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Design of Lotteries and Wait-Lists for Affordable Housing Allocation

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Abstract. We study a setting in which dynamically arriving items are assigned to waiting agents, who have heterogeneous values for distinct items and heterogeneous outside options. An ideal match would both *target* items to agents with the worst outside options and *match* them to items for which they have high value. Our first finding is that two common approaches—using independent lotteries for each item and using a waitlist in which agents lose priority when they reject an offer—lead to identical outcomes in equilibrium. Both approaches encourage agents to accept items that are marginal fits. We show that the quality of the match can be improved by using a common lottery for all items. If participation costs are negligible, a common lottery is equivalent to several other mechanisms, such as limiting participants to a single lottery, using a waitlist in which offers can be rejected without punishment, or using artificial currency. However, when there are many agents with low need, there is an unavoidable trade-off between matching and targeting. In this case, utilitarian welfare may be maximized by focusing on good matching (if the outside option distribution is light tailed) or good targeting (if it is heavy tailed). Using a common lottery achieves near-optimal matching, whereas introducing participation costs achieves near-optimal targeting.

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1. Introduction

Lotteries and wait-lists are commonly used to ration items for which demand exceeds supply. For example, New York City (NYC) allocates public housing using a waitlist and allocates newly built affordable housing by lottery. Many Broadway shows, musicians, and sports teams offer lotteries for discounted tickets. Organs from deceased donors are typically allocated using wait-lists. Occasionally, more complex allocation systems are employed, for example, Feeding America allows food banks to bid for donations using a virtual currency (Prendergast 2017).

Designers of these systems face many questions, such as the following:

1. Is it better to use lotteries, wait-lists, or an artificial currency system?
2. When using lotteries, should there be a limit on how many times each agent can apply?
3. In a waitlist, should agents who reject an offer keep their spot in line or lose it?

We address these questions by studying how different allocation systems perform according to the following objectives[1]:

- *Targeting individuals with the highest need*. Food banks and housing assistance programs target low-income individuals, and organs may be preferentially allocated to sicker patients.
- *Matching individuals with items that are well suited to their needs*. Food banks may need different types of food, housing units are in different locations of the city, and organs have different biological markers.

Targeting may be achieved *explicitly*, through eligibility and priority rules based on observable characteristics, or *implicitly*, because agents with different levels of need make different choices about where to apply and what to accept. In this paper, we focus on implicit targeting by studying anonymous mechanisms, that is, mechanisms that treat all eligible applicants identically. In many settings, anonymity is a reasonable approximation of current practice. In settings where agents are given priority based on observable characteristics, our study can be interpreted as analyzing the allocation within each priority group.

We use as a leading example the allocation of affordable housing in NYC, where developers receive a tax break if they offer a fraction of their newly built units to low- and middle-income residents. These units are awarded by lottery when the development is completed, and lotteries are independent across developments. More information about this system is available in Appendix A.

We capture the main features of this setting using a stylized model in which developments arrive over time and, upon arrival, are allocated to agents who are waiting for them. Each agent has an outside option, which is private information. Furthermore, agents have idiosyncratic values for each development. In this setting, *targeting* refers to matching agents with the worst outside options, who we also refer to as the agents with the greatest need. Meanwhile, effective *matching* requires that agents are assigned to developments for which their value is high.

We reach several high-level conclusions:

1. Several lottery-based approaches yield identical outcomes to waitlist-based alternatives. Thus, similar objectives can be achieved by superficially disparate systems.
2. Using a common lottery to determine priority for all items results in better matching than several existing systems and near-optimal matching if agents remain eligible for many periods.
3. When there are many eligible agents with low need, there is a trade-off between matching and targeting: improving one comes at the expense of the other.
4. The shape and support of the distribution of outside options determine whether it is more important to prioritize matching or targeting. In either case, simple mechanisms can attain near-optimal welfare.

We now introduce the mechanisms studied in this paper and discuss each of the four conclusions above in more detail. We begin with two systems that closely resemble common practice.[2]

- *Independent lotteries*. In each period, agents may enter a lottery for a unit at the current development. Tickets are drawn until the development fills or all tickets have been selected.
- *Wait-list without choice*. Entering agents are placed at the bottom of a waitlist. In each period, the current development is offered to agents at the top of the waitlist until it fills or is offered to every agent. Agents who reject an offer lose their spot and must reapply.

In addition, we consider the following approaches:

- *Common lottery*. Agents are assigned a random priority number upon entering the system. In each period, agents may apply for a unit at the current development. Units are offered to agents in the order of their priority numbers.
- *Wait-list with choice*. This is a waitlist in which agents keep their spot after rejecting an offer.[3]
- *Ticket-saving lottery*. Hold a lottery for every development. All agents receive a ticket each period, which can be used for the current development or for any future development. Agents can enter multiple tickets in a given lottery and are awarded housing if any of their tickets wins.
- *Single-entry lottery*. Allocate by a separate lottery for every development, but each agent can enter at most one lottery in his or her lifetime.

1.1. Equivalence of Mechanisms

We show in Theorem 1(a) that despite their very different descriptions, independent lotteries and the waitlist without choice lead to identical outcomes in equilibrium. By this, we mean that for each agent, the probability of matching, distribution of value conditioned on matching, and expected time in the system are identical across the two systems.

Analogously, Theorem 1(b) states that the ticket-saving lottery is equivalent to the waitlist with choice. If participation costs are negligible, Theorem 1(c) establishes that both of these mechanisms are equivalent to the common lottery and to the single-entry lottery.

1.2. Maximizing Match Quality

Independent lotteries and the waitlist without choice often fail to match agents to developments that fit their needs. The reason is that when lottery odds are low or waits are long, agents are willing to accept almost any development and are therefore assigned nearly at random. By contrast, when using a common lottery, agents with good priority numbers can get anything they choose and will therefore be selective in where they apply.

We formalize this intuition in Theorem 2(a), which shows that for every agent, a common lottery achieves better matching (i.e., higher utility per match) than any of the other systems considered in this paper. In fact, Theorem 2(b) states that when agents remain eligible for many periods, a common lottery approximately maximizes match quality over the space of all anonymous mechanisms.

1.3. Trade-Off Between Matching and Targeting

Although a common lottery matches effectively, it might fail to target need. The intuition, formalized in Theorem 2(b), is that in a common lottery, agents are selected at random, and therefore, all agents match at similar rates. With independent lotteries, meanwhile, agents with worse outside options enter more lotteries and match at higher rates.

When there are many agents with low need (good outside options), Theorem 3 establishes that the trade-off between matching and targeting is not unique to the mechanisms above but holds for *any* pair of anonymous mechanisms. This trade-off does not hold when all agents have high need and apply to all developments; in that case, a common lottery outperforms independent lotteries on both matching and targeting.

1.4. Maximizing Utilitarian Welfare

When there are many agents with low need so that there is a trade-off between matching and targeting, Theorem 4 gives guidance on which objective to prioritize to maximize utilitarian welfare. If the distribution of outside options has a light left tail, it is more important to match well. In this case, Theorem 2(b) implies that a common lottery is approximately optimal. If the distribution of outside options has a heavy left tail, it is most important to target effectively. In this case, Theorem 6 shows that it is approximately optimal to increase participation costs until only the most needy agents apply.[4]

When all agents have high need (poor outside options), it is not worthwhile to try to achieve targeting endogenously. Theorem 5 shows that if all applicants have sufficiently poor outside options, then a common lottery is approximately welfare optimal *regardless* of the shape of the outside option distribution.

2. Related Work

This paper contributes to the growing literature on dynamic matching markets. For reviews of the literature on static matching markets, see Roth and Sotomayor (1992) and Sönmez and Ünver (2011).

One strand of the dynamic matching literature focuses on generalizing the concept of stable matchings in static two-sided markets to dynamic settings. Papers that fall into this category include Damiano and Lam (2005), Kurino (2009), Pereyra (2013), Kennes et al. (2014), and Doval (2018). Contrasted with this work, the markets we study are one sided because items have no preferences. Thus, the concept of stability is not meaningful.

Another set of papers assumes that the social planner has all relevant information about the quality of each match. Much of this literature focuses on the application to kidney exchange, which started with the seminal paper by Roth et al. (2004). Representative recent works include those by Dickerson et al. (2012), Gurvich and Ward (2014), Akbarpour et al. (2017), Baccara et al. (2018), and Ashlagi et al. (2018). In earlier work, Kaplan (1987a, b, 1988) formulated the allocation of affordable housing as a queuing problem and studied waiting times and development diversity under various priority rules. In contrast with the preceding papers, we assume that most of the relevant match information is privately known and revealed strategically by agents.

Our paper falls into the category of dynamic matching with private, one-sided preferences. A series of papers in this category was motivated by the allocation of cadaver organs (Su and Zenios 2004, 2005, 2006, Schummer 2016, Agarwal et al. 2018). In this setting, items (organs) are perishable and thus can be offered to only a limited number of individuals, and agents agree on their relative preferences across organs. Su and Zenios (2004) advocate for switching from a first-come, first-served queue to something resembling a last-come, first-served queue to make agents less picky and increase the utilization of less desirable organs. Schummer (2016) notes that preventing agents from rejecting offers may decrease wastage, at the expense of reducing match quality for agents at the top of the queue. Agarwal et al. (2018) study the organ wastage problem from an empirical perspective and estimate agent preferences from data and simulate counterfactuals. In our setting, wastage is not a concern, and preference heterogeneity is horizontal rather than vertical. As a result, it is generally preferable to induce agents to be more (rather than less) selective.

