Main Appendices

A Proofs for Theoretical Foundations (Section 2)

A.1 Appendix for Problem Formulation

A.1.1 Recovering the Non-Normalized Problem

Letting nonbold variables be the boldface counterpart normalized by $\boldsymbol{p}_t$ (as with $m = \boldsymbol{m}/\boldsymbol{p}$), consider the problem in the second-to-last period:

$$\begin{aligned}\mathbf{v}_{T-1}(\boldsymbol{m}_{T-1}, \boldsymbol{p}_{T-1}) &= \max_{0<c_{T-1}\le m_{T-1}} \mathrm{u}(\boldsymbol{p}_{T-1}c_{T-1}) + \beta\, \mathbb{E}_{T-1}[\mathrm{u}(\boldsymbol{p}_T m_T)] \\ &= \boldsymbol{p}_{T-1}^{1-\gamma}\left\{\max_{0<c_{T-1}\le m_{T-1}} \mathrm{u}(c_{T-1}) + \beta\, \mathbb{E}_{T-1}[\mathrm{u}(\tilde{G}_T m_T)]\right\}.\end{aligned} \tag{1}$$

Since $\mathrm{v}_T(m_T) = \mathrm{u}(m_T)$, defining $\mathrm{v}_{T-1}(m_{T-1})$ from Problem ($\mathcal{P}_N$), we obtain:

$$\mathbf{v}_{T-1}(\boldsymbol{m}_{T-1}, \boldsymbol{p}_{T-1}) = \boldsymbol{p}_{T-1}^{1-\gamma}\mathrm{v}_{T-1}(\underbrace{\boldsymbol{m}_{T-1}/\boldsymbol{p}_{T-1}}_{=m_{T-1}}).$$

This logic induces to earlier periods; if we solve the normalized one-state-variable problem ($\mathcal{P}_N$), we will have solutions to the original problem for any $t < T$ from:

$$\begin{aligned}\mathbf{v}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) &= \boldsymbol{p}_t^{1-\gamma}\mathrm{v}_t(m_t), \\ \boldsymbol{c}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) &= \boldsymbol{p}_t \mathrm{c}_t(m_t).\end{aligned}$$

A.1.2 Challenges with Standard Dynamic Programming Approaches

A.1.3 Infinite Horizon Stochastic Dynamic Optimization Problem

How does the limiting nondegenerate solution connect to the solution of an infinite horizon stochastic dynamic optimization problem (Hernández-Lerma and Lasserre, 1995; Puterman, 1994; Stachurski, 2022)? The two problems are equivalent when the converged value function, v, is a fixed point of the stationary Bellman operator $\mathbb{T}$, and the nondegenerate consumption function is v-greedy, that is, Equation (24) holds. Given the particular approach taken by Theorem 2, and to aid the interpretation of our discussion on aggregate relationships, we state the standard result formally and present a proof Appendix **??**.

Let a sequence of shocks $\{\psi_k, \xi_k\}_{k=0}^{\infty}$ be defined on a common probability space, $(\Omega, \Sigma, \mathbb{P})$, and fix the problem primitives defined in Section 2.1. Consider the value function for stochastic infinite horizon sequence problem:

$$\tilde{\mathrm{v}}(m) = \max_{\{\tilde{c}_k\}_{k=0}^{\infty}} \mathbb{E}\sum_{k=0}^{\infty}\beta^k \Pi_{j=0}^{k}\tilde{G}_j \mathrm{u}(\tilde{c}_k), \qquad m \in S \tag{$\mathcal{P}_\infty$}$$

such that i) $\{\tilde{c}_k\}_{k=0}^{\infty}$ is a sequence of random variables defined on $(\Omega, \Sigma, \mathbb{P})$, progressively measurable with respect to the shocks $\{\psi_k, \xi_k\}_{k=0}^{\infty}$, ii) the inter-temporal budget constraint holds almost everywhere: $\tilde{m}_{k+1} = \tilde{R}_k(\tilde{m}_k - \tilde{c}_k) + \xi_k$, iii) the cannot die in debt condition holds almost everywhere in the limit: $\lim\limits_{k\to\infty} \tilde{m}_k \geq 0$ and iv) $\tilde{m}_0 = m$. The expectation $\mathbb{E}$ is taken with respect to $\mathbb{P}$.

Proposition 1. *Let the assumptions of Theorem 2 hold. If* v *and* c *are a limiting nondegenerate solution, them* $\mathrm{v} = \tilde{\mathrm{v}}$ *and the sequence* $\{\tilde{c}_k\}_{k=0}^{\infty}$ *generated by* $\tilde{c}_k = \mathrm{c}(\tilde{m}_k)$, *where* $\tilde{m}_{k+1} = \tilde{R}_k(\tilde{m}_k - \mathrm{c}(\tilde{m}_k)) + \xi_k$, *solves Problem* $(\mathcal{P}_\infty)$.

The proposition, an implication of the Bellman Principle of Optimality, says that an individual following the nondegenerate consumption rule has maximized the expected discounted sum of their future per-period utilities.

A.2 Perfect Foresight Benchmarks

PFBProofs

How do the finite value of human wealth, perfect foresight finite value of autarky and return impatience relate to each other? If the FHWC is satisfied, the PF-FVAC implies that the RIC is satisfied.[1] Likewise, if the FHWC and the GIC are both satisfied, PF-FVAC follows:

$$\begin{aligned} &\text{Þ} < G < R \\ &\frac{\text{Þ}}{R} < G/R < (G/R)^{1-1/\gamma} < 1 \end{aligned} \tag{2}$$

(the last line holds because FHWC $\Rightarrow 0 \leq (G/R) < 1$ and $\gamma > 1 \Rightarrow 0 < 1 - 1/\gamma < 1$).

Divide both sides of the second inequality in (9) by R:

$$\text{Þ}/R < (G/R)^{1-1/\gamma} \tag{3}$$

and FHWC $\Rightarrow$ the RHS is < 1 because $(G/R) < 1$ (and the RHS is raised to a positive power (because $\gamma > 1$)).

The first panel of Table 4 summarizes: The PF-Unconstrained model has a nondegenerate limiting solution if we impose the RIC and FHWC (these conditions are necessary as well as sufficient). Together the PF-FVAC and the FHWC imply the RIC. If we impose the GIC and the FHWC, both the PF-FVAC and the RIC follow, so GIC+FHWC are also sufficient. But there are circumstances under which the RIC and FHWC can hold while the PF-FVAC fails ('~~PF-FVAC~~'). For example, if $G = 0$, the problem is a standard 'cake-eating' problem with a nondegenerate solution under the RIC (when the consumer has access to capital markets).

[1] Divide both sides of the second inequality in (9) by R:

$$\text{Þ}/R < (G/R)^{1-1/\gamma}$$

and FHWC $\Rightarrow$ the RHS is < 1 because $(G/R) < 1$ (and the RHS is raised to a positive power (because $\gamma > 1$)).

A.3 Properties of the Consumption Function and Limiting MPCs

We start by stating some properties of the value functions generated by Problem ($\mathcal{P}_N$).

Lemma 1. *If* v_t *is strictly negative, strictly increasing, strictly concave,* $\mathbf{C}^3$ *and satisfies* $\lim_{m\to 0} \mathrm{v}_t(m) = -\infty$, *then* c_t *is in* $\mathbf{C}^2$.

Proof. Now define an end-of-period value function $\mathfrak{v}_t(a)$ as:

$$\mathfrak{v}_t(a) = \beta\, \mathbb{E}_t \left[\tilde{G}_{t+1}^{1-\gamma} \mathrm{v}_{t+1} \left(\tilde{R}_{t+1} a + \xi_{t+1} \right) \right]. \tag{4}$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\tilde{R}_{t+1} > 0$, it is clear that $\lim_{a\to 0} \mathfrak{v}_t(a) = -\infty$ and $\lim_{a\to 0} \mathfrak{v}'_t(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be satisfy the properties of the Proposition. (See Hiraguchi (2003).)

Next define $\underline{\mathrm{v}}_t(m, c)$ as:

$$\underline{\mathrm{v}}_t(m, c) = \mathrm{u}(c) + \mathfrak{v}_t(m - c). \tag{5}$$

Note that for fixed m, $c \mapsto \underline{\mathrm{v}}_t(m, c)$ is $\mathbf{C}^3$ on $(0, m)$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$. Next, observe that our problem's value function defined by Problem ($\mathcal{P}_N$) can be written as:

$$\mathrm{v}_t(m) = \max_{c} \ \underline{\mathrm{v}}_t(m, c), \tag{6}$$

where the function $\underline{\mathrm{v}}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c\to 0} \underline{\mathrm{v}}_t(m, c) = \lim_{c\to m} \underline{\mathrm{v}}_t(m, c) = -\infty$, $\frac{\partial^2 \underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\to 0} \frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\to m} \frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by:

$$\mathrm{c}_t(m) = \arg\max_{0<c<m} \underline{\mathrm{v}}_t(m, c) \tag{7}$$

exists and is unique and Problem ($\mathcal{P}_N$) has an interior solution. Moreover, by Berge's Maximum Theorem, c_t will be continuous on S. Next, note that c_t satisfies the first order condition:

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}'_t(m - \mathrm{c}_t(m)). \tag{8}$$

By the Implicit Function Theorem, we then have that c_t is differentiable and:

$$\mathrm{c}'_t(m) = \frac{\mathfrak{v}''_t(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}''_t(\mathrm{a}_t(m))}. \tag{9}$$

Since both u and $\mathfrak{v}_t$ are three times continuously differentiable and c_t is continuous, the RHS of the above equation is continuous and we can conclude that c'_t is continuous and c_t is in $\mathbf{C}^1$.

Finally, $\mathrm{c}'_t(m)$ is differentiable because $\mathfrak{v}''_t$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}''_t(\mathrm{a}_t(m)) < 0$. The second derivative $\mathrm{c}''_t(m)$ will be given by:

$$\mathrm{c}''_t(m) = \frac{a'_t(m)\mathfrak{v}'''_t(a_t)\left[\mathrm{u}''(c_t) + \mathfrak{v}''_t(a_t)\right] - \mathfrak{v}''_t(a_t)\left[c'_t\mathrm{u}'''(c_t) + a'_t\mathfrak{v}'''_t(a_t)\right]}{\left[\mathrm{u}''(c_t) + \mathfrak{v}''_t(a_t)\right]^2}. \tag{10}$$

Since $\mathfrak{v}''_t(\mathrm{a}_t(m))$ is continuous, $\mathrm{c}''_t(m)$ is also continuous.

□

Proposition 2. *For each t, v_t is strictly negative, strictly increasing, strictly concave, $\mathbf{C}^3$ and satisfies $\lim_{m\to 0} \mathrm{v}_t(m) = -\infty$.*

Proof. We will say a function is 'nice' if it satisfies the properties stated by the Proposition. Assume that for some $t+1$, v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and u, where $\mathrm{u}(m) = m^{1-\gamma}/(1-\gamma)$, is nice by inspection. By Lemma 1, if v_{t+1} is nice, c_t is in $\mathbf{C}^2$. Next, since both u and $\mathfrak{v}_t$ are strictly concave, both c_t and a_t, where $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$, are strictly increasing (Recall Equation (9)). This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = \mathrm{u}(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$. □

Proof for Proposition 3. By Proposition 2, each v_t is strictly negative, strictly increasing, strictly concave, $\mathbf{C}^3$ and satisfies $\lim_{m\to 0} \mathrm{v}_t(m) = -\infty$. As such, apply Lemma 1 to conclude the result.

