FBA | QUANTITATIVE FINANCE RESEARCH GROUP

Presentation Title : HFT session 1

Stochastic Calculus for Finance Chapter 1~6 & Dynamic Programming

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Outline

Lebesgue Measure and Lebesgue Integral

Def Let Ω be a set and $\mathcal{P}(\Omega) := \{A|A \subset \Omega\}$ its power set. $\mathcal{A} \in \mathcal{P}$ is called a σ-algebra if

(i) $\Omega \in \mathcal{A}$

(ii) $A \in \mathcal{A}$ implies $A^C := \Omega \setminus A \in \mathcal{A}$

(iii) $A_i \in \mathcal{A}, i \in \mathbb{N}$ implies $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{A}$

Def Let $\Omega \neq \emptyset$ and $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a σ-algebra. A mapping $\mathbb{P} : \mathcal{A} \to [0, \infty]$ is called a measure on $(\Omega, \mathcal{A})$ if :

(i) $\mathbb{P}(\emptyset) = 0$

(ii) $\mathbb{P}(\bigcup\limits_{i\in\mathbb{N}} A_i) = \sum\limits_{i_1}^{\infty} \mathbb{P}(A_i)$ for all pairwise disjoint $A_i \in \mathcal{A}, i \in \mathbb{N}$

Def Lebesgue measure $\mu : \mathfrak{B}(\mathbb{R}) \to [0, \infty]$ is a measure on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ which assigns the measure of each interval to be its length. $\mathfrak{B}(\mathbb{R})$ is the σ-algebra of Borel subsets of $\mathbb{R}$.

Def

(i) Indicator function $\chi : \mathbb{R} \to \mathbb{R}$ is a function which takes only the values 0 and 1.

Let $A = \{x \in \mathbb{R} : \chi(x) = 1\}$. Then Lebesgue integral of χ is defined as

$$\int_{\mathbb{R}} \chi d\mu = \mu(A)$$

(ii) Simple function $s : \mathbb{R} \to \mathbb{R}$ is a linear combination of indicators :

$$s(x) = \sum_{k=1}^{n} c_k \chi_k(x)$$

Then we define the Lebesgue interal of s as :

$$\int_{\mathbb{R}} s d\mu = \sum_{k=1}^{n} c_k \mu(A_k)$$

(iv) Let f be a function defined on $\mathbb{R}$, we define :

$$f^+(x) = \max\{f(x), 0\}, \quad f^-(x) = \max\{-f(x), 0\}$$

and defined Lebesgue Integral of f as :

$$\int_{\mathbb{R}} f d\mu = \int_{\mathbb{R}} f^+ d\mu + \int_{\mathbb{R}} f^- d\mu$$

(v) Let f be a function defined on $\mathbb{R}$ and $A \subset \mathbb{R}$. We define :

$$\int_A f d\mu = \int_{\mathbb{R}} \mathbb{I}_A f d\mu$$

Conditional Expectation

Def $(\Omega, \mathcal{A}, \mathbb{P})$ is called a measure space, $(\Omega, \mathcal{A})$ is called a measurable space and $A \in \mathcal{A}$ is called a measurable set. $\mathbb{P}$ is called a probability measure if $\mathbb{P}(\Omega) = 1$. In this case $(\Omega, \mathcal{A}, \mathbb{P})$ is called a probability space.

Def Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces. A map $X : \Omega \to \Omega'$ is called $\mathcal{A}/\mathcal{A}'$-measurable if

$$\{X \in A'\} := \{\omega \in \Omega | X(\omega) \in A'\} \in \mathcal{A}, \quad \forall A' \in \mathcal{A}'$$

A Random Variable on $(\Omega, \mathcal{A})$ is a $\mathcal{A}/\mathfrak{B}(\mathbb{R})-$measurable map $X : \Omega \to \mathbb{R}$.

