Appendices

Below are the paper's appendices referenced in the text.

A Appendix for Section 2

{sec:ApndxConcave

A.1 Recovering the Non-Normalized Problem

{sec:recoverLevels}

Letting nonbold variables be the boldface counterpart normalized by $\boldsymbol{p}_t$ (as with $m = \boldsymbol{m}/\boldsymbol{p}$), consider the problem in the second-to-last period:

$$\begin{aligned} \mathbf{v}_{T-1}(\boldsymbol{m}_{T-1}, \boldsymbol{p}_{T-1}) &= \max_{0<c_{T-1}<m_{T-1}} \mathrm{u}(\boldsymbol{p}_{T-1}c_{T-1}) + \beta\mathbb{E}_t[\mathrm{u}(\boldsymbol{p}_T m_T)] \\ &= \boldsymbol{p}_{T-1}^{1-\gamma}\left\{\max_{0<c_{T-1}\leq m_{T-1}} \mathrm{u}(c_{T-1}) + \beta\mathbb{E}_t[\mathrm{u}(\tilde{\mathcal{G}}_T m_T)]\right\}. \end{aligned} \tag{1}$$

{eq:vBold}

Since $\mathrm{v}_T(m_T) = \mathrm{u}(m_T)$, defining $\mathrm{v}_{T-1}(m_{T-1})$ from Problem ($\mathcal{P}_N$), we obtain:

$$\mathbf{v}_{T-1}(\boldsymbol{m}_{T-1}, \boldsymbol{p}_{T-1}) = \boldsymbol{p}_{T-1}^{1-\gamma}\mathrm{v}_{T-1}(\underbrace{\boldsymbol{m}_{T-1}/\boldsymbol{p}_{T-1}}_{=m_{T-1}}).$$

This logic induces to earlier periods; if we solve the normalized one-state-variable problem ($\mathcal{P}_N$), we will have solutions to the original problem for any $t < T$ from:

$$\begin{aligned} \mathbf{v}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) &= \boldsymbol{p}_t^{1-\gamma}\mathrm{v}_t(m_t), \\ \boldsymbol{c}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) &= \boldsymbol{p}_t\mathrm{c}_t(m_t). \end{aligned}$$

A.2 Perfect Foresight Benchmarks

{subsec:PFBProofs}

Proof of Claim 2. First we show that if finite limiting human wealth (Assumption I.3) and growth impatience (Assumption S.1) are both satisfied, perfect foresight finite value of autarky (Equation (9)). In particular, note that:

$$\begin{aligned} \text{Þ} &< \mathcal{G} < \mathsf{R} \\ \text{Þ}/\mathsf{R} &< \mathcal{G}/\mathsf{R} < (\mathcal{G}/\mathsf{R})^{1-1/\gamma} < 1. \end{aligned} \tag{2}$$

{eq:GICandFHWCi

The last line above holds because finite human wealth implies $0 \leq (\mathcal{G}/\mathsf{R}) < 1$ and $\gamma > 1 \Rightarrow 0 < 1 - 1/\gamma < 1$.

Next, we show that if finite limiting human wealth is satisfied, perfect foresight finite value of autarky (Equation (9)) implies return impatience (Assumption L.3). To see why, divide both sides of the second inequality in Equation (9) by R, and after some straightforward algebra, arrive at:

$$\text{Þ}/\mathsf{R} < (\mathcal{G}/\mathsf{R})^{1-1/\gamma}. \tag{3}$$ {eq:FHWCandPFFV

Due to finite limiting human wealth, the RHS above is strictly less than 1 because $(\mathcal{G}/\mathsf{R}) < 1$ (and the RHS is raised to a positive power (because $\gamma > 1$)).

□

A.3 Properties of the Consumption Function and Limiting MPCs

{sec:MPCiterproofs

For the following, a function with k continuous derivatives is called a $\mathbf{C}^k$ function.

{lemm:consC2}

Lemma 1. *Let $t < T$. If v_t is strictly negative, strictly increasing, strictly concave, $\mathbf{C}^3$ and satisfies $\lim_{m\to 0} \mathrm{v}_t(m) = -\infty$, then c_t is $\mathbf{C}^2$.*

Proof. Start by defining an end-of-period value function $\mathfrak{v}_t$ as:

$$\mathfrak{v}_t(a)\colon = \beta\mathbb{E}_t\left[\tilde{\mathcal{G}}_{t+1}^{1-\gamma}\mathrm{v}_{t+1}\left(\tilde{\mathcal{R}}_{t+1}a + \boldsymbol{\xi}_{t+1}\right)\right], \qquad a \in \mathbb{R}_{++}. \tag{4}$$ {eq:vfFrackdefn}

Since there is a positive probability that $\boldsymbol{\xi}_{t+1}$ will attain its minimum of zero and since $\tilde{\mathcal{R}}_{t+1} > 0$, we will have that $\lim_{a\to 0}\mathfrak{v}_t(a) = -\infty$. Moreover, note that $\mathfrak{v}_t(a)$ is real-valued iff $a > 0$. As such, by Leibniz Rule, $\mathfrak{v}_t$ will be $\mathbf{C}^3$.

Next, define $\underline{\mathrm{v}}_t(m, c)$ as:

$$\underline{\mathrm{v}}_t(m, c)\colon = \mathrm{u}(c) + \mathfrak{v}_t(m - c), \qquad (m, c) \in \mathbb{R}_{++}.$$

Note that for fixed m, $c \mapsto \underline{\mathrm{v}}_t(m, c)$ is $\mathbf{C}^3$ on $(0, m)$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$. Observe that the value function defined by Problem $(\mathcal{P}_N)$ can be written as:

$$\mathrm{v}_t(m) = \max_{0<c<m} \underline{\mathrm{v}}_t(m, c), \qquad m \in \mathbb{R}_{++}$$

where the function $\underline{\mathrm{v}}_t$ is real-valued if and only if $0 < c < m$. Furthermore, $\lim_{c\to 0}\underline{\mathrm{v}}_t(m, c) = \lim_{c\to m}\underline{\mathrm{v}}_t(m, c) = -\infty$, $\frac{\partial^2 \underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\to 0}\frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\to m}\frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$.

Letting $\underline{\mathrm{v}}_t(m, 0) = -\infty$ and $\underline{\mathrm{v}}_t(m, m) = -\infty$, consider that $\mathrm{c}_t(m)$ is given by:

$$\mathrm{c}_t(m) = \arg\max_{0<c<m} \underline{\mathrm{v}}_t(m, c) = \arg\max_{0\leq c\leq m} \underline{\mathrm{v}}_t(m, c)$$

where the maximizer exists, is unique and an interior solution. As such, note that c_t satisfies the first order condition:

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(m - \mathrm{c}_t(m)).$$

By the Implicit Function Theorem, c_t is continuous and differentiable and:

$$\mathrm{c}_t'(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))},$$

where the function a_t is defined by the evaluation $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable and c_t is continuous, the RHS of the above equation is continuous and we can conclude that c_t' is continuous and c_t is in $\mathbf{C}^1$.

Finally, $\mathrm{c}_t'(m)$ is differentiable because $\mathfrak{v}_t''$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $\mathrm{u}''(\mathrm{c}_t(m))+\mathfrak{v}_t''(\mathrm{a}_t(m)) < 0$. The second derivative $\mathrm{c}_t''(m)$ will then be given by:

$$\mathrm{c}_t''(m) = \frac{\mathrm{a}_t'(m)\mathfrak{v}_t'''(a_t)\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right] - \mathfrak{v}_t''(a_t)\left[\mathrm{c}_t'(m)\mathrm{u}'''(c_t) + \mathrm{a}_t'(m)\mathfrak{v}_t'''(a_t)\right]}{\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right]^2},$$

where $a_t = \mathrm{a}_t(m)$ in the equation above. Since $\mathfrak{v}_t''(\mathrm{a}_t(m))$ is continuous, $\mathrm{c}_t''(m)$ is also continuous.

□

{prop:vfc3}

Claim 1. *For each t, v_t is strictly negative, strictly increasing, strictly concave, $\mathbf{C}^3$ and satisfies $\lim_{m\to 0} \mathrm{v}_t(m) = -\infty$.*

Proof. We will say a function is 'nice' if it satisfies the properties stated by the Proposition. Assume that for some $t+1$, v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and u, where $\mathrm{u}(m) = m^{1-\gamma}/(1-\gamma)$, is nice by inspection. By Lemma 1, if v_{t+1} is nice, c_t is in $\mathbf{C}^2$. Next, since both u and $\mathfrak{v}_t$ are strictly concave, both c_t and a_t, where $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$, are strictly increasing (Recall Equation (A.3)). This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = \mathrm{u}(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$. □

Proof for Proposition 3. By Claim 1, each v_t is strictly negative, strictly increasing, strictly concave, $\mathbf{C}^3$ and satisfies $\lim_{m\to 0} \mathrm{v}_t(m) = -\infty$. As such, apply Lemma 1 to conclude that c_t is in $\mathbf{C}^2$. To see that c_t is strictly increasing, note (A.3). To see that c_t is strictly concave, see Theorem 1. in Carroll and Kimball (1996). □

Proof of Lemma 1 (Limiting MPCs). *Part (1.): Minimal MPCs*

Fix any t and for any m_t with $m_t > 0$, we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$. The Euler equation, Equation (4), can be rewritten as:

$$\mathrm{e}_t(m_t)^{-\gamma} = \beta\mathsf{R}\mathbb{E}_t\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m_t) + \tilde{\mathcal{G}}_{t+1}\boldsymbol{\xi}_{t+1}}^{=m_{t+1}\tilde{\mathcal{G}}_{t+1}}}{m_t}\right)\right)^{-\gamma} \tag{5}$$

{eq:eFuncEuler}

where $m_{t+1} = \tilde{\mathcal{R}}_{t+1}(m_t - \mathrm{c}_t(m_t)) + \boldsymbol{\xi}_{t+1}$. The minimal MPC's are obtained by letting where $m_t \to \infty$. Note that $\lim_{m_t \to \infty} m_{t+1} = \infty$ almost surely and thus $\lim_{m_t \to \infty} \mathrm{e}_{t+1}(m_{t+1}) = \underline{\kappa}_{t+1}$ almost surely. Turning to the second term inside the marginal utility on the RHS, we can write:

$$\lim_{m_t \to \infty} \frac{\mathsf{R}\mathrm{a}_t(m_t) + \tilde{\mathcal{G}}_{t+1}\boldsymbol{\xi}_{t+1}}{m_t} = \lim_{m_t \to \infty} \frac{\mathsf{R}\mathrm{a}_t(m_t)}{m_t} + \lim_{m_t \to \infty} \frac{\tilde{\mathcal{G}}_{t+1}\boldsymbol{\xi}_{t+1}}{m_t} \tag{6}$$

$$= \mathsf{R}(1 - \underline{\kappa}_t) + 0, \tag{7}$$

since $\tilde{\mathcal{G}}_{t+1}\boldsymbol{\xi}_{t+1}$ is bounded. Thus, we can assert:

$$\lim_{m_t \to \infty} \left(\mathrm{e}_{t+1}(m_{t+1}) \left(\frac{\mathsf{R}\mathrm{a}_t(m) + \tilde{\mathcal{G}}_{t+1}\boldsymbol{\xi}_{t+1}}{m_t} \right) \right)^{-\gamma} = (\mathsf{R}\underline{\kappa}_{t+1}(1 - \underline{\kappa}_t))^{-\gamma}, \tag{8}$$

almost surely. Next, the term inside the expectation operator at Equation (5) is bounded above by $\left(\mathsf{R}\underline{\kappa}_{t+1}(1 - \bar{\kappa}_t)\right)^{-\gamma}$. Thus, by the Dominated Convergence Theorem, we have:

$$\lim_{m_t \to \infty} \beta \mathsf{R} \mathbb{E}_t \left(\mathrm{e}_{t+1}(m_{t+1}) \left(\frac{\mathsf{R}\mathrm{a}_t(m_t) + \tilde{\mathcal{G}}_{t+1}\boldsymbol{\xi}_{t+1}}{m_t} \right) \right)^{-\gamma} = \beta \mathsf{R}(R\underline{\kappa}_{t+1}(1 - \underline{\kappa}_t))^{-\gamma}. \tag{9}$$

{eq:eFuncEulerMPC

Again applying L'Hôpital's rule to the LHS of Equation (5), letting $\lim_{m \to \infty} \mathrm{e}_t(m) = \underline{\kappa}_t$ and equating limits to the RHS, we arrive at:

$$\text{Þ}/\mathsf{R}\underline{\kappa}_t = (1 - \underline{\kappa}_t)\underline{\kappa}_{t+1}$$

Thus the minimal marginal propensity to consume satisfies the following recursive formula:

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1}\text{Þ}/\mathsf{R}, \tag{10}$$

{eq:MPCminInvApr

which implies $\{\underline{\kappa}_{T-n}^{-1}\}_{n=0}^{\infty}$ is an increasing sequence. Define:

$$\underline{\kappa}^{-1} : = \lim_{n \to \infty} \kappa_{T-n}^{-1} \tag{11}$$

as the limiting (inverse) marginal MPC. If return impatience(Assumption L.3) does *not* hold, then $\lim_{n \to \infty} \underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$. Otherwise if return impatience (Assumption L.3) holds, then $\underline{\kappa} > 0$.

