Last Time:

- Euler Integration
- Energy
- Lyapunov Stability
- Local Stability

Today:

- Finish Local Stability
- Taylor Integration
- Runge-Kutta Methods

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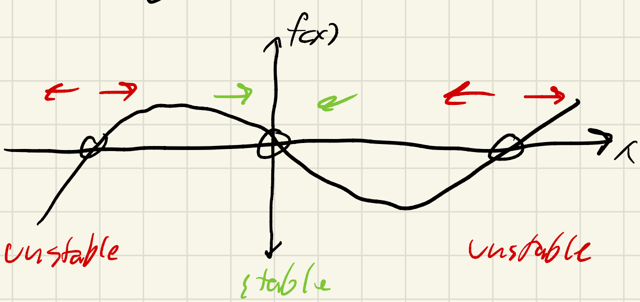
* Local Stability

- 1D Picture



$$\left.\frac{\partial f}{\partial x}\right|_{x^*} < 0 \Rightarrow \text{locally stable}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x^*} > \Rightarrow \text{unstable}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x^*} = 0 \Rightarrow \text{inconclusive}$$

- This picture generalizes easily to $\mathbb{R}^n$
$\frac{\partial f}{\partial x}$ is now a Jacobian matrix $\in \mathbb{R}^{n\times n}$ and we look at its eigenvalues:

if <u>all</u> $Re[eig(\frac{\partial f}{\partial x}|_{x^*})] < 0 \Rightarrow$ locally stable

if <u>any</u> $\quad$ '' $\quad > 0 \Rightarrow$ unstable

if <u>any</u> $\quad$ '' $\quad = 0 \Rightarrow$ inconclusive

- For the Pendulum

$$f(x) = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin(\theta) \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(\theta) & 0 \end{bmatrix}$$

- For the upward ($\theta = \pi$) equilibrium we get:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix} \Rightarrow eig\left(\frac{\partial f}{\partial x}\right) = \pm\sqrt{g/\ell} \Rightarrow \text{unstable}$$

- For the downward ($\theta = 0$) equilibrium we get:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -g/\ell & 0 \end{bmatrix} \Rightarrow eig\left(\frac{\partial f}{\partial x}\right) = 0 \pm i\sqrt{g/\ell} \Rightarrow \text{inconclusive}$$

(but we already know its Lyapunov stable)

★ Local Stability in Discrete Time:

- We can do similar analysis in discrete time
- For a discrete-time system:

$$x_{n+1} = f(x_n)$$

- Equilibrium $\Rightarrow$ $x^* = f(x^*)$ (fixed point)
- Think about an itterated map

$$x_N = f(f(f(f \cdots f(x_0)$$

- Stable $\Rightarrow$ $\lim_{N\to\infty} x_N = f(f(f \cdots f(x^* + \underbrace{\varepsilon}_{\text{"small"}}) = x^*$

- Assuming small ε, use the chain rule:

$$f(x^* + \varepsilon) \approx f(x^*) + \frac{\partial f}{\partial x}\Big|_{x^*} \varepsilon$$

$$\Rightarrow x_N \approx x^* + \left(\frac{\partial f}{\partial x}\Big|_{x^*}\right)^{N-1} \varepsilon = x^*$$

$$\lim_{N\to\infty} \left(\frac{\partial f}{\partial x}\Big|_{x^*}\right)^{N-1} \varepsilon = 0 \quad \forall \varepsilon$$

$\Rightarrow$ if <u>all</u> $\left|\operatorname{eig}\left(\frac{\partial f}{\partial x}\Big|_{x^*}\right)\right| < 1 \Rightarrow$ locally stable

if <u>any</u> " $> 1 \Rightarrow$ unstable

if <u>any</u> " $= 1 \Rightarrow$ inconclusive

* Higher-Order Integrators

- Explicit Euler is equivalent to taking a 1st order Taylor expansion for the solution $x(t_{n+1})$ about $x(t_n)$:

$$x_{n+1} = x(t_n + h) \approx x_n + \underbrace{\left.\frac{dx}{dt}\right|_{t_n} h}_{hf(x_n)} + \ldots$$

