Last Time:

- History
- Newtonian Review
- Coordinate Invariance
- Pendulum

Today:

- State Space
- Euler Integration
- Energy
- Lyapunov Stability

---

* State Space

- Pendulum from last time:

$$\ddot{\theta} = \frac{-g}{l} \sin(\theta)$$

- This is a 2^{nd}-order ODE:

$$\ddot{\theta} = f(\theta, \dot{\theta})$$

- To predict the system's motion we need θ and $\dot{\theta}$. Let's combine these into a state vector:

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \quad \dot{x} = f(x)$$

- We can always write ODEs in this 1^{st} order form

- This is the form that standard ODE solvers use.

* Our First Simulator

- We usually can't solve these ODEs analytically, so we need numerical methods.

- The simplest one is Explicit Euler

$$x_{n+1} \approx x_n + \underbrace{h}_{\text{(small) time step}} \dot{x}_n = x_n + h f(x_n)$$

- Intuition: take small steps along tangent vector to solution curve.

* Take-Aways from Simulation

- Simulators can produce unphysical behavior

- Structural things can change about the system due to discretization.

- You should probably never use explicit Euler

* Energy

- A scalar quantity that comes in 2 flavors

- Kinetic (how much I'm moving)

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v} \quad \text{for a particle}$$

- Potential (how much could I move based on where I am)

$$U_{grav} = mg\underbrace{z}_{\text{height}} \quad \text{for constant gravity}$$

- For a particle $m = 1$ that falls $\Delta z = 1$ from rest $\Rightarrow$ $\Delta U = -g$, gain $T = g$ kinetic

$$\Rightarrow T = g = \frac{1}{2} m v^2 \Rightarrow v = \sqrt{2g}$$

$$U_{spring} = \frac{1}{2} \underbrace{k}_{\text{stiffness}} x^2 \quad \text{for a spring}$$

- Note that $U(x)$ is only ever a function of position (never velocity)

- Total Energy

$$E = T + U$$

- Forces can be calculated from a potential

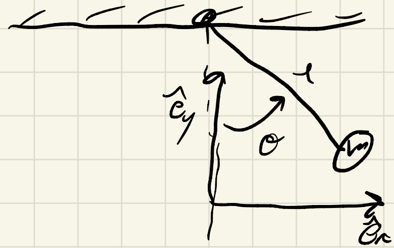
$$F = -\nabla U(x)$$

$$\Rightarrow F_{spring} = \frac{-\partial}{\partial x} U_{spring} = -kx$$

- Forces that can be derived from a potential are "conservative" since they don't change total energy (just trade for kinetic).

- Friction, damping, and drag are all non-conservative: they depend on velocity and change E.

* Lyapunov Stability



- Pendulum Energy:

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v} = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = mgy = mgl(1-\cos(\theta))$$

$$\Rightarrow E = T + U = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1-\cos(\theta))$$

- Look at $\dot{E}$

$$\dot{T} = \frac{\partial T}{\partial \dot{\theta}} \ddot{\theta} = m l^2 \dot{\theta} \underbrace{\left[-\frac{g}{l} \sin(\theta)\right]}_{\text{plug in dynamics for } \ddot{\theta}} = -mgl\sin(\theta)\dot{\theta}$$

$$\dot{U} = \frac{\partial U}{\partial \theta} \dot{\theta} = mgl \sin(\theta) \dot{\theta}$$

$$\dot{E} = \dot{T} + \dot{U} = 0 \Rightarrow \text{energy is conserved}$$

- Based on this, we can bound the region the system can move in in the state space.

Given E_0 at $t = 0$

$$T = \frac{1}{2} m l^2 \dot{\theta}^2 \Rightarrow \|\dot{\theta}(t)\| \leq \sqrt{\frac{2}{ml} E_0} \quad \underbrace{\forall t}_{\text{for all time}}$$

$$U = mgl(1-\cos(\theta)) \Rightarrow \|\theta(t)\| \leq \cos^{-1}\left(\frac{E_0}{mgl} - 1\right) \quad \forall t$$

- When a system is guaranteed to stay within some region we call it "Lyapunov stable"

- We can generalize/formalize this:

  * If we can find a function $V(x) : \mathbb{R}^n \to \mathbb{R}$

    1) $V(x) \geq 0 \quad \forall x$, $V(0) = 0$

    2) $V(x) \to \infty$ as $\|x\| \to \infty$

    3) $\dot{V}(x) = \frac{\partial V}{\partial x} \underbrace{f(x)}_{\dot{x}} \leq 0 \quad \forall x$

  $\Rightarrow$ The system is stable about the origin

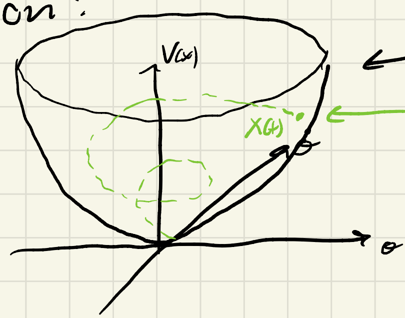
  - We can always change coordinates to make some other point x^* the origin:

$$\tilde{x} = x - x^*$$

  - $\dot{V}(x) = 0 \Rightarrow$ "Lyapunov stable"