Part (2.): Maximal MPCs

The Euler Equation (4) can be rewritten as:

$$
\begin{aligned}
\mathrm{e}_t(m_t)^{-\gamma} &= \beta\mathsf{R}\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m)+\tilde{\mathcal{G}}_{t+1}\boldsymbol{\xi}_{t+1}}^{=m_{t+1}\tilde{\mathcal{G}}_{t+1}}}{m_t}\right)\right)^{-\gamma}\right] \\
&= (1-\wp)\beta\mathsf{R}m_t^{\gamma}\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\tilde{\mathcal{G}}_{t+1}\right)^{-\gamma}\mid \boldsymbol{\xi}_{t+1}>0\right] \\
&\quad+\wp\beta\mathsf{R}^{1-\gamma}\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\tilde{\mathcal{R}}_{t+1}\mathrm{a}_t(m))\frac{m_t-\mathrm{c}_t(m)}{m_t}\right)^{-\gamma}\mid \boldsymbol{\xi}_{t+1}=0\right]
\end{aligned} \tag{12}
$$

{eq:eFuncEulerMPC

Now consider the first conditional expectation in the second line of Equation (12). Recall that if $\boldsymbol{\xi}_{t+1}>0$, then $\boldsymbol{\xi}_{t+1}=\theta_{t+1}/(1-\wp)$ by Assumption I.1. Since $\lim_{m_t\to 0}\mathrm{a}_t(m_t)=0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\tilde{\mathcal{G}}_{t+1})^{-\gamma}\mid \boldsymbol{\xi}_{t+1}>0]$ is contained in the bounded interval $[(\mathrm{e}_{t+1}(\underline{\theta}/(1-\wp))\mathcal{G}\underline{\psi}\underline{\theta}/(1-\wp))^{-\gamma},(\mathrm{e}_{t+1}(\bar{\theta}/(1-\wp))\mathcal{G}\bar{\psi}\bar{\theta}/(1-\wp))^{-\gamma}]$. As such, the first term after the second equality above converges to zero as m_t^{γ} converges to zero.

Turning to the second term after the second equality above, once again apply Dominated Convergence Theorem as noted above at Equation (9). As $m_t\to 0$, the expectation converges to $\overline{\kappa}_{t+1}^{-\gamma}(1-\overline{\kappa}_t)^{-\gamma}$.

Equating the limits on the LHS and RHS of Equation (12), we have $\overline{\kappa}_t^{-\gamma}=\beta\wp\mathsf{R}^{1-\gamma}\overline{\kappa}_{t+1}^{-\gamma}(1-\overline{\kappa}_t)^{-\gamma}$. Exponentiating by γ on both sides, we can conclude:

$$
\overline{\kappa}_t=\wp^{-1/\gamma}(\beta\mathsf{R})^{-1/\gamma}\mathsf{R}(1-\overline{\kappa}_t)\overline{\kappa}_{t+1}
$$

and,

$$
\underbrace{\wp^{1/\gamma}\overbrace{\mathsf{R}^{-1}(\beta\mathsf{R})^{1/\gamma}}^{\text{Þ}/\mathsf{R}}}_{\equiv\wp^{1/\gamma}\text{Þ}/\mathsf{R}}\overline{\kappa}_t=(1-\overline{\kappa}_t)\overline{\kappa}_{t+1} \tag{13}
$$

{eq:MPSminDef}

The equation above yields a recursive formula for the maximal marginal propensity to consume after some algebra:

$$
\begin{aligned}
(\wp^{1/\gamma}\text{Þ}/\mathsf{R}\overline{\kappa}_t)^{-1}&=(1-\overline{\kappa}_t)^{-1}\overline{\kappa}_{t+1}^{-1} \\
\Rightarrow \overline{\kappa}_t^{-1}(1-\overline{\kappa}_t)&=\wp^{1/\gamma}\text{Þ}/\mathsf{R}\overline{\kappa}_{t+1}^{-1} \\
\Rightarrow \overline{\kappa}_t^{-1}&=1+\wp^{1/\gamma}\text{Þ}/\mathsf{R}\overline{\kappa}_{t+1}^{-1}
\end{aligned}
$$

As noted in the main text, we need weak return impatience(Assumption L.4) for this to be a convergent sequence:

$$
0\le\wp^{1/\gamma}\text{Þ}/\mathsf{R}<1, \tag{14}
$$

{eq:WRICapndx}

Since $\overline{\kappa}_T = 1$, iterating (14) backward to infinity, we obtain:

$$\lim_{n\to\infty} \overline{\kappa}_{T-n} = \overline{\kappa} \equiv 1 - \wp^{1/\gamma}\mathbf{Þ}/\mathsf{R} \tag{15}$$

{eq:MPCmaxDef}

□

A.4 Existence of Limiting Solutions

{sec:Tcontractionm

We state Boyd's contraction mapping Theorem (Boyd,1990) for completeness.

Theorem 1. *(Boyd's Contraction Mapping) Let $\mathbb{B} : \mathcal{C}_\varphi(S, Y) \to \mathcal{C}_\varphi(S, Y)$ with $S \subset \mathbb{R}$ and $Y \subset \mathbb{R}$.* {thm:Boyd}

If,

1. *the operator $\mathbb{B}$ is non-decreasing, i.e. $\mathrm{x} \le \mathrm{y} \Rightarrow \mathbb{B}\mathrm{x} \le \mathbb{B}\mathrm{y}$,*
2. *we have $\mathbb{B}\mathbf{0} \in \mathcal{C}_\varphi(S, Y)$, where $\mathbf{0}$ is the null vector,*
3. *there exists α with $0 < \alpha < 1$ such that for all λ with $\lambda > 0$, we have:*

$$\mathbb{B}(\mathrm{x} + \lambda\varphi) \le \mathbb{B}\mathrm{x} + \lambda\alpha\varphi,$$

then $\mathbb{B}$ defines a contraction with a unique fixed point.

{claim:MPCMAX

Claim 2. *If weak return impatience (Assumption L.4) holds, then there exists k such that for all $0 \le \overline{\nu} \le \overline{\kappa}_{T-k}$, we have:*

$$\wp\beta(\mathsf{R}(1 - \overline{\nu}))^{1-\gamma} < 1 \tag{16}$$

{eq:MPCMAXKlec

Proof. By straightforward algebra and Equation (19) from the main text, we have:

$$\begin{aligned} \wp\beta(\mathsf{R}(1 - \overline{\kappa}))^{1-\gamma} &= \wp\beta\mathsf{R}^{1-\gamma}\left(\wp^{1/\gamma}\frac{(\mathsf{R}\beta)^{1/\gamma}}{\mathsf{R}}\right)^{1-\gamma} \\ &= \wp^{1/\gamma}\frac{(\mathsf{R}\beta)^{1/\gamma}}{\mathsf{R}} < 1, \end{aligned} \tag{17}$$

where the inequality holds by weak return impatience (Assumption L.4). Finally, the expression $\overline{\nu} \mapsto \wp\beta(\mathsf{R}(1 - \overline{\nu}))^{1-\gamma}$ is continuous and increasing in $\overline{\nu}$, and we have $1 > \overline{\kappa} > 0$ and $\overline{\kappa}_{T-n} \to \overline{\kappa}$ as $n \to \infty$. As such, there exists k such that $\wp\beta(\mathsf{R}(1 - \overline{\kappa}_{T-k}))^{1-\gamma} < 1$ and Equation (16) holds for all $\overline{\nu} \le \overline{\kappa}_{T-n}$.

□

{rem:shnkrdef}

Remark 1. *By the finite value of autarky (Assumption L.1) and for k large enough, fix α such that:*

$$\alpha = \max\{\wp\beta(\mathsf{R}(1 - \overline{\kappa}_k))^{1-\gamma}, \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\} < 1 \tag{18}$$

{eq:shnkrdef}

Note that this implies

$$\alpha(1-\alpha^{-1}\beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma})>0. \tag{19}$$ {eq:shrnkrCond}

We define the constant ζ as follows:

$$\zeta=\frac{\beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}(1-\wp)^{\gamma}\underline{\theta}^{1-\gamma}}{\alpha(1-\alpha^{-1}\beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma})}, \tag{20}$$ {eq:Mbarddef}

and the bounding function, φ, as follows $\varphi(x)=\zeta+x^{1-\gamma}$.

{clm:hiraguchi_co

Claim 3. *If* $\mathrm{x}\in\mathcal{C}_{\varphi}(S,Y)$, *then* $\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{x}\in\mathcal{C}_{\varphi}(\mathbb{R}_{++},\mathbb{R}_{+})$.

Proof. By definition, we have

$$\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{x}(m_t)=\max_{c_t\in[\underline{\nu}m_t,\overline{\nu}m_t]}\left\{\mathrm{u}(c_t)+\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{x}\left(m_{t+1}\right)\right]\right\},\qquad m_t\in\mathbb{R}_{++} \tag{21}$$

where $m_{t+1}=\tilde{\mathcal{R}}\left(m_t-c_t\right)+\boldsymbol{\xi}$.

First we verify that the mapping $c_t\mapsto\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{x}\left(m_{t+1}\right)\right]$, which we denote as g, is continuous. To proceed define the mapping $\tilde{\mathrm{g}}\colon\mathbb{R}_{++}\times\Omega\to\mathbb{R}$ by $c,\omega\mapsto\left[\tilde{\mathcal{G}}(\omega)^{1-\gamma}\mathrm{x}\left(\tilde{\mathcal{R}}(\omega)\left(m_t-c_t\right)+\boldsymbol{\xi}(\omega)\right)\right]$ and the mapping $\mathrm{g}\colon\mathbb{R}_{++}\times[\underline{\psi},\overline{\psi}]\times[0,\overline{\theta}]\to\mathbb{R}$ by $c,\psi,\boldsymbol{\xi}\mapsto\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{x}\left(\tilde{\mathcal{R}}\left(m_t-c_t\right)+\boldsymbol{\xi}\right)\right]$. Fix c and note that for any compact interval $[\bar{c},\underline{c}]$ such that $c\in[\bar{c},\underline{c}]\subset\mathbb{R}_{++}$, $c\in\mathbb{R}_{++}$, $\mathrm{g}(c,\bullet,\bullet)$ is continuous on $[\bar{c},\underline{c}]\times[\underline{\psi},\overline{\psi}]\times[0,\overline{\theta}]$. Thus, g is bounded above and below by $\bar{\Xi}$ and $\underline{\Xi}$ for any $c\in[\bar{c},\underline{c}]$ (where $\bar{\Xi}$ and $\underline{\Xi}$ do not depend on c). To show continuity of $\mathbb{E}\tilde{\mathrm{g}}(c,\bullet)$ for any $c\in\mathbb{R}_{++}$, note there exists $[\bar{c},\underline{c}]$ such that $c\in[\bar{c},\underline{c}]\subset\mathbb{R}_{++}$. Thus consider $\{c^i\}_i$, let $c^i\to c$ and we can assume $c^i\in[\bar{c},\underline{c}]$ for all i. Since for each i, $\tilde{\mathrm{g}}(c^i,\omega)$ is bounded above and below by $\bar{\Xi}$ and $\underline{\Xi}$, by the Dominated Convergence Theorem, we must have $\lim_{i\to\infty}\mathbb{E}\tilde{\mathrm{g}}(c_i,\bullet)=\mathbb{E}\tilde{\mathrm{g}}(c,\bullet)$.

Next, by Berge's Maximum Theorem (Theorem 17.31 in Aliprantis and Border (2006)), since the feasibilty correspondence $m_t\mapsto[\underline{\nu}m_t,\overline{\nu}m_t]$ has a closed graph and is and compact valued, $\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{x}$ must be continuous.

Finally, to show that $\|\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{x}\|_{\varphi}<\infty$. We have:

$$\|\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{x}\|_{\varphi}=\sup_{m}\left\{\frac{\left|\mathrm{u}(\mathrm{c}(m))+\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{x}\left(m^{\texttt{next}}\right)\right]\right|}{\zeta+m^{1-\gamma}}\right\} \tag{22}$$

$$\leq\sup_{m}\left\{\frac{\left|\frac{m^{1-\gamma}}{1-\gamma}+\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{x}\left(m^{\texttt{next}}\right)\right]\right|}{\zeta+m^{1-\gamma}}\right\} \tag{23}$$

$$\leq\sup_{m}\left\{\frac{\frac{m^{1-\gamma}}{1-\gamma}}{\zeta+m^{1-\gamma}}\right\}+\sup_{m}\left\{\frac{\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}|\mathrm{x}\left(m\right)|\right]}{\zeta+m^{1-\gamma}}\right\} \tag{24}$$

$$<\infty, \tag{25}$$

where $m^{\texttt{next}} = \tilde{\mathcal{R}}\,(m - c) + \boldsymbol{\xi}$ and the final inequality follows from the triangle inequality and the fact that x is φ-bounded.

$\square$

Proof of Theorem 1. Fix k such that Equation (16) holds. By Claim 3, $\mathbb{T}^{\underline{\nu},\overline{\nu}}\,\mathbb{T}^{\underline{\nu},\overline{\nu}}$ maps from $\mathcal{C}_\varphi(\mathbb{R}_{++}, \mathbb{R})$ to $\mathcal{C}_\varphi(\mathbb{R}_{++}, \mathbb{R})$. We now verify conditions (1)-(3) of Boyd's Theorem (1).

Condition (1). By definition of $\mathbb{T}^{\underline{\nu},\overline{\nu}}$, we have:

$$\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{x}(m_t) = \max_{c_t \in [\underline{\nu} m_t, \overline{\nu} m_t]} \left\{ \mathrm{u}(c_t) + \beta \mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma} \mathrm{x}\left(m_{t+1}\right)\right]\right\}, \tag{26}$$ {eq:condition1}

where $m_{t+1} = \tilde{\mathcal{R}}\,(m_t - c_t) + \boldsymbol{\xi}$. As such, $\mathrm{x} \leq \mathrm{y}$ implies $\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{x}(m_t) \leq \mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{y}(m_t)$ by inspection.

Condition (2.) Condition (2.) requires that $\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathbf{0} \in \mathcal{C}_\varphi\left(\mathcal{A}, \mathcal{B}\right)$. By definition,

$$\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathbf{0}(m_t) = \max_{c_t \in [\underline{\nu} m_t, \overline{\nu} m_t]} \left\{ \left(\frac{c_t^{1-\gamma}}{1-\gamma}\right) + \beta 0 \right\}$$

the solution to which implies $\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathbf{0}(m_t) = \mathrm{u}(\overline{\nu} m_t)$. Thus, Condition (2) will hold if $(\overline{\nu} m_t)^{1-\gamma}$ is φ-bounded, which it is if we use the bounding function

$$\varphi(x) = \zeta + x^{1-\gamma}, \tag{27}$$

defined in Remark 1.

