Repeated Matching Games
An Empirical Framework

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Today's Goal: Make Static Matching Games **Dynamic**

- Plenty of existing work on **estimating static matching games** (cited below)
- Make matching models **dynamic**
- **Estimate** models using panel data on **changing relationships over time**
- Study **transferable utility** matching games

Relationships Change Over Time

- Relationships often **change over time** in matching markets
- **Labor markets**
  - Tech workers switch firms frequently
  - Professional athletes switch teams
- **Supplier / assembler** matching
  - Assemblers can switch suppliers for a part
  - Apple often switches firms fabricating its phone/tablet processors
- Funding of **startups**
  - Many rounds of funding: accelerators, seed, early, later
  - Different **venture capitalists** fund startup each round
- Personal relationships
  - Dating, **marriage**, divorce, remarriage

Today's Relationship Affects Future Matches

- Match today affects agent **state variables** and therefore benefits and probabilities of attractive matches in future
- Labor markets
  - Worker may accept lower wage today to gain **on-the-job training**, better matches in future
  - Entrepreneurs might be generalists who have past **experiences** in many areas (Lazear 2009)
- Supplier / assembler matching
  - Lower quality car parts supplier participating in Toyota's Supplier Development Program raises **quality** for future matches (Fox 2018)
- Funding of startups
  - Participating in accelerator improves **business plan**, raises probability of seed round of funding
- Personal relationships
  - Divorce may cause **stigma**, lower value on marriage market
  - **Learn** from mistakes in past relationships
  - **Children** affect future partner preferences

Dynamic vs. Static Matching Games

- **Static** matching games capture market forces
  - Many startups relative to funders, hard to get startup funded (or quality VC for management advice)
  - High quality suppliers take matches, profits away from low quality suppliers
- In static model, agents do not account for how **matches today affect future matches**
- Our dynamic model has

1. Agent **state variables**
   - Matches today affect **evolution of agent state variables**
   - Agent state is also **type of agent** in language of static matching games
2. **Forward looking** agents
   - Agents consider how matches today **affect matches in the future**

Repeated Matching Game

- To our knowledge, first to introduce definition of **repeated matching game**
- Dynamic version of Shapley & Shubik (1971) (plus **continuum** of agents)
- **Matching market** in each period
  - **Transferable utility**, one-to-one matching
- Matching market **clears each period**
  - Matches, transfers, flow profits
- Matches affect evolution of **state variables** for each agent
- Next period, matching market forms and agents match again
- Agents **forward looking**: maximize present value of profits
- Stickiness of matches: **switching costs**
  - Not in estimation part of current paper: time persistent unobservable state variables

Not a Search Model

- Influential literature on **search** with forward looking agents
  - e.g., Burdett & Mortensen (1998)
- **Repeated matching game**
  - Agents match each period and **state variables** evolve
  - **Market clearing prices** differ each period
  - **Complete information**
  - No frictions unless modeled (switching costs)
  - Solution concept from TU matching games: **competitive equilibrium**
- Search may envision shorter time periods than our repeated matching game

Matching Theory Contribution

- Papers by Nobel-quality scholars on **static**, transferable utility, one-to-one **matching games**
  - Koopmans & Beckmann (1957), Gale (1960), Shapley & Shubik (1971), Becker (1973)
- Feel our model is natural extension of these static models to **dynamic matching**
- Prove that a competitive equilibrium **exists** and can be computed using a **social planner** problem
  - Like Shapley & Shubik (1971)
- Distinguish between
  1. Full competitive equilibrium with a **time-varying aggregate state**
  2. Stationary equilibrium with a **constant aggregate state**
- Existence results for **both**

Econometric Errors: Rust Meets Choo and Siow

- Also prove results for **repeated matching game** with **econometric errors**
- Concisely explain model with **econometric errors** as combination of two influential papers
- **Rust** (1987) on estimating single agent **dynamic discrete choice models**
  - Forward looking agents, state variables
- **Choo & Siow** (2006) on estimating **static matching games**
  - One-to-one, two-sided matching
- **Type of agent** in Choo & Siow is **agent state variable** in Rust
- Both frameworks build on multinomial choice (often **logit**) and we exploit commonality
- In our repeated matching game, discrete choice is **partner chosen each period**

Related Literatures

- 1980's classic papers on estimating **dynamic discrete choice models**
  - Eckstein (1984), Miller (1984), Wolpin (1984), Pakes (1986), Rust (1987), others
- Estimating static, transferable utility **matching games** with continuum of agents under assumption that **errors reflect preferences for type of partner**
  - Dagsvik (2000), Choo & Siow (2006), Fox (2010, 2018), Chiappori, Salanie & Weiss (2017), Galichon & Salanie (2021), others
- **MPEC** approach to **equilibrium computation** & **estimation**
  - Su & Judd (2012), Dube, Fox & Su (2012)
- **Max-Min** computational & estimation approach
  - Chambolle & Pock (2011)

