Econometric Society Dynamic Structural Econometrics Summer School: Market Design MIT, 2022

Structural Econometrics of Auctions

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Readings

Most of this lecture is based on the following:

> *Gentry, Matthew L., Timothy P. Hubbard, Denis Nekipelov, and Harry J . Paarsch, "Structural Econometrics of Auctions: A Review,"* Foundations and Trends® in Econometrics, *9 (2018), 79–302.*

Code

The appendices of the following book contain some useful MATLAB code that can be modified to implement what is suggested in the lecture:

> *Paarsch, Harry J. and Han Hong.* An Introduction to the Structural Econometrics of Auction Data. *Cambridge, Massachusetts: MIT Press, 2006. ISBN: 9780262162357*

Implementing Numerical Methods

For those of you with little (or no) experience implementing numerical methods on a computer, the following book may be helpful:

> *Paarsch, Harry J. and Konstantin Golyaev.* A Gentle Introduction to to Effective Computing in Quantitative Research: What Every Research Assistant Should Know. *Cambridge, Massachusetts: MIT Press, 2016. ISBN: 9780262034111*

Historical Background

- One of the great success stories in economics during the latter half of the twentieth century has been the systematic theoretical investigation of incomplete information in various economic environments—for example, moral hazard in insurance markets and adverse selection in such market institutions as auctions to name just two.
- Developed in tandem with the incredible advances in game theory since the Second World War, an important by-product of this research program was the formulation of many incomplete information problems as the design of optimal mechanisms—subsequently, the subfield of market design.

From the Econometrician's Perspective

- Theoretical models of incomplete information are particularly compelling because they provide richer explanations of the observed heterogeneity in data than ones relying on measurement error or productivity shocks.
- For example, when individual agents use private information strategically to make decisions, the equilibrium interactions of those decisions with the decisions of others, can generate new explanations of observed phenomena, particularly in the field of industrial organization.

Classic, Canonical Problem in Economic Theory

- A seller of an object is faced by several potential buyers.
- Although the seller probably knows the object's value to himself, he typically has little or no information concerning what value any one of the potential buyers places on the object.
- Similarly, each potential buyer may have a good notion of the object's value to himself, but probably knows little about the seller's valuation or the valuations of the other potential buyers.
- In short, the seller as well as the potential buyers face incomplete, asymmetric information.

How Should the Seller Dispose of the Object?

- One commonly-used, but relatively new, method of sale involves announcing a take-it-or-leave-it price and then selling the object to the first who accepts that price.
- Another involves the seller's engaging in pair-wise negotiations with individual potential buyers, either sequentially or simultaneously.
- Yet a third involves selling the object at auction.
- In short, a set of different selling mechanisms exists, from which the seller must choose, guided by some objective—an auction being one of those choices.
- The choice of mechanism by the seller typically depends on many factors, for example, the objective of the seller and transaction costs to name just two.

George Washington as the Executor of a Will

In his book *An Imperfect God: George Washington, His Slaves, and the Creation of America*, Henry Wiencek recounted an anecdote about George Washington that is helpful in understanding comparative institutional analysis—market design.

Basic Story

- In the mid-eighteenth century, before the Revolutionary War, George Washington was made the executor of the estate of a friend. Like many upper-class Virginians at the time, the friend was land rich, but cash poor.
- In fact, the friend died with debts that could only be met by selling his livestock, which at the time, in Virginia, included his slaves.
- The standard practice to sell slaves from estates was at auction. Washington chose an alternative mechanism—a raffle—and it raised considerable funds for the friend's heirs.

Why would a raffle do better than an auction?

- In eighteenth century Virginia, distance was a tyrant and information was scant.
- Thus, the participation rates at auctions were low, even those that were well advertised.
- Auctions, particularly multiunit or multiobject auctions, are potentially susceptible to collusion.
- Also, as mentioned above, most Virginian were cash poor and could not typically afford to buy high-priced assets.

Attractive Properties of a Raffle

- Now, a raffle ticket, which was reasonably priced, was another matter.
- Most Virginian, even poor ones, could afford a small wager—a flutter.
- In fact, many purchased one in Washington's case.
- Thus, the change in the participation rate between the auction and the raffle raised more money for the estate than perhaps an auction would have.
- It is, of course, difficult to say for sure, today, as the counter-factual did not actually happen.

How Can One Construct the Counter-Factual?

- Using the theory of mechanism design, market design.
- In that theory, there are two types of agents: the principals (for example, the sellers at auctions) and the agents (for example, the potential buyers at auctions).
- The incentives of the principals and the agents are typically not aligned: the seller wants to get the highest price, while the potential buyers want to pay the least.
- In the theory of mechanism (market) design, the principal seeks to choose the rules of the game to maximize some objective, knowing full well that the agents will try to thwart his efforts and do the best for themselves.

Early Example of Market Design

- Milton Friedman was perhaps the first economist to discuss market design seriously in his book *A Program for Monetary Stability*, which was published in 1960.
- In that book, Friedman focused on how best to sell treasury bills at sealed-tender auctions.
- Specifically, whether the pay-your-bid or highest-loosing-bid pricing rule should be used.
- In response to that proposal, William S. Vickrey investigated different auction formats and pricing rules within a particular informational paradigm—independent private-values.

Features of Auctions

- Auctions are ubiquitous and ancient, their durability suggesting that the market institution serves an important allocational role.
- Perhaps the most important feature defining environments within which auctions are used is the existence of incomplete information of some sort—the asymmetry of information between the seller and the potential buyers.
- Because the potential buyers have no incentive to tell the seller or any of their competitors anything about those valuations, the auction plays an important allocative role by inducing the potential buyers to reveal to the seller information concerning their valuations of the object.

Two Basic Roles of Auctions

1. Allocating the good for sale.
2. Discovering the price of a good.

How Do Valuations Obtain?

- The way potential buyers form their valuations remains an open question in economics.
- Suffice it to say, however, when economic theorists come to modelling the asymmetry in information as well as the heterogeneity in valuations across agents, they employ random variables.
- Typically, each potential buyer is assumed to demand at most one unit of the object in question.
- In the simplest model, for each potential buyer, the marginal utility of the one unit is assumed an independent realization of a continuous random variable.
- By and large, the budget constraint and issues of substitution are ignored.