Condition (3). Finally, we turn to condition (3), which requires us to show $\mathbb{T}^{\underline{\nu},\overline{\nu}}(\mathrm{z} + \lambda\varphi)(m_t) \leq \mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{z}(m_t) + \lambda\alpha\varphi(m_t)$ for $0 < \alpha < 1$ and $\lambda > 0$.

To proceed, define $\check{\mathrm{c}}$ as the consumption function[1] associated with $\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{z}$ and $\hat{\mathrm{c}}$ as the consumption function associated with $\mathbb{T}^{\underline{\nu},\overline{\nu}}(\mathrm{z} + \zeta\varphi)$; using this notation, Condition (3.) can be rewritten as:

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta \mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}(\mathrm{z} + \zeta\varphi) \circ \hat{\mathrm{m}}^{\texttt{next}} \leq \mathrm{u}(\check{\mathrm{c}}) + \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{z} \circ \check{\mathrm{m}}^{\texttt{next}} + \zeta\alpha\varphi,$$

where $\check{\mathrm{m}}^{\texttt{next}}(m) = \tilde{\mathcal{R}}(m - \check{\mathrm{c}}(m)) + \boldsymbol{\xi}$ and $\hat{\mathrm{m}}^{\texttt{next}}(m) = \tilde{\mathcal{R}}(m - \hat{\mathrm{c}}(m)) + \boldsymbol{\xi}$. If we now force the consumer facing z as the next period value function to consume the amount optimal for the consumer facing $\mathrm{z} + \zeta\varphi$, the value for the z consumer must be weakly lower. That is,

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{z} \circ \hat{\mathrm{m}}^{\texttt{next}} \leq \mathrm{u}(\check{\mathrm{c}}) + \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{z} \circ \check{\mathrm{m}}^{\texttt{next}}.$$

[1] Note that the maximand on the RHS of Equation (26) is continuous (Claim 3) and the feasible set of consumption choices is compact-valued. As such, a solution to the maximization problem exists for any m_t. Thus, letting Θ be the solution correspondence for the maximization problem, $\Theta(m_t)$ will be non-empty and will admit a selector function $\check{\mathrm{c}}$. See Section 17.11 in Aliprantis and Border (2006).

Thus, condition (3.) will certainly hold under the stronger condition

$$\begin{aligned} \mathrm{u}\circ\hat{\mathrm{c}} + \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}(\mathrm{z}+\lambda\varphi)\circ\hat{\mathrm{m}}^{\texttt{next}} &\leq \mathrm{u}\circ\hat{\mathrm{c}} + \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{z}\circ\hat{\mathrm{m}}^{\texttt{next}} + \lambda\alpha\varphi \\ \Leftrightarrow \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}(\mathrm{z}+\lambda\varphi)\circ\hat{\mathrm{m}}^{\texttt{next}} &\leq \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{z}\circ\hat{\mathrm{m}}^{\texttt{next}} + \lambda\alpha\varphi \\ \Leftrightarrow \beta\lambda\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\varphi\circ\mathrm{m}^{\widehat{\texttt{next}}} &\leq \lambda\alpha\varphi \\ \Leftrightarrow \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\varphi\circ\hat{\mathrm{m}}^{\texttt{next}} &\leq \alpha\varphi \qquad (28) \end{aligned}$$

{eq:reqCondWeak}

To show (28) holds, recall by Claim 2 that $\wp\beta(\mathsf{R}(1-\overline{\kappa}_{T-k}))^{1-\gamma} < 1$ for k large enough. As such, define α by Equation (18) and note that $\wp\beta(\mathsf{R}(1-\overline{\kappa}_{k}))^{1-\gamma} < \alpha < 1$ and $\alpha \geq \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}$. Letting $\hat{a} = m - \hat{\mathrm{c}}(m)$, Equation (28) will be satisfied if:

$$\beta\mathbb{E}[\tilde{\mathcal{G}}^{1-\gamma}(\hat{a}\tilde{\mathfrak{R}}+\boldsymbol{\xi})^{1-\gamma}] - \alpha m^{1-\gamma} < \alpha\zeta(1-\alpha^{-1}\beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}),$$

which, by imposing finite value of autarky (Assumption L.1) and Equation (19) can be rewritten as:

$$\zeta > \frac{\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}(\hat{a}\tilde{\mathfrak{R}}+\boldsymbol{\xi})^{1-\gamma}\right] - \alpha m^{1-\gamma}}{\alpha(1-\alpha^{-1}\beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma})} =: \bar{\bar{M}}. \qquad (29)$$

{eq:KeyCondition}

Thus, the proof reduces to showing Equation (29) holds. To proceed, consider that the numerator of (29) is bounded above as follows:

$$\begin{aligned} \beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}(\hat{a}\tilde{\mathfrak{R}}+\boldsymbol{\xi})^{1-\gamma}\right] - \alpha m^{1-\gamma} &= (1-\wp)\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}(\hat{a}\tilde{\mathfrak{R}}+\theta/(1-\wp))^{1-\gamma}\right] \\ &\quad + \wp\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}(\hat{a}\tilde{\mathfrak{R}})^{1-\gamma}\right] - \alpha m^{1-\gamma} \\ &\leq (1-\wp)\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}((1-\overline{\nu})m\tilde{\mathfrak{R}}+\theta/(1-\wp))^{1-\gamma}\right] \\ &\quad + \wp\beta\mathsf{R}^{1-\gamma}((1-\overline{\nu})m)^{1-\gamma} - \alpha m^{1-\gamma} \\ &= (1-\wp)\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}((1-\overline{\nu})m\tilde{\mathfrak{R}}+\theta/(1-\wp))^{1-\gamma}\right] \qquad (30) \\ &\quad + m^{1-\gamma}\left(\underbrace{\wp\beta(\mathsf{R}(1-\overline{\nu}))^{1-\gamma}}_{<\alpha \text{ by Claim 2}} - \alpha\right) \\ &< (1-\wp)\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}(\underline{\theta}/(1-\wp))^{1-\gamma}\right] \\ &= \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}(1-\wp)^{\gamma}\underline{\theta}^{1-\gamma}. \end{aligned}$$

Using Claim 2, we have that $\wp\beta(\mathsf{R}(1-\overline{\nu}))^{1-\gamma} < \alpha$ since $\alpha = \max\{\wp\beta(\mathsf{R}(1-\overline{\kappa}_{T-k}))^{1-\gamma}, \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\}$ and $\overline{\nu} \leq \overline{\kappa}_k$. We can thus conclude that equation (29) will hold since we have

$$\zeta \geq \frac{\beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}(1-\wp)^{\gamma}\underline{\theta}^{1-\gamma}}{\alpha(1-\alpha^{-1}\beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma})} > \bar{\bar{M}}. \qquad (31)$$

The proof that $\mathbb{T}^{\underline{\nu},\overline{\nu}}$ defines a contraction mapping under the conditions (L.4) and (L.1) is now complete. $\square$

Proof of Theorem 2 (continued). *Proof of part (ii).* We next establish the pointwise convergence of consumption the functions $\{\mathrm{c}_{t_n}\}_{n=0}^{\infty}$ along a sub-sequence. Fix any $m \in S$ and consider a convergent subsequence $\{\mathrm{c}_{t_{n(i)}}(m)\}_{i=0}^{\infty}$ of $\{\mathrm{c}_{t_n}(m)\}_{n=0}^{\infty}$. Let the function c denote the mapping from m to the limit of $\{\mathrm{c}_{t_{n(i)}}(m)\}_{i=0}^{\infty}$. Since $\mathrm{c}_{t_{n(i)}}(m)$ solves the time $t_{n(i)}$ finite horizon problem, we have:

$$\begin{aligned}\mathrm{u}(\mathrm{c}_{t_{n(i)}}(m)) + &\beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}_{t_{n(i)+1}}(m_{t_{n(i)+1}})\right] \\ &\geq \mathrm{u}(c) + \beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}_{t_{n(i)}+1}(\hat{m}^{\texttt{next}})\right],\end{aligned} \tag{32}$$

for any $c \in (0, \overline{\kappa}m]$, where $m_{t_{n(i)}+1} = \tilde{\mathcal{R}}(m - \mathrm{c}_{t_{n(i)}}(m)) + \boldsymbol{\xi}_{t_{n(i)}+1}$ and $\hat{m}^{\texttt{next}} = \tilde{\mathcal{R}}(m - c) + \boldsymbol{\xi}_{t_{n(i)}+1}$. Allowing $n(i)$ to tend to infinity, the left-hand side converges to:

$$\mathrm{u}(\mathrm{c}(m)) + \beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}(m^{\texttt{next}})\right], \tag{33}$$

where $m^{\texttt{next}} = \tilde{\mathcal{R}}(m - \mathrm{c}(m)) + \boldsymbol{\xi}$. Moreover, the right-hand side converges to:

$$\mathrm{u}(c) + \beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}(\hat{m}^{\texttt{next}})\right]. \tag{34}$$

Hence, as $n(i)$ tends to infinity, the following inequality is implied:

$$\mathrm{u}(\mathrm{c}(m)) + \beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}(m^{\texttt{next}})\right] \geq \mathrm{u}(c) + \beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}(\hat{m}^{\texttt{next}})\right]. \tag{35}$$

Since the c above was arbitrary, we have:

$$\mathrm{c}(m) \in \underset{c\in(0,\overline{\kappa}m]}{\arg\max}\left\{\mathrm{u}(c) + \beta\mathbb{E}\left[\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}(\hat{m}^{\texttt{next}})\right]\right\}. \tag{36}$$

{eq:statCbellman}

Next, since $\mathrm{c}_{t_{n(i)}} \to \mathrm{c}$ pointwise, and $\mathrm{v}_{t_{n(i)}} \to \mathrm{v}$ pointwise, we have:

$$\mathrm{v}(m) = \lim_{i\to\infty}\left[\mathrm{u}(\mathrm{c}_{t_{n(i)}}(m)) + \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}_{t_{n(i)}+1}(m_{t_{n(i)}+1})\right] = \mathrm{u}(\mathrm{c}(m)) + \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}(m^{\texttt{next}}). \tag{37}$$

{eq:convgcvftni}

where $m_{t_n} = \tilde{\mathcal{R}}(m - \mathrm{c}_{t_n}(m))$ and $m^{\texttt{next}} = \tilde{\mathcal{R}}(m - \mathrm{c}(m))$. The first equality stems form the fact that $\mathrm{v}_{t_n} \to \mathrm{v}$ pointwise, and because pointwise convergence implies pointwise convergence along a sub-sequence. To see why $\lim_{i\to\infty}\mathrm{u}(\mathrm{c}_{t_{n(i)}}(m)) = \mathrm{u}(\mathrm{c}(m))$, note the continuity of u and the convergence of $\mathrm{c}_{t_{n(i)}}$ to c point-wise. Turning to the second inequality, to see why $\lim_{i\to\infty}\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}_{t_{n(i)}+1}(m_{t_{n(i)}+1}) = \mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}(m^{\texttt{next}})$, note that $\mathrm{v}_{t_{n(i)}+1}$ converges in the φ-norm, hence converges uniformly over compact sets in $\mathbb{R}_{++}$ (Fact 1, Appendix

F). Thus, by Fact 2 in Appendix F, $\mathrm{v}_{t_{n(i)}+1}(m_{t_{n(i)}+1})$ converges almost surely. Applying Dominated Convergence Theorem gives us $\lim_{i\to\infty} \mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}_{t_{n(i)}+1}(m_{t_{n(i)}+1}) = \mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}(m^{\texttt{next}})$.

This completes the proof of part (ii) of the Theorem.

Proof of part (iii). The limits at Equation (37) immediately imply:

$$\mathrm{v}(m) = \lim_{n\to\infty}\left[\mathrm{u}(\mathrm{c}_{t_n}(m)) + \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}_{t_n+1}(m_{t_n+1})\right] = \mathrm{u}(\mathrm{c}(m)) + \beta\mathbb{E}\tilde{\mathcal{G}}^{1-\gamma}\mathrm{v}(m^{\texttt{next}}), \quad (38)$$

since a real valued sequence can have at most one limit.

Finally, applying Fact 1 from Appendix F, we get $\mathrm{c}_{t_n}(m) \to \mathrm{c}(m)$, thus establishing that c_{t_n} converges point-wise to c. Since $\mathrm{v} \in \mathcal{C}_{\varphi}(\mathbb{R}_{++}, \mathbb{R})$, we must have that $\mathrm{c}(m) > 0$ for any $m > 0$, allowing us to conclude that v and c is a non-degenerate limiting solution. □

A.5 Properties of the Converged Consumption Function

{sec:appxconvgC}

Let c be the limiting non-degenerate consumption function.