□

Proof of Lemma 3 (Limiting MPCs). *Proof of (i): Minimal MPC*

Fix any t and for any m_t with $m_t > 0$, we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$. The Euler equation, Equation (4), can be rewritten as:

$$\mathrm{e}_t(m_t)^{-\gamma} = \beta R\, \mathbb{E}_t\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{R\mathrm{a}_t(m) + \tilde{G}_{t+1}\xi_{t+1}}^{=m_{t+1}\tilde{G}_{t+1}}}{m_t}\right)\right)^{-\gamma} \tag{11}$$

where $m_{t+1} = \tilde{R}_{t+1}(m_t - \mathrm{c}_t(m_t)) + \xi_{t+1}$. The minimal MPC's are obtained by letting where $m_t \to \infty$. Note that $\lim_{m_t\to\infty} m_{t+1} = \infty$ almost surely and thus $\lim_{m_t\to\infty} \mathrm{e}_{t+1}(m_{t+1}) = \underline{\kappa}_{t+1}$ almost surely. Turning to the second term inside the marginal utility on the RHS, we can write:

$$\begin{aligned}\lim_{m_t\to\infty}\frac{R\mathrm{a}_t(m_t) + \tilde{G}_{t+1}\xi_{t+1}}{m_t} &= \lim_{m_t\to\infty}\frac{R\mathrm{a}_t(m_t)}{m_t} + \lim_{m_t\to\infty}\frac{\tilde{G}_{t+1}\xi_{t+1}}{m_t} &(12)\\ &= R(1-\underline{\kappa}_t) + 0, &(13)\end{aligned}$$

since $\tilde{G}_{t+1}\xi_{t+1}$ is bounded. Thus, we can assert:

$$\lim_{m_t\to\infty}\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{R\mathrm{a}_t(m) + \tilde{G}_{t+1}\xi_{t+1}}{m_t}\right)\right)^{-\gamma} = (R\underline{\kappa}_{t+1}(1-\underline{\kappa}_t))^{-\gamma}, \tag{14}$$

almost surely. Next, the term inside the expectation operator at Equation (79) is bounded above by $\left(R\underline{\kappa}_{t+1}(1-\bar{\kappa}_t)\right)^{-\gamma}$. Thus, by the Dominated Convergence Theorem, we have:

$$\lim_{m_t\to\infty} \beta R\,\mathbb{E}_t\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{R\mathrm{a}_t(m_t)+\tilde{G}_{t+1}\xi_{t+1}}{m_t}\right)\right)^{-\gamma} = \beta R(R\underline{\kappa}_{t+1}(1-\underline{\kappa}_t))^{-\gamma}. \quad (15)$$

Again applying L'Hôpital's rule to the LHS of Equation (79), letting $\lim_{m\to\infty}\mathrm{e}_t(m) = \underline{\kappa}_t$ and equating limits to the RHS, we arrive at:

$$\frac{\text{Þ}}{R}\underline{\kappa}_t = (1-\underline{\kappa}_t)\underline{\kappa}_{t+1}$$

The minimal marginal propensity to consume satisfies the following recursive formula:

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1}\frac{\text{Þ}}{R}, \quad (16)$$

which implies $(\{\underline{\kappa}_{T-n}^{-1}\})_{n=0}^{\infty}$ is an increasing convergent sequence. Define:

$$\underline{\kappa}^{-1}: = \lim_{n\to\infty}\kappa_{T-n}^{-1} \quad (17)$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n\to\infty}\underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$. Otherwise if RIC holds, then $\underline{\kappa} > 0$.

Proof of (ii): Maximal MPC

The Euler Equation (4) can be rewritten as:

$$\begin{aligned}
\mathrm{e}_t(m_t)^{-\gamma} &= \beta R\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{R\mathrm{a}_t(m)+\tilde{G}_{t+1}\xi_{t+1}}^{=m_{t+1}\tilde{G}_{t+1}}}{m_t}\right)\right)^{-\gamma}\right] \\
&= \quad (1-q)\beta R m_t^{\gamma}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\tilde{G}_{t+1}\right)^{-\gamma} \mid \xi_{t+1}>0\right] \\
&\quad + q\beta R^{1-\gamma}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\tilde{R}_{t+1}\mathrm{a}_t(m))\frac{m_t-\mathrm{c}_t(m)}{m_t}\right)^{-\gamma} \mid \xi_{t+1}=0\right]
\end{aligned} \quad (18)$$

Now consider the first conditional expectation in the second line of Equation (18). Recall that if $\xi_{t+1} > 0$, then $\xi_{t+1} = \theta_{t+1}/(1-q)$ by Assumption I.1. Since $\lim_{m_t\to 0}\mathrm{a}_t(m_t) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_t{}')m_t{}'G_{t+1})^{-\gamma} \mid \xi_{t+1}>0]$ is contained in the bounded interval $[(\mathrm{e}_{t+1}(\underline{\theta}/(1-q))G\underline{\psi}\underline{\theta}/(1-q))^{-\gamma}, (\mathrm{e}_{t+1}(\bar{\theta}/(1-q))G\bar{\psi}\bar{\theta}/(1-q))^{-\gamma}]$. As such, the first term after the second equality above converges to zero as m_t^{γ} converges to zero.

Turning to the second term after the second equality above, once again apply Dominated Convergence Theorem as noted above at Equation (15). As $m_t \to 0$, the expectation converges to $\bar{\kappa}_{t+1}^{-\gamma}(1-\bar{\kappa}_t)^{-\gamma}$.

Equating the limits on the LHS and RHS of Equation (18), we have $\bar{\kappa}_t^{-\gamma} =$

$\beta q R^{1-\gamma}\bar{\kappa}_{t+1}^{-\gamma}(1-\bar{\kappa}_t)^{-\gamma}$. Exponentiating by γ on both sides, we can conclude:

$$\bar{\kappa}_t = q^{-1/\gamma}(\beta R)^{-1/\gamma}R(1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

and,

$$\underbrace{q^{1/\gamma}\overbrace{R^{-1}(\beta R)^{1/\gamma}}^{\frac{\text{Þ}}{R}}}_{\equiv q^{1/\gamma}\frac{\text{Þ}}{R}}\bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1} \tag{19}$$

The equation above yields a useful recursive formula for the maximal marginal propensity to consume after some algebra:

$$\begin{aligned}
(q^{1/\gamma}\frac{\text{Þ}}{R}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\
\Rightarrow \bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= q^{1/\gamma}\frac{\text{Þ}}{R}\bar{\kappa}_{t+1}^{-1} \\
\Rightarrow \bar{\kappa}_t^{-1} &= 1 + q^{1/\gamma}\frac{\text{Þ}}{R}\bar{\kappa}_{t+1}^{-1}
\end{aligned}$$

As noted in the main text, we need the WRIC (**??**) for this to be a convergent sequence:

$$0 \leq q^{1/\gamma}\frac{\text{Þ}}{R} < 1, \tag{20}$$

Since $\bar{\kappa}_T = 1$, iterating (79) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty}\bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1 - q^{1/\gamma}\frac{\text{Þ}}{R} \tag{21}$$

□

A.4 Existence of Limiting Solutions

We state Boyd's contraction mapping Theorem (Boyd,1990) for completeness.

Theorem 1. *(Boyd's Contraction Mapping) Let $\mathbb{B} : \mathcal{C}_\varphi(S,Y) \to \mathcal{C}_\varphi(S,Y)$. If,*

1. *the operator $\mathbb{B}$ is non-decreasing, i.e. $x \leq y \Rightarrow \mathbb{B}x \leq \mathbb{B}y$,*
2. *we have $\mathbb{B}\mathbf{0}$ in $\mathcal{C}_\varphi(S,Y)$, where $\mathbf{0}$ is the null vector,*
3. *there exists some real $0 < \alpha < 1$ such that for all ζ with $\zeta > 0$, we have:*

$$\mathbb{B}(x+\zeta\varphi) \leq \mathbb{B}x + \zeta\alpha\varphi,$$

then $\mathbb{B}$ defines a contraction with a unique fixed point.

To prepare for the main contraction mapping proof, the following claim will allow us to employ the WRIC (Assumption L.4) to show $\mathbb{T}^{\underline{b},\bar{b}}f$ maps $\varphi-$bounded functions to $\varphi-$bounded for k large enough and $\bar{\kappa}_k \geq \bar{b}$, with $\bar{\kappa}_k$ close enough to $\bar{\kappa}$.

Claim 1. *If WRIC (Assumption L.4) holds, then there exists k such that:*

$$q\beta(R(1-\bar{\kappa}_k))^{1-\gamma} < 1 \tag{22}$$

Proof. By straight-forward algebra, we have:

$$\begin{aligned} q\beta(R(1-\bar{\kappa}))^{1-\gamma} &= q\beta R^{1-\gamma}\left(q^{1/\gamma}\frac{(R\beta)^{1/\gamma}}{R}\right)^{1-\gamma} \\ &= q^{1/\gamma}\frac{(R\beta)^{1/\gamma}}{R} < 1 \end{aligned} \tag{23}$$

where the inequality holds by the WRIC (Assumption L.4). Finally, since the expression $q\beta(R(1-\bar{\kappa}_k))^{1-\gamma}$ is continuous as a function of $\bar{\kappa}_k$, and we have $\bar{\kappa} > 0$ and $\bar{\kappa}_t \to \bar{\kappa}$, by the definition of continuity, there exists k such that Equation (22) holds. □

Proof of Theorem 2. Fix k such that Equation (22) holds. To show $\mathbb{T}^{\underline{b},\bar{b}}f$ satisfies the condition of Theorem 1, we first need to show $\mathbb{T}^{\underline{b},\bar{b}}f$ maps from $\mathcal{C}_{\varphi}(S,Y)$ to $\mathcal{C}_{\varphi}(S,Y)$. A preliminary requirement is therefore that $\mathbb{T}^{\underline{b},\bar{b}}\mathrm{f}$ be continuous for any $\varphi-$bounded f, $\mathbb{T}^{\underline{b},\bar{b}}\mathrm{f} \in \mathcal{C}(S,\mathbb{R})$. This is not difficult to show; see Hiraguchi (2003).

Proof of Condition 1

Consider condition (1). For this problem,

$$\begin{aligned} \mathbb{T}^{\underline{b},\bar{b}}\mathrm{f}(m) &= \max_{c\in[\underline{b}m,\bar{b}m]}\left\{\mathrm{u}(c)+\beta\,\mathbb{E}\,\tilde{G}\mathrm{f}\left(m'\right)\right\} \\ \mathbb{T}^{\underline{b},\bar{b}}\mathrm{g}(m) &= \max_{c\in[\underline{b}m,\bar{b}m]}\left\{\mathrm{u}(c)+\beta\,\mathbb{E}\,\tilde{G}'\mathrm{g}\left(m'\right)\right\}, \end{aligned}$$

so $\mathrm{f} \le \mathrm{g}$ implies $\mathbb{T}^{\underline{b},\bar{b}}\mathrm{g} \le \mathbb{T}^{\underline{b},\bar{b}}\mathrm{g}$ pointwise by inspection.[2]

Proof of Condition 2

Condition (2) requires that $\mathbb{T}^{\underline{b},\bar{b}}\mathbf{0} \in \mathcal{C}_{\varphi}\left(\mathcal{S},\mathcal{Y}\right)$. By definition,

$$\mathbb{T}^{\underline{b},\bar{b}}\mathbf{0}(m) = \max_{c\in[\underline{b},m,\bar{b}m]}\left\{\left(\frac{c^{1-\gamma}}{1-\gamma}\right)+\beta 0\right\}$$

the solution to which is $\mathrm{u}(\bar{b}m)$. Thus, condition (2) will hold if $(\bar{b}m)^{1-\gamma}$ is φ-bounded, which it is if we use the bounding function

$$\varphi(m) = \eta + m^{1-\gamma}, \tag{24}$$

defined in the main text.