- An obvious way to get a better answer is to take more terms in this expansion:

$$x_{n+1} \approx x(t_n) + h\left.\frac{dx}{dt}\right|_{x_n} + \frac{h^2}{2}\left.\frac{d^2x}{dt^2}\right|_{t_n} + \frac{h^3}{6}\frac{d^3x}{dt^3} + \ldots$$

- Expanding the derivatives in terms of $f(x)$:

$$\frac{dx}{dt} = f(x)$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}f(x) = \frac{\partial f}{\partial x}\frac{dx}{dt} = \frac{\partial f}{\partial x}f(x)$$

$$\frac{d^3x}{dt^3} = \frac{d}{dt}\left[\frac{\partial f}{\partial x}f(x)\right] = \frac{\partial f}{\partial x}\frac{\partial f}{\partial x}f(x) + \left[\underbrace{\frac{\partial^2 f}{\partial x^2}}_{\text{3rd-rank tensor}}f(x)\right]f(x)$$

- In theory, as long as $f(x)$ is smooth, we can compute solutions to arbitrary accuracy this way
- Not very popular due to need to compute high-order derivatives of $f(x)$ (high-rank tensors)

- Definition: "Order of accuracy" of an ODE solver is the number of Taylor-series terms that its approximate solution matches.

$\Rightarrow$ Euler is a 1^{st}-order method

* Runge-Kutta Methods:

  - We want to achieve high order without computing higher derivatives of $f(x)$

  - Key idea: use multiple evaluations of $f(x)$ over the time step to fit a polynomial to $x(t)$

  - Number of $f(x)$ evaluations per time step = ~~number~~ number of "stages"

- What's the best we can do with one stage (linear interpolation over the time step)?

$$x_{n+1} \approx x_n + hf((1-\alpha)x_n + \alpha x_{n+1}), \quad 0 \leq \alpha \leq 1$$

* $\alpha = 0 \Rightarrow$ explicit (forward) Euler

* $\alpha = 1 \Rightarrow$ implicit (backward) Euler

- Let's Taylor expand about x_n:

$$x_{n+1} \approx x_n + hf((1-\alpha)x_n + \alpha[x_n + hf(x_n) + \tfrac{h^2}{2}\tfrac{\partial f}{\partial x}f(x_n)])$$

$$\approx x_n + hf(x_n) + \alpha h^2 \frac{\partial f}{\partial x} f(x) + O(h^3)$$

$\Rightarrow \alpha = 1/2$ gives a 2^{nd}-order method!

* Implicit Midpoint:

$$x_{n+1} = x_n + hf(\tfrac{1}{2}x_n + \tfrac{1}{2}x_{n+1})$$

- x_{n+1} appears on RHS inside $f(x)$ ⇒ "implicit"
- We need to solve for x_{n+1} with Newton's method:

$$r(x_{n+1}) = x_{n+1} - x_n - hf(\tfrac{1}{2}x_n + \tfrac{1}{2}x_{n+1}) = 0$$

$$r(x_{n+1} + \Delta x_{n+1}) \approx r(x_{n+1}) + \frac{\partial r}{\partial x}\Delta x_{n+1} = 0$$

$$\Rightarrow \Delta x_{n+1} = -\left(\left.\frac{\partial r}{\partial x}\right|_{x_{n+1}}\right)^{-1} r(x_{n+1})$$

$$x_{n+1} \leftarrow x_{n+1} + \Delta x_{n+1}$$

(repeate until convergence)

- Downside: solving root-finding problem is expensive.

* Explicit Midpoint

  - Avoid root finding
  - Lets evaluate the midpoint using an Euler step:

$$x_{n+1} = x_n + hf(\underbrace{x_n + \tfrac{h}{2}f(x_n)}_{\text{midpoint approximation}})$$

- Taylor expand:

$$x_{n+1} \approx x_n + hf(x_n) + \frac{h^2}{2}\frac{\partial f}{\partial x} f(x) + O(h^3)$$

$\Rightarrow$ Explicit midpoint is 2^{nd} order

- Explicit midpoint has 2 stages.