Cutoff Characterization of Matching Mechanisms

Irene Lo
Stanford University

DSE Summer School, August 2022

Cutoffs and Large Matching Markets

- Matching mechanisms in school choice:
  - School choice: Many-to-one matching
  - Deferred Acceptance (DA) and Top Trading Cycles (TTC)

- Cutoff characterizations:
  - **Azevedo & Leshno (2016)**: Cutoff characterization of DA
  - **Leshno & Lo (2018):** Cutoff characterization of TTC
  - Cutoffs can be used to define propensity scores

- Two simplifications:
  - A large matching market – many students, each college has many seats
  - Cutoffs via non-combinatorial equations

Cutoff Characterizations

- **Recap**
- Model
- Deferred Acceptance
- Top Trading Cycles

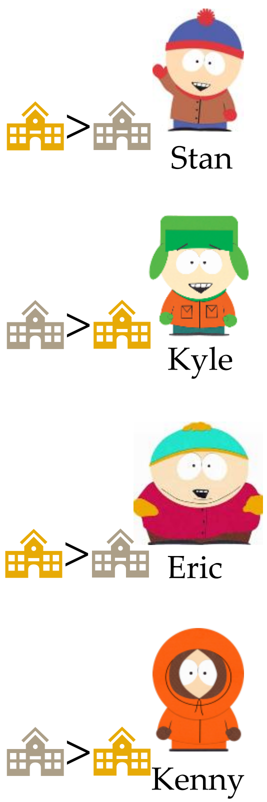
School Choice

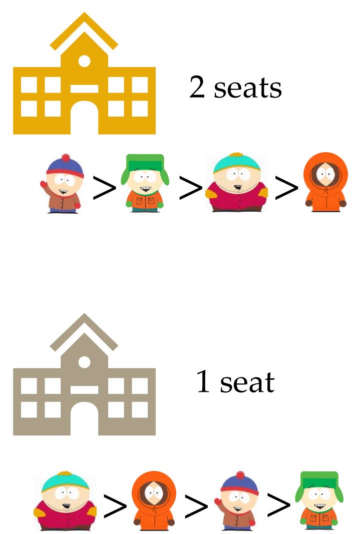
- Finite set of students Θ
  - Student θ has preferences $\succ^{\theta}$ over schools
- Finite set of schools C
  - School c can admit q_c students
  - $\succ^{c}$ a strict ranking over students, responsive preferences
- Assignment $\mu: \Theta \rightarrow C \cup \phi$ of students to schools
  - Feasibility constraint: $\mu^{-1}(c) \leq q_c$

- Abdulkadiroğlu & Sönmez 2013:
  - School choice is a mechanism design problem
  - Candidate mechanisms:
    Deferred Acceptance (DA) and Top Trading Cycles (TTC)

Deferred Acceptance

[Gale & Shapley '62]



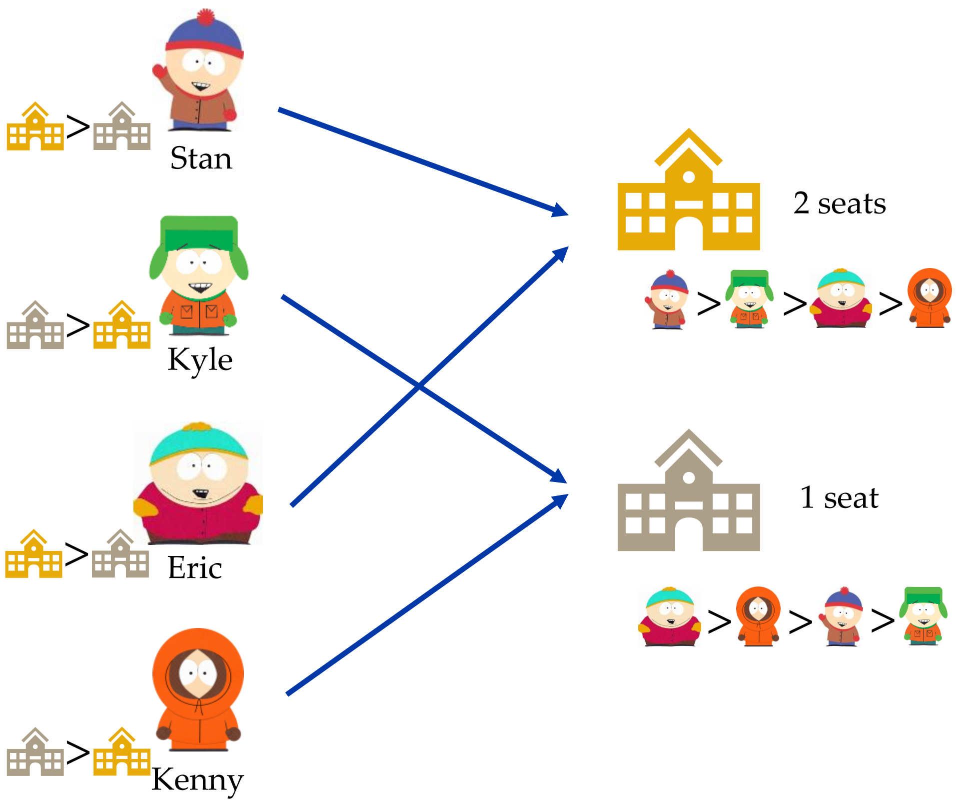


Student-Proposing Deferred Acceptance

```
while some student can still
propose:
    all students propose to
    favorite school that has
    not rejected them  before
    all schools tentatively
    accept top students (up
    to capacity) who have
    proposed to them and
    reject the rest
each student assigned to
favorite school that has still
tentatively accepted them
```

Deferred Acceptance

[Gale & Shapley '62]

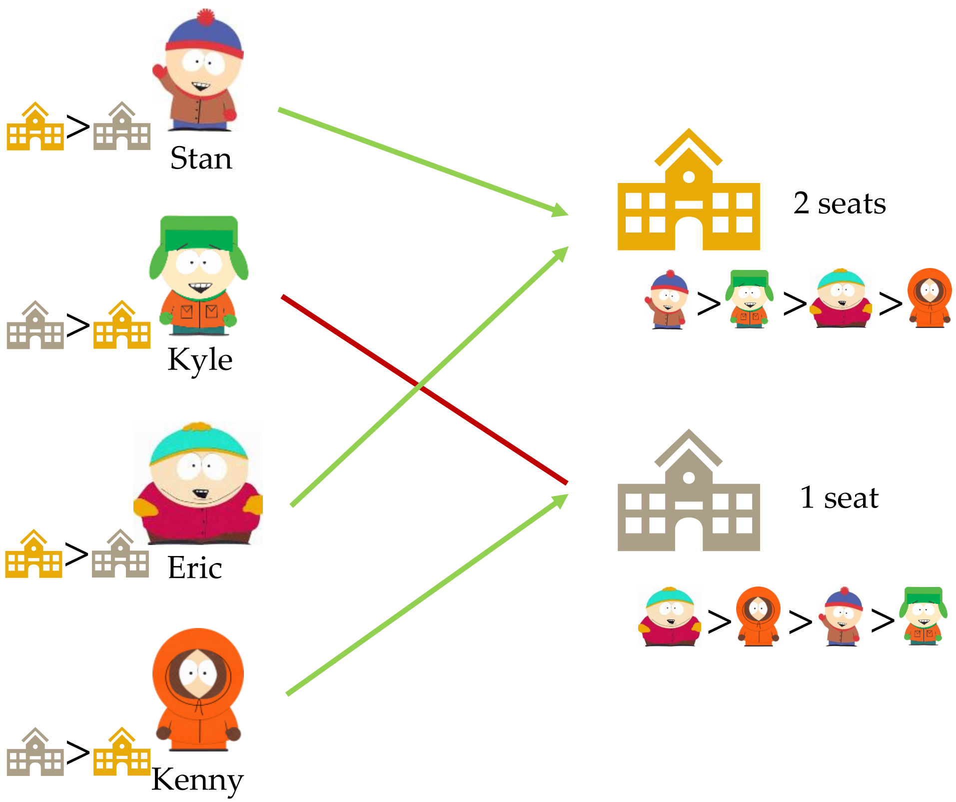



```
Student-Proposing Deferred
Acceptance

while some student can still
propose:
    all students propose to
    favorite school that has
    not rejected them  before
    all schools tentatively
    accept top students (up
    to capacity) who have
    proposed to them and
    reject the rest
each student assigned to
favorite school that has still
tentatively accepted them
```

Deferred Acceptance

[Gale & Shapley ’62]

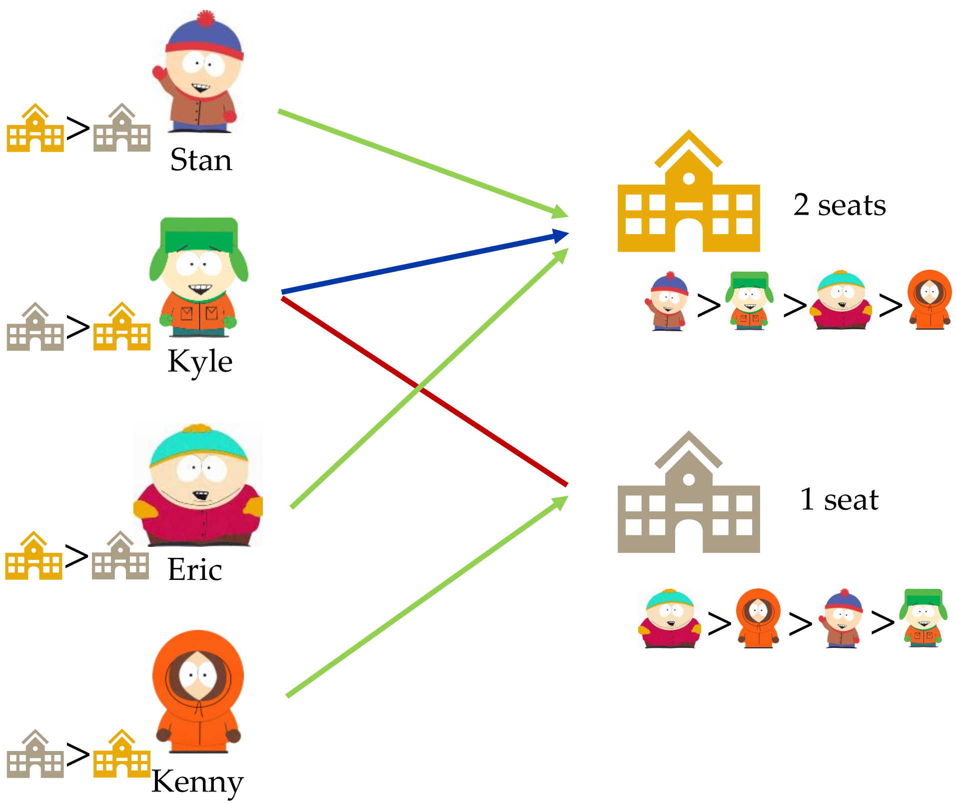



```
Student-Proposing Deferred
Acceptance

while some student can still
propose:
    all students propose to
    favorite school that has
    not rejected them before
    all schools tentatively
    accept top students (up
    to capacity) who have
    proposed to them and
    reject the rest
each student assigned to
favorite school that has still
tentatively accepted them
```

Deferred Acceptance

[Gale & Shapley ’62]

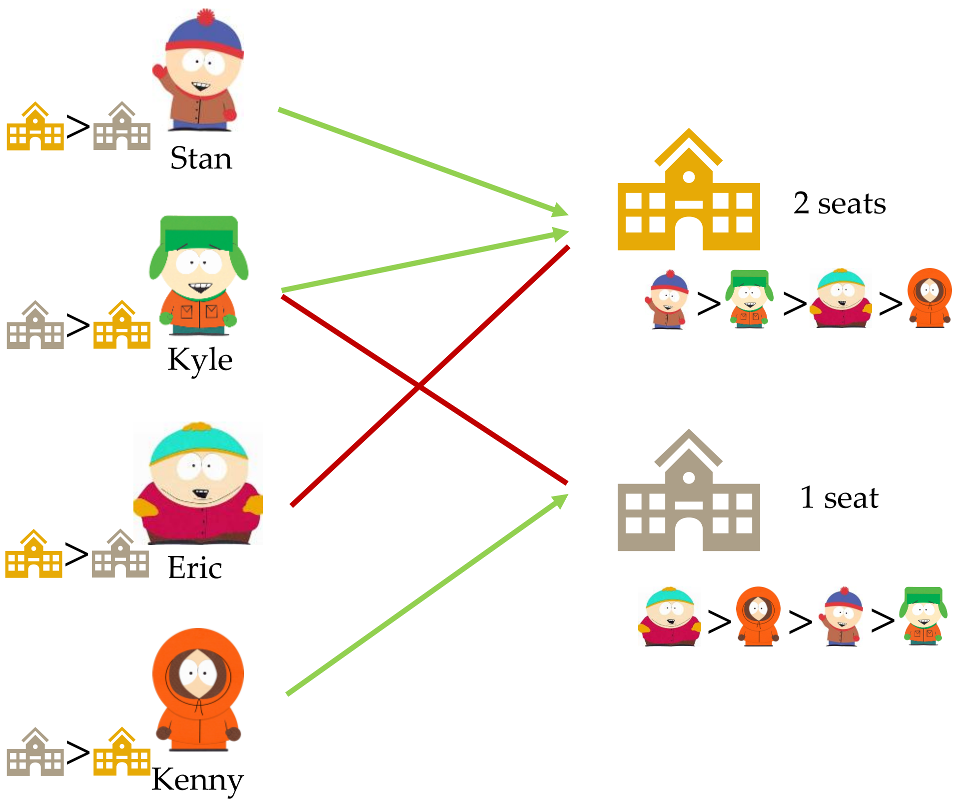



```
Student-Proposing Deferred
Acceptance

while some student can still
propose:
      all students propose to
      favorite school that has
      not rejected them  before
      all schools tentatively
      accept top students (up
      to capacity) who have
      proposed to them and
      reject the rest
each student assigned to
favorite school that has still
tentatively accepted them
```

Deferred Acceptance

[Gale & Shapley ’62]

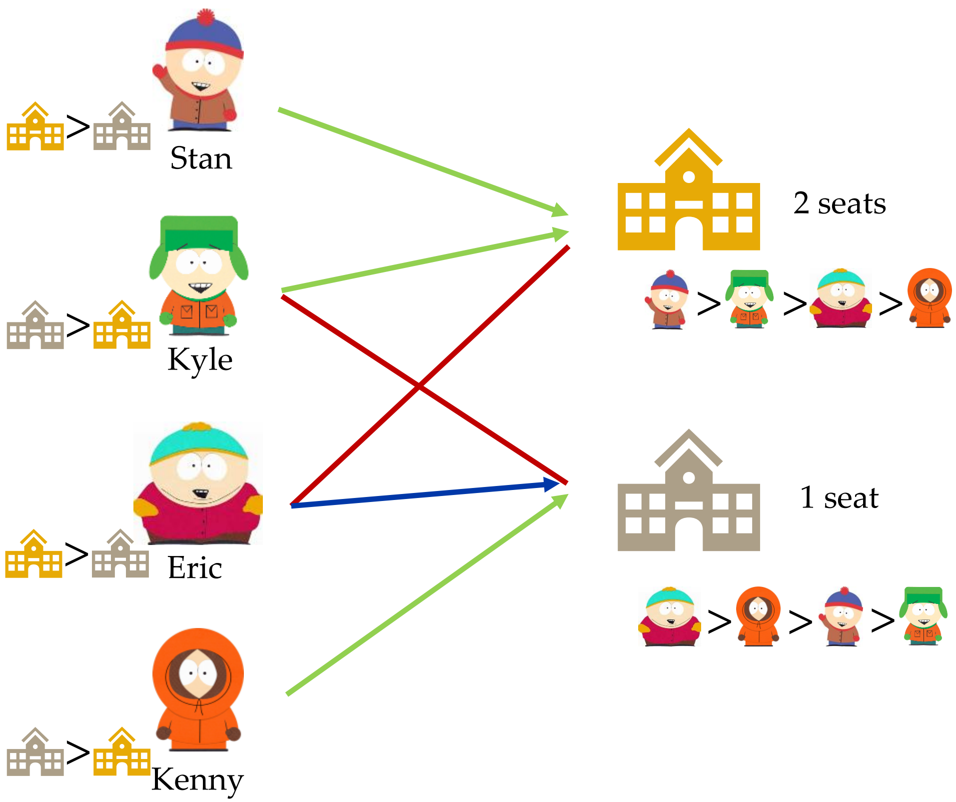



```
Student-Proposing Deferred
Acceptance

while some student can still
propose:
      all students propose to
      favorite school that has
      not rejected them  before
      all schools tentatively
      accept top students (up
      to capacity) who have
      proposed to them and
      reject the rest
each student assigned to
favorite school that has still
tentatively accepted them
```

Deferred Acceptance

[Gale & Shapley ’62]

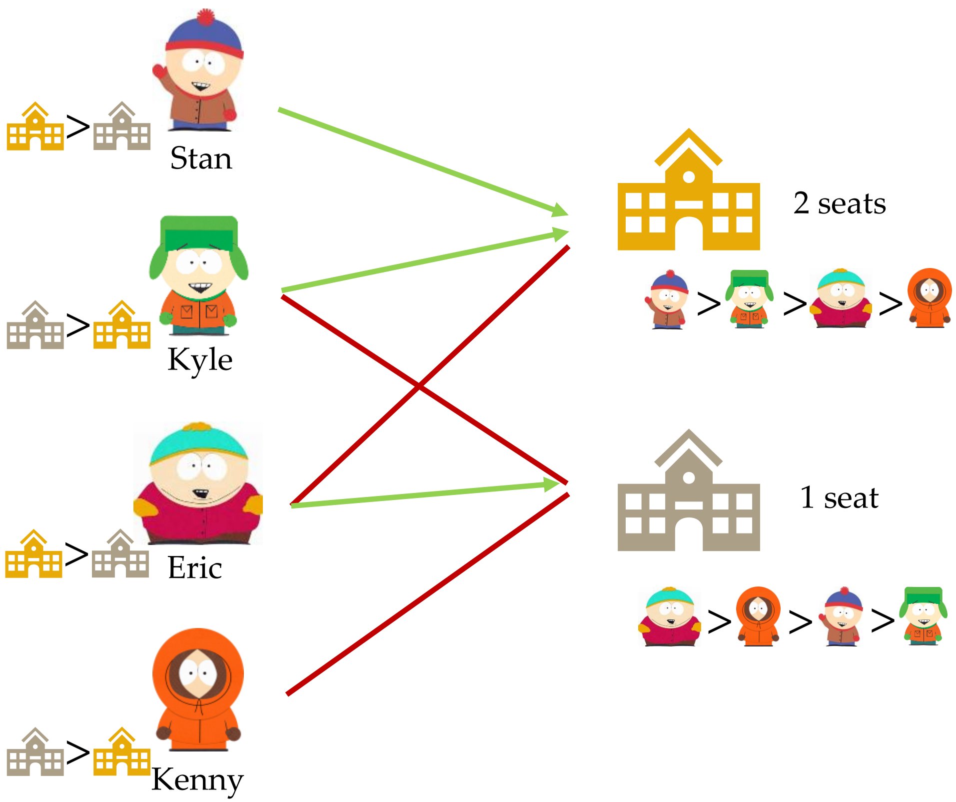



```
Student-Proposing Deferred
Acceptance

while some student can still
propose:
      all students propose to
      favorite school that has
      not rejected them  before
      all schools tentatively
      accept top students (up
      to capacity) who have
      proposed to them and
      reject the rest
each student assigned to
favorite school that has still
tentatively accepted them
```

Deferred Acceptance

[Gale & Shapley ’62]

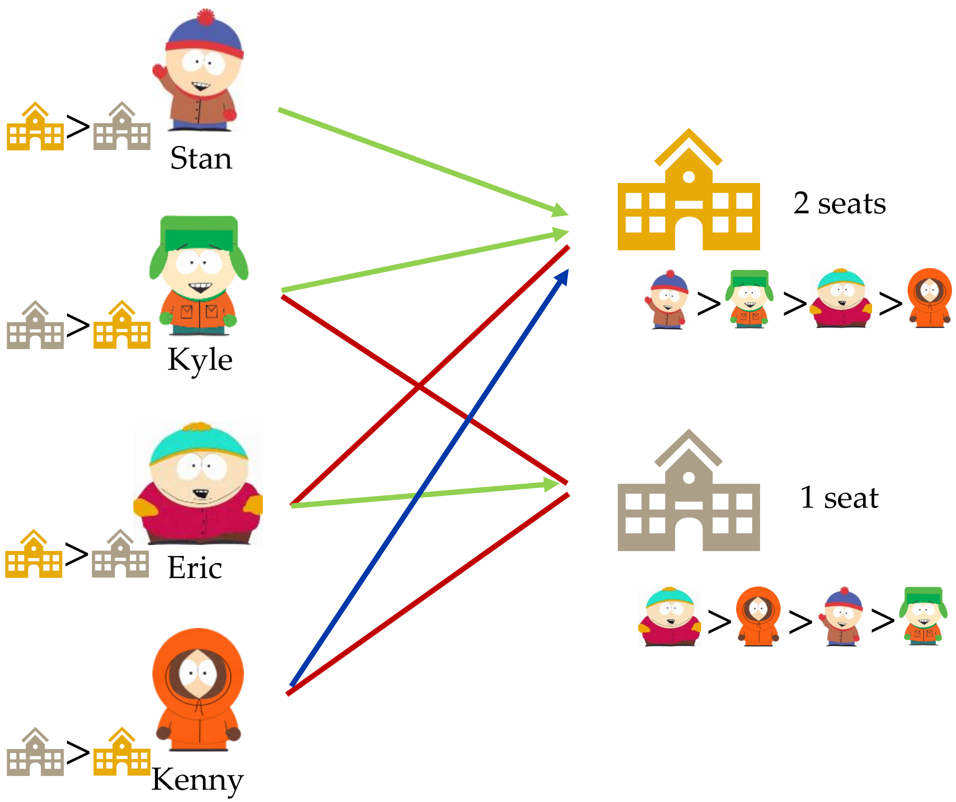



```
Student-Proposing Deferred
Acceptance

while some student can still
propose:
      all students propose to
      favorite school that has
      not rejected them  before
      all schools tentatively
      accept top students (up
      to capacity) who have
      proposed to them and
      reject the rest
each student assigned to
favorite school that has still
tentatively accepted them
```

Deferred Acceptance

[Gale & Shapley ’62]