Closer to our work are the papers of Bloch and Cantala (2017) and Leshno (2017), which are motivated by the allocation of subsidized housing units and focus on how to match people with the right places. Bloch and Cantala (2017) find, as we do, that the waitlist with choice induces agents to be pickier than under independent lotteries, resulting in higher match quality. Leshno (2017) notes that agents who have a middling position in a waitlist with choice would be more selective under a hybrid mechanism, which makes offers randomly among all agents with sufficiently high positions on the waitlist. The biggest difference between our work and these papers is that our agents have heterogeneous outside options, and thus the efficiency of a matching depends crucially on *which* agents match. Additionally, by studying a continuum model, we are able to consider richer environments (rather than assuming that values for a development are binary) and develop new insights about the equivalence of various mechanisms.

The problem of targeting aid to certain subpopulations has been considered in the public finance literature on the design of subsidies. Nichols and Zeckhauser (1982) and Blackorby and Donaldson (1988) use a simple model with two agent types to illustrate that one can target the type with higher need by restricting the flexibility of the subsidies or by adding friction. A similar idea appears in a series of papers on "money-burning auctions" (Hartline and Roughgarden 2008, Hoppe et al. 2009, Condorelli 2012, Chakravarty and Kaplan 2013), in which a social planner allocating a homogeneous good cannot use monetary payments to determine who values it the most but may screen agents based on how much wasteful effort they are willing to incur. Several of these papers have results resembling our Theorem 4: when the valuation distribution is heavy tailed, the designer should use wasteful effort to improve targeting; when it is light tailed, it is more efficient to allocate randomly. We extend this insight to a setting where agents care about *which* good they receive and illustrate that reducing match quality, instead of requiring wasteful effort, is an alternative way of targeting agents with greater need.

Finally, there is a growing empirical literature on the allocation of affordable housing. Glaeser and Luttmer (2003) provide evidence on the misallocation of rent-controlled housing in NYC and argue that it is caused by the random matching that arises from rationing. We show that in spite of the reality of scarce supply, there are mechanisms that can improve the matching. Geyer and Sieg (2013), Sieg and Yoon (2017), and Waldinger (2018) estimate random utility models of development choice using public housing data from Pittsburgh, NYC, and Cambridge, MA, respectively. All three papers assume a certain parametric form for the outside option of agents and use this to separately identify agent values for various developments and agent outside options; these two entities, respectively, correspond to our F and G distributions in Section 3.1, except that the empirical papers allow for a richer correlation structure through the use of agent and development characteristics. Thakral (2016) simulates the demand model of Geyer and Sieg (2013) and reports significant welfare gains by switching from the waitlist without choice to alternatives that encourage greater selectivity. Waldinger (2018) performs simulations using Cambridge data and reports that increasing choice leads to better matching but worse targeting (similar to our Theorem 3), but the net benefit in social welfare is positive (similar to our Theorem 4(a)). Both papers estimate economically significant welfare gains from switching to a mechanism that improves matching: Thakral (2016) estimates gains equivalent to a cash transfer of $2,572 per unit per year in Pittsburgh, and Waldinger (2018) estimates gains equivalent to a transfer of $1,557 per unit per year in Cambridge.

Our theoretical analysis complements the empirical work by showing that the insights of poor match quality from the waitlist without deferral, matching–targeting trade-off, and positive net benefit of better matching on welfare are not particular to the data from these cities but also hold for a wide variety of distributions and allocation mechanisms. Furthermore, our theory can also guide empirical researcher on what functional forms to explore for robustness checks. For example, Geyer and Sieg (2013), Sieg and Yoon (2017), and Waldinger (2018) all parameterize outside options within each demographic group as a linear function of the logarithm of household income, which makes it likely to be approximately light left tailed because household income is known to be approximately log-normally distributed for low- to middle-income groups (Clementi and Gallegati 2005). It is possible that the simulation result from Thakral (2016) and Waldinger (2018) that mechanisms that encourage selectivity have better utilitarian welfare is an artifact of the parametric form. One robustness check suggested by our Theorem 4 for this researcher is to also explore heavy-tailed parameterizations of outside options.[5]

3. Model

Section 3.1 describes the timing of agent arrivals and our assumptions about agent utilities. Section 3.2 discusses the dynamic decision problem facing each agent. Section 3.5 defines our equilibrium concept, which builds on a definition of optimal agent strategies (Section 3.3) and match outcomes (Section 3.4). Section 3.6 introduces the metrics that we use to evaluate equilibria. For clarity of exposition, we refer to all agents using female pronouns.

3.1. Agents, Outcomes, Utilities, and Timing

Time is discrete. In every period j, a continuum of agents of unit mass arrives, as does a new development, which can house a mass μ of agents and must be allocated immediately. We refer to μ as the *supply–demand ratio*.

Each agent will eventually either be matched to a single development or depart from the system unmatched. Before being matched, agent i receives payoff α_i in each period, and after being matched to development j, the agent receives payoff v_{ij} in each period. We refer to α_i as the agent's *outside option* and v_{ij} as her *value* for development j. We sometimes refer to α_i as the *type* of agent i.

Each agent (matched or not) has a life event with probability $1 - \delta$ in each period, after which she becomes ineligible for future allocations and leaves her affordable unit if she has been awarded one.[6] We normalize an agent's utility after her life event to be zero. We assume that the timing of the life event is independent of the agent's past actions. The expected number of periods before an agent's life event is $\frac{1}{1-\delta}$. We refer to this quantity as her *expected eligibility time*.

The timing within each period is as follows:

1. *Arrival and participation choice*. A unit mass of new agents arrives, and each is given a *state* (typically, a lottery number or waitlist position). Unmatched agents who remain choose whether to continue to participate in the mechanism or exit forever. Those who participate incur participation cost $c \geq 0$. For convenience, we assume that agents who are exactly indifferent between participating and exiting will choose to participate.[7]

2. *Life event*. Every agent (matched or not) has a major life event with probability $1 - \delta$, in which case she leaves her current housing. Her utility after this point is normalized to zero.

3. *New development and matching*. A development of mass μ arrives, labeled j. Each agent i observes v_{ij}.

Agents participate in a matching rule as described in Section 3.2. Those who are matched become ineligible for future matches.

4. *Payoff.* Every agent who remains receives a payout that depends on her current housing (v_{ik} if in development k and α_i if unmatched).

The outside options α_i are distributed according to cumulative distribution function (CDF) F, and the values v_{ij} are drawn independent and identically distributed (across agents and developments) from CDF G. We refer to F as the *outside option distribution* and G as the *value distribution*. For convenience, we assume that distributions F and G are continuous, with strictly positive density on their domains $(\underline{\alpha}, \overline{\alpha})$ and $(\underline{v}, \overline{v})$, respectively.[8] We allow for the possibility that $\underline{\alpha}$ or $\underline{v}$ may be $-\infty$ or that $\overline{\alpha}$ or $\overline{v}$ may be ∞ and assume without loss of generality that $\overline{\alpha} \le \overline{v}$ (agents with outside options exceeding $\overline{v}$ will never choose to participate, so we can exclude them and normalize F appropriately). We denote the density of F by the function $f(\alpha)$ and define $\overline{G}(v) = 1 - G(v)$.

3.2. Actions

The values v_{ij} and the outside option α_i are privately known to agent i. Thus, they cannot be directly used to determine an allocation. Instead, agents participate in a *matching rule*, which asks them to take an action in each period and uses the actions to determine who will match to the current development.

Before giving our formal definition of a matching rule, we motivate this definition: although agents are in principle playing a dynamic game, we restrict attention to designs in which agents are affected only by the *aggregate* profile of actions selected by others, and we assume that agents respond to this aggregate (rather than to actions of specific other agents). This implies that no single agent can directly influence the market or the future behavior of others. Therefore, each agent perceives herself not as playing a dynamic game but rather as facing a Markov decision process (MDP). She begins each period in some state, which determines the set of actions available to her.[9] Her action, in turn, influences whether she matches and which state she transitions to in the event that she does not match.

Definition 1 (Matching Rule). A *matching rule* $R = (S, D, A, T)$ specifies a countable set of states S and a distribution D over S specifying the probability of assigning each state $s \in S$ as the initial state. There is a countable set of actions $A = \bigcup_{s \in S} A_s$, where A_s is a finite set of actions for state $s \in S$. For each action $a \in A_s$, there is a transition function $T_a : S \times (S \cup \{m\}) \to [0, 1]$, where $T_a(s, s')$ is the probability of transitioning to state s' after taking action a in state s, and m corresponds to being matched to the current development.

Implicit in this definition is the assumption that the mechanism is *anonymous*: it can differentiate agents based on the history of actions taken but not based on the identity of the agent. Another implicit assumption is that the mechanism is *stationary*, meaning that in our continuum model, the aggregate profile of agent types and actions is deterministic and constant across periods. For this reason, the transition function T_a is not indexed by the period j.

Below we describe how the allocation systems described in the Introduction can be encoded as matching rules. The *lottery-based* rules are fully characterized by a success probability $p \in (0, 1]$, which is the chance that any given ticket will win a lottery. The *waitlist-based* rules are characterized by an average idle time $\tau \ge 0$, which is the expected number of periods a newly arrived agent must wait before receiving an offer.

3.2.1. Lottery Matching Rules.

- *Independent lotteries.* The state space S consists of a single state. In it, the agent chooses from the action set $\{Enter, Abstain\}$. If she abstains, she is not matched. If she enters, she is matched with probability p. When $p = 1$, we refer to this as the *guaranteed choice* matching rule.
- *Common lottery.* $S = \{0, 1\}$. The initial state of an agent is 1 with probability p, and an agent's state remains the same in every period. In both states, agents choose from the action set $\{Enter, Abstain\}$. An agent is matched if and only if she is in state 1 and chooses to enter. (Agents in state 0 will never match.)
- *Single-entry lottery.* $S = \{0, 1\}$. All agents start in state 1, from which they can choose from actions $\{Enter, Abstain\}$. If an agent abstains, she does not match and remains in state 1. If she enters, she matches with probability p and otherwise transitions to state 0, from which she will never be matched.
- *Ticket-saving lottery.* $S = \mathbb{N}$ represents the number of tickets possessed by the agent. The agent starts in state 1. From state s, the agent chooses an action $j \in \{0, \ldots, s\}$ (the number of tickets to use this period). An agent in state s choosing action j matches with probability $1 - (1 - p)^j$ and otherwise transitions to state $s - j + 1$.

