Last Time:

- Local Stability
- Taylor Integrators
- Runge-Kutta

Today:

- Higher-Order RK Methods
- Stiffness + Stability of RK methods

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* Higher-Order RK Methods:

- We've seen how to build 2^{nd}-order RK methods. The general strategy is:

  1) Choose a number of stages
  2) Choose implicit/explicit
  3) Write down integrator with undetermined weights
  4) Write down Taylor expansion of integrator
  5) Choose weights to achieve desired order

- Steps 4-5 become very messy past ~4^{th} order

- Weight choices are not unique - can optimize for accuracy, efficiency, etc.

- Most famous explicit RK methods are 4^{th} order w/ 4 stages:

$$\begin{aligned}
f_1 &= f(x_n) \\
f_2 &= f(x_n + ha_{21}f_1) \\
f_3 &= f(x_n + ha_{31}f_1 + ha_{32}f_2) \\
f_4 &= f(x_n + ha_{41}f_1 + ha_{42}f_2 + ha_{43}f_3)
\end{aligned}$$

$$x_{n+1} = x_n + h[b_1f_1 + b_2f_2 + b_3f_3 + b_4f_4]$$

- Can organize coefficients into a matrix A and vector b called "Butcher tableau"

$$\text{times when } \dot{x}=f(t,x)\left\{\begin{array}{c|cccc}
0 & & & & \\
c_2 & a_{21} & & & \\
c_3 & a_{31} & a_{32} & & \\
c_4 & a_{41} & a_{42} & a_{43} & \\
\hline
 & b_1 & b_2 & b_3 & b_4
\end{array}\right.$$

- A is lower triangular for explicit methods, full for implicit.

- Two examples of 4^{th} order methods:

$$\underbrace{\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
1/2 & 0 & 1/2 & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}}_{\text{"Classic RK4"}}
\qquad
\underbrace{\begin{array}{c|cccc}
0 & & & & \\
1/3 & 1/3 & & & \\
2/3 & -1/3 & 1 & & \\
1 & 1 & -1 & 1 & \\
\hline
 & 1/8 & 3/8 & 3/8 & 1/8
\end{array}}_{\text{"3/8 rule"}}$$

- Also common to use a pair of RK methods of different orders (e.g. 2/3, 4/5, 7/8) together and adapt h to ensure solutions match within some tolerance. Provides automatic error control.

* Stiff ODEs and Stability

  - Some systems are difficult to simulate: they require very small steps to not "blow up", even if the true system is smooth + stable

  - Often due to large mechanical stiffness/damping but not always

  - No precise mathematical definition

  - Let's look at a scalar test problem:

$$\dot{x} = ax \quad , \quad a \in \mathbb{C}$$

  - Remember: system is stable whenever $Re[a] < 0$ and exact solution is:

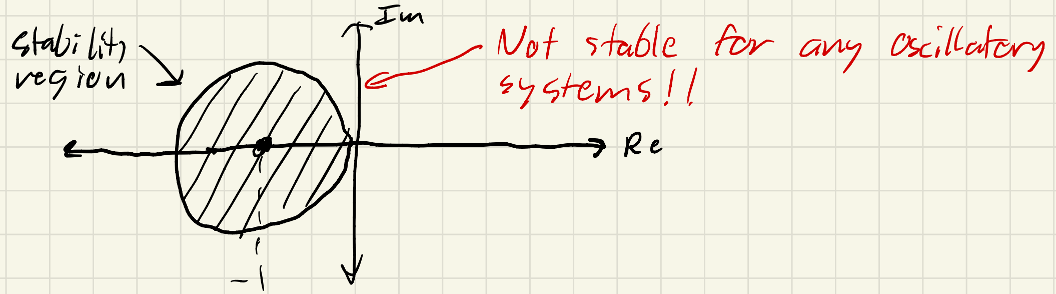
$$x(t) = e^{at} x_0$$

- Let's plug this into our RK methods and check stability of the discretized system.

