

# 1 The Limiting MPC's

For  $m_t > 0$  we can define  $e_t(m_t) = c_t(m_t)/m_t$  and  $a_t(m_t) = m_t - c_t(m_t)$  and the Euler equation (4) can be rewritten

$$\begin{aligned} e_t(m_t)^{-\gamma} &= \beta R \mathbb{E}_t \left[ \left( e_{t+1}(m_{t+1}) \left( \frac{\overbrace{Ra_t(m_t) + G_{t+1}\xi_{t+1}}^{=m_{t+1}G_{t+1}}}{m_t} \right) \right)^{-\gamma} \right] \\ &= (1-q)\beta R m_t^\gamma \mathbb{E}_t [(e_{t+1}(m_{t+1})m_{t+1}G_{t+1})^{-\gamma} | \xi_{t+1} > 0] \\ &\quad + q\beta R^{1-\gamma} \mathbb{E}_t \left[ \left( e_{t+1}(\mathcal{R}_{t+1}a_t(m_t)) \frac{m_t - c_t(m_t)}{m_t} \right)^{-\gamma} | \xi_{t+1} = 0 \right]. \end{aligned}$$

Consider the first conditional expectation in (4), recalling that if  $\xi_{t+1} > 0$  then  $\xi_{t+1} \equiv \theta_{t+1}/(1-q)$ . Since  $\lim_{m \downarrow 0} a_t(m) = 0$ ,  $\mathbb{E}_t[(e_{t+1}(m_{t+1})m_{t+1}G_{t+1})^{-\gamma} | \xi_{t+1} > 0]$  is contained within bounds defined by  $(e_{t+1}(\underline{\theta}/(1-q))G\underline{\psi}\underline{\theta}/(1-q))^{-\gamma}$  and  $(e_{t+1}(\bar{\theta}/(1-q))G\bar{\psi}\bar{\theta}/(1-q))^{-\gamma}$  both of which are finite numbers, implying that the whole term multiplied by  $(1-q)$  goes to zero as  $m_t^\gamma$  goes to zero. As  $m_t \downarrow 0$  the expectation in the other term goes to  $\bar{\kappa}_{t+1}^{-\gamma}(1-\bar{\kappa}_t)^{-\gamma}$ . (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting  $\bar{\kappa}_t$  satisfies  $\bar{\kappa}_t^{-\gamma} = \beta q R^{1-\gamma} \bar{\kappa}_{t+1}^{-\gamma} (1-\bar{\kappa}_t)^{-\gamma}$ . Exponentiating by  $\gamma$ , we can conclude that

$$\begin{aligned} \bar{\kappa}_t &= q^{-1/\gamma} (\beta R)^{-1/\gamma} R (1-\bar{\kappa}_t) \bar{\kappa}_{t+1} \\ \underbrace{q^{1/\gamma} R^{-1} (\beta R)^{1/\gamma}}_{\equiv q^{1/\gamma} \frac{\mathbf{p}}{R}} \bar{\kappa}_t &= (1-\bar{\kappa}_t) \bar{\kappa}_{t+1} \end{aligned}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned} (q^{1/\gamma} \frac{\mathbf{p}}{R} \bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1} \bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1} (1-\bar{\kappa}_t) &= q^{1/\gamma} \frac{\mathbf{p}}{R} \bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1} &= 1 + q^{1/\gamma} \frac{\mathbf{p}}{R} \bar{\kappa}_{t+1}^{-1}. \end{aligned}$$

As noted in the main text, we need the **WRIC** (??) for this to be a convergent sequence:

$$0 \leq q^{1/\gamma} \frac{\mathbf{p}}{R} < 1, \tag{1}$$

Since  $\bar{\kappa}_T = 1$ , iterating (1) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n \rightarrow \infty} \bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1 - q^{1/\gamma} \frac{\mathbf{p}}{R} \tag{2}$$

and we will therefore call  $\bar{\kappa}$  the ‘limiting maximal MPC.’

The minimal MPC’s are obtained by considering the case where  $m_t \uparrow \infty$ . If the **FHWC** holds, then as  $m_t \uparrow \infty$  the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving  $\xi_{t+1}$  in (1) can be neglected, leading to a revised limiting Euler equation

$$(m_t e_t(m_t))^{-\gamma} = \beta R \mathbb{E}_t \left[ \left( e_{t+1}(a_t(m_t) \tilde{R}_{t+1}) (Ra_t(m_t)) \right)^{-\gamma} \right]$$

and using L’Hôpital’s rule  $\lim_{m_t \rightarrow \infty} e_t(m_t) = \underline{\kappa}_t$ , and  $\lim_{m_t \rightarrow \infty} e_{t+1}(a_t(m_t) \tilde{R}_{t+1}) = \underline{\kappa}_{t+1}$  so a further limit of the Euler equation is

$$\begin{aligned} (m_t \underline{\kappa}_t)^{-\gamma} &= \beta R (\underline{\kappa}_{t+1} R (1 - \underline{\kappa}_t) m_t)^{-\gamma} \\ \underbrace{R^{-1} \mathbf{P}}_{\underline{\kappa}_t} \underline{\kappa}_t &= (1 - \underline{\kappa}_t) \underline{\kappa}_{t+1} \\ &\equiv \frac{\mathbf{P}}{R} = (1 - \underline{\kappa}) \end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the **RIC**  $0 \leq \frac{\mathbf{P}}{R} < 1$  holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1} \frac{\mathbf{P}}{R} \quad (3)$$

so that  $(\{\underline{\kappa}_{T-n}^{-1}\})_{n=0}^{\infty}$  is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n \uparrow \infty} \underline{\kappa}_{T-n}^{-1} \quad (4)$$

as the limiting (inverse) marginal MPC. If the **RIC** does *not* hold, then  $\lim_{n \rightarrow \infty} \underline{\kappa}_{T-n}^{-1} = \infty$  and so the limiting MPC is  $\underline{\kappa} = 0$ .

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$c_t \underbrace{\left( 1 + \frac{\mathbf{P}}{R} + \frac{\mathbf{P}^2}{R} + \cdots \right)}_{= 1 + \frac{\mathbf{P}}{R} (1 + \frac{\mathbf{P}}{R} \underline{\kappa}_{t+2}^{-1}) \dots} = c_t \underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t) \underline{\kappa}_t \quad (5)$$