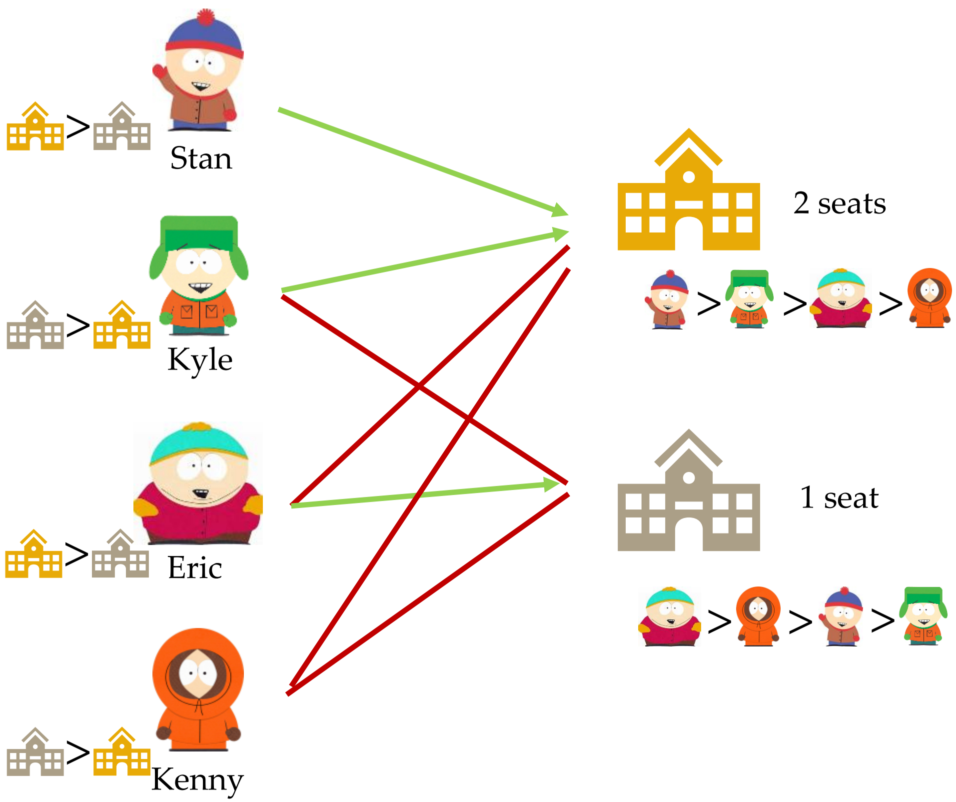



```
Student-Proposing Deferred
Acceptance

while some student can still
propose:
    all students propose to
    favorite school that has
    not rejected them before
    all schools tentatively
    accept top students (up
    to capacity) who have
    proposed to them and
    reject the rest
each student assigned to
favorite school that has still
tentatively accepted them
```

Deferred Acceptance

[Gale & Shapley ’62]

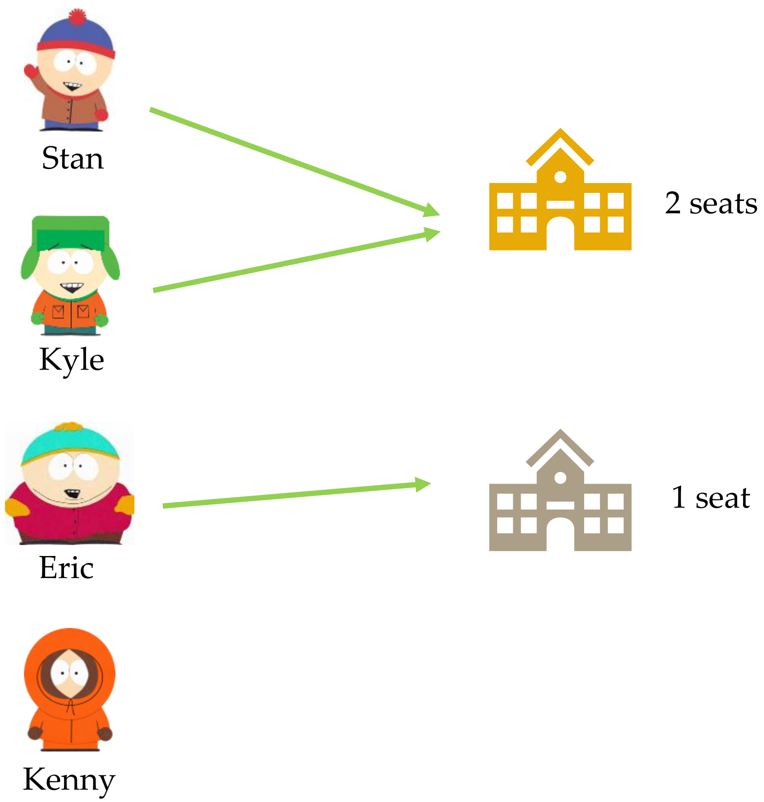



```
Student-Proposing Deferred
Acceptance

while some student can still
propose:
    all students propose to
    favorite school that has
    not rejected them before
    all schools tentatively
    accept top students (up
    to capacity) who have
    proposed to them and
    reject the rest
each student assigned to
favorite school that has still
tentatively accepted them
```

Deferred Acceptance

[Gale & Shapley '62]

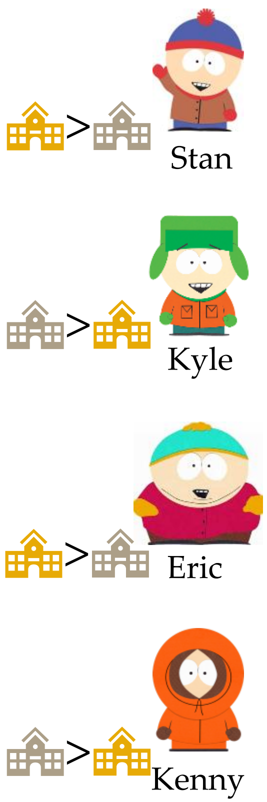


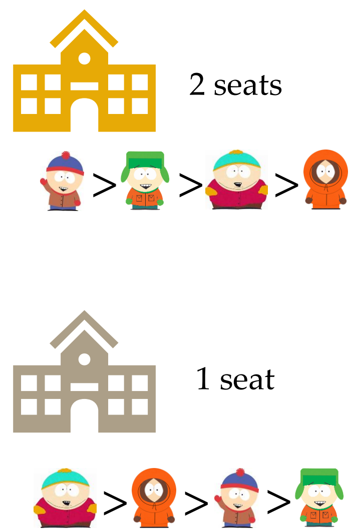

```
Student-Proposing Deferred
Acceptance

while some student can still
propose:
     all students propose to
     favorite school that has
     not rejected them  before
     all schools tentatively
     accept top students (up
     to capacity) who have
     proposed to them and
     reject the rest
each student assigned to
favorite school that has still
tentatively accepted them
```

Top Trading Cycles

[Shapley & Scarf ’74]



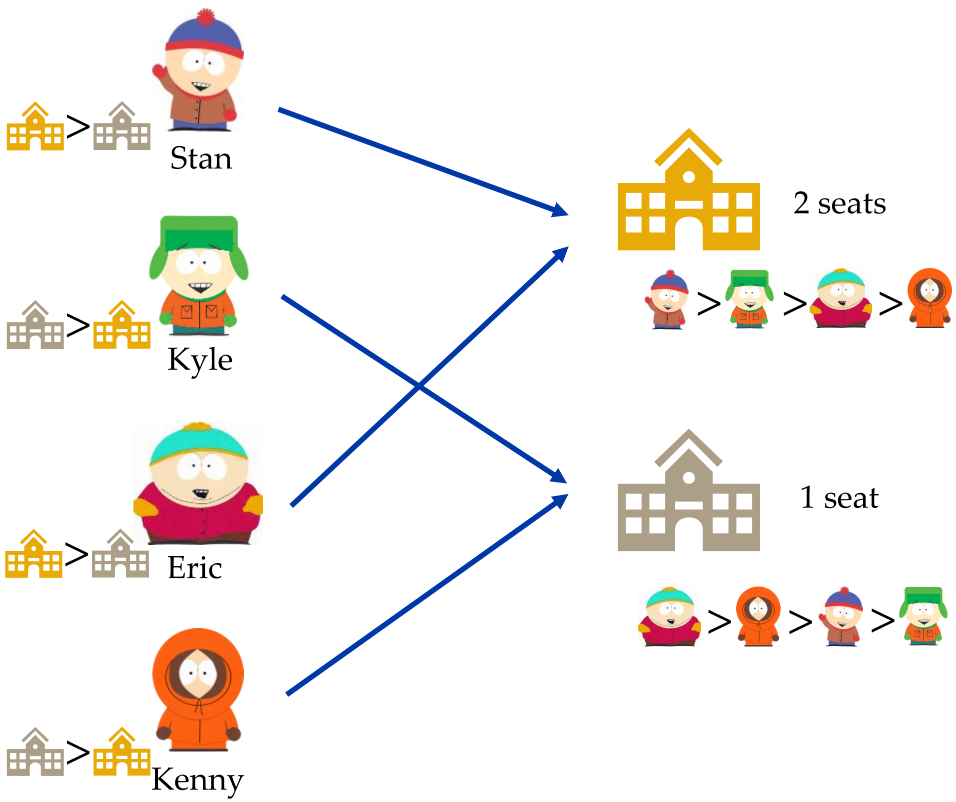



```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
    all remaining students
    point to favorite
    remaining school
    all remaining schools
    point to favorite
    remaining students
    select a cycle, assign
    students in cycle to
    school they point to
```

Top Trading Cycles

[Shapley & Scarf '74]

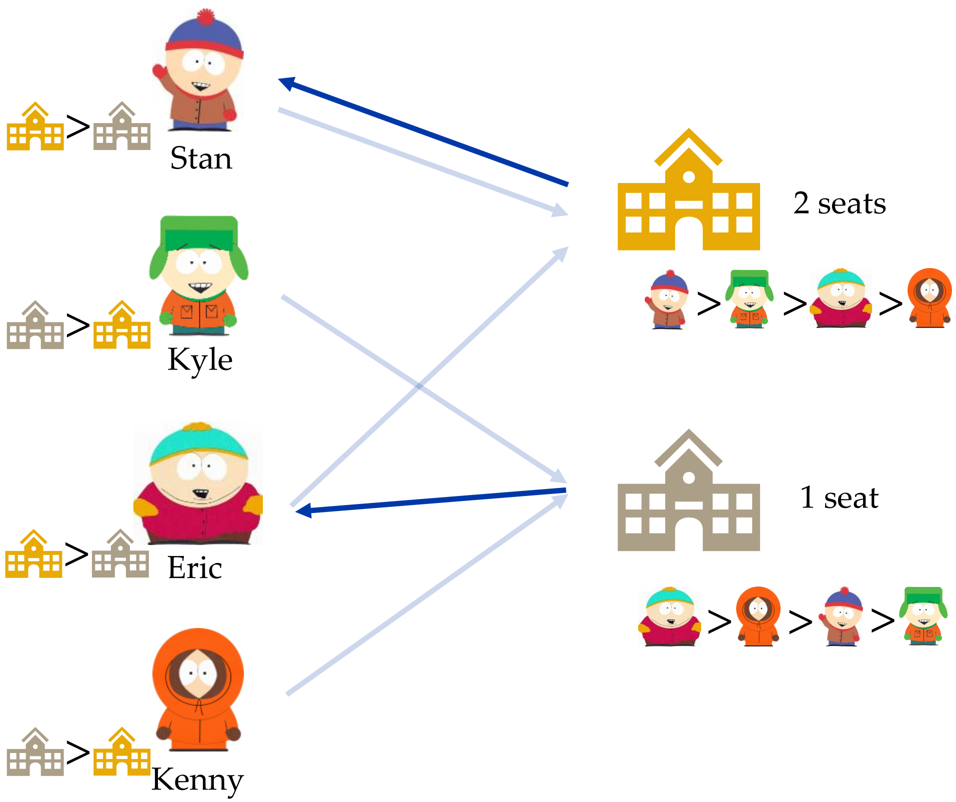



```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
    all remaining students
    point to favorite
    remaining school
    all remaining schools
    point to favorite
    remaining students
    select a cycle, assign
    students in cycle to
    school they point to
```

Top Trading Cycles

[Shapley & Scarf '74]

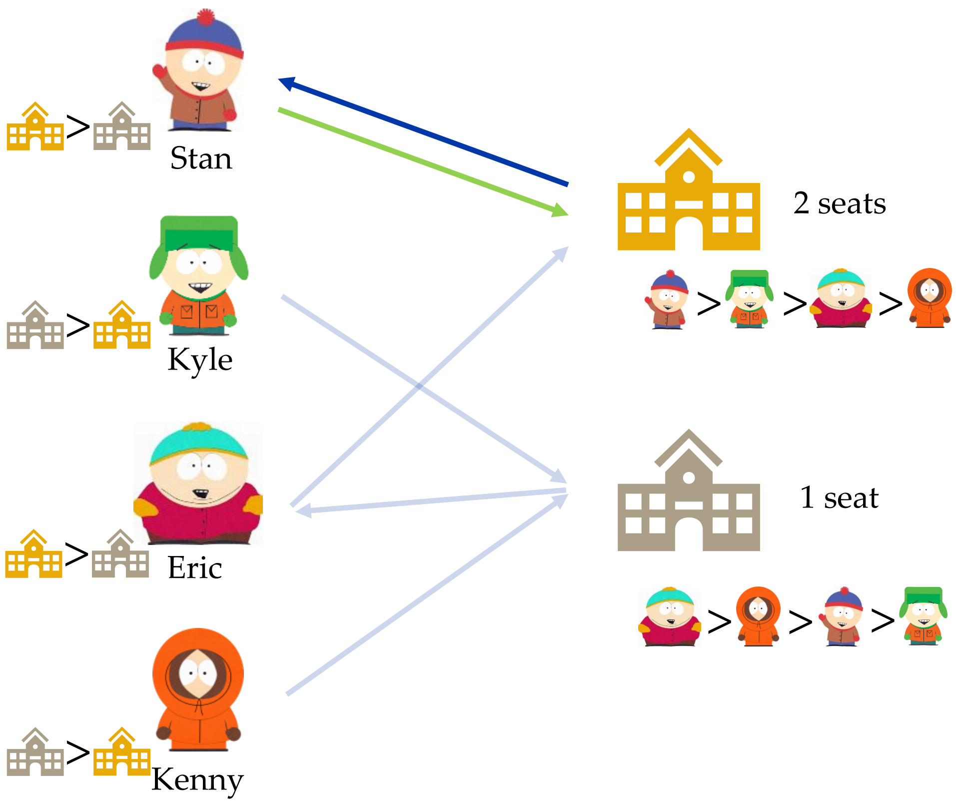



```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
       all remaining students
       point to favorite
       remaining school
       all remaining schools
       point to favorite
       remaining students
       select a cycle, assign
       students in cycle to
       school they point to
```

Top Trading Cycles

[Shapley & Scarf '74]

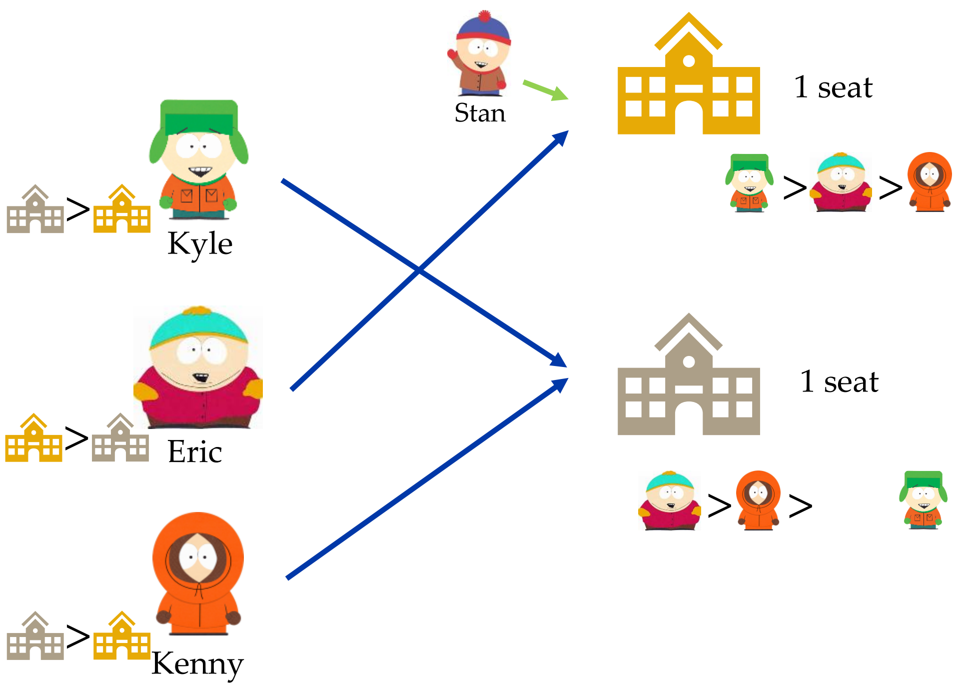



```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
      all remaining students
      point to favorite
      remaining school
      all remaining schools
      point to favorite
      remaining students
      select a cycle, assign
      students in cycle to
      school they point to
```

Top Trading Cycles

[Shapley & Scarf '74]

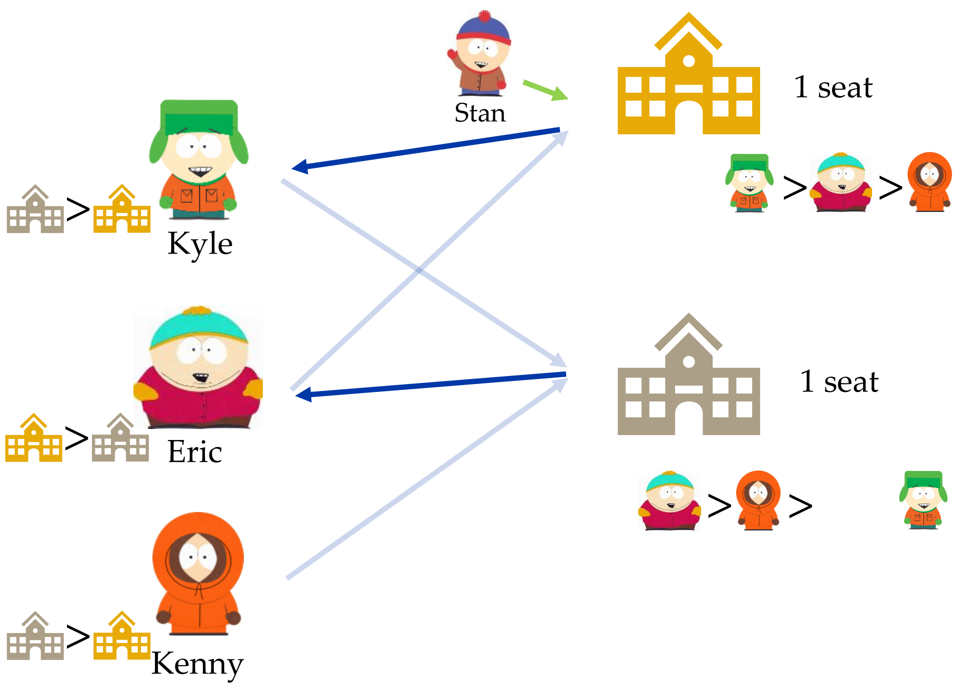



```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
    all remaining students
    point to favorite
    remaining school
    all remaining schools
    point to favorite
    remaining students
    select a cycle, assign
    students in cycle to
    school they point to
```

Top Trading Cycles

[Shapley & Scarf ’74]

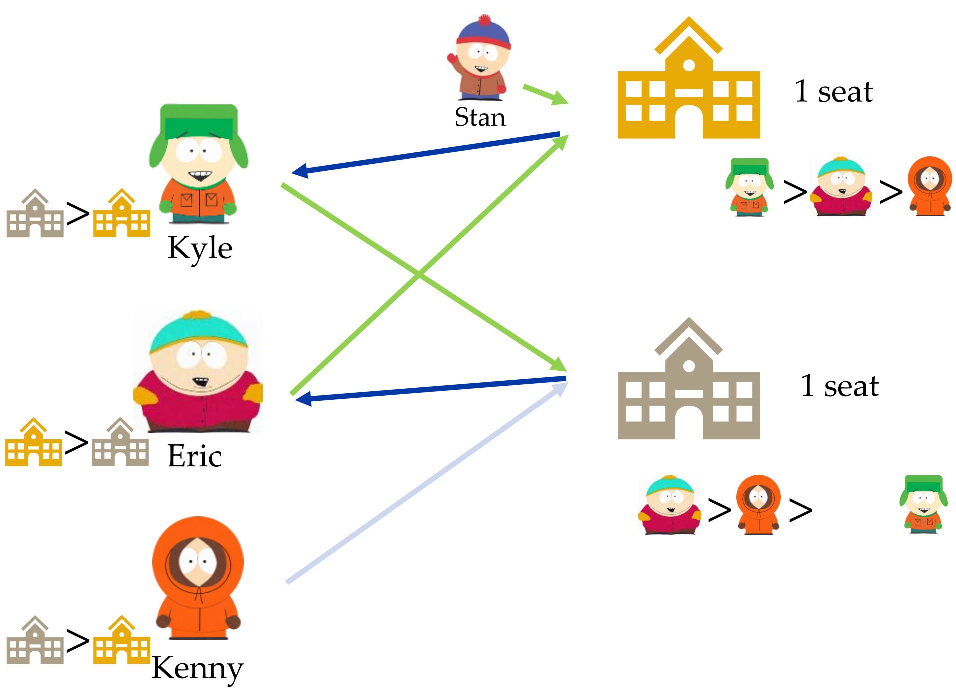



```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
    all remaining students
    point to favorite
    remaining school
    all remaining schools
    point to favorite
    remaining students
    select a cycle, assign
    students in cycle to
    school they point to
```

Top Trading Cycles

[Shapley & Scarf '74]

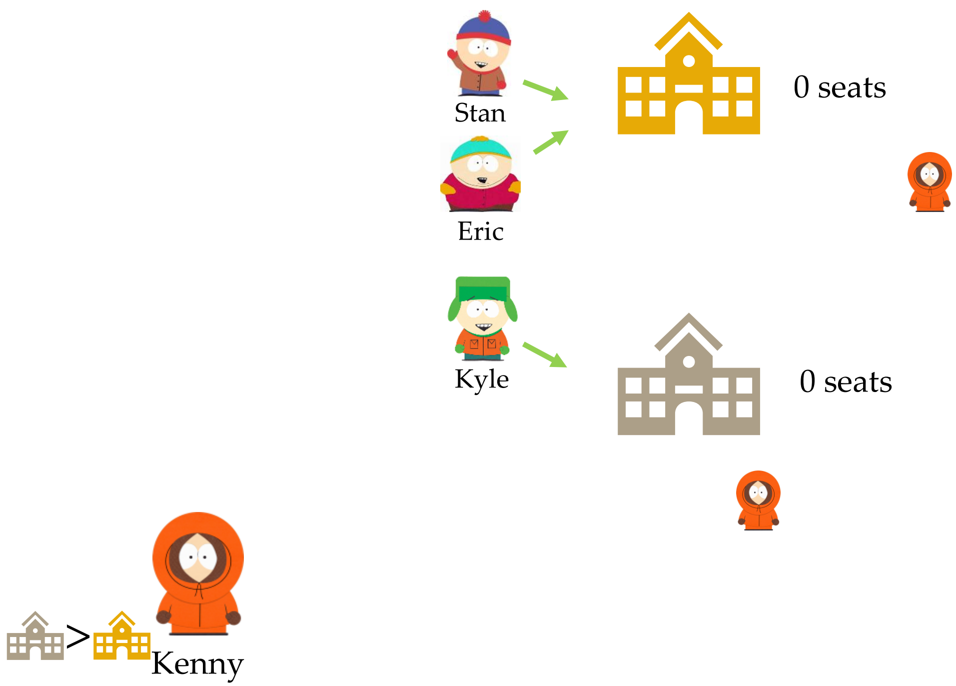



```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
    all remaining students
    point to favorite
    remaining school
    all remaining schools
    point to favorite
    remaining students
    select a cycle, assign
    students in cycle to
    school they point to
```

Top Trading Cycles

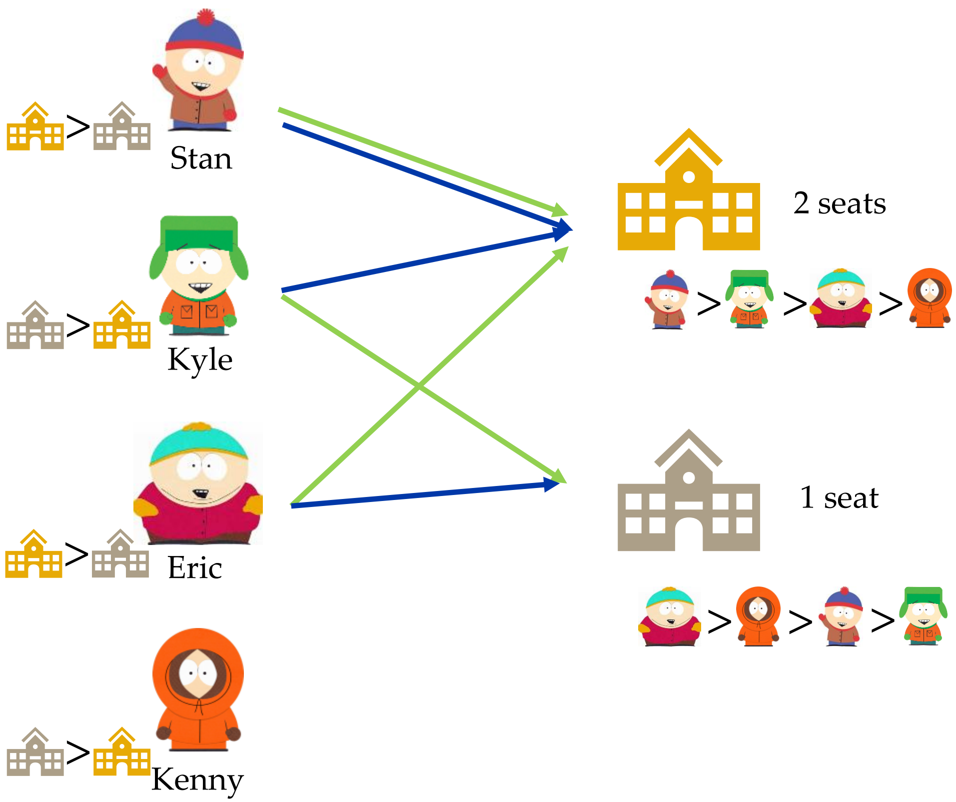
[Shapley & Scarf '74]




```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
    all remaining students
    point to favorite
    remaining school
    all remaining schools
    point to favorite
    remaining students
    select a cycle, assign
    students in cycle to
    school they point to
```

DA vs TTC



Desirable properties.

