

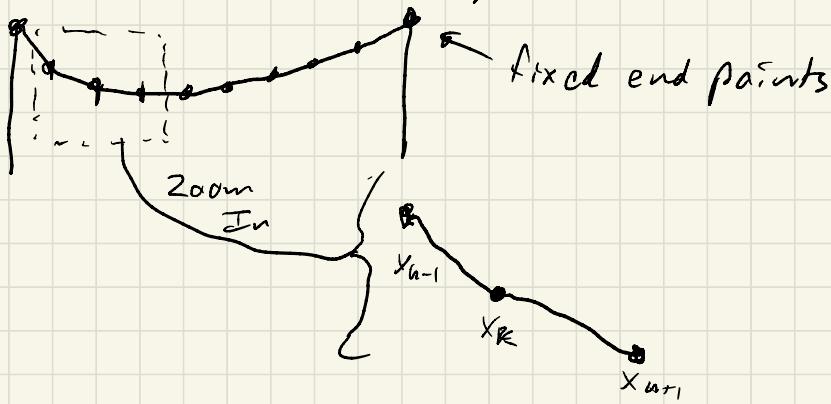
Last Time:

- Calculus of Variations
- Hanging Cable

Today:

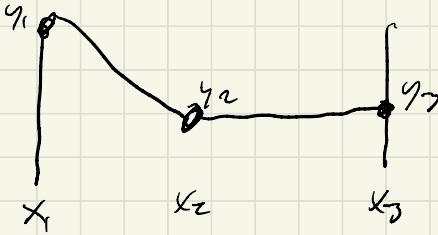
- Calculus of Variations Pt. 2
 - Dynamics from Energy
 - Lagrangian Mechanics
 - = Least Action Principle
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Another Look at the Hanging Cable:



- If we zoom in, a small sub-section also has to be optimal (minimum potential energy)
- We can take any subsection and fix its end points to the true solution
- Looks exactly like the original problem, just smaller

- Let's look at a sub-section with only 3 discrete points:



$$\begin{aligned} S_h^2 &= \Delta x^2 + \Delta y^2 \\ &= \Delta x^2 + (y_{n+1} - y_n)^2 \\ \Rightarrow S_n &= \sqrt{1 + \left(\frac{(y_{n+1} - y_n)}{\Delta x}\right)^2} \Delta x \end{aligned}$$

$$U_n = \frac{1}{2}(\gamma_{n+1} + \gamma_n) \rho g s_n = \frac{1}{2}(\gamma_{n+1} + \gamma_n) \rho g \sqrt{1 + \left(\frac{\gamma_{n+1} - \gamma_n}{\alpha x}\right)^2} \Delta x$$

$$V = U(y_1, y_2) + U(y_2, y_3)$$

- ## - Length Constraint:

$$S = S(\gamma_1, \gamma_2) + S(\gamma_2, \gamma_3) = l_{1:3}$$

- Now we only have to minimize w.r.t. y_2

- ## - KKT Conditions:

$$\frac{\partial}{\partial y_2} [V(\gamma_1, y_2, y_3) + \lambda S(\gamma_1, y_2, y_3)] = 0$$

$$\Rightarrow \frac{\partial}{\partial y_2} \left[U(y_1, y_0) + U(y_2, y_2) + \lambda S(y_1, y_2) + \lambda S(y_2, y_3) \right] = 0$$

$$\Rightarrow D_2 U(y_1, y_2) + D_1 U(y_2, y_3)$$

$$+ \lambda D_2 S(y_1, y_2) + \lambda D_1 S(y_2, y_3) = 0$$

discrete Euler-Lagrange Equations

- This matches what we got last time by maximizing the whole sum.
- We got this by minimizing $\text{var. } \gamma_2$, but I can use $D_{\bar{\gamma}}^L$ to solve for any $\gamma_1, \gamma_2, \gamma_3$ & even the other two.
- Now take a continuum limit $\Delta x \rightarrow 0$

