \leq \max_{k \leq K} w_k \leq \bar{w}$.

We say that a buffer-queue policy (K, φ) is a *load-independent expected-wait* (LIEW$_K$) policy if $w_k = w_{k'}$ for any positions $1 \leq k, k' \leq K$. That is, an agent joining an LIEW$_K$ buffer-queue faces the same expected wait independently of the number of agents already in the buffer-queue.[32] Because $w_k = \mathbb{E}[w_{\tilde{k}}]$ for any k, a LIEW$_K$ is IC iff $K \leq \kappa^*$.

Theorem 4. *Let $\mathcal{M}^* = \left(K^A, \varphi^{LIEW_{K^A}}, K^B, \varphi^{LIEW_{K^B}}\right)$ be an LIEW buffer-queue mechanism with $K^A = \kappa^*(\bar{w}, p_\alpha, p_A)$, $K^B = \kappa^*(\bar{w}, p_\beta, p_B)$. Then, $\mathcal{M}^*$ is incentive compatible*

A buffer-queue is not full, the agent can choose whether to truthfully reveal he is mismatched and wait for a future A, or misreport his type and take the current B item. Given expected wait w, the α agent prefers joining the A buffer-queue if $v - c \cdot w \geq 0$ or $w \leq v/c = \bar{w}$.

[32]See Ben-Neria et al. (2020) for a construction of a LIEW$_K$ policy.

and achieves weakly lower welfare loss than any incentive compatible buffer-queue mechanism.

Online Appendix D considers a wider class of mechanisms that are allowed to flexibly decide whether an agent should be assigned a mismatched item or wait for their preferred item and can use a general prioritization. The LIEW buffer-queue mechanism achieves the highest possible welfare among the wider class, up to the discreteness of K.

The LIEW buffer-queue mechanism in Theorem 4 achieves the same welfare as the optimal partial information mechanism described in Section 4 without withholding information. If fact, the LIEW$_K$ policy makes the agent's position immaterial, and eliminates the need to hide it. However, it is also subject to similar concerns: the planner will need to design the mechanism for the specific parameters, and agents who hold incorrect beliefs may refuse to join the buffer-queue.

6.2   The Robust SIRO Queueing Policy

The service-in-random-order (SIRO) policy is a simple alternative to FCFS. By giving all agents in the buffer-queue equal probability of receiving an arriving item, the SIRO policy reduces the variation in expected wait. Figure 5 provides an illustration. Intuitively, the expected wait w_K for the agent joining the last position K is lower when the last agent is given equal priority. But while there is less variation in w_k under SIRO than under FCFS, the expected waits w_k under SIRO are strictly increasing in k because the probability of getting assigned is lower when there are more agents in the buffer-queue.

This section gives a characterization showing that the SIRO policy maximizes welfare subject to a robust incentive compatibility constraint. A rough intuition is that the policy wants to give higher assignment priority to agents in the last positions to reduce w_k for high k. This requires reducing the priority of agents who entered the buffer-queue from the first positions, and increases the expected wait of such agents in the event that many agents join the buffer-queue behind them.[33] On the other hand, agents who join the first positions may believe that additional agents will join after them, and to be robustly incentive compatible the mechanism needs to ensure that such pessimistic agents are willing to join the first positions. Giving agents in all positions equal assignment priority balances the two considerations.

Formally, given a mechanism $\mathcal{M} = \left(K^A, \varphi^A, K^B, \varphi^B\right)$, let $w_{k,\sigma}^A$ denote the *subjective expected wait* of an agent with belief σ who declines a B and joins position k in the A

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[33]For example, the first agent to join a Last-Come First-Served queue faces a long expected wait if many subsequent agents join.

buffer-queue ($w_{k,\sigma}^B$ is defined symmetrically). Define a general belief σ as follows. Label the following agents by the order in which they are approached and offered the option to join the A buffer-queue. The agent's belief $\sigma : \mathbb{N} \times \mathbb{N} \rightarrow [0,1]$ specifies the probability $\sigma_\ell(k)$ that the ℓ-th agent will report to be of type α and join the A buffer-queue conditional on being offered position k.[34] The belief that corresponds to all agents being truthful is given by $\sigma_\ell(k) \equiv p_\alpha$ for all $\ell \geq 1$ and $k \leq K^A$. The subjective expected wait $w_{k,\sigma}^A$ is calculated by drawing future agents independently according to σ.

The following lemma generalizes Lemma 6.

Lemma 6'. *The expected waits $\{w_{k,\sigma}^A\}_{k=1}^{K^A}$ depend only on (K^A, φ^A), the belief σ, and p_A (and symmetrically for $w_{k,\sigma}^B$).*

Therefore, the following is well defined.

Definition 5. A buffer-queue policy (K, φ) with expected waits $\{w_k\}_{k=1}^K$ is *belief-free incentive compatible (BF-IC)* if $w_{k,\sigma} \leq \bar{w}$ for all $k \leq K$ and any belief σ.

A buffer-queue mechanism $\mathcal{M} = (K^A, \varphi^A, K^B, \varphi^B)$ is *BF-IC* if both (K^A, φ^A) and (K^B, φ^B) are BF-IC.

In other words, a mechanism is BF-IC if agents who wait for their preferred item face an acceptable subjective expected wait regardless of their beliefs. In particular, BF-IC implies that an agent should not regret joining the buffer-queue even if subsequent agents join the buffer-queue after him. The following theorem shows that the simple SIRO buffer-queue mechanism obtains the lowest possible welfare loss of any BF-IC mechanism.

Theorem 5. *Let $\mathcal{M}^\star = (K^A, \varphi^{SIRO}, K^B, \varphi^{SIRO})$ be the SIRO buffer-queue mechanism given by $\varphi(k, i) = \frac{1}{k}$ for any $i \leq k$, $K^A = \kappa^*(\bar{w}, 1, p_A)$, and $K^B = \kappa^*(\bar{w}, 1, p_B)$. Then, $\mathcal{M}^\star$ is BF-IC and achieves a weakly lower welfare loss than any BF-IC buffer-queue mechanism.*

As stated above, the intuition for the result is that SIRO balances the priority of agents. Agents may hold a belief $\sigma \equiv 1$, which is equivalent to the belief $\hat{p}_\alpha = 1$. Agents with this belief who enter position i believe that a large number of agents will join the buffer-queue after them, and they will end up in position i in a buffer-queue that holds the maximal number of agents K. To maintain BF-IC, it must be that such an agent does not regret joining the buffer-queue. On the other hand, prioritizing agents that joined

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[34]That is, $\sigma_1(k)$ gives the probability that the next agent approached will report to be an α agent and join the buffer-queue conditional on seeing the state where $k - 1$ agents are in the buffer-queue. This formulation that indexes agents by the order in which they are approach (instead of referring to agent by name) allows a more tractable formulation of belief updates.

earlier over the agents that join later prevents the policy from offering a low expected wait to agents who join an almost full buffer-queue. The SIRO policy balances these two goals by giving agents who join the buffer-queue the maximal priority such that none of the agents already in the buffer-queue regret joining, which is to give them all equal priority.

Figure 5 presents the expected waits under SIRO and other policies for $p_\alpha = p_A = 1/2$. It illustrates how the SIRO policy improves upon the FCFS policy. Any IC FCFS policy is also BF-IC, because expected waits under FCFS are independent of whether future agents join the buffer-queue. But SIRO reduces the variation in w_k and maintains an acceptable expected wait under a larger range of states than FCFS. In addition, the SIRO policy is simpler than the FCFS policy in that it does not require tracking positions of agents within the buffer-queue. Figure 5 also shows that w_k increases with k, implying that SIRO does not generate a LIEW$_K$ policy.

The variance of realized wait is higher under SIRO than under FCFS, and some agents can wait significantly longer than w_k before being assigned.[35] In the context of our model, any agent waiting in the buffer-queue will prefer to keep waiting for their preferred item over taking an immediate mismatched item, because past waiting costs are sunk and the expected wait of any agent in the buffer-queue is at most $w_K \leq \bar{w}$. However, the increased variance of waiting times may be undesirable for agents who wish to plan ahead. Additionally, although agents are offered more equitable expected waits when they make their choice, the realized wait may be less equitable. Section 7 presents simulation results showing that a modification of the SIRO policy can mitigate this concern.
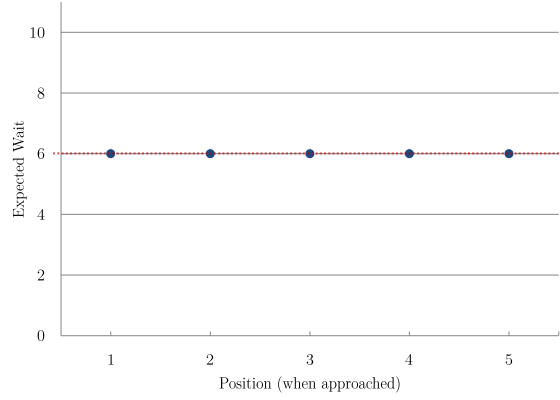
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[35]Vasicek (1977) shows that FCFS minimizes the variance of waiting times while LCFS maximizes it.
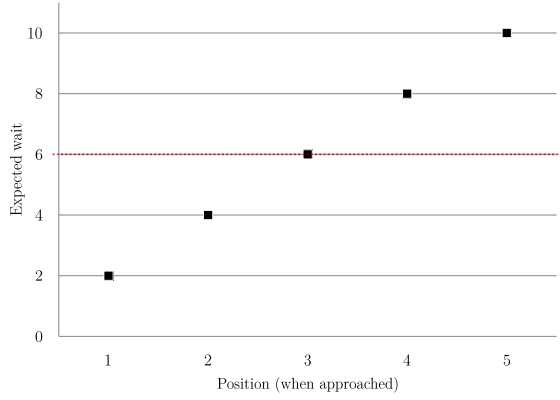
(a) FCFS
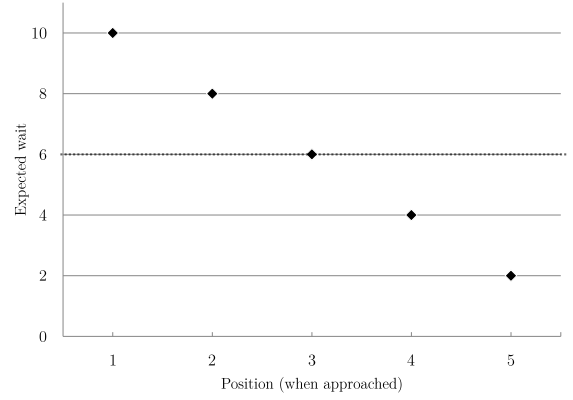
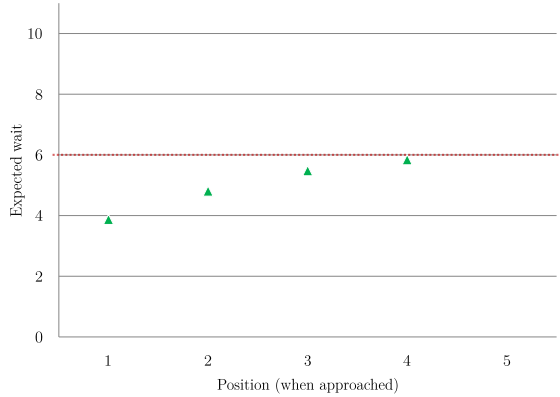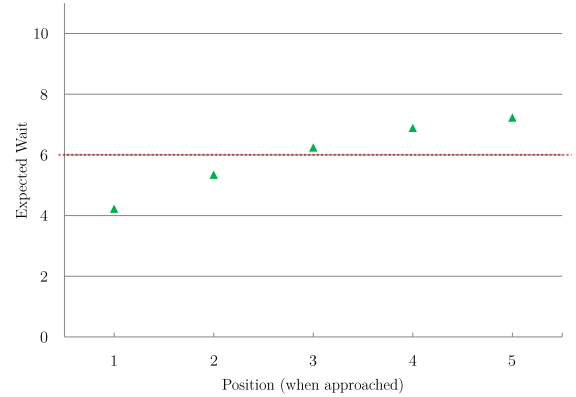(b) LCFS, agents join positions 1 to 5.

(c) LIEW$_5$ or information disclosure Υ^*, agents join positions 1 to 5.

(d) SIRO, agents join positions 1 to 4.

(e) SIRO, agents join positions 1 to 5.

Figure 5: Expected wait for agents who join the buffer-queue to wait for their preferred item under various queueing policies for $p_\alpha = p_A = 1/2$. The dotted line indicates $\bar{w} = 6$.

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6.3 Indirect Parameter-Free SIRO Mechanism

The SIRO mechanism $\mathcal{M}^\star = \left(K^A, \varphi^{\text{SIRO}}, K^B, \varphi^{\text{SIRO}}\right)$ is almost a prior-free mechanism. The queueing policy φ^{SIRO} is prior-free, but the planner needs information on the agent's valuations to appropriately set K^A, K^B to determine in which states agents decline mismatched items. If K^A, K^B are too high, the mechanism $\mathcal{M}^\star$ is not incentive compatible. If K^A, K^B are too low, the mechanism $\mathcal{M}^\star$ forces mismatched agents approached in the states K^A and $-K^B$ to take the current item even though they would have preferred to wait for their preferred item.[36] This section presents a prior-free indirect mechanism in which K^A, K^B are determined in equilibrium. This mechanism is a simple indirect mechanism that improves upon the waiting list with declines.

Consider $\mathcal{M}^\circ = \left(\infty, \varphi^{\text{SIRO}}, \infty, \varphi^{\text{SIRO}}\right)$ as an indirect mechanism. In contrast to a direct mechanism $\mathcal{M}^\star = \left(K^A, \varphi^{\text{SIRO}}, K^B, \varphi^{\text{SIRO}}\right)$ with finite K^A, K^B, all agents are offered the option to join the buffer-queue and wait for their preferred item. The mechanism $\mathcal{M}^\circ$ differs from the waiting list with declines only in that all agents who declined items are equally likely to be assigned a future arrival of their preferred item.

Restrict attention to strategies in which agents take an immediate matching item. Denote a mixed strategy of an α agent under $\mathcal{M}^\circ$ by $s : \mathbb{N} \to [0, 1]$, where $s(k)$ is the probability that the α agent in position k declines a mismatched B item and waits for an A. Because of the SIRO queueing policy, the expected waits $\{w_k\}$ depend[37] on s (for an illustration, compare Figures 5d and 5e). A strategy s constitutes a Nash equilibrium if for the corresponding $\{w_k\}$ it holds that $s(k) > 0 \implies w_k \leq \bar{w}$ and $s(k) < 1 \implies w_k \geq \bar{w}$.