- Strategyproof for students:
  Truthtelling is a dominant strategy
- Stable: No blocking pairs.
  i.e. no school and student who prefer each other to their match
- Pareto efficient for students:
  No other assignment is weakly preferred by all students

DA is strategyproof & stable.

TTC is strategyproof & Pareto efficient.

Theorem.
No mechanism is strategyproof, stable, and Pareto efficient.

DA and TTC via Cutoffs

Is there a non-combinatorial way of understanding DA and TTC?

- Azevedo, Eduardo M., and Jacob D. Leshno. "**A supply and demand framework for two-sided matching markets.**" *Journal of Political Economy* 124.5 (2016): 1235-1268.

  The Deferred Acceptance outcome can be characterized by one cutoff for every school. These cutoffs solve supply and demand equations.

- Leshno, Jacob D. and Irene Y. Lo. "**The cutoff structure of Top Trading Cycles in school choice.**"

  The Top Trading Cycles outcome can be characterized by one cutoff for every **pair** of schools. These cutoffs solve trade balance equations.

Cutoff Characterizations

- Recap
- **Large Market Model**
- Deferred Acceptance
- Top Trading Cycles

Large Matching Markets

- A large matching market – many students, each college has many seats

- A simpler matching model
  - Continuum of students (Aumann 1964)
  - Supply and demand characterization of stable matching
  - Trade balance characterization of TTC

- Allows for
  - Simpler derivation of outcomes and comparative statics
  - Complex preferences and no transfers (like Gale-Shapely 1962)

School Choice Model

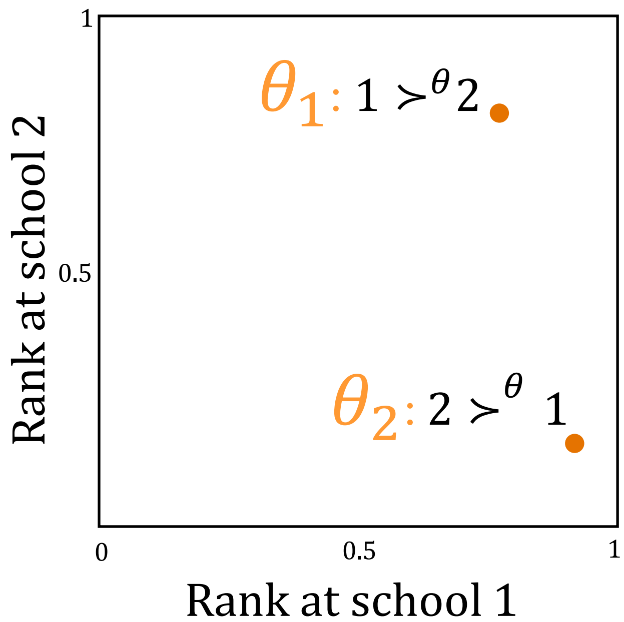
- Finite number of students $\theta = (\succ^{\theta}, r^{\theta})$
  - Student θ has preferences $\succ^{\theta}$ over schools
  - $r_c^{\theta} \in [0,1]$ is the rank of student θ at school c (percentile in c's priority list)
- Finite number of schools c
  - School c can admit q_c students
  - $\succ^{c}$ a strict ranking over students

School Choice **Continuum** Model

- **Continuous mass** of students $\theta = (\succ^\theta, r^\theta)$
  - Student θ has preferences $\succ^\theta$ over schools
  - $r_c^\theta \in [0,1]$ is the rank of student θ at school c (percentile in c's priority list)
  - **Distribution specified by measure η**

- Finite number of schools c
  - School c can admit q_c students

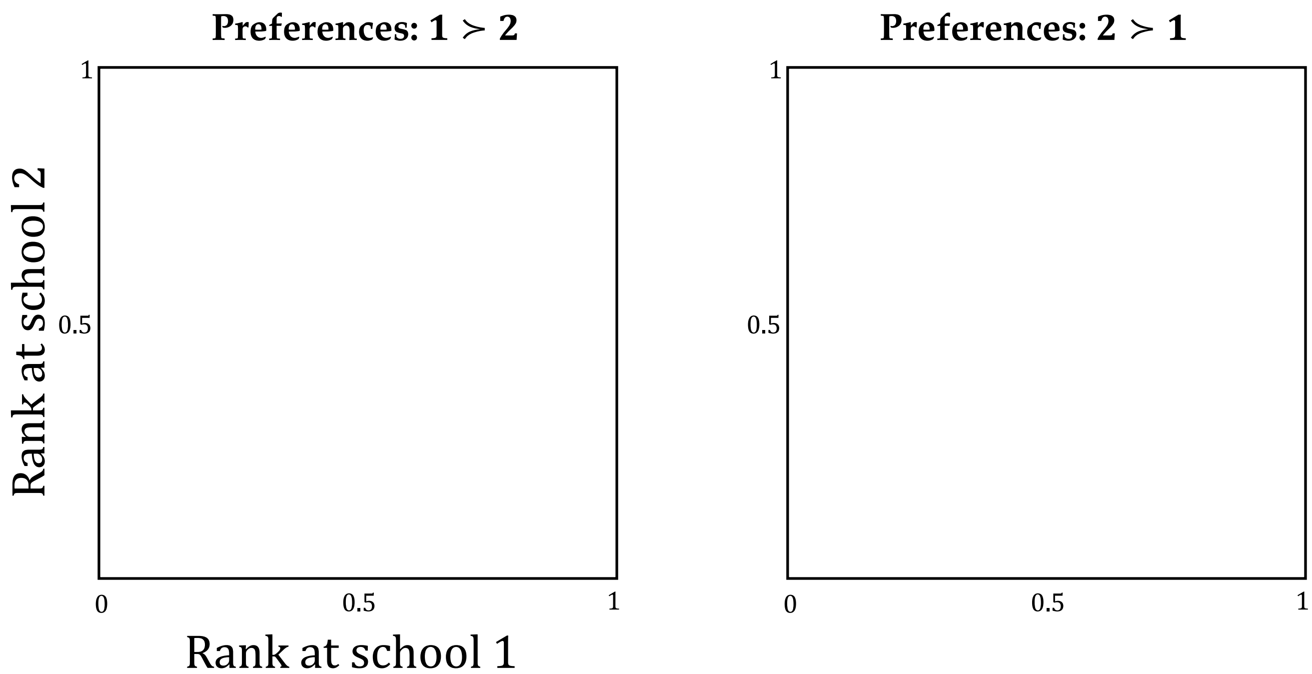
School Choice Visualization



Student θ_1
- prefers 1 to 2
- highly ranked at 1
- highly ranked at 2

Student θ_2
- prefers 2 to 1
- highly ranked at 1
- poorly ranked at 2

School Choice Visualization



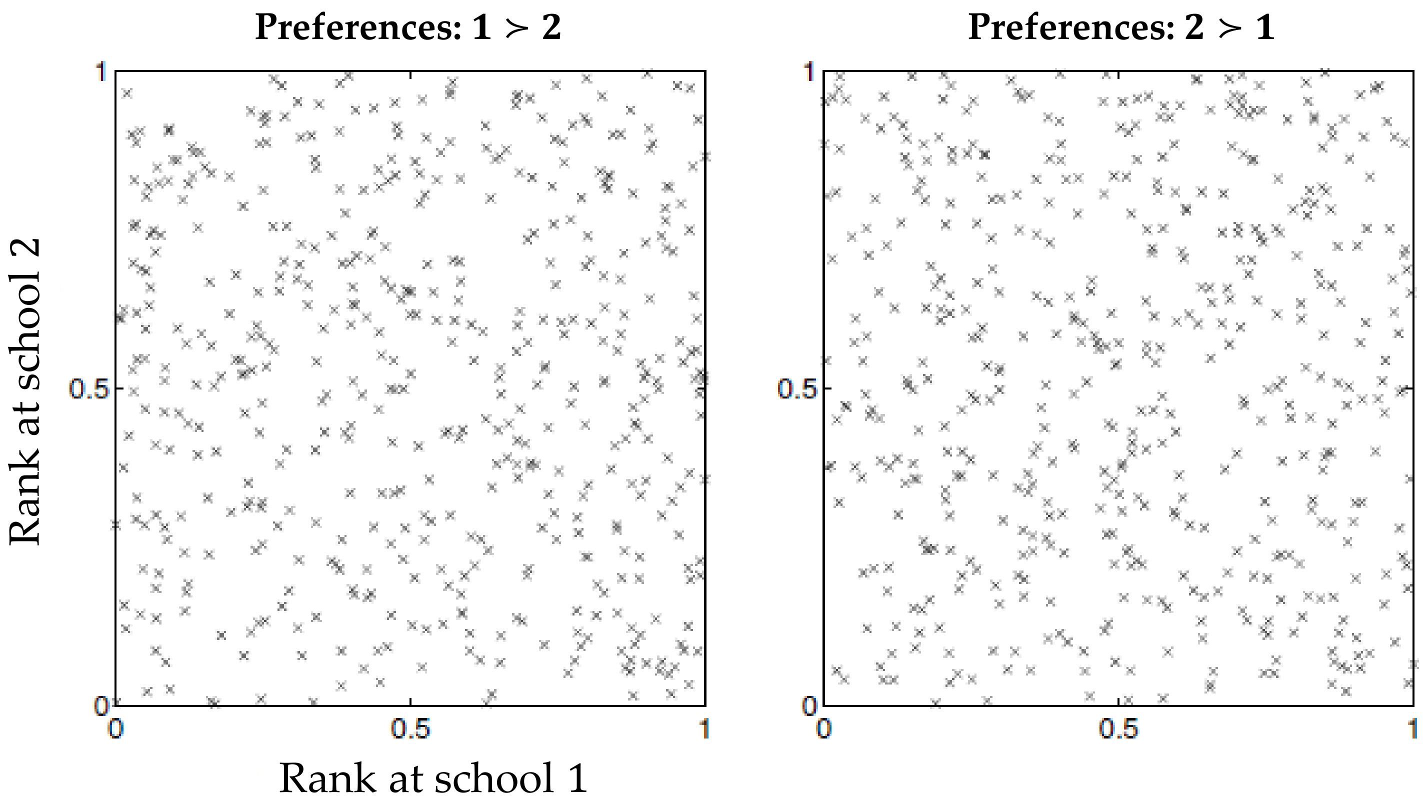
A Supply and Demand Framework for Two-Sided Matching Markets

Eduardo Azevedo & Jacob Leshno

Main Takeaways

The Deferred Acceptance outcome can be characterized by **cutoffs**.
These cutoffs solve **supply and demand equations**.

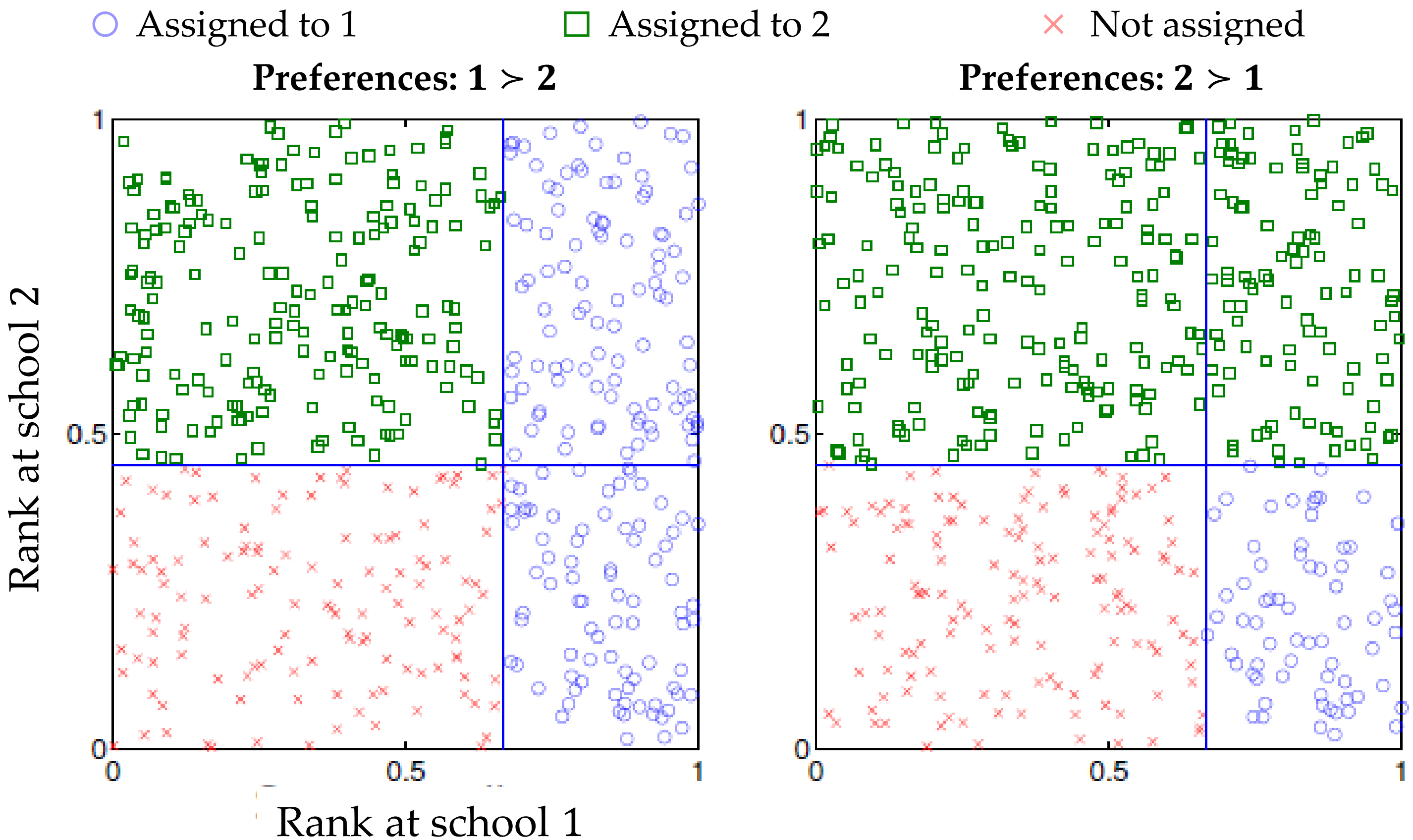
Example



1,000 students, 500 prefer 1, 500 prefer 2
$q_1 = 250, q_2 = 500$

Q. What is the outcome of DA on this economy?

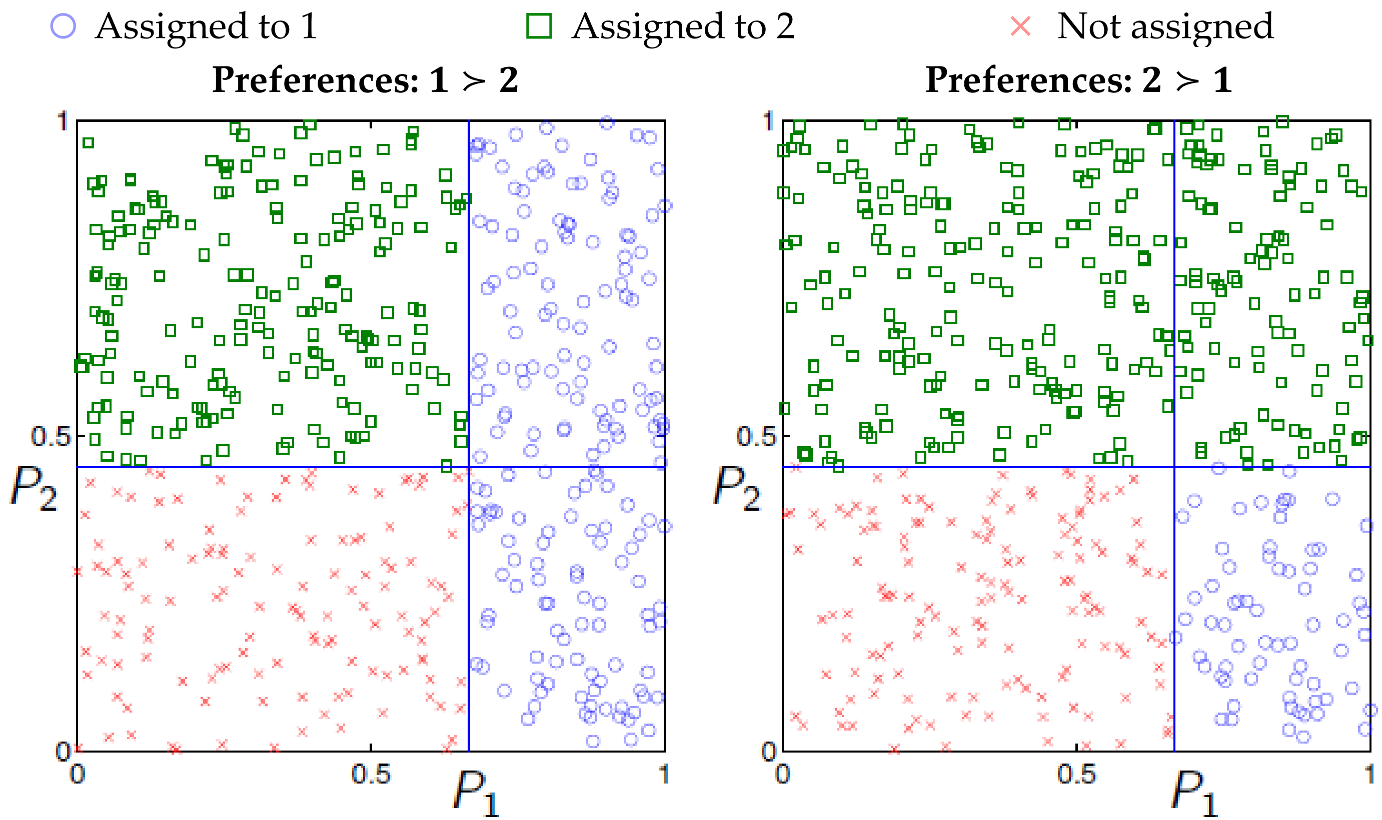
Example



1,000 students, 500 prefer 1, 500 prefer 2

$q_1 = 250, q_2 = 500$

A Simpler description



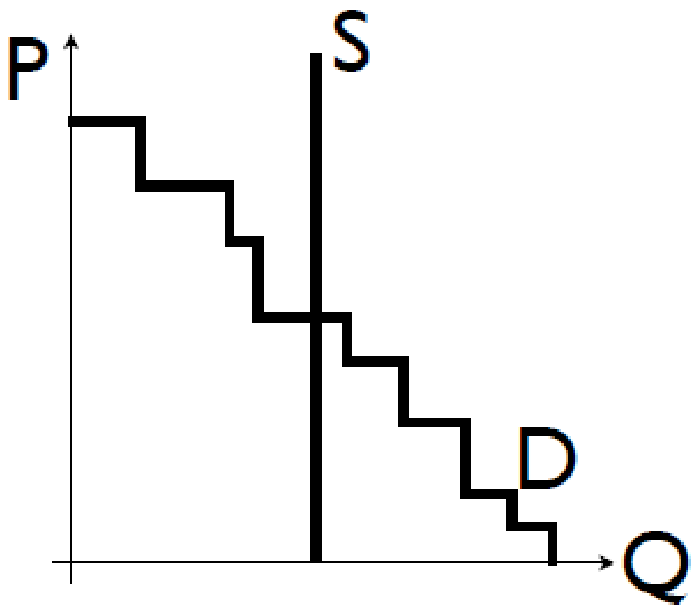
$$D(P) = S$$

Demand given cutoffs

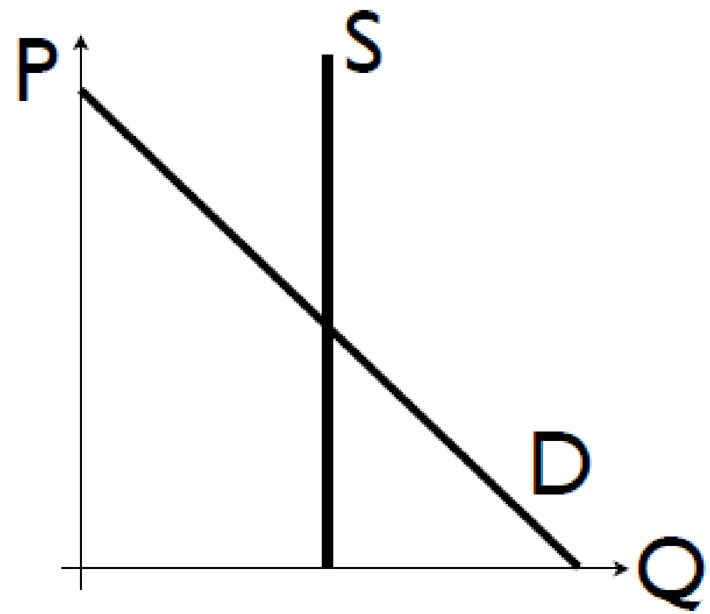
Supply of seats

Two Simplifications

1. Characterization of stable matching in terms of supply and demand



2. Continuum of agents



DA via Cutoffs (Azevedo Leshno)