3.2.2. Wait-List Matching Rules. The state space is $\mathbb{N}$, representing the number of periods that the agent has waited. The initial state is zero. In states $s < \lfloor \tau \rfloor$, the agent has a single action, $\{Wait\}$, and transitions deterministically to state $s + 1$. In states $s > \lfloor \tau \rfloor$, the agent selects an action from $\{Accept, Reject\}$. From state $s = \lfloor \tau \rfloor$, the agent is offered the action $\{Wait\}$ with probability $\tau - \lfloor \tau \rfloor$ and otherwise offered the actions $\{Accept, Reject\}$.[10] An agent matches if and only if she chooses *Accept*. The two variants are as follows:

- *Wait-list with choice.* Agents who reject retain their position (increment their state).

- *Wait-list without choice*. Agents who reject lose their position (go back to state 0).

We define a mechanism M to be a class of matching rules. For example, the *independent lottery mechanism* is the set of all independent lottery matching rules, parameterized by all possible values of success probability p. Analogously define the mechanisms for the common lottery, single-entry lottery, ticket-saving lottery, and wait-lists with and without choice.

3.3. Strategies

A matching rule $R = (S, D, A, T)$ (along with the probability δ of remaining eligible, a value α for going unmatched, the value distribution G, and the participation cost c) induces an MDP for each agent. A *strategy profile* Σ consists of a Markovian strategy $\Sigma(\alpha)$ for every agent type α. This strategy specifies, for each state s, whether to exit and what action to take as a function of the value v for the current development. A strategy is optimal if the continuation value of being in each state satisfies the following Bellman equation:

$$V(s) = \max\left(0, \delta\mathbb{E}_{v\sim G}\left[\max_{a\in A_s}\left\{T_a(s,m)\left(\frac{v-\alpha}{1-\delta}\right) + \sum_{s'\in S} T_a(s,s')V(s')\right\}\right] - c\right). \tag{1}$$

This equation can be interpreted as follows: the value of being in state $s \in S$ is the maximum of the value of exiting (normalized to zero) and staying. The value of staying is based on choosing the best action $a \in A_s$, and each action determines a probability $T_a(s,m)$ of matching in state s to the current development, as well as a probability $T_a(s,s')$ of transitioning to state s'. If the agent matches with the current development with value $v \sim G$, her value is $\frac{v-\alpha}{1-\delta}$ because she would receive a net benefit (over her current situation) of $v - \alpha$ in each period, and she is expected to be able to enjoy this benefit for $\frac{1}{1-\delta}$ periods. We multiply the term within the expectation by δ and subtract c because agents receive value only if they pay the participation cost and do not have a life event in the current period. Appendix C formally defines the MDP facing each agent and shows that for any $\delta < 1$, a solution $V(s)$ to the above Bellman equation exists and is unique.

3.4. Outcomes

An *outcome* specifies a distribution of payoffs for each agent type. Formally, an outcome E specifies an *outcome function* $P^E : [\underline{\alpha}, \overline{\alpha}] \times [\underline{v}, \overline{v}] \to [0,1]$ and a *waiting time function* $t^E : [\underline{\alpha}, \overline{\alpha}] \to [0, \infty)$, where $P^E(\alpha, v)$ specifies the probability that an agent with outside value α matches to a development for which her value is at most v, and $t^E(\alpha)$ specifies the expected number of periods she participates in the mechanism before leaving or being matched. For a given matching rule R, every strategy profile Σ induces a unique outcome, which we refer to as the *outcome corresponding to R and* Σ.[11] Proposition 1 in Section 4.3 implies that for outcomes that correspond to *optimal* strategy profiles, the waiting time function t^E can be expressed in terms of the outcome function.[12] For any such outcome E, it suffices to specify only the outcome function, and our proofs in the online appendix abuse notation and use $E(\alpha, v)$ to refer to the outcome function instead of $P^E(\alpha, v)$.

Given an outcome $E = (P^E, t^E)$, we define the following.

- The *allocation function*

$$\pi^E(\alpha) = P^E(\alpha, \overline{v}) \tag{2}$$

specifies the probability that an agent with outside option α matches to some development.

- The *match rate*

$$\overline{\pi}^E = \mathbb{E}_{\alpha\sim F}[\pi^E(\alpha)] \tag{3}$$

specifies the fraction of agents who match.

- The *expected utility* function

$$u^E(\alpha) = \int_{\underline{v}}^{\overline{v}} (v-\alpha)\, dP^E(\alpha, v) - (1-\delta)ct^E(\alpha) \tag{4}$$

specifies each agent's expected total benefit from participation multiplied by $1/(1-\delta)$. This scaling factor keeps the utility in the same scale as each period's value and cost because the expected eligibility time of an agent is $\frac{1}{1-\delta}$ periods.

3.5. Equilibrium

For simplicity, we first rule out the trivial case in which supply is so abundant that the market designer can offer every development to every agent—if this were feasible, then it would be clearly optimal to do so. Define GC to be the outcome when agents play optimally in the MDP induced by the guaranteed choice matching rule.[13] We assume the following for the remainder of this paper.

Assumption 1. *It is infeasible to offer guaranteed choice to all agents ($\mu < \overline{\pi}^{GC}$).*

Under this assumption, an equilibrium can be defined as a matching rule and an optimal strategy profile that exactly clear the market.

Definition 2 (Equilibrium Outcome). An outcome E is an *equilibrium outcome* of matching rule R if it can be expressed as the outcome corresponding to R and a strategy profile Σ such that

(a) for every α, the strategy $\Sigma(\alpha)$ is optimal for an agent with outside option α;

(b) the average match rate of E equals the supply–demand ratio $\overline{\pi}^E = \mu$.[14]

We sometimes refer to E simply as an *equilibrium outcome*, without mentioning the matching rule R. If E satisfies (a) but not (b), we refer to it as a *partial equilibrium outcome.*

In our model, the exogenous parameters are the market primitives F, G, δ, c, and μ and the designer's choice of mechanism M. The matching rule $R \in M$ is endogenously determined in equilibrium, as are the strategy profile Σ and the outcome E. The pair (R, Σ) corresponds to an equilibrium outcome E if and only if aggregate demand is equal to aggregate supply.[15] When M is a lottery-based matching rule, the endogenous quantities are characterized by the success probability p, which uniquely determines the matching rule R and the corresponding strategies and outcome. When M is waitlist based, the endogenous quantities are characterized by the average idle time τ.

3.6. Metrics for Evaluating Equilibria

Below we define several metrics used to evaluate outcomes. Given outcome E, we define the following.

- The *match distribution*

$$F^E(\alpha) = \frac{\int_{-\infty}^{\alpha} \pi^E(x) dF(x)}{\int_{-\infty}^{\infty} \pi^E(x)\, dF(x)} \tag{5}$$

gives the distribution of outside options conditional on matching. That is, $F^E(\alpha)$ is the fraction of matched agents who have outside options no better than α.

- The *value per match*

$$\nu^E(\alpha) = \frac{u^E(\alpha)}{\pi^E(\alpha)} \tag{6}$$

gives the expected utility per unit of housing allocated to each type α. For convenience, define the value per match to be zero when the denominator is zero.

- The *utilitarian welfare*

$$W^E = \mathbb{E}_{\alpha \sim F}[u^E(\alpha)]/\overline{\pi}^E \tag{7}$$

is the aggregate benefit per allocated housing unit over all types.

In the introduction, we discussed two objectives: ensuring that matched individuals receive a desirable development with minimal participation cost and targeting the most needy individuals. The first of these objectives, which we refer to as *matching*, is captured by the value per match $\nu^E(\alpha)$, whereas the second, which we refer to as *targeting*, is captured by the match distribution F^E. We now define what it means for one outcome to result in better matching or targeting than another. The definitions have a strong requirement of pointwise or stochastic dominance, but we will show that such relationships exist among the mechanisms we study.

Definition 3. Let E and E' be arbitrary outcomes. We say that

- *E match dominates E'* if $\nu^E(\alpha) \geq \nu^{E'}(\alpha)$ for all α;
- *E' targeting dominates E* if the match distribution of E first-order stochastic dominates the match distribution of E', that is, $F^{E'}(\alpha) \geq F^E(\alpha)$ for all $\alpha \in (\underline{\alpha}, \overline{\alpha})$.

4. Results

4.1. Equivalence of Mechanisms

We first show that mechanisms that look very different can achieve equivalent outcomes. In fact, when participation costs are negligible compared with the value of being awarded housing, all six mechanisms defined in Section 3.2 are equivalent to either independent lotteries or the common lottery.

To state the result formally, we say that mechanisms M and M' are *outcome equivalent* if the set of equilibrium outcomes are equal: $\mathscr{E}^M = \mathscr{E}^{M'}$, where

$$\mathscr{E}^M = \{(P^E, t^E) : E \text{ is an equilibrium outcome of some matching rule } R \in M\}. \tag{8}$$

In other words, there is a one-to-one correspondence between the equilibrium outcomes of the two mechanisms, such that in each pair of equilibrium outcomes, the distribution of matches and the expected waiting times are equal for every agent type.

Theorem 1 (Equivalence of Mechanisms).

(a) *Independent lotteries are outcome equivalent to the waitlist without choice.*

(b) *The ticket-saving lottery is outcome equivalent to the waitlist with choice.*

(c) *When $c = 0$, the ticket-saving lottery, waitlist with choice, and the single-entry lottery are outcome equivalent to the common lottery.*

The proof is in Section D.2 of the online appendix. We give the intuition below.