Proof of Condition 3

Finally, we turn to condition (3). We wish to show that there exists $\alpha \in (0,1)$ such that $\mathbb{T}^{\underline{b},\bar{b}}(\mathrm{f}+\zeta\varphi) \le \mathbb{T}^{\underline{b},\bar{b}}\mathrm{f}+\zeta\alpha\varphi$ holds for any $\bar{b}$ with $\bar{b} \le \bar{\kappa}_k$. Let f be given and let

[2] For a fixed m, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

$\mathrm{g} = \mathrm{f} + \zeta\varphi$. The proof will be more compact if we define $\bar{\mathrm{c}}$ as the consumption function[3] associated with $\mathbb{T}^{\underline{b},\bar{b}}\mathrm{f}$ and $\hat{\mathrm{c}}$ as the consumption functions associated with $\mathbb{T}^{\underline{b},\bar{b}}\mathrm{g}$. Using this notation, condition (3) requires that there exist some $\alpha \in (0,1)$ such that for all $\zeta > 0$, we have:

$$\mathrm{u} \circ \hat{\mathrm{c}} + \beta\,\mathbb{E}\,\tilde{G}\mathrm{g} \circ \hat{\mathrm{m}}^{\mathrm{next}} \leq \mathrm{u} \circ \bar{\mathrm{c}} + \beta\,\mathbb{E}\,\tilde{G}\mathrm{f} \circ \bar{\mathrm{m}}^{\mathrm{next}} + \zeta\alpha\varphi.$$

where $\bar{\mathrm{m}}^{\mathrm{next}}(m) = \tilde{R}(m - \bar{\mathrm{c}}(m)) + z^{\mathrm{next}}$ and $\hat{\mathrm{m}}^{\mathrm{next}}(m) = \tilde{R}(m - \hat{\mathrm{c}}(m)) + z^{\mathrm{next}}$. If we now force the consumer facing f as the next period value function to consume the amount optimal for the consumer facing g, the value for the f consumer must be weakly lower. That is,

$$\mathrm{u} \circ \hat{\mathrm{c}} + \beta\,\mathbb{E}\,\tilde{G}\mathrm{f} \circ \hat{\mathrm{m}}^{\mathrm{next}} \leq \mathrm{u}(\bar{\mathrm{c}}) + \beta\,\mathbb{E}\,\tilde{G}\mathrm{f} \circ \bar{\mathrm{m}}^{\mathrm{next}}$$

Thus, condition (3) will hold if there exists α with $\alpha \in (0,1)$ such that:

$$\begin{aligned}
\mathrm{u} \circ \hat{\mathrm{c}} + \beta\,\mathbb{E}\,\tilde{G}\mathrm{g} \circ \hat{m}^{\mathrm{next}} &\leq \mathrm{u} \circ \hat{\mathrm{c}} + \beta\,\mathbb{E}\,G\mathrm{f} \circ \hat{m}^{\mathrm{next}} + \zeta\alpha\varphi \\
\beta\,\mathbb{E}\,\tilde{G}(\mathrm{f} + \zeta\varphi)(\hat{m}^{\mathrm{next}}) &\leq \beta\,\mathbb{E}\,\tilde{G}\mathrm{f} \circ \hat{m}^{\mathrm{next}} + \zeta\alpha\varphi \\
\beta\zeta\,\mathbb{E}\,\tilde{G}\varphi \circ \hat{m}^{\mathrm{next}} &\leq \zeta\alpha\varphi \\
\beta\,\mathbb{E}\,\tilde{G}\varphi \circ \hat{m}^{\mathrm{next}} &\leq \alpha\varphi
\end{aligned}$$

Recall by Claim 1, we have $q\beta(R(1-\bar{\kappa}_k))^{1-\gamma} < 1$. As such, use FVAC (Equation (9), which says $\beta\,\mathbb{E}\,G < 1$) and fix α such that α satisfies $q\beta(R(1-\bar{\kappa}_k))^{1-\gamma} < \alpha < 1$ and $\alpha > \beta\,\mathbb{E}\,\tilde{G}$. Next, use $\varphi(m) = \bar{M} + m^{1-\gamma}$ and let $\hat{a}^{\mathrm{next}} = m - \hat{\mathrm{c}}(m)$. The condition above will be satisfied if:

$$\beta\,\mathbb{E}[\tilde{G}^{\mathrm{next}}(\hat{a}^{\mathrm{next}}\tilde{R} + \xi)^{1-\gamma}] - \alpha m^{1-\gamma} < \alpha\bar{M}(1 - \alpha^{-1}\beta\,\mathbb{E}\,\tilde{G})$$

which by the construction of α ($\beta\,\mathbb{E}\,\tilde{G} < \alpha$), can be rewritten as:

$$\bar{M} > \frac{\beta\,\mathbb{E}\left[\tilde{G}(a^{\mathrm{next}}\tilde{R}^{\mathrm{next}} + \xi_{t+1})^{1-\gamma}\right] - \alpha m^{1-\gamma}}{\alpha(1 - \alpha^{-1}\beta\,\mathbb{E}\,\tilde{G})}. \tag{25}$$

Since $\bar{M}$ is an arbitrary constant that we can pick, the proof reduces to showing the

[3] Section **??** proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

numerator of (25) is bounded from above:

$$
\begin{aligned}
&(1-q)\beta\,\mathbb{E}_t\left[\tilde{G}(\hat{a}^{\text{next}}\tilde{R}^{\text{next}}+\theta^{\text{next}}/(1-q))^{1-\gamma}\right]\\
&\quad+q\beta\,\mathbb{E}_t\left[\tilde{G}(\hat{a}^{\text{next}}\tilde{R}^{\text{next}})^{1-\gamma}\right]-\alpha m^{1-\gamma}\\
&\leq(1-q)\beta\,\mathbb{E}_t\left[\tilde{G}((1-\bar{\kappa}_k)m\tilde{R}^{\text{next}}+\theta^{\text{next}}/(1-q))^{1-\gamma}\right]\\
&\quad+q\beta R^{1-\gamma}((1-\bar{\kappa}_k)m)^{1-\gamma}-\alpha m^{1-\gamma}\\
&=(1-q)\beta\,\mathbb{E}_t\left[\tilde{G}((1-\bar{\kappa}_k)m\tilde{R}^{\text{next}}+\theta^{\text{next}}/(1-q))^{1-\gamma}\right]\\
&\quad+m^{1-\gamma}\left(\underbrace{q\beta(R(1-\bar{\kappa}_k))^{1-\gamma}}_{<\alpha\text{ by construction}}-\alpha\right)\\
&<(1-q)\beta\,\mathbb{E}_t\left[\tilde{G}(\underline{\theta}/(1-q))^{1-\gamma}\right]=\beta\,\mathbb{E}\,\tilde{G}(1-q)^{\gamma}\underline{\theta}^{1-\gamma}.
\end{aligned}
\tag{26}
$$

The first inequality holds since $\bar{b}\leq\bar{\kappa}_k$. We can thus conclude that equation (25) will certainly hold for any $\bar{M}$ such that:

$$
\bar{M}>\bar{\bar{M}}:\;=\frac{\beta\,\mathbb{E}\,\tilde{G}(1-q)^{\gamma}\underline{\theta}^{1-\gamma}}{\alpha(1-\alpha^{-1}\beta\,\mathbb{E}\,\tilde{G})}
\tag{27}
$$

which is a positive finite number under our assumptions. Noting that with the construction of α, the above holds for any $\bar{b}\geq\bar{\kappa}_k$. Thus $\mathbb{T}^{\underline{b},\bar{b}}$ defines a contraction mapping with modulus α for any $\bar{b}$ with $\bar{b}\geq\bar{\kappa}_k$ and $\underline{b}>0$. □

Proof of Theorem 2 (Continued). We continue the proof from the main text below.

Proof of (ii)

Given the proof that the value functions converge, we next establish the point-wise convergence of consumption the functions $\{\mathrm{c}_{t_n}\}_{n=0}^{\infty}$ along a sub-sequence which will allow us to show that v satisfies the Bellman operator. Fix any $m\in S$ and consider a convergent subsequence $\{\mathrm{c}_{t_{n(i)}}(m)\}_{i=0}^{\infty}$ of $\{\mathrm{c}_{t_n}(m)\}_{n=0}^{\infty}$. Let the function c denote the mapping from m to the limit of $\{\mathrm{c}_{t_{n(i)}}(m)\}_{i=0}^{\infty}$. By the definition of $\mathrm{c}_{t_n(i)}(m)$, we have:

$$
\begin{aligned}
\mathrm{u}(\mathrm{c}_{t_{n(i)}}(m))+&\beta\,\mathbb{E}_{t_{n(i)}}\left[\tilde{G}^{1-\gamma}_{t_{n(i)}+1}\mathrm{v}_{t_{n(i)+1}}(m_{t_{n(i)+1}})\right]\\
&\geq\mathrm{u}(c)+\beta\,\mathbb{E}_{t_{n(i)}}\left[\tilde{G}^{1-\gamma}_{t_{n(i)}+1}\mathrm{v}_{t_{n(i)+1}}(\hat{m}^{\text{next}})\right],
\end{aligned}
\tag{28}
$$

for any $c\in(0,\bar{\kappa}m]$, where $m_{t_{n(i)}+1}=\tilde{R}(m-\mathrm{c}_{t_{n(i)}}(m))+\xi_{t_{n(i)}+1}$ and $\hat{m}^{\text{next}}=\tilde{R}(m-c)+\xi_{t_{n(i)}+1}$. Allowing $n(i)$ to tend to infinity, the left-hand side converges to:

$$
\mathrm{u}(\mathrm{c}(m))+\beta\,\mathbb{E}\left[\tilde{G}^{1-\gamma}\mathrm{v}(m^{\text{next}})\right],
\tag{29}
$$

where $m^{\text{next}}=\tilde{R}(m-\mathrm{c}(m))+\xi$. Moreover, the right-hand side converges to:

$$\mathrm{u}(c) + \beta\,\mathbb{E}\left[\tilde{G}^{1-\gamma}\mathrm{v}(\hat{m}^{\text{next}})\right]. \tag{30}$$

Hence, as $n(i)$ tends to infinity, the following inequality is implied:

$$\mathrm{u}(\mathrm{c}(m)) + \beta\,\mathbb{E}\left[\tilde{G}^{1-\gamma}\mathrm{v}(m^{\text{next}})\right] \geq \mathrm{u}(c) + \beta\,\mathbb{E}\left[\tilde{G}^{1-\gamma}\mathrm{v}(\hat{m}^{\text{next}})\right]. \tag{31}$$

Since the c above was arbitrary, we have:

$$\mathrm{c}(m) \in \underset{c\in(0,\bar{\kappa}m]}{\arg\max}\left\{\mathrm{u}(c) + \beta\,\mathbb{E}_t\left[\tilde{G}_{t+1}^{1-\gamma}\mathrm{v}(\hat{m}^{\text{next}})\right]\right\}. \tag{32}$$

Next, since $\mathrm{c}_{t_{n(i)}} \to \mathrm{c}$ point-wise, and $\mathrm{v}_{t_{n(i)}} \to \mathrm{v}$ point-wise, we have:

$$\mathrm{v}(m) = \lim_{i\to\infty} \mathrm{u}(\mathrm{c}_{t_{n(i)}}(m)) + \beta\,\mathbb{E}\,\tilde{G}\mathrm{v}_{t_{n(i)}+1}(m_{t_{n(i)}+1}) = \mathrm{u}(\mathrm{c}(m)) + \beta\,\mathbb{E}\,\tilde{G}\mathrm{v}(m^{\text{next}}). \tag{33}$$

where $m_{t_n} = \tilde{R}(m - \mathrm{c}_{t_n}(m))$ and $m^{\text{next}} = \tilde{R}(m - \mathrm{c}(m))$. The first equality stems form the fact that $\mathrm{v}_{t_n} \to \mathrm{v}$ point-wise, and because point-wise convergence implies point-wise convergence along a sub-sequence. To see why $\lim_{i\to\infty} \mathrm{u}(\mathrm{c}_{t_{n(i)}}(m)) = \mathrm{u}(\mathrm{c}(m))$, note the continuity of u and the convergence of $\mathrm{c}_{t_{n(i)}}$ to c point-wise. To see why $\lim_{i\to\infty} \mathbb{E}\,\tilde{G}\mathrm{v}_{t_{n(i)}+1}(m_{t_{n(i)}+1}) = \mathbb{E}\,\tilde{G}\mathrm{v}(m^{\text{next}})$, note that $\mathrm{v}_{t_{n(i)}+1}$ converges in the φ-norm, hence converges uniformly over compact sets in S and apply Fact 3 from the standard mathematical results presented in Appendix D. This completes the proof of part (ii) of the Theorem.