- **Continuum Model**
- Cutoff Characterization
- Uniqueness and Convergence

Continuum Model

- Finite set of schools $c \in C = \{1, \dots, n\}$
  - School c can admit a mass q_c of students
- Measure η specifying a distribution of a continuous mass of students
  - A student $\theta \in \Theta$ is given by $\theta = \left(\succ^{\theta}, r^{\theta}\right)$
  - Student θ has preferences $\succ^{\theta}$ over schools
  - $r_c^{\theta} \in [0,1]$ is the student's rank at school c *(percentile in c's priority list)*

- An **economy** is $E = [\eta, q]$ where η is a distribution over student types and q is a vector of capacities
- **Assumption.** *(Strict preferences)* Schools' indifference curves have measure 0

$$\eta\left(\left\{\theta \in \Theta \mid r_c^{\theta} = x\right\}\right) = 0 \quad \forall x, c$$

Matchings

Def. A **matching** is a function

$$\mu: \Theta \to C \cup \{\phi\}$$

such that

1. Each student is matched to a school or the empty set
2. *(Feasibility)* Each school is matched to a set of students $\mu^{-1}(c)$ such that $\eta\left(\mu^{-1}(c)\right) \leq q_c$
3. *(Right continuity)* For $\theta^k = \left(\succ, r^k\right), r^{k+1} \leq r^k$ then

$$\mu\left(\lim \theta^k\right) = \lim \mu\left(\theta^k\right).$$

Stable Matchings

Def. A matching μ is **stable** if there are no blocking pairs, i.e. a student-school pair (θ, c) such that

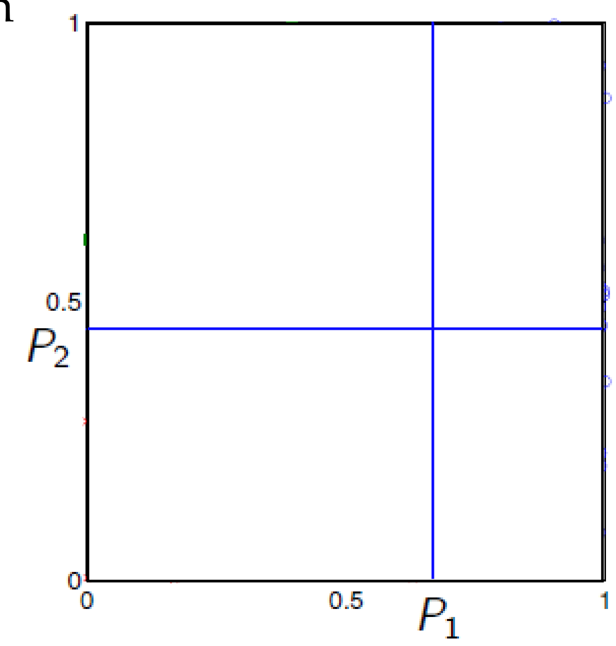
- Student θ prefers c over $\mu(\theta)$
- c either:
  I. did not fill its quota $\eta\left(\mu^{-1}(c)\right) < q_c$
  II. is matched to some less preferred student $\theta' \in \mu^{-1}(c)$ where $r_c^{\theta'} < r_c^{\theta}$

Source: Karlin & Peres, "Game Theory Alive"

Supply and Demand

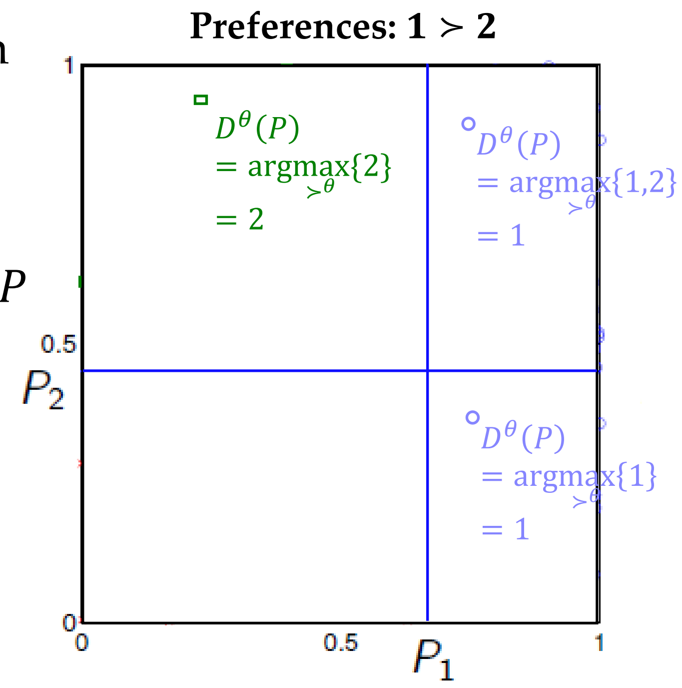
- A vector of **cutoffs *P*** is a vector $P \in [0,1]^C$ specifying a minimal score (cutoff) P_c for each school
  *How much does a school need to like a student for them to have that school as an **option**?*

Preferences: 1 ≻ 2



Supply and Demand

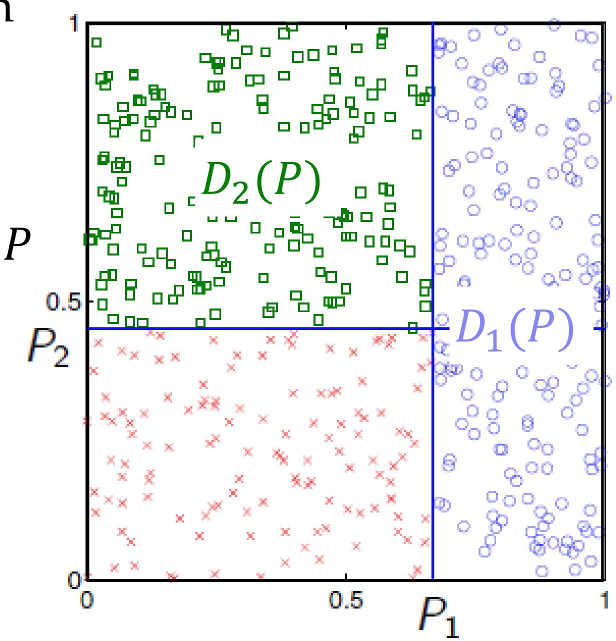
- A vector of **cutoffs P** is a vector $P \in [0,1]^C$ specifying a minimal score (cutoff) P_c for each school
  How much does a school need to like a student for them to have that school as an ***option****?*
- The **demand $D^\theta(P)$** of student θ given cutoff P is her most preferred school where she meets the cutoff: $D^\theta(P) = \arg\max_{\succ^\theta}\{c \mid P_c \leq r_c^\theta\}$
  What is student θ's favorite option?



Supply and Demand

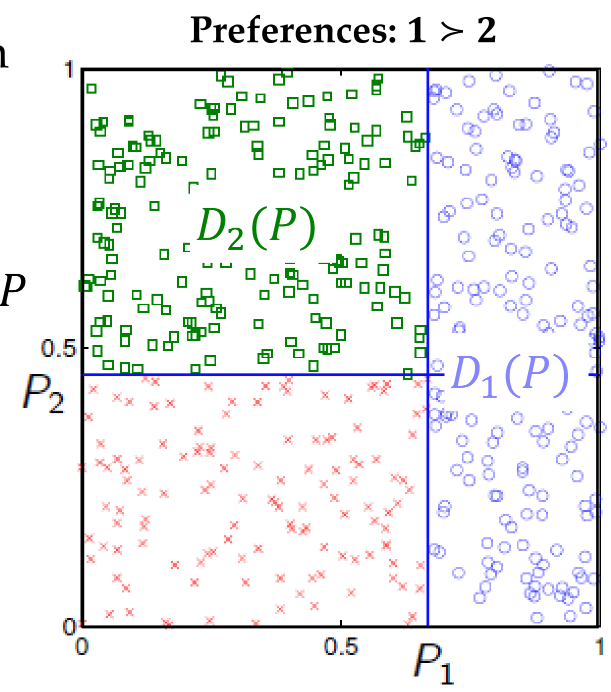
- A vector of **cutoffs P** is a vector $P \in [0,1]^C$ specifying a minimal score (cutoff) P_c for each school
  *How much does a school need to like a student for them to have that school as an **option**?*
- The **demand $D^\theta(P)$** of student θ given cutoff P is her most preferred school where she meets the cutoff: $D^\theta(P) = \arg\max_{\succ^\theta}\{c \mid P_c \leq r_c^\theta\}$
  What is student θ's favorite option?
- **Aggregate demand $D(P)$** is the mass of students demanding each school
  $$D_c(P) = \eta(\{\theta \mid D^\theta(P) = c\})$$

Preferences: 1 ≻ 2



Market Clearing Cutoffs

- A vector of **cutoffs** $\boldsymbol{P}$ is a vector $P \in [0,1]^C$ specifying a minimal score (cutoff) P_c for each school
  How much does a school need to like a student for them to have that school as an ***option****?*
- The **demand** $\boldsymbol{D^\theta(P)}$ of student θ given cutoff P is her most preferred school where she meets the cutoff: $D^\theta(P) = \arg\max_{\succ^\theta}\{c \mid P_c \leq r_c^\theta\}$
  What is student θ's favorite option?
- **Aggregate demand** $\boldsymbol{D(P)}$ is the mass of students demanding each school
  $$D_c(P) = \eta\left(\{\theta \mid D^\theta(P) = c\}\right)$$



Def. P^* is a vector of **market clearing cutoffs** if
$$D_c(P^*) \leq q_c \text{ for all } c \qquad \text{with equality if } P_c^* > 0$$

DA via Cutoffs (Azevedo Leshno)

- Continuum Model
- **Cutoff Characterization**
- Uniqueness and Convergence

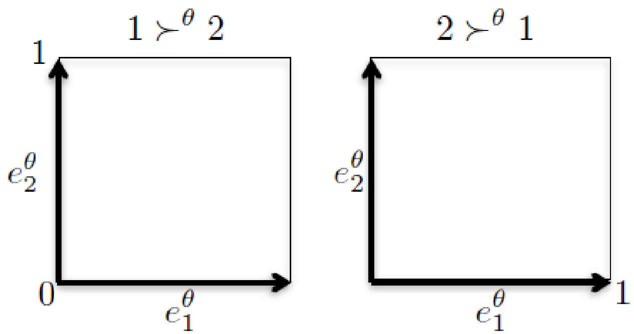
Matching and Cutoffs

- **$\mathcal{P}$: Matchings → Cutoffs:**
  Given a matching μ let $P = \mathcal{P}\mu$ be the scores of the marginal accepted students
  $$P_c = \inf_{\theta \in \mu(c)} e_c^\theta$$
- **$\mathcal{M}$: Cutoffs → Matchings:**
  Given cutoffs P let $\mu = \mathcal{M}P$ be the match resulting from the demand under P
  $$\mu(\theta) = D^\theta(P)$$

Lemma. $\mathcal{M}$ and $\mathcal{P}$ take stable matchings into market clearing cutoffs, and are inverses of each other

i.e. Stable matchings can be characterized using market-clearing cutoffs!

Intuition



$\frac{1}{2}$ students prefer 1, $\frac{1}{2}$ prefer 2

$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

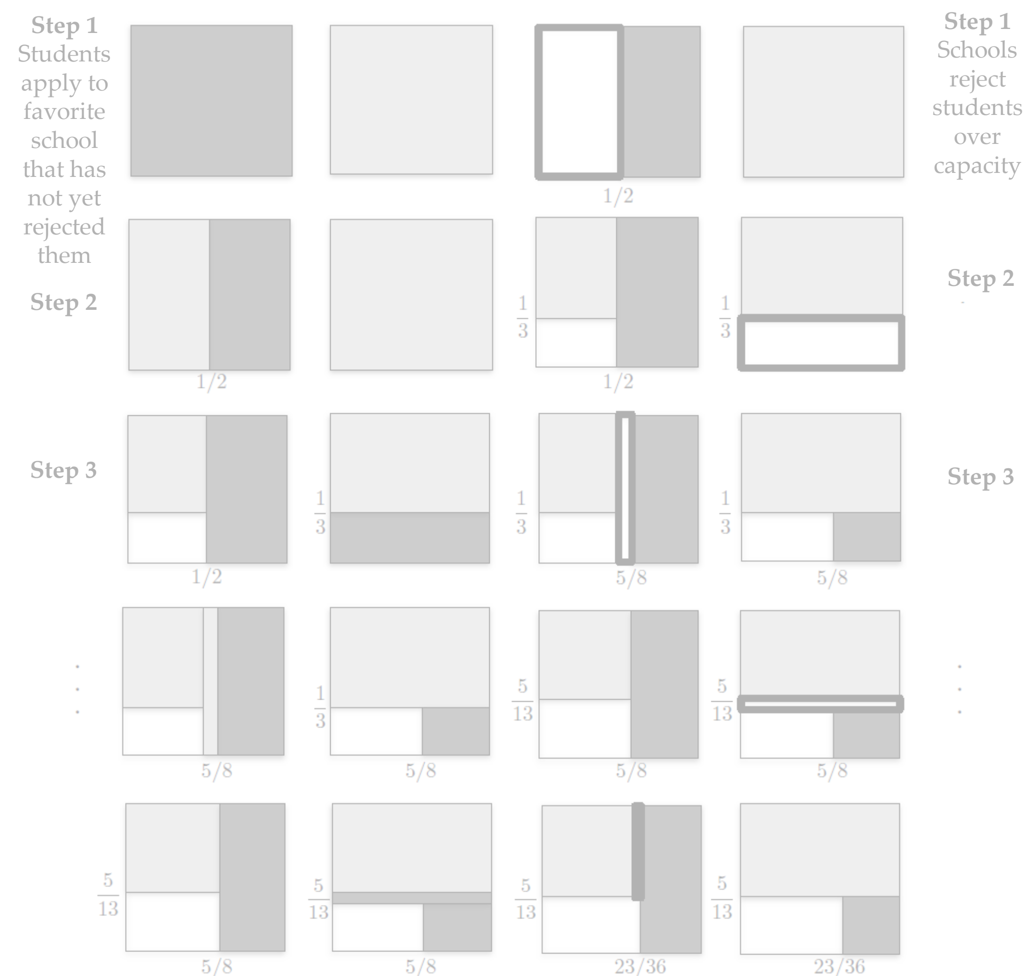
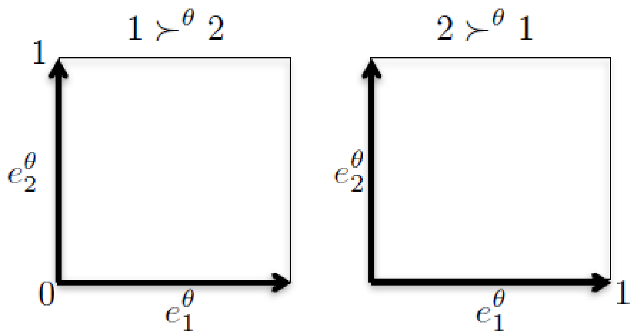


FIGURE 1. The Gale and Shapley algorithm

Intuition



$\frac{1}{2}$ students prefer 1, $\frac{1}{2}$ prefer 2

$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

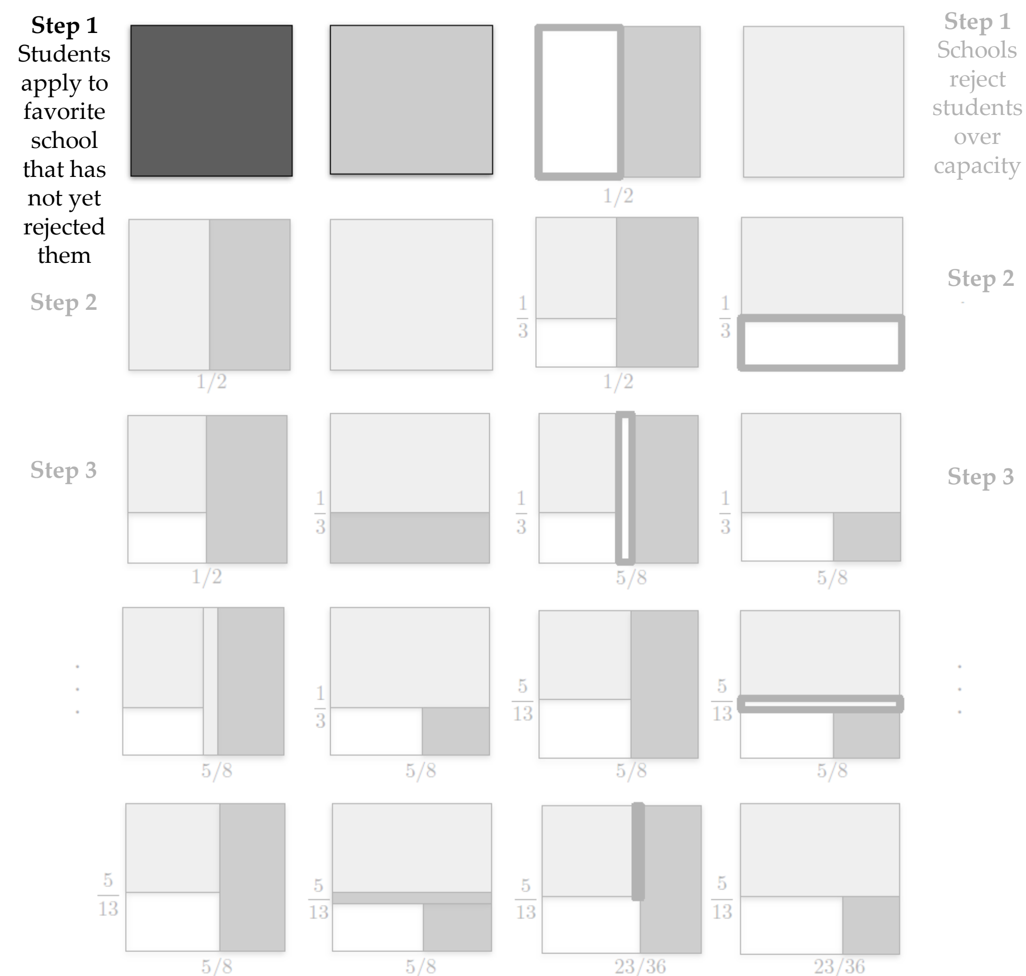
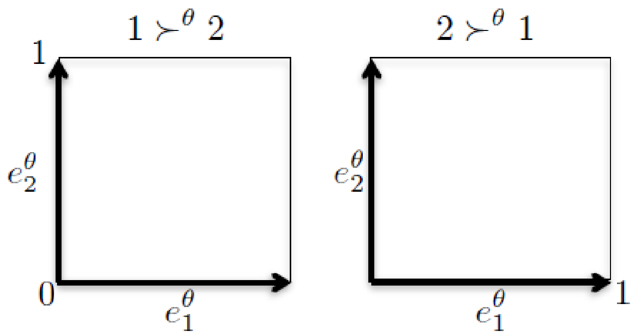


FIGURE 1. The Gale and Shapley algorithm

Intuition



$\frac{1}{2}$ students prefer 1, $\frac{1}{2}$ prefer 2

$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

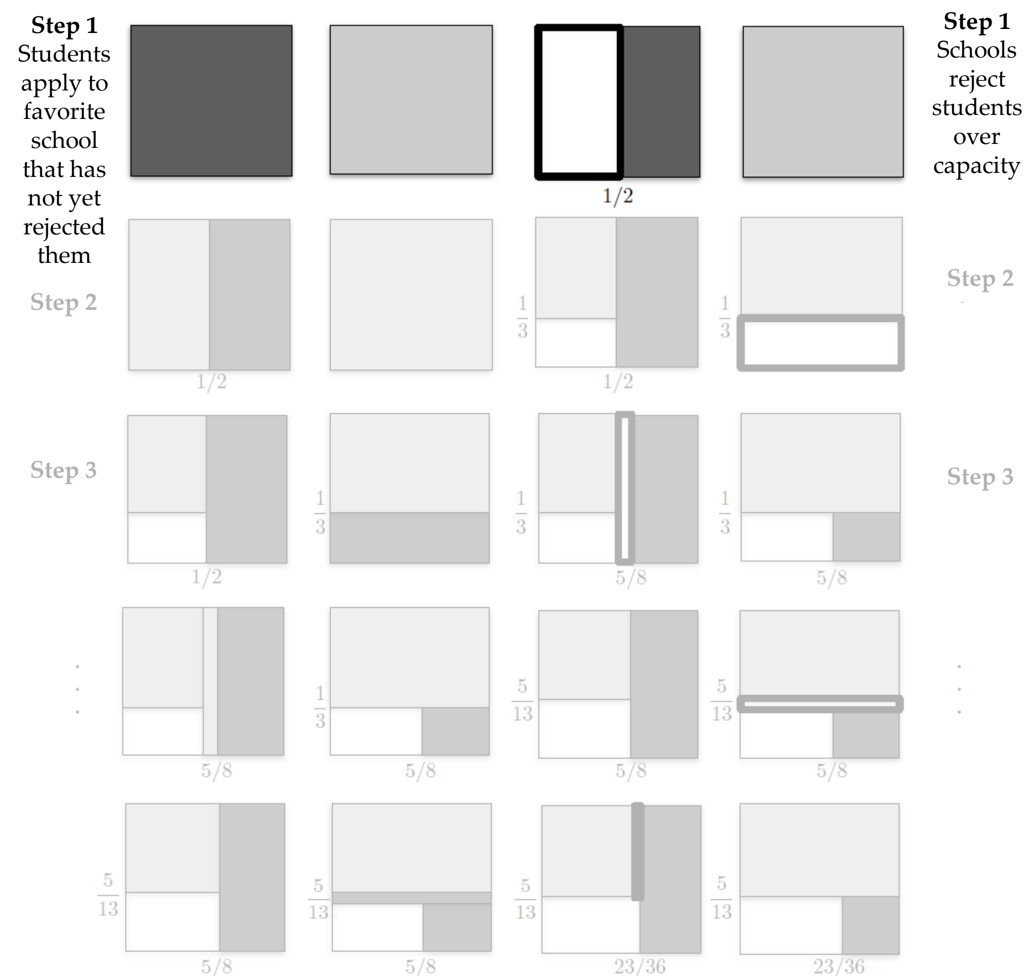
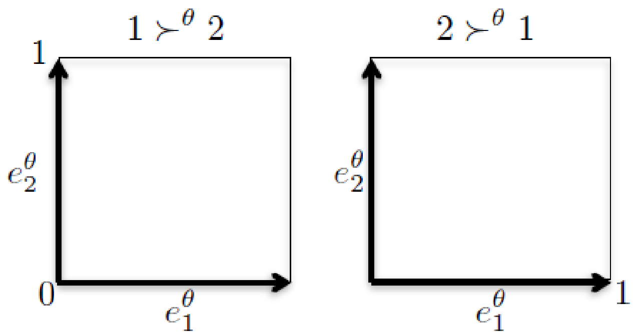