Prior Dynamic Matching Games

- Current paper supersedes unpublished Fox (2007)
- Estimating **two period models** of matching
  - Erlinger, McCann, Shi, Siow & Wolthoff (2015), McCann, Shi, Siow & Wolthoff (2015)
- **Marriage & remarriage** with divorce not a function of outside options
  - Choo (2015)
- **Search model** with separation also not a function of outside options
  - Peski (2021)
- **Dynamic equilibrium used car** model has some overlap with our model with econometric errors
  - Gillingham, Ishakov, Munk-Nielsen, Rust & Schjerning (forthcoming)

Outline

1. **Repeated matching game** without econometric errors
2. Repeated matching games with **econometric unobservables**
3. **Computing** time-varying equilibrium
4. Computing stationary equilibrium and **structural estimation** using match data
5. **Empirical** application to geographic mobility for Swedish engineers

Workers & Firms

- Repeated matching game without econometric errors
- Workers & firms as leading example
- $x \in \mathcal{X}$ **worker state variable**
  - Also call x **type of worker**, following matching games
  - Finite types
  - Could be scalar or vector of underlying state concepts
- $y \in \mathcal{Y}$ **firm state variable**
- One-to-one: each firm has one job
- Option 0: being **unmatched** for both workers, firms
- $\mathcal{Y}_0 = \mathcal{Y} \cup \{0\}$ firm type partners & option of being unmatched for worker
- $\mathcal{X}_0 = \mathcal{X} \cup \{0\}$ worker type partners & option of being unmatched for firm

Time Subscripts Often Dropped

- Infinite horizon
- Drop time t subscripts where possible
- **All agent-specific objects allowed to vary by time for given agent**

Evolution of Agent States

- Sates evolve as function of current **match** (x, y)
- $P_{x'|xy} = P(x' \mid x, y)$ transition mass function for **worker state** if worker x matches to firm y
- $Q_{y'|xy} = Q(y' \mid x, y)$ transition mass function for **firm state** if worker x matches to firm y
- Worker, firm examples
  - x tracks **general work human capital**, evolves if not unemployed $y \neq 0$
  - x tracks **experience in different occupations**, evolves according to occupation in y
- Venture capitalist examples
  - y tracks number of **previous investment deals**, evolves if deal is made $x \neq 0$
  - y tracks **experience in different startup sectors** (high tech, biotech, retail, etc), depends on sector of startup x

Aggregate State: Masses of Workers, Firms

- **Continuums** of workers, firms
- M **mass of all workers**, N **mass of all firms**
  - Time invariant in baseline model, could add entry, exit
- m_x **mass of workers** of state/type x
  - m: vector of masses m_x for all x
- n_y **mass of firms** of state/type y
  - n: vector of masses n_y for all y
- (m, n) **aggregate state** at beginning of time period

Outcomes: Matches, Transfers

- μ_{xy} **mass of matches** between types x, y
  - μ_{x0} unmatched mass for worker type x
  - μ_{0y} unmatched mass for firm type y
  - μ vector of masses μ_{xy} for all pairs x, y (and unmatched)
- w_{xy} monetary **transfer** (**wage**) paid by y to x
  - Collected into w for all matches (x, y)
  - Estimation: today **no data on transfers** w
- Infinite time horizon
- Competitive equilibrium: index matches, wages by **aggregate state** (m, n)

$$\mu(m, n),\ w(m, n)$$

- **Aggregate state transition** $(P\mu, Q\mu)$
- Could add stochastic aggregate transition, economywide demand shifters

Worker Flow and PDV Profits

- **Flow profit** of worker x matched to firm y and paid $w_{xy}(m,n)$

$$\alpha_{xy} + w_{xy}(m,n)$$

- α_{xy} **non-wage utility** (in money terms) from job y
- Say current period is 0
- Worker x picks current partner $y \in \mathcal{Y}_0$ to maximize expected, **present discounted value of profit**

$$E\left[\sum_{t=0}^{\infty} \beta^t \left(\alpha_{x^t,y^t} + w_{x^t,y^t}\left(m^t, n^t\right)\right) \mid x\right]$$

  - β **discount factor** < 1
  - x^t future worker state variable
  - y^t profit-maximizing choice of firm in future period t