Lemma 8. *If s^* is a symmetric equilibrium of the indirect mechanism $\mathcal{M}^\circ = \left(\infty, \varphi^{SIRO}, \infty, \varphi^{SIRO}\right)$, then $s^*(k) = 1$ for $1 \leq k \leq \kappa^*(\bar{w}, 1, p_A)$.*

In other words, if agents will decline a mismatched item in a state k under a BF-IC direct SIRO mechanism $\mathcal{M}^\star$, agents will decline a mismatched item in a state k under a symmetric equilibrium s^* of the indirect SIRO mechanism $\mathcal{M}^\circ$. An immediate corollary is that equilibrium welfare under the indirect mechanism is higher than under the optimal BF-IC SIRO mechanism $\mathcal{M}^\star$, and therefore also higher than the welfare under the waiting list with declines.

Under the indirect mechanism $\mathcal{M}^\circ$, any strategy s that declines mismatched items in some position k is not a dominant strategy, because an agent who joins the SIRO

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[36]An extreme example is the sequential assignment without choice mechanism, which is equivalent to $K^A = K^B = 0$.

[37]By Lemma 6', the expected waits $\{w_k\}$ are independent of the strategy chosen by β agents if under any strategy β agents always take an immediate B item.

buffer-queue faces arbitrarily long expected wait if sufficiently many subsequent agents join after him. However, the expected wait of any agent in the SIRO buffer-queue equals the expected wait of the last agent who joins the buffer-queue in the same period. If all α agents share identical preferences and beliefs, the decision of the last agent to join indicates that all agents in the buffer-queue also prefer to have joined.

Lemma 9. *Let s^* be a symmetric equilibrium of the indirect mechanism $\mathcal{M}^\circ = \left(\infty, \varphi^{SIRO}, \infty, \varphi^{SIRO}\right)$. Then at the end of each period, any agent in the buffer-queue prefers staying in the buffer-queue to being immediately assigned a mismatched item.*

The assumption that all agents in the buffer-queue have identical preferences is necessary for Lemma 9. For example, if agents have heterogeneous mismatch values, an agent with a high mismatch value may regret joining the buffer-queue if many agents with low mismatch values (who are willing to wait longer for their preferred item) join the buffer-queue after him.

Under the indirect mechanism, the burden of deciding in which states to decline mismatched items falls on the agents. If agents are provided with historical expected wait estimates, their decision is simple, because they need only to decide whether the offered expected wait is acceptable. Altman and Shimkin (1998) prove simple learning dynamics converge to equilibrium for a similar SIRO queueing game.
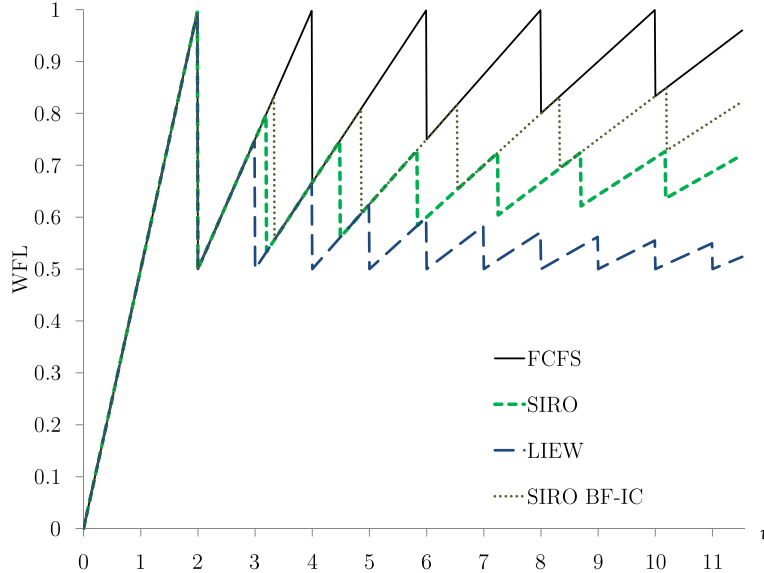

Figure 6: Welfare loss for varying values of v and $p_\alpha = p_A = 1/2, c = 1$.

Figure 6 depicts welfare loss under different mechanisms for varying values of v. The

figure shows the equilibrium welfare loss under the waiting list with declines (FCFS) and the indirect SIRO buffer-queue mechanism. Figure 6 also shows the minimal welfare loss under any BF-IC buffer-queue mechanism (labeled SIRO BF-IC) which is achieved by a direct SIRO buffer-queue mechanism that appropriately sets K^A, K^B to restrict entry to the buffer-queue. In addition, Figure 6 also depicts the minimal welfare loss under any IC buffer-queue mechanism (labeled LIEW). Note that SIRO captures more than half of the difference between FCFS and the minimal possible welfare loss.

7   Limiting Realized Envy

A potential challenge in implementing the SIRO buffer-queue mechanism is that the random assignment can cause some agents to wait significantly longer than expected. Moreover, agents who keep waiting and see others assigned before them experience realized envy and may be understandably aggravated. This section uses simulation to quantify realized envy and evaluate heuristics that limit realized envy.

To quantify realized envy, define the overtaking count of an agent who joins position k in the buffer-queue[38] and assigned the ℓ-th item to arrive[39] to be $\max \{\ell - k, 0\}$. That is, the overtaking count measures whether an agent experiences a longer wait than the agent would have faced under FCFS. Under FCFS, the overtaking count of all agents is 0. Figure 7 shows the distribution of overtaking counts for agents who join a SIRO buffer-queue with $K \in \{4, 10\}$, for $p_\alpha = p_A = 1/2$. The figure presents the distribution for all agents, as well as the distribution for agents who join an empty buffer-queue. Both distributions show a small but non-negligible probability that agents will experienced significant realized envy.

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[38]That is, there were $k - 1$ agents in the buffer-queue just before the agent joined.
[39]That is, ℓ items arrived while the agent was waiting in the buffer-queue.
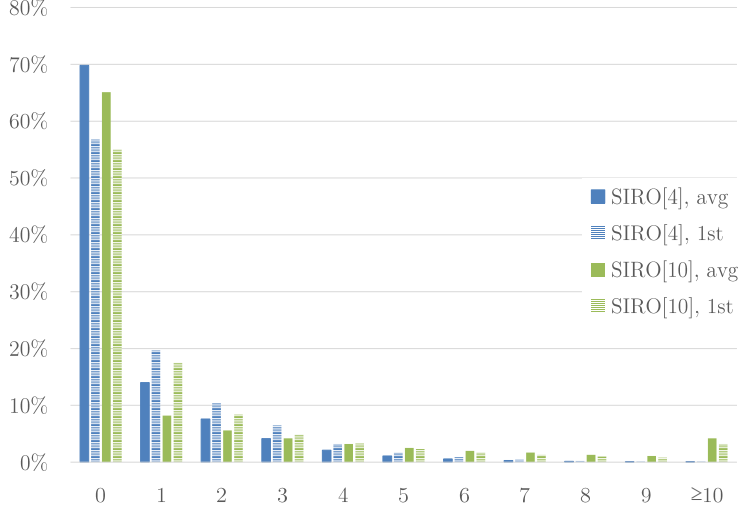
Figure 7: The distribution of the overtaking counts for agents who join a SIRO buffer-queue. Parameters used are $K \in \{4, 10\}$ and $p_\alpha = p_A = 1/2$.

We simulate a simple heuristic that deviates from the SIRO policy by prioritizing agents that reach a specified overtaking limit. This simple heuristic bounds the possible overtaking counts an agent can experience, and thus avoids extreme realized envy. Setting an overtaking limit equal to 0 is equivalent to using a FCFS queueing priority. Setting an overtaking limit equal to ∞ is equivalent to using a SIRO queueing priority. By choosing an appropriate intermediate overtaking limit the planner can strike a compromise between equalizing the expected waits and avoiding realized envy.

Figure 8 shows the performance of the SIRO with limited overtaking policy for various buffer-queue sizes and overtaking limits. The figure shows the expected wait w_K for an agent joining the last position in the buffer-queue given the maximal buffer-queue size K and the overtaking limit. Each buffer-queue policy is incentive-compatible if $\bar{w} = v/c$ is higher than the depicted w_K. The figure shows that even a mild overtaking limit can significantly reduce the expected wait w_K, which enables the planner to implement a larger incentive-compatible buffer-queue size and lower misallocation. The figure also shows that a moderately high overtaking limit does not hinder the performance of the SIRO policy. An overtaking limit of 20 is unlikely to bind and therefore yields essentially the same w_K as SIRO.
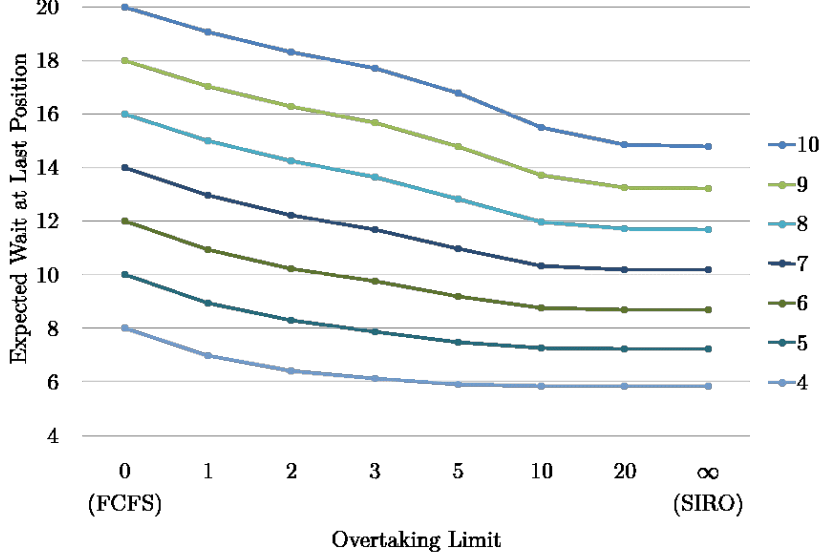
Figure 8: Expected wait for an agent who joins the last position of a SIRO buffer-queue with an overtaking limit for $p_\alpha = p_A = 1/2$. Each line corresponds to a different buffer-queue size K and various overtaking limits. The FCFS policy corresponds to an overtaking limit of 0.

Figure 9 presents the result of the evaluation of another heuristic, the Tiered-SIRO policy. This queueing policy groups the positions into tiers and uses FCFS between tiers and SIRO within a tier. That is, when an item arrives a randomly drawn agent from the first tier is assigned the item, a randomly drawn agent from the second tier advances to the first tier, and so on. SIRO is equivalent to a Tiered-SIRO with a single tier, while FCFS is equivalent to a Tiered-SIRO with tiers of size 1. Figure 9 shows the percent of agents that experience an overtaking count greater or equal to 5 and w_K for Tiered-SIRO with different equally-sized tiers. A larger tier size generates a policy that is closer to SIRO and has a lower w_K (and therefore requires a lower $\bar{w}$ to be incentive compatible). On the other hand, a larger tier size increases the likelihood that agents experience realized envy. The figure shows possible trade-offs between reducing realized envy and equalizing expected wait that can be achieved by a Tiered-SIRO policy. For comparison, Figure 9 also displays the SIRO with overtaking limit of 5 policy. Setting an overtaking limit eliminates the possibility of an extreme overtaking count, but results in more agents experiencing an overtaking count equal to the limit. In contrast, a Tiered-SIRO lowers the cumulative distribution of overtaking counts more evenly.
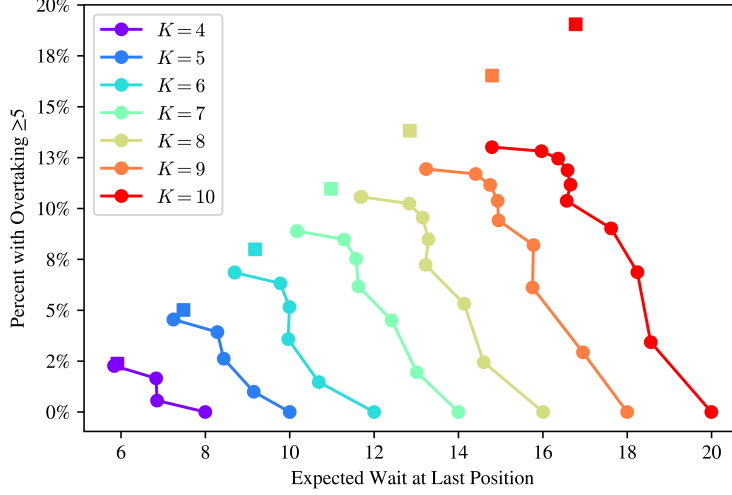
Figure 9: Probability that agents experience an overtaking count equal or greater to 5 and the expected wait at the last position w_K. Each color corresponds to a different value of $K \in \{4, \ldots, 10\}$. The connected circles correspond to Tiered-SIRO policies with equally sized tiers of size 1 to 10. The squares correspond to SIRO with an overtaking limit equal to 5.

Finally, we evaluate FCFS, SIRO and SIRO with limited overtaking in a more elaborate setting with heterogeneous values in which the value of the preferred item v is drawn from $U[0, 2]$ independently across agents.[40] In this setting, the welfare-maximizing assignment assigns all items to matching agents and attains an average assigned value of 1. Because an agent assigned to a mismatched item decreases total welfare by v, the mechanism can reduce welfare loss either by reducing the misallocation rate or by replacing misallocations of agents with high v with misallocations of agents with low v.

Figure 10 depicts the equilibrium welfare loss from misallocation given each policy for various delay costs $c \in \{0.05, 0.1, \ldots, 0.25\}$ and $p_\alpha = p_A = 0.5$. Equilibrium welfare was calculated by initializing estimates for $\{\hat{w}_k\}$, simulating the system with agents following the strategy of joining position k to wait for the preferred item iff $v - c \cdot \hat{w}_k > 0$ and calculating new expected wait estimates $\{\hat{w}_k\}$ from the simulation, and iterating until a fixed point is reached. This process converged to a fixed point quickly. Once a fixed point was found, an additional simulation used the equilibrium $\{w_k\}$ to calculate welfare loss.

