1 Supporting Standard Results in Real Analysis

Proposition 1. *Let* $\mathrm{f} : \mathbb{R}_{++} \to \mathbb{R}_{+}$ *be a continuous function. Consider sequences* x^n *in* $\mathbb{R}_{++}$ *and* $\mathrm{f}^n(x^n)$ *in* $\mathbb{R}_{+}$*. If* $\mathrm{f}^n(x^n) \to \mathrm{f}(x)$ *as* $n \to \infty$*, then* $x^n \to x$ *as* $n \to \infty$*.*

Proof. Given that f is continuous at x (with $x \in \mathbb{R}_{++}$), for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all y in $\mathbb{R}_{++}$ with $|y - x| < \delta$, we have $|\mathrm{f}(y) - \mathrm{f}(x)| < \epsilon$.

Given $\mathrm{f}^n(x^n) \to \mathrm{f}(x)$, for the above ϵ, there exists an N such that for all $n > N$, $|\mathrm{f}^n(x^n) - \mathrm{f}(x)| < \epsilon$.

Assume for the sake of contradiction that x^n doesn't converge to x. This implies that there exists a $\delta > 0$ such that for infinitely many terms of the sequence x^n, $|x^n - x| \geq \delta$.

By the continuity of f at x, if $|x^n - x| \geq \delta$ for infinitely many n, then $|\mathrm{f}^n(x^n) - \mathrm{f}(x)| \geq \epsilon$ for those n, contradicting our assumption that $\mathrm{f}^n(x^n) \to \mathrm{f}(x)$.

Therefore, our assumption for contradiction is false, and it follows that $x^n \to x$ as $n \to \infty$. □

Fact 1. *Let* $\mathrm{g} : X \to \mathbb{R}_{+}$ *be a continuous function, where* $X \subseteq \mathbb{R}^n$ *is an open convex set. Define the weighted supremum norm* $\|\cdot\|_{\mathrm{g}}$ *of a real-valued function* $\mathrm{f} : X \to \mathbb{R}$ *by*

$$\|\mathrm{f}\|_{\mathrm{g}} := \sup_{x \in X} \frac{|\mathrm{f}(x)|}{\mathrm{g}(x)}. \tag{1}$$

If $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_{\mathrm{g}} = 0$*,* f_n *converges to* $\mathrm{f}^\star$ *uniformly on compact sets.*

Proof. Let $\tilde{X}$ be an arbitrary compact subset of X. Since $\tilde{X}$ is compact, there exists a positive lower bound for g on this subset, denoted as

$$\bar{\mathrm{g}} = \min_{x \in \tilde{X}} \mathrm{g}(x) > 0. \tag{2}$$

Hence, on $\tilde{X}$, if $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_{\mathrm{g}} = 0$, then $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_{\infty} = 0$ on $\tilde{X}$, where $\|\cdot\|_\infty$ denotes the supremum norm.

Now, let K be a compact subset of X. Given the continuity of g, there exists a positive maximum value for g on K, denoted as M_K. Then, we have

$$\sup_{x \in K} |\mathrm{f}_n(x) - \mathrm{f}(x)| \leq M_K \sup_{x \in K} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)} \leq M_K \sup_{x \in X} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)}. \tag{3}$$

Thus, $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}\|_{=}0$ implies that f_n converges uniformly to f on the compact set K. It's also worth noting that the convexity and openness of X aren't strictly necessary for this argument. □

Fact 2. *Let* $\{\mathrm{f}_n\}$ *be a sequence of continuous functions defined on a subset of the real line and converging uniformly to a function*f *on compact sets. If*$\{x_n\}$ *is a convergent sequence of real numbers with limit* x*, then* $\mathrm{f}_n(x_n)$ *converges to* $\mathrm{f}(x)$*.*

Proof. Let $\tilde{X}$ be an arbitrary compact subset of X. Since $\tilde{X}$ is compact, there exists a positive lower bound for g on this subset, denoted as

$$\bar{\mathrm{g}} = \min_{x \in \tilde{X}} \mathrm{g}(x) > 0. \tag{4}$$

Hence, on $\tilde{X}$, if $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_{\mathrm{g}} = 0$, then $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_\infty = 0$ on $\tilde{X}$, where $\|\cdot\|_\infty$ denotes the supremum norm.

Now, let K be a compact subset of X. Given the continuity of g, there exists a positive maximum value for g on K, denoted as M_K. Then, we have

$$\sup_{x\in K} |\mathrm{f}_n(x) - \mathrm{f}(x)| \leq M_K \sup_{x\in K} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)} \leq M_K \sup_{x\in X} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)}. \tag{5}$$

Thus, $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}\|_= 0$ implies that f_n converges uniformly to f on the compact set K. It's also worth noting that the convexity and openness of X aren't strictly necessary for this argument. □

Fact 3. *Let $\{\mathrm{f}_n\}$ be a sequence of continuous functions defined on a subset of the real line and converging uniformly to a function* f *on compact sets. If $\{x_n\}$ is a convergent sequence of real numbers with limit x, then $\mathrm{f}_n(x_n)$ converges to $\mathrm{f}(x)$.*

Proof. Since x_n converges to x, the sequence $\{x_n\}$ is bounded. Therefore, there exists a compact set K (specifically, a closed interval in the real line) that contains all the x_n as well as x.

Given the uniform convergence of f_n to f on K, for every $\epsilon > 0$, there exists an N such that for all $n \geq N$ and for all $y \in K$, we have

$$|\mathrm{f}_n(y) - \mathrm{f}(y)| < \epsilon.$$

In particular, for $y = x_n$, we have

$$|\mathrm{f}_n(x_n) - \mathrm{f}(x_n)| < \epsilon$$

for all $n \geq N$.

Now, each f_n being continuous and x_n converging to x implies that $\mathrm{f}(x_n)$ converges to $\mathrm{f}(x)$. Thus, for sufficiently large n, $\mathrm{f}(x_n)$ can be made arbitrarily close to $\mathrm{f}(x)$.

Combining the two inequalities and taking n large enough, we deduce that $|\mathrm{f}_n(x_n) - \mathrm{f}(x)|$ can be made smaller than any given ϵ. Hence, $\mathrm{f}_n(x_n)$ converges to $\mathrm{f}(x)$. □