{eq:EuelrStatC}

Claim 4. *If weak return impatience (Assumption L.4) holds, then* c *satisfies* $\mathrm{c}(m)^{-\gamma} = \mathsf{R}\beta\mathbb{E}_t[\tilde{\mathcal{G}}_{t+1}^{-\gamma}\mathrm{c}(m^{\texttt{next}})^{-\gamma}]$*, where* $m^{\texttt{next}} = \tilde{\mathcal{R}}(m - \mathrm{c}(m)) + \boldsymbol{\xi}$*.*

Proof. By Theorem 2, c_{T-n} converges point-wise to c as $n \to \infty$. Since c_{T-n} is the optimal consumption function for time $T-n$, $\mathrm{c}_{T-n}(m)^{-\gamma} = \mathsf{R}\beta\mathbb{E}_t[\tilde{\mathcal{G}}_{t+1}^{-\gamma}\mathrm{c}_{T-n+1}(m_{t+1})^{-\gamma}]$, where $m_{t+1} = \tilde{\mathcal{R}}(m - \mathrm{c}_{T-n}(m)) + \boldsymbol{\xi}$. Fixing $m > 0$, $\tilde{\mathcal{R}}(m - \mathrm{c}_{T-n}(m)) + \boldsymbol{\xi}$ converges almost surely to $\tilde{\mathcal{R}}(m - \mathrm{c}(m)) + \boldsymbol{\xi}$. Making use of the Dominated Converge (see proof of Claim 3), $\mathsf{R}\beta\mathbb{E}_t[\tilde{\mathcal{G}}_{t+1}^{-\gamma}\mathrm{c}_{T-n+1}(m_{t+1})^{-\gamma}]$ converges to $\mathsf{R}\beta\mathbb{E}_t[\tilde{\mathcal{G}}_{t+1}^{-\gamma}\mathrm{c}(m^{\texttt{next}})^{-\gamma}]$. Since $\mathrm{c}_{T-n}(m)^{-\gamma}$ converges to $\mathrm{c}(m)$ and $m \in \mathbb{R}_{++}$, the result follows. □

Proof of Lemma 2. First, we verify c is concave. Since weak return impatience (Assumption L.4) holds, by Theorem 2, $\mathrm{c}_{T-n} \to \mathrm{c}$ point-wise on $\mathbb{R}_{++}$ as $n \to \infty$. Moreover, since $\mathbb{R}_{++}$ is open, we can apply Theorem 10.8 by Rockafellar (1972), which confirms that c is concave on $\mathbb{R}_{++}$.

Next, note that $\mathrm{c}(m) > 0$ on $\mathbb{R}_{++}$ (recall Remark 4). Thus, we must have that $\frac{\mathrm{c}(m)}{m}$ is non-increasing (see Claim 5 in Appendix F) and since $\mathrm{c}(m)$ is feasible (Equation 36), $0 \leq \frac{\mathrm{c}(m)}{m} \leq 1$. Because $\frac{\mathrm{c}(m)}{m}$ is non-increasing and bounded above and below on $\mathbb{R}_{++}$, we can define $\bar{\bar{\kappa}} \colon= \lim_{m\downarrow 0}\frac{\mathrm{c}(m)}{m}$ and $\underline{\underline{\kappa}} \colon= \lim_{m\to\infty}\frac{\mathrm{c}(m)}{m}$ where $0 \leq \underline{\underline{\kappa}} \leq \bar{\bar{\kappa}} \leq 1$.

We fist show $\bar{\bar{\kappa}} = \bar{\kappa}$ and then show $\underline{\underline{\kappa}} = \underline{\kappa}$. Since c satisfies the Euler equation by Claim 4, we have

$$\mathrm{e}(m)^{-\gamma} = \beta \mathsf{R} \mathbb{E}_t \left(\mathrm{e}(m) \left(\frac{\overbrace{\mathsf{R}\mathrm{a}(m) + \tilde{\mathcal{G}}\boldsymbol{\xi}}^{=m\tilde{\mathcal{G}}}}{m} \right) \right)^{-\gamma} \tag{39}$$ {eq:eFuncEulerStat

where $m^{\texttt{next}} = \tilde{\mathcal{R}}(m - \mathrm{c}(m)) + \boldsymbol{\xi}$. The minimal MPC's are obtained by letting $m \to \infty$. Note that $\lim_{m_t \to \infty} m^{\texttt{next}} = \infty$ almost surely and thus $\lim_{m_t \to \infty} \mathrm{e}_{t+1}(m_{t+1}) = \underline{\underline{\kappa}}$ almost surely. Turning to the second term inside the marginal utility on the RHS, we can write

$$\begin{aligned} \lim_{m \to \infty} \frac{\mathsf{R}\mathrm{a}(m) + \tilde{\mathcal{G}}\boldsymbol{\xi}}{m_t} &= \lim_{m \to \infty} \frac{\mathsf{R}\mathrm{a}(m)}{m} + \lim_{m \to \infty} \frac{\tilde{\mathcal{G}}\boldsymbol{\xi}}{m} && (40) \\ &= \mathsf{R}(1 - \underline{\underline{\kappa}}) + 0, && (41) \end{aligned}$$

since $\tilde{\mathcal{G}}\boldsymbol{\xi}$ is bounded. Thus, as m tends to ∞, we have

$$\lim_{m \to \infty} \mathrm{e}(m)^{-\gamma} = \underline{\underline{\kappa}}^{-\gamma} = \beta \mathsf{R} \underline{\underline{\kappa}}^{-\gamma} \mathsf{R}^{-\gamma} (1 - \underline{\underline{\kappa}})^{-\gamma}. \tag{42}$$

Re-arranging the terms above gives us $\underline{\underline{\kappa}} = 1 - \text{Þ}/\mathsf{R} = \underline{\kappa}$ as required. Finally, analogously following the steps before Equation (13) and noting $\overline{\overline{\kappa}} = \lim_{m \downarrow 0} \frac{\mathrm{c}(m)}{m}$, we can conclude $\overline{\overline{\kappa}} = \wp^{-1/\gamma} (\beta \mathsf{R})^{-1/\gamma} \mathsf{R}(1 - \overline{\overline{\kappa}})\overline{\overline{\kappa}}$. Whence $\overline{\overline{\kappa}} = 1 - \wp^{1/\gamma} \text{Þ}/\mathsf{R} = \overline{\kappa}$.

□

A.6 The Liquidity Constrained Solution as a Limit

{sec:LiqConstrAsLi

Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned} \wp &= 0 \\ c_t &\leq m_t, \end{aligned}$$

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed $\wp$ as $\mathrm{c}_t(m; \wp)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and $\wp$, which is not. The proposition we wish to demonstrate is

$$\lim_{\wp \downarrow 0} \mathrm{c}_t(m; \wp) = \grave{\mathrm{c}}_t(m). \tag{43}$$ {eq:RestrEqUnrestr

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = \mathsf{R} = \mathcal{G} = 1$, and there are no permanent shocks, $\psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ \mathrm{u}(m-a) + \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{v}_T(a+\theta) d\mathcal{F}_\theta \right\}. \tag{44}$$ {eq:vUnconstr}

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{u}'(\mathrm{c}_T(a+\theta)) d\mathcal{F}_\theta,$$

and the solution to (44) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{45}$$ {eq:uPConstr}

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $\wp > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (1991)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\gamma}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (45), defining

$$\mathfrak{v}'_{T-1}(a;\wp) \equiv \left[\wp a^{-\gamma} + (1-\wp) \int_{\underline{\theta}}^{\bar{\theta}} \left(\mathrm{c}_T(a+\theta/(1-\wp))\right)^{-\gamma} d\mathcal{F}_\theta \right], \tag{46}$$ {eq:vFrakPrime}

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\gamma} = \mathfrak{v}'_{T-1}(a;\wp) \tag{47}$$ {eq:uPUnconstr}

with solution $\mathrm{a}^*_{T-1}(m;\wp)$. Now note that for any fixed $a > 0$, $\lim_{\wp \downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (45) and (47) are identical, this means that $\lim_{\wp \downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) =$

$\grave{\mathrm{a}}^{*}_{T-1}(m)$. That is, for any fixed value of $m > m^{1}_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^{1}_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (46) is $\lim_{a \downarrow 0} \wp a^{-\gamma} = \infty$, while $\lim_{a \downarrow 0} (m-a)^{-\gamma}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^{1}_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp \downarrow 0} \mathfrak{v}'_{T-1}(a; \wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^{*}_{T-1}(m) < 0$, and we showed earlier that $\lim_{\wp \downarrow 0} \mathrm{a}^{*}_{T-1}(m; \wp) = \grave{\mathrm{a}}^{*}_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^{1}_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp \downarrow 0} \mathrm{c}_{T-1}(m; \wp) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (43). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (15) for the maximal marginal propensity to consume satisfies

$$\lim_{\wp \downarrow 0} \overline{\kappa} \ \ = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of ‘constrained’ and ‘restrained.’

References

B Appendix for Section 3

{sec:ApndxMTarget

B.1 Asymptotic Consumption Growth Factors

{subsec:AppxCgrow

Proof for Proposition 4. For consumption growth, as $m \to 0$ we have:

$$
\begin{aligned}
\lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) \tilde{\mathcal{G}}_{t+1} \right] &> \lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\tilde{\mathcal{R}}_{t+1} \mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1})}{\overline{\kappa} m_t} \right) \tilde{\mathcal{G}}_{t+1} \right] \\
&= \wp \lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\tilde{\mathcal{R}}_{t+1} \mathrm{a}(m_t))}{\overline{\kappa} m_t} \right) \mathcal{G}_{t+1} \right] \qquad (48) \\
&\quad + (1-\wp) \lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\tilde{\mathcal{R}}_{t+1} \mathrm{a}(m_t) + \theta_{t+1}/(1-\wp))}{\overline{\kappa} m_t} \right) \tilde{\mathcal{G}}_{t+1} \right] \qquad (49) \\
&> (1-\wp) \lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/(1-\wp))}{\overline{\kappa} m_t} \right) \tilde{\mathcal{G}}_{t+1} \right] \\
&= \infty
\end{aligned}
$$

{eq:consGrowth}

where the second-to-last line follows because $\lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\tilde{\mathcal{R}}_{t+1} \mathrm{a}(m_t))}{\overline{\kappa} m_t} \right) \tilde{\mathcal{G}}_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.

Next we establish the limit of the expected consumption growth factor as $m_t \to \infty$:

$$
\lim_{m_t \to \infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] = \lim_{m_t \to \infty} \mathbb{E}_t[\tilde{\mathcal{G}}_{t+1} c_{t+1}/c_t].
$$

But

$$
\mathbb{E}_t[\tilde{\mathcal{G}}_{t+1} \underline{c}_{t+1}/\bar{c}_t] \leq \mathbb{E}_t[\tilde{\mathcal{G}}_{t+1} c_{t+1}/c_t] \leq \mathbb{E}_t[\tilde{\mathcal{G}}_{t+1} \bar{c}_{t+1}/\underline{c}_t]
$$

and

$$
\lim_{m_t \to \infty} \tilde{\mathcal{G}}_{t+1} \underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t \to \infty} \tilde{\mathcal{G}}_{t+1} \bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t \to \infty} \tilde{\mathcal{G}}_{t+1} m_{t+1}/m_t,
$$

while (for convenience defining $\mathrm{a}(m_t) = m_t - \mathrm{c}(m_t)$),

$$
\begin{aligned}
\lim_{m_t \to \infty} \tilde{\mathcal{G}}_{t+1} m_{t+1}/m_t &= \lim_{m_t \to \infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \tilde{\mathcal{G}}_{t+1} \boldsymbol{\xi}_{t+1}}{m_t} \right) \qquad (50) \\
&= (\mathsf{R}\beta)^{1/\gamma} = \text{Þ}
\end{aligned}
$$

{eq:xtp1toinfty}

because $\lim_{m_t \to \infty} \mathrm{a}'(m) = \text{Þ}/\mathsf{R}$[2] and $\tilde{\mathcal{G}}_{t+1}\boldsymbol{\xi}_{t+1}/m_t \leq (\mathcal{G}\bar{\psi}\bar{\theta}/(1-\wp))/m_t$ which goes to zero as m_t goes to infinity. Hence we have:

$$\text{Þ} \leq \lim_{m_t \to \infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] \leq \text{Þ}$$

so as cash goes to infinity, consumption growth approaches its value Þ in the perfect foresight model. □

This appendix proves Theorems 3-4 and:

{lemma:orderingP

Lemma 2. *If $\breve{m}$ and $\hat{m}$ both exist, then $\breve{m} \leq \hat{m}$.*

B.2 Existence of Buffer Stock Target

B.2.1 Existence of Individual Buffer Stock Target

{subsubsec:AppxInd

Proof of Theorem 3. First, observe that $\mathbb{E}_t[m_{t+1}/m_t] = \frac{\mathbb{E}_t\left((m_t - \mathrm{c}(m_t))\tilde{\mathcal{R}}_{t+1} + \boldsymbol{\xi}_{t+1}\right)}{m_t}$. Note that c is continuous since c is concave on $\mathbb{R}_{++}$ by Lemma 2. Thus for any convergent sequence $\left\{m_t^j\right\}_{j=0}^{\infty}$, with $m_t^j \in \mathbb{R}_{++}$, $(m_t^j - \mathrm{c}(m_t^j))\tilde{\mathcal{R}}_{t+1} + \boldsymbol{\xi}_{t+1}$ will be bounded above and below. It follows that, using the Dominated Convergence Theorem, $\mathbb{E}_t[m_{t+1}/m_t]$ will be continuous in m_t.

The remainder of the proof proceeds as follows. To establish Equation (25), we will show (i) that there exists a point $\breve{m}_t$ where $\mathbb{E}_t[\breve{m}^\star_{t+1}/\breve{m}^\star_t] < 1$ and (ii) a point $\grave{m}$ where $\mathbb{E}_t[\grave{m}_{t+1}/\grave{m}_t] > 1$. By continuity of $\mathbb{E}[m_{t+1}/m_t]$ in m_t and the Intermediate Value Theorem, there will exist $\hat{m}$ such that $\mathbb{E}_t[\hat{m}_{t+1}/\hat{m}_t] = 1$. In turn, to establish that $\hat{m}$ is a point of stability, Equation (26), we will show that (iii) $\mathbb{E}_t[m_{t+1}] - m_t$ is decreasing.

Part (i). Existence of $\breve{m}_t$ where $\mathbb{E}_t[\breve{m}_{t+1}/\breve{m}_t] < 1$.

To proceed, first suppose return impatience holds and take the steps analogous to those leading to Equation (50) in the proof of proof for Proposition 4, but dropping the $\mathcal{G}_{t+1}$ from the RHS:

$$\begin{aligned}
\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \to \infty} \mathbb{E}_t\left[\frac{\tilde{\mathcal{R}}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1}}{m_t}\right] \\
&= \mathbb{E}_t[(\mathsf{R}/\tilde{\mathcal{G}}_{t+1})\text{Þ}/\mathsf{R}] \\
&= \mathbb{E}_t[\text{Þ}/\tilde{\mathcal{G}}_{t+1}] \qquad (51) \\
&< 1,
\end{aligned}$$ {eq:emgro}

where the inequality follows from strong growth impatience. By continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ in m_t, there exists $\breve{m}_t$ large enough such that $\mathbb{E}_t[\breve{m}_{t+1}/\breve{m}_t] < 1$.