Figure 10 shows that SIRO reduces welfare loss in this setting as well. The FCFS policy (given by an overtaking limit of 0) eliminates all realized envy but results in higher welfare loss. SIRO with limited overtaking reduces welfare loss, even with a small overtaking limit; in fact, most of the welfare gains can be attained by allowing a small overtaking limit.
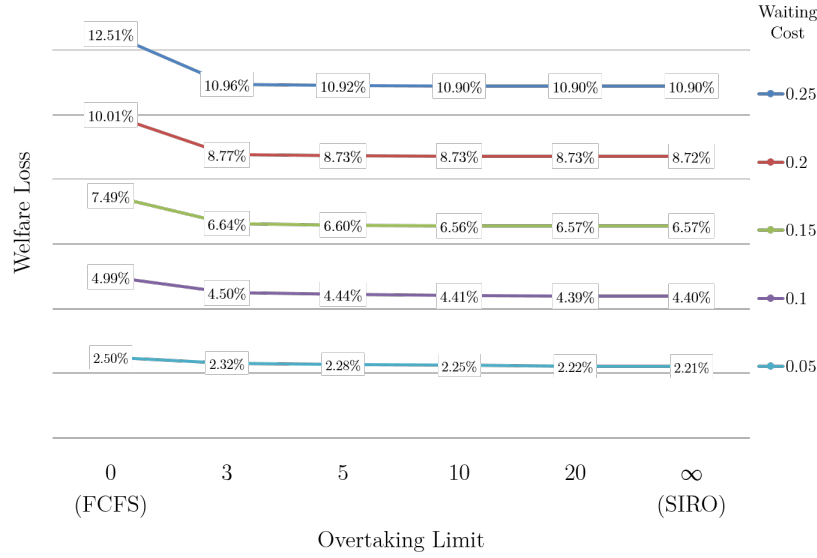
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[40]The value of the mismatched item is zero.

Figure 10: Welfare loss under SIRO with various overtaking limits for an economy with $p_\alpha = p_A = 0.5$, v drawn from $U[0,2]$ independently for each agent and $c \in \{0.05, 0.1, 0.15, 0.2, 0.25\}$.

8 Concluding Remarks

SIRO may raise concerns of fairness, in that agents are not assigned in order. First, we note that SIRO is more fair than FCFS in that agents are offered a more consistent expected wait for their preferred items. Second, the ordering of agents on a waiting list may be arbitrary, and agents who sign up earlier may not have higher assignment value. For example, local parents may be able to register for daycare centers years in advance, whereas advanced registration is not possible for recently moved parents who may have a greater need for daycare. Constraining the mechanism to prioritize agents who made their choice earlier is equivalent to requiring the FCFS policy, which generates lower welfare than SIRO.

Under our assumptions, agents exert a positive externality when declining items because they are essentially letting other agents pass them in line. However, waiting-list policies commonly discourage agents from declining items.[41] One possible justification is that agents who decline items are an administrative burden. Although our analysis does not explicitly account for the time and costs required to administer the offers, under

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[41]We surveyed the waiting-list policies of the New York City Housing Authority, Newark Housing Authority, Boston Housing Authority, Atlanta Housing Authority, Philadelphia Housing Authority, the Housing Authority of Los Angeles, Miami-Dade County Public Housing, the Housing Authority of Baltimore City, and the Chicago Housing Authority. All of these authorities penalize agents if they decline apartments. In several authorities, agents who decline an apartment will not be offered another one.

buffer-queue mechanisms, each agent is approached only once, limiting the administrative burden. Furthermore, these mechanisms require storing preference information for relatively few agents. We therefore believe overloaded waiting lists should encourage rather than discourage agents to decline mismatched items.

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Appendices

A   Nonlinear Waiting Costs

This appendix extends the model to allow for discounting. Agents pay a linear waiting cost c per period until they are assigned, and discount the value of the assigned item by $\delta < 1$ per period. Formally, the period t_0 utility of an agent of type θ who is assigned item x_t item in period $t \geq t_0$ is

$$u_\theta^{t_0}(X, t) = -c \cdot (t - t_0) + \delta^{t - t_0} v_\theta(x).$$

where the valuation $v_\theta(x)$ is $v > 0$ for a matching item and 0 for mismatched items:

$$v_\theta(x) = \begin{cases} v & (\theta, X) \in \{(\alpha, A), (\beta, B)\} \\ 0 & (\theta, X) \in \{(\alpha, B), (\beta, A)\} \end{cases}.$$

Note that past waiting costs are sunk costs, and that $u_\theta^{t_0}$ is independent of the time when the agent joined the waiting-list.

To analyze the misallocation rate under the FCFS buffer-queue policy, denote by $U_\alpha(A^{(k)})$ the utility of an α agent who receives the k-th future A item to arrive. The probability that the k-th item will arrive in exactly t periods is $(p_A)^k (1 - p_A)^{t-k} \binom{t-1}{k-1}$, and we have that

$$U_\alpha(A^{(k)}) = \sum_{t=k}^{\infty} (p_A)^k (1 - p_A)^{t-k} \binom{t-1}{k-1} u_\alpha^{t_0}(A, t_0 + t)$$

$$= -c \cdot k/p + \left(\frac{\delta p_A}{1 - \delta(1 - p_A)} \right)^k \cdot v.$$

Thus, an α agent will be willing to decline a B item if

$$\left(\frac{\delta p_A}{1 - \delta(1 - p_A)} \right)^k \geq \frac{c}{v} \cdot k/p,$$

or, equivalently,

$$k \leq \frac{W \left(\frac{pv \log(\lambda_A)}{c} \right)}{\log(\lambda_A)},$$

where $\lambda_A = \frac{1 - \delta(1 - p_A)}{\delta p_A}$ and $W(\cdot)$ is the Lambert W function. These derivations allows us to extend our results from Section 3 to settings in which agents discount future periods.

Theorem 6. *Let agents have a linear cost of waiting of $c \geq 0$ per period and discount*

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the value of assigned items by $\delta < 1$ per period. Assume $p_A = p_\alpha = p$. The misallocation rate under the FCFS buffer-queue mechanism is

$$\xi^{FCFS} = \frac{2p(1-p)}{(1-p)K^A + pK^B + 1},$$

where $K^A = \left\lfloor W\left(\frac{pv\log(\lambda_A)}{c}\right)/\log(\lambda_A)\right\rfloor$ and $K^B = \left\lfloor W\left(\frac{pv\log(\lambda_B)}{c}\right)/\log(\lambda_B)\right\rfloor$ with $\lambda_A = \frac{1-\delta(1-p_A)}{\delta p_A}, \lambda_B = \frac{1-\delta(1-p_B)}{\delta p_B}$.

B   The Disjoint-Queues Mechanism

This section considers the disjoint-queues (DQ) mechanism, which asks agents to report their preferences as soon as they join the waiting-list. The mechanism holds two separate queues, one for A items and one for B items, and asks agents to select and join a single queue. Agents observe the length of both queues when they make their choice. Both queues follow an FCFS policy. Once agents join a queue, they wait in that queue until they are assigned to the item of that queue.[42]

Misallocation happens under the DQ mechanism for similar reasons that misallocation happens under the FCFS buffer-queue mechanism: the random arrivals of agents and items may result in a temporary imbalance between the demand from agents and the available supply of items. Under the DQ mechanism, this imbalance is realized in the form of a difference between the expected wait in the two queues. When this difference grows too large, agents will join the queue with a shorter wait regardless of their type, possibly resulting in misallocation.

In contrast to buffer-queue mechanisms, the analysis of the DQ mechanism requires a specification of the agent arrival process. For simplicity, we assume that in each period, one new agent joins the waiting-list and then one item arrives. In addition, assume that initially both queues are of equal length.

We capture the dynamics of the DQ mechanism using techniques similar to those used in section 5 and Appendix C.1. Denote by ℓ_A the number of agents in the A queue and by ℓ_B the number of agents in the B queue. By assumption, each period one agent joins one of the queues and one agent is assigned, and therefore the total number of agents remains a constant we denote by $2M$. We assume $2M$ is sufficiently large so that neither queue is ever empty. The Markov chain used to capture the dynamics of the DQ mechanism is

---

[42]This mechanism is a simplified version of mechanisms commonly used by public housing authorities, where applicants are asked to select a single project-specific waiting-list they would like to join (e.g., the New York City Housing Authority (New York City Public Housing Authority, 2015)).

described within the proof of the following Lemma.

Lemma 10. *Assume $p_A = p_\alpha = p$. For all sufficiently large M, the misallocation rate under the disjoint-queues mechanism is*

$$\xi^{DQ} = \frac{2p(1-p)}{\lfloor p(1-p)\,\bar{w} - (1-2p)M \rfloor + \lfloor p(1-p)\,\bar{w} + (1-2p)\,M \rfloor + 2}.$$

Notice $\xi^{DQ} \approx \frac{2p(1-p)}{2p(1-p)\bar{w}+2}$ (by ignoring the integer constraints), which is similar to the misallocation rate under an FCFS buffer-queue mechanism. The difference is due to the assumption that in each period, one item and one agent arrives. A different arrival process can lead to a different misallocation probability.

Proof. When an agent joins the waiting-list, he observes the length of both queues. The number of misallocated items is equal to the number of agents who join their mismatched queue. To calculate the latter, we establish a Markov chain that captures the dynamics of this system.

Let $\Delta = \ell_A/p - \ell_B/(1-p)$ be the difference between the expected wait of the two queues. An α agent will prefer to join the A queue if

$$v - c \cdot \frac{\ell_A}{p} \geq -c \cdot \frac{\ell_B}{1-p},$$

or

$$\Delta \leq v/c = \bar{w}.$$

Similarly, a β agent will prefer to join the B queue if $-\Delta \leq \bar{w}$. Thus, for sufficiently large M, neither queue is ever empty.

We can capture the dynamics of the mechanism using a Markov chain whose states are possible values of Δ. The value of Δ changes when either an agent joins the waiting-list or an item arrives and an agent is assigned. If a period begins with $\Delta \in [-\bar{w}, \bar{w}]$ with probability p_α, the new agent joins the A queue, increasing ℓ_A by 1, and with probability p_β, the new agent joins the B queue, increasing ℓ_B by 1. If $\Delta \notin [-\bar{w}, \bar{w}]$, the new agent always joins the shorter queue. Immediately after the agent joins, the item arrives; with probability p_A, an A arrives and ℓ_A decreases by 1, and with probability p_B, a B arrives and ℓ_B decreases by 1. Thus, in every period, the value of Δ can change by $0, -\gamma$, or $+\gamma$ where $\gamma = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$. We assume that initially both queues hold the same number of agents, and therefore the initial value of Δ is given by $\Delta_0 = \frac{M}{p} - \frac{M}{1-p}$.

We have that $\Delta - \Delta_0$ is always a multiple of γ and write $\Delta = \Delta_0 + k \cdot \gamma$. Denote the maximal value of k such that $\Delta \in [-\bar{w}, \bar{w}]$ by

$$k^A = \max\left\{k \in \mathbb{Z} \mid, \Delta_0 + k \cdot \gamma \le \bar{w}\right\}$$
$$= \max\left\{k \in \mathbb{Z} \mid, k \le (\bar{w} - \Delta_0)/\gamma\right\}$$
$$= \lfloor (\bar{w} - \Delta_0)/\gamma \rfloor,$$

and denote the minimal value of k such that $\Delta \in [-\bar{w}, \bar{w}]$ by

$$k^B = \min\left\{k \in \mathbb{Z} \mid, v - c \cdot \frac{\ell_B}{p_B} \ge -c \cdot \frac{\ell_A}{p_A}\right\}$$
$$= \min\left\{k \in \mathbb{Z} \mid, -\Delta_0 - k \cdot \gamma \le \bar{w}\right\}$$
$$= \min\left\{k \in \mathbb{Z} \mid, k \ge (-\bar{w} - \Delta_0)/\gamma\right\}$$
$$= \lceil (-\bar{w} - \Delta_0)/\gamma \rceil$$
$$= -\lfloor (\bar{w} + \Delta_0)/\gamma \rfloor.$$

We capture the dynamics of the system by a Markov chain whose states S are possible values of k at the beginning of a period

$$S = \{k^B - 1, k^B, \ldots, \ldots, k^A, k^A + 1\},$$

and transition probabilities are given by

$$P\left(s_t | s_{t-1}\right) = \begin{cases} p_\alpha p_B & s_{t-1} \in \left[k^B, k^A\right], \ s_t = s_{t-1} + 1 \\ p_\beta p_A & s_{t-1} \in \left[k^B, k^A\right], \ s_t = s_{t-1} - 1 \\ p_\alpha p_A + p_\beta p_B & s_{t-1} \in \left[k^B, k^A\right], \ s_t = s_{t-1} \\ p_A & s_{t-1} = k^B - 1, \ s_t = s_{t-1} \\ p_B & s_{t-1} = k^B - 1, \ s_t = s_{t-1} + 1 \\ p_B & s_{t-1} = k^A + 1, \ s_t = s_{t-1} \\ p_A & s_{t-1} = k^A + 1, \ s_t = s_{t-1} - 1 \ , \end{cases}$$

We next solve for the stationary distribution π. Equating the flow between $s = k$ and $s' = (k+1)$ for $k, (k+1) \in \left[k^B, k^A\right]$, we get that

$$\pi(k) p_\alpha p_B = \pi(k+1) p_\beta p_A,$$

and given that $p_\alpha = p_A$, we get

$$\pi(k) = \pi(k+1).$$

Denote $\tau = \pi(k)$ for any $k \in [k^B, k^A]$. For $s = k^A + 1$ we get the flow equation

$$\pi(k^A + 1)p_A = \pi(k^A)p_\alpha p_B,$$

giving

$$\pi(k^A + 1) = (1 - p)\tau$$
$$\pi(k^B - 1) = p\tau.$$

Equating the sum of probabilities to 1, we get that

$$1 = \sum_{k \in [k^B, k^A]} \pi(k) + \pi(k^A + 1) + \pi(k^B - 1)$$
$$= \left(k^A - k^B + 1\right)\tau + (1 - p)\tau + p\tau$$
$$= \left(k^A - k^B + 2\right)\tau,$$

and therefore

$$\tau = \frac{1}{k^A - k^B + 2}.$$

Misallocation happens when the state is either $k^A + 1$ or $k^B - 1$ and the new agent is mismatched; therefore, the misallocation rate is

$$\xi^{DQ} = p_\alpha \pi(k^A + 1) + p_\beta \pi(k^B - 1)$$
$$= \frac{2p(1 - p)}{k^A - k^B + 2}$$
$$= \frac{2p(1 - p)}{\lfloor (\bar{w} - \Delta_0)/\gamma \rfloor + \lfloor (\bar{w} + \Delta_0)/\gamma \rfloor + 2}$$
$$= \frac{2p(1 - p)}{\lfloor p(1 - p)\bar{w} - (1 - 2p)M \rfloor + \lfloor p(1 - p)\bar{w} + (1 - 2p)M \rfloor + 2}.$$

$\square$

C   Omitted Proofs

C.1   The Buffer-Queue Markov Chain

In this appendix, we describe the details of the Markov chain that captures the dynamics of the BQ mechanism. The Markov chain captures changes in the buffer-queues from one period to another. Its states are $S = \{-K^B, \ldots, -1, 0, 1, 2, \ldots, K^A\}$, where $k \geq 0$ indicates k agents of type α waiting in the A BQ and $k \leq 0$ indicates $|k|$ agents of type β waiting in the B BQ. To see no other possible states of the system are possible, notice that at any time one of the buffer-queues must be empty.