Workhorse Model due to Vickrey [1961]

- Each of a known number N of potential buyers draws an individual-specific random valuation independently from the same differentiable CDF $F_V(v)$ that has corresponding PDF $f_V(v) = \mathrm{d}F_V(v)/\mathrm{d}v$.
- Although Vickrey assumed that V was distributed uniformly on the unit interval, that is unnecessary. In short, the specific value of his draw is that potential buyer's private information; it represents the monetary value of the object to him.
- Economic theorists refer to this as the symmetric independent private-values (IPV) model because the draws are independent and the valuations are bidder specific; because the valuations are drawn from the same law, the bidders are *ex ante* symmetric.

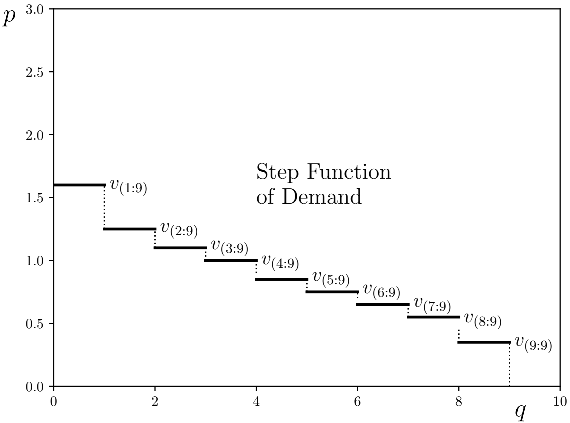
Interpreting the Valuation Draws

- Order the N realizations $\{v_n\}_{n=1}^{N}$ from largest to smallest:

$$v_{(1:N)} \geq v_{(2:N)} \geq \cdots \geq v_{(n:N)} \geq \cdots \geq v_{(N-1:N)} \geq v_{(N:N)}.$$

- Adopting the economists' convention of writing price p on the ordinate, and demand on the abscissa, plot a representative sequence when $N = 9$.

Step Demand Function

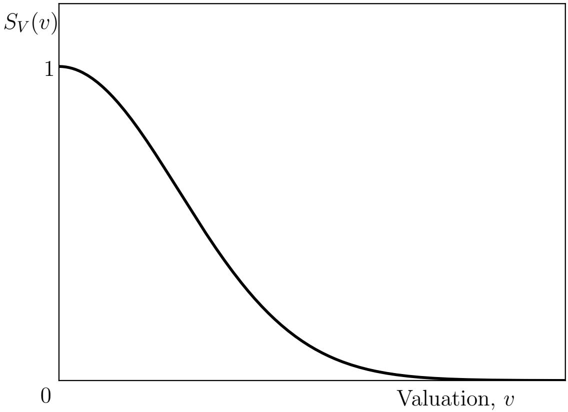


Interpreting this in terms of $F_V(v)$

- A function related to $F_V(v)$ is the survivor function $S_V(v)$.
- Where the CDF is the proportion of the population less than or equal to some value, the survivor function is the proportion of the population greater than that value.
- In symbols,

$$S_V(v) = \Pr(V > v) = 1 - \Pr(V \leq v) = [1 - F_V(v)].$$

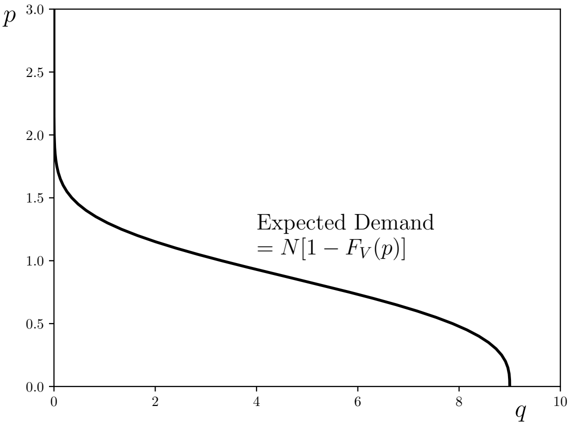
Survivor Function $S_V(v)$



Constructing Expected Demand

- Imagine a population of N potential buyers (consumers), each of whom has some valuation draw from $F_V(v)$.
- What is the equivalent of the demand function?
- For any price p, the fraction of the population who would purchase at that price is $S_V(p) = [1 - F_V(p)]$.
- Thus, the expected demand is $NS_V(p) = N[1 - F_V(p)]$.

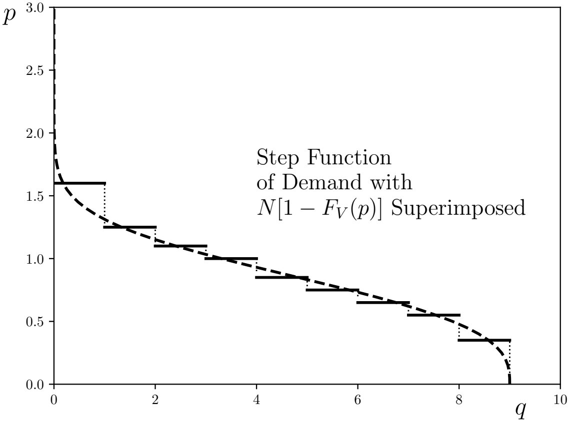
Expected Demand $NS_V(v)$



Using the Sample Analog

- From the realized valuations, were they observed, one can create the sample analog of $S_V(v)$, the empirical survivor function $\hat{S}_V(v)$—for instance, using the Kaplan–Meier product-limit estimator.
- An estimate of expected demand, based on the estimated survivor function, generated from a sample of size $N = 9$, along with the population expected-demand function when $N = 9$, is depicted below.
- In short, sample data, when available, can provide enough information to estimate expected demand.

