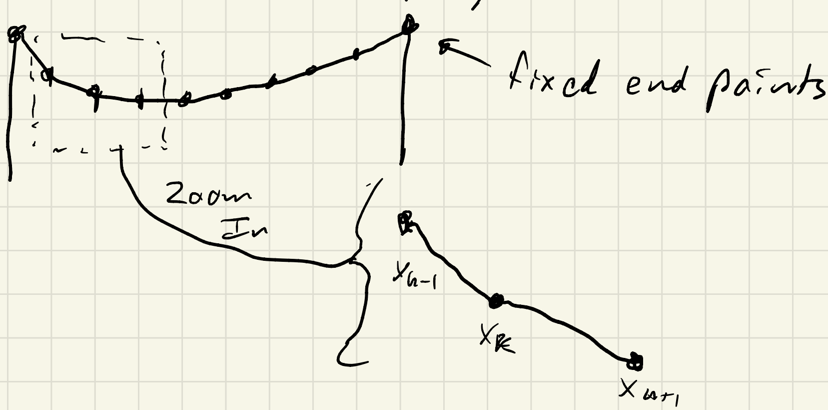
Last Time:

- Calculus of Variations
- Hanging Cable

Today:

- Calculus of Variations Pt. 2
- Dynamics from Energy
- Lagrangian Mechanics
- Least Action Principle

Another Look at the Hanging Cable:

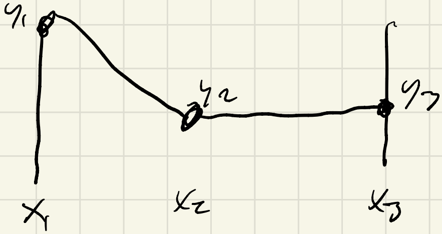


- If we zoom in, a small sub-section also has to be optimal (minimum potential energy)

- We can take any subsection and fix its endpoints to the true solution

- Looks exactly like the original problem, just smaller

- Let's look at a sub-section with only 3 discrete points:



$$s_n^2 = \Delta x^2 + \Delta y^2$$
$$= \Delta x^2 + (y_{n+1} - y_n)^2$$
$$\Rightarrow s_n = \sqrt{1 + \left(\frac{(y_{n+1} - y_n)}{\Delta x}\right)^2}\, \Delta x$$

$$u_n = \frac{1}{2}(y_{n+1} + y_n)\rho g s_n = \frac{1}{2}(y_{n+1} + y_n)\rho g \sqrt{1 + \left(\frac{y_{n+1} - y_n}{\Delta x}\right)^2}\, \Delta x$$

$$U = u(y_1, y_2) + u(y_2, y_3)$$

- Length Constraint:

$$S = s(y_1, y_2) + s(y_2, y_3) = l_{1:3}$$

- Now we only have to minimize w.r.t. y_2
- KKT Conditions:

$$\frac{\partial}{\partial y_2}\left[U(y_1, y_2, y_3) + \lambda S(y_1, y_2, y_3)\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial y_2}\left[u(y_1, y_2) + u(y_2, y_3) + \lambda s(y_1, y_2) + \lambda s(y_2, y_3)\right] = 0$$

$$\Rightarrow \underbrace{D_2 u(y_1, y_2) + D_1 u(y_2, y_3) + \lambda D_2 s(y_1, y_2) + \lambda D_1 s(y_2, y_3) = 0}_{\text{discrete Euler-Lagrange Equation}}$$

- This matches what we got last time by minimizing the whole sum.

- We got this by minimizing w.r.t. y_2, but I can use DEL to solve for any y_1, y_2, y_3 given the other two.

- Now take a continuum limit $\Delta x \to 0$

$$U(y_n, y_{n+1}) + \lambda S(y_n, y_{n+1}) =$$

$$\left(\lambda + \tfrac{1}{2}(y_n + y_{n+1})\rho g\right)\sqrt{1 + \left(\frac{y_{n+1} - y_n}{\Delta x}\right)^2}\,\Delta x$$

- Limit $\Rightarrow$ $\quad g(y, \dot{y}) = (\lambda + y\rho g)\sqrt{1 + y'^2}$

gravity, not function $g(y,\dot{y})$

$$\Rightarrow U(y_n, y_{n+1}) + \lambda S(y_n, y_{n+1}) = g\left(\frac{y_{n+1} + y_n}{2}, \frac{y_{n+1} - y_n}{\Delta x}\right)\Delta x$$

$$\frac{\partial}{\partial y_2}\left[U + \lambda S\right] = \frac{\Delta x}{2} D_1 g\left(\frac{y_1 + y_2}{2}, \frac{y_2 - y_1}{\Delta x}\right) + D_2 g\left(\frac{y_1 + y_2}{2}, \frac{y_2 - y_1}{\Delta x}\right)$$

$$+ \frac{\Delta x}{2} D_1 g\left(\frac{y_2 + y_3}{2}, \frac{y_3 - y_2}{\Delta x}\right) - D_1 g\left(\frac{y_2 + y_3}{2}, \frac{y_3 - y_2}{\Delta x}\right) = 0$$

- Divide through Δx

$$\frac{1}{2} D_1 g\left(\frac{y_1 + y_2}{2}, \frac{y_2 - y_1}{\Delta x}\right) + \frac{1}{2} D_1 g\left(\frac{y_2 + y_3}{2}, \frac{y_3 - y_2}{\Delta x}\right)$$

$$- \frac{1}{\Delta x}\left[D_2 g\left(\frac{y_2 + y_3}{2}, \frac{y_3 - y_2}{\Delta x}\right) - D_2 g\left(\frac{y_1 + y_2}{2}, \frac{y_2 - y_1}{\Delta x}\right)\right]$$

$$= 0$$

- Take $\lim \Delta x \to 0$

$$D_1 g(y, \dot{y}) - \frac{d}{dx}\left[D_2 g(y, \dot{y})\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial g}{\partial \dot{y}}\right) - \frac{\partial g}{\partial y} = 0$$

Euler-Lagrange Equation

- The fact that every sub-segment has to be optimal lets us derive these local optimality results (Euler-Lagrange equation).