FIGURE 1. The Gale and Shapley algorithm

Intuition



$\frac{1}{2}$ students prefer 1, $\frac{1}{2}$ prefer 2

$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

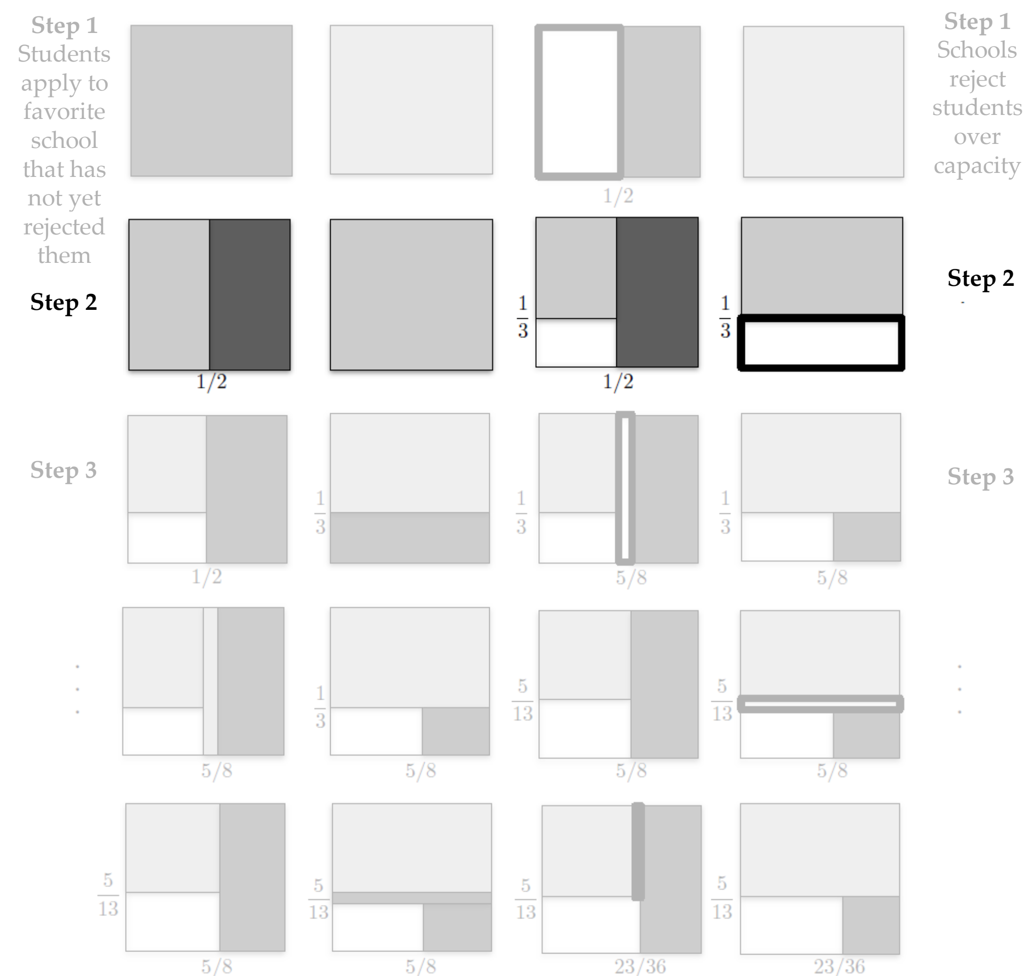
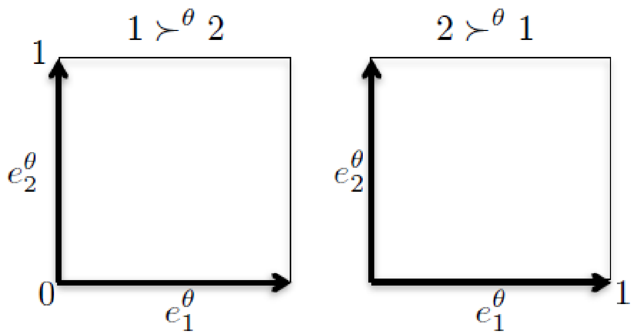


FIGURE 1. The Gale and Shapley algorithm

Intuition



$\frac{1}{2}$ students prefer 1, $\frac{1}{2}$ prefer 2

$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

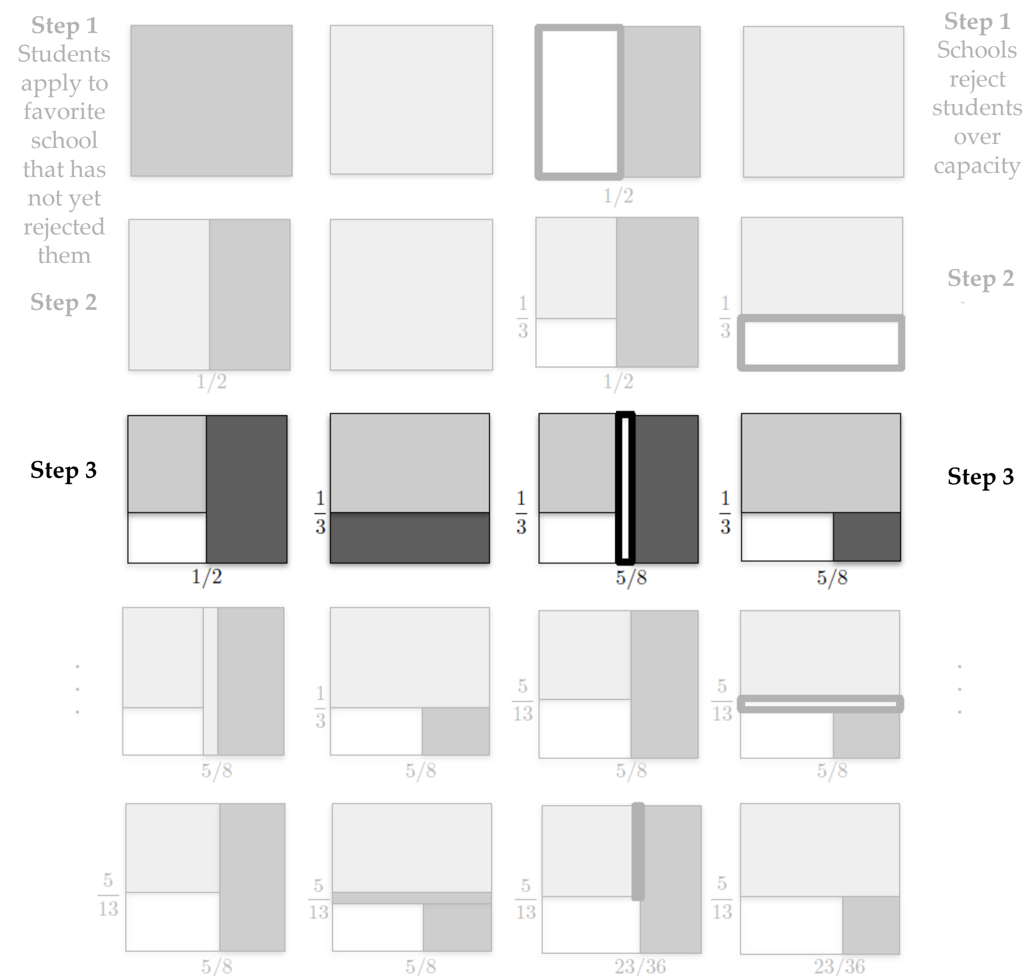
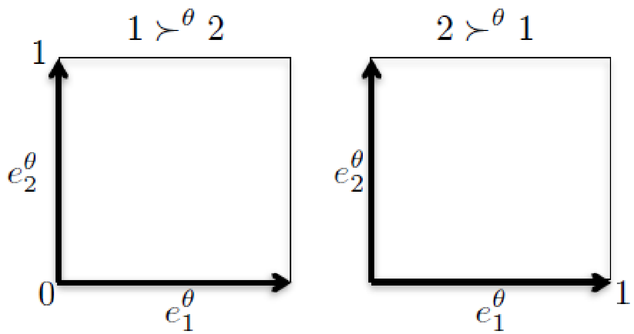


FIGURE 1. The Gale and Shapley algorithm

Intuition



$\frac{1}{2}$ students prefer 1, $\frac{1}{2}$ prefer 2

$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

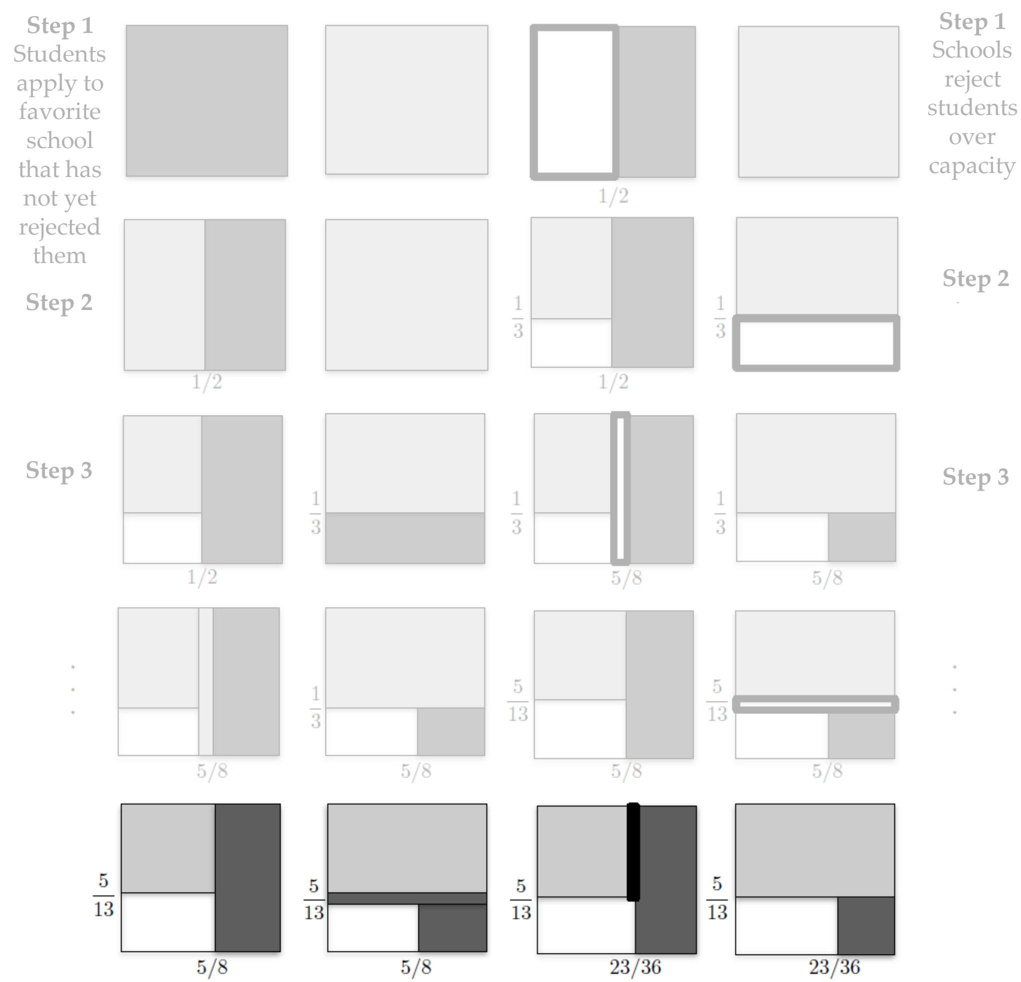
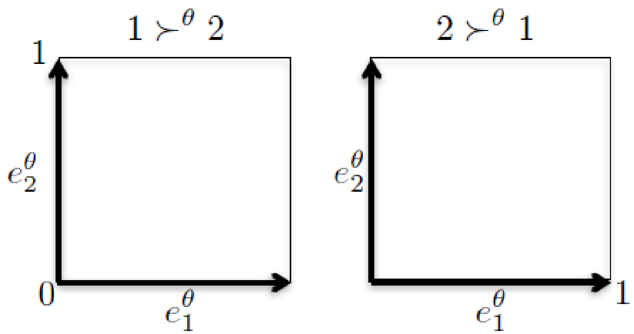


FIGURE 1. The Gale and Shapley algorithm

Intuition



$\frac{1}{2}$ students prefer 1, $\frac{1}{2}$ prefer 2

$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

Proof. Process converges to market-clearing cutoffs:

- Convergence: Monotonicity
- $D_c(P^*) \leq q_c$: True at every step
- If $P_c^* > 0$ then $D_c(P^*) = q_c$: School rejected some students, so school is full

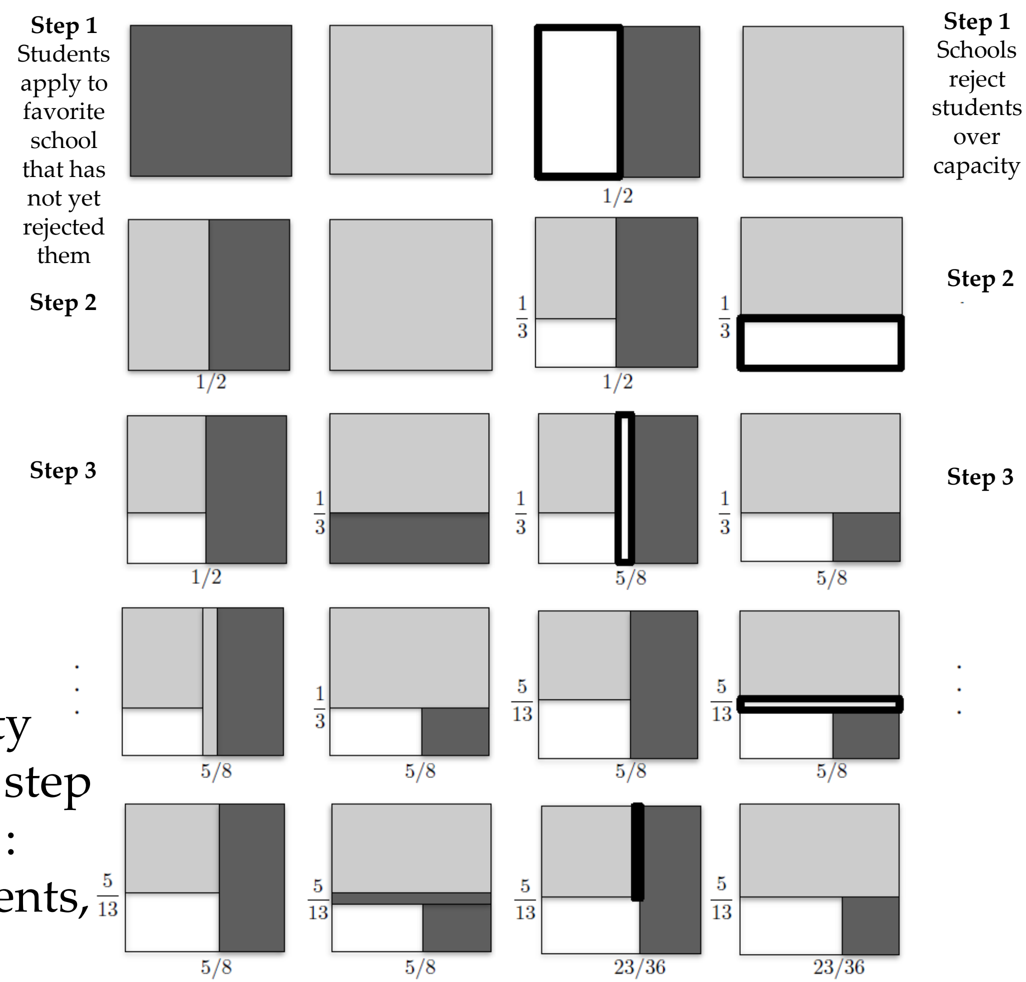


FIGURE 1. The Gale and Shapley algorithm

Standard Results Still Hold

- **Existence.** The deferred acceptance algorithm converges to a stable matching
- **Lattice.** The set of stable matchings is a lattice
  - The set of market clearing cutoffs is a lattice
- **Rural Hospitals Theorem (Roth 1986).**
  - The measure of students matched to each school is the same in all stable matchings
  - The set of students matched to each under-demanded school is the same in all stable matchings

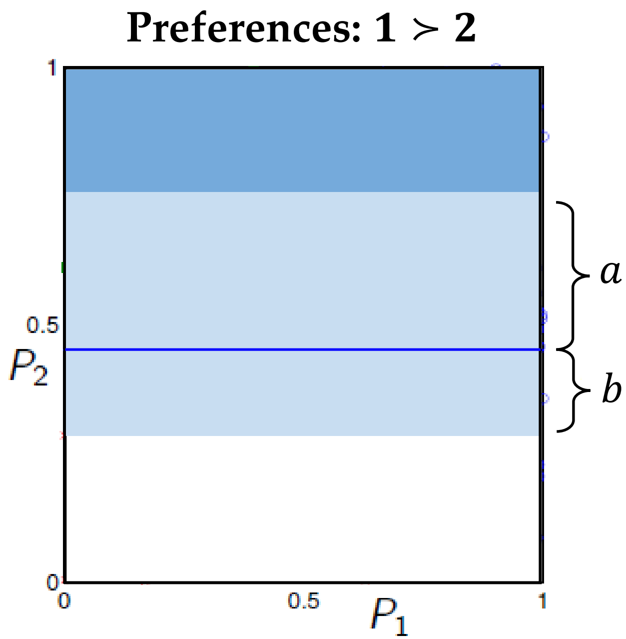
Cutoffs and Inference

- In school choice, schools often have weak priorities
- Strict priorities are generated using random tiebreaking:
    - Each student gets a lottery number $U[0,1]$ at each school
    - **Single tie-breaking:** Same lottery used at every school
    - **Multiple tie-breaking:** New lottery drawn at every school
- Students are ordered first by priority, then by lottery number.
- **Main idea:** The lottery induces a quasi-experiment!
(see, e.g., Abdulkadiroglu et al. 2017, "Research Design Meets Market Design" – i.e. Lecture 16 on Thursday 8/18)

Cutoffs and Inference

Example

- All students prefer $1 \succ 2$
- School 1 has strict priorities
- School 2 gives priority to siblings, then neighborhood, and breaks remaining ties randomly



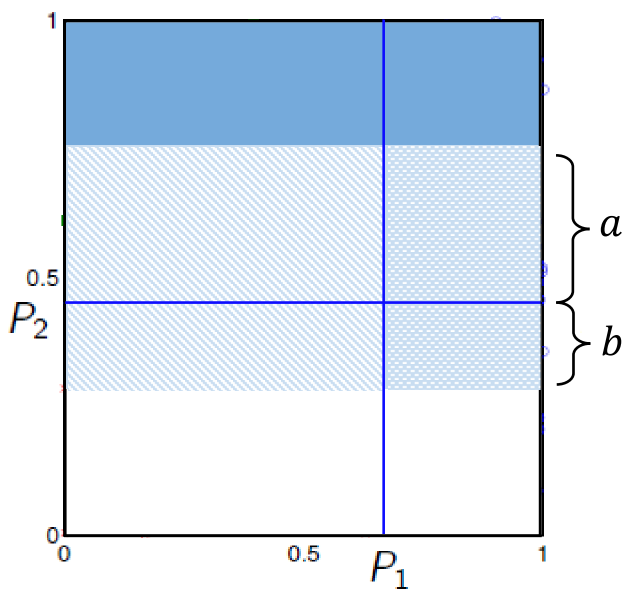
- Students in the neighborhood have the *option* to go to school 2 w.p. $\frac{a}{a+b}$

Cutoffs and Inference

Example

- All students prefer $1 \succ 2$
- School 1 has strict priorities
- School 2 gives priority to siblings, then neighborhood, and breaks remaining ties randomly

Preferences: $1 \succ 2$



- Students in the neighborhood have the *option* to go to school 2 w.p. $\frac{a}{a+b}$
- Since students all prefer $1 \succ 2$, students in the neighborhood will *take the option* only if they don't also have school 1 as an option
- We can use the cutoffs to define separate propensity scores for students in &

DA via Cutoffs (Azevedo Leshno)

- Continuum Model
- Cutoff Characterization
- **Uniqueness and Convergence**

Uniqueness Theorem

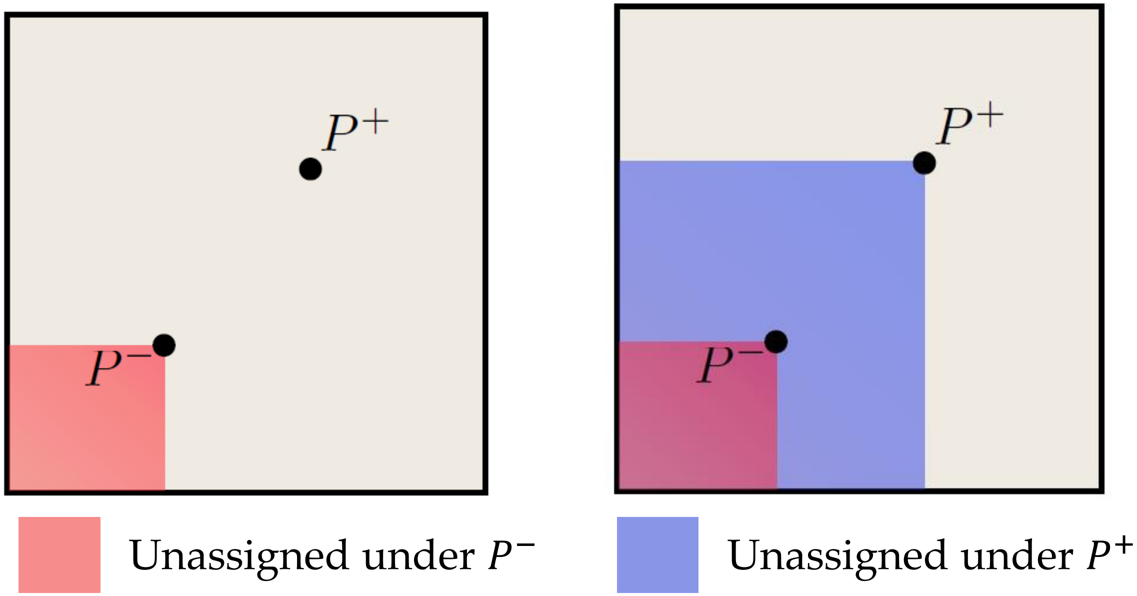
Theorem. Let $E = [\eta, q]$ be a continuum economy

I. If η has full support then there is a unique stable matching

II. If $D(\cdot)$ is continuously differentiable then for almost any q there is a unique stable matching.

Proof by picture:

I. Full support



Continuity and Convergence

Def. A **discrete economy** is $F = [\eta, q]$ where η is composed of a finite number of atoms.

Lemma. For a discrete economy $\mathcal{M}$ and $\mathcal{P}$ take stable matchings into market clearing cutoffs, and $\mathcal{MP}$ is the identity.