Estimated Expected Demand: Estimated Survivor Function $S_V(v)$



Auction Formats and Pricing Rules

- Different auction formats (open-outcry versus sealed-bid) and different pricing rules (pay-your-bid versus second-price) provide potential buyers with different incentives concerning how to bid.
- For example, under the pay-your-bid rule, a bidder's action (his bid) determines what he pays should he win, whereas under the second-price rule, the action (bid) of his nearest rival determines what the winner pays.

Implications of Different Formats and Rules

- In equilibrium, different functions map the private information of potential buyers (their values) into their actions (their bids).
- For example, open-outcry (oral) auctions can be conducted in two different ways: First, the price is set very low, perhaps at zero, and then allowed to rise more or less continuously until only one participant remains active in the auction.
- That remaining active bidder is the winner, and he pays what the last other active bidder was willing to pay—sometimes plus a small increment.
- In short, the observed bids at auctions are not necessarily the realized valuations that one would like to use to estimate the survivor function—expected demand.

Some Technical Points

- As a technical aside, within the IPV model, the equilibrium at an oral, ascending-price auction (sometimes referred to as an English auction) has a special structure.
- It is a weakly dominant-strategy equilibrium: each participant has an incentive to reveal his private information, that is, to tell the truth concerning his value by continuing to bid up to his value, regardless of what his rivals do.

Modelling English Auctions

- Economic theorists have typically chosen to model oral auctions as clocks, where the price rises continuously with the movement of a clock hand.
- In this case, the winner of the auction is the participant with the highest valuation and he pays what his nearest rival (that participant with the second highest value) was willing to pay.
- Thus, the oral, ascending-price auction guarantees the efficient allocation of the object: the participant with highest valuation wins the auction.
- Such an auction is sometimes referred to as a second-price auction because in the absence of bid increments the winning bid is the second-highest bid, which happens to be the second-highest valuation as well.

Important Interpretation in Terms of an Economic Concept

- In economics, this outcome has special meaning: the second-highest valuation represents the opportunity cost of the object for sale—its value in the next-best alternative.
- Thus, one can see why economists are naturally attracted to mechanisms that have this property.

Dutch Auctions

- The second way to conduct an oral auction involves initially setting the price very high, and then allowing it to fall continuously; the winner is the first participant to cry out a bid, and he pays his bid.
- In practice, these oral auctions are typically implemented using a clock, where the hand (or a digital panel) lists the current price.
- Participants affirm their willingness to pay the current price by pushing a button which stops the clock at that price.
- Such auctions are often referred to as Dutch auctions, perhaps because they are frequently used in the Netherlands to sell fish and flowers.

Importance of the Pricing Rule

- Because the price that the winner pays is related to his action (crying out or pushing the button), he has an incentive to shave his bid—to wait longer before pushing the button to stop the clock.
- In theoretical models of Dutch auctions within the symmetric IPV model, the equilibrium is not a dominant-strategy equilibrium, but rather a Bayes–Nash equilibrium, which is a much stronger notion of equilibrium.
- Although the Bayes–Nash equilibrium bid function is an increasing function of a bidder's value, it has a slope that is less than one: each bidder is deceptive when bidding; he does not tell the truth, but rather bids less than his value.
- Again, however, the winner is the participant with the highest valuation, so objects are allocated efficiently at Dutch auctions.

Symmetric IPV Model

- In general, economic theorists have found that the seller's expected revenue depends first and foremost on the information available to potential buyers and then on the auction format and pricing rules employed, the amount of competition, and the attitudes of potential buyers toward risk.
- Within the symmetric IPV model, however, under the assumption that potential buyers are risk neutral with respect to winning the object for sale, a remarkable result obtains: revenue equivalence (in expectation).
- That is, if the same object were sold under the two different institutions, then the average winning bid at the English auction would equal the average winning bid at the Dutch auction.

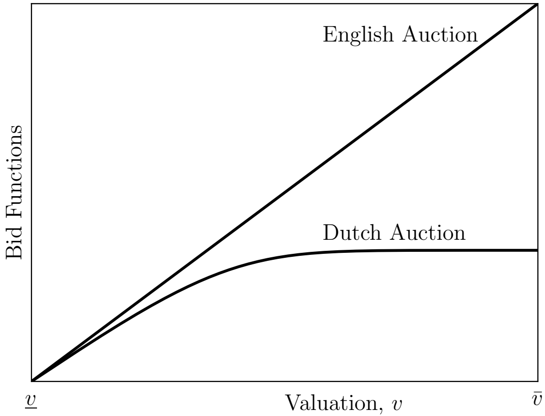
Why Revenue Equivalence?

- To most people, revenue equivalence is at first somewhat surprising because at English auctions considerable information is revealed during the course of bidding, whereas at Dutch auctions no information is revealed until the winner has been determined.
- Within the IPV model, information plays no extra role in determining the average winning price since each bidder's private information (his value) is, by assumption, statistically independent of the private information of his rivals (their values): knowing something about the values of his rivals provides no extra information to a bidder concerning his own valuation, or likelihood of winning the auction.

Implications

- No bidder at an English auction can learn anything more about his valuation from the actions (bids) of his rivals.
- In other words, learning cannot really obtain with the IPV model.
- Once one realizes this fact, the equivalence of average winning bids is clear: at a Dutch auction, assuming he wins because he has the highest value, a representative participant forms his bid so that he will, on average, just beat his nearest rival, the bidder with the second-highest valuation.

First- and Second-Price Equilibrium Bid Functions



What about Sealed Formats?

- Similar analyses have been performed for the sealed format under different pricing rules.
- In fact, theorists have shown that sealed auctions at which the highest bidder wins the auction and pays what he bid are strategically equivalent to Dutch auctions.
- Consequently, the Bayes–Nash equilibrium bid function at a sealed, pay-your-bid, auction is identical to that at a Dutch auction.
- Also, sealed auctions where the highest bidder wins, but pays the bid of his closest rival, are strategically equivalent to English auctions, so it is a dominant strategy at these auctions for bids to tell the truth, too.

Revenue Equivalence Proposition

- In its full generality, the REP states that any combination of auction format and pricing rule that has the same probability of assigning a winning bidder generates the same expected revenue to the seller.

What is the Best Selling Mechanism?

