HFT session homework 3

1)

Consider the following stochastic differential equations:

$$dx = a\,dt + b\,dz_1$$
$$dy = f\,dt + g\,dz_2$$

where z_1 and z_2 are correlated Winner processes with correlation coefficient ρ. (i.e. $E(dz_1\, dz_2)=\rho dt$)

(a) Use Ito's lemma to find d(xy).

(b) Use Ito's lemma to find d(x/y).

(a) let $h(x,y) = xy$

- $h_x = y$
- $h_y = x$
- $h_{xx} = h_{yy} = 0$
- $h_{xy} = 1$

so,

$$dh = h_x dx + h_y dy + h_{xy} dx dy + \cancel{\frac{h_{xx} dx^2}{2}} + \cancel{\frac{h_{yy} dy^2}{2}}$$
$$= y(adt + bdz_1) + x(fdt + gdz_2) + (adt + bdz_1)(fdt + gdz_2)$$
$$= (ay + xf)dt + yb\,dz_1 + xg\,dz_2 + \rho bg\,dt \qquad (\because dtdt = 0,\ dtdz = 0,\ dz_1 dz_2 = \rho dt)$$

(b) let $h(x,y) = x/y$

- $h_x = 1/y$
- $h_y = -x/y^2$
- $h_{xx} = 0$
- $h_{yy} = 2x/y^3$
- $h_{xy} = -1/y^2$

so,

$$dh = h_x dx + h_y dy + \frac{1}{2} h_{xx} (dx)^2 + \frac{1}{2} h_{yy} (dy)^2 + h_{xy} dx dy$$
$$= \frac{1}{y}(adt + bdz_1) - \frac{x}{y^2}(fdt + gdz_2) + \frac{1}{2}\frac{2x}{y^3} g^2 dt - \frac{1}{y^2}\rho bg\,dt$$
$$= \left(\frac{a}{y} - \frac{xf}{y^2} + \frac{x}{y^3} g^2 - \frac{1}{y^2}\rho bg\right)dt + \frac{b}{y} dz_1 - \frac{x}{y^2} g\,dz_2$$

2)

Derive a PDE for the price of a European call option on a non-dividend paying stock when interest rates are random, and the short rate follows Cox-Ingersoll-Ross dynamics. That is:

$$dS = \mu S dt + \sigma S dz_1$$
$$dr = a(b-r)dt + c\sqrt{r}dz_2$$

where z_1 and z_2 are correlated Winner processes with correlation coefficient ρ. (i.e. $E(dz_1\, dz_2)=\rho dt$)

(Hint: Your answer may contain a market price of risk.)

" Expected profit − risk free profit ; w.r.t μ, r

Let $C(S,r,t)$ be the price process for the European Call option.

By Itô's Lemma,

$$dC = \left(C_t + \mu S C_s + a(b-r)C_r + \tfrac{1}{2}\sigma^2 S^2 C_{ss} + \tfrac{1}{2}c^2 r C_{rr} + \rho\sigma c S\sqrt{r}C_{rS}\right)dt + \sigma S C_s dz_1 + c\sqrt{r}C_r dz_2$$

- Risk Free : $\frac{dB}{B} = r dt$
- Stock : $\frac{dS}{S} = \mu dt + \sigma dz_1$
- European Call : $\frac{dC}{C} = \mu_c dt + \frac{\sigma S C_s}{C} dz_1 + \frac{c\sqrt{r}C_r}{C} dz_2$

Which can be written as

$$\begin{bmatrix} \frac{dB}{B} \\ \frac{dS}{S} \\ \frac{dC}{C} \end{bmatrix} = \begin{bmatrix} r & & \\ \mu & \sigma & \\ \mu_c & \frac{\sigma S C_s}{C} & \frac{c\sqrt{r}C_r}{C} \end{bmatrix} \begin{bmatrix} dt \\ dz_1 \\ dz_2 \end{bmatrix}$$

From Arbitrage Pricing Theory, we can assume the expected return μ as linear summation of Risk Free and Risk. $(\underset{\text{risk free}}{\mu = 1\,\lambda_0} + \underset{\text{risk}}{\sigma}\,\underset{\text{stochastic factor}}{\lambda_1})$

Which implies,

$$\begin{bmatrix} r \\ \mu \\ \mu_c \end{bmatrix} = \begin{bmatrix} 1 & & \\ 1 & \sigma & \\ 1 & \frac{\sigma S C_s}{C} & \frac{c\sqrt{r}C_r}{C} \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \begin{array}{l} \text{risk free} \\ \text{stochastic factor} \\ \text{'' (market price of risk)} \end{array}$$

Thus,

$$\lambda_0 = r$$
$$\lambda_1 = \frac{\mu - r}{\sigma}$$

Market price of risk with regard to the random interest rate r.

$$\mu_c = \lambda_0 + \lambda_1 \frac{\sigma S C_s}{C} + \lambda_2 \frac{c\sqrt{r}C_r}{C} \quad : \text{linear sum of risk free \& stochastic factors.}$$
$$= r + \frac{\mu - r}{\sigma}\frac{\sigma S C_s}{C} + \lambda_2 \frac{c\sqrt{r}C_r}{C} \quad : \mu_c = \mu_c(\mu, r)$$

So,

$$C_t + \mu S C_s + a(b-r)C_r + \tfrac{1}{2}\sigma^2 S^2 C_{ss} + \tfrac{1}{2}c^2 r C_{rr} + \rho\sigma c S\sqrt{r}C_{rs}$$
$$= \mu_c C dt$$
$$= rC + (\mu - r)SC_s + \lambda_2 c\sqrt{r}C_r$$

$$\therefore\ rC = C_t + rSC_s + \left(a(b-r) - \lambda_2 c\sqrt{r}\right)C_r + \tfrac{1}{2}\sigma^2 S^2 C_{ss} + \tfrac{1}{2}c^2 r C_{rr} + \rho\sigma c S\sqrt{r}C_{rs}$$