Proof of (iii)

The limits at Equation (33) immediately imply:

$$\mathrm{v}(m) = \lim_{n\to\infty} \mathrm{u}(\mathrm{c}_{t_n}(m)) + \beta\,\mathbb{E}\,\tilde{G}\mathrm{v}_{t_n+1}(m_{t_n+1}) = \mathrm{u}(\mathrm{c}(m)) + \beta\,\mathbb{E}\,\tilde{G}\mathrm{v}(m^{\text{next}}), \tag{34}$$

since a real valued sequence can have at most one limit. Finally, applying Fact 8 from Appendix D, we get $\mathrm{c}_{t_n}(m)) \to \mathrm{c}(m)$, thus establishing that c_{t_n} converges point-wise to c.

□

A.5 The Liquidity Constrained Solution as a Limit

Proof of Proposition 4. Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned} q &= 0 \\ c_t &\leq m_t, \end{aligned}$$

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed q as $\mathrm{c}_t(m;q)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and q, which is not. The proposition we wish to demonstrate is

$$\lim_{q\downarrow 0} \mathrm{c}_t(m;q) = \grave{\mathrm{c}}_t(m). \tag{35}$$

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = R = G = 1$, and there are no permanent shocks, $\psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ \mathrm{u}(m-a) + \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{v}_T(a+\theta) d\mathcal{F}_\theta \right\}. \tag{36}$$

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{u}'(\mathrm{c}_T(a+\theta)) d\mathcal{F}_\theta,$$

and the solution to (63) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{37}$$

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $q > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (1991)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\gamma}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (64), defining

$$\mathfrak{v}'_{T-1}(a;q) \equiv \left[qa^{-\gamma} + (1-q) \int_{\underline{\theta}}^{\bar{\theta}} \left(\mathrm{c}_T(a+\theta/(1-q))\right)^{-\gamma} d\mathcal{F}_\theta \right], \tag{38}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\gamma} = \mathfrak{v}'_{T-1}(a;q) \tag{39}$$

with solution $\mathrm{a}^*_{T-1}(m;q)$. Now note that for any fixed $a > 0$, $\lim_{q\downarrow 0} \mathfrak{v}'_{T-1}(a;q) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (64) and (66) are identical, this means that $\lim_{q\downarrow 0} \mathrm{a}^*_{T-1}(m;q) = \grave{\mathrm{a}}^*_{T-1}(m)$. That is, for any fixed value of $m > m^1_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $q \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^1_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (65) is $\lim_{a\downarrow 0} qa^{-\gamma} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\gamma}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^1_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{q\downarrow 0} \mathfrak{v}'_{T-1}(a;q) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^*_{T-1}(m) < 0$, and we showed earlier that $\lim_{q\downarrow 0} \mathrm{a}^*_{T-1}(m;q) = \grave{\mathrm{a}}^*_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^1_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{q\downarrow 0} \mathrm{c}_{T-1}(m;q) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (62). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (80) for the maximal marginal propensity to consume satisfies

$$\lim_{q\downarrow 0} \bar{\kappa} \quad = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of 'constrained' and 'restrained.'

□

B Convergence in Euclidian Space

B.1 Convergence of v_t

Boyd's theorem shows that $\mathbb{T}$ defines a contraction mapping in an φ-bounded space. We now show that $\mathbb{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathbb{T}$, since $\mathrm{v}^*(m) = \mathbb{T}\mathrm{v}^*(m)$,

$$\left\|\mathrm{v}_{T-n+1} - \mathrm{v}^*\right\|_{\varphi} \leq \alpha^{n-1} \left\|\mathrm{v}_T - \mathrm{v}^*\right\|_{\varphi}. \tag{40}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_\varphi(\mathcal{A},\mathcal{B})$ and $\kappa = \|\mathrm{v}_T - \mathrm{v}^*\|_\varphi < \infty$ because v_T and v^* are in $\mathcal{C}_\varphi(\mathcal{A},\mathcal{B})$. It follows that

$$|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)| \leq \kappa\alpha^{n-1}\,|\varphi(m)|\,. \tag{41}$$

Then we obtain

$$\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{42}$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\gamma}}{1-\gamma}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\gamma}}{1-\gamma} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathbb{T}\mathrm{v}_{T-1} \leq \mathbb{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

B.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

Consider any convergent subsequence $\{\mathrm{c}_{T-n(i)}(m)\}$ of $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to c^*. By the definition of $\mathrm{c}_{T-n}(m)$, we have

$$\begin{aligned}\mathrm{u}(c_{T-n(i)}(m)) + &\beta\,\mathbb{E}_{T-n(i)}[G_{T-n(i)+1}^{1-\gamma}\mathrm{v}_{T-n(i)+1}(m)]\\ &\geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_{T-n(i)}[G_{T-n(i)+1}^{1-\gamma}\mathrm{v}_{T-n(i)+1}(m)],\end{aligned} \tag{43}$$

for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[G_t^{1-\gamma}\mathrm{v}(m)]$, and the right hand side converges to $\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[G_t^{1-\gamma}\mathrm{v}(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[G_{t+1}^{1-\gamma}\mathrm{v}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[G_{t+1}^{1-\gamma}\mathrm{v}(m)]. \tag{44}$$

Hence, $c^* \in \underset{c_{T-n(i)}\in[\underline{\kappa}m,\bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[G_{t+1}^{1-\gamma}\mathrm{v}(m)]\right\}$. By the uniqueness of $\mathrm{c}(m)$, $c^* = \mathrm{c}(m)$.

C Perfect Foresight Liquidity Constrained Solution

We briefly interpret FVAC before turning to how all the conditions relate under uncertainty. Analogously to (45), the value for a consumer who spent exactly their permanent income every period would reflect the product of the expectation of the (independent) future shocks to permanent income:

$$= \mathrm{u}(\boldsymbol{p}_t)\left(\frac{1-\left(\beta G^{1-\gamma}\,\mathbb{E}[\psi^{1-\gamma}]\right)^{T-t+1}}{1-\beta G^{1-\gamma}\,\mathbb{E}[\psi^{1-\gamma}]}\right),$$

The function $\mathbf{v}_t$ will be finite as T approaches ∞ if the FVAC holds. In the case without uncertainty, Because $\mathrm{u}(xy) = x^{1-\gamma}\mathrm{u}(y)$, the value the consumer would achieve is:

$$\begin{aligned}\mathbf{v}_t^{\text{autarky}} &= \mathrm{u}(\boldsymbol{p}_t) + \beta\mathrm{u}(\boldsymbol{p}_t G) + \beta^2\mathrm{u}(\boldsymbol{p}_t G^2) + \dots \\ &= \mathrm{u}(\boldsymbol{p}_t)\left(\frac{1-\left(\beta G^{1-\gamma}\right)^{T-t+1}}{1-\beta G^{1-\gamma}}\right)\end{aligned}$$

which (for $G > 0$) asymptotes to a finite number as n, with $n = T - t$, approaches $+\infty$.

C.1 Perfect Foresight Unconstrained Solution

The first result relates to the perfect foresight case without liquidity constraints.

Proof. (Proof of Proposition 1) Consider a sequence of consumption $\{\boldsymbol{c}_{T-n}\}_{n=t}^{T}$ and a sequence of income $\{\boldsymbol{p}_{T-n}\}_{n=t}^{T}$ and let $\mathrm{PDV}_t^T(\boldsymbol{c})$ and $\mathrm{PDV}_t^T(\boldsymbol{p})$ denote the present discounted value of the consumption sequence and permanent income sequence respectively. The dynamic budget constraint, strictly positive marginal utility, and the can't-die-in-debt condition, Equation (1), imply an exactly-holding intertemporal budget constraint (IBC):

$$\mathrm{PDV}_t^T(\boldsymbol{c}) = \overbrace{\boldsymbol{m}_t - \boldsymbol{p}_t}^{\boldsymbol{b}_t} + \overbrace{\mathrm{PDV}_t^T(\boldsymbol{p})}^{\mathbf{h}_t}, \tag{45}$$

where $\boldsymbol{b}$ is beginning-of-period 'market' balances; with $\bar{R}\colon = R/G$ 'human wealth' can be written as:

$$\begin{aligned}\mathbf{h}_t &= \boldsymbol{p}_t + \bar{R}^{-1}\boldsymbol{p}_t + \bar{R}^{-2}\boldsymbol{p}_t + \dots + \bar{R}^{t-T}\boldsymbol{p}_t \\ &= \underbrace{\left(\frac{1-\bar{R}^{-(T-t+1)}}{1-\bar{R}^{-1}}\right)}_{\equiv h_t}\boldsymbol{p}_t. \end{aligned}\tag{46}$$

Let h denote the limiting value of normalized human wealth as the planning horizon recedes, we have $h\colon = \lim_{n\to\infty} h_{t_n}$.

Next, since consumption is growing by $\text{\TH}$ but discounted by R:

$$\mathrm{PDV}_t^T(\boldsymbol{c}) = \left(\frac{1-\frac{\text{\TH}}{R}^{T-t+1}}{1-\frac{\text{\TH}}{R}}\right)\boldsymbol{c}_t$$

from which the IBC (45) implies

$$\boldsymbol{c}_t = \overbrace{\left(\frac{1-\frac{\text{\TH}}{R}}{1-\frac{\text{\TH}}{R}^{T-t+1}}\right)}^{\equiv\underline{\kappa}_t}(\boldsymbol{b}_t + \mathbf{h}_t) \tag{47}$$

defining a normalized finite-horizon perfect foresight consumption function:

$$\bar{\mathrm{c}}_{T-n}(m_{T-n}) = (\overbrace{m_{T-n} - 1}^{\equiv b_{T-n}} + h_{T-n})\underline{\kappa}_{t-n}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC). (The overbar signifies that $\bar{\mathrm{c}}$ will be an upper bound as we modify the problem to incorporate constraints and uncertainty; analogously, $\underline{\kappa}$ is the MPC's lower bound).

The horizon-exponentiated term in the denominator of (47) is why, for $\underline{\kappa}$ to be strictly positive as n goes to infinity, we must impose the RIC. The RIC thus implies that the consumer cannot be so pathologically patient as to wish, in the limit as the horizon approaches infinity, to spend nothing today out of an increase in current wealth (the RIC rules out the degenerate limiting solution $\bar{\mathrm{c}}(m) = 0$).

Given that the RIC holds, and (as before) defining limiting objects by the absence of a time subscript, the limiting upper bound consumption function will be

$$\bar{\mathrm{c}}(m) = (m + h - 1)\underline{\kappa}, \tag{48}$$

and so in order to rule out the degenerate limiting solution $\bar{\mathrm{c}}(m) = \infty$ we need h to be finite; that is, we must impose the Finite Human Wealth Condition (FHWC), eq. (**??**).

$\square$

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\grave{\mathrm{c}}(m)$ that arise under various parametric conditions.

Results are summarized in table 5.