- From a policymaker's perspective, an important issue involves choosing the selling mechanism that obtains the most revenue for the seller, on average.
- To a large extent, the structure of the optimal selling mechanism depends on the informational environment.
- Within the symmetric IPV model, given the REP, one question arises naturally: Can one still improve on the structure of the four combinations of auction format and pricing rule?

Optimal Reserve Price

- Roger B. Myerson [1981] as well as John G. Riley and William F. Samuelson [1981] showed that devising a selling mechanism that maximizes the seller's expected gain involves choosing a reserve price r, the minimum price that must be bid, optimally, where the optimal reserve price r^* solves the following equation:

$$r^* = v_0 + \frac{[1 - F_V(r^*)]}{f_V(r^*)},$$

where v_0 denotes the seller's valuation of the object at auction.

Some Implications of r^*

- The presence of a binding reserve price means that some fraction of the time the object at auction goes unsold.
- In other words, an inefficiency is introduced because $r^* > v_0$, meaning some bidders might value the object more than the seller, but the object goes unsold.
- This inefficiency highlights the tension between expected revenue maximization on the part of the seller in particular and allocative efficiency in the economy in general.

Practical Value of Mechanism Design

- Historically, the literature concerned with mechanism design was sometimes criticized as lacking practical value because the optimal selling mechanism (in this case, the optimal reserve price r^*) typically depends on a primitive like $F_V(\cdot)$, the distribution of the valuations, which is often unknown to the designer.
- In the past, because the distribution of valuations has been unknown, calculating the optimal reserve price, the optimal selling mechanism, for a real-world auction seemed impossible.

From a Structural Econometrician's Perspective

- Auctions are particularly attractive because the rules of an auction govern how the potential buyers must behave during the selling process—specifically, how bids must be tendered, who wins the auction, what the winner pays, and so forth.
- These rules place incredible structure on the data generating process, unlike in some other economic applications.

Value of Structural Approach

In particular, under certain conditions, the twin hypotheses of optimization and equilibrium allow an econometrician to accomplish several things:

1. Identify the unobserved distribution of valuations from the observed distribution bids. In other words, part of the structural econometric approach to auctions is an identification strategy.
2. Reverse-engineer an estimate of the distribution of latent types (for example, valuations) from the observed distribution of actions (the bids).
3. Conduct comparative institutional design: use the estimate of the distribution of latent valuations to improve on auction design.

Why is this important?

- For example, at auctions within the IPV model, the equilibrium bidding strategies of potential buyers are increasing functions of their valuations.
- At English auctions, under the clock model, for instance, the dominant strategy of bidders who lose the auction is to bid their valuations.
- Thus, in principle, it is possible to estimate the underlying probability law of valuations using the empirical distribution of bids from a cross-section of auctions.
- Because a researcher can recover the primitives of the economic model, the Lucas critique is circumvented; the researcher can then also entertain comparative institutional design, for instance, comparing outcomes under alternative market institutions not observed in the data.

Some Problems Still Encountered

- First, at English auctions, as shown later, the winning bid does not reveal complete information concerning the winner's actual valuation of the object for sale.
- Next, in the presence of a binding reserve price, the empirical distribution of observed bids represents a truncated sample of data: only those potential buyers whose valuations exceeded the reserve price chose to bid.
- Finally, in the presence of a binding reserve price, the joint distribution of bidding and nonparticipation depends on the number of potential buyers, but finding a measure of potential competition is often impossible; when it can be done, the specific proxy is often inaccurate.

Equilibrium Bid Functions within IPV Model

- Within the IPV model, under risk neutrality, with known N bidders, the dominant-strategy, equilibrium-bid function at English and Vickrey auctions is the following:

$$\beta(v) = \begin{cases} v & \text{if } v \geq r; \\ 0 & \text{otherwise.} \end{cases}$$

- The Bayes–Nash equilibrium-bid function at Dutch and first-price, sealed-bid auctions is the following:

$$\sigma(v) = \begin{cases} v - \frac{\int_r^v F_V(u)^{N-1}\ \mathrm{d}u}{F_V(v)^{N-1}} & \text{if } v \geq r; \\ 0 & \text{otherwise.} \end{cases}$$

Transformations of Random Variables

Three important transformations:

1. indicators, like $\mathbf{1}(V \geq r)$;
2. monotonic ones, like $\sigma(v)$;
3. order statistics, like $Z = \max(V_1, V_2, \ldots, V_N)$.

Indicator Transformation of a Random Variable

- For a sequence of N independent and identically-distributed random variables $\{V_n\}_{n=1}^{N}$, consider the following indicator transformation:

$$P_n = \begin{cases} 1 & \text{if } V \geq r; \\ 0 & \text{otherwise.} \end{cases}$$

- In this case, $\{P_n\}_{n=1}^{N}$ is a sequence of independent and identically-distributed Bernoulli random variables having mean $[1 - F_V(r)]$ and variance $F_V(r)[1 - F_V(r)]$,

Also, ...

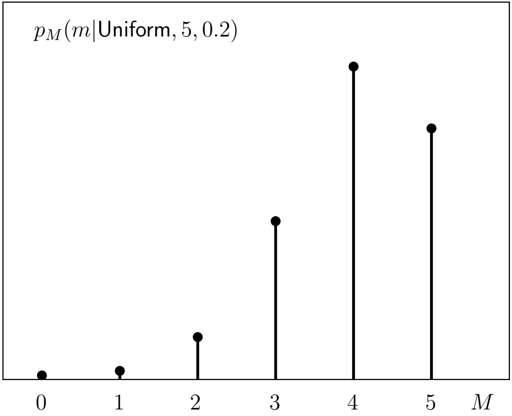
- The sum

$$M = \sum_{n=1}^{N} P_n,$$

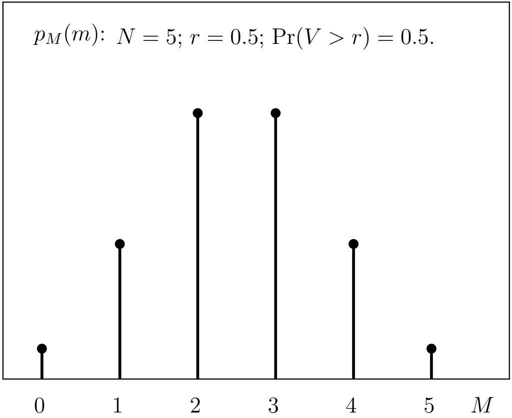
which represents the number of participants at the auction, follows a binomial distribution having mean $N[1 - F_V(r)]$ and variance $NF_V(r)[1 - F_V(r)]$.