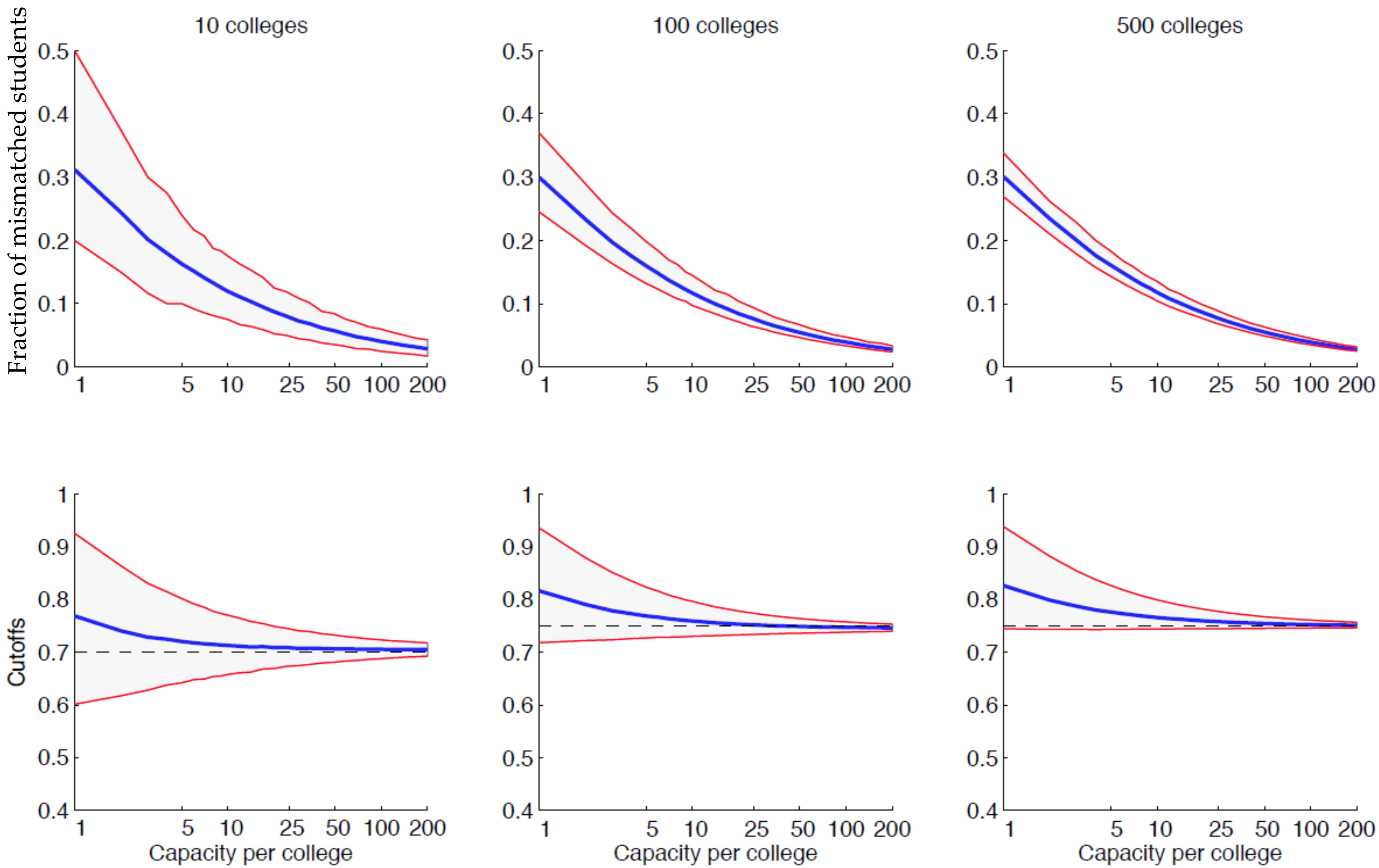
i.e. Stable matchings in discrete economies can also be characterized using market-clearing cutoffs:

1. $\mathcal{M}$: Find the marginal students assigned to each school c
2. $\mathcal{P}$: Let students more prioritized at c than marginal students have the *option* of attending c.

Theorem (*Informal*). If discrete economies F^k converge to an economy E with a unique stable matching μ, then stable matchings and cutoffs converge to μ and $\mathcal{P}(\mu)$. Moreover, the stable matching correspondence is continuous at E

Implication: *Results for continuum economies should hold for large finite economies, and randomly sampled economies*

Simulations



Summary: DA via Cutoffs

- New model of matching allowing for complex preferences but simple derivation

- Cutoff characterization of DA outcome
  - Can be solved using supply/demand equations
  - Enables inference

- The continuum model approximates large markets

BREAK

In small groups:

- Share:
  - Name
  - Affiliation
  - Where you're from
- Find:
  - A fun fact for each person that is unique to them in the group

The Cutoff Structure of Top Trading Cycles in School Choice

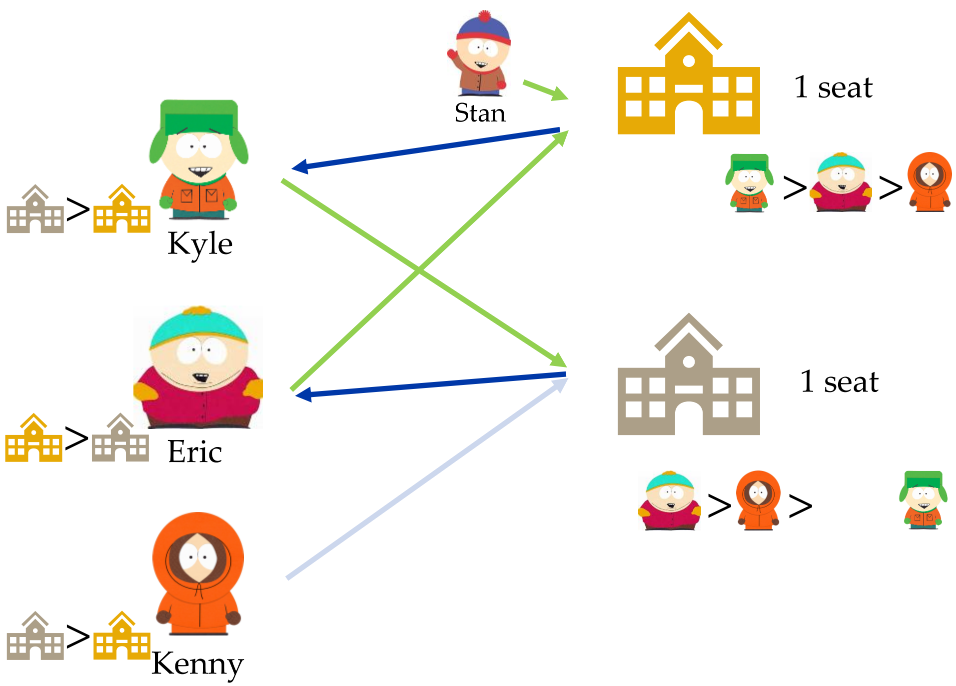
Jacob Leshno & Irene Lo

Main Takeaways

The Top Trading Cycles outcome can be characterized by **cutoffs**.
These cutoffs solve **trade balance equations**.

Top Trading Cycles

[Shapley & Scarf '74]

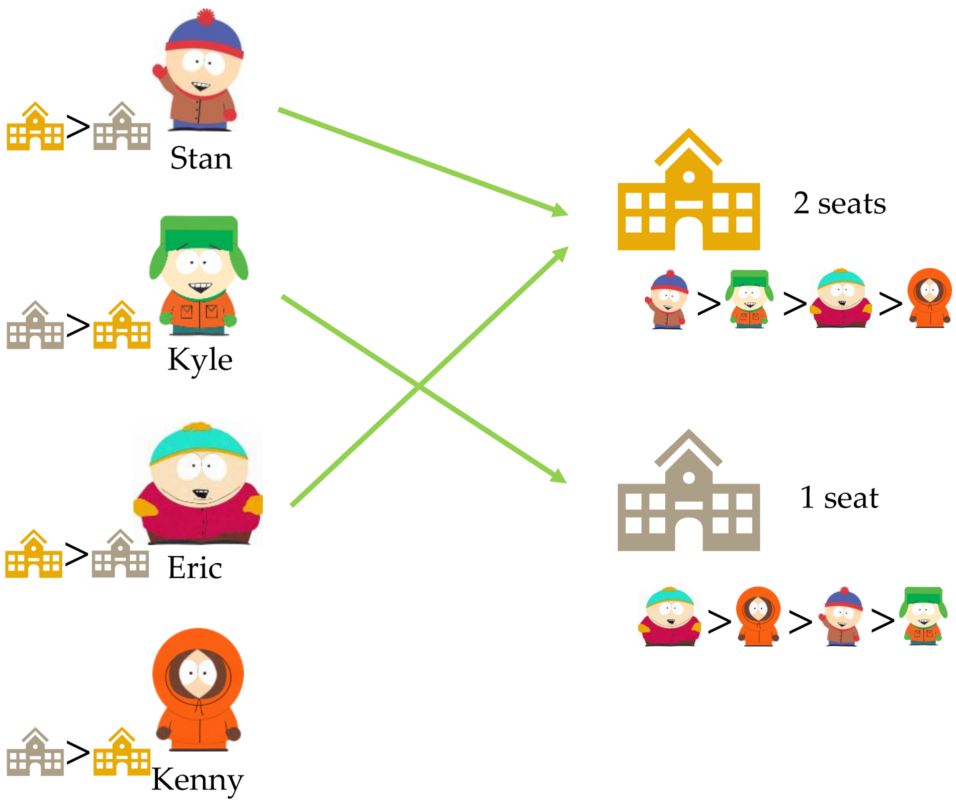



```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
    all remaining students
    point to favorite
    remaining school
    all remaining schools
    point to favorite
    remaining students
    select a cycle, assign
    students in cycle to
    school they point to
```

Top Trading Cycles

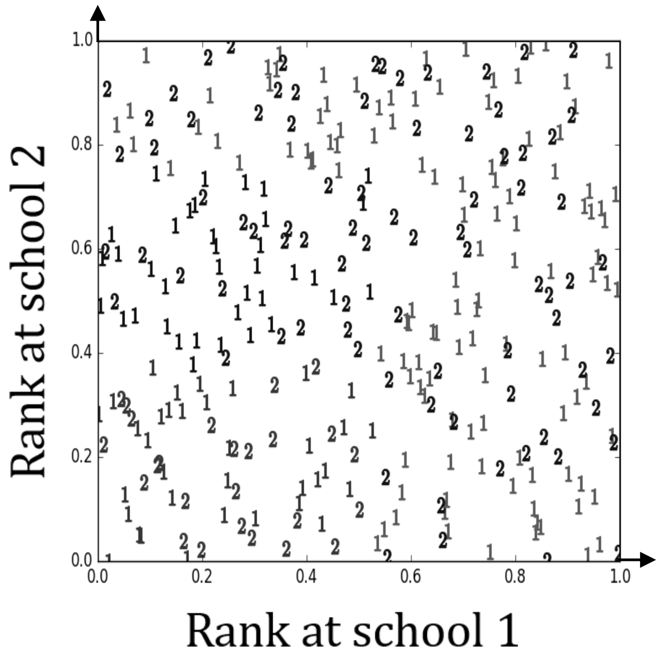
[Shapley & Scarf '74]




```
Top Trading Cycles

while some students unassigned
or some schools unfilled:
      all remaining students
      point to favorite
      remaining school
      all remaining schools
      point to favorite
      remaining students
      select a cycle, assign
      students in cycle to
      school they point to
```

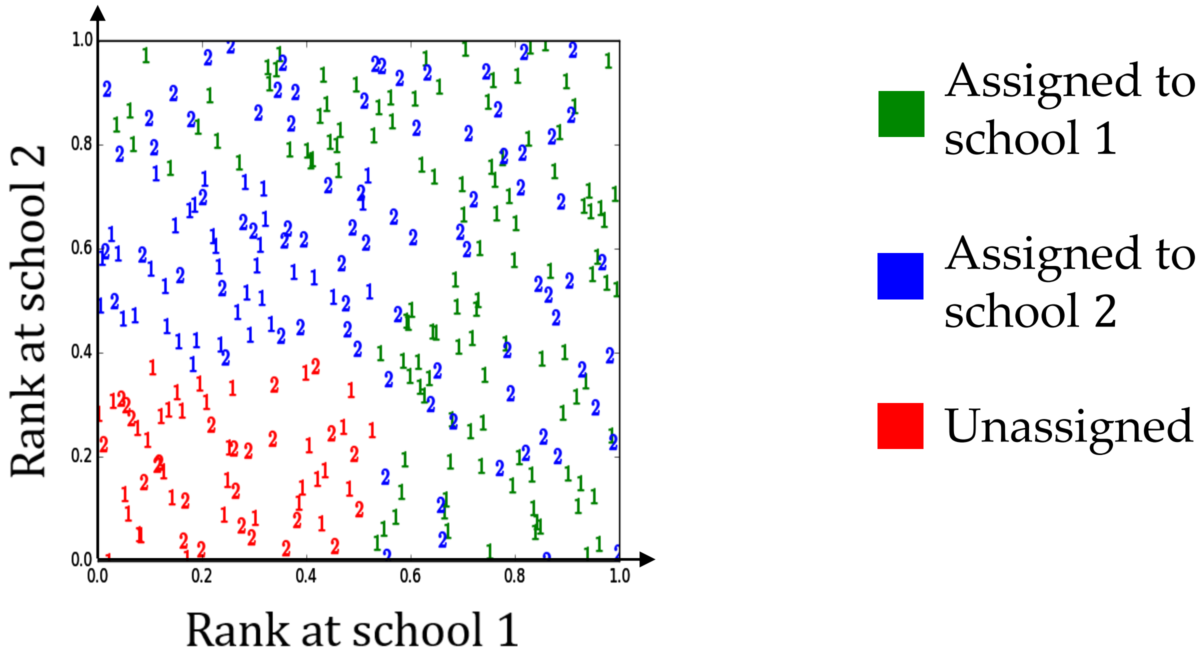
Example



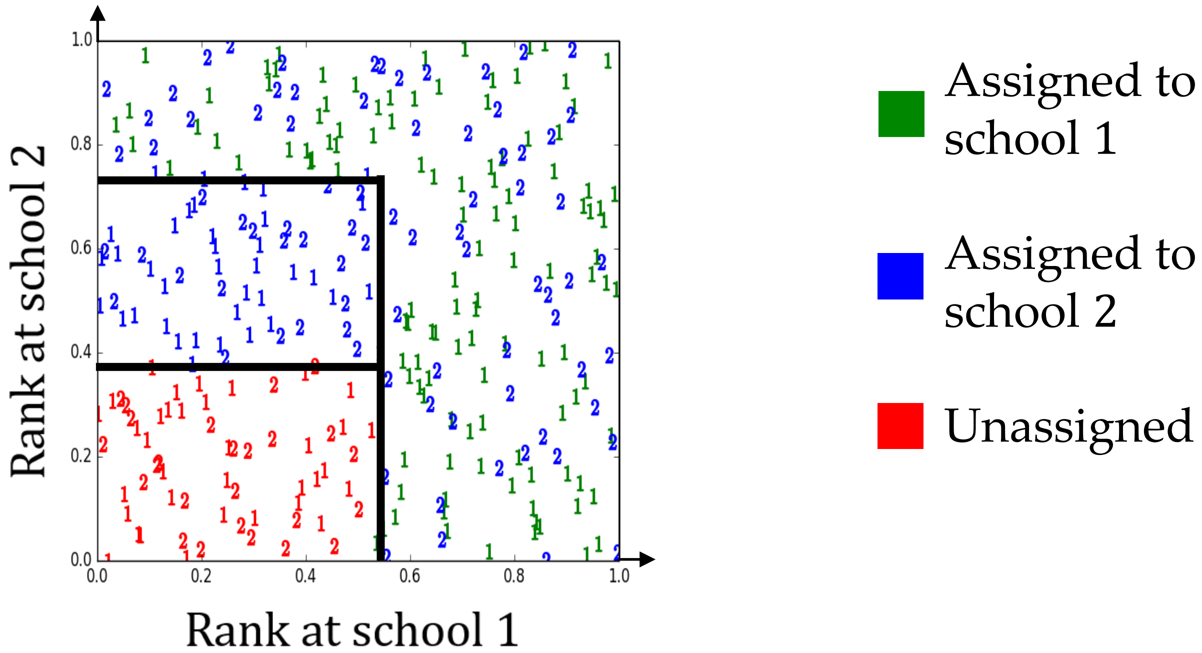
- 2/3 students prefer school 1
- Students marked with preferred school
- Ranks are uniformly i.i.d. across schools
- $q_1 = q_2$

Q. What is the outcome of TTC on this economy?

Example: TTC Assignment



Example: TTC Assignment



The TTC assignment can (like DA) also be described using cutoffs!

Q. How are TTC cutoffs different to DA cutoffs?

TTC via Cutoffs (Leshno Lo)

- **Cutoff Characterization**
- Continuum Model
- Uniqueness, Convergence, and Welfare

TTC Assignment via Cutoffs

Theorem.
The TTC assignment is given by cutoffs $\{p_b^c\}$ where:

- Each student θ has a budget set

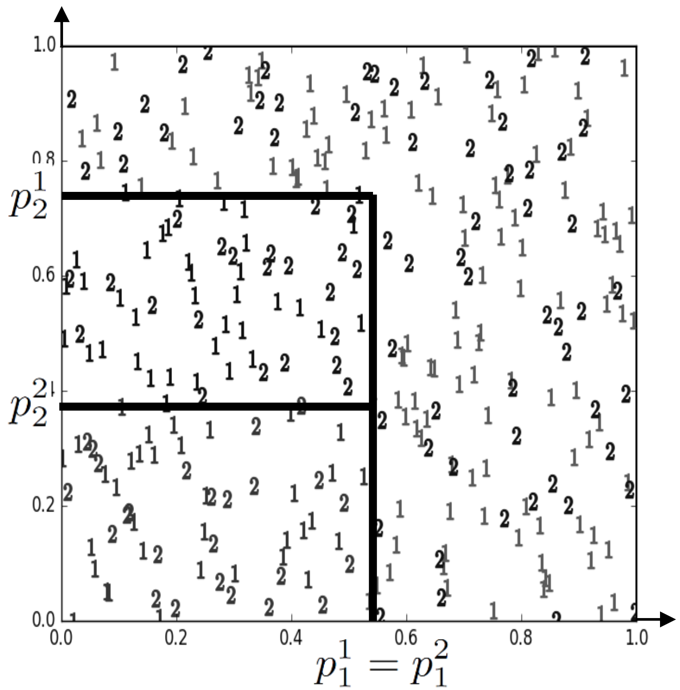
$$B(p, \theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}$$

- Students assigned to their favorite school in their budget set

$$\mu(\theta) = \max_{\succ^\theta}(B(p, \theta))$$

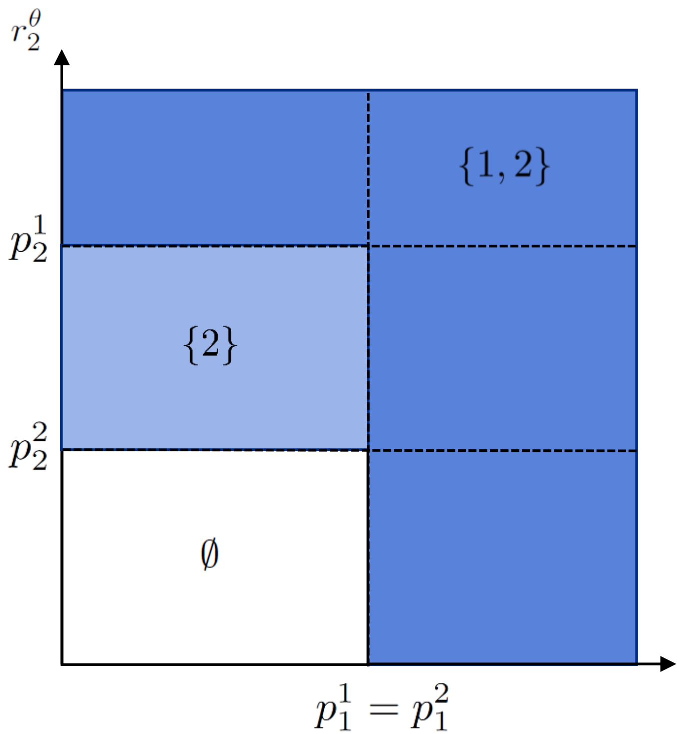
Interpretation. p_b^c is the minimal priority at school b that allows trading a seat at school b for a seat at school c

Example: Assignment via Cutoffs

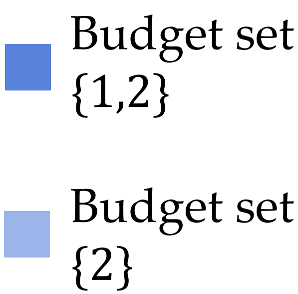


- 2/3 students prefer school 1
- Students marked with preferred school
- Ranks are uniformly i.i.d. across schools
- $q_1 = q_2$

