* Newtonian Mechanics Review

- Describes the motion of "particles" or "point mass"
- Particle ⇒ configuration is completely described by a position $\vec{r}$ ← "abstract vector"
- In "coordinate-free" form:

$$\vec{F} = \frac{d}{dt}\Big(\underbrace{m}_{\text{mass (scalar)}} \frac{d\vec{r}}{dt}\Big) = \frac{d}{dt}(m \underbrace{\vec{v}}_{\text{velocity}}) = \frac{d\vec{p}}{dt}$$

← momentum

- Assuming constant mass:

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$$

← acceleration

- Note that we can't actually do any computation until we choose coordinates.

- 2D Cartesian Coordinates

unit basis vectors

$$\vec{r} = r_x \hat{e}_x + r_y \hat{e}_y = \begin{bmatrix} r_x \\ r_y \end{bmatrix}^T \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \end{bmatrix}$$

scalar components

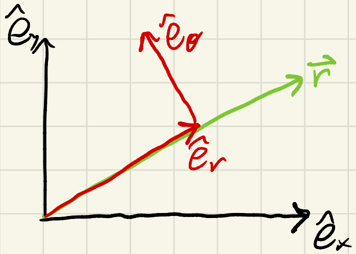
$$\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \dot{r}_x \hat{e}_x + \cancelto{0}{r_x \dot{\hat{e}}_x} + \dot{r}_y \hat{e}_y + \cancelto{0}{r_y \dot{\hat{e}}_y}$$

$$= \dot{r}_x \hat{e}_x + \dot{r}_y \hat{e}_y$$

$$\vec{a} = \ddot{r}_x \hat{e}_x + \ddot{r}_y \hat{e}_y$$

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix}^T \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \end{bmatrix} = m \begin{bmatrix} \ddot{r}_x \\ \ddot{r}_y \end{bmatrix}^T \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} F_x \\ F_y \end{bmatrix} = m \begin{bmatrix} \ddot{r}_x \\ \ddot{r}_y \end{bmatrix}}$$

- 2D Polar Coordinates



$$\hat{e}_r = \cos(\theta)\,\hat{e}_x + \sin(\theta)\,\hat{e}_y$$

$$\hat{e}_\theta = -\sin(\theta)\,\hat{e}_x + \cos(\theta)\,\hat{e}_y$$

$$\vec{r} = r\,\hat{e}_r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\,\hat{e}_r + r\,\underbrace{\dot{\hat{e}}_r}_{\text{not zero!}}$$

$$\frac{d\hat{e}_r}{dt} = \frac{d}{dt}\left(\cos(\theta)\,\hat{e}_x + \sin(\theta)\,\hat{e}_y\right) = -\sin(\theta)\dot{\theta}\,\hat{e}_x + \cos(\theta)\dot{\theta}\,\hat{e}_y$$
$$= \dot{\theta}\,\hat{e}_\theta$$

$$\Rightarrow \vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

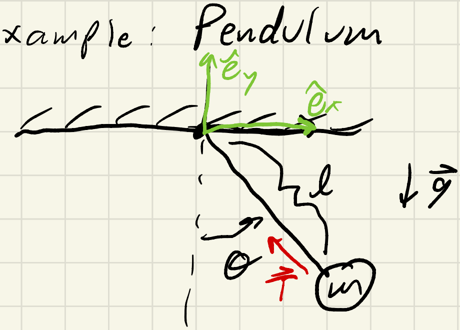
$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\,\hat{e}_r + \dot{r}\,\dot{\hat{e}}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,\dot{\hat{e}}_\theta$$
$$= \ddot{r}\,\hat{e}_r + 2\dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta - r\dot{\theta}^2\,\hat{e}_r$$

$$\Rightarrow \boxed{\begin{bmatrix} F_r \\ F_\theta \end{bmatrix} = m \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{bmatrix}}$$

* Take-Away: the Newtonian formulation is <u>not</u> coordinate invariant!

- Becomes super impractical for complex systems

* Example: Pendulum



- System has 2DOF and 1 constraint:

$$\|\vec{r}(t)\| = l$$

$$\vec{F} = \vec{T} + m\vec{g} = m\dot{\vec{v}}$$

- Polar Coordinates:

$$F_r = m(\cancelto{0}{\ddot{r}} - r\dot{\theta}^2)$$

$$F_\theta = m(r\ddot{\theta} + 2\cancelto{0}{\dot{r}}\dot{\theta})$$
$$= ml\ddot{\theta}$$

$$\vec{T} = -T\hat{e}_r$$

$$\vec{g} = -g\,\hat{e}_y = -g\left[\cos(\theta)\,\hat{e}_r + \sin(\theta)\,\hat{e}_\theta\right]$$

$$\Rightarrow F_\theta = -mg\sin(\theta) = ml\ddot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} = \frac{-g}{l}\sin(\theta)}$$

* Important Take-Aways

- Constraint was non-trivial in Cartesian coordinates but simple in Polar coordinates

- We could ignore the r-direction dynamics and only care about θ

- We also never had to explicitly calculate T (the cable tension / constraint force)

- This system has 1 DOF. If we can use 1 coordinate, we call that "minimal" coordinates. Often called "joint" coordinates in robotics and "generalized" coordinates in physics.

- Most current robotics simulators use minimal/joint coordinates. Not always necessarily the best choice.

- The opposite approach (full cartesian coordinates + explicit constraints) is often called "maximal" coordinates.