[2] $\lim_{m_t \to \infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t \to \infty} \mathrm{c}(m_t)/m_t = 1 - \lim_{m_t \to \infty} \mathrm{c}'(m_t) = \text{Þ}/\mathsf{R}$.

Next, suppose return impatience fails. The fact that $\lim_{m_t\to\infty} \frac{\mathrm{c}(m_t)}{m_t} = 0$ (Lemma 2) means the limit of the RHS of (51) as $m_t \to \infty$ is $\bar{\tilde{\mathcal{R}}} = \mathbb{E}_t[\tilde{\mathcal{R}}_{t+1}]$. Equations (55)-(56) below show that when strong growth impatience holds and return impatience fails $\bar{\tilde{\mathcal{R}}} < 1$.

Thus, we have $\lim_{m\to\infty} \mathbb{E}[m_{t+1}/m_t] < 1$ whether the return impatience holds or fails.

Part (ii). Existence of $\grave{m}_t > 1$ where $\mathbb{E}_t[\grave{m}_{t+1}/\grave{m}_t] > 1$.

Analogous to Equation (48), the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \to 0$ because $\lim_{m_t\to 0} \mathbb{E}[m_{t+1}] > 0$. Thus, if $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous in m_t, and takes on values above and below one at $\grave{m}_t$ and $\breve{m}_t$, by the Intermediate Value Theorem, there must be at least one point at which it is equal to one.

Part (iii). $\mathbb{E}_t[m_{t+1}] - m_t$ is strictly decreasing.

Finally to show $\mathbb{E}_t[m_{t+1}] - m_t$ is strictly decreasing m_t, define $\boldsymbol{\zeta}(m_t) \colon = \mathbb{E}_t[m_{t+1}] - m_t$ and note that:

$$
\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,,
\end{aligned}
\tag{52}
$$

{eq:difNrmioEquiv}

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$. Let Δ_ϵ be the finite forward difference for spacing $\epsilon > 0$. Fixing $\epsilon > 0$, we will have:

$$
\begin{aligned}
\Delta_\epsilon \boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\Delta_\epsilon\left(\tilde{\mathcal{R}}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1} - m_t\right)\right] \\
&= \bar{\tilde{\mathcal{R}}}\left(\epsilon - \Delta_\epsilon \mathrm{c}(m_t)\right) - \epsilon = \epsilon\left(\bar{\tilde{\mathcal{R}}}\left[1 - \frac{\Delta_\epsilon \mathrm{c}(m_t)}{\epsilon}\right] - 1\right).
\end{aligned}
\tag{53}
$$

{eq:finiteDiff2}

Notice that $\frac{\Delta_\epsilon \mathrm{c}(m_t)}{\epsilon} \le \frac{\mathrm{c}(m_t)}{m_t} < 1$ since $\frac{\mathrm{c}(m_t)}{m_t}$ is decreasing in m_t by Claim 5 in Appendix F. Consider the case when return impatience holds. Equation (15) and Lemma 2 indicate $0 < \underline{\kappa} \le \frac{\mathrm{c}(m_t)}{m_t} < 1$. It follows that:

$$
\begin{aligned}
\bar{\tilde{\mathcal{R}}}\left[1 - \frac{\Delta_\epsilon \mathrm{c}'(m_t)}{\epsilon}\right] - 1 &\le \bar{\tilde{\mathcal{R}}}(1 - \underbrace{(1 - \text{Þ}/\mathsf{R})}_{\underline{\kappa}}) - 1 \\
&= \bar{\tilde{\mathcal{R}}}\text{Þ}/\mathsf{R} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\mathcal{G}\psi_{t+1}}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\mathcal{G}\psi_{t+1}}\right]}_{=\text{Þ}/\mathcal{G}\mathbb{E}[\psi^{-1}]} - 1
\end{aligned}
$$

which is negative because the strong growth impatience says $\text{Þ}/\mathcal{G}\mathbb{E}[\psi^{-1}] < 1$. Conversely, when return impatience holds fails, recall $\lim_{m_t \to \infty} \frac{\mathrm{c}(m_t)}{m_t} = 0$. This means $\Delta_\epsilon \boldsymbol{\zeta}(m_t)$ from (53) is guaranteed to be negative if:

$$\bar{\bar{\mathcal{R}}} = \mathbb{E}_t \left[\frac{\mathsf{R}}{\mathcal{G}\psi_{t+1}} \right] < 1. \tag{54}$$

{eq:RbarBelowOne}

But the combination of the strong growth impatience holding and the return impatience failing can be written:

$$\overbrace{\mathbb{E}_t \left[\frac{\text{Þ}}{\mathcal{G}\psi_{t+1}} \right]}^{\text{Þ}/\mathcal{G}\mathbb{E}[\psi^{-1}]} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}/\mathsf{R}}, \tag{55}$$

{eq:GICStrRICfails

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives:

$$\mathbb{E}_t \left[\frac{\mathsf{R}}{\mathcal{G}\psi_{t+1}} \right] < \mathsf{R}/\text{Þ} < 1, \tag{56}$$

{eq:GICStrRICfails

which satisfies our requirement in (54), thus completing the proof.

□

B.2.2 Existence of Pseudo-Steady-State

{subsubsec:AppxPs

Proof of Theorem 4. Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in B.2.1 that demonstrated existence and continuity of $\mathbb{E}[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}[\psi_{t+1}m_{t+1}/m_t]$.

Part (i). Existence of a stable point

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in Subsection B.2.1 that the ratio of $\mathbb{E}[m_{t+1}]$ to m_t is unbounded as $m_t \to 0$ implies that the ratio $\mathbb{E}[\psi_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \to 0$. The limit of the expected ratio as $m_t \to \infty$ goes to infinity is can be found as follows:

$$\begin{aligned}
\lim_{m_t \to \infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t \to \infty} \mathbb{E}_t \left[\frac{\tilde{\mathcal{G}}_{t+1} \left((\mathsf{R}/\tilde{\mathcal{G}}_{t+1})\mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1} \right) / \mathcal{G}}{m_t} \right] \\
&= \lim_{m_t \to \infty} \mathbb{E}_t \left[\frac{(\mathsf{R}/\mathcal{G})\mathrm{a}(m_t) + \psi_{t+1}\boldsymbol{\xi}_{t+1}}{m_t} \right] \\
&= \lim_{m_t \to \infty} \left[\frac{(\mathsf{R}/\mathcal{G})\mathrm{a}(m_t) + 1}{m_t} \right] \\
&= (\mathsf{R}/\mathcal{G})\text{Þ}/\mathsf{R} \qquad (57) \\
&= \text{Þ}/\mathcal{G} \\
&< 1,
\end{aligned}$$

{eq:emgro2}

where the last two lines are merely a restatement of growth impatience.

To conclude Part (i) of the proof, the Intermediate Value Theorem says that if $\mathbb{E}[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

Part (ii). $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ *is monotonically decreasing.*

Define $\boldsymbol{\zeta}(m_t) := \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that:

$$\begin{aligned} \boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1, \end{aligned} \tag{58}$$ {eq:difLvlEquiv}

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$. Letting Δ_ϵ be the forward difference operator, we have:

$$\begin{aligned} \Delta_\epsilon \boldsymbol{\zeta}(m_t) &= \mathbb{E}\left[\Delta_\epsilon\left(\frac{\mathsf{R}}{\mathcal{G}}(m_t - \mathrm{c}(m_t)) + \psi_{t+1}\boldsymbol{\xi}_{t+1} - m_t\right)\right] \\ &= \frac{\mathsf{R}}{\mathcal{G}}\left(\epsilon - \Delta_\epsilon \mathrm{c}'(m_t)\right) - \epsilon = \epsilon\left(\frac{\mathsf{R}}{\mathcal{G}}\left[1 - \frac{\Delta_\epsilon \mathrm{c}(m_t)}{\epsilon}\right] - 1\right). \end{aligned} \tag{59}$$ {eq:finiteDiff}

for any given $\epsilon > 0$. Notice that $\frac{\Delta_\epsilon \mathrm{c}'(m_t)}{\epsilon} \leq \frac{\mathrm{c}(m_t)}{m_t} < 1$ since $\frac{\mathrm{c}(m_t)}{m_t}$ is decreasing in m_t by Claim 5 in Appendix. Now, we show that $\boldsymbol{\zeta}(m)$ is decreasing when return impatience holds and when return impatience fails. When return impatience holds, Equation (15) and Lemma 2 indicate that $\underline{\kappa} > 0$ and $0 < \underline{\kappa} \leq \frac{\mathrm{c}(m_t)}{m_t} < 1$. It follows that:

$$\begin{aligned} \frac{\mathsf{R}}{\mathcal{G}}\left(1 - \mathrm{c}'(m_t)\right) - 1 &< \frac{\mathsf{R}}{\mathcal{G}}(1 - \underbrace{(1 - \text{Þ}/\mathsf{R})}_{\underline{\kappa}}) - 1 \\ &= (\mathsf{R}/\mathcal{G})\text{Þ}/\mathsf{R} - 1, \end{aligned}$$

which is negative because growth impatience says $\text{Þ}/\mathcal{G} < 1$. Conversely, when return impatience holds fails, recall $\lim\limits_{m_t \to \infty} \frac{\mathrm{c}(m_t)}{m_t} = 0$. In turn, this means $\Delta_\epsilon \boldsymbol{\zeta}(m_t)$ from (59) is guaranteed to be negative if:

$$(\mathsf{R}/\mathcal{G}) < 1. \tag{60}$$ {eq:FHWCFails}

But we showed in Section 2.3.1, Equation (44), that the only circumstances under which the problem has a non-degenerate solution while return impatience fails were ones where the finite limiting human wealth also fails. Thus, $(\mathsf{R}/\mathcal{G}) < 1$, completing the proof.

□

C Appendix for Section 4

C.1 Apparent Balanced Growth in $\bar{c}$ and $\text{cov}(c, \boldsymbol{p})$

{sec:ApndxBalance

Section 4.2 demonstrates some propositions under the assumption that, when an economy satisfies the GIC, there will be constant growth factors $\Omega_{\bar{c}}$ and Ω_{cov} respectively for $\bar{c}$ (the average value of the consumption ratio) and $\text{cov}(c, \boldsymbol{p})$. In the case of a Szeidl-invariant economy, the main text shows that these are $\Omega_{\bar{c}} = 1$ and $\Omega_{\text{cov}} = \mathcal{G}$. If the economy is Harmenberg- but not Szeidl-invariant, no proof is offered that these growth factors will be constant.

C.2 $\log c$ and $\log\left(\text{cov}(c, \boldsymbol{p})\right)$ Grow Linearly

Figures 1 and 2 plot the results of simulations of an economy that satisfies Harmenberg- but not Szeidl-invariance with a population of 4 million agents over the last 1000 periods (of a 2000 period simulation).[3] The first figure shows that $\log \bar{c}$ increases apparently linearly. The second figure shows that $\log(-\,\text{cov}(c, \boldsymbol{p}))$ also increases apparently linearly. (These results are produced by the notebook `ApndxBalancedGrowthcNrmAndCov.ipynb`).

[3]For an exposition of our implementation of Harmenberg's method, see this supplemental appendix.

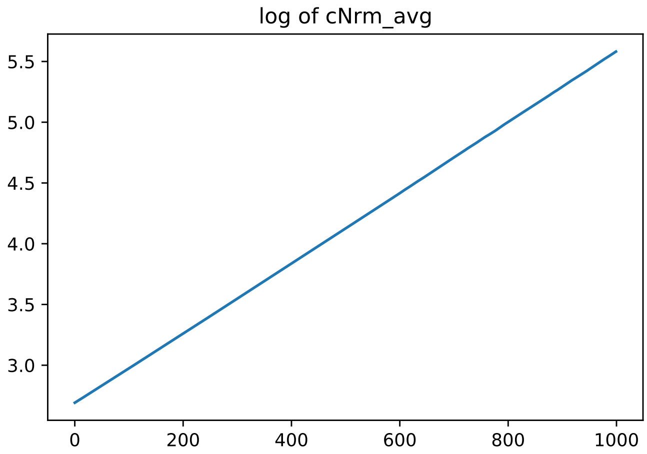

Figure 1 Appendix: log $\boldsymbol{\mathfrak{c}}$ Appears to Grow Linearly {fig:logcNrm}

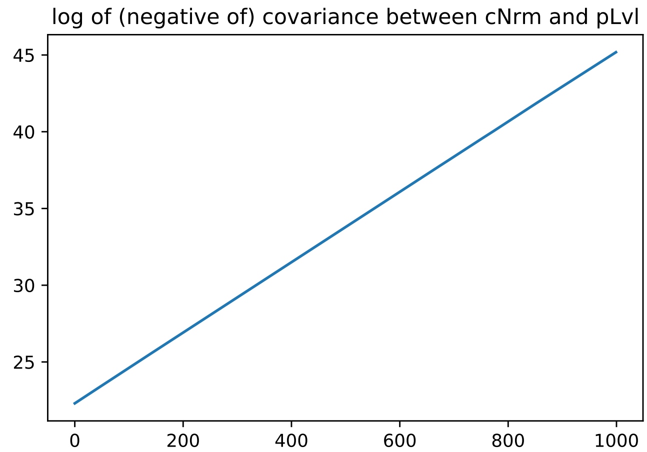

Figure 2 Appendix: $\log\left(-\operatorname{cov}(c, \boldsymbol{p})\right)$ Appears to Grow Linearly {fig:logcov}

D Appendix for Section 5

{sec:ApndxLiqCons

In this appendix, we use the following acronyms to refer to the consumer patience conditions identified in Section 2.2 using the acronyms from Table 3.