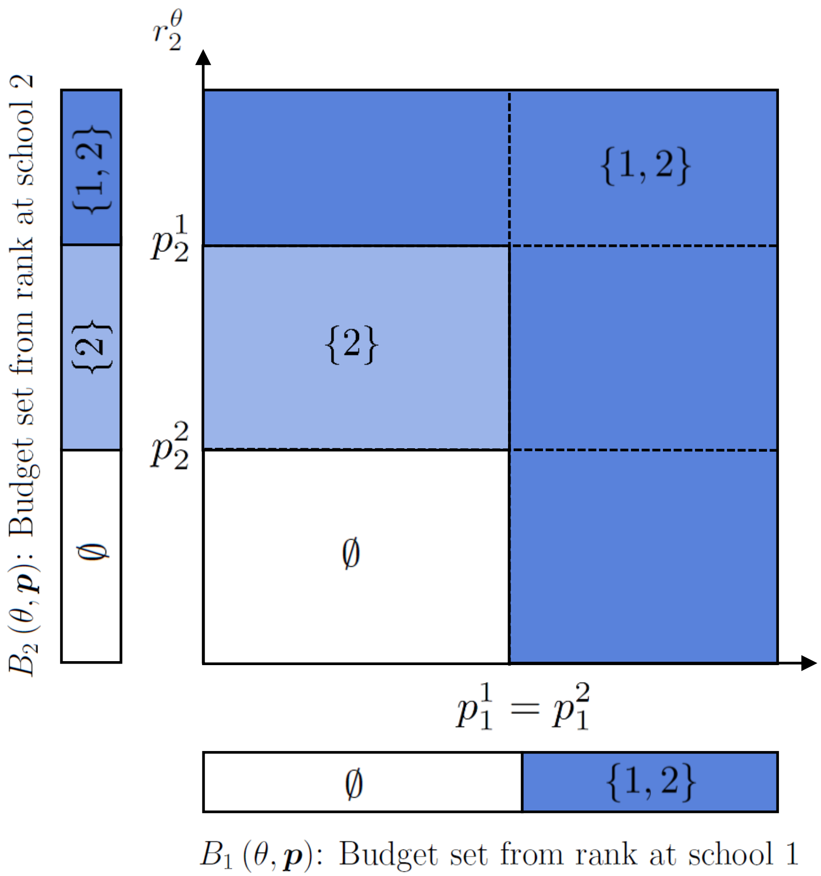
Example: Assignment via Cutoffs



$$B(p,\theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}$$



Example: Assignment via Cutoffs

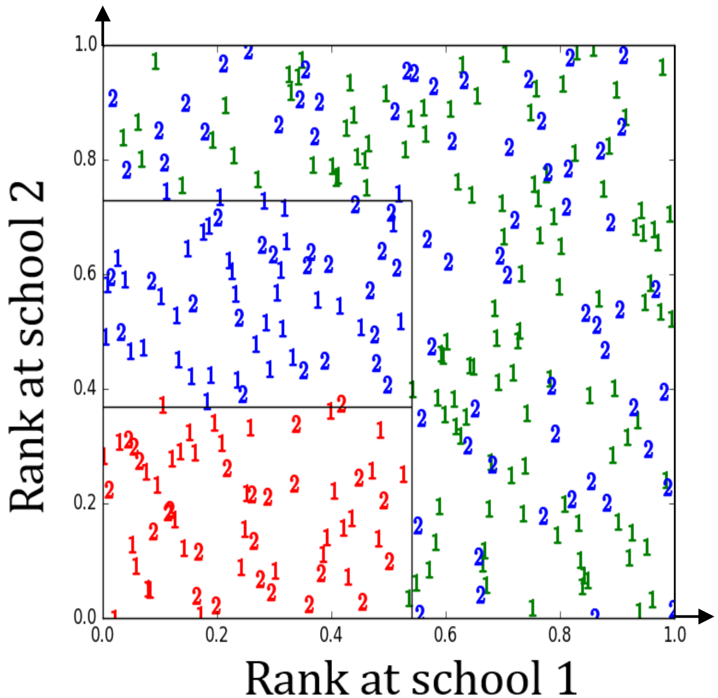


$$B(p, \theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}$$

Budget set {1,2}

Budget set {2}

Example: Assignment via Cutoffs



$$\mu(\theta) = \max_{\succ^\theta}(B(p, \theta))$$

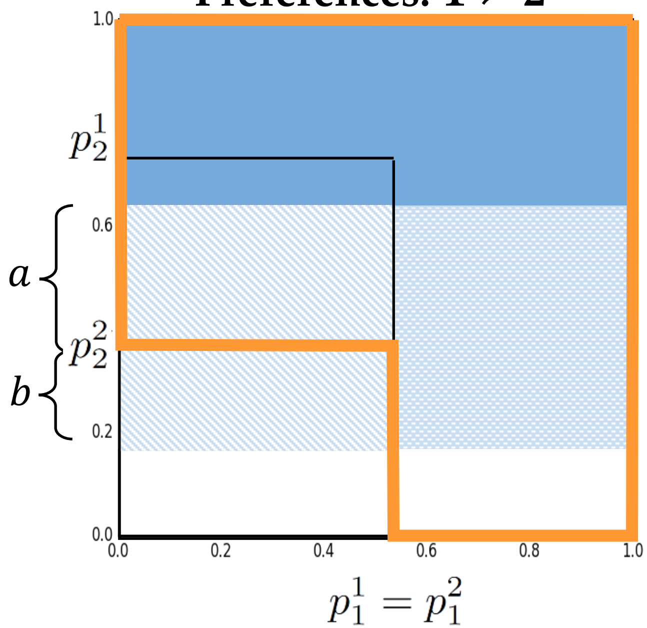
- Assigned to school 1
- Assigned to school 2
- Unassigned

Cutoffs and Inference

Example

- All students prefer $1 \succ 2$
- School 1 has strict priorities
- School 2 gives priority to siblings, then neighborhood, and breaks remaining ties randomly

Preferences: $1 \succ 2$



- Students in have the *option* to go to school 2
- Students in the neighborhood have the *option* to go to school 2

$$w.p. \begin{cases} \dfrac{a}{a+b} & if\ r^1 < p_1^2 \\ 1 & if\ r^1 \geq p_1^2 \end{cases}$$

- As with DA, can use the cutoffs to define propensity scores

TTC via Cutoffs (Leshno Lo)

- Cutoff Characterization
- **Continuum Model**
- Uniqueness, Convergence, and Welfare

Continuum Model

- Finite set of schools $c \in C = \{1, \dots, n\}$
  - School c can admit a mass q_c of students

- Measure η specifying a distribution of a continuous mass of students
  - A student $\theta \in \Theta$ is given by $\theta = (\succ^\theta, r^\theta)$
  - Student θ has preferences $\succ^\theta$ over schools
  - $r_c^\theta \in [0,1]$ is the student's rank at school c (percentile in c's priority list)

- TTC produces an allocation $\mu: \Theta \to C \cup \{\phi\}$

Continuum Model

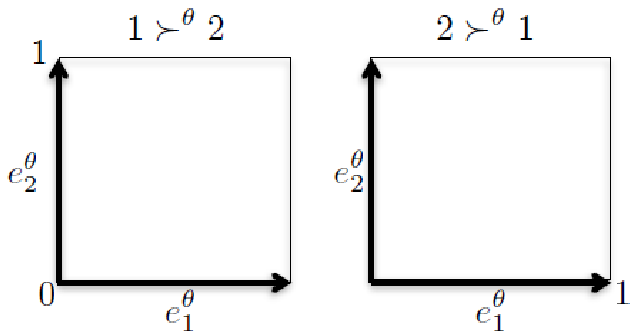
Assumption:

- (*Lipschitz density*) η has a density ν,
$$\eta(A) = \int_A \nu(\theta) d\theta \quad \forall A \subset \Theta,$$
where ν is Lipschitz continuous except on a finite grid, and bounded above

This implies:

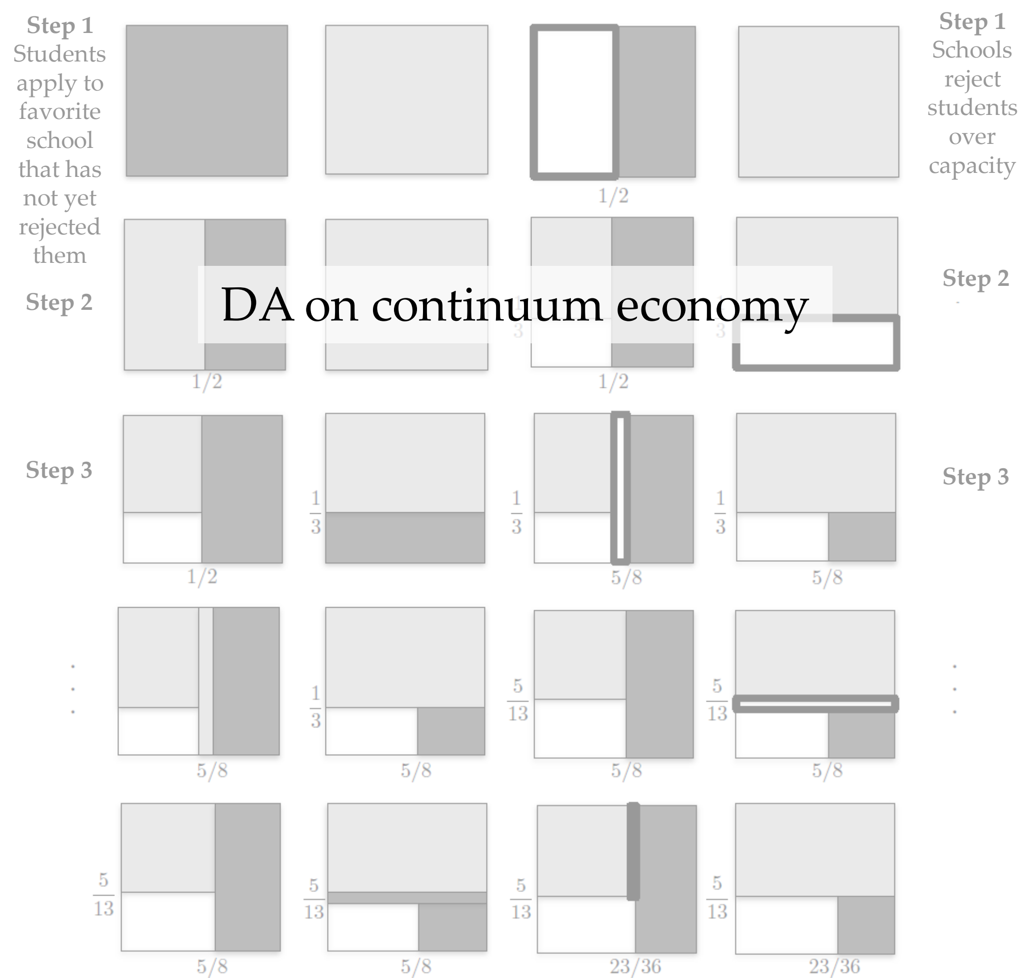
- (*Strict preferences*) Colleges' indifference curves have measure 0, that is,
$$\eta\left(\{\theta \in \Theta : r_c^\theta = x\}\right) = 0 \qquad \forall x, c$$

TTC in the Continuum

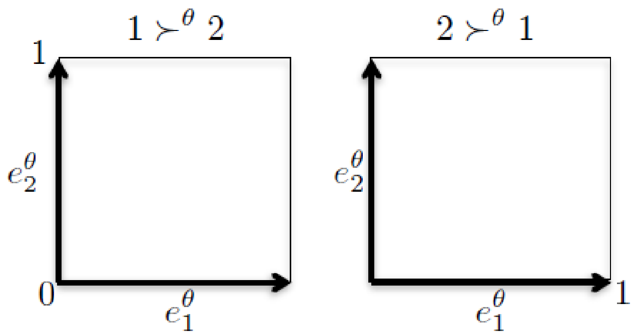


$\frac{1}{2}$ students prefer 1,
$\frac{1}{2}$ prefer 2
$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

Q. How does TTC run on this continuum economy?



TTC in the Continuum



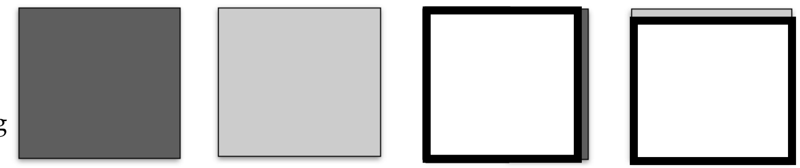
$\frac{1}{2}$ students prefer 1,

$\frac{1}{2}$ prefer 2

$q_1 = \frac{1}{4}, q_2 = \frac{1}{2}$

Q. How does TTC run on this continuum economy?

Step 1
Students point to favorite remaining school



Schools point to favorite remaining student

Problem. How to point to top student in a continuum?

Challenges

- Multiplicity in pointing
  - In continuum, colleges give priority ('point') to a **set** of students
  - In the discrete setting, when colleges 'point' to multiple students, outcome can depend on the cycle selection rule

- How to define the continuum TTC algorithm?
  - Continuous time instead of discrete steps
  - How do you select a continuum cycle?
  - How do you provide measure 0 capacity updates?

- How to track the progression of the algorithm?

Defining TTC in the Continuum Model

- 'Top' trading cycles → Measure zero cycles
  - Schools can only point to measure zero sets
  - Limit of infinitesimally small trading cycles

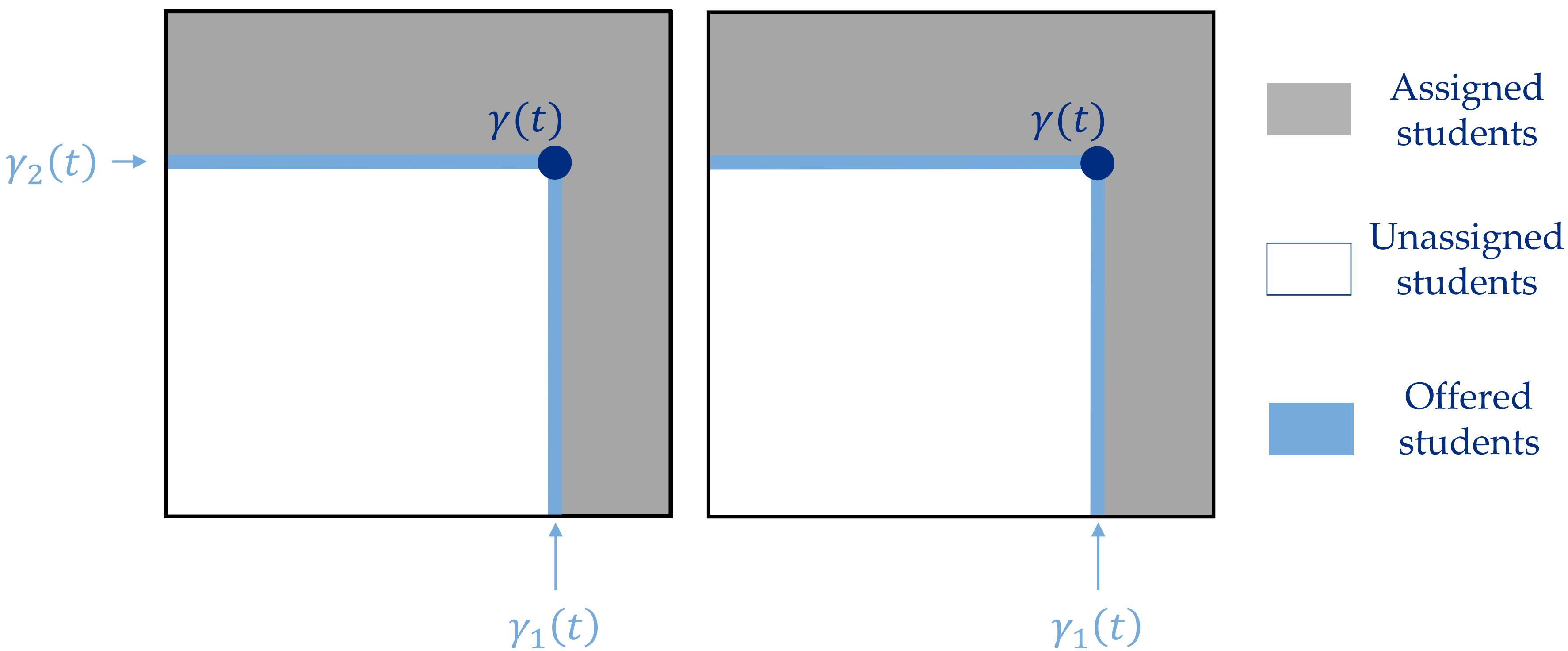
- Track progression of the algorithm using counter $\gamma_c(t)$

  $\gamma_c(t)$: Rank of students pointed to by school c at time t

- Aggregated cycles satisfy trade balance
  - The number of students offered a seat by a school is the same as the number of students assigned a seat at the school

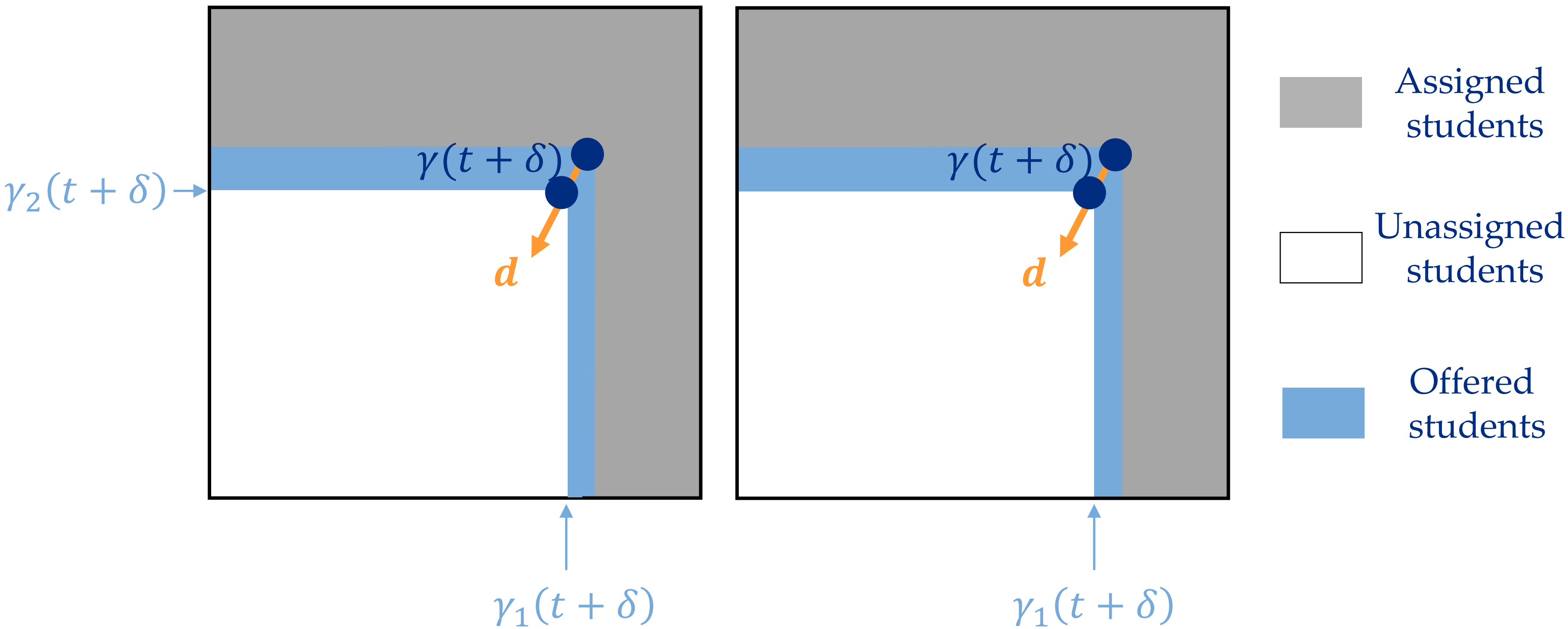
Trade Balance - Visualization

$\gamma_c(t)$: Rank of students pointed to by school c at time t



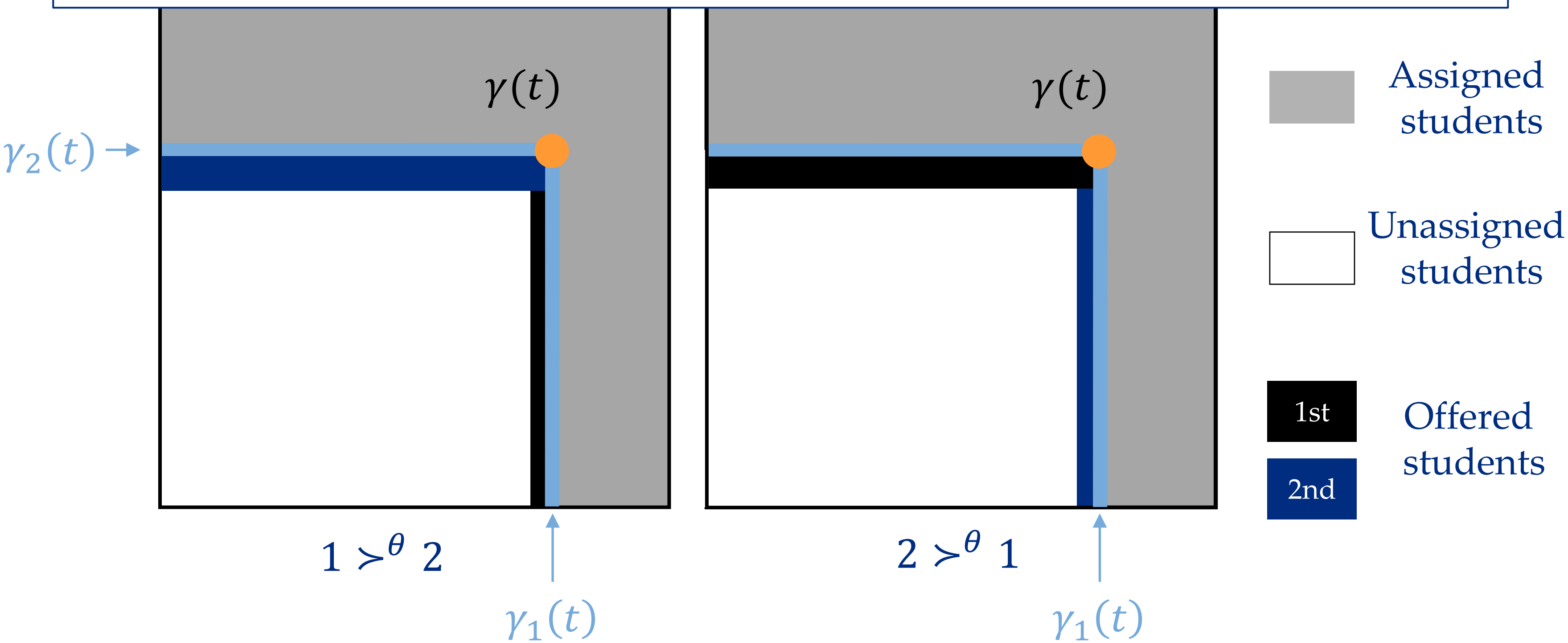
Trade Balance - Visualization

$\gamma_c(t+\delta)$: Rank of students pointed to by school c at time $t+\delta$



Trade Balance - Visualization

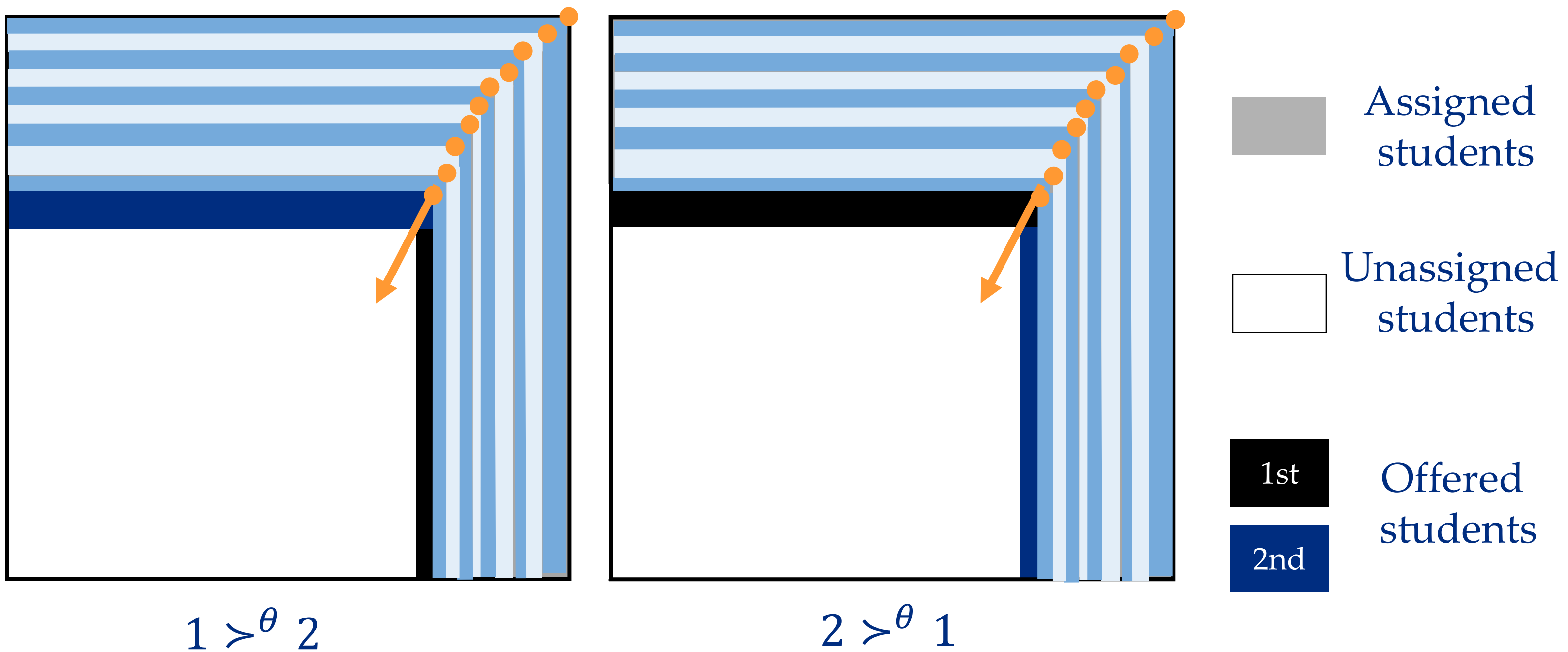
Trade balance from time t to $t + \delta$: *Offered a seat by 2* = *Assigned a seat at 2*



