Last Time:

- Higher-Order RK
- Stiff ODEs
- Stability of RK methods

Today:

- Rigid Bodies
- Reference Frames
- Attitude + Representations
- Rotation Matrices

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* Rigid Bodies

- So far we've talked about particles

- Next step: "rigid bodies" that have finite volume

- Definition: A collection of N particles whose relative distances are fixed:

$$\underbrace{\|r_i}_{\text{position } \in \mathbb{R}^3} - r_j\| = \underbrace{c_{ij}}_{\text{distance } \mathbb{R}_+} \quad , \quad i,j \in [1 \ldots N]$$

- Never true in reality. Approximation is valid $\underbrace{\omega_{flex}}_{\text{Natural frequencies of flexible dynamics}} \gg \underbrace{\omega_{RB}}_{\text{Natural frequencies of rigid-body dynamics}}$

- Example:

1) Car on highway ⇒ rigid body model OK vs. race car where roll dynamics, suspension, and tire deformation become important

2) Rocket has bending modes at 10~20 Hz so control signals for thruster must be filtered / band-limited to avoid exciting flexible dynamics (Not rigid).

- If flexible dynamics are much faster than control inputs, rigid body motion ⇒ can ignore

- Rigid body models are extremely useful across robotics, control, aerospace, etc.

* Transformations + Configuration / Pose

- Particle only has position but rigid bodies have more configuration/pose information

- We could just keep track of N particles, but thats inneficient.

- What are all the transformations I can do to the N particles that will respect the relative distance constraints?

Translation: $r_i' = r_i + v \quad \forall i, \; v \in \mathbb{R}^3$

$$\| r_i' - r_j' \| = \| r_i + v - r_j - v \| = \| r_i - r_j \| = c_{ij}$$

Rotation: $r_i' = Q r_i \ \forall i, \ Q \in SO(3)$

$$\|r_i' - r_j'\| = \|Q r_i - Q r_j\| = \|\underbrace{Q(r_i - r_j)}_{\text{preserves lengths}}\|$$

$$= \|r_i - r_j\| = c_{ij}$$

- Now we can just reason about a single translation + rotation instead of N positions.

* Reference Frames + Attitude:

  - Set of mutually-orthogonal basis vectors that form a right-handed coordinate system

  - Two kinds:

    - "Inertial/Newtonian" ⇒ Newton's 2nd law holds

    - "Body-fixed"/"Body" ⇒ attached to a moving rigid body

  - For our purposes, ridid body always comes with a reference frame

What is Attitude?

- Rotation between body-fixed frame and inertial frame

- A Lie group $SO(3)$ (more on this later)

* How do we write down Attitude?

- Euler Angles:

    3 numbers, nonlinear kinematics/composition
    singularities at 90°, 180°

- Quaternions:

    4 numbers, 1 constraints, no singularities
    linear/bilinear kinematics + composition
    Double-cover (2 quaternions for every rotation)

- Rotation:

    9 numbers, 6 constraints, no singularities
    linear/bilinear kinematics.

- Gibbs/Rodrigues Vector:

    3 numbers, singularities at 180°,
    quadratic composition/kinematics

- Axis-Angle Vector:

    3 numbers, singularities at 180°, nonlinear
    kinematics/composition

- Modified Rodrigues Parameters:

    3 numbers, singularity at 360°,
    quartic composition/kinematics

- We will primarily use quaterions + rotation matrices.

- Don't use Euler angles!

* Rotation Matrices:

  - Linear transformation between body frame and inertial frame

$$\underset{\text{Newtonian frame}}{{}^{N}V} = Q\,\underset{\text{body frame}}{{}^{B}V}$$

  - Write $\vec{v}$ explicity in terms of components

$$\vec{V} = \begin{bmatrix} \vec{n}_1 \\ \vec{n}_2 \\ \vec{n}_3 \end{bmatrix}^T \begin{bmatrix} {}^{N}V_1 \\ {}^{N}V_2 \\ {}^{N}V_3 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix}^T \begin{bmatrix} {}^{B}V_1 \\ {}^{B}V_2 \\ {}^{B}V_3 \end{bmatrix}$$

$${}^{N}V = \begin{bmatrix} \vec{n}_1 \cdot \vec{V} \\ \vec{n}_2 \cdot \vec{V} \\ \vec{n}_3 \cdot \vec{V} \end{bmatrix} = \vec{n} \cdot \vec{V} = \vec{n} \cdot (\vec{b}^{T}\,{}^{B}V)$$

$$= \underbrace{(\vec{n} \cdot \vec{b}^{T})}_{Q}\,{}^{B}V$$

$$\Rightarrow Q = \begin{bmatrix} \vec{n}_1 \\ \vec{n}_2 \\ \vec{n}_3 \end{bmatrix} \cdot [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3] = \begin{bmatrix} \vec{n}_1 \cdot \vec{b}_1 & \vec{n}_1 \cdot \vec{b}_2 & \vec{n}_1 \cdot \vec{b}_3 \\ \vec{n}_2 \cdot \vec{b}_1 & \vec{n}_2 \cdot \vec{b}_2 & \vec{n}_2 \cdot \vec{b}_3 \\ \vec{n}_3 \cdot \vec{b}_1 & \vec{n}_3 \cdot \vec{b}_2 & \vec{n}_3 \cdot \vec{b}_3 \end{bmatrix}$$

$$= \underbrace{[{}^{N}b_1 \ {}^{N}b_2 \ {}^{N}b_3]}_{\text{B basis in N components}} = \underbrace{\begin{bmatrix} {}^{B}n_1^T \\ {}^{B}n_2^T \\ {}^{B}n_3^T \end{bmatrix}}_{\text{N basis in B components}}$$

* What is the inverse of Q

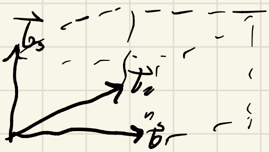
$$Q^T Q = \underbrace{\begin{bmatrix} \tilde{b}_1^T \\ \tilde{b}_2^T \\ \tilde{b}_3^T \end{bmatrix} \begin{bmatrix} \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \end{bmatrix}}_{\text{Orthogonal basis}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q^{-1} Q = I \Rightarrow \boxed{Q^T = Q^{-1}}$$

"Orthogonal Matrix

* What is the determinant of Q?

- Measures stretching
- Gives signed volume spanned by 3 columns of Q



Vol = 1

- Negative determinant implies reflection

$$\det(Q) = 1$$