- Most important, the number of participants M is an endogenous random variable, having pmf $p_M(m|F_V, N, r)$.
- In short, using the observed number of participants in a linear regression violates one of the maintained orthogonality assumptions of least-squares estimation.

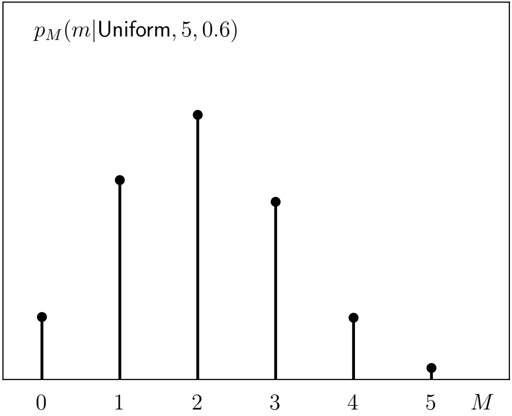
PMF of M



Another PMF of M



Yet Another PMF of M



Monotonic Transformation of Random Variable

- Under a monotonic transformation, $S = \sigma(V)$

$$\mathsf{B} = f_S(s)\ \mathrm{d}s = f_V(v)\ \mathrm{d}v = \mathsf{A}.$$

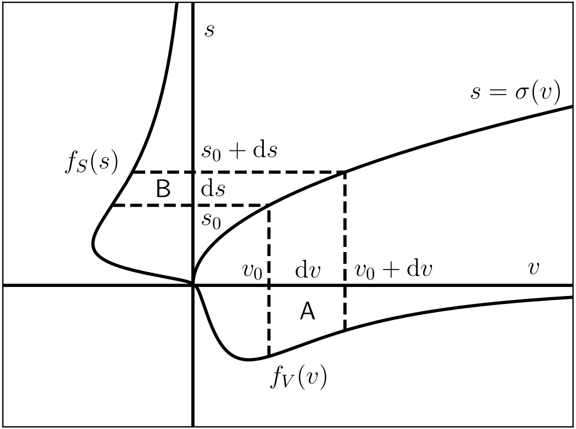
- When $\sigma(v)$ is monotonic, there exists a unique function $\sigma^{-1}(\cdot)$ such that

$$\sigma^{-1}(s) = \sigma^{-1}\left[\sigma(v)\right] = v,$$

where

$$\frac{\mathrm{d}\sigma^{-1}(s)}{\mathrm{d}s} = \left[\frac{\mathrm{d}\sigma(v)}{\mathrm{d}v}\right]^{-1} = \frac{\mathrm{d}v}{\mathrm{d}s}.$$

Monotonic Transformation of Random Variable



Deriving the CDF of S

- When V is defined on the interval $[\underline{v}, \bar{v}]$, S is defined on the interval $[\underline{v}, \sigma(\bar{v})]$, where

$$\bar{s} \equiv \sigma(\bar{v}) = \bar{v} - \frac{\int_{\underline{v}}^{\bar{v}} F_V(u)^{N-1} \, \mathrm{d}u}{F_V(\bar{v})^{N-1}} = \bar{v} - \int_{\underline{v}}^{\bar{v}} F_V(u)^{N-1} \, \mathrm{d}u < \bar{v}.$$

- That noted, the CDF of S, which is the probability of V's being less than some value v, is

$$\begin{aligned} F_S(s) &= \Pr(S \leq s) \\ &= \Pr[\sigma(V) \leq s] \\ &= \Pr[V \leq \sigma^{-1}(s)] \\ &= F_V\left[\sigma^{-1}(s)\right]. \end{aligned}$$

Deriving the PDF of S

- Differentiating $F_S(s)$ or $F_V\left[\sigma^{-1}(s)\right]$ with respect to s yields the PDF of S

$$
\begin{aligned}
f_S(s) &= \frac{\mathrm{d}F_S(s)}{\mathrm{d}s} \\
&= \frac{\mathrm{d}}{\mathrm{d}s} F_V\left[\sigma^{-1}(s)\right] \\
&= f_V\left[\sigma^{-1}(s)\right] \frac{\mathrm{d}\sigma^{-1}(s)}{\mathrm{d}s} \\
&= f_V\left[\sigma^{-1}(s)\right] \frac{\mathrm{d}v}{\mathrm{d}s}.
\end{aligned}
$$

Implications

- Because $\sigma(\cdot)$ is a monotonic function,

$$f_S(s|F_V, N) = \frac{f_V\left[\sigma^{-1}(s)\right]}{\sigma'\left[\sigma^{-1}(s)\right]} \quad s \in [\underline{v}, \bar{s}(N, F_V)],$$

  where I have stressed the dependence of $f_S(\cdot)$ on both $F_V(\cdot)$ and N by conditioning on them.
- Note, too, I have also stressed that $\bar{s}$, the support of S, depends on the number of potential buyers N, not the number of actual bidders M, which can be important when a binding reserve price r exists, as well as $F_V(\cdot)$.
- Because $\bar{s}$ depends on $F_V(\cdot)$, one of the regularity conditions typically assumed when defining the maximum-likelihood estimator is violated.

Brute-Force Calculation Yields ...

