HFT session homework 4

1)

In this problem you will use the simple binomial approximation of Brownian motion to help understand Girsanov's theorem. Consider a simple binary random variable

$$X = \begin{cases} \sqrt{dt} & \text{with probability } 0.5 + \mu(dt)^n/2 \\ -\sqrt{dt} & \text{with probability } 0.5 - \mu(dt)^n/2 \end{cases}$$

(a) For this Brownian motion with drift to have a drift term of order dt, what must n be?

(Hint: Compute the mean of X and choose n so that the mean is of order dt.)

(b) Given that you computed n corrected from part (a), compute the variance of X.

(c) Can you use the results of (a) and (b) to argue that an equivalent change of measure will change the mean of a Brownian motion, but not the instantaneous variance?

$$X = \begin{cases} dt^{1/2}, & P = \frac{1}{2} + \frac{\mu}{2}(dt)^n \\ -dt^{1/2}, & P = \frac{1}{2} - \frac{\mu}{2}(dt)^n \end{cases}$$

(a) $E[X] = dt^{1/2}\left(\frac{1}{2} + \frac{\mu}{2} dt^n\right) - dt^{1/2}\left(\frac{1}{2} - \frac{\mu}{2} dt^n\right)$

$= \mu \, dt^{1/2} dt^n$

$\therefore \; n = 1/2$

(b) $V[X] = E[X^2] - E[X]^2$ with $n = 1/2$

$= dt\left(\frac{1}{2} + \frac{\mu}{2} dt^{1/2}\right) + dt\left(\frac{1}{2} - \frac{\mu}{2} dt^{1/2}\right) - \mu^2 dt^2$

$= dt - \mu^2 dt^2$

(c) Since $V[X] = dt - \mu^2 dt^2$ & $dt^2 \to 0$,

$V[X] = dt$ is independent of drift μ.

Change μ $\longrightarrow$ measure $\mathbb{P}$ changes (to $\tilde{\mathbb{P}}$)

$\longrightarrow$ probability variable X changes (to $\tilde{X}$)

$\longrightarrow$ Brownian motion B changes (to $\tilde{B}$)

$\longrightarrow$ But, not $V[B]/dt$ changes!

- $E[B] \neq E[\tilde{B}]$: mean of BM changes
- $V[B]/dt = V[\tilde{B}]/dt$: instantaneous variance of BM does not.

2)

Using principles from linear pricing (or risk-neutrality), derive a PDE for the price of a European call option on a non-dividend paying stock when interest rates are random, and the short rate follows Cox-Ingersoll-Ross dynamics. That is:

$$\text{Stock: } \quad dS = \mu S dt + \sigma S dz_1$$
$$\text{Interest Rate: } \quad dr = a(b-r)dt + c\sqrt{r}dz_2$$

where z_1 and z_2 are correlated Winner processes with correlation coefficient ρ. (i.e. $E(dz_1\, dz_2)=\rho dt$)

(Hint: Your answer may contain a market price of risk.)

Use risk-neutral pricing measure $\widetilde{\mathbb{P}}$

Under $\widetilde{\mathbb{P}}$,

- $d\tilde{z}_1 = dz_1$
- $E[d\tilde{z}_1, d\tilde{z}_2] = \rho$
- $d\tilde{z}_2 = \lambda dt + dz_2$
- $d\tilde{z}_1 \perp d\tilde{z}_2$

So, the Stock and Interest Rate becomes

$$dS = \mu S dt + \sigma S d\tilde{z}_1$$
$$dr = a(b-r)dt + c\sqrt{r}(d\tilde{z}_2 - \lambda dt)$$

Let $C(S,r,t)$ be the price of derivative on S. That means,

$$dC = rCdt + (?)d\tilde{z}_1 + (?)d\tilde{z}_2,$$

(?) term is not important.
We will only care the drift term.

By Ito's formula,

$$dC = C_t dt + C_S dS + \tfrac{1}{2}C_{SS}dS^2 + C_r dr + \tfrac{1}{2}C_{rr}dr^2 + C_{Sr}dSdr$$
$$= C_t dt + C_S[\mu S dt + \sigma S d\tilde{z}_1] + \tfrac{1}{2}C_{SS}[\mu S dt + \sigma S d\tilde{z}_1]^2$$
$$+ C_r[a(b-r)dt + c\sqrt{r}(d\tilde{z}_2 - \lambda dt)] + \tfrac{1}{2}C_{rr}[a(b-r)dt + c\sqrt{r}(d\tilde{z}_2 - \lambda dt)]^2$$
$$+ \tfrac{1}{2}C_{Sr}[\mu S dt + \sigma S d\tilde{z}_1][a(b-r)dt + c\sqrt{r}(d\tilde{z}_2 - \lambda dt)]$$
$$= dt\left\{C_t + rSC_S + \tfrac{1}{2}\sigma^2 S^2 C_{SS} + (a(b-r) - c\sqrt{r}\lambda)C_r + \tfrac{1}{2}c^2 r C_{rr} + \rho\sigma S c\sqrt{r} C_{Sr}\right\} + (?)d\tilde{z}_1 + (?)d\tilde{z}_2$$

Thus,

$$rC = C_t + rSC_S + \tfrac{1}{2}\sigma^2 S^2 C_{SS} + (a(b-r) - c\sqrt{r}\lambda)C_r + \tfrac{1}{2}c^2 r C_{rr} + \rho\sigma S c\sqrt{r} C_{Sr}$$