Recall that the period starts when the mechanism learns the type of the current period's item. If matching agents are in the BQ for the current item,[43] the mechanism assigns the item to the first agent in that BQ; the period ends and the next period starts with one less agent in that BQ.[44] If the BQ of the current item is empty,[45] the mechanism starts offering the current item to new agents. The mechanism continues to approach new agents until either a matching agent is found or a BQ reaches its maximal size. If the BQ reaches its maximal size, the mechanism assigns the item to the next new agent, regardless of his type. If the period started with $|k|$ agents in the BQ, the next period starts with $|\ell|$ agents in the BQ, where $|\ell - k|$ is the number of agents who declined the current item and joined the BQ. The possible transitions between s_t and s_{t+1} are depicted in Figure 3.

Each random transition of the Markov chain corresponds to a period, and transition probabilities P are given by:

$$P\left(s_t = \ell \,|\, s_{t-1} = k\right) = \begin{cases} p_A & k > 0,\ \ell = k - 1 \\ p_B p_\alpha^{\ell-k} p_\beta & k \geq 0,\ k \leq \ell < K^A, \ell \neq 0 \\ p_B p_\alpha^{\ell-k} (p_\alpha + p_\beta) & k \geq 0,\ \ell = K^A \\ p_A p_\alpha + p_B p_\beta & k = \ell = 0 \\ p_B & k < 0,\ \ell = k + 1 \\ p_A p_\beta^{|\ell-k|} p_\alpha & k \leq 0,\ k \geq \ell > -K^B, \ell \neq 0 \\ p_A p_\beta^{|\ell-k|} (p_\beta + p_\alpha) & k \leq 0,\ \ell = -K^B. \end{cases}$$

For example, if the period begins in state $s_t = 1$, that is, one α agent is waiting in the BQ, the probability that the period will end in state $s_{t+1} = 2 < K^A$ is $p_B p_\alpha p_\beta$. To calculate this probability, observe that the system accumulates another agent in the BQ when the following occurs. First, a B item arrives, which occurs with probability p_B. The B item is offered to a new agent, who declines the item and joins the BQ, which occurs if the agent is of type α with probability p_α. The item is then offered to the following new agent, who accepts the item, which occurs if that agent is of type β with probability p_β. If $K^A = 2$, the last agent will accept the item even if he is of type α, and so the probability of this transition becomes $p_B p_\alpha (p_\alpha + p_\beta) = p_B p_\alpha$.

We refine this Markov chain to describe the mechanism within a period using the

---

[43]That is, an A item arrived and there are $s_t = k > 0$ agents of type α, or a B item arrived and $s_t = k < 0$.

[44]That is, if $k > 0$, then the next period starts with state $s_{t+1} = k - 1$, and if $k < 0$, the next period starts with state $s_{t+1} = k + 1$.

[45]That is, the current item is B and $k \geq 0$, or the current item is A and $k \leq 0$.

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extended set of states $\hat{S}$ defined by

$$
\begin{aligned}
\hat{S} &= S^\phi \cup S^A \cup S^B \\
&= \{(k, \phi) \mid -K^B \leq k \leq K^A\} \\
&\quad \cup \{(-k, A) \mid 0 \leq k \leq K^B\} \\
&\quad \cup \{(k, B) \mid 0 \leq k \leq K^A\} \,.
\end{aligned}
$$

The Markov chain on $\hat{S}$ is depicted in Figure 11. Using $\hat{S}$, we can generate the transitions over S by restricting attention to visits to S^ϕ states. Every period begins and ends in a state in S^ϕ. We move from a state in S^ϕ when an item arrives. For example, suppose the initial state is (k, ϕ) for $0 < k < K^A$. If an A item arrives, it is assigned to the first agent in the A BQ, the system transitions to state $(k-1, \phi)$, and the period ends. If a B item arrives, the system transitions to state (k, B). The transitions from a state (k, B) correspond to the mechanism approaching a new agent, and the following state depends on the type of the agent. If the new agent is of type β, he takes the current item, the system transitions to state (k, ϕ), and the period ends. If the new agent is of type α, he declines the item and joins the A BQ, the system transitions to state $(k+1, B)$, and the period continues.

For calculation purposes, the state space $\hat{S}$ has the advantage that all transitions are between adjacent states. Using the state space $\hat{S}$ and the flow equations of the Markov chain, we derive the stationary distribution that describes the long-run behavior of the system. Note that only two transitions in the chain imply misallocation: the transition from (K^A, B) to (K^A, ϕ) when an α agent is drawn, and the symmetric transition from

$(-K^B, A)$ to $(-K^B, \phi)$ when a β is drawn. Transition probabilities are given by

$$P\left(s|(k,\phi)\right) = \begin{cases} p_A & s = (k-1,\phi) \\ p_B & s = (k,B) \end{cases} \qquad k > 0$$

$$P\left(s|(k,\phi)\right) = \begin{cases} p_A & s = (k,A) \\ p_B & s = (k+1,\phi) \end{cases} \qquad k < 0$$

$$P\left(s|(0,\phi)\right) = \begin{cases} p_A & s = (0,A) \\ p_B & s = (0,B) \end{cases}$$

$$P\left(s|(k,B)\right) = \begin{cases} p_\alpha & s = (k+1,B) \\ p_\beta & s = (k,\phi) \end{cases} \qquad 0 \le k < K^A$$

$$P\left(s|(K^A,B)\right) = \begin{cases} p_\alpha + p_\beta & s = (K^A,\phi) \end{cases}$$

$$P\left(s|(k,A)\right) = \begin{cases} p_\alpha & s = (k,\phi) \\ p_\beta & s = (k-1,A) \end{cases} \qquad -K^B < k \le 0$$

$$P\left(s|(-K^B,A)\right) = \begin{cases} p_\alpha + p_\beta & s = (-K^B,\phi) \end{cases}.$$
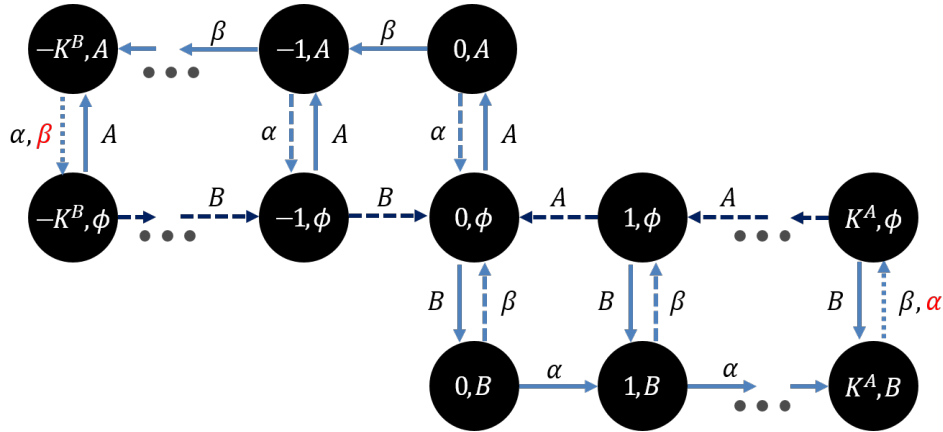
Figure 11: The Markov chain for the state space $\hat{S}$

Using the Markov chain on $\hat{S}$, we can calculate the stationary distribution for both S and $\hat{S}$.

Lemma 11. *The Markov chain is ergodic and its stationary distribution π over $\hat{S}$ is*

$$\pi\left(k,\phi\right) = \begin{cases} \left(\frac{p_\alpha}{p_A}\right)^k p_B\, \pi\left(0,\phi\right) & k > 0 \\ \left(\frac{p_\beta}{p_B}\right)^{|k|} p_A\, \pi\left(0,\phi\right) & k < 0 \end{cases}$$

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and

$$\pi(k, B) = \begin{cases} \pi(k, \phi) & k > 0 \\ p_B \pi(0, \phi) & k = 0 \end{cases} \qquad \pi(k, A) = \begin{cases} \pi(k, \phi) & k < 0 \\ p_A \pi(0, \phi) & k = 0 \end{cases}$$

with

$$\pi(0, \phi) = \begin{cases} \dfrac{1}{2} \dfrac{1}{p_B K^A + p_A K^B + 1} & p_A = p_\alpha \\[2ex] \dfrac{1}{2} \dfrac{p_A - p_\alpha}{p_A p_\beta \left(\frac{p_\beta}{p_B}\right)^{K^B} - p_B p_\alpha \left(\frac{p_\alpha}{p_A}\right)^{K^A}} & p_A \neq p_\alpha. \end{cases}$$

Proof of Lemma 11. It is clear that all states in the Markov chain on $\hat{S}$ are recurrent. Let us denote the stationary distribution by π, where $\pi(k)$ is the stationary probability of state (k, ϕ), and $\pi^B(k)$ is the stationary probability of (k, B) (and likewise for π^A). The balance equations for $0 \le k < K^A$ are

$$\begin{aligned} \pi(k) &= p_A \pi(k+1) + p_\beta \pi^B(k) \\ \pi^B(k) &= p_\alpha \pi^B(k-1) + p_B \pi(k). \end{aligned}$$

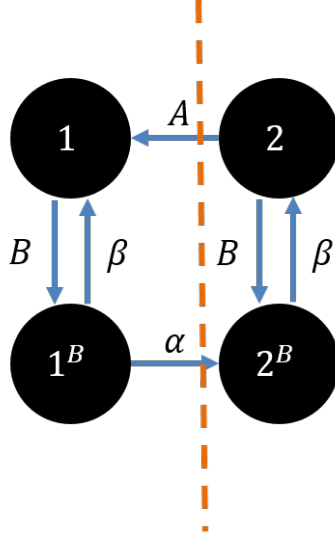The flow through the cut between $s \le k$ and $s \ge k+1$ must be zero (see Figure 12), and therefore[46]

Figure 12: The cut between $s \le 1$ and $s \ge 2$

$$p_A \pi(k+1) = p_\alpha \pi^B(k).$$

---

[46]See Caldentey et al. (2009).

Together we get that for $0 \leq k < K^A$,

$$
\begin{aligned}
\pi(k) &= p_A \pi(k+1) + p_\beta \frac{p_A}{p_\alpha} \pi(k+1) \\
&= p_A \frac{p_\alpha + p_\beta}{p_\alpha} \pi(k+1) \\
&= \frac{p_A}{p_\alpha} \pi(k+1),
\end{aligned}
$$

and

$$
\begin{aligned}
\pi^B(k) &= \frac{p_A}{p_\alpha} \pi(k+1) \\
&= \pi(k).
\end{aligned}
$$

By induction, for $0 < k \leq K^A$ we have

$$
\pi(k) = p_B \left(\frac{p_\alpha}{p_A} \right)^k \pi(0).
$$

The balance equations for $(0, B)$ yields

$$
\pi^B(0) = p_B \pi(0).
$$

We get $\pi(0)$ by normalizing the total probability to 1. When $p_A = p_\alpha = p$, we have

$$
\begin{aligned}
1 &= \sum_{k=1}^{K^A} \left(\pi(k) + \pi^B(k) \right) + \sum_{k=1}^{K^B} \left(\pi(-k) + \pi^A(-k) \right) + \pi(0) + \pi^B(0) + \pi^A(0) \\
&= 2 \sum_{k=1}^{K^A} p_B \pi(0) + 2 \sum_{k=1}^{K^B} p_A \pi(0) + \pi(0) + p_B \pi(0) + p_A \pi(0) \\
&= 2 \ \pi(0) \left(p_B K^A + p_A K^B + 1 \right),
\end{aligned}
$$

implying that

$$
\pi(0) = \frac{1}{2} \frac{1}{(1-p)K^A + pK^B + 1}.
$$

When $p_A \neq p_\alpha$,

$$
\begin{aligned}
1 &= \sum_{k=1}^{K^A} \left(\pi(k) + \pi^B(k) \right) + \sum_{k=1}^{K^B} \left(\pi(-k) + \pi^A(-k) \right) + \pi(0) + \pi^B(0) + \pi^A(0) \\
&= 2 \sum_{k=0}^{K^A} p_B \left(\frac{p_\alpha}{p_A} \right)^k \pi(0) + 2 \sum_{k=0}^{K^B} p_A \left(\frac{p_\beta}{p_B} \right)^k \pi(0) \\
&= 2\pi(0) \left(p_B \frac{p_\alpha \left(\frac{p_\alpha}{p_A} \right)^{K^A} - p_A}{p_\alpha - p_A} + p_A \frac{p_\beta \left(\frac{p_\beta}{p_B} \right)^{K^B} - p_B}{p_\beta - p_B} \right) \\
&= 2\pi(0) \frac{p_A p_\beta \left(\frac{p_\beta}{p_B} \right)^{K^B} - p_B p_\alpha \left(\frac{p_\alpha}{p_A} \right)^{K^A}}{p_A - p_\alpha},
\end{aligned}
$$

implying

$$
\pi(0) = \frac{1}{2} \frac{p_A - p_\alpha}{p_A p_\beta \left(\frac{p_\beta}{p_B} \right)^{K^B} - p_B p_\alpha \left(\frac{p_\alpha}{p_A} \right)^{K^A}},
$$

which converges to the former expression when $p_\alpha \to p_A$. $\qquad \square$