For part (a), think of the following implementation of independent lotteries: instead of asking agents to enter the lottery and *then* selecting winners, select winners among *all* eligible agents and offer these winners the opportunity to match to the development. This procedure is equivalent because the agents who choose to enter the lottery in the first description are exactly those who will accept the development in the second. Therefore, in both independent lotteries and the waitlist without choice, agents are periodically offered the chance to match to the current development. In the waitlist without choice, this occurs approximately every τ periods. Under independent lotteries, this occurs independently in each period, with some probability p. However, what matters to each agent is not the *distribution* of when she will next receive an offer, but rather the *probability* that she will receive at least one more offer (call this q).[16] This probability determines which developments she will accept, and thus her probability of matching. Because both mechanisms match the same number of agents, it follows that any value of q that arises in equilibrium of independent lotteries must also be an equilibrium of the waitlist without choice—and that in these equilibria, each agent sets the same threshold when determining which developments to accept and matches with the same probability.

For part (b), first consider the waitlist with choice. Because the equilibrium is stationary, once an agent is offered one development, that agent will be offered every future development. Therefore, agents in the waitlist with choice must wait (for approximately τ periods) before playing guaranteed choice. Now consider the ticket-saving lottery and recall that regardless of when it is used, each ticket wins with some fixed probability p. Consider a variant of the ticket-saving lottery in which each ticket, when given to a participant, is visibly labeled as a winning ticket (with probability p) or a losing ticket (with probability $1-p$). It is clear that in this variant, agents must wait a geometric number of periods before receiving a winning ticket, and from that point onward, will set an acceptance threshold as in guaranteed choice (and use all tickets when entering). As in part (a), the distribution of idle time does not matter to an agent, but only the probability q that she becomes eligible before her life event. Because both the waitlist with choice and the ticket-saving variant match the same number of agents, they must have the same value of q and therefore lead to equivalent outcomes.

Of course, in the actual ticket-saving lottery, the labels "winning ticket" and "losing ticket" are revealed only after the tickets are used. But this knowledge does not change an agent's optimal strategy because when she does not hold a winning ticket, her actions do not matter.[17] Therefore, she should always behave as though she holds a winning ticket: set an acceptance threshold as in guaranteed choice and use all tickets upon seeing such a development.

For part (c), note that when participation cost $c = 0$, delays are costless, so the delayed guaranteed mechanisms in part (b) are equivalent to selecting a random subset of agents to face guaranteed choice, which is the definition of the common lottery. Similarly, the single-entry lottery effectively selects some agents (those with winning tickets) to play guaranteed choice while eliminating others. Although agents in a single-entry lottery do not know whether they have been selected until after entering the lottery, the reasoning from part (b) implies that they will set acceptance thresholds as though they held a winning ticket.

Note that the arguments for part (c) no longer hold when $c > 0$. In particular, the waitlist with choice is no longer equivalent to the common lottery because agents in the former must incur a significant cost to be given the option to match to a development. Moreover, agents in the single-entry lottery have an incentive to use their ticket early so that they can exit and stop incurring participation costs.

4.2. Maximizing Match Quality

In our model, effective matching requires that agents are matched to items that are a good fit at minimal participation cost. The common lottery accomplishes both of these goals. Agents with good lottery numbers have high continuation values and therefore an incentive to be selective; agents with poor lottery numbers learn immediately that they will not match and therefore do not incur participation costs while clinging to a false hope. In fact, Theorem 2 shows that the common lottery not only match dominates all the other mechanisms we study but also converges to the best possible outcome in terms of matching when agents are eligible for many periods ($\delta \to 1$). We think that this is a reasonable limit to study for the application of affordable housing in NYC: there are lotteries for over 70 new developments each year, so if agents expect to remain eligible and interested for at least 18 months, then δ exceeds 0.99.

For the asymptotic limit to be defined, we require that values are bounded ($\overline{v} < \infty$).

Definition 4. When $\overline{v} < \infty$, define *perfect matching* (PM) to be the outcome in which agents of all types match with probability μ and, conditioned on matching, have the highest possible value $\overline{v}$ for their assignment with negligible participation cost: $P^{PM}(\alpha, v) = \mu \mathbb{1}(v \geq \overline{v} > \alpha)$, with $t^{PM}(\alpha) = 0$.

Definition 5. A sequence of equilibrium outcomes E^n converges to outcome E if the outcome functions converge pointwise; that is, $P^{E_n}(\alpha, v) \to P^E(\alpha, v)$, for all α, v.[18]

Theorem 2 (Match Dominance of the Common Lottery).

(a) *The unique equilibrium outcome of the common lottery match dominates any equilibrium outcome of the independent lotteries, single-entry lottery, ticket-saving lottery, waitlist with choice, and waitlist without choice.*

(b) *When $\overline{v} < \infty$, as $\delta \to 1$, the equilibrium outcome of the common lottery converges to perfect matching, which match dominates any equilibrium outcome.*

The proof of Theorem 2 is in Section D.3 of the online appendix. Part (a) is based on the structural results derived in the proof of Theorem 1, which allow us to derive explicit expressions for the value per match in each mechanism. Part (b) is based on showing that when δ is high, agents will accept only developments close to the maximum value of $\overline{v}$. Moreover, any utility loss they incur while waiting for such a development is negligible compared with the many periods they get to enjoy their apartment after matching. Finally, almost every agent who wins the common lottery eventually matches, so the probability of matching is nearly the same (and equal to μ) for all agents.

When participation cost is negligible, the equilibrium outcomes of the single-entry lottery, waitlist with choice, and ticket-saving lottery also converge to perfect matching by Theorem 1. When $c > 0$, Section D.3 of the online appendix shows that the single-entry lottery converges to perfect matching, but the waitlist with choice and ticket-saving lottery do not. The reason is that under these mechanisms, high values of δ result in long expected wait times, causing some agents to not participate and the remainder to experience significant participation costs. By contrast, in the single-entry lottery, agents can leave as soon as they use their ticket, so the participation cost they incur is minimal.

4.3. Trade-Off Between Matching and Targeting

An anonymous mechanism cannot benefit agents with the greatest need without allocating also to agents with less need because low-need agents can always copy the behavior of high-need agents. This is formalized in Proposition 1, which states that in any equilibrium outcome, the utility of an agent is equal to the integral of the match rate for all agents with better outside options. Hence, to increase the utility of agents with poor outside options, it is necessary to increase the match probability of those with better outside options.

Proposition 1. *For any partial equilibrium outcome E, the allocation function $\pi^E(\alpha)$ is weakly decreasing, and the expected utility function is given by*

$$u^E(\alpha) = \int_\alpha^\infty \pi^E(x)\,dx. \tag{9}$$

Proposition 1 can be used to show that when there are many low-need individuals, there is a trade-off between providing high-quality matches and targeting need effectively.

Definition 6. There are many *low-need individuals* if $\overline{\alpha} \geq \overline{v}$ and the density of outside options f is increasing on $(\underline{\alpha}, \overline{\alpha})$.

Theorem 3 (Matching vs. Targeting). *Let E and E' be equilibrium outcomes. If E match dominates E' and if there are many low-need individuals (see Definition 6), then E' targeting dominates E.*

We prove Proposition 1 in Section D.4.1 of the online appendix and prove Theorem 3 in Section D.4.4 of the online appendix.

4.4. Maximizing Utilitarian Welfare

When matching and targeting are in conflict with one another, it is natural to wonder which objective is more important. Theorem 4 shows that the answer to this question depends on both the shape of F and the support of F and G.

Definition 7. F has a *light left tail* if $F(x)/f(x)$ is weakly increasing in the domain $(\underline{\alpha}, \overline{\alpha})$, and F has a *heavy left tail* if $F(x)/f(x)$ is weakly decreasing in the domain $(\underline{\alpha}, \overline{\alpha})$.

Examples of distributions with light left tails include the uniform, the normal, and the negated exponential or Gumbel distributions. Translated or truncated versions of these distributions also have light left tails. The (negated) exponential distribution has the property that $F(x)/f(x)$ is constant (and thus is the dividing line between light- and heavy-tailed distributions). The negated versions of the Pareto and the log-normal distributions have heavy left tails.[19]

Theorem 4 (Welfare Comparisons). *Let E and E' be equilibrium outcomes. If $\overline{\alpha} \geq \overline{v}$ and E' targeting dominates E, then the following hold:*

(a) *If F has a light left tail, then $W^E \geq W^{E'}$.*
(b) *If F has a heavy left tail, then $W^E \leq W^{E'}$.*

Theorem 3 implies that the conditions of Theorem 4 are satisfied if E match dominates E' and there are many low-need individuals (Definition 6). Therefore, Theorem 4(b) implies that when there are many low-need individuals and the outside option distribution has a heavy left tail, optimizing for match quality is detrimental to aggregate welfare. In this case, a common lottery might lower utilitarian welfare compared with independent lotteries.

For affordable housing allocation in NYC, we believe that this is not the case. the reason is that those who qualify for housing already fall within a narrow income range, so it seems reasonable that many agents have similar outside options. Moreover, the developments being allocated in NYC are newly constructed and designed to be attractive to market-rate renters, so we expect that most eligible applicants consider many of these units preferable to their current living situation. Hence, the setting in NYC may be better approximated by the conditions of Theorem 5, which states that if outside options are light tailed or sufficiently poor,[20] then utilitarian welfare is maximized by prioritizing good matching. Under these conditions, it follows from Theorem 2(b) that when δ is high, a common lottery achieves near-optimal utilitarian welfare.

Theorem 5 (Optimality of Perfect Matching). *Perfect matching achieves weakly higher utilitarian welfare than any equilibrium outcome if either*

(a) *F has a light left tail, or*
(b) $\overline{v} - \overline{\alpha} \geq \overline{\alpha} - \underline{\alpha}$.

