1 Proofs for Individual Stability (Section ??)

{sec:ApndxMTarget

Proof for Proposition ??. *Proof.* For consumption growth, as $m \downarrow 0$ we have

$$\begin{aligned}
\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) \mathcal{G}_{t+1} \right] &> \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1})}{\bar{\kappa} m_t} \right) \mathcal{G}_{t+1} \right] \\
&= q \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \mathcal{G}_{t+1} \right] \\
&\quad + (1-q) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \theta_{t+1}/(1-q))}{\bar{\kappa} m_t} \right) \mathcal{G}_{t+1} \right] \\
&> (1-q) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/(1-q))}{\bar{\kappa} m_t} \right) \mathcal{G}_{t+1} \right] \\
&= \infty
\end{aligned}$$

where the second-to-last line follows because $\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \mathcal{G}_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.[1]

Next we establish the limit of the expected consumption growth factor as $m_t \uparrow \infty$:

$$\lim_{m_t \uparrow \infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] = \lim_{m_t \uparrow \infty} \mathbb{E}_t[\mathcal{G}_{t+1} c_{t+1}/c_t].$$

But

$$\mathbb{E}_t[\mathcal{G}_{t+1}\underline{c}_{t+1}/\bar{c}_t] \leq \mathbb{E}_t[\mathcal{G}_{t+1} c_{t+1}/c_t] \leq \mathbb{E}_t[\mathcal{G}_{t+1}\bar{c}_{t+1}/\underline{c}_t]$$

and

$$\lim_{m_t \uparrow \infty} \mathcal{G}_{t+1}\underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t \uparrow \infty} \mathcal{G}_{t+1}\bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t \uparrow \infty} \mathcal{G}_{t+1} m_{t+1}/m_t,$$

while (for convenience defining $\mathrm{a}(m_t) = m_t - \mathrm{c}(m_t)$),

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathcal{G}_{t+1} m_{t+1}/m_t &= \lim_{m_t \uparrow \infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \mathcal{G}_{t+1}\boldsymbol{\xi}_{t+1}}{m_t} \right) \\
&= (\mathsf{R}\beta)^{1/\gamma} = \text{Þ}
\end{aligned} \tag{1}$$

{eq:xtp1toinfty}

[1]None of the arguments in either of the two prior sections depended on the assumption that the consumption functions had converged. With more cumbersome notation, each derivation could have been replaced by the corresponding finite-horizon versions. This strongly suggests that it should be possible to extend the circumstances under which the problem can be shown to define a contraction mapping to the union of the parameter values under which {RIC,FHWC} hold and {FVAC,WRIC} hold. That extension is not necessary for our purposes here, so we leave it for future work.

because $\lim_{m_t \uparrow \infty} \mathrm{a}'(m) = \frac{\text{Þ}}{\mathsf{R}}$[2] and $\mathcal{G}_{t+1}\boldsymbol{\xi}_{t+1}/m_t \leq (\mathcal{G}\bar{\psi}\bar{\theta}/(1-q))/m_t$ which goes to zero as m_t goes to infinity.

Hence we have

$$\text{Þ} \leq \lim_{m_t \uparrow \infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] \leq \text{Þ}$$

so as cash goes to infinity, consumption growth approaches its value **Þ** in the perfect foresight model. □

□

This appendix proves Theorems **??**-2 and:

{lemma:orderingP

Lemma 1. *If $\check{m}$ and $\hat{m}$ both exist, then $\check{m} \leq \hat{m}$.*

1.1 Proof of Theorem ??

The elements of the proof of Theorem **??** are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

1.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

{subsubsec:RatExit

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem **??**).

Section **??** shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\tilde{\mathcal{R}}_t + \boldsymbol{\xi}_t$, even if $\boldsymbol{\xi}_t$ takes on its minimum value of 0, $a_{t-1}\tilde{\mathcal{R}}_t > 0$, since both a_{t-1} and $\tilde{\mathcal{R}}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

[2] $\lim_{m_t \uparrow \infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t \uparrow \infty} \mathrm{c}(m_t)/m_t = 1 - \lim_{m_t \uparrow \infty} \mathrm{c}'(m_t) = \frac{\text{Þ}}{\mathsf{R}}$.

1.3.1 *Existence of m where $\mathbb{E}_t[m_{t+1}/m_t] < 1$*

If RIC holds. Logic exactly parallel to that of Section **??** leading to equation (1), but dropping the $\mathcal{G}_{t+1}$ from the RHS, establishes that

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1}}{m_t} \right] \\
&= \mathbb{E}_t[(\mathsf{R}/\mathcal{G}_{t+1}) \frac{\text{\textbf{Þ}}}{\mathsf{R}}] \\
&= \mathbb{E}_t[\text{\textbf{Þ}}/\mathcal{G}_{t+1}] \qquad (2) \\
&< 1
\end{aligned}$$

{eq:emgro}

where the inequality reflects imposition of the GIC-Mod (**??**).

If RIC fails. When the RIC fails, the fact that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$ (see equation (**??**)) means that the limit of the RHS of (2) as $m \uparrow \infty$ is $\bar{\tilde{\mathcal{R}}} = \mathbb{E}_t[\tilde{\mathcal{R}}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Mod and ~~RIC~~ implies $\bar{\tilde{\mathcal{R}}} < 1$.