Worker Bellman Equation

- Bellman equation for worker x in aggregate state (m, n)

$$U_x(m,n) = \max_{y \in \mathcal{Y}_0} \left\{ \alpha_{xy} + w_{xy}(m,n) + \beta \sum_{x' \in \mathcal{X}} U_{x'}(P\mu, Q\mu) P_{x'|xy} \right\}$$

- $U_x(m,n)$ continuation profit for worker x in aggregate state (m,n)
- $\sum_{x' \in \mathcal{X}}$ sum over next period's individual state variables x'
- $P_{x'|xy}$ transition rule for individual state x
- $(P\mu, Q\mu)$ shorthand for transition of aggregate state (m,n)

Firm Flow Profits & Bellman Equation

- Firm y **flow profits** in aggregate state (m, n)

$$\gamma_{xy} - w_{xy}(m, n)$$

- γ_{xy} output / **pre-wage profit** of firm y from worker x
- Firms pay workers wages, "wages" not restricted to be positive
- **Bellman equation** for firm y

$$V_y(m, n) = \max_{x \in \mathcal{X}_0} \left\{ \gamma_{xy} - w_{xy}(m, n) + \delta \sum_{y' \in \mathcal{Y}} V_{y'}(P\mu, Q\mu)\, Q_{y'|xy} \right\}$$

Competitive Equilibrium

- Define **competitive equilibrium** as matches, wages as function of **aggregate state** (m, n)

$$\mu(m, n), w(m, n)$$

- If pair x, y observed, match should **maximize profits**
- If $\mu_{xy} > 0$, then

$$y \in \arg\max_{\tilde{y} \in \mathcal{Y}_0} \left\{ \alpha_{x\tilde{y}} + w_{x\tilde{y}}(m, n) + \beta \sum_{x' \in \mathcal{X}} U_{x'}(P\mu, Q\mu) P_{x'|x\tilde{y}} \right\}$$

$$x \in \arg\max_{\tilde{x} \in \mathcal{X}_0} \left\{ \gamma_{\tilde{x}y} - w_{\tilde{x}y}(m, n) + \beta \sum_{y' \in \mathcal{Y}} V_{y'}(P\mu, Q\mu) Q_{y'|\tilde{x}y} \right\}$$

- Competitive equilibrium induces a deterministic **time series of aggregate states** (m, n)

Social Planner Problem

Theorem

Matching policy $\mu(m,n)$ *in a competitive equilibrium maximizes the social planner's primal problem:*

$$\max_{\mu_{xy}^t \geq 0} \left\{ \sum_{t=0}^{\infty} \beta^t \sum_{xy \in \mathcal{X}_0 \mathcal{Y}_0} \mu_{xy}^t \left(\alpha_{xy} + \gamma_{xy} \right) \right\}$$

subject to constraints

- $\sum_{y \in \mathcal{Y}_0} \mu_{xy}^t = m_x^t$ *for all* x, t & $\sum_{x \in \mathcal{X}_0} \mu_{xy}^t = n_y^t$ *for all* x, t
- $\sum_{x'y' \in \mathcal{X}_0 \mathcal{Y}_0} P_{x|x'y'} \mu_{x'y'}^t = m_x^{t+1}$ *for all* t, x & $\sum_{x'y' \in \mathcal{X}_0 \mathcal{Y}_0} Q_{y|x'y'} \mu_{x'y'}^t = n_y^{t+1}$ *for all* t, y

- Similar to Shapley and Shubik (1971) for static models

Social Planner Problem Bellman

- **Bellman equation** for social planner's **primal problem** at aggregate state (m, n)

$$W(m,n) = \max_{\mu_{xy} \geq 0} \left\{ \sum_{xy \in \mathcal{X}_0 \mathcal{Y}_0} \mu_{xy} \left(\alpha_{xy} + \gamma_{xy} \right) + \beta W \left(P\mu, Q\mu \right) \right\}$$

- Subject to constraints
  - $\sum_{y \in \mathcal{Y}_0} \mu_{xy} = m_x$ for all x
  - $\sum_{x \in \mathcal{X}_0} \mu_{xy} = n_y$ for all y
- Right side of Bellman is a **contraction**

Theorem

*A competitive equilibrium **exists** and the economywide sum of future profit $W(m,n)$ is **uniquely determined**.*

- In many parameterizations, matches $\mu(m,n)$ uniquely determined across all competitive equilibria