$$U(\gamma_n, \gamma_{n+1}) + \lambda S(\gamma_n, \gamma_{n+1}) =$$

$$(A + \frac{1}{2}(\gamma_n + \gamma_{n+1})\rho g) \sqrt{1 + \left(\frac{\gamma_{n+1} - \gamma_n}{\Delta x}\right)^2} \Delta x$$

$$- \text{Limit} \Rightarrow g(\gamma, \dot{\gamma}) = (A + \gamma \rho g) \sqrt{1 + \gamma'^2}$$

gravity, cost function $g(\gamma, \dot{\gamma})$

$$\Rightarrow U(\gamma_n, \gamma_{n+1}) + \lambda S(\gamma_n, \gamma_{n+1}) = g\left(\frac{\gamma_{n+1} + \gamma_n}{2}, \frac{\gamma_{n+1} - \gamma_n}{\Delta x}\right) \Delta x$$

$$\begin{aligned} \frac{\partial}{\partial \gamma_n} [U + \lambda S] &= \frac{\Delta x}{2} D_1 g\left(\frac{\gamma_1 + \gamma_2}{2}, \frac{\gamma_2 - \gamma_1}{\Delta x}\right) + D_2 g\left(\frac{\gamma_1 + \gamma_2}{2}, \frac{\gamma_2 - \gamma_1}{\Delta x}\right) \\ &\quad + \frac{\Delta x}{2} D_1 g\left(\frac{\gamma_2 + \gamma_3}{2}, \frac{\gamma_3 - \gamma_2}{\Delta x}\right) - D_2 g\left(\frac{\gamma_2 + \gamma_3}{2}, \frac{\gamma_3 - \gamma_2}{\Delta x}\right) = 0 \end{aligned}$$

- Divide through Δx

$$\frac{1}{2} D_1 g\left(\frac{\gamma_1 + \gamma_2}{2}, \frac{\gamma_2 - \gamma_1}{\Delta x}\right) + \frac{1}{2} D_2 g\left(\frac{\gamma_2 + \gamma_3}{2}, \frac{\gamma_3 - \gamma_2}{\Delta x}\right)$$

$$- \frac{1}{\Delta x} \left[D_2 g\left(\frac{\gamma_1 + \gamma_2}{2}, \frac{\gamma_2 - \gamma_1}{\Delta x}\right) - D_2 g\left(\frac{\gamma_2 + \gamma_3}{2}, \frac{\gamma_3 - \gamma_2}{\Delta x}\right) \right]$$

$$= 0$$

- Take the lim $\Delta x \rightarrow 0$

$$D_1 g(\gamma, \dot{\gamma}) - \frac{d}{dx} [D_2 g(\gamma, \dot{\gamma})] = 0$$

$$\Rightarrow \underbrace{\frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial \dot{y}} \right) - \frac{\partial \mathcal{L}}{\partial y}}_{\text{Euler-Lagrange Equation}} = 0$$

Euler-Lagrange Equation

- The fact that every sub-segment has to be optimal lets us derive the (local) optimality results (Euler-Lagrange equation).
- This is called the "Principle of Optimality" or "Bellman's Principle" and underlies optimal control, dynamic programming, and RL.
- We can also view this as an optimal control problem:

$$\min_{y(x)} U = \int_0^l u(y, \dot{y}) dx \quad \leftarrow \text{potential energy}$$

s.t. $y(0) = y_0$

$$y(l) = y_n \quad \leftarrow \begin{array}{l} \text{end point} \\ \text{constraints} \end{array}$$



$$\min_{\substack{x(t) \\ u(t)}} J = \int_{t_0}^{t_f} l(x, u) dt \quad \leftarrow \text{cost}$$

s.t. $\dot{x} = f(x, u)$ \leftarrow dynamics

$x(0) = x_0$ \leftarrow initial state

$x(t_f) = x_f$ \leftarrow goal state

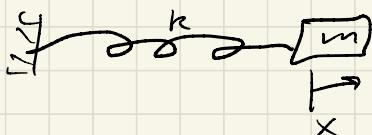
- Correspondence:

$$\begin{array}{ccc} U & \longrightarrow & T \\ x & \longrightarrow & t \\ y(t) & \longrightarrow & X(t) \\ \dot{y}(t) & \longrightarrow & U(t) \end{array}$$