We briefly interpret FVAC before turning to how all the conditions relate under uncertainty. Analogously to (61), the value for a consumer who spent exactly their permanent income every period would reflect the product of the expectation of the (independent) future shocks to permanent income:

$$= \mathrm{u}(\boldsymbol{p}_t)\left(\frac{1-\left(\beta\mathcal{G}^{1-\gamma}\mathbb{E}[\psi^{1-\gamma}]\right)^{T-t+1}}{1-\beta\mathcal{G}^{1-\gamma}\mathbb{E}[\psi^{1-\gamma}]}\right),$$

The function $\mathbf{v}_t$ will be finite as T approaches ∞ if the FVAC holds. In the case without uncertainty, Because $\mathrm{u}(xy) = x^{1-\gamma}\mathrm{u}(y)$, the value the consumer would achieve is:

$$\begin{aligned}
\mathbf{v}_t^{\text{autarky}} &= \mathrm{u}(\boldsymbol{p}_t) + \beta\mathrm{u}(\boldsymbol{p}_t\mathcal{G}) + \beta^2\mathrm{u}(\boldsymbol{p}_t\mathcal{G}^2) + \ldots \\
&= \mathrm{u}(\boldsymbol{p}_t)\left(\frac{1-\left(\beta\mathcal{G}^{1-\gamma}\right)^{T-t+1}}{1-\beta\mathcal{G}^{1-\gamma}}\right)
\end{aligned}$$

which (for $\mathcal{G} > 0$) asymptotes to a finite number as n, with $n = T - t$, approaches $+\infty$.

D.1 Perfect Foresight Unconstrained Solution

{subsec:ApndxUCP

The first result relates to the perfect foresight case without liquidity constraints.

Proof of Proposition 1. Consider a sequence of consumption $\{\boldsymbol{c}_{T-n}\}_{n=t}^{T}$ and a sequence of income $\{\boldsymbol{p}_{T-n}\}_{n=t}^{T}$ and let $\mathrm{PDV}_t^T(\boldsymbol{c})$ and $\mathrm{PDV}_t^T(\boldsymbol{p})$ denote the present discounted value of the consumption sequence and permanent income sequence respectively. The dynamic budget constraint, strictly positive marginal utility, and the can't-die-in-debt condition, Equation (1), imply an exactly-holding intertemporal budget constraint (IBC):

$$\mathrm{PDV}_t^T(\boldsymbol{c}) = \overbrace{\boldsymbol{m}_t - \boldsymbol{p}_t}^{\boldsymbol{b}_t} + \overbrace{\mathrm{PDV}_t^T(\boldsymbol{p})}^{\boldsymbol{h}_t}, \tag{61}$$ {eq:IBCFinite}

where $\boldsymbol{b}$ is beginning-of-period 'market' balances; with $\tilde{\mathcal{R}} := \mathsf{R}/\mathcal{G}$ 'human wealth' can be written as:

$$\begin{aligned}
\boldsymbol{h}_t &= \boldsymbol{p}_t + \tilde{\mathcal{R}}^{-1}\boldsymbol{p}_t + \tilde{\mathcal{R}}^{-2}\boldsymbol{p}_t + \cdots + \tilde{\mathcal{R}}^{t-T}\boldsymbol{p}_t \\
&= \underbrace{\left(\frac{1-\tilde{\mathcal{R}}^{-(T-t+1)}}{1-\tilde{\mathcal{R}}^{-1}}\right)}_{\equiv h_t}\boldsymbol{p}_t.
\end{aligned} \tag{62}$$ {eq:HDefAppx}

Let h denote the limiting value of normalized human wealth as the planning horizon recedes, we have $h := \lim_{n\to\infty} h_{t_n}$.

Next, since consumption is growing by Þ but discounted by R:

$$\mathrm{PDV}_t^T(\boldsymbol{c}) = \left(\frac{1-\text{Þ}/\mathsf{R}^{T-t+1}}{1-\text{Þ}/\mathsf{R}}\right)\boldsymbol{c}_t$$

from which the IBC (61) implies

$$\boldsymbol{c}_t = \overbrace{\left(\frac{1 - \text{Þ}/\mathsf{R}}{1 - \text{Þ}/\mathsf{R}^{T-t+1}}\right)}^{\equiv \underline{\kappa}_t}(\boldsymbol{b}_t + \boldsymbol{h}_t) \tag{63}$$ {eq:WDef}

defining a normalized finite-horizon perfect foresight consumption function:

$$\bar{\mathrm{c}}_{T-n}(m_{T-n}) = (\overbrace{m_{T-n} - 1}^{\equiv b_{T-n}} + h_{T-n})\underline{\kappa}_{t-n}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC). (The overbar signifies that $\bar{\mathrm{c}}$ will be an upper bound as we modify the problem to incorporate constraints and uncertainty; analogously, $\underline{\kappa}$ is the MPC's lower bound).

The horizon-exponentiated term in the denominator of (63) is why, for $\underline{\kappa}$ to be strictly positive as n goes to infinity, we must impose the RIC. The RIC thus implies that the consumer cannot be so pathologically patient as to wish, in the limit as the horizon approaches infinity, to spend nothing today out of an increase in current wealth (the RIC rules out the degenerate limiting solution $\bar{\mathrm{c}}(m) = 0$).

Given that the RIC holds, and (as before) defining limiting objects by the absence of a time subscript, the limiting upper bound consumption function will be

$$\bar{\mathrm{c}}(m) = (m + h - 1)\underline{\kappa}, \tag{64}$$ {eq:cFuncPFUncAp

and so in order to rule out the degenerate limiting solution $\bar{\mathrm{c}}(m) = \infty$ we need h to be finite; that is, we must impose the Finite Human Wealth Condition (FHWC), Assumption (I.3).

□

D.2 Perfect Foresight Liquidity Constrained Solutions

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\grave{\mathrm{c}}(m)$ that arise under various parametric conditions.

Results are summarized in table 1.

D.2.1 If GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails ($\cancel{\text{GIC}}$, $1 < \text{Þ}/\mathcal{G}$). Under $\cancel{\text{GIC}}$ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\gamma}/\mathcal{G}$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[4]

$$
\begin{aligned}
1 &< (\mathsf{R}\beta)^{1/\gamma}\mathcal{G}^{-1} \\
1 &< \mathsf{R}\beta\mathcal{G}^{-\gamma} \\
\mathrm{u}'(1) &< \mathsf{R}\beta\mathrm{u}'(\mathcal{G}).
\end{aligned}
$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

RIC fails, FHWC holds. If the RIC fails ($1 < \text{Þ}/\mathsf{R}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \to \infty$ is the degenerate function

$$\grave{\mathrm{c}}_{T-n}(m) = 0(b_t + h). \tag{65}$$

(that is, consumption is zero for any level of human or nonhuman wealth).

RIC fails, FHWC fails. ~~FHWC~~ implies that human wealth limits to $h = \infty$ so the consumption function limits to either $\grave{\mathrm{c}}_{T-n}(m) = 0$ or $\grave{\mathrm{c}}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[5]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying ~~GIC~~ we must impose the RIC (and the FHWC can be shown to be a consequence of ~~GIC~~ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer would choose $c = m$ from Equation (14):

$$
\begin{aligned}
m_{\#} &= (m_{\#} - 1 + h)\underline{\kappa} \\
m_{\#}(1 - \underline{\kappa}) &= (h - 1)\underline{\kappa} \\
m_{\#} &= (h - 1)\left(\frac{\underline{\kappa}}{1 - \underline{\kappa}}\right)
\end{aligned}
\tag{66}
$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[6] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\grave{\mathrm{c}}(m) = m$.[7] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{\mathrm{c}}(m)$.

---

[4]The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_{\#}) = \mathsf{R}\beta\mathrm{u}'(\mathcal{G})$ which is $c_{\#} = \mathcal{G}/(\mathsf{R}\beta)^{1/\gamma}$ and the consumption function will be defined by $\grave{\mathrm{c}}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

[5]The knife-edge case is where $\text{Þ} = \mathcal{G}$, in which case the two quantities counterbalance and the limiting function is $\grave{\mathrm{c}}(m) = \min[m, 1]$.

[6]Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~GIC~~.

[7]As an illustration, consider a consumer for whom $\text{Þ} = 1$, $\mathsf{R} = 1.01$ and $\mathcal{G} = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\mathcal{G} < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

(Stachurski and Toda (2019) obtain a similar lower bound on consumption and use it to study the tail behavior of the wealth distribution.)

D.2.2 If GIC Holds

Imposition of the GIC reverses the inequality in (65), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b_{\#}^{n}$ as the b_t such that an unconstrained consumer holding $b_t = b_{\#}^{n}$ would behave so as to arrive in period $t+n$ with $b_{t+n} = 0$ (with $b_{\#}^{0}$ trivially equal to 0); for example, a consumer with $b_{t-1} = b_{\#}^{1}$ was on the 'cusp' of being constrained in period $t-1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m_{\#}^{0} = 1 + b_{\#}^{0} = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c_{\#}^{0} = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to $\text{Þ}/\mathcal{G}$, so consumption n periods in the past must have been

$$c_{\#}^{n} = \text{Þ}/\mathcal{G}^{-n} c_t = \text{Þ}/\mathcal{G}^{-n}. \tag{67}$$ {eq:cPreHist}

The PDV of consumption from $t-n$ until t can thus be computed as

$$\begin{aligned}
\mathbb{C}_{t-n}^{t} &= c_{t-n}(1 + \text{Þ}/\mathsf{R} + \cdots + (\text{Þ}/\mathsf{R})^{n}) \\
&= c_{\#}^{n}(1 + \text{Þ}/\mathsf{R} + \cdots + \text{Þ}/\mathsf{R}^{n}) \\
&= \text{Þ}/\mathcal{G}^{-n}\left(\frac{1 - \text{Þ}/\mathsf{R}^{n+1}}{1 - \text{Þ}/\mathsf{R}}\right) \\
&= \left(\frac{\text{Þ}/\mathcal{G}^{-n} - \text{Þ}/\mathsf{R}}{1 - \text{Þ}/\mathsf{R}}\right)
\end{aligned}$$

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h_{\#}^{n} = 1 + \cdots + \tilde{\mathcal{R}}^{-n} \tag{68}$$

while the intertemporal budget constraint says

$$\mathbb{C}_{t-n}^{t} = b_{\#}^{n} + h_{\#}^{n}$$

from which we can solve for the $b_{\#}^{n}$ such that the consumer with $b_{t-n} = b_{\#}^{n}$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t = 0$:

$$b_{\#}^{n} = \mathbb{C}_{t-n}^{t} - \overbrace{\left(\frac{1-\tilde{\mathcal{R}}^{-(n+1)}}{1-\tilde{\mathcal{R}}^{-1}}\right)}^{h_{\#}^{n}}. \tag{69}$$ {eq:bPound}

Defining $m_{\#}^{n} = b_{\#}^{n} + 1$, consider the function $\grave{\mathrm{c}}(m)$ defined by linearly connecting the points $\{m_{\#}^{n}, c_{\#}^{n}\}$ for integer values of $n \geq 0$ (and setting $\grave{\mathrm{c}}(m) = m$ for $m < 1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets $m = m_{\#}^{n} - \epsilon$ is discretely higher than for a consumer with assets $m = m_{\#}^{n} + \epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (69) for the entire domain of positive real values of b, we need $b_{\#}^{n}$ to become arbitrarily large with n. That is, we need

$$\lim_{n\to\infty} b_{\#}^{n} = \infty. \tag{70}$$ {eq:bToInfty}

If FHWC Holds The FHWC requires $\tilde{\mathcal{R}}^{-1} < 1$, in which case the second term in (69) limits to a constant as $n \to \infty$, and (70) reduces to a requirement that

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\text{Þ}/\mathcal{G}^{-n} - (\text{Þ}/\mathsf{R}/\text{Þ}/\mathcal{G})^{n}\text{Þ}/\mathsf{R}}{1-\text{Þ}/\mathsf{R}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}/\mathcal{G}^{-n} - \tilde{\mathcal{R}}^{-n}\text{Þ}/\mathsf{R}}{1-\text{Þ}/\mathsf{R}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}/\mathcal{G}^{-n}}{1-\text{Þ}/\mathsf{R}}\right) &= \infty.
\end{aligned}$$

Given the GIC $\text{Þ}/\mathcal{G}^{-1} > 1$, this will hold iff the RIC holds, $\text{Þ}/\mathsf{R} < 1$. But given that the FHWC $\mathsf{R} > \mathcal{G}$ holds, the GIC is stronger (harder to satisfy) than the RIC; thus, the FHWC and the GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \grave{\mathrm{c}}(m) - \bar{\mathrm{c}}(m) = 0. \tag{71}$$

If FHWC Fails If the FHWC fails, matters are a bit more complex.