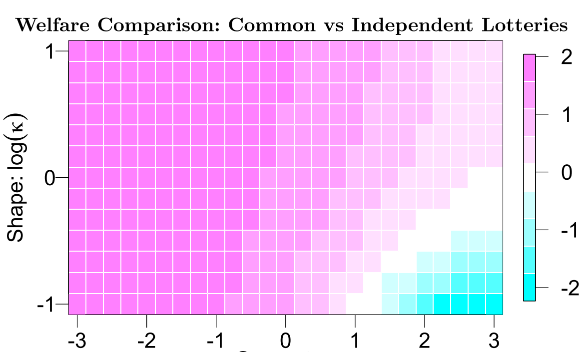
We interpret Theorems 4 and 5 to mean that matching is more important than targeting whenever outside options follow a light-tailed distribution or are sufficiently low. Figure 1 reinforces this point.[21] It displays the welfare difference between common and independent lotteries as the shape and support of the outside option distribution vary. The common lottery is superior unless the outside option distribution is heavy tailed and outside options are good (the lower-right region). Furthermore, the differences are significant: a welfare difference of 1.5 implies that the difference between the two mechanisms is equal to the difference between matching agents to random developments and matching them to something that they prefer to 93% of developments.

We prove a more general version of Theorem 4 in Section D.4.5 of the online appendix and prove Theorem 5 in Section D.4.6 of the online appendix.

4.5. Achieving Effective Targeting

Although a common lottery may not be effective at targeting need, the same is true of independent lotteries and the waitlist without choice. In fact, Section D.6.2 of the online appendix shows that in some cases these approaches result in no targeting at all. Even when there are many low-need individuals—in which case Theorems 2(a) and 3 jointly imply that these approaches targeting dominate a common lottery—there are generally more effective ways to target need.

Figure 1. (Color online) Difference in Welfare Between Common and Independent Lotteries for Various Outside Option Distributions



Notes. We take G to be $Normal(0,1)$ and let $\mu = 0.1$, $\delta = 0.99$, and $c = 0$. The outside option distribution F is a (negated) Weibull distribution, given by $F(\alpha) = \exp\left(-\left(\Gamma(1+\frac{1}{\kappa})(\overline{\alpha}-\alpha)\right)^{\kappa}\right)$. This distribution has expected value $\overline{\alpha}-1$. It is light tailed for $\kappa>1$ and heavy tailed for $\kappa<1$, with $\kappa=1$ corresponding to an exponential distribution. Positive values (pink online) correspond to higher welfare under the common lottery. Moving from bottom to top, the tail of F becomes lighter, with $\log(\kappa)=0$ corresponding to the exponential distribution. Moving from right to left, the outside option distribution shifts downward. A common lottery attains higher welfare whenever outside options are light tailed (top region) or sufficiently poor (left region). Independent lotteries are superior when many agents have good outside options and the distribution of outside options is heavy tailed (lower right). A difference of 2 means that the improvement in per-match welfare is equal to two standard deviations of the outside option distribution G.

A simple approach to achieve good targeting regardless of distributional assumptions is as follows: artificially increase participation cost until it is possible to match every agent who is willing to participate. In practice, this may mean requiring agents to undergo a costly ordeal to remain eligible, such as to complete endless paperwork or to physically line up at a central office every week.[22] Although we do not believe that this is a good solution for allocating affordable housing, such practices may be reasonable in settings with loose eligibility criteria, such as in the allocation of discounted Broadway tickets.

Precisely speaking, participation cost c is said to be *market clearing* if under this participation cost the average match rate under the guaranteed choice matching rule is equal to the supply–demand ratio μ. We show in Section D.5 of the online appendix that a market-clearing cost always exists; although market-clearing costs may not be unique, there always exists a *highest* market-clearing cost $\overline{c} < \infty$. Define the *costly guaranteed choice* outcome as the guaranteed choice outcome under the highest market-clearing cost $\overline{c}$. When participation cost is increased to $\overline{c}$, all the mechanisms studied in this paper implement the costly guaranteed choice outcome.

Theorem 6 shows that costly guaranteed choice always targeting dominates the common lottery. Furthermore, it converges to the best possible outcome in terms of targeting when agents are long lived ($\delta \to 1$) and values are bounded ($\overline{v} < \infty$).

Definition 8. Define *perfect targeting* (PT) to be the outcome in which agents with outside option $\alpha \le F^{-1}(\mu)$ are matched with certainty, and no other agents are matched: $P^{PT}(\alpha, v) = \mathbb{1}(\alpha \le F^{-1}(\mu))\mathbb{P}_{v' \sim G}(v \ge v' | v' \ge F^{-1}(\mu))$, and $t^{PT}(\alpha) = \frac{1}{c(1-\delta)}\mathbb{1}(\alpha \le F^{-1}(\mu))\mathbb{E}_{v' \sim G}[v' - F^{-1}(\mu) | v' \ge F^{-1}(\mu)]$.[23]

Theorem 6 (Targeting Dominance of Costly Guaranteed Choice).

(a) *The costly guaranteed choice outcome targeting dominates any equilibrium outcome of the common lottery, single-entry lottery, ticket-saving lottery, and waitlist with choice.*

(b) *When $\overline{v} < \infty$, as $\delta \to 1$, the sequence of costly guaranteed choice outcomes converges to perfect targeting, which targeting dominates any equilibrium outcome.*

The proof is given in Section D.5 of the online appendix. Part (a) is based on structural results derived in the proof of Theorem 1. To complement Theorem 6(a), we also show in Section D.6.1 of the online appendix that costly guaranteed choice targeting dominates independent lotteries and the waitlist without choice when the value distribution G is light tailed.

For part (b), we show that if agents remain in the system for many periods, then almost everyone who chooses to participate will eventually find a development that they are willing to accept. Moreover, the agents who choose to participate will be those with the greatest need. Because the most needy are matched with near certainty and everyone else is not matched at all, this is the best possible outcome in terms of targeting.

Hence, it is rarely a good idea to use independent lotteries with low participation cost: if matching is more important, the designer should adopt a common lottery. If targeting is more important, the designer should increase participation cost so that low-need agents do not participate at all.

5. Discussion

In this paper, we argue that two common systems for allocating affordable housing—independent lotteries and a waitlist in which applicants lose priority after declining an offer—incentivize prospective tenants to accept buildings that are only marginally better than their outside options. The resulting allocation is inefficient, in that many or all agents could be simultaneously made better off. We discuss several reforms that could improve the quality of the assignment, including limiting lottery entry, allowing applicants to keep their position in a waitlist after rejecting an offer, allowing applicants to save and combine lottery tickets, and using a common lottery to determine priority for all buildings. Our equivalence results suggest that authorities can select among these allocation procedures based on criteria such as ease of implementation.

In NYC, several features suggest that switching to a system that offers choice to applicants would improve outcomes. First, income eligibility limits prevent those with the best outside options from participating. Second, the available units are in new developments with good amenities, built to attract market-rate tenants but offered at much lower prices. Combined, these facts suggest that applicants may significantly prefer these units to their outside options—in which case Theorem 5(b) states that welfare is highest when applicants are offered choice. Furthermore, the size of the city and dispersion of units across boroughs suggest that the benefits of matching applicants to suitable apartments may be economically significant.

We believe that using a common lottery could improve outcomes while requiring only minor changes to current practice. As discussed in Appendix A, the allocation in NYC includes features not captured in our model. For example, eligibility is building specific, and certain groups, such as city employees or neighborhood residents, get priority for a certain number of units in each building. These practices could be maintained under a common lottery, if desired; the city could treat units that give priority to specific groups as separate buildings and, for each building, offer units to those eligible for them in the order determined by the (universal) ranking of applicants.

There are, of course, many ways in which our model oversimplifies reality. We conclude by discussing the robustness of our findings when our modeling assumptions are relaxed.

5.1. Multidimensional Agent Types

Consider a richer model in which agents differ not only in their outside option, but also in their value distribution, participation cost, and expected eligibility time. The type of an agent is represented by a tuple $\theta = (\alpha, G, c, \delta)$ and is distributed according to distribution Θ. This is a straightforward generalization of our current framework, and the Bellman equation for the optimal strategy of each agent remains the same as in (1). The only difference is that we must define the outcome E as a function of the tuple (θ, v) instead of only (α, v).

In this model, our result on the matching efficacy of the common lottery (Theorem 2) continues to hold because the proof is based on analyzing each agent type separately. For a similar reason, the equivalence results continue to hold if agents are homogeneous in δ. However, if the expected eligibility time $\frac{1}{1-\delta}$ varies across agents, then the equivalences break down: agents who are eligible for more periods are more likely to match in waitlist-based mechanisms, whereas short-lived agents prefer lottery-based mechanisms.

Analysis of targeting becomes nuanced under such a model. First, it is unclear whom to target: does someone with very high value for one development but not another have greater or lesser need than someone with a moderate value for all developments? Second, even if the market designer identifies which types to target, the answer to the question of how to target effectively will depend on the distribution of types. For example, if it happens that agents with the worst outside options also have higher participation costs, then it is possible that a common lottery simultaneously match dominates and targeting dominates independent lotteries, even if there are many agents with low need. The reason is that independent lotteries may require participation for many periods before matching (and therefore deter entry by those with high participation costs), whereas the winners of a common lottery are matched very quickly.

5.2. Vertical Differentiation of Developments

Our model assumes purely horizontal differentiation between developments, so all are equally popular in the aggregate. One might consider a model in which development j has quality q_j, and values are distributed as $v_{ij} = q_j + \epsilon_{ij}$, where $\epsilon_{ij} \sim G$ is the horizontal component of preferences. It is much harder to analyze an

Management Science, 2020, vol. 66, no. 6, pp. 2291–2307, © 2020 INFORMS

equilibrium under this model because it is no longer stationary: the type distribution of agents remaining in the system depends on the history of development qualities, and agents must reason about how this type distribution will evolve when making decisions.