Dual Problem and Equilibrium Transfers

- Linear programming theory: **dual problem**
- Dual problem can be used to directly compute **lifetime utilities** U_x and V_y
- **Bellman equation** for social planner's **dual problem** at aggregate state (m, n)

$$D(m,n) = \min_{U_x, V_y} \left\{ \sum_{x \in \mathcal{X}} m_x U_x(m,n) + \sum_{y \in \mathcal{Y}} n_y V_y(m,n) \right\}$$

- Subject to constraints for all pairs x, y

$$\begin{aligned} U_x(m,n) + V_y(m,n) \geq & (\alpha_{xy} + \gamma_{xy}) \\ & + \beta \sum_{x' \in \mathcal{X}} U_{x'}(P\mu, Q\mu) P_{x'|xy} + \beta \sum_{y' \in \mathcal{Y}} V_{y'}(P\mu, Q\mu) Q_{y'|xy} \quad \forall x, y \\ U_x(m,n) \geq & \beta \sum_{x' \in \mathcal{X}} U_{x'}(P\mu, Q\mu) P_{x'|x0} \quad \forall x \\ V_y(m,n) \geq & \beta \sum_{y' \in Y} V_{y'}(P\mu, Q\mu) Q_{y'|0y} \quad \forall y \end{aligned}$$

- Once matches μ_{xy} from primal and utilities U_x, V_y known, compute **equilibrium transfers** $w(m, n)$

Recap of Full Equilibrium

- In competitive equilibrium, **deterministic time series of aggregate states** (m, n)
  - m: masses m_x of workers of type x
  - n: masses n_y of firms of type y
- Track matches and state variables of all other agents in economy
- **Competition** affects wages, match opportunities in future
- Equilibrium depends on **structural parameters** β, α_{xy}, γ_{xy}, P, Q
- Previous theorems do **not restrict** model parameters

Stationary Equilibrium

- **Constant aggregate state** (m, n) satisfies

$$m = P\mu(m, n) \text{ and } n = Q\mu(m, n)$$

- (m, n) **remains next period's state** if current period state is (m, n)
- **Stationary equilibrium** is constant aggregate state plus (μ, w), matches & transfers
- In stationary equilibrium, workers & firms still maximize present-discounted value of profits
- If $\mu_{xy} > 0$,

$$y \in \arg\max_{\tilde{y} \in \mathcal{Y}_0} \left\{ \alpha_{x\tilde{y}} + w_{x\tilde{y}} + \beta \sum_{x' \in \mathcal{X}} U_{x'} P_{x'|x\tilde{y}} \right\}$$

$$x \in \arg\max_{\tilde{x} \in \mathcal{X}_0} \left\{ \gamma_{\tilde{x}y} - w_{\tilde{x}y} + \beta \sum_{y' \in \mathcal{Y}} V_{y'} Q_{y'|\tilde{x}y} \right\}$$

- Aggregate state (m, n) dropped from several pieces of notation

Individual Dynamics Only

- In **stationary equilibrium**, model has **individual agent dynamics** only
- Worker x Bellman equation **simplifies**

$$U_x = \max_{y \in \mathcal{Y}_0} \left\{ \alpha_{xy} + w_{xy} + \beta \sum_{x' \in \mathcal{X}} U_{x'} P_{x'|xy} \right\}$$

- Firm y Bellman equation

$$V_y = \max_{x \in \mathcal{X}_0} \left\{ \gamma_{xy} - w_{xy} + \beta \sum_{y' \in \mathcal{Y}} V_{y'} Q_{y'|xy} \right\}$$

- Wages w_{xy} still **equilibrium objects**
- Will discuss **computation** of stationary equilibrium only for model with **econometric errors**

Stationary Equilibrium Exists

- **Constant aggregate state**

$$m = P\mu\,(m, n) \text{ and } n = Q\mu\,(m, n)$$

Theorem

*A **stationary equilibrium exists** and hence a **constant aggregate state exists**.*

- Multiple constant aggregate states possible
- All workers, firms know that aggregate state will remain (m, n)

Applications & Stationary Equilibrium

- Researcher **modeling decision** to assume market at **stationary equilibrium**
- In **venture capital**, if assume at constant aggregate state then distributions of investment opportunities & funding sources do not change over time
- Model evolution of **individual state variables** such as the experience levels of individual venture capitalists and startups
- In IO literature on dynamic games, can **approximate Markov perfect equilibria** with model where firms optimize with respect to analog to a constant aggregate state
  - Weintraub, Benkard & Van Roy (2008)