  - $\dot{V}(x) < 0 \Rightarrow$ "Asymptotically stable"

- Intuition:



$\leftarrow$ V(x) is a bowl

$\leftarrow$ x(t) moves down hill

* Pendulum With Euler Integration

- We can do the same analysis in discrete time by looking at ΔE (or ΔV) instead of $\dot{E}$ (or $\dot{V}$):

$$\Delta T_n = T_{n+1} - T_n = \frac{1}{2} m l^2 (\dot{\theta}_{n+1}^2 - \dot{\theta}_n^2)$$

$$\Delta U_n = U_{n+1} - U_n = mgl(\cos(\theta_n) - \cos(\theta_{n+1}))$$

- Plug in Euler-integrated dynamics for x_{n+1}

$$\begin{bmatrix} \theta_{n+1} \\ \dot{\theta}_{n+1} \end{bmatrix} = \begin{bmatrix} \theta_n + h\dot{\theta}_n \\ \dot{\theta}_n - h\frac{g}{l}\sin(\theta_n) \end{bmatrix}$$

$$\Rightarrow \Delta T_n = \frac{1}{2} m l^2 \left(\cancel{\dot{\theta}_n^2} - 2h\frac{g}{l}\sin(\theta_n)\dot{\theta}_n + h^2 \frac{g^2}{l^2}\sin^2(\theta_n) - \cancel{\dot{\theta}_n^2}\right)$$

$$= -hmgl\sin(\theta_n)\dot{\theta}_n + \frac{1}{2}h^2 m g^2 \sin^2(\theta_n)$$

$$\Delta U_n = mgl[\cos(\theta_n) - \underbrace{\cos(\theta_n + h\dot{\theta}_n)}_{\substack{\text{Taylor expand assuming } h \ll 1 \\ \approx \cos(\theta_n) - \sin(\theta_n) h\dot{\theta}_n}}]$$

$$= hmgl\sin(\theta_n)\dot{\theta}_n$$

$$\Rightarrow \Delta E_n = \Delta T_n + \Delta U_n = \underbrace{\frac{1}{2}h^2 m g^2 \sin^2(\theta_n)}_{\text{Always Positive!}} \geq 0$$

$\Rightarrow$ The discretized system is <u>always</u> unstable!

★ Local Stability

- Often we can't find a Lyapunov function, but we can still get weaker local stability results.

- We can do this around points, trajectories, and orbits

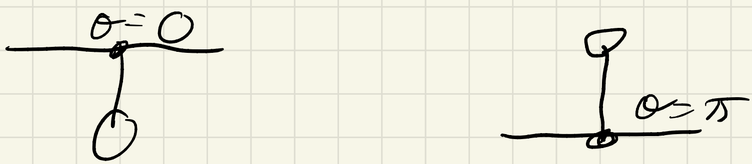
- Most commonly look at equilibrium points

- Equilibrium $\Rightarrow$ $\dot{x} = f(x) = 0$ (system won't move)

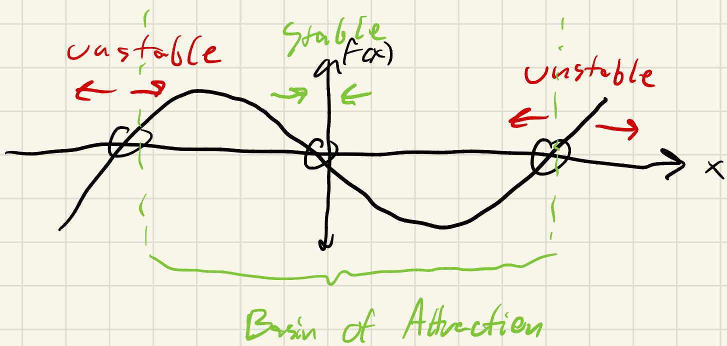
- For pendulum we have 2:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ if } x^* = \begin{bmatrix} 0 \\ \pi n \end{bmatrix}, \ k \in \mathbb{Z}$$

integer



- Let's look at 1D system ($x \in \mathbb{R}$)



$\left.\frac{\partial f}{\partial x}\right|_{x^*} < 0 \Rightarrow$ x^* is locally stable

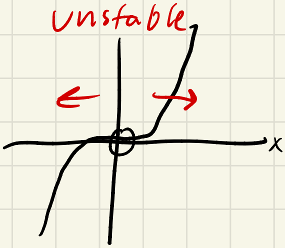
$\left.\frac{\partial f}{\partial x}\right|_{x^*} > 0 \Rightarrow$ x^* is unstable

$\left.\frac{\partial f}{\partial x}\right|_{x^*} = 0 \Rightarrow$ inconclusive

- Examples of inconclusive systems:

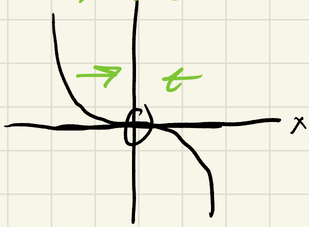
$f(x) = x^3$

unstable



$f(x) = -x^3$

stable



$x^* = 0$

$\left.\frac{\partial f}{\partial x}\right|_{x^*} = 0$