- This is called the "Principle of Optimality" or "Bellman's Priciple" and underlies optimal control, dynamic programming, and RL.

- We can also view this as an optimal control problem:

$$\min_{y(x)} U = \int_0^l u(y, \dot{y})\, dx$$ ← potentia energy

$$\text{s.t. } y(0) = y_0, \quad y(l) = y_n$$ } end point constraints

$$\Downarrow$$

$$\min_{\substack{x(t)\\u(t)}} J = \int_{t_0}^{t_f} l(x, u)\, dt$$ ← cost

$$\text{s.t. } \dot{x} = f(x, u)$$ ← dynamics

$$x(0) = x_0$$ ← initial state

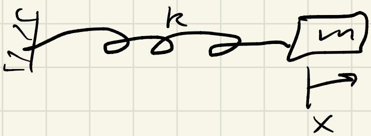
$$x(t_f) = x_g$$ ← goal state

- Correspondence:

$$U \longrightarrow J \qquad\qquad \underbrace{\dot{x} = f(x,u) = u}_{\text{dynamics constraint}}$$
$$x \longrightarrow t$$
$$y(x) \longrightarrow x(t)$$
$$\dot{y}(x) \longrightarrow u(t)$$

Dynamics from Energy:

- Let's look at a spring-mass system:



$$m\ddot{x} = -kx$$

$$\underbrace{T = \tfrac{1}{2} m \dot{x}^2}_{\text{kinetic Energy}} \qquad \underbrace{U = \tfrac{1}{2} k x^2}_{\text{Potential Energy}}$$

- No damping $\Rightarrow$ Total energy is conserved

$$T + U = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} k x^2 = \underbrace{H}_{\text{constant}}$$

$$\Rightarrow \frac{d}{dt} H = m\dot{x}\ddot{x} + kx\dot{x} = 0$$

- Divide by $\dot{x}$

$$\Rightarrow m\ddot{x} = -kx$$

- This works for any system of the form:

$$H = \frac{1}{2}\dot{q}^T \underbrace{M}_{\text{constant mass matrix}} \dot{q} + U(q)$$

$$\Rightarrow \dot{H} = \dot{q}^T M \ddot{q} + \frac{\partial U}{\partial q}\dot{q} = 0 = \dot{q}^T(M\ddot{q} + \nabla U(q))$$

$$\Rightarrow M\ddot{q} + \nabla U(q) = 0$$

- It turns out this doesn't work if $M(q)$ is not constant.
- This is not coordinate invariant
- A slightly different setup that does work:
  - Define $L = T - U$
  - knowing $M\ddot{q} + \nabla U(q) = 0$

$$\Rightarrow \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)}_{M\ddot{q}} - \underbrace{\frac{\partial L}{\partial q}}_{-\nabla U} = 0$$

- To show that this is coordinate invariant, $q' = f(q)$, $\dot{q}' = \frac{\partial f}{\partial q}\dot{q} = J(q)\dot{q}$

$$\Rightarrow L(q', \dot{q}') = L(f(q), J(q)\dot{q})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} J(q)\right) - \frac{\partial L}{\partial q} J(q) - \frac{\partial L}{\partial \dot{q}}\left(\frac{\partial}{\partial q} J(q)\, \dot{q}\right)$$

cancel

$$\Rightarrow \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) J(q) + \cancel{\frac{\partial L}{\partial \dot{q}}\left(\frac{\partial J}{\partial q}\dot{q}\right)} - \frac{\partial L}{\partial q} J(q) - \cancel{\frac{\partial L}{\partial \dot{q}}\left(\frac{\partial J}{\partial q}\dot{q}\right)} = 0$$

$$\Rightarrow \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right] J(q) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

- Euler-Lagrange equation is exactly the same in q and q' coordinates.
- We also know that this is the first-order necessary condition for an optimization problem:

$$\min_{q(t)} \underbrace{S}_{\text{"Action"}} = \int_{t_0}^{t_f} \overbrace{L(q, \dot{q})}^{\text{"Lagrangian"}}\, dt$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

- This approach to dynamics is called "Lagrangian Mechanics"

- Minimization problem is called "Least-Action principle" or "Hamilton's Principle"

- Least-Action is most general formulation of dynamics. Can derive all others by making particular choices of L, q, etc.

- Major advantage over F=ma: I can pick whatever coordinates I want for q.

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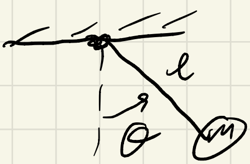
* Example: Pendulum



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = mgy = mgl(1-\cos(\theta))$$

$$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos(\theta))$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = ml\ddot{\theta} + mgl\sin(\theta) = 0$$