C.2 Proofs from Section 2

Proof of Lemma 1. Consider an arbitrary assignment μ and an arrival process specified by $\chi : \mathcal{I} \to \{t \geq 0\}$ where $\chi(i)$ is the arrival time of agent $i \in \mathcal{I}$. Assume the world ends after period T and let $\mathcal{I}_T = \{i \in \mathcal{I} \mid \chi(i) \leq T\}$ denote the set of agents that arrive before period T, and let $\mathcal{I}_T(\mu) = \{\mu(t) \mid t \leq T\}$ denote the set of agents that were assigned under μ before period T. Let $\xi_t \in \{0, 1\}$ be an indicator equal to 1 if the item x_t is misallocated under μ. The sum of agents utilities up to time T under μ is

$$
\mathrm{WF}_T = \sum_{t=0}^{T} \left((1 - \xi_t) \cdot v + \xi_t \cdot 0 - c \cdot (t - \chi(\mu(t))) \right) - \sum_{i \in \mathcal{I}_T \setminus \mathcal{I}_T(\mu)} c \cdot (T - \chi(i))
$$

where the first summation gives the total utility of agents in $\mathcal{I}_T(\mu)$ and the second summation gives the total utility of the remaining unassigned agents. Rewriting, we have that

$$
\mathrm{WF}_T = v \cdot \left(T - \sum_{t=0}^{T} \xi_t \right) + \sum_{t=0}^{T} c \cdot (T - t) - \sum_{i \in \mathcal{I}_T} c \cdot (T - \chi(i))
$$

Since the last two arguments do not depend on μ, they will cancel out when we take the difference between the welfare under the two assignments μ, μ'. Therefore the relative welfare of an assignment depends only on the number of misallocations $\sum_{t=0}^{T} \xi_t$. $\qquad\square$

Proof of Lemma 2. The lemma follows from the observation that the full information mechanism is equivalent to a buffer-queue mechanism with $K_\alpha = K_\beta = M - 1$ together with Corollary 1. $\qquad\square$

C.3 Proofs from Section 3

Proof of Lemma 3. Consider an α agent in position k. The α agent always accepts an A item. If the α agent is offered a B item, it must be that the agents in positions $1, \ldots, k-1$ declined the B item and are waiting for an A.[47] Thus, the agent's expected wait for an A is the expected number of periods until k copies of A arrive, which is k/p_A. Thus, the α agent will decline a B if and only if $v - c \cdot k/p_A \geq 0$, or $k \leq p_A v/c = p_A \bar{w}$. Symmetrically, a β agent in position k declines an A item if and only if $k \leq p_B \bar{w}$. $\qquad\square$

Proof of Theorem 1. Theorem 1 is a direct corollary of Theorem 3 and Lemma 5.

The expression for the WFL are obtained by substituting in the expression for the misallocation rate and the formulas for the maximal BQ sizes. When the system is balanced, the WFL is

$$
\begin{aligned}
\text{WFL}^{\text{FCFS}} &= v \xi^{\text{FCFS}} \\
&= v \frac{2p(1-p)}{(1-p)K^A + pK^B + 1} \\
&= v \frac{2p(1-p)}{(1-p)\lfloor p\bar{w} \rfloor + p \lfloor (1-p)\bar{w} \rfloor + 1} .
\end{aligned}
$$

When the system is unbalanced, the WFL is

$$
\begin{aligned}
\text{WFL}^{\text{FCFS}} &= v \left(\xi^{\text{FCFS}} - |p_A - p_\alpha| \right) \\
&= v \cdot (p_A - p_\alpha) \frac{(p_\beta/p_B)^{K^B+1} + (p_\alpha/p_A)^{K^A+1}}{(p_\beta/p_B)^{K^B+1} - (p_\alpha/p_A)^{K^A+1}} - v \cdot |p_A - p_\alpha| .
\end{aligned}
$$

If $p_\alpha < p_A$,

---

[47]To see this claim is true, observe that the problem of the agent in the first position is stationary, and he will either immediately take a mismatched B or wait for an A (by assumption, the agent waits for an A when both options give the same utility). This argument implies the problem of the agent in the second position is stationary, and the claim follows by induction.

$$v \cdot (p_A - p_\alpha) \frac{\left(p_\beta/p_B\right)^{K^B+1} + \left(p_\alpha/p_A\right)^{K^A+1}}{\left(p_\beta/p_B\right)^{K^B+1} - \left(p_\alpha/p_A\right)^{K^A+1}} - v \cdot (p_A - p_\alpha)$$

$$= v \cdot (p_A - p_\alpha) \frac{2\left(p_\alpha/p_A\right)^{K^A+1}}{\left(p_\beta/p_B\right)^{K^B+1} - \left(p_\alpha/p_A\right)^{K^A+1}}$$

$$= \frac{2v \cdot |p_A - p_\alpha|}{\left(p_\alpha/p_A\right)^{-(K^A+1)} \left(p_\beta/p_B\right)^{K^B+1} - 1}.$$

If $p_\alpha > p_A$, we perform similar operations and obtain

$$v \cdot (p_A - p_\alpha) \frac{\left(p_\beta/p_B\right)^{K^B+1} + \left(p_\alpha/p_A\right)^{K^A+1}}{\left(p_\beta/p_B\right)^{K^B+1} - \left(p_\alpha/p_A\right)^{K^A+1}} + v \cdot (p_A - p_\alpha)$$

$$= \frac{2v \cdot |p_A - p_\alpha|}{\left(p_\alpha/p_A\right)^{K^A+1} \left(p_\beta/p_B\right)^{-(K^B+1)} - 1}.$$

We can combine the results from these two cases into one expression:

$$\mathrm{WFL}^{\mathrm{FCFS}} = \frac{2v\,|p_A - p_\alpha|}{\left(\left(p_\alpha/p_A\right)^{K^A+1} \left(p_\beta/p_B\right)^{-(K^B+1)}\right)^{sgn(p_\alpha - p_A)} - 1}.$$

□

Proof of Corollary 1. The maximal sizes in the FCFS BQ system are $K^A = \lfloor p_A \bar{w} \rfloor = \lfloor p_A v/c \rfloor$ and $K^B = \lfloor p_B \bar{w} \rfloor = \lfloor p_B v/c \rfloor$. As $c \to 0$, we have that $K^A, K^B \to \infty$. The waiting cost c affects ξ and WFL only through K^A and K^B. Hence, by Theorem 3 we have that

$$\lim_{c \to 0} \xi^{\mathrm{FCFS}} = \lim_{K^A, K^B \to \infty} \xi^{\mathrm{FCFS}} = |p_A - p_\alpha|,$$

and

$$\lim_{c \to 0} \mathrm{WFL}^{\mathrm{FCFS}} = \lim_{K^A, K^B \to \infty} v\left(\xi^{\mathrm{FCFS}} - |p_A - p_\alpha|\right) = 0.$$

□

Proof of Corollary 2. If the system is balanced, we have that

$$\lim_{v \to \infty} \mathrm{WFL}^{\mathrm{FCFS}} = \lim_{v \to \infty} v \frac{2p(1-p)}{(1-p)\lfloor pv/c \rfloor + p\lfloor(1-p)v/c\rfloor + 1}.$$

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Because $x - 1 \le \lfloor x \rfloor \le x + 1$, we have that

$$(1 - p) \lfloor pv/c \rfloor + p \lfloor (1 - p)v/c \rfloor + 1 = 2p(1 - p)v/c + r(v)$$

for $r(v)$ satisfying $-10 \le r(v) \le 10$ for all $v > 0$. Therefore,

$$
\begin{aligned}
\lim_{v \to \infty} \text{WFL}^{\text{FCFS}} &= \lim_{v \to \infty} v \frac{2p(1 - p)}{2p(1 - p)v/c + r(v)} \\
&= \lim_{v \to \infty} c \cdot \frac{2p(1 - p)}{2p(1 - p) + r(v) \cdot c/v} \\
&= c.
\end{aligned}
$$

If $p_\alpha > p_A$, we have that

$$
\begin{aligned}
\text{WFL}^{\text{FCFS}} &= \frac{2v \cdot (p_\alpha - p_A)}{\left(p_\alpha/p_A\right)^{K^A + 1} \left(p_\beta/p_B\right)^{-K^B - 1} - 1} \\
&= 2 \cdot (p_\alpha - p_A) \frac{v \cdot b^{K^B + 1}}{a^{K^A + 1} - b^{K^B + 1}} \\
&= 2 \cdot (p_\alpha - p_A) \frac{v \cdot b^{\lfloor p_B v/c \rfloor + 1}}{a^{\lfloor p_A v/c \rfloor + 1} - b^{\lfloor p_B v/c \rfloor + 1}},
\end{aligned}
$$

where $a = p_\alpha/p_A > 1$ and $b = p_\beta/p_B < 1$. Because $\lim_{v \to \infty} v \cdot b^{\lfloor p_B v/c \rfloor + 1} = 0$ and $\lim_{v \to \infty} a^{\lfloor p_A v/c \rfloor + 1} - b^{\lfloor p_B v/c \rfloor + 1} = \infty$, we have that

$$
\begin{aligned}
\lim_{v \to \infty} \text{WFL}^{\text{FCFS}} &= \lim_{v \to \infty} 2 \cdot (p_\alpha - p_A) \frac{v \cdot b^{\lfloor p_B v/c \rfloor + 1}}{a^{\lfloor p_A v/c \rfloor + 1} - b^{\lfloor p_B v/c \rfloor + 1}} \\
&= 0.
\end{aligned}
$$

A symmetric argument shows that if $p_\alpha < p_A$, we have $\lim_{v \to \infty} \text{WFL}^{\text{FCFS}} = 0$. $\qquad \square$

C.4   Proofs from Section 4

Proof of Theorem 2. In the unique equilibrium under the information disclosure Υ^*, an α agent joins the buffer-queue if he is offered a B and informed that $k \in \{1, \ldots, 2p_A\bar{w} - 1\}$, and takes the current item otherwise. To see that this is an equilibrium, observe that $k \ge 2p_A\bar{w}$ implies the expected wait is above $\bar{w}$, and therefore any α agent best responds by taking the current item. By Lemma 11 in Appendix C.1, in equilibrium an α agent who is informed that $k < 2p_A\bar{w}$ is equally likely to be in either of the positions $1, \ldots, \lfloor 2p_A\bar{w} \rfloor - 1$. Thus, the agent believes his expected wait is $\lfloor 2p_A\bar{w} \rfloor / 2p_A \le \bar{w}$ and best responds by waiting for an A. By Little's law, the expected wait conditional on being informed that $k < 2p_A\bar{w}$ is lower than $\bar{w}$ if some agents do not join the buffer-queue after being informed that $k < 2p_A\bar{w}$. Therefore, it is a dominant strategy for an α agent to join the buffer-queue

if offered a B and informed that $k < 2p_A \bar{w}$.

The fact that information disclosure Υ^* gives the minimal welfare loss follows from the proof of Theorem 7 in online appendix D. To see that, observe that without loss we can restrict attention to information disclosures which send a recommendation whether mismatched agents should take the item or join the buffer-queue, and equilibria in which the agents follow the recommended action. The expected incentive compatibility constraint is captured by the optimization problem in equation 3 in the proof, and the optimality of LIEW implies the optimality of Υ^*. □

C.5 Proofs from Section 5

Proof of Lemma 4. Follows from Appendix C.1. □

Proof of Theorem 3. Note that every agent is approached exactly once, and every agent who declines an item will eventually receive his preferred item (after waiting in the BQ). Therefore, the misallocation rate is equal to the fraction of offers in which a mismatched agents takes the current item.

An α agent takes a B item if the system is in state (K^A, B) and the approached agent is of type α. Symmetrically, a β agent takes an A item if the system is in state $(-K^B, A)$ and the approached agent is of type β. To obtain the probability of these events conditional on the system making an offer to an agent, we use the stationary distribution calculated in Lemma 11 in Appendix C.1. Denote the conditional probability of a state where the system makes an offer to an agent by

$$\hat{\pi}^X(k) = \frac{\pi^X(k)}{\sum_k \pi^B(k) + \sum_k \pi^A(k)}$$
$$= 2\pi^X(k)$$

for $X \in \{A, B\}$. From Lemma 11, we have that if the system is balanced, the misallocation rate is

$$\xi = p \cdot \hat{\pi}^B(K^A) + (1 - p) \cdot \hat{\pi}^A(-K^B)$$
$$= \frac{2p(1 - p)}{(1 - p)K^A + pK^B + 1}.$$

If $p_\alpha \neq p_A$, the misallocation rate is

$$\xi = p_\alpha \cdot \hat{\pi}^B(K^A) + p_\beta \cdot \hat{\pi}^A(-K^B)$$

$$= p_\alpha \cdot p_B \left(\frac{p_\alpha}{p_A}\right)^{K^A} \cdot \frac{p_A - p_\alpha}{p_A p_\beta \left(\frac{p_\beta}{p_B}\right)^{K^B} - p_B p_\alpha \left(\frac{p_\alpha}{p_A}\right)^{K^A}}$$

$$+ p_\beta \cdot p_A \left(\frac{p_\beta}{p_B}\right)^{K^B} \cdot \frac{p_A - p_\alpha}{p_A p_\beta \left(\frac{p_\beta}{p_B}\right)^{K^B} - p_B p_\alpha \left(\frac{p_\alpha}{p_A}\right)^{K^A}}$$

$$= (p_A - p_\alpha) \frac{p_A p_\beta \left(\frac{p_\beta}{p_B}\right)^{K^B} + p_B p_\alpha \left(\frac{p_\alpha}{p_A}\right)^{K^A}}{p_A p_\beta \left(\frac{p_\beta}{p_B}\right)^{K^B} - p_B p_\alpha \left(\frac{p_\alpha}{p_A}\right)^{K^A}}$$

$$= (p_A - p_\alpha) \frac{\left(\frac{p_\beta}{p_B}\right)^{K^B+1} + \left(\frac{p_\alpha}{p_A}\right)^{K^A+1}}{\left(\frac{p_\beta}{p_B}\right)^{K^B+1} - \left(\frac{p_\alpha}{p_A}\right)^{K^A+1}}.$$

We now show that the misallocation rate ξ is monotonically decreasing in K^A, K^B. This is immediate if the system is balanced. If $p_\alpha \neq p_A$, consider the derivatives $\frac{d\xi}{dK^A}$ and $\frac{d\xi}{dK^B}$. To simplify notation, we substitute $a = p_\alpha/p_A$ and $b = p_\beta/p_B$:

$$\frac{d\xi}{dK^A} = \frac{d}{dK^A}\left[(p_A - p_\alpha)\frac{b^{K^B+1} + a^{K^A+1}}{b^{K^B+1} - a^{K^A+1}}\right]$$

$$= \frac{(p_A - p_\alpha) \cdot (\log a) \cdot a^{K^A+1}b^{K^B+1}}{\left(b^{K^B+1} - a^{K^A+1}\right)^2}$$

$$\frac{d\xi}{dK^B} = \frac{-(p_A - p_\alpha) \cdot (\log b) \cdot a^{K^A+1}b^{K^B+1}}{\left(b^{K^B+1} - a^{K^A+1}\right)^2}.$$

If $p_A > p_\alpha$ (or $a < 1$), we have that $p_A - p_\alpha > 0$ and $\log a < 0$, which implies $\frac{d\xi}{dK^A} < 0$. If $p_A < p_\alpha$ (or $a > 1$), $p_A - p_\alpha < 0$ and $\log a > 0$, which again implies $\frac{d\xi}{dK^A} < 0$. A symmetric argument shows $\frac{d\xi}{dK^B} < 0$.