So we have $\lim_{m \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

1.3.2 *Existence of $m > 1$ where $\mathbb{E}_t[m_{t+1}/m_t] > 1$*

Paralleling the logic for c in Section **??**: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.3.3 *$\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \qquad (3) \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned}$$

{eq:difNrmioEquiv}

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t} \right) \boldsymbol{\zeta}(m_t) &= \mathbb{E}_t \left[\left(\frac{d}{dm_t} \right) \left(\tilde{\mathcal{R}}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1} - m_t \right) \right] \qquad (4) \\
&= \bar{\tilde{\mathcal{R}}} \left(1 - \mathrm{c}'(m_t)\right) - 1.
\end{aligned}$$

{eq:difFuncmNrmD

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (**??**) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show in Section **??** that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$
\begin{aligned}
\bar{\bar{\mathcal{R}}}\,(1 - \mathrm{c}'(m_t)) - 1 &< \bar{\bar{\mathcal{R}}}(1 - \underbrace{(1 - \frac{\text{Þ}}{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= \bar{\bar{\mathcal{R}}}\frac{\text{Þ}}{\mathsf{R}} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\mathcal{G}\psi}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\mathcal{G}\psi}\right]}_{=\frac{\text{Þ}}{\mathcal{G}}\mathbb{E}[\psi^{-1}]} - 1
\end{aligned}
$$

which is negative because the GIC-Mod says $\frac{\text{Þ}}{\mathcal{G}}\mathbb{E}[\psi^{-1}] < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$
\bar{\bar{\mathcal{R}}}\,(1 - \mathrm{c}'(m_t)) < \bar{\bar{\mathcal{R}}}
$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (4) is guaranteed to be negative if

$$
\bar{\bar{\mathcal{R}}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\mathcal{G}\psi}\right] < 1. \tag{5}
$$

{eq:RbarBelowOne}

But the combination of the GIC-Mod holding and the RIC failing can be written:

$$
\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\mathcal{G}\psi}\right]}^{\frac{\text{Þ}}{\mathcal{G}}\mathbb{E}[\psi^{-1}]} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\frac{\text{Þ}}{\mathsf{R}}},
$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$
\mathbb{E}_t\left[\frac{\mathsf{R}}{\mathcal{G}\psi}\right] < \mathsf{R}/\text{Þ} < 1
$$

which satisfies our requirement in (5).

1.4 Proof of Theorem 2

Theorem 2. *(Individual Balanced-Growth 'Pseudo Steady State'). Consider the problem defined in Section* **??**. *If weak return impatience (Assumption* **??***), finite value of autarky (Assumption* **??***) and growth impatience (Assumption* **??***) hold, then there exists a unique pseudo-steady-state market resources to permanent income ratio* $\check{m} > 0$ *such* {thm:MSSBalExis *that:*

$$\mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = 1 \textit{ if } m_t = \check{m}. \tag{6}$$ {eq:mBalLvl}

Moreover, $\check{m}$ is a point of stability in the sense that:

$$\begin{aligned} \forall m_t \in (0, \check{m}),\ \mathbb{E}_t[\boldsymbol{m}_{t+1}]/\boldsymbol{m}_t > \mathcal{G} \\ \forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[\boldsymbol{m}_{t+1}]/\boldsymbol{m}_t < \mathcal{G}. \end{aligned} \tag{7}$$ {eq:stabilityStE}

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\psi_{t+1} m_{t+1} - m_t]$ is monotonically decreasing

1.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\psi} \le \psi_{t+1} \le \bar{\psi} < \infty$, our proof in 1.2 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t]$.

1.4.2 Existence of a stable point

Since by assumption $0 < \underline{\psi} \le \psi_{t+1} \le \bar{\psi} < \infty$, our proof in Subsection 1.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\psi_{t+1} m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{G}_{t+1} \left((\mathsf{R}/\mathcal{G}_{t+1}) \mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1} \right) / \mathcal{G}}{m_t} \right] \\ &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{(\mathsf{R}/\mathcal{G}) \mathrm{a}(m_t) + \psi_{t+1} \boldsymbol{\xi}_{t+1}}{m_t} \right] \\ &= \lim_{m_t \uparrow \infty} \left[\frac{(\mathsf{R}/\mathcal{G}) \mathrm{a}(m_t) + 1}{m_t} \right] \\ &= (\mathsf{R}/\mathcal{G}) \frac{\text{Þ}}{\mathsf{R}} \\ &= \frac{\text{Þ}}{\mathcal{G}} \\ &< 1 \end{aligned} \tag{8}$$ {eq:emgro2}

where the last two lines are merely a restatement of the GIC (**??**).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.4.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ is monotonically decreasing.

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1,
\end{aligned} \tag{9}$$

{eq:difLvlEquiv}

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\tilde{\mathcal{R}}(m_t - \mathrm{c}(m_t)) + \psi_{t+1}\boldsymbol{\xi}_{t+1} - m_t\right)\right] \\
&= (\mathsf{R}/\mathcal{G})\left(1 - \mathrm{c}'(m_t)\right) - 1.
\end{aligned} \tag{10}$$

{eq:difFuncDecrease

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails ($\cancel{\text{RIC}}$).

If RIC holds. Equation (**??**) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section **??** that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}
\tilde{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) - 1 &< \tilde{\mathcal{R}}(1 - \underbrace{(1 - \frac{\boldsymbol{\text{Þ}}}{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= (\mathsf{R}/\mathcal{G})\frac{\boldsymbol{\text{Þ}}}{\mathsf{R}} - 1
\end{aligned}$$

which is negative because the GIC says $\frac{\boldsymbol{\text{Þ}}}{\mathcal{G}} < 1$.