$\dot{x} = f(x, u) = U$
dynamics
constraint

Dynamics from Energy:

- Let's look at a spring-mass system:



$$m\ddot{x} = -kx$$

$$T = \frac{1}{2}m\dot{x}^2$$

Kinetic Energy

$$U = \frac{1}{2}kx^2$$

Potential Energy

- No damping \Rightarrow Total energy is conserved

$$T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = H$$

H constant

$$\Rightarrow \frac{dH}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = 0$$

- Divide by \dot{x}

$$\Rightarrow m\ddot{x} = -kx$$

- This works for any system of the form:

$$H = \frac{1}{2} \dot{q}^T M \dot{q} + U(q)$$

$\underbrace{\phantom{M \dot{q}}}_{\text{constant mass matrix}}$

$$\Rightarrow \ddot{H} = \ddot{q}^T M \ddot{q} + \frac{\partial U}{\partial q} \dot{q} = 0 = \ddot{q}^T (M \ddot{q} + \nabla U(q))$$

$$\Rightarrow M \ddot{q} + \nabla U(q) = 0$$

- It turns out this doesn't work if $M(q)$ is not constant.

- This is not coordinate invariant

- A slightly different setup that does work:

- Define $L = T - U$

- knowing $M \ddot{q} + \nabla U(q) = 0$

$$\Rightarrow \frac{d}{dt} \left(\underbrace{\frac{\partial L}{\partial \dot{q}}}_{M \dot{q}} \right) - \underbrace{\frac{\partial L}{\partial q}}_{-\nabla U} = 0$$

- To show that this is coordinate invariant,
 $q' = f(q)$, $\dot{q}' = \frac{df}{dq} \dot{q} = J(q) \dot{q}$

$$\Rightarrow L(\dot{q}', q') = L(f(q), J(q) \dot{q})$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} J(q) \right) - \frac{\partial L}{\partial q} J(q) \\ - \frac{\partial L}{\partial \dot{q}} \left(\frac{\partial}{\partial q} J(q) \dot{q} \right)$$

$$\Rightarrow \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) J(q) + \cancel{\frac{\partial L}{\partial \dot{q}} \left(\frac{\partial}{\partial q} \dot{q} \right)} - \cancel{\frac{\partial L}{\partial q} J(q)} - \cancel{\frac{\partial L}{\partial \dot{q}} \left(\frac{\partial}{\partial q} \dot{q} \right)} \\ = 0 \\ \Rightarrow \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \right] J(q) = 0$$

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

- Euler-Lagrange equation is exactly the same in q and q' coordinates.
- We also know that this is the first-order necessary condition for an optimization problem!

"Action"

$$\min_{q(t)} S = \int_{t_0}^{t_f} L(q, \dot{q}) dt$$

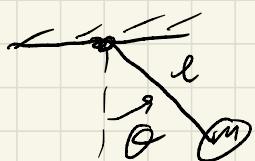
"Lagrangian"

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

- This approach to dynamics is called "Lagrangian Mechanics"

- Minimization problem is called "Least-Action principle" or "Hamilton's Principle"
 - Least-Action is most general formulation of dynamics. Can derive all others by making particular choices of L , q , etc.
 - Major advantage over $F = ma$: I can pick whatever coordinates I want for q .
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* Example: Pendulum



$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$U = mg y = mgl(1 - \cos(\theta))$$

$$L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos(\theta))$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = ml\ddot{\theta} + mgl\sin(\theta) = 0$$