Trade Balance - Visualization

$\gamma_c(t)$: Rank of students pointed to by school c at time t

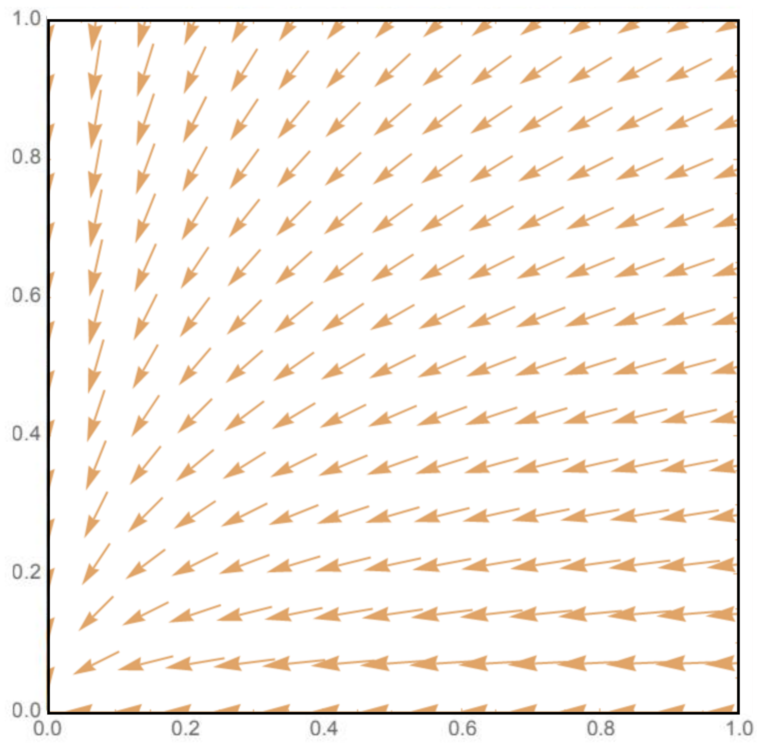
Trade balance: $\gamma_2'(t)(density\ of\ 1 \succ 2) = \gamma_1'(t)(density\ of\ 2 \succ 1)$



Trade Balance - Visualization

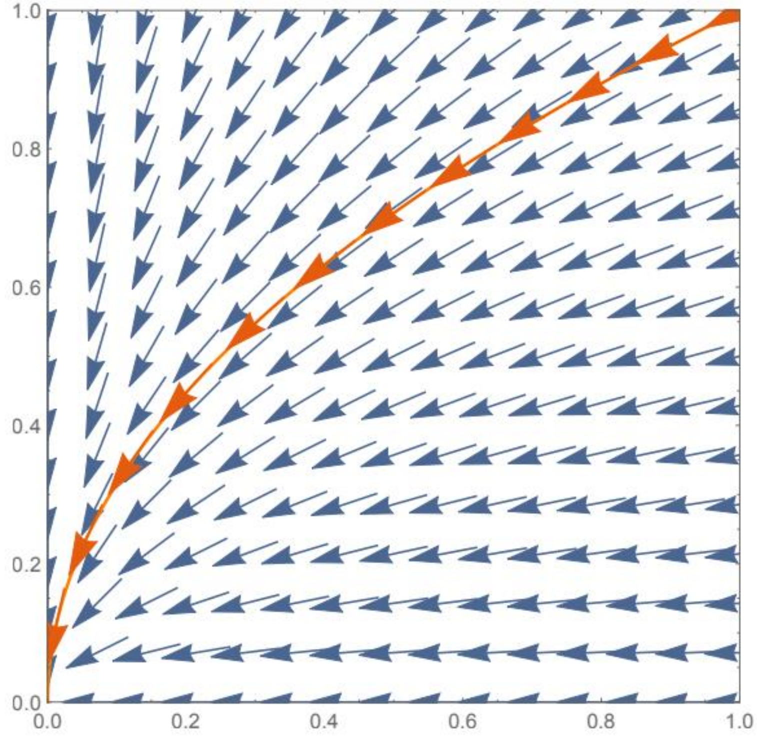
Trade balance: # Offered seat by 2 = # Assigned seat at 2

$\Rightarrow \gamma_2'(t)(density\ of\ 1 \succ 2) = \gamma_1'(t)(density\ of\ 2 \succ 1)$



TTC in the Continuum

The TTC path γ starts at $\gamma(0) = 1$ and follows $\gamma'(t) = d(\gamma(t))$, where d satisfies trade balance.



Trade Balance and Capacity Equations

Trade Balance Equation. For all times t, $\gamma'(t) = d(\gamma(t))$, where $\gamma_c(t)$ is the rank of students pointed to by school c at time t, and $d(x)$ balances the relative marginal densities.

- Necessary condition for aggregate trade
- Intuition: $\#\begin{Bmatrix}\text{Assigned at } c \\ \text{by time } t\end{Bmatrix} = \#\begin{Bmatrix}\text{Offered seat by } c \\ \text{by time } t\end{Bmatrix}$

Capacity Equation *(Informal)*.
Students stop being assigned to school c at

$$t^{(c)} = min\left\{t \colon \#\begin{Bmatrix}\text{Assigned at } c \\ \text{by time } t\end{Bmatrix} \geq q_c\right\}.$$

Calculating TTC Cutoffs

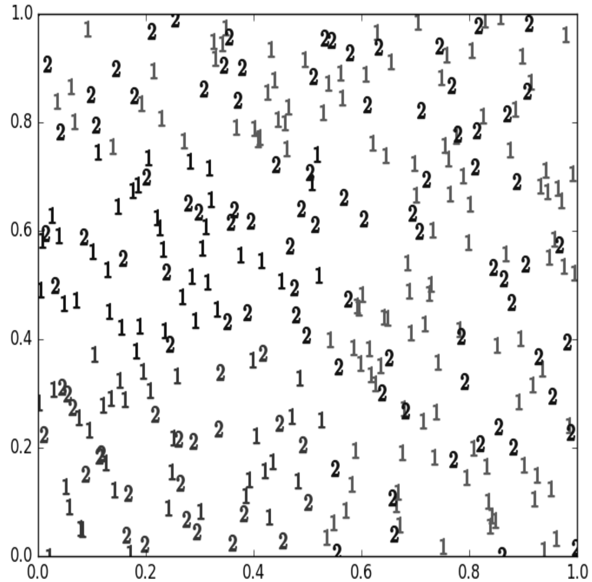
Theorem.

- (Existence) There exist a TTC path $\gamma(\cdot)$ and stopping times $\{t^{(c)}\}_{c\in C}$ that satisfy the trade balance and capacity equations.
- (Cutoffs) The n^2 TTC cutoffs $\{p_b^c\}$ are given by
$$p_b^c = \gamma_b\left(t^{(c)}\right).$$
- (Definition) The continuum TTC assignment is given by
$$\mu_{cTTC}(\theta) = \max_{\succ^\theta}\{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}.$$
- (Uniqueness) Any $\gamma(\cdot)$ and $\{t^{(c)}\}_{c\in C}$ that satisfy trade balance and capacity equations yield the same assignment μ_{cTTC}.

Calculating TTC Cutoffs

Recipe for calculating the TTC assignment
1. Compute 'trade balance' marginal densities $d(\cdot)$
2. Calculate the TTC path γ using trade balance equations
3. Calculate the TTC cutoffs p_b^c using capacity equations

Example



- Trade balance
$$d(x) = -\begin{bmatrix} \frac{x_1}{x_1 + 2x_2} & \frac{2x_2}{x_1 + 2x_2} \end{bmatrix}$$
- TTC path
$$\gamma(t) = \left(t^{1/3}, t^{2/3}\right)$$
- TTC cutoffs
$$p^1 = \Big((1 - 3q_1)^{1/3}, \big((1 - 3q_1)^{2/3}\big)\Big)$$

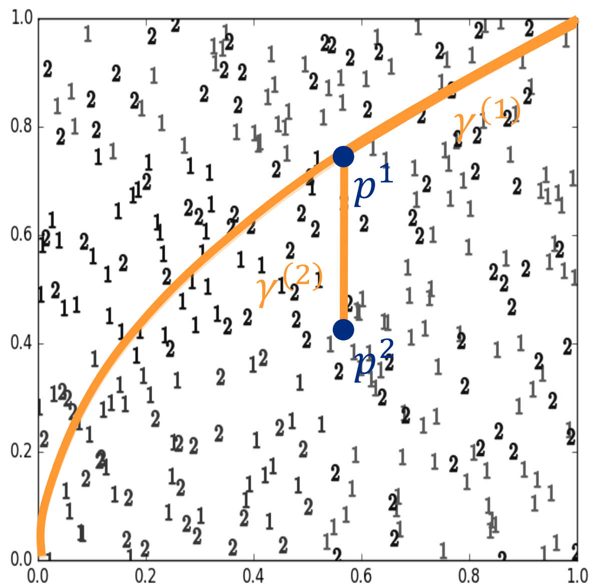
2/3 of students prefer school 1*, ranks are uniformly i.i.d. across schools,* $q_1 = q_2$

Calculating TTC Cutoffs

Recipe for calculating the TTC assignment

1. Compute 'trade balance' marginal densities $d(\cdot)$
2. Calculate the TTC path γ → *indicates the run of TTC*
3. Calculate the TTC cutoffs p_b^c → *when schools reach capacity*

Example



- Trade balance

$$d(x) = -\begin{bmatrix} \dfrac{x_1}{x_1 + 2x_2} & \dfrac{2x_2}{x_1 + 2x_2} \end{bmatrix}$$

- TTC path

$$\gamma(t) = \left(t^{1/3}, t^{2/3}\right)$$

- TTC cutoffs

$$p^1 = \left((1 - 3q_1)^{1/3}, \left((1 - 3q_1)^{2/3}\right)\right)$$

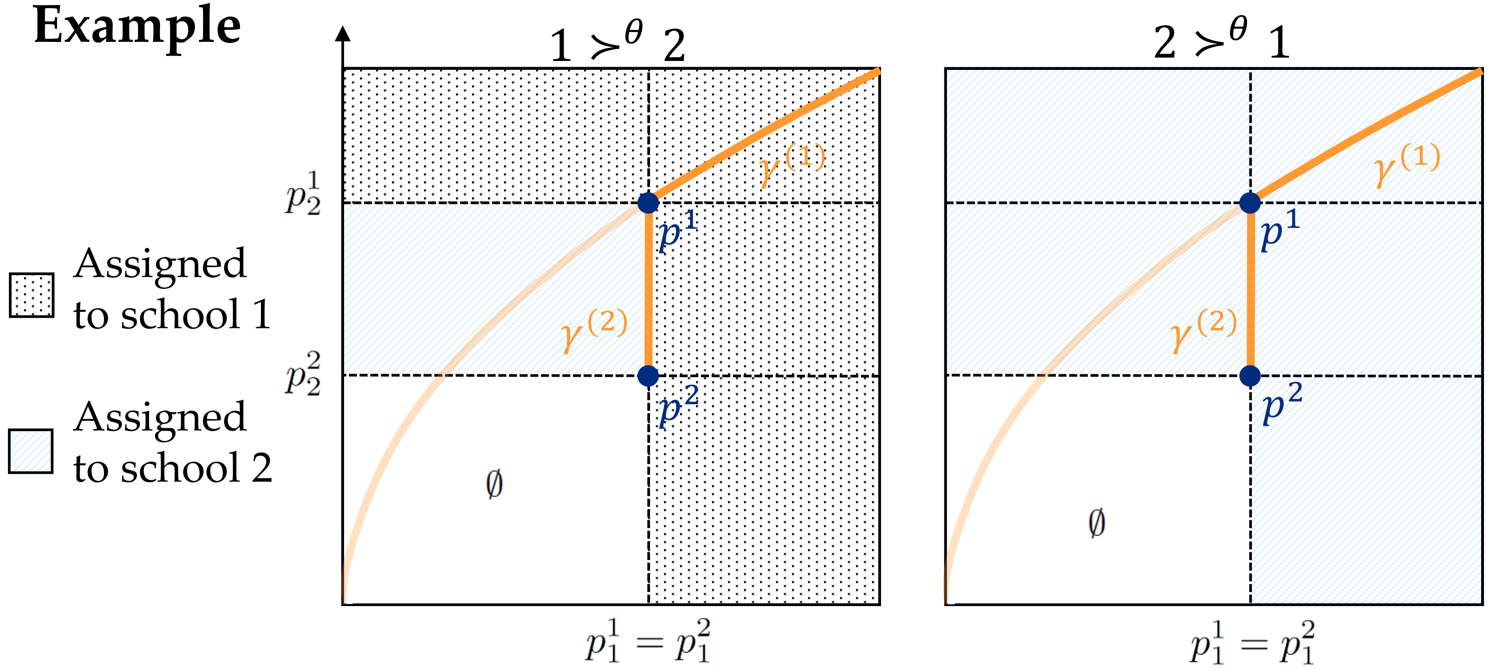
2/3 of students prefer school 1*, ranks are uniformly i.i.d. across schools,* $q_1 = q_2$

Calculating TTC Cutoffs

Recipe for calculating the TTC assignment

1. Compute ‘trade balance’ marginal densities $d(\cdot)$
2. Calculate the TTC path γ → *indicates the run of TTC*
3. Calculate the TTC cutoffs p_b^c → *when schools reach capacity*

Example



2/3 of students prefer school 1, ranks are uniformly i.i.d. across schools, $q_1 = q_2$

TTC via Cutoffs (Leshno Lo)

- Cutoff Characterization
- Continuum Model
- **Uniqueness, Convergence, and Welfare**

Continuum TTC Generalizes Discrete TTC

- **Trade Balance Uniquely Determines the Allocation**
  - Differential equation and TTC path may not be unique, but all give the same allocation

- **Consistent with Discrete TTC**
  - Can naturally embed discrete TTC in the continuum model
  - The continuum embedding gives the same allocation as TTC in the discrete model

- **Convergence**
  - If two distributions of students have full support and total variation distance ε, then the TTC allocations differ on a set of students of measure $O(\varepsilon|C|^2)$.

Welfare via Cutoffs

- Cutoffs determine **budget set**
  - $B(p, \theta) = \{c \mid \exists b \text{ s.t. } r_b^\theta \geq p_b^c\}$

- Budget sets determine assignment and **welfare**
  - **Assignment.** Favorite school in budget set
$$\mu(\theta) = \max_{\succ^\theta}(B(p, \theta)) = \underset{c \in B(p,\theta)}{\operatorname{argmax}}(u^\theta(c))$$
  - **Welfare.** Highest utility school in budget set
$$W(\theta) = \max_{c \in B(p,\theta)}(u^\theta(c))$$

Comparing TTC & DA: Welfare

2 schools, $q_1 = q_2 = 0.4$, distance-based utilities:

- Student s's preferences given by:

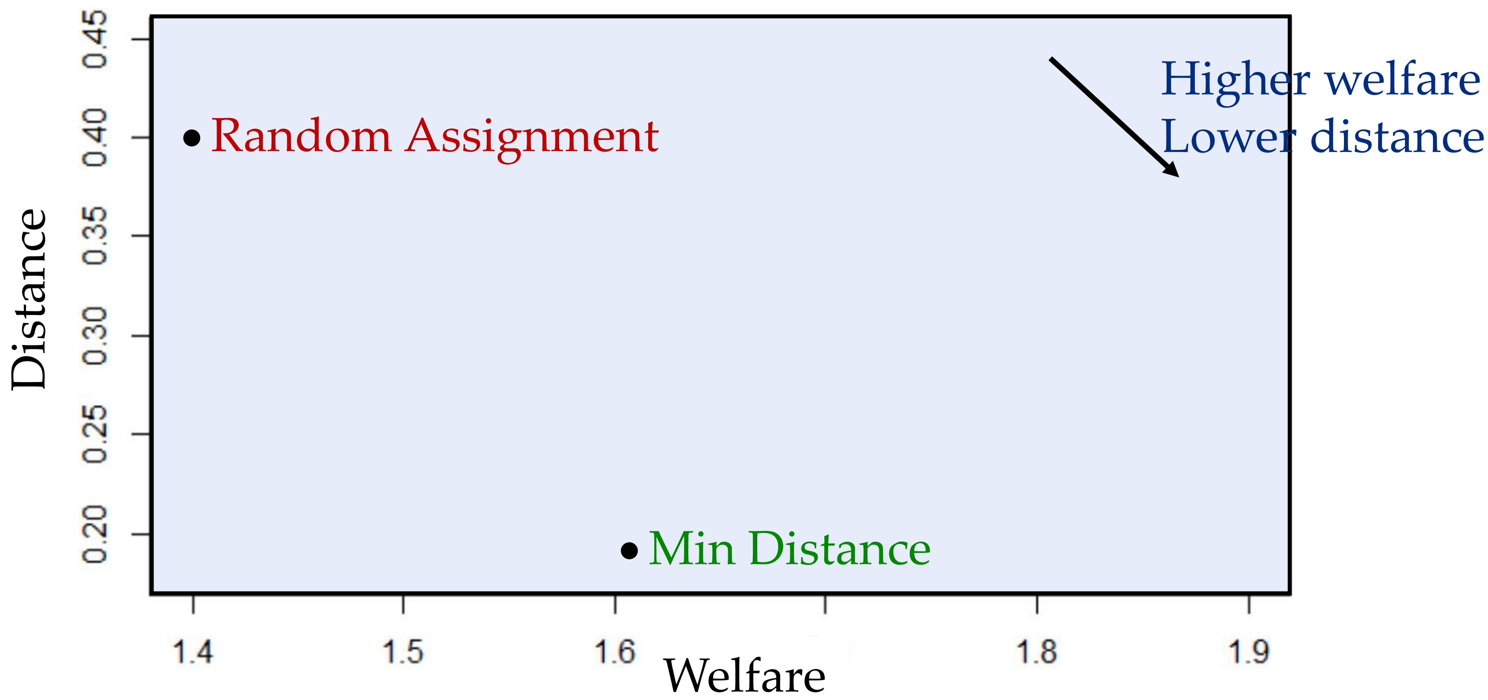
$$u_s(c) = 2 - \underbrace{d_{sc}}_{\text{distance}} + \underbrace{\varepsilon_{sc}}_{\text{idiosyncratic match value}}$$

  - $u_s(\phi) = 0$, $d_{sc} \sim U[0,1]$, $\varepsilon_{s1} \sim U[-1,2]$, $\varepsilon_{s2} \sim 0$.
  - 2/3 of students prefer school 1

- Welfare under TTC vs DA?
  - Uncorrelated priorities?
  - Distance-based priorities?

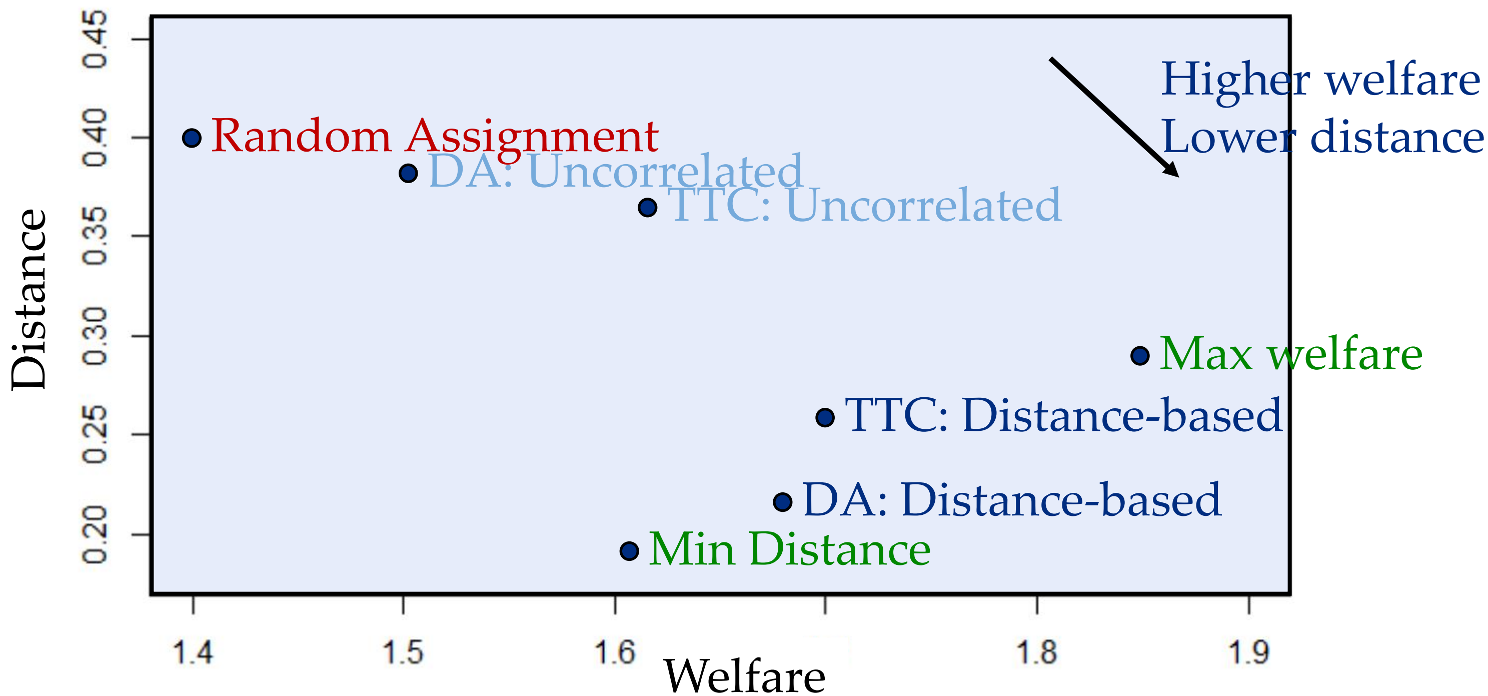
Comparing TTC & DA: Welfare

2 schools, $q_1 = q_2 = 0.4$, distance-based utilities:



Comparing TTC & DA: Welfare

2 schools, $q_1 = q_2 = 0.4$, distance-based utilities:



Priority design can have higher welfare effects than choice of mechanism

Summary: TTC via Cutoffs

- Cutoff description of TTC
  - n^2 admissions cutoffs
  - Enables inference
- Tractable framework for analyzing TTC
  - Trade balance equations
  - TTC cutoffs are a solution to a differential equation
  - Can give closed form expressions
- Quantifying Welfare
  - Priorities have larger welfare effects than DA vs TTC