Nevertheless, under such a model, we would still expect independent lotteries and waitlist without choice to yield low match quality when supply is scarce because agents' acceptance thresholds on the added value of a development ($v_j - \alpha$) will still equal their continuation value, which is approximately zero if μ is small. Meanwhile, we expect the common lottery to result in better match quality because agents offered a building j for which their idiosyncratic term ϵ_{ij} is small could wait for a building of similar quality that is better suited to their needs. Furthermore, the matching/targeting trade-off described in Theorems 3, 4, and 5 would continue to hold because the proofs rely only on anonymity of the mechanism and not on any assumptions about the nature of the dynamic game being played by agents.[24]

5.3. Partially Observable Outside Options

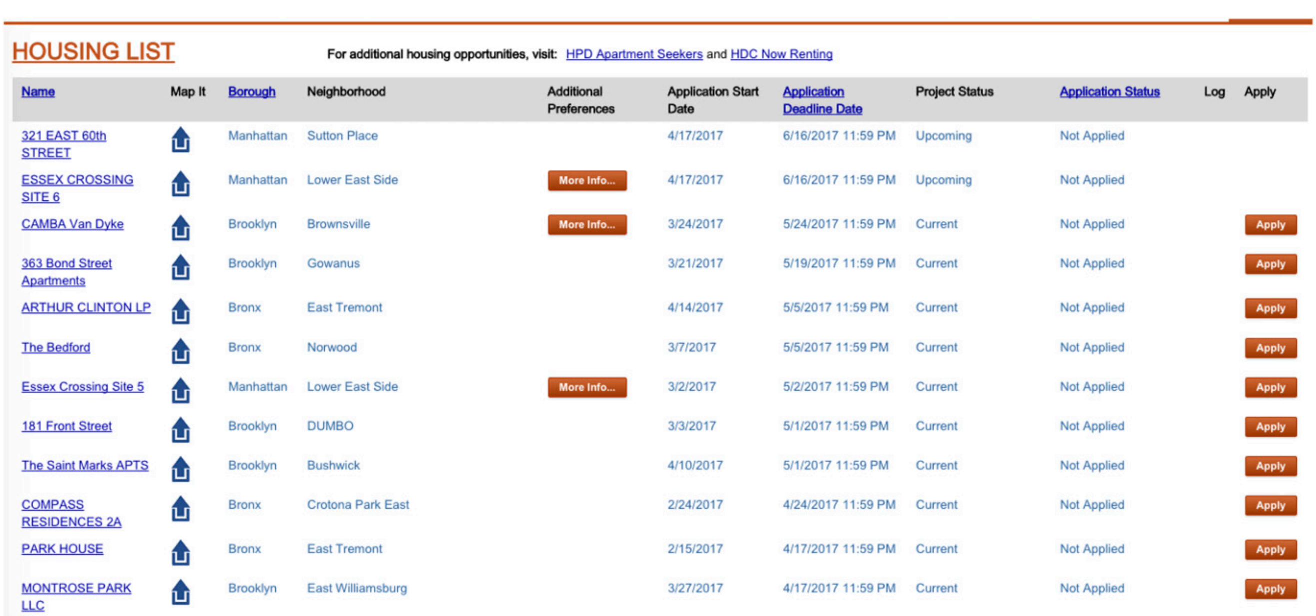
In practice, observable information is often used to prioritize certain agents. This can be captured by an extension of our model in which agents are classified into groups based on characteristics such as income, family status, current residence, etc. Within each group k, the arrival rate of agents is λ_k per period, and the primitives F, G, c, and δ may also be indexed by k. A natural extension of the common lottery to this setting is as follows: assign a priority to each group and a lottery number to each agent. Agent-level priorities are induced by the group priorities, and the lottery numbers are used to break ties. Under this mechanism, it will be the case that high-priority groups can choose whatever they want, and low-priority groups are never matched; agents in borderline priority groups are selected based on their lottery numbers.

Our results imply that this extension of the common lottery will work well if priority is given to groups with the lowest average outside option. In particular, if agents remain eligible for many periods, $\overline{v}$ is the same for each group and finite, and outside options are light tailed within each group, one can show that this version of the common lottery assigns every matched agent to a development where her value is close to the upper bound $\overline{v}$, thus generalizing Theorem 2(b). Moreover, this mechanism achieves near-optimal utilitarian welfare among all stationary mechanisms that are anonymous within each group, thus generalizing Theorem 5(a).

Appendix A. Affordable Housing Allocation in NYC

Since the mid-1980s, NYC has been granting private developers of rental apartments tax exemptions for setting aside a certain proportion of units at an affordable rate.[25] The newly built affordable units are allocated by the oversight of the NYC Department of Housing Preservation and Development (HPD) and the NYC Housing Development Corporation (HDC). Allocation of the newly built affordable units is by lottery, and an independent lottery is conducted for each building. Before 2012, the application was by paper only: applicants had to periodically check announcements for new buildings and mail in their requests to enter the lottery. In 2012, the HPD and HDC launched the NYC Housing Connect web portal, which

Figure A.1. (Color online) Screenshot of the NYC Housing Connect Web Portal in April 2017


HOUSING LIST

For additional housing opportunities, visit: HPD Apartment Seekers and HDC Now Renting

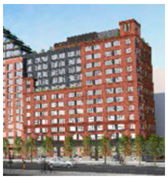
Name	Map It	Borough	Neighborhood	Additional Preferences	Application Start Date	Application Deadline Date	Project Status	Application Status	Log	Apply
321 EAST 60th STREET		Manhattan	Sutton Place		4/17/2017	6/16/2017 11:59 PM	Upcoming	Not Applied		
ESSEX CROSSING SITE 6		Manhattan	Lower East Side	More Info...	4/17/2017	6/16/2017 11:59 PM	Upcoming	Not Applied		
CAMBA Van Dyke		Brooklyn	Brownsville	More Info...	3/24/2017	5/24/2017 11:59 PM	Current	Not Applied		Apply
363 Bond Street Apartments		Brooklyn	Gowanus		3/21/2017	5/19/2017 11:59 PM	Current	Not Applied		Apply
ARTHUR CLINTON LP		Bronx	East Tremont		4/14/2017	5/5/2017 11:59 PM	Current	Not Applied		Apply
The Bedford		Bronx	Norwood		3/7/2017	5/5/2017 11:59 PM	Current	Not Applied		Apply
Essex Crossing Site 5		Manhattan	Lower East Side	More Info...	3/2/2017	5/2/2017 11:59 PM	Current	Not Applied		Apply
181 Front Street		Brooklyn	DUMBO		3/3/2017	5/1/2017 11:59 PM	Current	Not Applied		Apply
The Saint Marks APTS		Brooklyn	Bushwick		4/10/2017	5/1/2017 11:59 PM	Current	Not Applied		Apply
COMPASS RESIDENCES 2A		Bronx	Crotona Park East		2/24/2017	4/24/2017 11:59 PM	Current	Not Applied		Apply
PARK HOUSE		Bronx	East Tremont		2/15/2017	4/17/2017 11:59 PM	Current	Not Applied		Apply
MONTROSE PARK LLC		Brooklyn	East Williamsburg		3/27/2017	4/17/2017 11:59 PM	Current	Not Applied		Apply

Figure A.2. (Color online) Sample Information Page for a Development Allocated Through NYC Housing Connect

NYC Housing Connect

Affordable Housing for Rent

540 WEST 53RD STREET APARTMENTS



102 NEWLY CONSTRUCTED UNITS AT 540 WEST 53RD STREET, NEW YORK, NY 10019
CLINTON

Amenities: 12-hour attended lobby, on-site resident super, two residential terraces, children's splash pad, children's play room, fitness center, laundry room (card operated), adjacent public community garden.

Transit: Trains – A, B, C, D, E, 1 Buses – M11, M12, M20 M31, M50, M57, M104

No application fee • No broker's fee • Smoke-free building • More information: www.540w53.com

This building is being constructed through the Inclusionary Housing and Multifamily Rental Mixed Programs of the New York City Department of Housing Preservation and Development.

Who Should Apply?

Individuals or households who meet the income and household size requirements listed in the table below may apply. Qualified applicants will be required to meet additional selection criteria. Applicants who live in New York City receive a general preference for apartments.

- A percentage of units is set aside for applicants with disabilities:
  - Mobility (5%)
  - Vision/Hearing (2%).
- Preference for a percentage of units goes to:
  - Residents of **Manhattan Community Board 4** (50%)
  - Municipal employees (5%)

AVAILABLE UNITS AND INCOME REQUIREMENTS

Unit Size		Monthly Rent[1]	Units Available		Household Size[2]	Annual Household Income[3] *Minimum – Maximum[4]*		Monthly Rent[1]	Units Available		Household Size[2]	Annual Household Income[3] *Minimum – Maximum[4]*
Studio	80% AREA MEDIAN INCOME (AMI) UNITS	$1,091	2	→	1 person	$39,292 - $58,480	100% AREA MEDIAN INCOME (AMI) UNITS	$1,377	3	→	1 person	$49,098 - $73,100
1 bedroom		$1,375	9	→	1 person	$49,098 - $58,480		$1,733	9	→	1 person	$61,372 - $73,100
					2 people	$49,098 - $66,800					2 people	$61,372 - $83,500
2 bedroom		$1,660	10	→	2 people	$58,903 - $66,800		$2,089	13	→	2 people	$73,612 - $83,500
					3 people	$58,903 - $75,120					3 people	$73,612 - $93,900
					4 people	$58,903 - $83,440					4 people	$73,612 - $104,300
3 bedroom		$1,910	3	→	3 people	$68,023 - $75,120		$2,407	1	→	3 people	$85,063 - $93,900
					4 people	$68,023 - $83,440					4 people	$85,063 - $104,300
					5 people	$68,023 - $90,160					5 people	$85,063 - $112,700
					6 people	$68,023 - $96,800					6 people	$85,063 - $121,000
Unit Size		**Monthly Rent[1]**	**Units Available**		**Household Size[2]**	**Annual Household Income[3]** *Minimum – Maximum[4]*		**Monthly Rent[1]**	**Units Available**		**Household Size[2]**	**Annual Household Income[3]** *Minimum – Maximum[4]*
Studio	125% AREA MEDIAN INCOME (AMI) UNITS	$1,735	3	→	1 person	$61,372 - $91,375	165% AREA MEDIAN INCOME (AMI) UNITS	$2,165	6	→	1 person	$76,115 - $120,615
1 bedroom		$2,180	8	→	1 person	$76,698 - $91,375		$2,717	11	→	1 person	$95,109 - $120,615
					2 people	$76,698 - $104,375					2 people	$95,109 - $137,775
2 bedroom		$2,626	14	→	2 people	$92,023 - $104,375		$3,270	10	→	2 people	$114,103 - $137,775
					3 people	$92,023 - $117,375					3 people	$114,103 - $154,935
					4 people	$92,023 - $130,375					4 people	$114,103 - $172,095

