Last Time:

- Rigid Bodies
- Reference Frames
- Attitude Representations
- Rotation Matrices

Today:

- Linear Systems Review
- A little Group Theory
- Rotation Matrix Kinematics
- Quaternion Geometry

Linear Systems Review:

- Scalar Case:

$$\dot{x} = ax \Rightarrow x(t) = e^{at}x_0 \,, \quad a \in \mathbb{R}$$

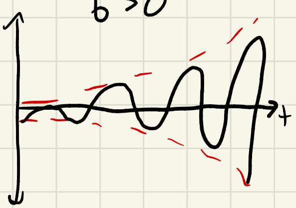
$$a > 0 \Rightarrow e^{at} \to \infty \Rightarrow \text{unstable}$$
$$a < 0 \Rightarrow e^{at} \to 0 \Rightarrow \text{stable}$$

- Complex Scalar Case:

$$\dot{x} = ax = (b + ic)x \Rightarrow x(t) = e^{(b+ci)t}x_0$$

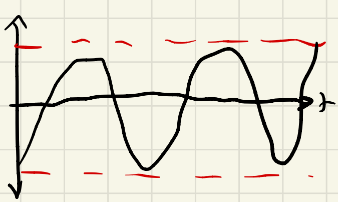
$$= \underbrace{e^{bt}}_{\text{"envelope"}} \underbrace{[\cos(ct) + i\sin(ct)]}_{\text{"oscillations"}} x_0$$

$b > 0$



$b < 0$

$b = 0$



$\Rightarrow$ real part determines stability
imaginary part determines oscillation frequency

(physically, simple harmonic oscillator)

- Some interesting 2×2 Matrices:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow J^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$\Rightarrow$ J and I act exactly like i and 1 (complex numbers)

$A = bI + cJ$ and $a = b + ci$ are $\underbrace{\text{"isomorphic"}}_{\text{"functionally identical"}}$

Unsurprisingly, $\text{eig}\left(\begin{bmatrix} b & c \\ -c & b \end{bmatrix}\right) = b \pm ci$

$$\underbrace{e^{At}}_{\text{matrix exponential}} = e^{bt}\left[\cos(ct)I + \sin(ct)J\right]$$

- General Case:

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n \Rightarrow x(t) = e^{At}x_0$$

formally define e^{At} by power series:

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \dots$$

- Assuming $A = VDV^{-1}$ (nice eigendecomposition):

$$e^{At} = I + VDV^{-1} + \frac{1}{2}\underbrace{VD\cancel{V^{-1}V}DV^{-1}}_{VD^2V^{-1}}t^2 + \frac{1}{6}VD^3V^{-1}t^3 + \dots$$

$$= Ve^{D}V^{-1}$$

$\Rightarrow$ In eigenbasis, system decouples into lots of simple harmonic oscillators

$\Rightarrow$ Eigenvalues tell us stability properties

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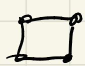
* A Little Group Theory

- A group of elements with:

  1) A multiplication operation
  2) An identity element
  3) An inverse

  that is closed under multiplication (multiplying two elements gives another group element).

- Examples:

  - Positive reals under standard multiplication
  - Integers under addition
  - Complex numbers
  - Invertible matrices $GL(n)$
  - Discret symmetry groups e.g. C_4 
  - Rotations of $\mathbb{R}^n$ $SO(n)$
  - Rigid body transformations $SE(3)$ (translation + rotation).

- All useful groups have matrix representations
- Formalizes what we saw with complex numbers and I/J matrices
- Separate representation from function/structure
- Continuous groups (as opposed to discrete like $\mathbb{Z}$ and C_4) are Lie groups
- 3D rotations are called SO(3)

$$\underbrace{\text{Special}}_{\det(Q)=1}\ \underbrace{\text{Orthogonal}}_{Q^TQ=I}\ \text{Group in}\ \underbrace{\text{3D}}$$

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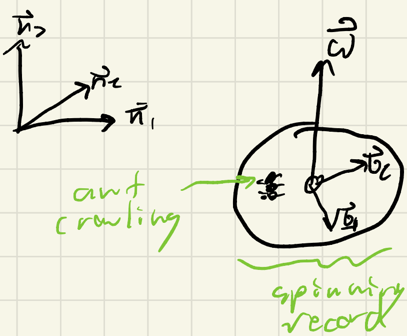
* Rotation Matrix Kinematics

- How do I integrate $\omega(t)$ from a gyro?

$$\omega(t) \xrightarrow[?]{} \dot{Q}(t) \longrightarrow Q(t)$$

- Velocities in a rotating reference frame



$${}^N\dot{X} = Q({}^B\dot{X} + {}^B\omega \times {}^BX)$$

$$\underbrace{{}^B\dot{X} = Q^{T}\,{}^N\dot{X} - {}^B\omega \times {}^BX}_{\text{"kinematic transport theorem"}}$$

(usually not written with Q)

- Think about a vector fixed in body frame

$$ {}^N x = Q\,{}^B x \Rightarrow {}^N\dot{x} = \dot{Q}\,{}^B x + \cancel{Q\,{}^B\dot{x}} $$

$$ {}^N\dot{x} = Q({}^B\omega \times {}^B x) = \dot{Q}\,{}^B x \Rightarrow \dot{Q} = Q\,{}^B\hat{\omega} $$

$$ \underbrace{\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}}_{\text{skew-symmetric} \Rightarrow \hat{\omega}^T = -\hat{\omega}}, \quad \underbrace{\hat{\omega}\, x}_{\text{“hat map”}} = \omega \times x $$

- Linear 1st order ODE

$$ \dot{Q} = Q\hat{\omega} \Longleftrightarrow \dot{x} = Ax $$

- For constant ω:

$$ Q(t) = Q_0 \underbrace{e^{\hat{\omega} t}}_{\text{matrix exponential}} $$

- For small Δt

$$ e^{\hat{\omega}\Delta t} \approx I + \hat{\omega}\Delta t $$

$$ \Rightarrow e^{\hat{\omega}\Delta t} x \approx x + \Delta t(\omega \times x) $$

- $\omega\Delta t$ is an axis-angle vector

$$ \phi = \omega \Delta t $$

- Exponential maps from axis-angle vectors to rotation matrices

- Useful for easily sampling rotation matrices

- You can also go the other way!

$$Q = e^{\hat{\phi}} \Longleftrightarrow \hat{\phi} = \log(Q)$$

* A Little More Group Theory

  - Axis-angle vectors / skew-symmetric matrices are Lie algebra $\underbrace{so(3)}_{\text{(lower case)}}$ corresponding to $SO(3)$

- From Taylor series:

$$e^{\hat{\omega} t} \approx \underbrace{I + \hat{\omega} t}_{\text{Lie algebra is the linearization of the group at the identity}} + \ldots$$

- While group is not a vector space, the Lie algebra is

- The Lie algebra $\longleftrightarrow$ group connection lets us use standard vector math ideas and translate them to the group.

* Quaternions

  - Standard rotation representation used in simulation.

  - No singularities + more efficient than rotation matrices

- Geometry:

  - Set of all possible axis-angle vectors

$$\|\phi\| \leq \pi \quad (\text{ball in } \mathbb{R}^3)$$

- Visualize as disk in 2D:

- There is a discontinuous jump at $\pm\pi$
- We want to get rid of the jump
- Stretch disk up out of plane into hemisphere

- Make a copy, flip it, glue it on underneath to form a sphere

- Now instead of jumping we can continue smoothly onto "southern hemisphere".