We next show the misallocation rate ξ converges to $|p_A - p_\alpha|$ as $K^A \to \infty$ or $K^B \to \infty$. If the system is balanced, this is immediate. For an unbalanced system, first consider the case in which $p_A > p_\alpha$, $a < 1$, and therefore $p_B < p_\beta$, $b > 1$:

$$\lim_{K^A\to\infty} \xi = \lim_{K^A\to\infty} (p_A - p_\alpha) \frac{b^{K^B+1} + a^{K^A+1}}{b^{K^B+1} - a^{K^A+1}}$$

$$= (p_A - p_\alpha) \frac{b^{K^B+1} + 0}{b^{K^B+1} - 0}$$

$$= p_A - p_\alpha = |p_A - p_\alpha|$$

$$\lim_{K^B\to\infty} \xi = \lim_{K^B\to\infty} (p_A - p_\alpha) \frac{b^{K^B+1} + a^{K^A+1}}{b^{K^B+1} - a^{K^A+1}}$$

$$= (p_A - p_\alpha) \cdot 1 = |p_A - p_\alpha|.$$

If $p_A < p_\alpha, a > 1$ and $p_B > p_\beta, b > 1$, we have that

$$\lim_{K^A\to\infty} \xi = (p_A - p_\alpha) \cdot -1$$

$$= p_\alpha - p_A = |p_A - p_\alpha|$$

$$\lim_{K^B\to\infty} \xi = (p_A - p_\alpha) \cdot \frac{0 + a^{K^A+1}}{0 - a^{K^A+1}}$$

$$= p_\alpha - p_A = |p_A - p_\alpha|.$$

Hence,

$$\lim_{K^A\to\infty} \xi = \lim_{K^B\to\infty} \xi = |p_A - p_\alpha|.$$

$\square$

Proof of Lemma 5. To characterize equilibrium behavior, consider an α agent in position k. The α agent always accepts an A item. If the α agent is offered a B item, it must be that the agents in positions $1, \ldots, k-1$ declined the B item and are waiting for an A.[48] Thus, the agent faces expected wait for an A equal to $w_k = k/p_A$, which is the expected number of periods until k copies of A arrive. Thus, the α agent will decline a B if and only if $w_k \leq \bar{w}$, or $k \leq p_A v/c = p_A \bar{w}$. Symmetrically, a β agent in position k declines an A item if and only if $k \leq p_B \bar{w}$.

Observe that we can equivalently describe the waiting list with declines as an IC BQ mechanism with a FCFS BQ policy. The choice of an α agent in position k to decline a B item is equivalent to that agent choosing to join the k-th position in the A FCFS BQ,

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[48]To see this claim is true, observe that the problem of the agent in the first position is stationary, and he will either immediately take a mismatched B or wait for an A (by assumption, the agent waits for an A when both options give the same utility). This argument implies the problem of the agent in the second position is stationary, and the claim follows by induction.

because under both the agent faces an expected wait of $w_k = k/p_A$, as for an A item they need to wait until k copies of A items arrive. The maximal size of the A BQ is given by the maximal K^A satisfying the IC constraint $w_K^A = K^A/p_A \leq \bar{w}$. $\qquad \square$

C.6 Proofs from Section 6

Proof of Lemma 6 and Lemma 6'. Observe that α agents must all be assigned before any agent joins the B BQ, and therefore their expected wait cannot be affected by the B BQ policy or the decisions of β agents to join the B BQ. $\qquad \square$

Proof of Theorem 4. By Theorem 3, the misallocation rate when agents are truthful is decreasing in K^A, K^B. Therefore, welfare is increasing in K^A, K^B. The sizes K^A, K^B are constrained by the IC requirement, which can be decomposed by Lemma 6 to requiring separately that $\left(K^A, \varphi^A\right)$ and $\left(K^B, \varphi^B\right)$ are each IC. This argument reduces the problem to characterizing the maximal K' for which some φ' exists such that (K', φ') is IC.

For ease of notation, we consider the policy for the A BQ and use p_A for the item arrival probability. Let (K, φ) be the policy of the BQ, and let $\{w_k\}_{k=1}^{K}$ be the implied expected waits when agents are truthful (note the expected waits may depend on both p_A and p_α). We establish an upper bound on K for any IC (K, φ) by looking at the expected wait of a randomly drawn agent and using Little's Law.

We consider the dynamics of the BQ, restricting attention to periods when the BQ is not empty. Little's Law (Little 1961) states that if L is the long-term average number of agents in the BQ (conditional on the BQ being non-empty), λ is the long-term average rate at which agents join the BQ, and w is the average time that an agent waits in the BQ, then it holds that

$$L = \lambda \mathsf{w}.$$

Because the number of agents in the BQ is independent of φ (i.e., independent of which agent is selected to receive an item and leave the BQ), we have that L is independent of φ. Using Lemma 11 of Appendix C.1, we can calculate that

$$L = \sum_{k=1}^{K} k \cdot \frac{\pi(k, \phi)}{\sum_{k=1}^{K} \pi(k, \phi)}$$

$$= \begin{cases} \frac{K+1}{2} & p_\alpha = p_A \\ K + \frac{p_A}{p_A - p_\alpha} + \frac{K}{(p_\alpha/p_A)^K - 1} & p_\alpha \neq p_A, \end{cases}$$

where $\pi(k, \phi) / \sum_{k=1}^{K} \pi(k, \phi)$ is the conditional probability that the BQ holds k agents.

We have that $\lambda = p_A$, because the average number of agents that leave the BQ in a period is equal to the probability that an A item arrives (and one agent leaves). Because the number of agents in the BQ is bounded by a constant, the long run average rate at which agents join the BQ is equal to the long-run average rate at which agents leave the BQ.

We now calculate w from $\{w_k\}_{k=1}^K$. Let $\tilde{k}$ be a random variable whose support is $1, \ldots, K$ and $P(\tilde{k} = k) > 0$ is equal to the probability that an agent in the BQ initially joined at position k. Because each agent who joins the BQ does so in some state (k, B), we have that

$$P(\tilde{k} = k) = \frac{\pi(k, B)}{\sum_{k=1}^K \pi(k, B)}.$$

By the law of iterated expectation, we have that

$$\mathsf{w} = \mathbb{E}[w_{\tilde{k}}] = \sum_{k=1}^K w_k \cdot P(\tilde{k} = k).$$

If a BQ policy $\langle K, \varphi \rangle$ is IC, we have that $w_k \leq \bar{w}$, and therefore $\mathsf{w} \leq \bar{w}$. Together with Little's Law, the IC constraint implies

$$\mathsf{w} = L/p_A \leq \bar{w},$$

and therefore $K \leq \kappa^*(\bar{w}, p_\alpha, p_A)$. In other words, for any BQ policy $\langle K, \varphi \rangle$ with $K > \kappa^*(\bar{w}, p_\alpha, p_A)$ at least one position k must exist such that $w_k > \bar{w}$, violating the IC constraint.

An LIEW_K policy satisfies $w_k = \mathsf{w}$ for any $k \leq K$, and is therefore IC for any $K \leq \kappa^*(\bar{w}, p_\alpha, p_A)$. Together, we have that an LIEW_K BQ policy with $K = \kappa^*(\bar{w}, p_\alpha, p_A)$ achieves the maximal K (and therefore, highest welfare) of any IC policy.

To derive an explicit expression for $\kappa^*(\bar{w}, p_\alpha, p_A)$ when $p_\alpha \neq p_A$, we first rewrite

$$\begin{aligned}
L(K) &= K + \frac{p_A}{p_A - p_\alpha} + \frac{K}{(p_\alpha/p_A)^K - 1} \\
&= K + \frac{1}{1 - a} + \frac{K}{a^K - 1} \\
&= \frac{Ka^K}{a^K - 1} + \frac{1}{1 - a},
\end{aligned}$$

where $a = p_\alpha/p_A$. To see that $L(K)$ is monotonically increasing in K, observe that

$$\frac{dL}{dK} = \frac{a^K(a^K - K\log a - 1)}{(a^K - 1)^2}.$$

To see that $a^K - K\log a - 1$ is positive, we can use the fact that for any $x > 0$, $\log x \leq x - 1$. Plugging in $x = a^K$, we obtain $\log a^K \leq a^K - 1$ and therefore $a^K - K\log a - 1 \geq 0$. Hence,

$\frac{dL}{dK} \geq 0$ and L is monotonically increasing in K.

If $p_\alpha > p_A$, we have that $a > 1$ and

$$\lim_{K \to \infty} L(K) = \infty.$$

If $p_\alpha < p_A$, we have that $a < 1$ and

$$\lim_{K \to \infty} L(K) = \frac{\lim_{K \to \infty} K a^K}{\lim_{K \to \infty} a^K - 1} + \frac{1}{1 - a}$$
$$= \frac{0}{-1} + \frac{1}{1 - a}$$
$$= \frac{p_A}{p_A - p_\alpha}.$$

Therefore, if $p_\alpha < p_A$ and $\bar{w} \geq 1/(p_A - p_\alpha)$, it holds that $L(K) \leq p_A \bar{w}$ for any $K \geq 1$, thus showing that if $p_\alpha < p_A$ and $\bar{w} \geq 1/(p_A - p_\alpha)$, we have that

$$\kappa^* (\bar{w}, p_\alpha, p_A) = \infty,$$

and the average expected wait for $K = \infty$ is

$$\mathsf{w} = \lim_{K \to \infty} L(K) / p_A.$$
$$= 1/(p_A - p_\alpha) \leq \bar{w}.$$

Next, we show that if $p_\alpha > p_A$, or $p_\alpha < p_A$ and $\bar{w} < 1/(p_A - p_\alpha)$, a unique solution $K \geq 1$ exists to

$$L(K) = p_A \bar{w}. \tag{1}$$

Observe that

$$L(1) < p_A \bar{w} < \lim_{K \to \infty} L(K),$$

where the left inequality follows because $L(1) = 1$ and by assumption $p_A \bar{w} > 1$. Because $L(K)$ is monotonic and continuous, a unique solution exists. To obtain an explicit expression for the solution, rewrite equation (1) as

$$\frac{K a^K}{a^K - 1} = \frac{p_A (\bar{w} (p_A - p_\alpha) - 1)}{p_A - p_\alpha}. \tag{2}$$

To simplify notation, we substitute $\gamma = \frac{p_A (\bar{w}(p_A - p_\alpha) - 1)}{p_A - p_\alpha}$. The equation can be rewritten as follows:

$$(\gamma - K) a^K = \gamma.$$

Under this notation, the solution to equation (1) is

$$K = \gamma + \frac{W\left(-\gamma a^{-\gamma} \cdot \log a\right)}{\log a},$$

where $W\left(\cdot\right)$ is the Lambert W function. To verify the solution, plug the suggested expression into the LHS of equation (2) and see that

$$(\gamma - K)a^K = - \frac{W\left(-\gamma a^{-\gamma} \cdot \log a\right)}{\log a} a^{\gamma + \frac{W\left(-\gamma a^{-\gamma} \cdot \log a\right)}{\log a}}$$

$$= - a^{\gamma} \cdot \frac{W\left(-\gamma a^{-\gamma} \cdot \log a\right)}{\log a} a^{\frac{W\left(-\gamma a^{-\gamma} \cdot \log a\right)}{\log a}} .$$

By the definition of the Lambert W function, we have that

$$\frac{W(x \log a)}{\log a} a^{\frac{W(x \log a)}{\log a}} = x .$$

Plugging $x = -\gamma a^{-\gamma}$, we obtain

$$a^{\gamma} \cdot \frac{W\left(-\gamma a^{-\gamma} \cdot \log a\right)}{\log a} a^{\frac{W\left(-\gamma a^{-\gamma} \cdot \log a\right)}{\log a}}$$

$$= - a^{\gamma} \cdot -\gamma a^{-\gamma}$$

$$= \gamma ,$$

and thus K solves equation (2). Therefore, if $p_\alpha > p_A$, or $p_\alpha < p_A$ and $\bar{w} < 1/\left(p_A - p_\alpha\right)$, we have that

$$\kappa^* \left(\bar{w}, p_\alpha, p_A\right) = \left\lfloor \gamma + \frac{W\left(-\gamma a^{-\gamma} \cdot \log a\right)}{\log a} \right\rfloor .$$

$\square$

Proof of Theorem 5. The first steps of the proof parallel the proof of Theorem 4. By Proposition 3, the misallocation rate when agents are truthful is decreasing in K^A, K^B. Therefore, welfare is increasing in K^A, K^B. The sizes K^A, K^B are constrained by the BF-IC requirement, which can be decomposed by Lemma 6 to requiring separately that $\left(K^A, \varphi^A\right)$ and $\left(K^B, \varphi^B\right)$ are each BF-IC. This argument reduces the problem to characterizing the maximal K' for which some φ' exists such that (K', φ') is BF-IC.

For ease of notation, we consider the policy for the A BQ and use p_A for the item arrival probability. Let (K, φ) be the policy of the BQ, and let $w_{k,\sigma}$ be the implied expected waits under the belief σ (note these expected waits are independent of p_α).