- The derivative of the equilibrium bid function can be found by brute-force calculation

$$
\begin{aligned}
\sigma'(v) &= 1 - \frac{F_V(v)^{N-1} F_V(v)^{N-1} - (N-1) F_V(v)^{N-2} f_V(v) \int_{\underline{v}}^{v} F_V(u)^{N-1} \, \mathrm{d}u}{[F_V(v)^{N-1}]^2} \\
&= \frac{(N-1) f_V(v) \int_{\underline{v}}^{v} F_V(u)^{N-1} \, \mathrm{d}u}{F_V(v)^N} \\
&= [v - \sigma(v)] \frac{(N-1) f_V(v)}{F_V(v)}.
\end{aligned}
$$

Implication

- Direct substitution yields,

$$f_S(s|N, F_V) = \frac{F_V\left[\sigma^{-1}(s)\right]^N}{(N-1)\int_{\underline{v}}^{\sigma^{-1}(s)} F_V(u)^{N-1}\,\mathrm{d}u},$$

  whence one can derive in a straightforward manner the joint density of all the bids $\boldsymbol{S}$.
- Clearly, being able to calculate $\sigma^{-1}(s)$ is key to calculating the PDF of S.

Guerre, Perrigne, and Vuong [2000]

- In a pathbreaking paper, Emmanuel Guerre, Isabel Perrigne, and Quand H. Vuong [2000], noted that the first-order condition of expected-profit maximization at a first-price, sealed-bid auction can be written as:

$$\sigma'(v) + \sigma(v)\frac{(N-1)f_V(v)}{F_V(v)} = \frac{v(N-1)f_V(v)}{F_V(v)}.$$

- Now,

$$F_S[\sigma^{-1}(s)] = F_V(v)$$

and

$$\frac{f_S[\sigma^{-1}(s)]}{\sigma'[\sigma^{-1}(s)]} = f_V(v).$$

GPV

- Thus, in equilibrium, under risk neutrality and independence,

$$v = \sigma^{-1}(s) = s + \frac{F_S(s)}{(N-1)f_S(s)}.$$

- In short, the latent distribution of valuations $F_V^0(\cdot)$ is identified by the distribution of bids $F_S^0(\cdot)$.
- This is a major accomplishment, but I shall wait to discuss its implications for empirical work for a bit.

Distributions of Order Statistics

- At Dutch auctions, only the winning bid W is observed, which is a monotonic function of $Z = \max(V_1, V_2, \ldots, V_N)$.
- Instead of deriving the distribution of each order statistic individually, let's derive that of the k^{th} highest order statistic from a sample of N independent, identically-distributed draws from $F_V(\cdot)$.
- From that distribution, one can obtain the distribution of any order statistic required.

Distributions of Order Statistics

- Let X denote $V_{(k:N)}$.
- For X to be the k^{th} highest order statistic and fall within the interval $[x, x+\Delta x)$, there must be $(N-k)$ below x and $(k-1)$ draws above $[x+\Delta x)$.
- The probability of this event is

$$\Pr\left\{X \in [x, x+\Delta x)\right\} = \frac{N!}{(N-k)!(1-1)!(k-1)!} F_V(x)^{N-k}$$
$$\left[F_V(x+\Delta x) - F_V(x)\right]\left[1 - F_V(x+\Delta x)\right]^{k-1},$$

Implications

- Here,
$$\frac{N!}{(N-k)!(1-1)!(k-1)!}$$
is the multinomial combinatoric formula for the number of ways that one can get $(N-k)$ draws below x, $(k-1)$ draws above $(x+\Delta x)$, and one between x and $(x+\Delta x)$, in N draws.
- Deriving $f_X(x)$ simply involves calculating the limit

$$\lim_{\Delta x \to 0} \frac{\Pr\left\{X \in [x, x+\Delta x)\right\}}{\Delta x} = \lim_{\Delta x \to 0} \frac{[F_X(x+\Delta x) - F_X(x)]}{\Delta x},$$

PDF and CDF of X

- Thus, the PDF of X is

$$f_X(x|F_V, N, k) = \frac{N!}{(N-k)!(k-1)!} F_V(x)^{N-k}[1-F_V(x)]^{k-1} f_V(x).$$

- One convenient way of summarizing the relationship between $F_X(x|F_V, N, k)$, the CDF of X, and $F_V(v)$ as well as N and k is

$$F_X(x|F_V, N, k) = \frac{N!}{(N-k)!(k-1)!} \int_0^{F_V(x)} u^{N-k}(1-u)^{k-1}\,\mathrm{d}u.$$

Implications

- Note that $F_X(\cdot)$ is a monotonic transformation of $F_V(\cdot)$, where (conditional on knowing N and k) the monotonic function has a known form—the incomplete Beta function multiplied by a known constant.
- In other words, knowing $F_X(\cdot)$ and N is tantamount to knowing $F_V(\cdot)$, which will be an important result for identification.

PDFs of Two Important Order Statistics

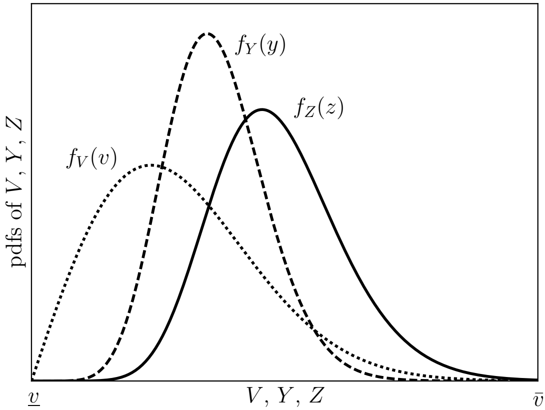
- We can now easily find the PDF of $Z = V_{(1:N)}$,

$$f_Z(z|F_V, N) = f_X(z|F_V, N, 1) = NF_V(z)^{N-1}f_V(z),$$

and $Y = V_{(2:N)}$,

$$\begin{aligned} f_Y(y|F_V, N) &= f_X(y|F_V, N, 2) \\ &= N(N-1)F_V(y)^{N-2}[1 - F_V(y)]f_V(y). \end{aligned}$$

