HFT session homework 2

1)

▶ **Exercise 3.2.** Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale. (Hint: For $0 \leq s \leq t$, write $W^2(t)$ as $(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s)$.)

def. Martingale

$$E[X_{n+1} \mid \mathcal{F}_n] = X_n$$

- $W^2(t) = \{W(t) - W(s)\}^2 + 2W(t)W(s) - W^2(s)$
- $E[W^2(t) - W^2(s) \mid \mathcal{F}(s)] = E[\{W(t) - W(s)\}^2 + 2W(t)W(s) - 2W^2(s) \mid \mathcal{F}(s)]$

$$= E[\{W(t) - W(s)\}^2 \mid \mathcal{F}(s)] + E[2W(s)\{W(t) - W(s)\} \mid \mathcal{F}(s)]$$

by Martingale Property

$E[B(t) \mid \mathcal{F}(s)] = B(s)$, so $E[W(t) - W(s) \mid \mathcal{F}(s)] = t(=s) - s = 0$

by property of Brownian

$E[W(t)] = 0$, $V[W(t)] = t$

$V[W \mid \mathcal{F}] = E[W^2 \mid \mathcal{F}] - E[W \mid \mathcal{F}]^2$ (조건부 분산의 성질)

so, $E[\{W(t) - W(s)\}^2 \mid \mathcal{F}(s)] = E[\{W(t) - W(s)\}^2 \mid \mathcal{F}(s)] - \underbrace{E[W(t) - W(s) \mid \mathcal{F}(s)]}_{=0}{}^2$

$$= V[W(t) - W(s) \mid \mathcal{F}(s)]$$

$$= t - s \quad (?)$$

Thus,

$$E[W^2(t) - W^2(s) \mid \mathcal{F}(s)] = t - s$$

and

$$E[W^2(t) - t \mid \mathcal{F}(s)] = E[W^2(t) \mid \mathcal{F}(s)] - t = E[W^2(s) \mid \mathcal{F}(s)] - s = W^2(s) - s$$

$\therefore$ $W^2(t) - t$ is martingale $\square$

2)

Let z_t be a standard Brownian motion (or Wiener process). Then, simplify the following expression:

$$\int_0^T z_t dz_t = ?$$

(Hint) Suppose $x_t = z_t$ and $f(x,t) = \frac{x^2}{2}$. Then, apply Ito's lemma on $f(x,t)$.

Suppose $X_t = Z_t$ & $f(x,t) = x^2/2$

Following Ito's Lemma,

$$df(X_t, t) = \partial_{X_t} f(X_t, t) dX_t + \frac{1}{2} \partial_{X_t}^2 f(X_t, t)\, dX_t^2 + \partial_t f(X_t, t) dt$$

where $f(X_t, t) = \frac{1}{2} Z^2$, $\partial_Z f = Z$, $\partial_Z^2 f = 1$, $\partial_t f = 0$

So,

$$d\left(\frac{1}{2} Z_t\right)^2 = Z_t dZ_t + \frac{1}{2} dZ_t^2$$

$$= Z_t dZ_t + \frac{1}{2} dt$$

Integral both sides,

$$\frac{1}{2} Z_T^2 = \int_{Z,T} Z_t dZ_t + \frac{1}{2} dt = \int_0^{Z_T} Z_t dZ_t + \underbrace{\int_0^T \frac{1}{2} dt}_{= \frac{1}{2} T}$$

$$\therefore \int_0^{T (\sim Z_T)} Z_t dZ_t = \frac{1}{2} Z_T^2 - \frac{1}{2} T \quad \square$$