C.2 If GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails (~~GIC~~, $1 < \mathbf{Þ}/G$). Under ~~GIC~~ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (R\beta)^{1/\gamma}/G$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[4]

$$\begin{aligned} 1 &< (R\beta)^{1/\gamma} G^{-1} \\ 1 &< R\beta G^{-\gamma} \\ \mathrm{u}'(1) &< R\beta \mathrm{u}'(G). \end{aligned}$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

RIC fails, FHWC holds. If the RIC fails ($1 < \frac{\mathbf{Þ}}{R}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$\grave{\mathrm{c}}_{T-n}(m) = 0(b_t + h). \tag{49}$$

[4]The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_{\#}) = R\beta\mathrm{u}'(G)$ which is $c_{\#} = G/(R\beta)^{1/\gamma}$ and the consumption function will be defined by $\grave{\mathrm{c}}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

(that is, consumption is zero for any level of human or nonhuman wealth).

RIC fails, FHWC fails. ~~FHWC~~ implies that human wealth limits to $h = \infty$ so the consumption function limits to either $\grave{c}_{T-n}(m) = 0$ or $\grave{c}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[5]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying ~~GIC~~ we must impose the RIC (and the FHWC can be shown to be a consequence of ~~GIC~~ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer would choose $c = m$ from Equation (48):

$$\begin{aligned} m_{\#} &= (m_{\#} - 1 + h)\underline{\kappa} \\ m_{\#}(1 - \underline{\kappa}) &= (h - 1)\underline{\kappa} \\ m_{\#} &= (h - 1)\left(\frac{\underline{\kappa}}{1 - \underline{\kappa}}\right) \end{aligned} \tag{50}$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[6] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\grave{c}(m) = m$.[7] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{c}(m)$.

(Stachurski and Toda (2019) obtain a similar lower bound on consumption and use it to study the tail behavior of the wealth distribution.)

C.3 If GIC Holds

Imposition of the GIC reverses the inequality in (49), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b_{\#}^{n}$ as the b_t such that an unconstrained consumer holding $b_t = b_{\#}^{n}$ would behave so as to arrive in period $t + n$ with $b_{t+n} = 0$ (with $b_{\#}^{0}$ trivially equal to 0); for example, a consumer with $b_{t-1} = b_{\#}^{1}$ was on the 'cusp' of being constrained in period $t - 1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m_{\#}^{0} = 1 + b_{\#}^{0} = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c_{\#}^{0} = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must

[5]The knife-edge case is where $\text{Þ} = G$, in which case the two quantities counterbalance and the limiting function is $\grave{c}(m) = \min[m, 1]$.

[6]Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~GIC~~.

[7]As an illustration, consider a consumer for whom $\text{Þ} = 1$, $R = 1.01$ and $G = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $G < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

have been growing according to $\frac{\text{Þ}}{G}$, so consumption n periods in the past must have been

$$c^n_{\#} = \frac{\text{Þ}}{G}^{-n} c_t = \frac{\text{Þ}}{G}^{-n}. \tag{51}$$

The PDV of consumption from $t-n$ until t can thus be computed as

$$\begin{aligned}
\mathbb{C}^t_{t-n} &= c_{t-n}(1 + \text{Þ}/R + \cdots + (\text{Þ}/R)^n) \\
&= c^n_{\#}(1 + \frac{\text{Þ}}{R} + \cdots + \frac{\text{Þ}}{R}^n) \\
&= \frac{\text{Þ}}{G}^{-n} \left(\frac{1 - \frac{\text{Þ}}{R}^{n+1}}{1 - \frac{\text{Þ}}{R}} \right) \\
&= \left(\frac{\frac{\text{Þ}}{G}^{-n} - \frac{\text{Þ}}{R}}{1 - \frac{\text{Þ}}{R}} \right)
\end{aligned}$$

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h^n_{\#} = 1 + \cdots + \tilde{R}^{-n} \tag{52}$$

while the intertemporal budget constraint says

$$\mathbb{C}^t_{t-n} \;\; = b^n_{\#} + h^n_{\#}$$

from which we can solve for the $b^n_{\#}$ such that the consumer with $b_{t-n} = b^n_{\#}$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t = 0$:

$$b^n_{\#} = \mathbb{C}^t_{t-n} - \overbrace{\left(\frac{1 - \tilde{R}^{-(n+1)}}{1 - \tilde{R}^{-1}} \right)}^{h^n_{\#}}. \tag{53}$$

Defining $m^n_{\#} = b^n_{\#} + 1$, consider the function $\grave{c}(m)$ defined by linearly connecting the points $\{m^n_{\#}, c^n_{\#}\}$ for integer values of $n \geq 0$ (and setting $\grave{c}(m) = m$ for $m < 1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets $m = m^n_{\#} - \epsilon$ is discretely higher than for a consumer with assets $m = m^n_{\#} + \epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (53) for the entire domain of positive real values of b, we need $b^n_{\#}$ to become arbitrarily large with n. That is, we need

$$\lim_{n \to \infty} b^n_{\#} = \infty. \tag{54}$$

C.3.1 If FHWC Holds

The FHWC requires $\tilde{R}^{-1} < 1$, in which case the second term in (53) limits to a constant as $n \uparrow \infty$, and (54) reduces to a requirement that

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{\frac{Þ}{G}^{-n} - (\frac{Þ}{R}/\frac{Þ}{G})^n \frac{Þ}{R}}{1-\frac{Þ}{R}}\right) &= \infty \\
\lim_{n\to\infty}\left(\frac{\frac{Þ}{G}^{-n} - \tilde{R}^{-n}\frac{Þ}{R}}{1-\frac{Þ}{R}}\right) &= \infty \\
\lim_{n\to\infty}\left(\frac{\frac{Þ}{G}^{-n}}{1-\frac{Þ}{R}}\right) &= \infty.
\end{aligned}$$

Given the GIC $\frac{Þ}{G}^{-1} > 1$, this will hold iff the RIC holds, $\frac{Þ}{R} < 1$. But given that the FHWC $R > G$ holds, the GIC is stronger (harder to satisfy) than the RIC; thus, the FHWC and the GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \grave{\mathrm{c}}(m) - \bar{\mathrm{c}}(m) = 0. \tag{55}$$

C.3.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex.

Given failure of FHWC, (54) requires

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{\tilde{R}^{-n}\frac{Þ}{R} - \frac{Þ}{G}^{-n}}{\frac{Þ}{R}-1}\right) + \left(\frac{1-\tilde{R}^{-(n+1)}}{\tilde{R}^{-1}-1}\right) &= \infty \\
\lim_{n\to\infty}\left(\frac{\frac{Þ}{R}}{\frac{Þ}{R}-1} - \frac{\tilde{R}^{-1}}{\tilde{R}^{-1}-1}\right)\tilde{R}^{-n} - \left(\frac{\frac{Þ}{G}^{-n}}{\frac{Þ}{R}-1}\right) &= \infty
\end{aligned}$$

If RIC Holds. When the RIC holds, rearranging (56) gives

$$\lim_{n\to\infty}\left(\frac{\frac{Þ}{G}^{-n}}{1-\frac{Þ}{R}}\right) - \tilde{R}^{-n}\left(\frac{\frac{Þ}{R}}{1-\frac{Þ}{R}} + \frac{\tilde{R}^{-1}}{\tilde{R}^{-1}-1}\right) = \infty$$

and for this to be true we need

$$\begin{aligned}
\frac{Þ}{G}^{-1} &> \tilde{R}^{-1} \\
G/Þ &> G/R \\
1 &> Þ/R
\end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty} \kappa_{\#}^{n} = \lim_{n\to\infty} \left(\frac{c_{\#}^{n}}{b_{\#}^{n}} \right) \tag{56}$$

which with a bit of algebra[8] can be shown to asymptote to the MPC in the perfect foresight model:[9]

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 1 - \frac{\mathbf{Þ}}{R}. \tag{58}$$

If RIC Fails. Consider now the $\cancel{\text{RIC}}$ case, $\frac{\mathbf{Þ}}{R} > 1$. We can rearrange (56)as

$$\lim_{n\to\infty} \left(\frac{\frac{\mathbf{Þ}}{R}(\tilde{R}^{-1}-1)}{(\tilde{R}^{-1}-1)(\frac{\mathbf{Þ}}{R}-1)} - \frac{\tilde{R}^{-1}(\frac{\mathbf{Þ}}{R}-1)}{(\tilde{R}^{-1}-1)(\frac{\mathbf{Þ}}{R}-1)} \right) \tilde{R}^{-n} - \left(\frac{\frac{\mathbf{Þ}}{G}^{-n}}{\frac{\mathbf{Þ}}{R}-1} \right) = \infty. \tag{59}$$

which makes clear that with $\cancel{\text{FHWC}} \Rightarrow \tilde{R}^{-1} > 1$ and $\cancel{\text{RIC}} \Rightarrow \frac{\mathbf{Þ}}{R} > 1$ the numerators and denominators of both terms multiplying $\tilde{R}^{-n}$ can be seen transparently to be positive. So, the terms multiplying $\tilde{R}^{-n}$ in (56) will be positive if

$$\begin{aligned} \frac{\mathbf{Þ}}{R}\tilde{R}^{-1} - \frac{\mathbf{Þ}}{R} &> \tilde{R}^{-1}\frac{\mathbf{Þ}}{R} - \tilde{R}^{-1} \\ \tilde{R}^{-1} &> \frac{\mathbf{Þ}}{R} \\ G &> \mathbf{Þ} \end{aligned}$$

which is merely the GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\tilde{R}^{-n}$ goes to $+\infty$ faster than the term involving $-\frac{\mathbf{Þ}}{G}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned} \tilde{R}^{-1} &> \frac{\mathbf{Þ}}{G}^{-1} \\ G/R &> G/\mathbf{Þ} \\ \mathbf{Þ}/R &> 1 \end{aligned}$$

which merely confirms the starting assumption that the RIC fails.

What is happening here is that the $c_{\#}^{n}$ term is increasing backward in time at rate dominated in the limit by $G/\mathbf{Þ}$ while the $b_{\#}$ term is increasing at a rate dominated by G/R term and

$$G/R > G/\mathbf{Þ} \tag{60}$$

because $\cancel{\text{RIC}} \Rightarrow \mathbf{Þ} > R$.

[8]Calculate the limit of

$$\left(\frac{\frac{\mathbf{Þ}}{G}^{-n}}{\frac{\mathbf{Þ}}{G}^{-n}/(1-\frac{\mathbf{Þ}}{R}) - (1-\tilde{R}^{-1}\tilde{R}^{-n})/(1-\tilde{R}^{-1})} \right) = \left(\frac{1}{1/(1-\frac{\mathbf{Þ}}{R}) + \tilde{R}^{-n}\tilde{R}^{-1}/(1-\tilde{R}^{-1})} \right) \tag{57}$$

[9]For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.