Given failure of FHWC, (70) requires

$$\lim_{n\to\infty}\left(\frac{\tilde{\mathcal{R}}^{-n}\text{Þ}/\mathsf{R}-\text{Þ}/\mathcal{G}^{-n}}{\text{Þ}/\mathsf{R}-1}\right)+\left(\frac{1-\tilde{\mathcal{R}}^{-(n+1)}}{\tilde{\mathcal{R}}^{-1}-1}\right)=\infty$$
$$\lim_{n\to\infty}\left(\frac{\text{Þ}/\mathsf{R}}{\text{Þ}/\mathsf{R}-1}-\frac{\tilde{\mathcal{R}}^{-1}}{\tilde{\mathcal{R}}^{-1}-1}\right)\tilde{\mathcal{R}}^{-n}-\left(\frac{\text{Þ}/\mathcal{G}^{-n}}{\text{Þ}/\mathsf{R}-1}\right)=\infty$$

If RIC Holds. When the RIC holds, rearranging (72) gives

$$\lim_{n\to\infty}\left(\frac{\text{Þ}/\mathcal{G}^{-n}}{1-\text{Þ}/\mathsf{R}}\right)-\tilde{\mathcal{R}}^{-n}\left(\frac{\text{Þ}/\mathsf{R}}{1-\text{Þ}/\mathsf{R}}+\frac{\tilde{\mathcal{R}}^{-1}}{\tilde{\mathcal{R}}^{-1}-1}\right)=\infty$$

and for this to be true we need

$$\begin{aligned}\text{Þ}/\mathcal{G}^{-1}&>\tilde{\mathcal{R}}^{-1}\\ \mathcal{G}/\text{Þ}&>\mathcal{G}/\mathsf{R}\\ 1&>\text{Þ}/\mathsf{R}\end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty}\kappa_{\#}^{n}=\lim_{n\to\infty}\left(\frac{c_{\#}^{n}}{b_{\#}^{n}}\right)\tag{72}$$ {eq:MPCConstrLim

which with a bit of algebra[8] can be shown to asymptote to the MPC in the perfect foresight model:[9]

$$\lim_{m\to\infty}\grave{\boldsymbol{\kappa}}(m)=1-\text{Þ}/\mathsf{R}.\tag{74}$$

If RIC Fails. Consider now the $\cancel{\text{RIC}}$ case, $\text{Þ}/\mathsf{R}>1$. We can rearrange (72) as

$$\lim_{n\to\infty}\left(\frac{\text{Þ}/\mathsf{R}(\tilde{\mathcal{R}}^{-1}-1)}{(\tilde{\mathcal{R}}^{-1}-1)(\text{Þ}/\mathsf{R}-1)}-\frac{\tilde{\mathcal{R}}^{-1}(\text{Þ}/\mathsf{R}-1)}{(\tilde{\mathcal{R}}^{-1}-1)(\text{Þ}/\mathsf{R}-1)}\right)\tilde{\mathcal{R}}^{-n}-\left(\frac{\text{Þ}/\mathcal{G}^{-n}}{\text{Þ}/\mathsf{R}-1}\right)=\infty.\tag{75}$$

which makes clear that with $\cancel{\text{FHWC}}\Rightarrow\tilde{\mathcal{R}}^{-1}>1$ and $\cancel{\text{RIC}}\Rightarrow\text{Þ}/\mathsf{R}>1$ the numerators and denominators of both terms multiplying $\tilde{\mathcal{R}}^{-n}$ can be seen transparently to be positive.

---

[8] Calculate the limit of

$$\left(\frac{\text{Þ}/\mathcal{G}^{-n}}{\text{Þ}/\mathcal{G}^{-n}/(1-\text{Þ}/\mathsf{R})-(1-\tilde{\mathcal{R}}^{-1}\tilde{\mathcal{R}}^{-n})/(1-\tilde{\mathcal{R}}^{-1})}\right)=\left(\frac{1}{1/(1-\text{Þ}/\mathsf{R})+\tilde{\mathcal{R}}^{-n}\tilde{\mathcal{R}}^{-1}/(1-\tilde{\mathcal{R}}^{-1})}\right)\tag{73}$$

[9] For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.

So, the terms multiplying $\tilde{\mathcal{R}}^{-n}$ in (72) will be positive if

$$\begin{aligned}
\mathbf{Þ}/\mathsf{R}\tilde{\mathcal{R}}^{-1} - \mathbf{Þ}/\mathsf{R} &> \tilde{\mathcal{R}}^{-1}\mathbf{Þ}/\mathsf{R} - \tilde{\mathcal{R}}^{-1} \\
\tilde{\mathcal{R}}^{-1} &> \mathbf{Þ}/\mathsf{R} \\
\mathcal{G} &> \mathbf{Þ}
\end{aligned}$$

which is merely the GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\tilde{\mathcal{R}}^{-n}$ goes to $+\infty$ faster than the term involving $-\mathbf{Þ}/\mathcal{G}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned}
\tilde{\mathcal{R}}^{-1} &> \mathbf{Þ}/\mathcal{G}^{-1} \\
\mathcal{G}/\mathsf{R} &> \mathcal{G}/\mathbf{Þ} \\
\mathbf{Þ}/\mathsf{R} &> 1
\end{aligned}$$

which merely confirms the starting assumption that the RIC fails.

What is happening here is that the $c^n_{\#}$ term is increasing backward in time at rate dominated in the limit by $\mathcal{G}/\mathbf{Þ}$ while the $b_{\#}$ term is increasing at a rate dominated by $\mathcal{G}/\mathsf{R}$ term and

$$\mathcal{G}/\mathsf{R} > \mathcal{G}/\mathbf{Þ} \tag{76}$$

because $\cancel{\text{RIC}} \Rightarrow \mathbf{Þ} > \mathsf{R}$.

Consequently, while $\lim_{n\to\infty} b^n_{\#} = \infty$, the limit of the *ratio* $c^n_{\#}/b^n_{\#}$ in (72) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It remains true that $\cancel{\text{RIC}}$ implies a limiting MPC of zero,

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 0, \tag{77}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\grave{\mathrm{c}}(m) = 0$. (Figure 3 presents an example for $\gamma = 2$, $\mathsf{R} = 0.98$, $\beta = 1.00$, $\mathcal{G} = 0.99$; note that the horizontal axis is bank balances $b = m - 1$; the part of the consumption function below the depicted points is uninteresting — $c = m$ — so not worth plotting).

We can summarize as follows. Given that the GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into the solution to the unconstrained problem as $m \to \infty$. But even if the FHWC fails, the problem has a well-defined and nondegenerate solution, whether or not the RIC holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

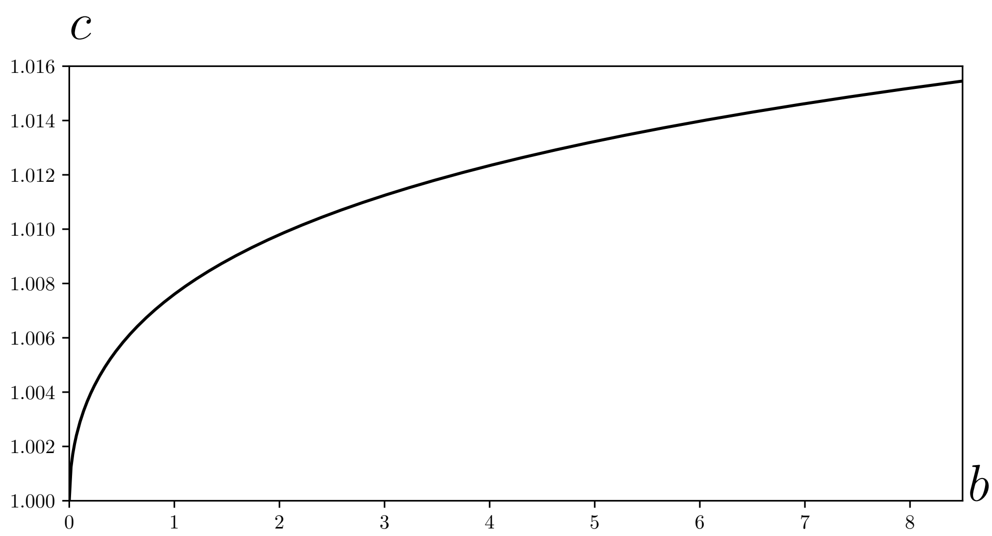

Figure 3 Appendix: Nondegenerate c Function with ~~FHWC~~ and ~~RIC~~ {fig:PFGICHoldsFH

Ma and Toda (2020) characterize the limits of the MPC in a more general framework that allows for capital and labor income risks in a Markovian setting with liquidity constraints, and find that in that much more general framework the limiting MPC is also zero.

References

E Relational Diagrams for the Inequality Conditions

{sec:ApndxConditic

This appendix explains in detail the paper's 'inequalities' diagrams (Figures 6, 7).

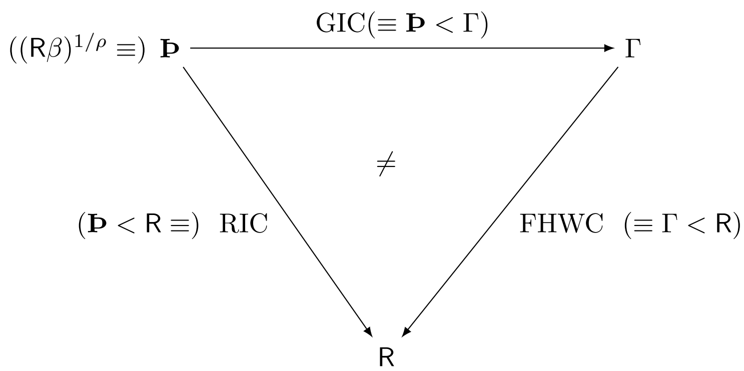

Figure 4 Appendix: Inequality Conditions for Perfect Foresight Model (Start at a node and follow arrows) {fig:InequalityPFGI

E.1 The Unconstrained Perfect Foresight Model

A simple illustration is presented in Figure 4, whose three nodes represent values of the absolute patience factor Þ, the permanent-income growth factor $\mathcal{G}$, and the riskfree interest factor R. The arrows represent imposition of the labeled inequality condition (like, the uppermost arrow, pointing from Þ to $\mathcal{G}$, reflects imposition of the GIC condition (clicking GIC should take you to its definition; definitions of other conditions are also linked below)).[10] Annotations inside parenthetical expressions containing $\equiv$ are there to make the diagram readable for someone who may not immediately remember terms and definitions from the main text. (Such a reader might also want to be reminded that R, β, and Γ are all in $\mathbb{R}_{++}$, and that $\gamma > 1$).

Navigation of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes. Traversal must stop upon arrival at a node with no exiting arrows. So, for example, we can start at the Þ node and impose the GIC and then the FHWC, and see that imposition of these conditions allows us to conclude that $\text{Þ} < \mathsf{R}$.

One could also impose $\text{Þ} < \mathsf{R}$ directly (without imposing GIC and FHWC) by following the downward-sloping diagonal arrow exiting Þ. Although alternate routes from one node to another all justify the same core conclusion ($\text{Þ} < \mathsf{R}$, in this case), $\neq$ symbol in the center is meant to convey that these routes are not identical in other respects. This notational convention is used in category theory diagrams,[11] to indicate that the diagram is not commutative.[12]

Negation of a condition is indicated by the reversal of the corresponding arrow. For example, negation of the RIC, $\cancel{\text{RIC}} \equiv \text{Þ} > \mathsf{R}$, would be represented by moving the arrowhead from the bottom right to the top left of the line segment connecting Þ and R.

If we were to start at R and then impose $\cancel{\text{FHWC}}$, that would reverse the arrow connecting R and $\mathcal{G}$, but the $\mathcal{G}$ node would then have no exiting arrows so no further deductions could be made. However, if we *also* reversed GIC (that is, if we imposed $\cancel{\text{GIC}}$), that would take us to the Þ node, and we could deduce $\mathsf{R} > \text{Þ}$. However, we would have to stop traversing the diagram at this point, because the arrow exiting from the Þ node points back to our starting point, which (if valid) would lead us to the conclusion that $\mathsf{R} > \mathsf{R}$. Thus, the reversal of the two earlier conditions (imposition of $\cancel{\text{FHWC}}$ and $\cancel{\text{GIC}}$) requires us also to reverse the final condition, giving us $\cancel{\text{RIC}}$.[13]

---

[10] For convenience, the equivalent ($\equiv$) mathematical statement of each condition is expressed nearby in parentheses.

[11] For a popular introduction to category theory, see Riehl (2017).

[12] But the rest of our notation does not necessarily abide by the other conventions of category theory diagrams.

[13] The corresponding algebra is

$$\begin{aligned} \cancel{\text{FHWC}}: &\quad \mathsf{R} < \mathcal{G} \\ \cancel{\text{GIC}}: &\quad \mathcal{G} < \text{Þ} \\ \Rightarrow \cancel{\text{RIC}}: &\quad \mathsf{R} < \text{Þ}, \end{aligned}$$

.

Under these conventions, Figure 6 in the main text presents a modified version of the diagram extended to incorporate the PF-FVAC.

This diagram can be interpreted, for example, as saying that, starting at the Þ node, it is possible to derive the PF-FVAC[14] by imposing both the GIC and the FHWC; or by imposing RIC and ~~FHWC~~. Or, starting at the $\mathcal{G}$ node, we can follow the imposition of the FHWC (twice — reversing the arrow labeled ~~FHWC~~) and then ~~RIC~~ to reach the conclusion that $\text{Þ} < \mathcal{G}$. Algebraically,

$$
\begin{aligned}
\text{FHWC}: \quad & \mathcal{G} < \mathsf{R} \\
\cancel{\text{RIC}}: \quad & \mathsf{R} < \text{Þ} \\
& \mathcal{G} < \text{Þ}
\end{aligned} \tag{78}
$$

{eq:cnclGICRaw}

which leads to the negation of both of the conditions leading into Þ. ~~GIC~~ is obtained directly as the last line in (78) and ~~PF-FVAC~~ follows if we start by multiplying the Return Patience Factor (RPF=$\text{Þ}/\mathsf{R}$) by the FHWF ($=\mathcal{G}/\mathsf{R}$) raised to the power $1/\gamma-1$, which is negative since we imposed $\gamma > 1$. FHWC implies FHWF < 1 so when FHWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because ~~RIC~~) by another number greater than one yields a product that must be greater than one:

$$
\begin{aligned}
1 &< \overbrace{\left(\frac{(\mathsf{R}\beta)^{1/\gamma}}{\mathsf{R}}\right)}^{>1 \text{ from } \cancel{\text{RIC}}} \overbrace{(\mathcal{G}/\mathsf{R})^{1/\gamma-1}}^{>1 \text{ from FHWC}} \\
1 &< \left(\frac{(\mathsf{R}\beta)^{1/\gamma}}{(\mathsf{R}/\mathcal{G})^{1/\gamma}\mathsf{R}\mathcal{G}/\mathsf{R}}\right) \\
\mathsf{R}^{1/\gamma}\mathcal{G}^{1-1/\gamma} = (\mathsf{R}/\mathcal{G})^{1/\gamma}\mathcal{G} &< \text{Þ}
\end{aligned}
$$

which is one way of writing ~~PF-FVAC~~.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure 7 in the paper (reproduced here).

