1 The Limiting MPC's

{sec:MPCLimits}

For $m_t > 0$ we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$ and the Euler equation (**??**) can be rewritten

$$
\begin{aligned}
\mathrm{e}_t(m_t)^{-\gamma} &= \beta\mathsf{R}\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m_t)+\mathcal{G}_{t+1}\boldsymbol{\xi}_{t+1}}^{=m_{t+1}\mathcal{G}_{t+1}}}{m_t}\right)\right)^{-\gamma}\right] \\
&= \quad (1-q)\beta\mathsf{R}m_t^{\gamma}\mathbb{E}_t\left[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\mathcal{G}_{t+1})^{-\gamma} \mid \boldsymbol{\xi}_{t+1} > 0\right] \\
&\quad + q\beta\mathsf{R}^{1-\gamma}\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\mathcal{R}_{t+1}\mathrm{a}_t(m_t))\frac{m_t-\mathrm{c}_t(m_t)}{m_t}\right)^{-\gamma} \mid \boldsymbol{\xi}_{t+1} = 0\right].
\end{aligned}
$$

Consider the first conditional expectation in (**??**), recalling that if $\boldsymbol{\xi}_{t+1} > 0$ then $\boldsymbol{\xi}_{t+1} \equiv \theta_{t+1}/(1-q)$. Since $\lim_{m\downarrow 0}\mathrm{a}_t(m) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\mathcal{G}_{t+1})^{-\gamma} \mid \boldsymbol{\xi}_{t+1} > 0]$ is contained within bounds defined by $(\mathrm{e}_{t+1}(\underline{\theta}/(1-q))\mathcal{G}\underline{\psi}\underline{\theta}/(1-q))^{-\gamma}$ and $(\mathrm{e}_{t+1}(\bar{\theta}/(1-q))\mathcal{G}\bar{\psi}\bar{\theta}/(1-q))^{-\gamma}$ both of which are finite numbers, implying that the whole term multiplied by $(1-q)$ goes to zero as m_t^{γ} goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\overline{\kappa}_{t+1}^{-\gamma}(1-\overline{\kappa}_t)^{-\gamma}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\overline{\kappa}_t$ satisfies $\overline{\kappa}_t^{-\gamma} = \beta q\mathsf{R}^{1-\gamma}\overline{\kappa}_{t+1}^{-\gamma}(1-\overline{\kappa}_t)^{-\gamma}$. Exponentiating by γ, we can conclude that

$$
\overline{\kappa}_t = q^{-1/\gamma}(\beta\mathsf{R})^{-1/\gamma}\mathsf{R}(1-\overline{\kappa}_t)\overline{\kappa}_{t+1}
$$

$$
\underbrace{q^{1/\gamma}\overbrace{\mathsf{R}^{-1}(\beta\mathsf{R})^{1/\gamma}}^{\frac{\text{Þ}}{\mathsf{R}}}}_{\equiv q^{1/\gamma}\frac{\text{Þ}}{\mathsf{R}}}\overline{\kappa}_t = (1-\overline{\kappa}_t)\overline{\kappa}_{t+1}
$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$
\begin{aligned}
\left(q^{1/\gamma}\frac{\text{Þ}}{\mathsf{R}}\overline{\kappa}_t\right)^{-1} &= (1-\overline{\kappa}_t)^{-1}\overline{\kappa}_{t+1}^{-1} \\
\overline{\kappa}_t^{-1}(1-\overline{\kappa}_t) &= q^{1/\gamma}\frac{\text{Þ}}{\mathsf{R}}\overline{\kappa}_{t+1}^{-1} \\
\overline{\kappa}_t^{-1} &= 1 + q^{1/\gamma}\frac{\text{Þ}}{\mathsf{R}}\overline{\kappa}_{t+1}^{-1}.
\end{aligned}
$$

As noted in the main text, we need the WRIC **??** for this to be a convergent sequence:

$$
0 \le q^{1/\gamma}\frac{\text{Þ}}{\mathsf{R}} < 1, \tag{1}
$$

{eq:WRICapndx}

Since $\overline{\kappa}_T = 1$, iterating (1) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty} \overline{\kappa}_{T-n} = \overline{\kappa} \equiv 1 - q^{1/\gamma}\frac{\text{Þ}}{\mathsf{R}} \tag{2}$$ {eq:MPCmaxDef}

and we will therefore call $\overline{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving $\boldsymbol{\xi}_{t+1}$ in (1) can be neglected, leading to a revised limiting Euler equation

$$(m_t \mathrm{e}_t(m_t))^{-\gamma} \;\; = \beta \mathsf{R}\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\tilde{\mathcal{R}}_{t+1})\left(\mathsf{R}\mathrm{a}_t(m_t)\right)\right)^{-\gamma}\right]$$

and using L'Hôpital's rule $\lim_{m_t\to\infty} \mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t\to\infty} \mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\tilde{\mathcal{R}}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$\begin{aligned} (m_t\underline{\kappa}_t)^{-\gamma} &= \beta\mathsf{R}\big(\underline{\kappa}_{t+1}\mathsf{R}(1-\underline{\kappa}_t)m_t\big)^{-\gamma} \\ \underbrace{\mathsf{R}^{-1}\text{Þ}}_{\equiv\frac{\text{Þ}}{\mathsf{R}}=(1-\underline{\kappa})}\ \underline{\kappa}_t &= (1-\underline{\kappa}_t)\underline{\kappa}_{t+1} \end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \leq \frac{\text{Þ}}{\mathsf{R}} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1}\frac{\text{Þ}}{\mathsf{R}} \tag{3}$$ {eq:MPCminInvApr

so that $(\{\underline{\kappa}_{T-n}^{-1}\})_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n\uparrow\infty} \kappa_{T-n}^{-1} \tag{4}$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n\to\infty} \underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$\underbrace{c_t\left(1 + \frac{\text{Þ}}{\mathsf{R}} + \frac{\text{Þ}}{\mathsf{R}}^2 + \cdots\right)}_{=1+\frac{\text{Þ}}{\mathsf{R}}(1+\frac{\text{Þ}}{\mathsf{R}}\underline{\kappa}_{t+2}^{-1})\ldots} = c_t\underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t)\underline{\kappa}_t \tag{5}$$