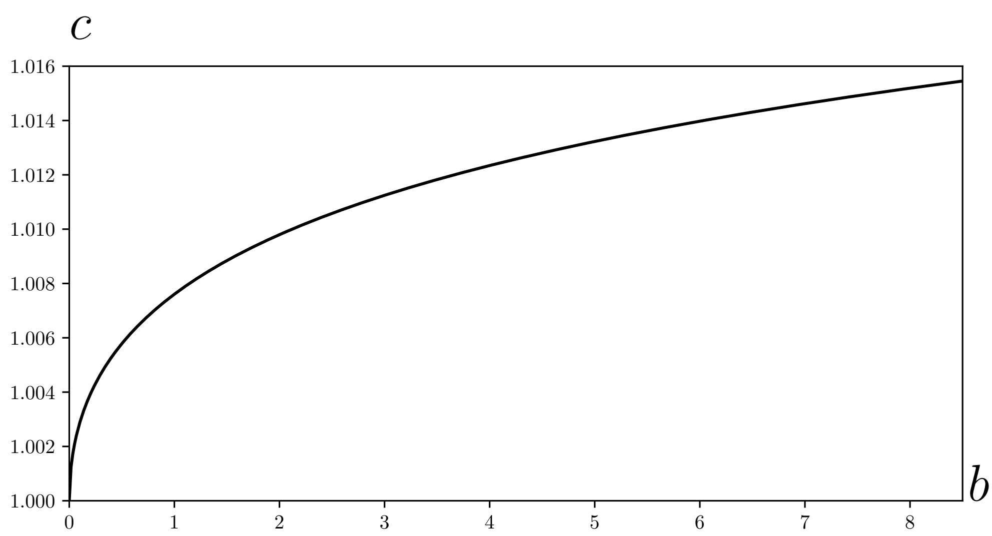

Figure 1 Appendix: Nondegenerate c Function with $\cancel{\text{FHWC}}$ and $\cancel{\text{RIC}}$

Consequently, while $\lim_{n\uparrow\infty} b^n_{\#} = \infty$, the limit of the *ratio* $c^n_{\#}/b^n_{\#}$ in (56) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It remains true that $\cancel{\text{RIC}}$ implies a limiting MPC of zero,

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 0, \tag{61}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\grave{\mathrm{c}}(m) = 0$. (Figure 1 presents an example for $\gamma = 2$, $R = 0.98$, $\beta = 1.00$, $G = 0.99$; note that the horizontal axis is bank balances $b = m-1$; the part of the consumption function below the depicted points is uninteresting — $c = m$ — so not worth plotting).

We can summarize as follows. Given that the GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined and nondegenerate solution, whether or not the RIC holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

Ma and Toda (2020) characterize the limits of the MPC in a more general framework that allows for capital and labor income risks in a Markovian setting with liquidity constraints, and find that in that much more general framework the limiting MPC is also zero.

D The Liquidity Constrained Solution as a Limit

Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned} q &= 0 \\ c_t &\leq m_t, \end{aligned}$$

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed q as $\mathrm{c}_t(m; q)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and q, which is not. The proposition we wish to demonstrate is

$$\lim_{q \downarrow 0} \mathrm{c}_t(m; q) = \grave{\mathrm{c}}_t(m). \tag{62}$$

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = R = G = 1$, and there are no permanent shocks, $\psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ \mathrm{u}(m-a) + \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{v}_T(a+\theta) d\mathcal{F}_\theta \right\}. \tag{63}$$

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{u}'(\mathrm{c}_T(a+\theta)) d\mathcal{F}_\theta,$$

and the solution to (63) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{64}$$

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $q > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (1991)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\gamma}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (64), defining

$$\mathfrak{v}'_{T-1}(a;q) \equiv \left[qa^{-\gamma} + (1-q)\int_{\underline{\theta}}^{\bar{\theta}} \left(\mathrm{c}_T(a+\theta/(1-q))\right)^{-\gamma} d\mathcal{F}_\theta \right], \tag{65}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\gamma} = \mathfrak{v}'_{T-1}(a;q) \tag{66}$$

with solution $\mathrm{a}^*_{T-1}(m;q)$. Now note that for any fixed $a > 0$, $\lim_{q\downarrow 0} \mathfrak{v}'_{T-1}(a;q) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (64) and (66) are identical, this means that $\lim_{q\downarrow 0} \mathrm{a}^*_{T-1}(m;q) = \grave{\mathrm{a}}^*_{T-1}(m)$. That is, for any fixed value of $m > m^1_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $q \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^1_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (65) is $\lim_{a\downarrow 0} qa^{-\gamma} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\gamma}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^1_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{q\downarrow 0} \mathfrak{v}'_{T-1}(a;q) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^*_{T-1}(m) < 0$, and we showed earlier that $\lim_{q\downarrow 0} \mathrm{a}^*_{T-1}(m;q) = \grave{\mathrm{a}}^*_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^1_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{q\downarrow 0} \mathrm{c}_{T-1}(m;q) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (62). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (80) for the maximal marginal propensity to consume satisfies

$$\lim_{q\downarrow 0} \bar{\kappa} \;\; = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of 'constrained' and 'restrained.'

E Proofs for Individual Stability (Section 3)

Proof for Proposition 5. *Proof.* For consumption growth, as $m \downarrow 0$ we have

$$\begin{aligned}
\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) G_{t+1} \right] &> \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})}{\bar{\kappa} m_t} \right) G_{t+1} \right] \\
&= q \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) G_{t+1} \right] \\
&\quad + (1-q) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \theta_{t+1}/(1-q))}{\bar{\kappa} m_t} \right) G_{t+1} \right] \\
&> (1-q) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/(1-q))}{\bar{\kappa} m_t} \right) G_{t+1} \right] \\
&= \infty
\end{aligned}$$

where the second-to-last line follows because $\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) G_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.[10]

Next we establish the limit of the expected consumption growth factor as $m_t \uparrow \infty$:

$$\lim_{m_t \uparrow \infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] = \lim_{m_t \uparrow \infty} \mathbb{E}_t[G_{t+1} c_{t+1}/c_t].$$

But

$$\mathbb{E}_t[G_{t+1}\underline{c}_{t+1}/\bar{c}_t] \leq \mathbb{E}_t[G_{t+1}c_{t+1}/c_t] \leq \mathbb{E}_t[G_{t+1}\bar{c}_{t+1}/\underline{c}_t]$$

and

$$\lim_{m_t \uparrow \infty} G_{t+1}\underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t \uparrow \infty} G_{t+1}\bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t \uparrow \infty} G_{t+1}m_{t+1}/m_t,$$

while (for convenience defining $\mathrm{a}(m_t) = m_t - \mathrm{c}(m_t)$),

$$\begin{aligned}
\lim_{m_t \uparrow \infty} G_{t+1}m_{t+1}/m_t &= \lim_{m_t \uparrow \infty} \left(\frac{R\mathrm{a}(m_t) + G_{t+1}\xi_{t+1}}{m_t} \right) \\
&= (R\beta)^{1/\gamma} = \text{Þ}
\end{aligned} \tag{67}$$

because $\lim_{m_t \uparrow \infty} \mathrm{a}'(m) = \frac{\text{Þ}}{R}$[11] and $G_{t+1}\xi_{t+1}/m_t \leq (G\bar{\psi}\bar{\theta}/(1-q))/m_t$ which goes to zero as m_t goes to infinity.

Hence we have

$$\text{Þ} \leq \lim_{m_t \uparrow \infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] \leq \text{Þ}$$

[10]None of the arguments in either of the two prior sections depended on the assumption that the consumption functions had converged. With more cumbersome notation, each derivation could have been replaced by the corresponding finite-horizon versions. This strongly suggests that it should be possible to extend the circumstances under which the problem can be shown to define a contraction mapping to the union of the parameter values under which {RIC,FHWC} hold and {FVAC,WRIC} hold. That extension is not necessary for our purposes here, so we leave it for future work.

[11] $\lim_{m_t \uparrow \infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t \uparrow \infty} \mathrm{c}(m_t)/m_t = 1 - \lim_{m_t \uparrow \infty} \mathrm{c}'(m_t) = \frac{\text{Þ}}{R}$.

so as cash goes to infinity, consumption growth approaches its value **Þ** in the perfect foresight model. ☐

☐

This appendix proves Theorems 6-7 and:

Lemma 2. *If $\check{m}$ and $\hat{m}$ both exist, then $\check{m} \leq \hat{m}$.*

E.1 Proof of Theorem 6

Theorem 2. *(Individual Market-Resources-to-Permanent-Income Ratio Target). Consider the problem defined in Section 2.1. If weak return impatience (Assumption L.4), finite value of autarky (Assumption L.1) and strong growth impatience (Assumption S.2) hold, then there exists a unique market resources to permanent income ratio, $\hat{m}$, with $\hat{m} > 0$, such that:*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \tag{68}$$

Moreover, $\hat{m}$ is a point of 'stability' in the sense that:

$$\begin{aligned} \forall m_t \in (0, \hat{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\ \forall m_t \in (\hat{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t. \end{aligned} \tag{69}$$

The elements of the proof of Theorem 6 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

E.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem ??).

Section ?? shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\tilde{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\tilde{R}_t > 0$, since both a_{t-1} and $\tilde{R}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

E.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

E.3.1 Existence of m where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

If RIC holds. Logic exactly parallel to that of Section **??** leading to equation (67), but dropping the G_{t+1} from the RHS, establishes that

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1}}{m_t} \right] \\
&= \mathbb{E}_t[(R/G_{t+1}) \frac{\text{Þ}}{R}] \\
&= \mathbb{E}_t[\text{Þ}/G_{t+1}] \\
&< 1
\end{aligned} \tag{70}$$

where the inequality reflects imposition of the GIC-Mod (**??**).

If RIC fails. When the RIC fails, the fact that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$ (see equation (19)) means that the limit of the RHS of (70) as $m \uparrow \infty$ is $\bar{\tilde{R}} = \mathbb{E}_t[\tilde{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Mod and ~~RIC~~ implies $\bar{\tilde{R}} < 1$.

So we have $\lim_{m \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

E.3.2 Existence of $m > 1$ where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

Paralleling the logic for c in Section **??**: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

E.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned} \tag{71}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right) \boldsymbol{\zeta}(m_t) &= \mathbb{E}_t \left[\left(\frac{d}{dm_t}\right) \left(\tilde{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1} - m_t\right) \right] \\
&= \bar{\tilde{R}} \left(1 - \mathrm{c}'(m_t)\right) - 1.
\end{aligned} \tag{72}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (**??**) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section **??** that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}
\bar{\bar{\mathcal{R}}}\,(1 - \mathrm{c}'(m_t)) - 1 &< \bar{\bar{\mathcal{R}}}(1 - \underbrace{(1 - \frac{\mathbf{Þ}}{R})}_{\underline{\kappa}}) - 1 \\
&= \bar{\bar{\mathcal{R}}}\frac{\mathbf{Þ}}{R} - 1 \\
&= \mathbb{E}_t\left[\frac{R}{G\psi}\frac{\mathbf{Þ}}{R}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\mathbf{Þ}}{G\psi}\right]}_{=\frac{\mathbf{Þ}}{G}\,\mathbb{E}[\psi^{-1}]} - 1
\end{aligned}$$

which is negative because the GIC-Mod says $\frac{\mathbf{Þ}}{G}\,\mathbb{E}[\psi^{-1}] < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\bar{\mathcal{R}}}\,(1 - \mathrm{c}'(m_t)) < \bar{\bar{\mathcal{R}}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (72) is guaranteed to be negative if

$$\bar{\bar{\mathcal{R}}} \equiv \mathbb{E}_t\left[\frac{R}{G\psi}\right] < 1. \tag{73}$$

But the combination of the GIC-Mod holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\mathbf{Þ}}{G\psi}\right]}^{\frac{\mathbf{Þ}}{G}\,\mathbb{E}[\psi^{-1}]} < 1 < \overbrace{\frac{\mathbf{Þ}}{R}}^{\frac{\mathbf{Þ}}{R}},$$

and multiplying all three elements by $R/\mathbf{Þ}$ gives

$$\mathbb{E}_t\left[\frac{R}{G\psi}\right] < R/\mathbf{Þ} < 1$$

which satisfies our requirement in (73).