[1] Tenant Responsible for electricity.
[2] Household size includes everyone who will live with you, including parents and children. Subject to occupancy criteria.
[3] Household earnings includes salary, hourly wages, tips, Social Security, child support, and other income. Income guidelines subject to change.
[4] Minimum income listed may not apply to applicants with Section 8 or other qualifying rental subsidies. Asset limits also apply.

allows prospective applicants to browse the list of upcoming lotteries and apply to as many as they want with a few clicks of a button (see Figure A.1). In 2018, there were 7,857 units offered through NYC Housing Connect (Satow 2019).[26]

To be eligible for a given unit, an applicant's household income must not exceed a certain fraction of New York's area median income. Furthermore, there is a minimum income that depends on the size of the unit, to make sure that the applicant can afford the reduced rent. (Because of this requirement, the system does not target the lowest-income individuals, which differentiates it from the government-owned public housing system administered by the NYC Housing Authority.) A certain proportion of units in each building is set aside for people with disabilities, for community board residents, and for municipal employees. Once the application deadline passes, the NYC Housing Connect platform generates a randomized log number for each application and gives these to the developer, who must process applications in the order of the log numbers subject to first filling all of the set-aside units with applicants from the corresponding groups.[27] Figure A.2 shows an information page for a particular development, specifying the income restrictions and the amount set aside for special groups. For more information on the NYC Housing Connect system, go to https://a806-housingconnect.nyc.gov/nyclottery/lottery.html#faq.

In the allocation system just outlined, an applicant's chance of being selected for a given unit depends on the location, quality, and number of units of the requested size at the development and whether the applicant is part of a special group (i.e., disabled, a community board resident, or a municipal employee). By contrast, our stylized model assumes that developments are of equal size and desirability and that all applicants are treated identically. Furthermore, working in the continuum eliminates stochasticity in aggregate demand. These simplifications allow us to focus on the first-order effect of generating the randomized log numbers independently across developments and to analytically describe the effects of alternative designs. Having a simplified model also allows us to pinpoint the driving forces behind our results and produce insights that generalize beyond the idiosyncrasies of the NYC system.[28]

Appendix B. Alternate Payout Models

In this appendix, we give two alternative ways to formulate our model in Section 3 that are mathematically equivalent. These alternative formulations enrich the interpretation of our results.

B.1. One-Time Payoffs

In the first formulation, payoffs are incurred on matching or exit, instead of in each period. This model is more natural for allocating Broadway tickets or other experience goods such as hiking, camping, and hunting permits. The modified timeline is as follows:

1. *Arrival and participation choice*. This occurs as in Section 3.
2. *Life event.* Every agent exits exogenously with probability $1-\delta$ and receives her outside option α_i.
3. *New development and matching*. This occurs as in Section 3.
4. *Payoff.* Every matched agent i exits with a one-time payoff of v_{ij}. The unmatched agents continue to the next period.

Agents seek to maximize their expected payout before matching or exiting, minus any participation costs. The updated Bellman equation is as follows:

$$V(s) = \max\left(0, \delta\mathbb{E}_{v\sim G}\left[\max_{a\in A_s}\left\{T_a(s,m)(v-\alpha) + \sum_{s'\in S} T_a(s,s')V(s')\right\}\right] - c\right). \tag{B.1}$$

The only change from the original Bellman equation (1) is that there is no longer a multiplicative factor of $\frac{1}{1-\delta}$ before the $(v-\alpha)$ term, which does not change the mathematical structure. Correspondingly, we remove the $(1-\delta)$ scaling term in an agent's expected utility, so $u^E(\alpha) = \int_{\underline{v}}^{\overline{v}}(v-\alpha)\cdot dP^E(\alpha, v) - ct^E(\alpha)$. The definitions for outcome, match distribution, value per match, and utilitarian welfare are unchanged. All the results are unchanged, except that the asymptotic results in which $\delta \to 1$ also requires scaling c so that $\frac{c}{1-\delta}$ is bounded.

B.2. Reward for Voluntary Exit

Instead of incurring a participation cost c for each period before exiting, agents get a one-time bonus[29] of $r := \frac{c}{1-\delta}$ for voluntarily exiting, and the outside options are all shifted downward by c.

To see that this formulation is equivalent, note that a function $V(s)$ satisfies the Bellman equation (1) if and only if the function $\tilde{V}(s) := V(s) + r$ satisfies

$$\tilde{V}(s) = \max\left(r, \delta\mathbb{E}_{v\sim G}\left[\max_{a\in A_s}\left\{T_a(s,m)\left(\frac{v-(\alpha-c)}{1-\delta}\right) + \sum_{s'\in S} T_a(s,s')\tilde{V}(s')\right\}\right]\right), \tag{B.2}$$

which is the Bellman equation for the formulation with exit reward r, no participation cost, and outside option α shifted down by c.

Appendix C. Formal Definition of Matching MDP

Given a matching rule $R = (S, D, A, T)$ and an outside option α, value distribution G, persistence δ, and participation cost c, a *matching MDP* $\Psi(R) = (S', A', T', \Gamma)$ is a Markov decision process with the following parameters:

- State space $S' = S \cup (S \times \mathbb{R}) \cup \{m, e\}$. The states $\{m, e\}$ are terminal states, corresponding, respectively, to matching and exiting without a match.
- For every state $s \in S$, $A_s = \{l, r\}$, where … to remaining. For every state $(s, v) \in S \times \mathbb{R}$, $A_{(s,v)'} = A_s$.
- Transition probability function $T' : S' \times A' \times S' \to \mathbb{R}$ and reward function $\Gamma : S' \times A' \to \mathbb{R}$ are as follows:
  - If the current state is $s \in S$, the action l results in transition to state e with no reward, and the action r results in transition with probability $1-\delta$ to e with reward $-c$, and transition with probability δ to (s, v) with reward $-c$, where v is a new draw from G.
  - If the current state is (s, v), the action $a \in A_s$ results in transition with probability $T_a(a, m)$ to state m with reward $\frac{v-\alpha}{1-\delta}$ and transitions with probability $T_a(s, s')$ to state $s' \in S$ and no reward.

A strategy σ to the preceding MDP is represented by functions $a : S \times \mathbb{R} \to A$ and $b : S \to \{0, 1\}$, where $a(s, v) \in A_s$ is what action to take in state (s, v), and $b(s)$ is whether to take action r in state s.

Every strategy $\sigma = (a, b)$ defines a Markov chain with state space $S'' = S \cup \{m, e_1, e_2\}$, where m, e_1, and e_2 are absorbing. The state e_1 corresponds to a voluntary exit, and e_2 to a forced exit owing to the life event. The transition probabilities are $p_{se_1} = 1 - b(s)$, $p_{se_2} = (1-\delta)b(s)$, and $p_{ss'} = \delta b(s) E_{v\sim G}[T_{a(s,v)}(s, s')]$ for all $s \in S$ and $s' \in S \cup \{m\}$. This is an absorbing Markov chain with a countable state space, in which the chance of transitioning to an absorbing state from any state is at least $1-\delta > 0$. Hence, given the initial state distribution D, the expected number of times a transition occurs between each pair of states $(s, s') \in S'' \times S''$ before the chain reaches an absorbing state is well defined. Denote this quantity by $\chi(s, s')$. The outcome function and waiting time function corresponding to this strategy for this agent type α are given by

$$P(\alpha, v) := \sum_{s: p_{sm} > 0} \frac{\chi(s, m)}{p_{sm}} E_{v'\sim G}[T_{a(s,v')}(s, m)\mathbb{1}(v' \le v)], \tag{C.1}$$

$$t(\alpha) := \sum_{s\in S} \sum_{s'\in S\cup\{m, e_2\}} \chi(s, s'). \tag{C.2}$$

A strategy (a, b) is optimal if and only if

$$a(s,v) \in \arg\max_{a \in A_s}\{Q(s,v,a)\}$$
$$\text{and } b(s) \in \arg\max(0, \delta\mathbb{E}_{v\sim G}[Q(s,v,a(s,v))] - c),$$

where $Q(s,v,a) = T_a(s,m)\left(\frac{v-\alpha}{1-\delta}\right) + \sum_{s'\in S} T_a(s,s')V(s')$ is the term inside the max of the Bellman equation (1), and $V(s)$ is the unique solution to (1).

The claim that the Bellman equation (1) has a unique solution $V(s)$ whenever $\delta < 1$ follows from proposition 1.6.1 of Bertsekas (2012). The claim that $V(s)$ represents the maximum attainable continuation value in state s from any strategy follows from proposition 1.6.2 of Bertsekas (2012). The conditions needed to apply these two propositions are verified in Lemma C.1.

