HFT session homework 1

1)

Exercise 2.2. Consider the stock price S_3 in Figure 2.3.1.

(i) What is the distribution of S_3 under the risk-neutral probabilities $\tilde{p} = \frac{1}{2}$, $\tilde{q} = \frac{1}{2}$.

(ii) Compute $\tilde{\mathbb{E}}S_1$, $\tilde{\mathbb{E}}S_2$, and $\tilde{\mathbb{E}}S_3$. What is the average rate of growth of the stock price under $\tilde{\mathbb{P}}$?

(iii) Answer (i) and (ii) again under the actual probabilities $p = \frac{2}{3}$, $q = \frac{1}{3}$.

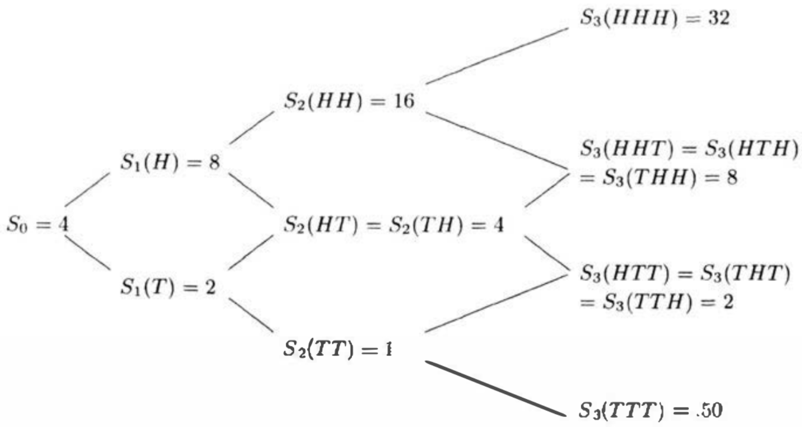

Fig. 2.3.1. A three-period model.

(i) $\left\{\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right\}$

(ii) $\tilde{E}X = \sum X(\omega)\mathbb{P}(\omega)$

- $\tilde{E}S_1 = pS_1(H) + qS_1(T) = \frac{1}{2}\cdot 8 + \frac{1}{2}\cdot 2 = 5$
- $\tilde{E}S_2 = p^2 S_2(HH) + 2pq S_2(HT) + q^2 S_2(TT) = \frac{1}{4}\cdot 16 + \frac{2}{4}\cdot 4 + \frac{1}{4}\cdot 1 = 7$
- $\tilde{E}S_3 = \frac{1}{8}32 + \frac{3}{8}\cdot 8 + \frac{3}{8}\cdot 2 + \frac{1}{8}\cdot\frac{1}{2} = \frac{125}{16}$

(iii)

– $\left\{\frac{8}{27}, \frac{12}{27}, \frac{6}{27}, \frac{1}{27}\right\}$

– $\cdot\ \tilde{E}S_1 = \frac{2}{3}8 + \frac{1}{3}\cdot 2 = \frac{18}{3} = 6$

$\cdot\ \tilde{E}S_2 = \frac{4}{9}\cdot 16 + 2\cdot\frac{2}{9}\cdot 4 + \frac{1}{9}\cdot 1 = \frac{81}{9} = 9$

$\cdot\ \tilde{E}S_3 = \frac{8}{27}\cdot 32 + 3\cdot\frac{4}{27}\cdot 8 + 3\cdot\frac{2}{27}\cdot 2 + \frac{1}{27}\cdot\frac{1}{2} =$ 계산 귀찮..

2)

Exercise 2.3. Show that a convex function of a martingale is a submartingale. In other words, let $M_0, M_1, \ldots, M_N$ be a martingale and let φ be a convex function. Show that $\varphi(M_0), \varphi(M_1), \ldots, \varphi(M_N)$ is a submartingale.

Martingale $(M_n, \mathcal{F}_n)$

Convex function $\varphi : \mathbb{R} \to \mathbb{R}$

Recall Jensen's inequality

$$E[\varphi(M) \mid G] \geq \varphi(E[M \mid G]) \quad \text{for } G \subset \mathcal{F}$$

$$E[\varphi(M_{n+1}) \mid \mathcal{F}_n] \geq \varphi(E[M_{n+1} \mid \mathcal{F}_n]) = \varphi(M_n) \quad \because \text{property } 2$$

By definition, $\forall n \in \mathbb{N}$, $\varphi(M_n)$ is submartingale.