E.4 Proof of Theorem 7

Theorem 3. *(Individual Balanced-Growth 'Pseudo Steady State'). Consider the problem defined in Section 2.1. If weak return impatience (Assumption L.4), finite value of autarky (Assumption L.1) and growth impatience (Assumption S.1) hold, then there exists a unique pseudo-steady-state market resources to permanent income ratio $\check{m} > 0$ such that:*

$$\mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{74}$$

Moreover, $\check{m}$ is a point of stability in the sense that:

$$\begin{aligned}\forall m_t \in (0,\check{m}),\ \mathbb{E}_t[\boldsymbol{m}_{t+1}]/\boldsymbol{m}_t > G \\ \forall m_t \in (\check{m},\infty),\ \mathbb{E}_t[\boldsymbol{m}_{t+1}]/\boldsymbol{m}_t < G.\end{aligned} \tag{75}$$

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\psi_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

E.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in E.2 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$.

E.4.2 Existence of a stable point

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in Subsection E.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\psi_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}\lim_{m_t\uparrow\infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{G_{t+1}\left((R/G_{t+1})\mathrm{a}(m_t)+\xi_{t+1}\right)/G}{m_t}\right] \\ &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{(R/G)\mathrm{a}(m_t)+\psi_{t+1}\xi_{t+1}}{m_t}\right] \\ &= \lim_{m_t\uparrow\infty}\left[\frac{(R/G)\mathrm{a}(m_t)+1}{m_t}\right] \\ &= (R/G)\frac{\text{Þ}}{R} \\ &= \frac{\text{Þ}}{G} \\ &< 1\end{aligned} \tag{76}$$

where the last two lines are merely a restatement of the GIC (**??**).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

E.4.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ is monotonically decreasing.

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that

$$\boldsymbol{\zeta}(m_t) < 0 \leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1$$

$$\begin{aligned} \boldsymbol{\zeta}(m_t) = 0 \leftrightarrow \mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = 1 \\ \boldsymbol{\zeta}(m_t) > 0 \leftrightarrow \mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] > 1, \end{aligned} \tag{77}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned} \boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\tilde{R}(m_t - \mathrm{c}(m_t)) + \psi_{t+1}\xi_{t+1} - m_t\right)\right] \\ &= (R/G)\left(1 - \mathrm{c}'(m_t)\right) - 1. \end{aligned} \tag{78}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (**??**) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section **??** that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned} \tilde{R}\left(1 - \mathrm{c}'(m_t)\right) - 1 &< \tilde{R}(1 - \underbrace{(1 - \frac{\text{Þ}}{R})}_{\underline{\kappa}}) - 1 \\ &= (R/G)\frac{\text{Þ}}{R} - 1 \end{aligned}$$

which is negative because the GIC says $\frac{\text{Þ}}{G} < 1$.

F The Limiting MPC's

For $m_t > 0$ we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$ and the Euler equation (4) can be rewritten

$$\begin{aligned} \mathrm{e}_t(m_t)^{-\gamma} &= \beta R\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{R\mathrm{a}_t(m_t) + G_{t+1}\xi_{t+1}}^{=m_{t+1}G_{t+1}}}{m_t}\right)\right)^{-\gamma}\right] \\ &= \quad (1-q)\beta R m_t^{\gamma}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}G_{t+1}\right)^{-\gamma} \mid \xi_{t+1} > 0\right] \\ &\quad + q\beta R^{1-\gamma}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\mathcal{R}_{t+1}\mathrm{a}_t(m_t))\frac{m_t - \mathrm{c}_t(m_t)}{m_t}\right)^{-\gamma} \mid \xi_{t+1} = 0\right]. \end{aligned}$$

Consider the first conditional expectation in (4), recalling that if $\xi_{t+1} > 0$ then $\xi_{t+1} \equiv \theta_{t+1}/(1-q)$. Since $\lim_{m \downarrow 0} \mathrm{a}_t(m) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}G_{t+1})^{-\gamma} \mid \xi_{t+1} > 0]$ is contained within bounds defined by $(\mathrm{e}_{t+1}(\underline{\theta}/(1-q))G\underline{\psi}\underline{\theta}/(1-q))^{-\gamma}$ and $(\mathrm{e}_{t+1}(\bar{\theta}/(1-q))G\bar{\psi}\bar{\theta}/(1-q))^{-\gamma}$ both of which are finite numbers, implying that the whole term multiplied by $(1-q)$ goes to zero as m_t^{γ} goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\gamma}(1-\bar{\kappa}_t)^{-\gamma}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting

$\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\gamma} = \beta q R^{1-\gamma} \bar{\kappa}_{t+1}^{-\gamma}(1-\bar{\kappa}_t)^{-\gamma}$. Exponentiating by γ, we can conclude that

$$\bar{\kappa}_t = q^{-1/\gamma}(\beta R)^{-1/\gamma} R(1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

$$\underbrace{q^{1/\gamma} \overbrace{R^{-1}(\beta R)^{1/\gamma}}^{\frac{\text{Þ}}{R}}}_{\equiv q^{1/\gamma}\frac{\text{Þ}}{R}} \bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned} (q^{1/\gamma}\frac{\text{Þ}}{R}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= q^{1/\gamma}\frac{\text{Þ}}{R}\bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1} &= 1 + q^{1/\gamma}\frac{\text{Þ}}{R}\bar{\kappa}_{t+1}^{-1}. \end{aligned}$$

As noted in the main text, we need the WRIC (**??**) for this to be a convergent sequence:

$$0 \le q^{1/\gamma}\frac{\text{Þ}}{R} < 1, \tag{79}$$

Since $\bar{\kappa}_T = 1$, iterating (79) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1 - q^{1/\gamma}\frac{\text{Þ}}{R} \tag{80}$$

and we will therefore call $\bar{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving ξ_{t+1} in (79) can be neglected, leading to a revised limiting Euler equation

$$(m_t \mathrm{e}_t(m_t))^{-\gamma} \;\; = \beta R\, \mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\tilde{R}_{t+1})\,(R\mathrm{a}_t(m_t))\right)^{-\gamma}\right]$$

and using L'Hôpital's rule $\lim_{m_t\to\infty} \mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t\to\infty} \mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\tilde{R}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$\begin{aligned} (m_t\underline{\kappa}_t)^{-\gamma} &= \beta R\big(\underline{\kappa}_{t+1}R(1-\underline{\kappa}_t)m_t\big)^{-\gamma} \\ \underbrace{R^{-1}\text{Þ}}_{\equiv\frac{\text{Þ}}{R}=(1-\underline{\kappa})} \underline{\kappa}_t &= (1-\underline{\kappa}_t)\underline{\kappa}_{t+1} \end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \le \frac{\text{Þ}}{R} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1}\frac{\text{Þ}}{R} \tag{81}$$

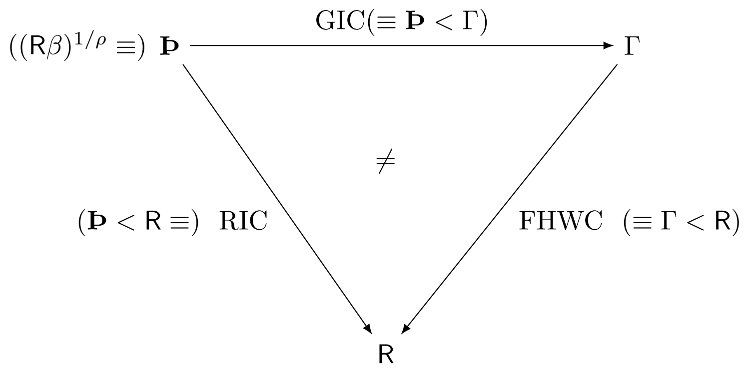

Figure 2 Appendix: Inequality Conditions for Perfect Foresight Model
(Start at a node and follow arrows)

so that $(\{\underline{\kappa}_{T-n}^{-1}\})_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n \uparrow \infty} \kappa_{T-n}^{-1} \tag{82}$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n\rightarrow\infty} \underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$c_t \underbrace{\left(1 + \frac{\mathbf{Þ}}{R} + \frac{\mathbf{Þ}}{R}^2 + \cdots \right)}_{=1+\frac{\mathbf{Þ}}{R}(1+\frac{\mathbf{Þ}}{R}\underline{\kappa}_{t+2}^{-1})\ldots} = c_t \underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t)\underline{\kappa}_t \tag{83}$$

G Relational Diagrams for the Inequality Conditions

This appendix explains in detail the paper's 'inequalities' diagrams (Figures 6, 7).

G.1 The Unconstrained Perfect Foresight Model

A simple illustration is presented in Figure 2, whose three nodes represent values of the absolute patience factor $\mathbf{Þ}$, the permanent-income growth factor G, and the riskfree interest factor R. The arrows represent imposition of the labeled inequality condition (like, the uppermost arrow, pointing from $\mathbf{Þ}$ to G, reflects imposition of the GIC condition (clicking GIC should take you to its definition; definitions of other conditions are also linked below)).[12] Annotations inside parenthetical expressions containing $\equiv$ are there to make the diagram readable for someone who may not immediately remember terms and definitions from the main text. (Such a reader might also want to be reminded that R, β, and Γ are all in $\mathbb{R}_{++}$, and that $\gamma > 1$).

Navigation of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes. Traversal must stop upon arrival at a node with no exiting arrows. So, for example, we can start at the **Þ** node and impose the GIC and then the FHWC, and see that imposition of these conditions allows us to conclude that **Þ** $< R$.

One could also impose **Þ** $< R$ directly (without imposing GIC and FHWC) by following the downward-sloping diagonal arrow exiting **Þ**. Although alternate routes from one node to another all justify the same core conclusion (**Þ** $< R$, in this case), $\neq$ symbol in the center is meant to convey that these routes are not identical in other respects. This notational convention is used in category theory diagrams,[13] to indicate that the diagram is not commutative.[14]

Negation of a condition is indicated by the reversal of the corresponding arrow. For example, negation of the RIC, ~~RIC~~ $\equiv$ **Þ** $> R$, would be represented by moving the arrowhead from the bottom right to the top left of the line segment connecting **Þ** and R.

If we were to start at R and then impose ~~FHWC~~, that would reverse the arrow connecting R and G, but the G node would then have no exiting arrows so no further deductions could be made. However, if we *also* reversed GIC (that is, if we imposed ~~GIC~~), that would take us to the **Þ** node, and we could deduce $R >$ **Þ**. However, we would have to stop traversing the diagram at this point, because the arrow exiting from the **Þ** node points back to our starting point, which (if valid) would lead us to the conclusion that $R > R$. Thus, the reversal of the two earlier conditions (imposition of ~~FHWC~~ and ~~GIC~~) requires us also to reverse the final condition, giving us ~~RIC~~.[15]

Under these conventions, Figure 6 in the main text presents a modified version of the diagram extended to incorporate the PF-FVAC (reproduced here for convenient reference).

This diagram can be interpreted, for example, as saying that, starting at the **Þ** node, it is possible to derive the PF-FVAC[16] by imposing both the GIC and the FHWC; or by imposing RIC and ~~FHWC~~. Or, starting at the G node, we can follow the imposition

---

[12]For convenience, the equivalent ($\equiv$) mathematical statement of each condition is expressed nearby in parentheses.

[13]For a popular introduction to category theory, see Riehl (2017).

[14]But the rest of our notation does not necessarily abide by the other conventions of category theory diagrams.

[15]The corresponding algebra is

$$\begin{aligned} \cancel{\text{FHWC}}: &\quad R < G \\ \cancel{\text{GIC}}: &\quad G < \textbf{Þ} \\ \Rightarrow \cancel{\text{RIC}}: &\quad R < \textbf{Þ}, \end{aligned}$$

.