Problem Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{A}$ be a given sub-σ-algebra. Let $X \in \mathfrak{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, we want to find the random variable $Y \in \mathfrak{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ that minimizes the mean squared error, i.e.

$$\mathbb{E}(X - Y)^2 \leq \mathbb{E}(X - Y_0)^2, \quad \forall Y_0 \in \mathfrak{L}^2(\Omega, \mathcal{A}.\mathbb{P})$$

Def Let $X \in \mathfrak{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. The conditional expectation of X given $\mathcal{A}$ is any r.v. Y with
(i) Y is $\mathcal{A}$-measurable
(ii) $\mathbb{E}(X 1_A) = \mathbb{E}(Y 1_A)$, $\forall A \in \mathcal{A}$

Prop Tower property : Let $\mathcal{A}_1, \mathcal{A}_2$ be two sub-σ-algebras of $\mathcal{F}$ with $\mathcal{A}_1 \subset \mathcal{A}_2$. Then we have

$$\mathbb{E}(\mathbb{E}(X|\mathcal{A}_1)|\mathcal{A}_2) = \mathbb{E}(\mathbb{E}(X|A_2)|A_1) = \mathbb{E}(X|A_1)$$

Prop Jensen's Inequality : let ϕ is convex function and $\phi(X) \in \mathfrak{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Then we have following property :

$$\phi(E(X|\mathcal{A}) \leq \mathbb{E}(\phi(X)|\mathcal{A})$$

Martingales, Risk-Neutral Probability Measure

Def A map $X : \Omega \times \mathbb{T} \to \mathbb{R}, (X = (X_t)_{x \in \mathbb{T}})$ is called a stochastic process and (i) with fixed $\omega \in \Omega$, $X.(\omega) : \mathbb{T} \to \mathbb{R}$ is a sequence
(ii) with fixed $t \in \mathbb{T}$, $X_t(.) : \Omega \to \mathbb{R}$ is a random variable

Def With measurable space $(\Omega, \mathcal{F})$, filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$ is a family of σ-algebras with
(i) $\mathcal{F}_t \subseteq \mathcal{F} \quad \forall t \in \mathbb{T}$
(ii) $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \quad \forall t \in \mathbb{T}$
Also we call $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$ a filtered probability space.

Def With filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$ and $X : \Omega \times \mathbb{T} \to \mathbb{R}$ be a stochastic process. We say X is adapted if $X_t : \Omega \to \mathbb{R}$ is $\mathcal{F}_t$-measurable.

Def With filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$ and $M : \Omega \times \mathbb{T} \to \mathbb{R}$ be a stochastic process. We say M is a Martingale if
(i) M is adapted
(ii) $\mathbb{E}^P(M_{t+1} | F_t) = M_t \quad \forall t \in \mathbb{T}$

Def With filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ and $G, S : \Omega \times \mathbb{T} \to \mathbb{R}$ be two adapted & positive stochastic process. A Risk-Neutral measure is a probability measure Q on Ω s.t.

(i) $Q(\{\omega\}) > 0 \quad \forall \omega \in \Omega$

(ii) $\left(\frac{S_t}{G_t}\right)_{t\in\mathbb{T}}$ is martingale under Q

Remark For binomial model, a pair of probabilities (q_u, q_d) is a risk-neutral measure if

(i) $q_u > 0$, $q_d > 0$, $q_u + q_d = 1$

(ii) $S = q_u * \frac{S_u}{1+R} + q_d * \frac{S_d}{1+R}$

(expectation of present value of future price equals to present price)

Markov Process, Stopping Times

Def Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

A Brownian motion is a stochastic process $B : \Omega \times [0, \infty) \to \mathbb{R}$ s.t.