PDFs of V, Y, and Z

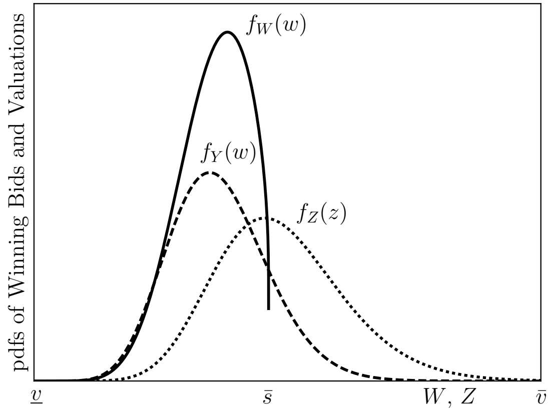


PDF of Winning Bid at Dutch Auction

- The winning bid at a Dutch auction is the equilibrium bid function $\sigma(\cdot)$, evaluated at Z, so the pdf of W, which is defined on the interval $[\underline{v}, \bar{s}]$, is the following:

$$
\begin{aligned}
f_W(w|F_V, N) &= \frac{f_Z\left[\sigma^{-1}(w)|F_V, N\right]}{\sigma'\left[\sigma^{-1}(w)\right]} \\
&= \frac{N F_V\left[\sigma^{-1}(w)\right]^{N-1} f_V\left[\sigma^{-1}(w)\right]^{N-1}}{\sigma'\left[\sigma^{-1}(w)\right]} \\
&= \frac{N F_V\left[\sigma^{-1}(w)\right]^{2N-1}}{(N-1)\int_{\underline{v}}^{\sigma^{-1}(w)} F_V(u)^{N-1}\,\mathrm{d}u}.
\end{aligned}
$$

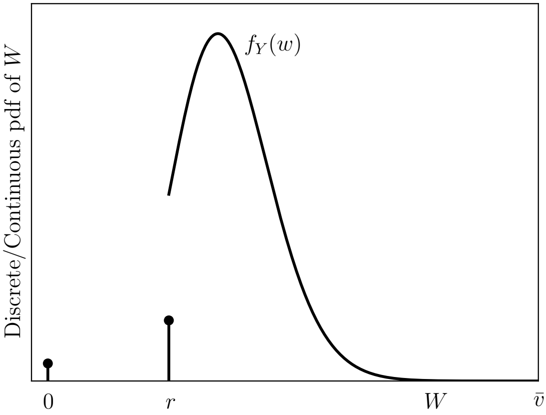
PDFs of W: Different Auctions Formats and Pricing Rules



Binding Reserve Price and Endogenous Participation

- In the presence of a binding reserve price, only those potential buyers whose valuations exceed the reserve price participate at the auction.
  1. The object goes unsold (so $w = 0$) when no potential buyer participates, the probability of which is
  $$\Pr(W = 0|F_V, N, r) = \Pr(M = 0|F_V, N, r) = F_V(r)^N.$$
  2. The object is sold at the reserve r when only one potential buyer bids, so
  $$\begin{aligned}\Pr(W = r|F_V, N, r) &= \Pr(M = 1|F_V, N, R) \\ &= NF_V(r)^{N-1}[1 - F_V(r)].\end{aligned}$$
  3. When M exceeds one, the winning bid density follows $f_Y(w)$.

Discrete-Continuous PMF/PDF of W: English Auction



Two Implications

- Auctions formats and pricing rules will imply very different shapes for the distributions of bids—the winning bids, in particular.
- Since the objects of interest in the calculation of the optimal reserve price are the PDF and CDF of valuations, methods that recover those objects are particularly appropriate empirical tools to use.

Vickrey Auction without a Reserve Price

- At Vickrey auction t, when N_t potential buyers exist, all of whom participate, the joint density of all bids $\boldsymbol{B}_t = [b_{1,t}, \ldots, b_{N,t}]$ is the following:

$$f_{\boldsymbol{B}}(\boldsymbol{b}_t) = \prod_{n=1}^{N_t} f_V(b_{n,t}).$$

- In short, one can identify $F_V^0(\cdot)$, the distribution of types (valuations), from $F_B^0(\cdot)$, the distribution of actions (bids).
- Moreover, one can estimate the distribution of valuations nonparametrically using your favorite kernel and an appropriate bandwidth parameter from the observed bids.

Vickrey Auction with a Reserve Price

- At Vickrey auction t, with a reserve price r_t, when N_t potential buyers exist, of which m participate, the joint density of all bids $\boldsymbol{B}_t = [b_{1,t}, \ldots, b_{N,t}]$ is the following:

$$f_{\boldsymbol{B}}(\boldsymbol{b}_t) = F_V(r_t)^{N_t - m_t} \prod_{n=1}^{m_t} f_V(b_{n,t}).$$

- In short, one only identify $F_V^0(\cdot)$ from $F_B^0(\cdot)$ above the reserve price r_t.
- Thus, without some additional assumption(s), one can only estimate $F_V^0(\cdot)$ nonparametrically above r_t; also, in the neighborhood of r_t, nonparametric estimators are severely biased.

Vickrey with Unobserved Heterogeneity and No Reserve

- Suppose each auction t differs by some random component ε_t that is the same for all potential bidders.
- Assume ε_t enters either additively or multiplicatively.
- In symbols, either

$$\beta(v|\varepsilon) = \beta(v) + \varepsilon_t$$

or

$$\beta(v|\varepsilon) = \beta(v)\varepsilon_t.$$

What do We Know from Measurement Error Models?

- Suppose
$$Y_{n,t} = V_n + \varepsilon_t,$$
where V_n and ε_t are independent.
- Now, the characteristic function of Y can be written in terms of those for V and ε as follows:
$$\mathbb{C}_Y(\tau) = \mathbb{C}_V(\tau) \times \mathbb{C}_\varepsilon(\tau).$$
- Also, the characteristic function of the difference any two random pairs $D_{nm,t} = (V_{n,t} - V_{m,t})$ is
$$\mathbb{C}_D(\tau) = \mathbb{C}_V(\tau)^2,$$
which can be estimated from data, and inverted to get the distribution of V.

So . . .

- When no reserve price exists, one can deal with unobserved heterogeneity in a straightforward way.
- In the presence of a binding reserve price, problems arise.
- To address those problems, one must make an assumption concerning the form of the unobserved heterogeneity—for example, it follows the Gaussian law, or some other parametric family.
- Perhaps not so obvious, but nevertheless relevant: If observed heterogeneity is going to be introduced in a numerically tractable way, then it will have to enter either additively or multiplicatively.