Both Individual, Aggregate Dynamics

- **Full competitive equilibrium** allows **time-varying aggregate state** (m, n)
- Workers, firms track masses and hence **matches of other workers, firms**
- In **macro**, models with both heterogeneous agents and aggregate dynamics common
  - e.g., Rios-Rull (1995), Krusell and Smith (1998)
- In **IO dynamic games** agents best respond to all other agents, keep track of individual states of all firms in an industry
  - e.g., Ericson and Pakes (1995)
- Can add exogenous aggregate state variables to model
- Can add exogenous model of agents entering, leaving matching game

Econometric Unobservables

- Prior model often implies $\mu_{xy} = 0$ for some types x and y
- In data, agents with same x match to many y's
- Add **econometric unobservables** to fit data

Econometric Preference Shocks

- Worker i of type x gets **flow profit**

$$\alpha_{xy} + w_{xy}(m, n) + \epsilon_{iy}$$

- Unmeasured preference ϵ_{iy} of i over **type y of partner**
- Payoff to being **unmatched**

$$\alpha_{x0} + \epsilon_{i0}$$

- Firm j of type y has **similar econometric errors** $(\eta_{jx})_{x \in \mathcal{X}_0}$
- Follow Rust (1987) and Choo & Siow (2006)
- Parameterize distributions of $(\varepsilon_{iy})_{y \in \mathcal{Y}_0}$ & $(\eta_{jx})_{x \in \mathcal{X}_0}$, often iid logit

Econometric Preference Shocks

- Recall

$$\alpha_{xy} + w_{xy}(m, n) + \epsilon_{iy}$$

Assumption

*$(\epsilon_{iy})_{y \in \mathcal{Y}_0}$ & $(\eta_{jx})_{x \in \mathcal{X}_0}$ **independent over time** for each agent*

- *Also **independent across agents**, conditional on measured types x or y*

- **Conditional independence** follows Rust (1987)
- Relax in ongoing work

Social Planner Problem with Econometric Errors

- **Social planner** problem with **econometric errors**

$$\max_{\mu_{xy}^t \geq 0} \left\{ \sum_{t=0}^{\infty} \beta^t \left(\sum_{x,y \in \mathcal{X}_0 \mathcal{Y}_0} \mu_{xy}^t \left(\alpha_{xy} + \gamma_{xy}\right) - \mathcal{E}(\mu^t, m^t, n^t) \right) \right\}$$

- Subject to **transition rules**

$$\sum_{x'y' \in \mathcal{X}_0 \mathcal{Y}_0} P_{x|x'y'} \mu_{x'y'}^t = m_x^{t+1} \, \forall \, t, x \; \& \sum_{x'y' \in \mathcal{X}_0 \mathcal{Y}_0} Q_{y|x'y'} \mu_{x'y'}^t = n_y^{t+1} \, \forall \, t, y$$

- **Entropy** $\mathcal{E}(\mu, m, n)$ expectation of **sum of econometric errors** for all realized matches
  - Galichon & Salanie (2021)
- Entropy built from McFadden **social surplus** terms like

$$G_x(u) = \mathbb{E}_\epsilon \left[\max_{y \in \mathcal{Y}_0} \left\{ u_{xy} + \varepsilon_y \right\} \right]$$

- See our paper for full entropy expression

Social Planner Bellman with Unobservables

- Can derive Bellman equation for social planner with time independent unobservables
- **Bellman equation** for social planner's **primal problem** at aggregate state (m, n)

$$W(m, n) =$$

$$\max_{\mu_{xy}} \left\{ \sum_{xy \in \mathcal{X}_0 \mathcal{Y}_0} \mu_{xy} (\alpha_{xy} + \gamma_{xy}) - \mathcal{E}(\mu; m, n) + \beta W(P\mu, Q\mu) \right\}$$

- **No explicit constraints** on μ as $\mathcal{E}(\mu; m, n)$ equals $+\infty$ when constraints not satisfied

- Assume **logit errors** to analyze **stationary equilibrium** with econometric errors
- iid standard **type one extreme value** (Gumbel)

Theorem

A ***stationary equilibrium*** *and hence a* ***constant aggregate state exist*** *in the model with* ***logit*** *econometric errors.*

Logit Errors & Stationary Equilibrium

- Establish following **social-plannerish problem** gives matches in stationary equilibrium

$$\max_{\mu_{xy}\geq 0} \sum_{t=0}^{\infty} \beta^t \left(\sum_{x,y \in \mathcal{X}_\mathbf{0}\mathcal{Y}_\mathbf{0}} \mu_{xy}^t \left(\alpha_{xy} + \gamma_{xy}\right) - \right.$$

$$\left. \sum_{xy \in \mathcal{X}\mathcal{Y}} \mu_{xy} \log \mu_{xy} - \sum_{x \in X} \mu_{x0} \log \mu_{x0} - \sum_{y \in Y} \mu_{0y} \log \mu_{0y} \right)$$

$$\text{s.t} \sum_{y \in \mathcal{Y}_\mathbf{0}} \mu_{xy} = \sum_{x'y' \in \mathcal{X}\mathcal{Y}_\mathbf{0}} P_{x|x'y'} \mu_{x'y'} \quad \forall x$$

$$\sum_{x \in \mathcal{X}_\mathbf{0}} \mu_{xy} = \sum_{x'y' \in \mathcal{X}\mathcal{Y}} Q_{y|x'y'} \mu_{x'y'} \quad \forall y$$