We establish an upper bound on K for any BF-IC (K, φ) by looking at the expected wait $w_{k,\hat{\sigma}}$ under the belief $\hat{\sigma} \equiv 1$. That is, $w_{k,\hat{\sigma}}$ is the expected wait when every agent

who is offered a B item declines the B item and joins the A BQ. Equivalently, $w_{k,\hat{\sigma}}$ is the expected wait when $p_\alpha = 1$ and all agents are truthful. As in the proof of Theorem 4, we have that the average number of agents in the BQ is given by

$$L = \sum_{k=1}^{K} k \cdot \frac{\pi(k,\phi)}{\sum_{k=1}^{K} \pi(k,\phi)}$$
$$= \frac{K}{1 - p_A^K} - \frac{p_A}{1 - p_A},$$

which is monotonically increasing in K. Let $\mathsf{w}_{\hat{\sigma}}$ denote the average time an agent waits in the BQ. Using Little's Law, we have that

$$\mathsf{w}_{\hat{\sigma}} = L/p_A.$$
$$= \frac{K}{p_A(1 - p_A^K)} - \frac{1}{1 - p_A}.$$

Let K^* be defined by

$$K^* = \max\{K \mid \mathsf{w}_{\hat{\sigma}} \leq \bar{w}\}$$
$$= \kappa^*(\bar{w}, 1, p_A).$$

By the law of iterated expectation, we have

$$\mathsf{w}_{\hat{\sigma}} = \mathbb{E}\left[w_{\tilde{k},\hat{\sigma}}\right] = \sum_{k=1}^{K} w_{k,\hat{\sigma}} \cdot P(\tilde{k} = k).$$

Therefore, if (K, φ) is a BF-IC policy, we have that $\mathsf{w}_{\hat{\sigma}} \leq \bar{w}$ and $K \leq K^*$.

We proceed to show that $\left(K^*, \varphi^{\mathrm{SIRO}}\right)$ is BF-IC. First, observe that under the belief $\hat{\sigma} \equiv 1$ for any positions $k, k' \leq K^*$, we have that $w_{k,\hat{\sigma}} = w_{k',\hat{\sigma}} = \mathsf{w}_{\hat{\sigma}}$. Because SIRO treats agents at all positions equally, the expected wait at the end of the period under SIRO is independent of the agent's position in the BQ. Under $\hat{\sigma}$, any agent joining the BQ believes the BQ will hold K^* agents at the end of the period. Therefore, all agents who join the BQ receive the same expected wait, and because $w_{k,\hat{\sigma}} = w_{k',\hat{\sigma}}$ for any $k, k' \leq K^*$, we have that $w_{k,\hat{\sigma}} = \mathsf{w}_{\hat{\sigma}} \leq \bar{w}$.

Second, we show that under the BQ policy $\left(K, \varphi^{\mathrm{SIRO}}\right)$ for any belief σ and position k, we have that $w_{k,\sigma} \leq w_{k,\hat{\sigma}}$. To see that this inequality holds, consider an agent joining the BQ at position k, and fix the realized sequence of items arriving in future periods. Given the sequence, let $\ell(n)$ be the number of agents in the BQ when the n-th A item in the sequence arrives. The agent waits until an assignment, and conditional on not being assigned earlier, he is assigned the n-th item with probability $1/\ell(n)$. The agent's expected wait increases as $1/\ell(n)$ decreases, and therefore as $\ell(n)$ increases. The belief

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$\hat{\sigma}$ implies the maximal possible $\ell(n)$ for each n, regardless of the item process, as the number of agents $\ell(n)$ is maximized when the BQ reaches its maximal size each time a B item arrives. Averaging over all item arrival sequences, we thus have that $w_{k,\sigma} \leq w_{k,\hat{\sigma}}$.

In summary, we have that $(K^A, \varphi^{\mathrm{SIRO}})$ with $K^A = \kappa^*(\bar{w}, 1, p_A)$ is BF-IC, and for any BF-IC BQ policy (K', φ'), it holds that $K' \leq \kappa^*(\bar{w}, 1, p_A)$. $\hfill\square$

Proof of Lemma 8. Consider a symmetric equilibrium in which an α agent who is offered position k in the A BQ joins with probability $s(k) \in [0, 1]$. Let $\sigma(k) = p_\alpha \cdot s(k)$ denote the correct equilibrium beliefs. We first show that for any s, the expected wait $w_{k,\sigma}$ is increasing in k. Fix the sequence of items arriving in future periods and compare two trajectories starting in period t, the first starting with k agents and the second with $k' > k$ agents. We couple the two trajectories and consider the number of agents in the BQ in each period until the BQ empties under the first trajectory. Let $\hat{t} > t+1$ be the first period in which the two trajectories have an equal number of agents, or the BQ empties under the first trajectory. Note that in every period from $t+1$ to $\hat{t}$, the BQ has strictly fewer agents under the first trajectory. The probability that a given agent in the BQ is assigned the n-th arriving A item is $1/\ell(n)$ if $\ell(n)$ agents are in the BQ at the beginning of period when the n-th item arrives, which is strictly decreasing in $\ell(n)$. If an A item arrives between $t+1$ and $\hat{t}$, a given agent in the BQ has higher assignment probability under the first trajectory. If any agents remain in the BQ under the first trajectory in period $\hat{t}$, their future assignment probabilities are equal under the two trajectories. Averaging over all sequences of item arrivals, we find that under under the first trajectory, a given agent in the BQ has a weakly higher probability of getting assigned in all periods under the first trajectory, and a strictly higher probability in some periods. Therefore, agents face a higher expected wait under the second trajectory, and the expected wait $w_{k,\sigma}$ is strictly monotonically decreasing in k.

The agent's best response implies $s(k) > 0$ only if $w_{k,\sigma} \leq \bar{w}$ and $s(k) < 1$ only if $w_{k,\sigma} \geq \bar{w}$. Together with the monotonicity of $w_{k,\sigma}$, the best response implies the existence of x^* such that[49] $s(k) = \mathbf{1}_{\{k \leq x^*\}} + \mathbf{1}_{\{\lfloor x^* \rfloor < k < \lfloor x^* \rfloor + 1\}}[x^*]$, that is, $s(k) = 1$ for $k \leq \lfloor x^* \rfloor$, $s(k) = 0$ for $k > \lfloor x^* \rfloor + 1$ and $s(k) = [x^*] = x^* - \lfloor x^* \rfloor$ for $k = \lfloor x^* \rfloor + 1$.

The same coupling argument shows $w_{k,\sigma}$ is strictly increasing in x^* (and therefore, a unique equilibrium exists). Let $(K^*, \varphi^{\mathrm{SIRO}})$ be the BF-IC SIRO BQ policy with the maximal $K^* = \kappa^*(\bar{w}, 1, p_A)$, and let $w_{k,\hat{\sigma}}$ be the implied wait under the belief $\hat{\sigma} \equiv 1$. Let $\sigma'(k) = p_\alpha \cdot \mathbf{1}_{\{k \leq K^*\}}$ be the belief that agents join position 1 to K^*. From the proof of

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[49]The fractional part of x^* is denoted by $[x^*] = x^* - \lfloor x^* \rfloor$.

Theorem 5, we have that

$$w_{k,\sigma'} \leq w_{k,\hat{\sigma}} \leq \bar{w} \,,$$

and therefore we must have that $x^* \geq K^*$. $\qquad\square$

Proof of Lemma 9. At the end of a period, all agents in the BQ have the same expected utility. The monotonicity arguments used in the proof of Lemma 8 show an agent is weakly better off if no agents join after him. Thus, the last agent would have chosen to join the BQ if he knew no agents would join after him. Because the last agent prefers joining the BQ to being assigned immediately to a mismatched item, all other agents prefer to stay in the BQ as well. $\qquad\square$

Dynamic Matching in Overloaded Waiting Lists –

Online Appendixes

(not for publication)

D General Dynamic Mechanisms

In this section, we define a more general class of mechanisms, and present assumptions that characterize the class of BQ mechanisms. We also allow for a more general agent arrival process, and define $\mathcal{I}_t$ to be the set of agents who arrived by period t and were not assigned. We restrict attention to mechanisms that operate sequentially as agents and items arrive over time. At any point, the mechanism can take one of two actions: assign the current item to an agent, or collect more information by approaching an agent and asking him to report his type.[50]

A few observations allow us to give a concise representation. Partition $\mathcal{I}_t = \mathcal{W} \cup \mathcal{N}$ as follows. Let $\mathcal{N}$ be the set of agents who were not yet approached by the mechanism, which we refer to as *new agents*. The set $\mathcal{W} = \mathcal{I}_t \setminus \mathcal{N}$ includes the agents who already reported their types (but have not been assigned), whom we refer to as *approached agents*. All new agents are interchangeable to the mechanism,[51] and therefore, when the mechanism approaches a new agent to report his type, which new agent is asked is immaterial. Likewise, the mechanism can assign the current item to a particular approached agent or to an arbitrary new agent.

We assume the arrival process satisfies $|\mathcal{I}_t| \geq M$ for sufficiently large M. In that case, we say the system is overloaded. This assumption allows us to abstract away from the agent's arrival process. The reason is that the number of agents on the waiting-list is immaterial as long as $\mathcal{N} \neq \varnothing$ whenever the mechanism seeks to approach a new agent (because all agents in $\mathcal{N}$ are interchangeable).

Finally, we encode the relevant part of the history known to the mechanism using a state space. The state of the world in period t is encoded as $\omega_t = (x_t, \omega_t^{\mathcal{I}})$, where x_t is the kind of item that arrives in period t and $\omega_t^{\mathcal{I}}$ is a sequence specifying the types of new

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[50]When only two types are possible, any deterministic choice either fully reveals the agent's type or discloses no information.

[51]Recall that past waiting costs are sunk, and agents' preferences in period t are independent of the time spent on the waiting-list.

agents.[52]

Definition 6. A deterministic[53] dynamic matching mechanism $\mathcal{M}$ for an overloaded system is given by a state space S, an initial state $s_0 \in S$, and a transition function

$$\mathcal{T} : s_t \times \omega_t \mapsto s_{t+1} \times a \times k,$$

where $s_t \in S$ is the state at the beginning of period t, $s_{t+1} \in S$ is the state at the end of the period, $\omega_t = (x_t, \omega_t^{\mathcal{I}}) \in \{A, B\} \times \{\alpha, \beta\}^{\mathbb{N}}$ is the state of the world, a is the agent assigned the item, and $k \in \mathbb{N}$ is the number of new agents the mechanism approached during the period.

The definition above abstracts away from the arrival process of agents, allowing any arrival process such that the system is overloaded and the mechanism can always find a new agent when seeking for one. Note that an arrival process may be sufficiently overloaded for some mechanisms but not others. For BQ mechanisms, this assumption is easily satisfied; BQ mechanisms ask at most K agents (where K is the largest possible size of any of the BQs), and any arrival process guaranteeing at least $M = K + 1$ agents are in the system will suffice.

We impose several assumptions to restrict the class of mechanisms we consider. The first assumption requires that the mechanism uses only a finite number of states. It ensures the number of approached agents is bounded, allowing for the assumption that the system is overloaded to be satisfied.

Assumption 1. *The state space S is finite.*

The following assumptions require that the mechanism does not ask for more information than is needed for assigning the current item, and that it does not unnecessarily delay the assignment of items. These assumption ensure the mechanism determines assignments sequentially over time as items arrive.

Assumption 2. *If an item is available for assignment and the mechanism knows of a matching agent, the mechanism assigns the item to a matching agent and ends the period.*

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[52]Specifically, encode ω_t to be the types of agents in the order the mechanism approaches them. For example, $\omega_0 = (\alpha, \beta, \beta, ...)$ denotes that the first agent mechanism approaches is of type α, the second agent approached is of type β, and so on.

[53]The transitions of a deterministic Markovian mechanism are random because of the random realization of agent types and item kinds, but its choice of action given a state is deterministic.

Assumption 3. *When an item arrives and the mechanism does not know of a matching agent, the mechanism either asks an agent to report his type or assigns the item to a new agent (of an unknown type).*[54]

These assumptions imply agents cannot be asked to report their type ahead of time; agents are asked to report their type only if they can be immediately assigned. These assumptions are appealing in that they ensure agents report their preferences close to the actual assignment, and that the mechanism stores a relatively small amount of preference information.

Definition 7. A *dynamic Markovian mechanism* is a dynamic mechanism that satisfies assumptions 1, 2, and 3.

We analyze dynamic Markovian mechanisms by using an extension of the methodology used to analyze BQ mechanisms.

Lemma 12. *The dynamics of any dynamic Markovian mechanism can be described by a Markov chain over a finite state space $\hat{S}$ that can be partitioned to*

$$\hat{S} = \left(\cup_{k \geq 0} S_k^B \cup S_k^\phi \right) \bigcup \left(\cup_{k \leq 0} S_k^A \cup S_k^\phi \right),$$

and we have that:

- *Any transition from a state $s \in S_k^B$ is to a state in $S_{k+1}^B \cup S_k^\phi$.*

- *Any transition from a state $s \in S_{-k}^A$ is to a state in $S_{-k-1}^A \cup S_{-k}^\phi$.*

- *Any transition from a state $s \in S_k^\phi$ for $k > 0$ is to a state in $S_k^B \cup S_{k-1}^\phi$.*

- *Any transition from a state $s \in S_k^\phi$ for $k < 0$ is to a state in $S_k^A \cup S_{k+1}^\phi$.*

- *Any transition from a state $s \in S_0^\phi$ is to a state in $S_0^A \cup S_0^B$.*

- *A period starts and ends with a state in $\cup_k S_k^\phi$.*

In addition, all approached agents are of the same type and are eventually assigned their preferred item.

Proof. By Assumption 2, at the beginning of a period, all approached agents must be of the same type. By Assumption 3, an approached agent is never assigned a mismatched item.

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[54]Assigning the item to a new agent of unknown type is equivalent to asking the agent to report his type and assigning him the item regardless of his answer.

As in Appendix C.1, we can take the Markov chain whose states S describe the mechanism at the end of the period and decompose it to a state space $\hat{S}$ to describe the transitions within a period. Let $\hat{S}$ be the Markov chain that transitions with any possible events within a period: arrival of the current item, assignment of the item, or an agent reporting his type. Because the state space S is finite, it must be that in each period, the mechanism asks a finite number of agents for their type. Therefore, the set $\hat{S}$ of all possible states within a period is finite as well.