English Auction with No Reserve

- Suppose that w_t, the winning bid at auction t, is only observed.
- Assume that the number of potential bidders N is the same across the $t = 1, 2, \ldots, T$ auctions.
- Now, the CDF of W, which is the $V_{(2:N)}$ under the clock model of an English auction, is related to the CDF $F_V(\cdot)$ according to the following formula:

$$F_W(w) = N(N-1) \int_0^{F_V(w)} u^{N-2}(1-u)\, \mathrm{d}u$$
$$\equiv \varphi[F_V(w); N]$$

where $\varphi(\cdot)$ is a known, monotonic function.

English Auction with No Reserve (continued)

- Thus, $F^0_W(w)$ identifies $F^0_V(v)$, when N is known.
- Also, when N is fixed across T auctions, one can estimate $F_V(v)$ nonparametrically based on the empirical distribution function (EDF)

$$\hat{F}_W(v) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}(w_t \leq v)$$

using

$$\hat{F}_V(v) = \varphi^{-1}\left[\hat{F}_W(v)\right].$$

First-Price, Sealed-Bid Auction, No Reserve, Fixed N

- In this case, one can estimate $f_S(s)$ and $F_S(s)$ nonparametrically, and then form the pseudo-valuations according to the GPV formula:

$$\tilde{v}_{n,t} = s_{n,t} + \frac{\hat{F}_S(s_{n,t})}{(N_t - 1)\hat{f}_S(s_{n,t})},$$

  which can then be used to estimate $F_V(\cdot)$ nonparametrically.
- What about in the presence of a binding reserve price? Well, again, the distribution of valuations is only identified above r.
- Moreover, nonparametric estimates in the neighborhood of the binding reserve price are severely biased, but this time at both ends: at r and at $\bar{s}$.

First-Price, Sealed-Bid Auction, Varying N_t and r_t

- If one has enough data, when no reserve exists, then one can estimate $f_S(s)$ and $F_S(s)$ nonparametrically, and then form the pseudo-valuations for each N_t according to the GPV formula:
$$\tilde{v}_{n,t} = s_{n,t} + \frac{\hat{F}_S(s_{n,t})}{(N-1)\hat{f}_S(s_{n,t})}.$$
- What about when a binding reserve price exists that varies across auctions?
- Well, it really depends on the richness of the data.
- Historically, auction data sets, because they were collected by hand, typically only had a few hundred observations, so nonparametric analyses were infeasible.

English Auction with Binding Reserve

- Some fraction of the time, no potential buyers attend the auction, so the good goes unsold, which one can define as $W = 0$; the probability of this event is

$$p_M(0|F_V, N, r) = F_V(r)^N.$$

- Another fraction of the time, only one potential buyer attends the auction, in which case the winning bid is the reserve price r, and the probability of this event is

$$p_M(1|F_V, N, r) = NF_V(r)^{N-1}\left[1 - F_V(r)\right].$$

- Finally, when two or more potential buyers attend the auction, the winning bid is determined by the second-highest order statistic from a sample of size N.

English Auction with Binding Reserve (continued)

Thus, the discrete/continuous "density" is then

$$\begin{aligned} f_W(w|F_V,N,r) = & \Big\{F_V(r)^N\Big\}^{\mathbf{1}(W=0)} \times \\ & \Big\{NF_V(r)^{N-1}\left[1-F_V(r)\right]\Big\}^{\mathbf{1}(W=r)} \times \\ & \Big\{N(N-1)F_V(w)^{N-2}\left[1-F_V(w)\right]f_V(w)\Big\}^{\mathbf{1}(W>r)}, \end{aligned}$$

where $\mathbf{1}(W=0)$ is an indicator function of a winning bid of zero, $\mathbf{1}(W=r)$ is an indicator function of a winning bid of the reserve price r exactly, and $\mathbf{1}(W>r)$ is an indicator function of a winning bid greater than the reserve price.

Inexorably . . .

- When r_t and N_t vary across auctions, only parts of the distribution of values are identified.
- In the presence of observed covariates, collected in the vector $\mathbf{x_t}$, it must be obvious that nonparametric methods are incapable of delivering reliable estimates, even when potentially much data exist.
- What to do? Make a parametric assumption concerning $f_V(\cdot;\boldsymbol{\gamma})$ and introduce observed heterogeneity $\mathbf{x_t}$ through a single-index structure such as $\mathbf{x_t}\boldsymbol{\delta}$, where the unknown parameter vector $\boldsymbol{\delta}$ is conformable to $\mathbf{x}$.
- Collect γ and $\boldsymbol{\delta}$ in the vector $\boldsymbol{\theta}$ and employ the method of maximimum likelihood—or the generalized method of moments, or the method of simulated moments, if those are appropriate, and easier.

What Does All This Have to Do with Dynamics?

- Well, the IPV model, with single-unit demand, is the place to start.
- If you would like to admit multi-unit demand, then do so according to a Poisson process, since that process is memoryless.
- Without the memoryless property, a sequential auction becomes horrendously asymmetric, and potentially impossible to solve.

Learning

- Within the IPV model, no real role exists for learning from other bidders.
- In order to introduce such learning, a richer model must be built. Paul R. Milgrom and Robert J. Weber [1982] constructed such a model and coined the term "affiliation" to describe the dependence between the signals of potential buyers.
- In the mathematics literature, affiliation is referred to by a term coined by Samuel Karlin [1968]: multivariate total positivity of order two, or MTP2 for short.

Wilson Critique

- Robert B. Wilson [1987] criticized mechanism for focusing on the problem of designing a mechanism explicitly for each specific problem (for example, as determined by the specification of an environment and a market size).
- An economic consultant asked for advice on the selection of a mechanism may not know all the parameters that specify the problem, and the parameters may change over time.
- Thus, theoretical results that describe how the mechanism should be chosen assuming detailed knowledge of the problem may have little value to the consultant.
- A more meaningful task for mechanism design is to establish the sense in which a simple mechanism performs reasonably well across the variety of problems that might be encountered in practice.