Overview of Equilibrium Computation with Logit Errors

- Full competitive equilibrium with **time-varying aggregate state**
  - Social planner problem is **dynamic program** with **continuous state variables**
- **Stationary equilibrium** computation
  1. **MPEC**
  2. **Max-Min** problem
- Extend both approaches for stationary equilibrium to **structural estimation**

- **Bellman equation** for social planner's **primal problem** at aggregate state (m, n)

$$W(m,n) = \max_{\mu_{xy}} \left\{ \sum_{xy \in \mathcal{X}_0 \mathcal{Y}_0} \mu_{xy} (\alpha_{xy} + \gamma_{xy}) - \mathcal{E}(\mu; m, n) + \beta W(P\mu, Q\mu) \right\}$$

- Vectors of agent type masses (m, n) are continuous states
- Benchmarking two approaches to dynamic programming

1. **Value function iteration** with Anderson acceleration (Walker & Ni 2011)
2. **Deep learning** using Flux, PyTorch, TensorFlow

$$\min_{\theta} \frac{1}{N} \sum_{i} (\mathrm{ANN}(m_i, n_i | \theta) - H(\mathrm{ANN}(m_i, n_i | \theta)))^2$$

- H is right side of Bellman's equation

MPEC for Computing Stationary Equilibrium

- Mathematical program with equilibrium constraints
- Solve **system of equations**
- **Nonlinear programming** with dummy objective function (max 0)
- Prefer solver **MadNLP** in Julia
- Solve following equations for for unknowns (U, V), (m, n)

$$\sum_{y \in \mathcal{Y}_0} \mu_{xy}(U, V, m, n) = m_x \;\&\; \sum_{x \in \mathcal{X}_0} \mu_{xy}(U, V, m, n) = n_y$$

$$\sum_{xy \in \mathcal{X}_0\mathcal{Y}} P_{x'|xy}\mu_{xy}(U, V, m, n) = m_{x'} \;\&\; \sum_{xy \in \mathcal{X}\mathcal{Y}_0} Q_{y'|xy}\mu_{xy}(U, V, m, n) = n_{y'}$$

$$2\sum_{xy \in \mathcal{X}\mathcal{Y}} \mu_{xy}(U, V, m, n) + \sum_{x \in \mathcal{X}} \mu_{x0}(U, V, m, n) + \sum_{y \in \mathcal{Y}} \mu_{0y}(U, V, m, n) = M$$

- **Definition** of terms like $\mu_{xy}(U, V, m, n)$ for **logit** errors

$$\mu_{xy}(U, V, m, n) =$$

$$\sqrt{m_x n_y} \exp\left(\frac{\Phi_{xy} + \beta \sum_{x' \in \mathcal{X}} U_x P_{x'|xy} + \beta \sum_{y' \in \mathcal{Y}} V_y Q_{y'|xy} - U_x - V_y}{2}\right)$$

$$\mu_{x0}(U, V, m, n) = m_x \exp\left(\beta \sum_{x' \in \mathcal{X}} U_x P_{x'|xy} - U_x\right)$$

$$\mu_{0y}(U, V, m, n) = n_y \exp\left(\beta \sum_{y' \in \mathcal{Y}} V_y Q_{y'|xy} - V_y\right)$$

Max-Min Program for Stationary Equilibrium

- **Saddle-point** program

$$\max_{m,n} \min_{U,V} Z(U, V, U, V, m, n, \beta)$$

- Where **objective function** is

$$Z(U, V, U', V', m, n, \beta) = -\sum_{x \in X} m_x - \sum_{y \in Y} n_y$$
$$+ 2 \sum_{xy \in \mathcal{X}\mathcal{Y}} \mu_{xy}(U, V, U', V', m, n, \beta)$$
$$+ \sum_{x \in \mathcal{X}} \mu_{x0}(U, V, U', V', m, n, \beta) + \sum_{y \in \mathcal{Y}} \mu_{0y}(U, V, U', V', m, n, \beta)$$