Finally, the next diagram substitutes the values of the various objects in the diagram under the baseline parameter values and verifies that all of the asserted inequality conditions hold true.

[14] in the form $\text{Þ} < (\mathsf{R}/\mathcal{G})^{1/\gamma}\mathcal{G}$

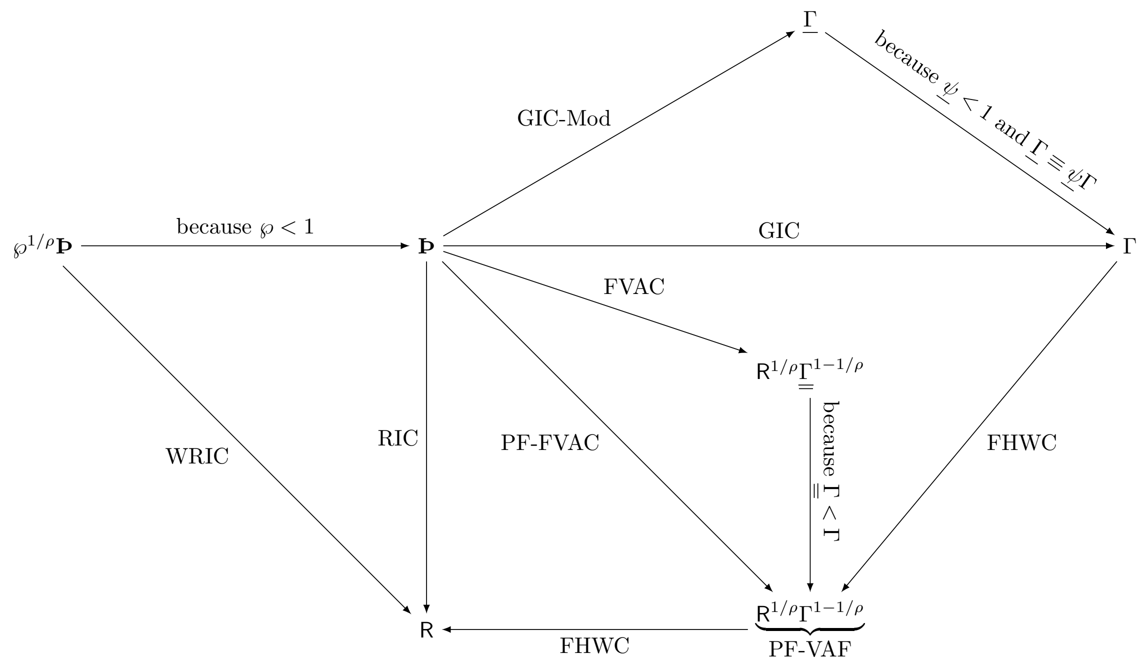

Figure 5 Appendix: Relation of All Inequality Conditions {fig:InequalitiesApp

References

F Additional Standard Results

Proposition 1. *Let* $\mathrm{f} : \mathbb{R}_{++} \to \mathbb{R}_{+}$ *be a continuous function. Consider sequences* x^n *in* $\mathbb{R}_{++}$ *and* $\mathrm{f}^n(x^n)$ *in* $\mathbb{R}_{+}$. *If* $\mathrm{f}^n(x^n) \to \mathrm{f}(x)$ *as* $n \to \infty$, *then* $x^n \to x$ *as* $n \to \infty$.

Proof. Given that f is continuous at x (with $x \in \mathbb{R}_{++}$), for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all y in $\mathbb{R}_{++}$ with $|y - x| < \delta$, we have $|\mathrm{f}(y) - \mathrm{f}(x)| < \epsilon$.

Given $\mathrm{f}^n(x^n) \to \mathrm{f}(x)$, for the above ϵ, there exists an N such that for all $n > N$, $|\mathrm{f}^n(x^n) - \mathrm{f}(x)| < \epsilon$.

Assume for the sake of contradiction that x^n doesn't converge to x. This implies that there exists a $\delta > 0$ such that for infinitely many terms of the sequence x^n, $|x^n - x| \geq \delta$.

By the continuity of f at x, if $|x^n - x| \geq \delta$ for infinitely many n, then $|\mathrm{f}^n(x^n) - \mathrm{f}(x)| \geq \epsilon$ for those n, contradicting our assumption that $\mathrm{f}^n(x^n) \to \mathrm{f}(x)$.

Therefore, our assumption for contradiction is false, and it follows that $x^n \to x$ as $n \to \infty$. □

{fact:normimplies

Fact 1. *Let* $\mathrm{g} : X \to \mathbb{R}_{+}$ *be a continuous function, where* $X \subseteq \mathbb{R}^n$ *is an open convex set. Define the weighted supremum norm* $\|\cdot\|_{\mathrm{g}}$ *of a real-valued function* $\mathrm{f} : X \to \mathbb{R}$ *by*

$$\|\mathrm{f}\|_{\mathrm{g}} := \sup_{x \in X} \frac{|\mathrm{f}(x)|}{\mathrm{g}(x)}. \tag{79}$$

If $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_{\mathrm{g}} = 0$, f_n *converges to* $\mathrm{f}^\star$ *uniformly on compact sets.*

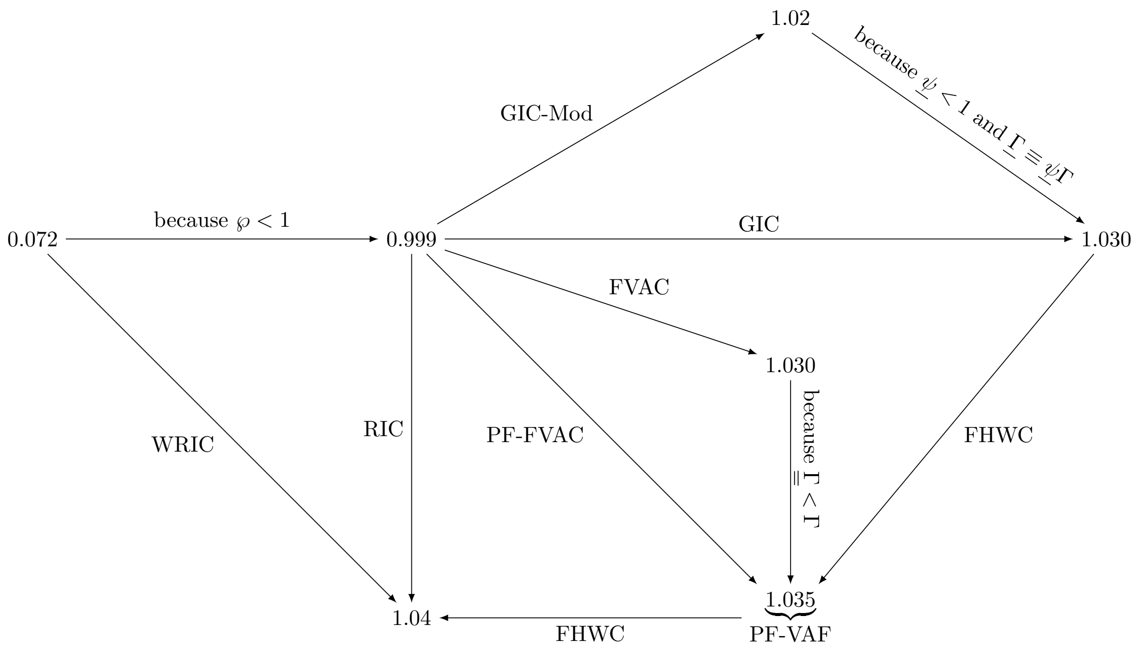

Figure 6 Appendix: Numerical Relation of All Inequality Conditions

{fig:InequalitiesApp

Proof. Let $\tilde{X}$ be an arbitrary compact subset of X. Since $\tilde{X}$ is compact, there exists a positive lower bound for g on this subset, denoted as

$$\bar{\mathrm{g}} = \min_{x \in \tilde{X}} \mathrm{g}(x) > 0. \tag{80}$$

Hence, on $\tilde{X}$, if $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_{\mathrm{g}} = 0$, then $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_\infty = 0$ on $\tilde{X}$, where $\|\cdot\|_\infty$ denotes the supremum norm.

Now, let K be a compact subset of X. Given the continuity of g, there exists a positive maximum value for g on K, denoted as M_K. Then, we have

$$\sup_{x\in K} |\mathrm{f}_n(x) - \mathrm{f}(x)| \leq M_K \sup_{x\in K} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)} \leq M_K \sup_{x\in X} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)}. \tag{81}$$

Thus, $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}\|_= 0$ implies that f_n converges uniformly to f on the compact set K. It's also worth noting that the convexity and openness of X aren't strictly necessary for this argument. □

{fact:compactnt}

Fact 2. *Let $\{\mathrm{f}_n\}$ be a sequence of continuous functions defined on a subset of the real line and converging uniformly to a function*f *on compact sets. If$\{x_n\}$ is a convergent sequence of real numbers with limit x, then $\mathrm{f}_n(x_n)$ converges to $\mathrm{f}(x)$.*

Proof. Let $\tilde{X}$ be an arbitrary compact subset of X. Since $\tilde{X}$ is compact, there exists a positive lower bound for g on this subset, denoted as

$$\bar{\mathrm{g}} = \min_{x \in \tilde{X}} \mathrm{g}(x) > 0. \tag{82}$$

Hence, on $\tilde{X}$, if $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_{\mathrm{g}} = 0$, then $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_\infty = 0$ on $\tilde{X}$, where $\|\cdot\|_\infty$ denotes the supremum norm.

Now, let K be a compact subset of X. Given the continuity of g, there exists a positive maximum value for g on K, denoted as M_K. Then, we have

$$\sup_{x\in K} |\mathrm{f}_n(x) - \mathrm{f}(x)| \leq M_K \sup_{x\in K} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)} \leq M_K \sup_{x\in X} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)}. \tag{83}$$

Thus, $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}\|_{=}0$ implies that f_n converges uniformly to f on the compact set K. It's also worth noting that the convexity and openness of X aren't strictly necessary for this argument. ☐

{claim:rationonde

Claim 5. *If* f *is convex and* $\mathrm{f} < 0$ *on* $(0, \lambda)$*, then* $\frac{\mathrm{f}(s)}{s}$ *is increasing on* $(0, \lambda)$*.*

Proof. Let f be convex on $(0, \lambda)$ and $\mathrm{f} < 0$ on $(0, \lambda)$. Let x_1 and x_2 be two points in $(0, \lambda)$. Choose $0 < \alpha < x_1$. Then, any point (in particular, x_1) in (α, x_2) can be written as $x_1 = t\alpha + (1-t)x_2$ for some $0 < t < 1$.

Now, define a new function F on $[\alpha, x_2]$ as:

$$F(x) = \mathrm{f}(x) - \mathrm{f}(\alpha).$$

Since f is convex, $F(x)$ is also convex on $[\alpha, x_2]$. To see this, observe that:

$$F(t\alpha+(1-t)x_2) = \mathrm{f}(t\alpha+(1-t)x_2) - \mathrm{f}(\alpha) \leq t\mathrm{f}(\alpha)+(1-t)\mathrm{f}(x_2) - \mathrm{f}(\alpha) = tF(\alpha)+(1-t)F(x_2).$$

Since $F(\alpha) = 0$, the inequality simplifies to $F(x_1) \leq (1-t)F(x_2)$. This implies that $\frac{F(s)}{s}$ is increasing. And thus, if $y_1 \leq y_2$, then:

$$\frac{F(y_1)}{y_1} \leq \frac{F(y_2)}{y_2}.$$

Now, using the definition of $F(x)$, we have:

$$\frac{\mathrm{f}(y_1)}{y_1} = \frac{F(y_1)}{y_1} + \frac{\mathrm{f}(\alpha)}{y_1}.$$

Similarly, for y_2:

$$\frac{\mathrm{f}(y_2)}{y_2} = \frac{F(y_2)}{y_2} + \frac{\mathrm{f}(\alpha)}{y_2}.$$

Since $\frac{F(y_1)}{y_1} \leq \frac{F(y_2)}{y_2}$ and $\mathrm{f}(\alpha) < 0$, we conclude that:

$$\frac{\mathrm{f}(y_1)}{y_1} \leq \frac{\mathrm{f}(y_2)}{y_2}.$$

Thus, $\frac{\mathrm{f}(s)}{s}$ is increasing on $(0, \lambda)$, completing the proof. ☐

References

Table 1 Appendix: Perfect Foresight Liquidity Constrained Taxonomy

{table:LiqConstrSce

For constrained $\grave{c}$ and unconstrained $\bar{\mathrm{c}}$ consumption functions

Main Condition Subcondition	Math	Outcome, Comments or Results
$\cancel{\text{GIC}}$	$1 < \mathbf{Þ}/\mathcal{G}$	Constraint never binds for $m \geq 1$
and RIC	$\mathbf{Þ}/\mathsf{R} < 1$	FHWC holds ($\mathsf{R} > \mathcal{G}$);
		$\grave{\mathrm{c}}(m) = \bar{\mathrm{c}}(m)$ for $m \geq 1$
and $\cancel{\text{RIC}}$	$1 < \mathbf{Þ}/\mathsf{R}$	$\grave{\mathrm{c}}(m)$ is degenerate: $\grave{\mathrm{c}}(m) = 0$
GIC	$\mathbf{Þ}/\mathcal{G} < 1$	Constraint binds in finite time $\forall\ m$
and RIC	$\mathbf{Þ}/\mathsf{R} < 1$	FHWC may or may not hold
		$\lim_{m\uparrow\infty} \bar{\mathrm{c}}(m) - \grave{\mathrm{c}}(m) = 0$
		$\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
and $\cancel{\text{RIC}}$	$1 < \mathbf{Þ}/\mathsf{R}$	$\cancel{\text{FHWC}}$
		$\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~GIC~~ and RIC both hold, while the third row indicates that when the GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the GIC holds, the constraint will bind in finite time.