(i) $B_0 = 0$

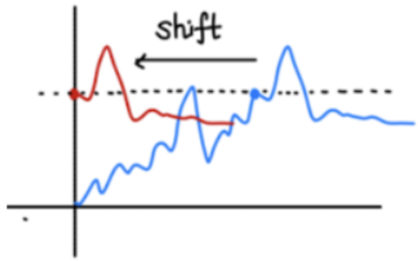
(ii) $0 = t_0 \leq t_1 \leq \cdots \leq t_m \Rightarrow B_{t_1} - B_{t_0}, \cdots, B_{t_m} - B_{t_{m-1}}$ are mutually independent

(iii) $0 \leq s < t \Rightarrow B_t - B_s \sim N(0, t-s)$

(iv) sample paths are continuous (i.e. $t \mapsto B_t(\omega)$ is continuous $\forall \omega \in \Omega$)

Def Consider the shift transformations $\theta_s : \Omega \to \Omega, s \geq 0$ defined by

$$\theta_s(\omega)(t) := \omega(s+t), t \geq 0$$



and let $Y : \Omega \to \mathbb{R}$ then we can say $Y \circ \theta_s$ is a function of the future after time s. For instance, if $f : \mathbb{R} \to \mathbb{R}$ is bounded and measurable and $Y = f(B_t)$ then $Y(\theta_s(\omega)) = f(B_{t+s}(\omega))$ and so $Y \circ \theta_s = f(B_{t+s})$, similarly we have $B_t \circ \theta_s = B_{t+s} \quad \forall s, t \geq 0$.
Brownian motion $((B_t)_{t \geq 0}, \Omega, \mathcal{F}, (\mathbb{P}_x)_{x \in \mathbb{R}})$ has the following Markov Property :

$$\mathbb{E}(Y \circ \theta_s | \mathcal{F}_s) = \mathbb{E}_{B_s}(Y) \, (= \mathbb{E}(Y|B_s)) \quad \forall s \geq 0, x \in \mathbb{R}$$

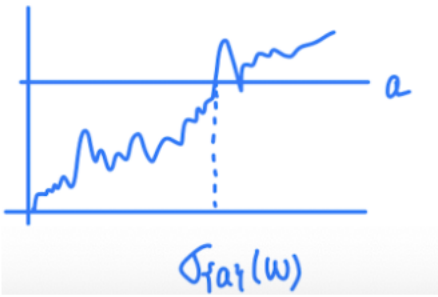
or in sense of probability, we can write as : $\mathbb{P}(X_{t+s} \in A | \mathcal{F}_s) = \mathbb{P}(X_{t+s} \in A | X_s) \quad \forall A \in \mathfrak{B}(\mathbb{R})$

Def We say X is Time-homogeneous Markov Process if :
$\mathbb{E}(f(X_{t+s})|X_s = x)$ is independent of s (only depends on time-difference)

Def A map $T : \Omega \to [0, \infty]$ is called an $(\mathcal{F}_t)$-stopping time if

$$\{T \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0$$

Example $\sigma_{\{a\}} := \inf\{t > 0 | B_t > a\}$ is a stopping time.



Dynamic Programming; Category: 최적화이론, 알고리즘 …

큰 문제를 작은 문제로 나누어 푸는 것

ex) 수학적 귀납법

$$F(0) \wedge \left(\forall n, F(n) \Rightarrow F(n+1)\right) \Rightarrow \forall n, F(n+1)$$

Def. DP := 목적함수(Optimal Object, $W(x)$)를 최대화/최소화하는 관계식을 찾아내는 것

$$W_n = \sup_{w_{\mathrm{n}}}\{\, f(W_{n+1}, a_{n+1})\}$$: top-down

$$W_n = \max\{\, f(W_{n-1}, a_{n-1})\}$$: bottom-up

Find f that maximizes W

Def. Plant eq.

- $x_t \in X$: state at time t
- $a_t \in A_t$: action at time t

$$f_t: X \times A_t \to X \;\; that \;\; is \;\; f_t(x_t, a_t) = x_{t+1}$$

Topic 2 Dynamic Programming

Def. DP

- r_t: reward at time t
- $\tilde{a} \in (a_0, \dots a_T)$: path of actions through time

Maximize sum of rewards

$$R(\tilde{a}) = \sum_{t=0}^{T} r_t(x_t, a_t)$$

$$R_\tau(\tilde{a}_\tau) = \sum_{t=\tau}^{T} r_t(x_t, a_t) = r_t(x_t, a_t) + R_{\tau+1}(x_{\tau+1}, a_{\tau+1})$$

Maximize

$$W_\tau(\tilde{a}_\tau) = \max_{\tilde{a}_\tau} R_\tau(\tilde{a}_\tau)$$

Topic 2 Dynamic Programming

Def. Bellman eq.