Max-Min Algorithm for Stationary Equilibrium

- **Definitions** used on previous slide for **logit** errors

$$\mu_{xy}(U, V, U', V', m, n, \beta) =$$

$$\sqrt{m_x n_y} \exp\left(\frac{\Phi_{xy} + \beta \sum_{x' \in \mathcal{X}} U'_x P_{x'|xy} + \beta \sum_{y' \in \mathcal{Y}} V'_y Q_{y'|xy} - U_x - V_y}{2}\right)$$

$$\mu_{x0} = m_x \exp\left(\beta \sum_{x' \in \mathcal{X}} U'_x P_{x'|xy} - U_x\right)$$

$$\mu_{0y} = n_y \exp\left(\beta \sum_{y' \in \mathcal{Y}} V'_y Q_{y'|xy} - V_y\right)$$

Stationary Equilibrium: Chambolle-Pock

- Above **min/max** problem formally works when discount factor $\beta = 1$
- Meaning **Chambolle-Pock** algorithm converges to solution
- For usual case of $\beta < 1$, we **modify** Chambolle-Pock algorithm as discussed in our paper
- No formal results
- In practice our modified algorithm **converges** to **stationary equilibrium** for $\beta < 1$

Stationary Equilibrium: Speed Comparison

Method	$nbx = 2,\ nby = 2$	$nbx = 10,\ nby = 10$	$nbx = 30,\ nby = 30$
MPEC	0.004s	1.168s	408.770s
MaxMin	0.048s	1.805s	35.871s

- Vary **number** nbx **of worker types**, **number** nby **of firm types**
- Times are on a laptop

Structural Estimation Overview

- Assume **data** come from **stationary equilibrium**
- Data on **matches** and **agent types / states**
- n **observations** on

$$(x, x', y, y')$$

  - x worker state
  - x' worker state next period
  - y firm state
  - y' firm state next period
- In practice, often longer panels on each agent
- As in Rust (1987), estimate **transition rules** P and Q in **first stage**
- Simple estimators, like empirical frequencies as x, y finite

Structural Estimation of Flow Match Production

- Now estimate **structural parameters** θ in sum of **flow profit terms**

$$\Phi_{xy}^{\theta} = \alpha_{xy}^{\theta} + \gamma_{xy}^{\theta}$$

- **Log likelihood**

$$\sum_{xy} 2\hat{\mu}_{xy} \log \mu_{xy}(\theta) + \sum_{x} \hat{\mu}_{x0} \log \mu_{x0}(\theta) + \sum_{y} \hat{\mu}_{0y} \log \mu_{0y}(\theta)$$

- $\hat{\mu}_{xy}$ is **estimated mass of matches** of x to y in **data** (frequency)
- **Maximum likelihood nested fixed point**
  - For each θ, compute stationary equilibrium matches like $\mu_{xy}(\theta)$
  - **Statistically efficient** up to first stage error in P and Q
  - **Multiple matches μ in stationary equilibria**: use max of **distinct equilibria** likelihoods

MPEC: Stationary Equilibrium & Estimation

- **Computing stationary equilibrium**

$$\max_{\mu,w,U,V} 0$$

  - Subject to **constraints** from **equilibrium computation**
- **Estimating** while **imposing stationary equilibrium**

$$\max_{\theta,\mu,w,U,V} \sum_{xy} 2\hat{\mu}_{xy} \log \mu_{xy} (\theta) + \sum_{x} \hat{\mu}_{x0} \log \mu_{x0} (\theta) + \sum_{y} \hat{\mu}_{0y} \log \mu_{0y} (\theta)$$

  - Subject to **same constraints** as computing stationary equilibrium
- MPEC numerically **same estimator** as **nested fixed point MLE** if **unique** matchings μ in all stationary equilibria
- **No max over likelihoods** if **multiple** matchings μ in stationary equilibria

Min/Max: Stationary Equilibrium & Estimation

- Recall **min/max program** to compute stationary equilibrium
- Add **moment conditions** equating weighted sums of matches in model, data
- Augment min/max $Z(U, V, U, V, m, n, \beta)$ by an extra term so that the **first order conditions** contain **estimation moment conditions**
- Then modified (for $\beta < 1$) **Chambolle-Pock algorithm estimates** structural parameters θ

Stationary Equilibrium: Estimation Speed Comparison

Method	$nbk = 2$	$nbk = 10$	$nbk = 30$
MPEC	3.077s	11.918s	35.910s
MaxMin	0.751s	2.044s	9.211s

$nbx = 10$, $nby = 10$

- Vary **number of structural parameters** nbk

Unobserved Heterogeneity & State Dependence

- Key issue in empirical work (e.g., Heckman 1981)
- Say **data** show startups with **several previous rounds of successful funding** x often match to venture capitalists with **high measured experience** y
- **State dependence**: previous rounds of funding x improve startup
- **Unobserved heterogeneity**: startups with better **unmeasured business plans** ν have previously gotten more rounds of funding x
- Goal: **consistent estimator** of θ in flow profits $\alpha_{xy}^{\theta} + \gamma_{xy}^{\theta}$ when ν also in flow profits

Instrumental Variables?