We partition $\hat{S}$ into $\{S_k^B\}_k, \{S_k^A\}_k, \{S_k^\phi\}_k$, as follows. For each state $s \in \hat{S}$, let $k = k(s)$ indicate the number of approached agents, as in Appendix C.1: $k \geq 0$ indicates k approached agents of type α, and $k \leq 0$ indicates $|k|$ approached agents of type β. If in state s the mechanism holds an A or B item that is yet to be assigned, we classify state s by $s \in S_k^A$ or $s \in S_k^B$, respectively. States in which the current item was assigned and the type of the next item is not yet revealed are classified by $s \in S_k^\phi$.

Using Assumptions 2 and 3, we can eliminate some of the states in $\hat{S}$. Consider a state $s \in S_k^B$ in which a current B item is to be assigned, and the mechanism does not know of any β approached agents. Suppose the mechanism transitions from s to s' by asking an agent for his type and learning this agent is of type β. By Assumption 2, the mechanism must continue to assign the item to that β agent and end the period in state $s'' \in S_k^\phi$. Thus, we can dispense with such intermediate states s' and replace them by a direct transition to s''.

After the elimination of such intermediate states, the possible transitions from a state $s \in S_k^B$ are either to a state S_{k+1}^B (by asking an α agent) or to a state S_k^ϕ (by either asking a β agent or by assigning the item to a new agent of unknown type). The possible transitions from $s \in S_{-k}^A$ or $s \in S_k^\phi$ are analogous. Any reachable state is in $\left(\cup_{k \geq 0} S_k^B \cup S_k^\phi\right) \bigcup \left(\cup_{k \leq 0} S_k^A \cup S_k^\phi\right)$ and the period always ends with an assignment that leads to a state in $\cup_k S_k^\phi$. $\qquad\square$

Randomized Markovian mechanisms are naturally defined by allowing the mechanism to randomize over possible transitions. The class of BQ mechanisms is characterized by two additional assumptions.

Assumption 4. *The mechanism's decisions depend only on the number and type of approached agents.*

Assumption 5. *The mechanism's decisions are deterministic, except for possibly a random selection of an approached agent when assigning an item.*

Lemma 13. *A dynamic Markovian mechanism that satisfies Assumptions 4 and 5 is a buffer-queue mechanism. Conversely, any buffer-queue mechanism is a dynamic Markovian mechanism that satisfies Assumptions 4 and 5.*

Proof. By Assumption 4, for any $k \geq 0$, we have $|S_k^B| = |S_k^\phi| = 1$, and for $k \leq 0$ we have $|S_k^A| = |S_k^\phi| = 1$. Define K^A to be the minimal k such that at state S_k^B, the mechanism assigns the item to a new agent, and similarly for K^B. Such k must exist by the definition of dynamic Markovian mechanisms and Assumption 5. Thus, the state space and transitions of a dynamic Markovian mechanism that satisfies the two assumptions are identical to those of a BQ mechanism. For the converse part of the statement, observe that BQ mechanisms satisfy all the stated assumptions. $\square$

The following example gives a dynamic Markovian mechanism that is not a BQ mechanism. This example considers agents with non-linear waiting costs, as in Appendix A. We naturally extend the definition of BF-IC to the requirement that agents receive higher expected utility by joining the BQ regardless of their belief regarding whether future agents will join the BQ. This dynamic Markovian mechanism can hold up to four approached agents, and is BF-IC. No BF-IC BQ mechanism can hold four agents.

Example 1. Let agents have non-linear waiting costs (as defined in Appendix A) with $c = 0$, $\delta = 0.8$ and $v = 65/162 \approx 0.4$. Let $p_A = p_\alpha = p = 1/2$. Consider the following dynamic Markovian mechanism. The mechanism uses a policy that behaves like an SIRO BQ policy with a maximum size of 4, except that if a forth agent joins the BQ the policy does not allow any agents to join until all four agents are assigned.

The policy can be described by a Markov chain whose states (in which α agents are waiting for A items) are $\{0, 1, 2, 3, 4, 3l, 2l, 1l\}$, where $s \in \{0, 1, 2, 3, 4\}$ are the regular states corresponding to the number of agents on the BQ, and $s \in \{3l, 2l, 1l\}$ are the states in which new agents are not allowed to join the BQ. This policy is not a BQ policy, because agents are allowed to join the BQ in state $s = 3$ but not in state $s = 3l$, despite both having three approached agents.

This Markovian mechanism is BF-IC. When four approached agents are present, the mechanism assigns the next four items to these four approached agents in random order, giving each of them an expected utility of exactly $65/162 = v$. This utility is the minimal utility for an agent who joins the BQ under any beliefs, and therefore the mechanism is BF-IC.

Under any BQ policy, the average utility of four agents that are in the BQ is strictly lower than $65/162 = v$, because either the BQ policy is FCFS or new agents join and get

assigned before these four agents with a positive probability. Therefore, no BQ policy that is BF-IC can hold four approached agents.

When agents have linear waiting costs, we find the LIEW_K BQ mechanism is optimal among all Markovian policies, apart from the constraint that BQ size must be a deterministic integer.

Theorem 7. *Let $p_\alpha = p_A = p, \bar{w}$ such $K^A = 2p\bar{w} - 1$, $K^B = 2(1-p)\bar{w} - 1$ are integers. Then, an LIEW BQ mechanism $\mathcal{M} = \left(K^A, \varphi^{LIEW_{K^A}}, K^B, \varphi^{LIEW_{K^B}}\right)$ achieves a weakly higher welfare than any IC Markovian mechanism.*

Proof. Consider a Markovian policy. By Lemma 12, the dynamics can be described by a Markov chain on a finite set of states $\hat{S} = \left(\cup_{k \geq 0}^{K^A} S_k^B \cup S_k^\phi\right) \bigcup \left(\cup_{k \leq 0}^{K^B} S_k^A \cup S_k^\phi\right)$, where K^A, K^B are the maximal number of approached agents in the mechanism. When the current state is $s \in S_k^B$, the mechanism can either assign the B item to the next agent (regardless of his type) or ask the next agent for his type. If the mechanism learns the next agent is of type β, he will be assigned the item, whereas if the agent is of type α, the mechanism will make that agent wait and keep searching. Thus, whenever the next agent is of type β, the mechanism assigns the item to a matching agent and transitions to a state in S_k^ϕ. If the next agent is of type α, the mechanism can randomize between two possible transitions. With probability $1 - f(s)$, the mechanism asigns the B item to the α agent (misallocating the item) and transitions to a state in S_k^ϕ. With probability $f(s)$, the α agent becomes an approached agent and the mechanism transitions to a state in S_{k+1}^B. We likewise define $1 - f(s)$ for $s \in S_{-k}^A$ to be the probability that the mechanism assigns the A item to the approached β agent conditional on the mechanism being in state s and the approached agent being of type β.

Denote the stationary distribution of the finite Markov chain by $\pi(s)$ for each $s \in \hat{S}$. For $k \geq 0$, denote the following:

$$\pi^\phi(k) = \sum_{s \in S_k^\phi} \pi(s)$$

$$\pi^B(k) = \sum_{s \in S_k^B} \pi(s)$$

$$f^B(k) = \sum_{s \in S_k^B} f(s) \times \frac{\pi(s)}{\pi^B(k)}$$

$$F^B(k) = \prod_{i=0}^{k-1} f^B(i),$$

and the analogous definitions for $k \leq 0$ for π^A, f^A, F^A. For notational convenience, we define $F^B(0) = F^A(0) = 1$.

We start by calculating the stationary distribution π. For any $k > 0$, the mechanism must visit S_k^B between every two visits to S_k^ϕ, and vice versa. Therefore, we have that

$$\pi^B(k) = \pi^\phi(k).$$

For $k = 0$, we have that the set S_0^B can only be reached from a state in S_0^ϕ when a B item arrives, and all such transitions end in S_0^B. Therefore,

$$\pi^B(0) = p_B \pi^\phi(0) = (1 - p)\pi^\phi(0).$$

Consider the cut between the states $\cup_{j>k} S_j^B \cup S_j^\phi$ and the rest of $\hat{S}$. The flow out of the set is the sum of transitions from states in S_{k+1}^ϕ that correspond to an arrival of a matching A item. The flow into the set is the sum of transitions from states in S_k^B that correspond to the next agent being a mismatched α who becomes an approached agent. Therefore, to balance the flow across the cut, we must have that

$$\sum_{s \in S_{k+1}^\phi} \pi(s) p_A = \sum_{s \in S_k^B} \pi(s) f(s) p_\alpha$$

or

$$\pi^\phi(k + 1) = f^B(k) \pi^B(k),$$

giving us that

$$\pi^\phi(k) = \pi^B(k) = (1 - p)\pi(0) F^B(k).$$

We solve for $\pi(0)$ using that the total probability is equal to 1. We get that

$$\pi^\phi(0) \;\; = \;\; \frac{1}{2} \frac{1}{1 + (1 - p)\sum_{k=1}^{K^A} F^B(k) + p \sum_{k=1}^{K^B} F^A(-k)}.$$

Given the stationary distribution, we can calculate the misallocation rate. The misallocation rate is the fraction of mismatched agents out of all assigned agents, which is the fraction of transitions out of states $\cup_{k \geq 0}^{K^A} S_k^B \bigcup \cup_{k \leq 0}^{K^B} S_k^A$ that are misallocations. A transition results in misallocation if the mechanism assigns the item to the next agent and the next agent is mismatched. Therefore,

$$\xi \;\; = \;\; \frac{\sum_{k=0}^{K^A} \sum_{s \in S_k^B} p_\alpha (1 - f(s))\pi(s) + \sum_{k=0}^{K^B} \sum_{s \in S_k^A} p_\beta (1 - f(s))\pi(s)}{\sum_{k=0}^{K^A} \sum_{s \in S_k^B} \pi(s) + \sum_{k=0}^{K^B} \sum_{s \in S_k^A} \pi(s)}.$$

To simplify this expression, note

$$\sum_{k=0}^{K^A} \sum_{s \in S_k^B} p_\alpha (1 - f(s)) \pi(s) = \sum_{k=0}^{K^A} p_\alpha (1 - f^B(k)) \pi^B(k)$$

$$= \sum_{k=0}^{K^A} p(1 - f^B(k))(1-p)\pi(0) \prod_{i=1}^{k-1} f^B(i)$$

$$= p(1-p)\pi(0) \times \sum_{k=0}^{K^A} \left(\prod_{i=1}^{k-1} f^B(i) - \prod_{i=1}^{k} f^B(i) \right)$$

$$= p(1-p)\pi(0) \times (1 - F^B(K^A))$$

$$= p(1-p)\pi(0).$$

where the last line follows because, by definition, $F^B(K^A) = 0$. The denominator, which is the probability that the next transition will approach a new agent, is equal to $1/2$ by the previous derivation. Therefore, we get that

$$\xi = \frac{2p(1-p)}{1 + (1-p)\sum_{k=1}^{K^A} F^B(k) + p \sum_{k=1}^{K^B} F^A(-k)}.$$

We now consider the IC constraint. Matching agents who are truthful receive their preferred item immediately, and therefore have no incentive to deviate. When a mismatched agent is asked to report his type, he has two options: if he reports his type truthfully, he becomes an approached agent and waits further until he receives his preferred item. If he falsely reports that he is a matching agent, he is assigned a mismatched item immediately. Thus, for the mechanism to be incentive compatible, we must have that the expected wait w_s given state s is acceptable. That is, $w_s \leq \bar{w} = v/c$ for all s.

Instead of looking at the individual incentive constraints for each state s, we consider the expected incentive constraint. We derive the expected wait for a random approached agent by using Little's Law, which states that the average wait is equal to the average number of agents in the system divided by the arrival rate. We consider the part of the mechanism in which α agents are waiting. The arrival rate of α approached agents is equal to the rate at which α approached agents are assigned, which is p_A. We calculate the average number of α approached agents by taking the stationary distribution restricted to the states in which α agents are waiting. Therefore, the expected wait of a random α

agent is

$$\mathbb{E}_{s \sim \pi|_{\cup S_k^B}}[w_s] = \frac{1}{p_A} \frac{\sum_{k=1}^{K^A} \pi^B(k) \times k}{\sum_{k=1}^{K^A} \pi^B(k)}$$

$$= \frac{1}{p} \frac{\sum_{k=1}^{K^A} F^B(k) \times k}{\sum_{k=1}^{K^A} F^B(k)},$$

and we get that any IC mechanism must satisfy the following expected-IC constraint:

$$\sum_{k=1}^{K^A} F^B(k) \times k \leq \bar{w} \times p \sum_{k=1}^{K^A} F^B(k).$$

Taking these results together, we find the minimal misallocation rate out of all IC Markovian mechanisms is weakly lower than the solution to the following optimization problem:

$$\text{Minimize}_{F^A, F^B} \quad \frac{2p(1-p)}{1 + (1-p)\sum_{k=1}^{K^A} F^B(k) + p\sum_{k=1}^{K^B} F^A(-k)}$$

$$s.t. \quad \sum_{k=1}^{K^A} (k - \bar{w}p) \times F^B(k) \leq 0$$

$$\sum_{k=1}^{K^B} (k - \bar{w}p) \times F^A(-k) \leq 0$$

$$1 \geq F^B(k) \geq F^B(k+1) \qquad \forall k > 0$$

$$1 \geq F^A(-k) \geq F^A(-k-1) \quad \forall k > 0$$

which can be decomposed to two separate optimization problems, one for the domain where α approached agents are present:

$$\text{Maximize}_{F^B} \quad \sum_{k=1}^{K^A} F^B(k) \tag{3}$$

$$s.t. \quad \sum_{k=1}^{K^A} (k - \bar{w}p) \times F^B(k) \leq 0$$

$$1 \geq F^B(k) \geq F^B(k+1) \qquad \forall k > 0$$

and the analogous problem for the domain in which β approached agents are present.

The optimal solution to (3) is given by

$$F^*(k) = \begin{cases} 1 & k < L^* \\ x^* & k = L^* \end{cases}$$

for $L^* \in \mathbb{N}$ and $x^* \in [0, 1)$, and the first constraint must bind. Thus, the optimal solution corresponds to $L^* = \lfloor 2p\bar{w} \rfloor - 1$ and $x^* = [2p\bar{w}] = 2p\bar{w} - \lfloor 2p\bar{w} \rfloor$. In particular, $L^* = K^A$ and $x^* = 0$ when $2p\bar{w} - 1$ is an integer. Therefore, we find that any dynamic Markovian mechanism that satisfies the expected IC constraint achieves a weakly worse misallocation

rate than the LIEW BQ mechanism, proving the result. $\qquad\square$