Definition C.1. Let $R(S)$ denote the set of functions $S \to \mathbb{R}$, and let $B(S) \subseteq R(S)$ denote the set of functions with a finite sup-norm. A mapping $\Pi : R(S) \to R(S)$ is monotonic if for any two functions $J, J' \in R(S)$, $J \le J'$ implies that $\Pi J \le \Pi J'$, where comparisons of functions are defined pointwise. It is a contraction with modulus ρ if $J \in B(S)$ implies $\Pi J \in B(S)$, and for any $J, J' \in B(S)$, $\|\Pi J - \Pi J'\| \le \rho\|J - J'\|$, where $\|\cdot\|$ is the sup-norm, with $\|J\| := \sup_{s\in S} J(s)$.

Lemma C.1. *Building on the notation of* Definition C.1, *define the mapping* $\Pi : R(S) \to R(S)$, *where* $(\Pi J)(s)$ *is given by the right-hand side of the Bellman* equation (1) *with* V *replaced by* J. *For a given strategy* $\sigma = (a, b)$, *define the mapping* $\Pi_\sigma : R(S) \to R(S)$, *where*

$$(\Pi_\sigma J)(s) = b(s)\left(\delta\mathbb{E}_{v\sim G}\left[T_{a(s,v)}(s,m)\left(\frac{v-\alpha}{1-\delta}\right) + \sum_{s'\in S} T_{a(s,v)}(s,s')J(s')\right] - c\right). \tag{C.3}$$

Both Π *and* Π_σ *are monotonic contraction mappings with modulus* $\delta < 1$.

Proof of Lemma C.1. The monotonicity of Π_σ follows from its linearity, and that of Π follows from $\Pi(s) = \sup_\sigma \Pi_\sigma(s)$. Define $H(x) = \mathbb{E}_{v\sim G}[\max(v - x, 0)]$. For any $J \in B(S)$,

$$\|\Pi_\sigma J\| \le \|\Pi J\| \le \delta\mathbb{E}_{v\sim G}\left[\max\left(\|J\|, \frac{v-\alpha}{1-\delta}\right)\right]$$
$$\le \delta(\|J\| + H(\alpha + (1-\delta)\|J\|)) < \infty.$$

Finally, we have $\|\Pi_\sigma J - \Pi_\sigma J'\| \le \delta\|J - J'\|$ because $\sum_s T_{a(s,v)}(s,s') \le 1$, and $\|\Pi J - \Pi J'\| \le \sup_\sigma \|\Pi_\sigma J - \Pi_\sigma J'\| \le \delta\|J - J'\|$. □

Endnotes

[1] These objectives appear to be fairly universal. For example, a policy document released by the UK government lists "[s]upport for those in greatest housing need" as an "outcome which allocation policies must achieve" (p. 10) and lists "Greater choice and wider options" as an "outcome which the Government believes allocation policies should achieve" (p. 13; see http://webarchive.nationalarchives.gov.uk/20120919214909/http://www.communities.gov.uk/documents/housing/pdf/1403131.pdf).

[2] As in NYC, a new lottery is used in Toronto to allocate affordable units in each new development (Pelley 2018). In Providence, public housing is allocated using a waitlist in which people who reject an offer lose their position on the list (Providence Housing Authority 2018). Minneapolis uses a similar system but waits until a second rejection before removing an applicant (Minneapolis Public Housing Authority 2017).

[3] This system is used to allocate public housing in Amsterdam (Van Ommeren and Van der Vlist 2016).

[4] This resembles the approach commonly used to allocate discounted tickets to popular sports events or shows, where agents engage in a costly competition by physically waiting in long queues.

[5] If income Y is log-normally distributed, then parameterizing outside options using $-a/Y^b$ for any positive constants a and b results in outside options which are increasing in income but with a heavy left tail.

[6] Life events can be thought of as capturing scenarios such as marrying and moving in with a partner, receiving a big promotion and moving to a nicer apartment close to work, or relocating to another city to take care of an elderly family member. Mathematically, the presence of these life events ensures that the system is stable; that is, the number of unmatched agents does not grow indefinitely. All of our results continue to apply in a model where the probability of a life event in a given period depends on whether the agent is currently matched. For simplicity, we do not model the reallocation of units that open up when previously matched agents leave because of their life events.

[7] The assumption that indifferent agents will participate is not substantial because the set of such agents has measure zero in our model. This assumption allows us to rule out mixed strategies for these agents, simplifying the characterization of equilibrium outcomes (see Proposition 2 in Section D.1 and Proposition 5 in Section D.2 in the online appendix).

[8] The assumptions of continuity and positive density for F and G are not necessary for any of our results in the main body. They are included only to simplify the notation in the proofs and remove uninteresting technicalities, such as allowing mixed strategies for agents who are exactly indifferent to ensure that the market clears exactly.

[9] For example, her state may be the number of periods that she has waited. If she has just arrived, she can only continue to wait, whereas if she has waited for a long time, she may be offered the current development and asked to accept or reject the offer. Alternately, her state may represent the number of lotteries that she has entered so far or her priority as determined by a common lottery.

[10] Although our formal definition does not allow for the action set A_s to be randomized, it is straightforward to encode an equivalent matching rule with a deterministic action set: in state $s = \lfloor\tau\rfloor$, the agent is always offered the actions $\{Accept, Reject\}$. With probability $\tau - \lfloor\tau\rfloor$, the agent's action is ignored and her state incremented, and otherwise she transitions as defined earlier. We choose the description with a randomized action set for its conceptual clarity.

[11] See Appendix C for more details about how to compute the outcome induced by the matching rule R, strategy profile Σ, and market primitives.

[12] Specifically, it holds that $t^E(\alpha) = \frac{1}{c(1-\delta)}\left(\int_\alpha^\infty P^E(x,\overline{v})\,dF(x) - \int_{\underline{v}}^{\overline{v}}(v-\alpha)\,dP^E(\alpha,v)\right)$.

[13] Proposition 2 in Section D.1 of the online appendix shows that this outcome is unique.

[14] In ruling out the case $\overline{\pi}^E < \mu$, we eliminate mechanisms that intentionally withhold supply. Nevertheless, one can study the effect of withholding supply by performing comparative statics with respect to μ.

[15] This is analogous to the definition of a Walrasian equilibrium. A price vector can arise in Walrasian equilibrium if when agents respond optimally, the market clears. In our model, a matching rule $R \in M$ can arise in equilibrium if, when agents respond optimally, the average match rate equals μ.

[16] In both mechanisms, the number of offers received by an agent who decides to reject all offers follows a geometric distribution on $\{0, 1, 2, \ldots\}$ with a mean of $\frac{q}{1-q}$.

[17] One crucial but subtle point is that agents in a ticket-saving lottery are never worse off than they are upon entry, and therefore, any agent who chooses to participate will not quit even if told that she does not hold a winning ticket.

[18] Definition 5 does not mention waiting times because, for any equilibrium outcome E, the waiting time function t^E is determined by the outcome function P^E by Proposition 1 in Section 4.3. Convergence of outcome function as $\delta \to 1$ does not imply convergence of the waiting time function but does imply convergence of the scaled participation cost, that is, $(1-\delta_n)c|t^{E_n}(\alpha) - t^E(\alpha)| \to 0$. This is sufficient to imply that utilities, the value per match, and the match distribution all converge pointwise: $u^{E_n}(\alpha) \to u^E(\alpha)$, $v^{E_n}(\alpha) \to v^E(\alpha)$, and $F^{E_n}(\alpha) \to F^E(\alpha)$ for all α.

[19] Typically, the tail of a distribution refers to the right tail. Definition 7 refers to the left tail because the agents with the highest need for being matched are those with the worst outside options.

[20] In particular, condition (b) of Theorem 5 is that the value of matching any agent well is greater than the difference in need between any two agents.

[21] The findings in Figure 1 are in alignment with our interpretation, despite the fact that the example violates the conditions of Theorems 3, 4 and 5: G follows a normal distribution, for which $\overline{v} = \infty$, and F follows a (negated) Weibull distribution, for which the density is not increasing whenever $\log(k) > 0$.

[22] Our model assumes that the participation cost c is identical for all agents and therefore that the agents who participate will be those with the greatest need (worst outside options). In practice, an important caveat when adding participation costs is that the designer should ensure that the type of cost added does not disproportionately impact high-need agents. For example, if wealthier applicants are systematically more adept at filling out forms or more able to take an afternoon off of work, then adding paperwork to the application process or mandating physical presence may have the opposite of the desired effect. Alatas et al. (2016) and Deshpande and Li (2017) explore this concern using (quasi-)random experiments to empirically estimate the effects of certain types of friction on various subpopulations. We discuss this point further in Section 5.1.

[23] This waiting time ensures that agents with outside option $\alpha = F^{-1}(\mu)$ are indifferent about whether to participate.

[24] In particular, Theorems 3, 4, and 5 also apply in settings where agents have information about future developments, where the designer delays allocation of some units, or where the designer observes agent values and uses this information to determine the allocation.

[25] Most units are financed under legislation 421-a, under which at least 20% of units must be certified to be affordable by the NYC HPD. Since 2017, the proportion requirement increased to 25%–30% of units. A lesser used program is called Article IX, under which at least two-thirds of units must be affordable. Go to https://www1.nyc.gov/site/hpd/developers/tax-incentives.page for a listing of all HPD tax incentive programs.

[26] For a map of developments currently accepting applications, go to http://hpd.maps.arcgis.com/apps/webappviewer/index.html?id=b06672e7899b40d0b0f8e1c81f9a5f58.

[27] A detailed description of the required process developers follow in allocating units is available at https://www1.nyc.gov/assets/hpd/downloads/pdf/developers/marketing-handbook.pdf.

[28] To properly account for the aforementioned complications of the NYC system, one would need to apply a different research methodology based on empirical estimation and counterfactual simulations. This would yield more precise estimates for NYC, but the findings would be particular to the data used to estimate the model's parameters, and the key factors driving any conclusions would be more opaque.

[29] Instead of a one-time bonus, agents who voluntarily exit can equivalently receive a subsidy of c per period until their life event.

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