- Berry & Compiani (2020) study **instrumental variables** in dynamic noncooperative games of private information
- Repeated matching game is fully specified model, **need to prove** which variables can be instruments for current agent states like x and y

1. Lags of current x or lags of past partners y may be **independent** of current time persistent, **unmeasured state** ν or independent of **one period change** in ν, $\nu_t - \nu_{t-1}$
    - Standard argument in linear panel data models (e.g., Arellano & Bond 1991)
2. **Aggregate state** (m, n) changes over **time** (or across **geography**)
    - More or fewer agents of say high types affect probability that particular worker x matches to particular firm y
    - Previous period aggregate states (m, n) affect previous matches of worker and hence evolution of state variable of worker
    - Static matching: Sørensen (2007) & Ackerberg & Botticini (2002)

Trading Networks Model

- Often agents **both buy and sell** simultaneously
- Supplier of metal car parts buys steel, sells car parts to assembler
- **Trading networks** allows simultaneous buying, selling
  - Static: Hatfield, Kominers, Nichifor, Ostrovsky & Westkamp (2013), Azevedo & Hatfield (2015)
- Per period flow profit for type x

$$u_x(\Phi, \Psi) - \sum_{\omega \in \Phi} p_\omega + \sum_{\omega \in \Psi} p_\omega$$

- $\Phi \subseteq \Omega_x$ trades ω where x **buys**
- $\Psi \subseteq \Omega_x$ trades ω where x **sells**
- Previous theorems should extend to trading networks

Identification of Repeated Matching Games

- Many fascinating issues in **identification of static, transferable utility matching games**
  - e.g., Fox (2010) and Fox, Yang, & Hsu (2018)
- In current dynamic model, **different transition rules** P and Q of workers and firms do **not** allow **separate identification of flow profits** of workers and firms
- Many originally surprising issues in **identification of single-agent dynamic discrete choice models**
  - e.g., Rust (1994) and Kalouptsidi, Scott & Souza-Rodrigues (2020)
- Highlight role of a **strong normalization**: value of being unmatched is **zero plus a logit shock** for all worker types x and firm types y

Geography and Employer Switching

- Matched employer/employee data on most **Swedish engineers** from 1970–1990
- Used in Fox (2009, 2010)
- Today focus on **geographic** aspects of switching employers
- Moving costs

Geography and Employer Switching

- Five worker **age bins**
- Define four **regions in Sweden**
  1. Stockholm
  2. Counties adjacent to Stockholm
  3. Counties in south Sweden
  4. Counties in the center and north of the country.
- **Worker types** and **firm types**

$$x = (\text{age}, \text{previous location}) \ \text{ and } y = (\text{location})$$

- Skipping some details about entry/exit, **worker transition rule** P is deterministic given firm choice y
- Have to move to region where new employer is

- Match production function with switching costs in geographic distance

$$\Phi^{\theta}_{xy} = \alpha^{\theta}_{xy} + \gamma^{\theta}_{xy} = \sum_{a=1}^{5} \theta_a \mathbb{1}_{[x_{\mathrm{age}}=a]} \mathrm{dist}(x,y)$$

- Cannot identify whether switching costs accrue to workers or firms
    - Likely workers as workers move in real life

Method	Age bin 1	Age bin 2	Age bin 3	Age bin 4	Age bin 5
Min/Max	-51.81	-49.86	-49.10	-47.28	-52.97
MPEC	-48.42	-48.88	-51.22	-48.31	-50.32

Summary

- Introduce **repeated matching game**
  - Each period, **all agents match**
  - Market clears each period
  - Agent **state variables** evolve according to **current matches**
- Natural extension of say Shapley and Shubik (1971) to dynamic matching
- Show equilibrium **existence**, **social planner** property
- **Stationary equilibrium** exists
- Model with **econometric errors**
  - **Social planner** problem
  - Show **stationary equilibrium exists**
  - **Two numerical algorithms** to compute stationary equilibrium
  - Both algorithms can also be used for **structural estimation**
- **Empirical** application to Swedish engineers