- $W_T(x) = r_T(x)$
- $W_t(x_t) = \sup_{a_t \in A_t} \{r_t(x_t, a_t) + W_{t+1}(x_{t+1})\}$

  where $x_t \in X \ \wedge \ x_{t+1} = f_t(x_t, a_t)$

Maximize

$$W_\tau(\tilde{a}_\tau) = \max_{\tilde{a}_\tau} R_\tau(\tilde{a}_\tau)$$

$$\begin{aligned} W_t(x_t) &= \max_{\tilde{a}_t} R_t(\tilde{a}_t) \\ &= \max_{a_t} \max_{\tilde{a}_{t+1}} \{r_t(x_t, a_t) + R_{t+1}(x_{t+1}, a_{t+1})\} \\ &= \max_{a_t} r_t(x_t, a_t) + \max_{\tilde{a}_{t+1}} R_{t+1}(x_{t+1}, a_{t+1}) \\ &= \max_{a_t} r_t(x_t, a_t) + W_{t+1}(x_{t+1}) \end{aligned}$$

Topic 2 Dynamic Programming

Problem.

Plant eq.

$$x_{t+1} = x_t + rx_t(1 - a_t)$$

- $x_0 = x,\ r$: constant
- $0 \le a_t \le 1$: variable

Total Rewards

$$W_0 = ra_0 + ra_1 + \cdots ra_{T-1} = \sum_{t=0}^{T-1} ra_t = R(\tilde{a})$$

Maximize

$$W_\tau(\tilde{a}_\tau) = \max_{\tilde{a}_\tau} R_\tau(\tilde{a}_\tau)$$

Partial Total Rewards

$$W_\tau := \sum_{t=t}^{T-1} ra_t = R_\tau(\tilde{a}_\tau) = r_t(x_t, a_t) + R_{\tau+1}(x_{\tau+1}, a_{\tau+1})$$

- $t = T-1$,

$$W_{T-1} = \max\{rx_{T-1}a_{T-1}\}, \text{ so } a_{T-1} = 1$$

- $t = T-2$,

$$\begin{aligned} W_{T-2} &= \max_{0 \le a_{T-2} \le 1}\{rx_{T-1}a_{T-1} + W_{T-1}(X_{T-1})\} \\ &= \max\{rx_{T-1}a_{T-1} + r[x_{T-2} + rx_{T-2}(1 - a_{T-2})]\} \\ &= rx_{T-2}\max\{(1+r) + (1-r)a_{T-2}\} \\ &= rx_{T-2}\max(1+r, 2) \end{aligned}$$

Topic 2 Dynamic Programming

Partial Total Rewards

$$W_\tau := \sum_{t=t}^{T-1} ra_t = R_\tau(\tilde{a}_\tau) = r_t(x_t, a_t) + R_{\tau+1}(x_{\tau+1}, a_{\tau+1})$$

- $t = T - \mathrm{S}$,

Assume, $W_{T-S+1}(x_{T-S+1}) = rx_{T-S+1} \cdot \rho_{T-S+1}$

$$W_{T-S}(x_{T-S}) = \max_{a_{T-S}}\{rx_{T-S}a_{T-S} + rx_{T-S+1}\rho_{T-S+1}\}$$
$$= \max\{rx_{T-S}a_{T-S} + r\rho_{T-S+1}[x_{T-S} + rx_{T-S}(1 - a_{T-S})]\}$$
$$= rx_{T-S}\max\{(1+r)\rho_{T-S+1} + (1 - r\rho_{T-S+1})a_{T-S}\}$$
$$= rx_{T-S}\max((1+r)\rho_{T-S+1}, 1 + \rho_{T-S+1})$$

- $Thus,$

$$\therefore \rho_{T-S} = \max\big((1+r)\rho_{T-S+1}, 1 + \rho_{T-S+1}\big)$$

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