- Explicit Euler:

$$x_{n+1} = x_n + h a x_n = (1 + ha) x_n$$

$$\text{stability} \Rightarrow |1 + ha| < 1$$

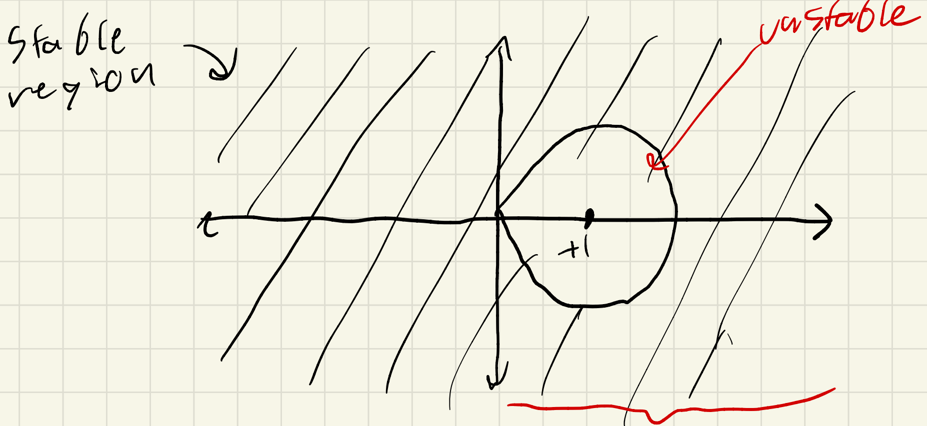


- Implicit Euler:

$$x_{n+1} = x_n + ha x_{n+1} \Rightarrow (1-ha)x_{n+1} = x_n$$

$$\Rightarrow x_{n+1} = \left(\frac{1}{1-ha}\right) x_n$$

$$\text{stability} \Rightarrow \left|\frac{1}{1-ha}\right| < 1 \Rightarrow |1-ha| > 1$$



artificially stabilizes unstable systems! (adds damping)

* Implicit Midpoint

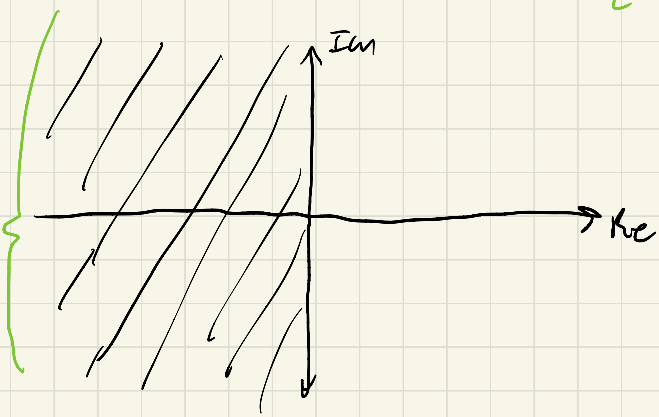
$$x_{n+1} = x_n + h\left(\tfrac{1}{2} a x_n + \tfrac{1}{2} a x_{n+1}\right) = \left(1 + \tfrac{1}{2} ha\right) x_n + \tfrac{1}{2} a h x_{n+1}$$

$$\Rightarrow \quad x_{n+1} = \left(\frac{1 + \frac{1}{2} ha}{1 - \frac{1}{2} ha}\right) x_n$$

$$\text{Stability} \Rightarrow \left| \frac{1 + \frac{1}{2} ha}{1 - \frac{1}{2} ha} \right| < 1$$

$$\Rightarrow \underbrace{\left|1 + \tfrac{1}{2} ha\right| < \left|1 - \tfrac{1}{2} ha\right|}_{\text{true whenever } Re[ha] < 0}$$

Stable region is $Re[a] < 0$ Just like the real system!



- If an integrator's stability region contains the entire LHP it is called "A-stable" (A = "absolute")

- Can show that explicit RK methods can <u>not</u> be A-stable

- Stability condition can be written in terms of Butcher tableau:

$$\frac{\det(I - zA + z e b^T)}{\det(I - zA)} < 1$$

$e = [1\,1\,1 \ldots 1]^T$

$z \in \mathbb{C} = ha$

* Definitive reference:

Hairer, Norsett, Wanner, "Solving ordinary differential equations."