Last Time:

- Discrete mechanics for rigid bodies
- Multi-body dynamics in maximal coordinates

Today:

- Contact dynamics
- Discrete Mechanics with Impacts
- Coulomb Friction

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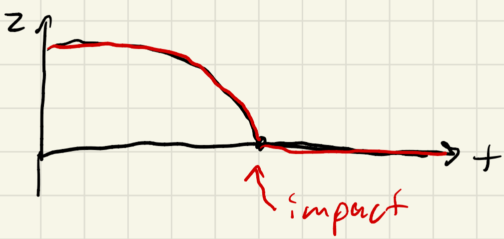
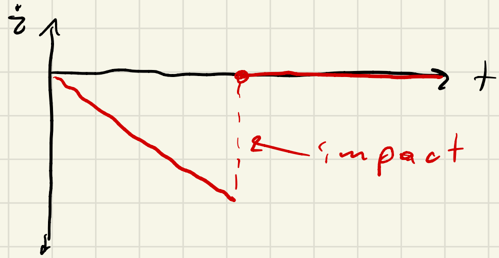
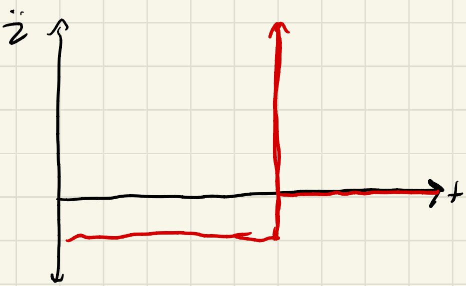
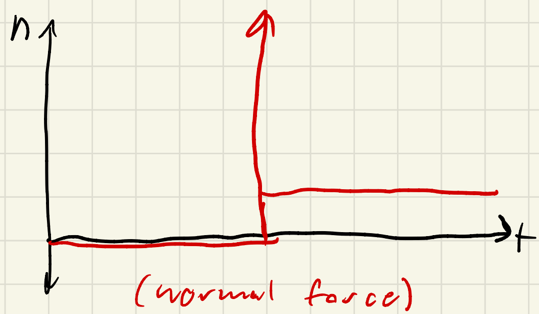
Contact Dynamics:

- Why is this hard?
- Imagine a falling particle that hits the ground (no friction)




- Impacts (and stiction) cause velocity discontinuities
For ideal rigid bodies this leads to infinit forces, accelerations.

- Strictly speaking, F=ma isn't well defined in these scenarios.

- In reality, things are not perfectly rigid, but practically this is still difficult.

→ Makes modeling, simulation, control, learning very difficult.

- Three broad classes simulation techniques:

1) Smooth Contact Models

   - Smooth all contact forces with e.g. nonlinear spring damper model

   - Pros: Easy to implement, differentiable

   - Cons: Not very accurate: inter-penetration + creep, stiff ODEs

   - MuJoCo does this

2) Hybrid/Event Based Methods

   - Explicitly detect impact events while integrating ODEs. Perform discontinuous jump update, then proceed with smooth integration.

   - Pros: Can use standard ODE tools, not stiff, captures qualitatively correct physics.

   - Cons: Scales poorly with number of contacts, issues with simultaneous impacts, not differentiable.

- Common in control

3) Time-Stepping Methods

- Solve an optimization problem at each time step to compute contact forces that satisfy interpenetration constraints.

- Pros: Scales well in number of contact points, No issues with simultaneous impacts, qualitatively correct physics.

- Cons: Computationally expensive. Not differentiable.

- Used by Bullet.

* We're going to focus on 3)

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Discrete Mechanics with Impacts:

- Review KKT conditions for an inequality constrained optimization problem:

$$\left.\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & c(x) \geq 0 \end{aligned}\right\} \Rightarrow \underbrace{L(x,\lambda) = f(x) - \lambda^T c(x)}_{\text{"Lagrangian" from optimization}}$$

- KKT conditions:

$$\frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} - \lambda^T \frac{\partial c}{\partial x} = 0 \quad \text{(stationarity)}$$

"complementarity condition"

$$\begin{cases} c(x) \geq 0 & \text{(primal feasibility)} \\ \lambda \geq 0 & \text{(dual feasibility)} \\ \lambda \odot c(x) = 0 & \text{(compl. slackness)} \end{cases}$$

- Complementarity conditions define switching behavior for active vs. inactive constraints.

- Now let's apply this to the falling particle

- Let's define a "signed distance function" that returns distances between closest points. For us its just the height of the particle


$$\phi(q) = [0 \ 1]\, q$$

- Note $\phi(q) = 0$ indicates contact and $\phi(q) < 0$ indicates interpenetration (bad).

- Recall Lagrangian for a point particle:

$$L(q, \dot{q}) = \frac{1}{2} m \dot{q}^T \dot{q} - mg [0 \ 1]\, q$$

$$L_d(q_n, q_{n+1}) = h L\left(\frac{q_{n+1} + q_n}{2}, \frac{q_{n+1} - q_n}{h}\right)$$

- Now we write down least action s.t. interpenetration constraint:

$$\min_{q_{1:N}} \sum_{n=1}^{N-1} L_d(q_n, q_{n+1})$$

$$\text{s.t.} \quad \phi(q_n) \geq 0$$

- Looks almost like joint constraints. KKT conditions:

$$D_2 L_d(q_{n-1}, q_n) + D_1 L_d(q_n, q_{n+1}) + h n_n^T D_1 \phi(q_n) = 0$$

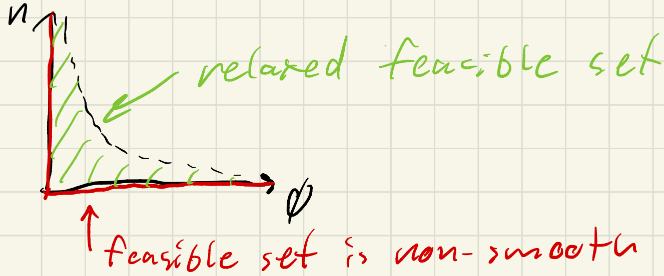
stay above ground $\quad \phi(q_{n+1}) \geq 0$

normal force can only push $\quad n_n \geq 0$

no force unless in contact $\quad n_n \phi(q_n) = 0$

★ Quick Notes on Solving this:

- Feasible region for complementarity:




- Relax with a slack variable:

$$n_n \phi(q_n) = 0 \Rightarrow n_n \phi_n \leq s_n \quad , \quad s_n \geq 0$$