[16]in the form $\textbf{Þ} < (R/G)^{1/\gamma} G$

$$
\begin{tikzcd}
\textbf{Þ} \arrow[rr, "\text{GIC}"] \arrow[dd, "\text{RIC}"'] \arrow[ddrr, "\text{PF-FVAC}"] & & \Gamma \arrow[dd, "\text{FHWC}"] \\
& & \arrow[l, phantom, "\neq"] \\
\mathsf{R} \arrow[rr, "\cancel{\text{FHWC}}"'] \arrow[ur, phantom, "\neq"] & & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}
$$

Figure 3 Appendix: Relation of GIC, FHWC, RIC, and PFVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\textbf{Þ} < R^{1/\gamma}G^{1-1/\gamma}$, which is an alternative way of writing the PF-FVAC, (9)

of the FHWC (twice — reversing the arrow labeled ~~FHWC~~) and then ~~RIC~~ to reach the conclusion that $\textbf{Þ} < G$. Algebraically,

$$
\begin{aligned}
\text{FHWC}: &\quad G < R \\
\cancel{\text{RIC}}: &\quad R < \textbf{Þ} \\
&\quad G < \textbf{Þ}
\end{aligned}
\tag{84}
$$

which leads to the negation of both of the conditions leading into $\textbf{Þ}$. ~~GIC~~ is obtained directly as the last line in (84) and ~~PF-FVAC~~ follows if we start by multiplying the Return Patience Factor (RPF=$\textbf{Þ}/R$) by the FHWF (=G/R) raised to the power $1/\gamma - 1$, which is negative since we imposed $\gamma > 1$. FHWC implies FHWF < 1 so when FHWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because ~~RIC~~) by another number greater than one yields a product that must be greater than one:

$$
\begin{aligned}
1 &< \overbrace{\left(\frac{(R\beta)^{1/\gamma}}{R}\right)}^{>1 \text{ from } \cancel{\text{RIC}}} \overbrace{(G/R)^{1/\gamma-1}}^{>1 \text{ from FHWC}} \\
1 &< \left(\frac{(R\beta)^{1/\gamma}}{(R/G)^{1/\gamma} RG/R}\right) \\
R^{1/\gamma}G^{1-1/\gamma} = (R/G)^{1/\gamma}G &< \textbf{Þ}
\end{aligned}
$$

which is one way of writing ~~PF-FVAC~~.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure 7 in the paper (reproduced here).

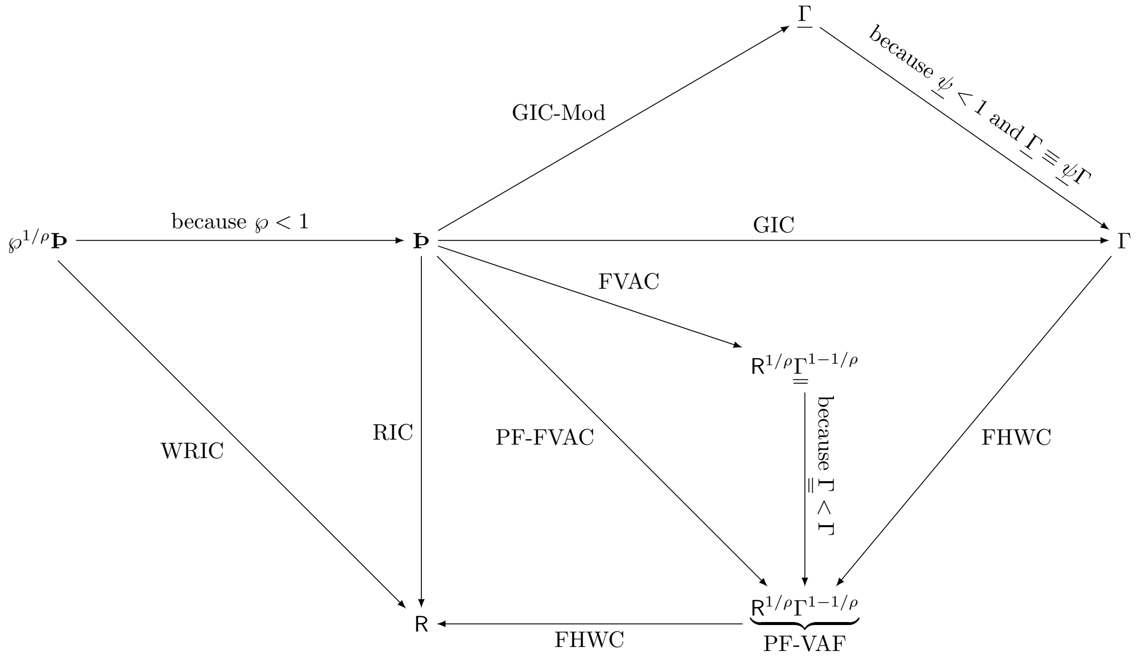

Figure 4 Appendix: Relation of All Inequality Conditions

Finally, the next diagram substitutes the values of the various objects in the diagram under the baseline parameter values and verifies that all of the asserted inequality conditions hold true.

H Apparent Balanced Growth in $\mathfrak{c}$ and $\text{cov}(c, \boldsymbol{p})$

Section 4.2 demonstrates some propositions under the assumption that, when an economy satisfies the GIC, there will be constant growth factors $\Omega_{\mathfrak{c}}$ and Ω_{cov} respectively for $\mathfrak{c}$ (the average value of the consumption ratio) and $\text{cov}(c, \boldsymbol{p})$. In the case of a Szeidl-invariant economy, the main text shows that these are $\Omega_{\mathfrak{c}} = 1$ and $\Omega_{\text{cov}} = G$. If the economy is Harmenberg- but not Szeidl-invariant, no proof is offered that these growth factors will be constant.

H.1 $\log c$ and $\log(\text{cov}(c, \boldsymbol{p}))$ Grow Linearly

Figures 6 and 7 plot the results of simulations of an economy that satisfies Harmenberg- but not Szeidl-invariance with a population of 4 million agents over

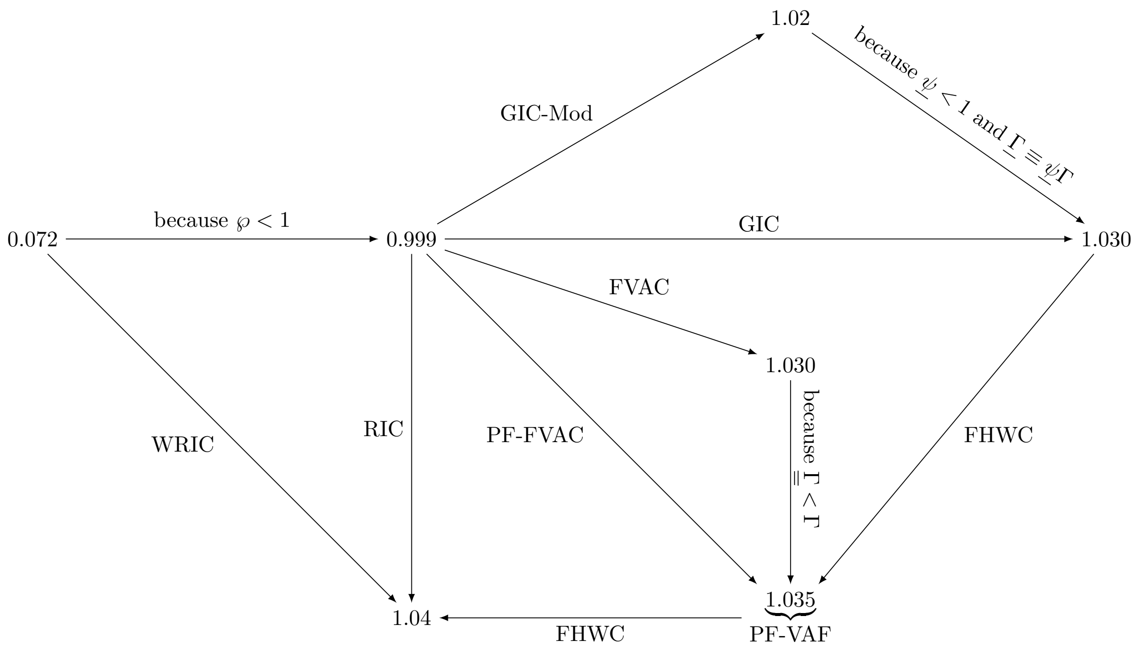

Figure 5 Appendix: Numerical Relation of All Inequality Conditions

the last 1000 periods (of a 2000 period simulation).[17] The first figure shows that $\log \mathfrak{c}$ increases apparently linearly. The second figure shows that $\log(-\mathrm{cov}(c, \boldsymbol{p}))$ also increases apparently linearly. (These results are produced by the notebook `ApndxBalancedGrowthcNrmAndCov.ipynb`).

[17]For an exposition of our implementation of Harmenberg's method, see this supplemental appendix.

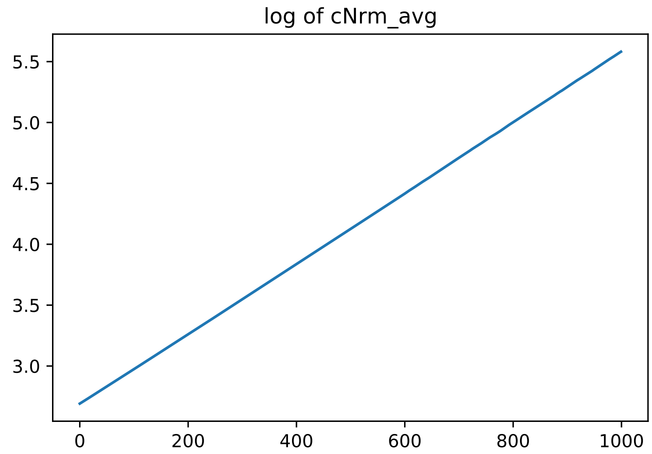

Figure 6 Appendix: log $\mathfrak{c}$ Appears to Grow Linearly

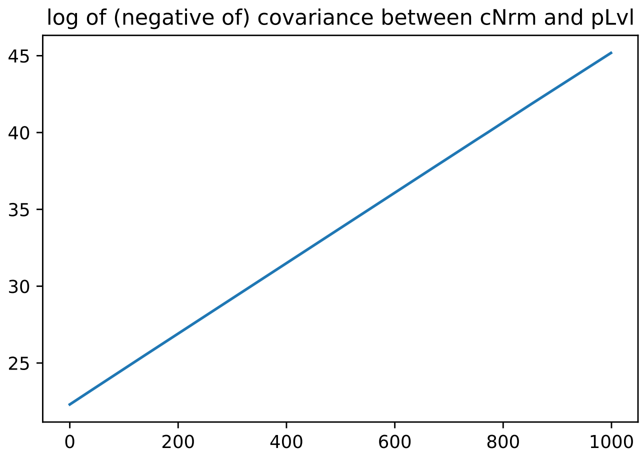

Figure 7 Appendix: log $(-\text{cov}(c, \boldsymbol{p}))$ Appears to Grow Linearly

Supplemental Appendices

I Equality of $\boldsymbol{c}$ and $\boldsymbol{p}$ Growth with Transitory Shocks

Section 4.1 asserted that in the absence of permanent shocks it is possible to prove that the growth factor for aggregate consumption approaches that for aggregate permanent income. This section establishes that result.

First define $\mathrm{a}(m)$ as the function that yields optimal end-of-period assets as a function of m.

Suppose the population starts in period t with an arbitrary value for $\text{cov}_t(a_{t+1,i}, \boldsymbol{p}_{t+1,i})$. Then if $\hat{m}$ is the invariant mean level of m we can define an 'average marginal propensity to save away from $\hat{m}$' function:

$$\bar{\mathrm{a}}'(\Delta) = \Delta^{-1} \int_{\hat{m}}^{\hat{m}+\Delta} \mathrm{a}'(z)dz$$

where the combination of the bar and the ´ are meant to signify that this is the average value of the derivative over the interval. Since $\psi_{t+1,i} = 1$, $\tilde{R}_{t+1,i}$ is a constant at $\tilde{R}$, so if we define $\hat{a}$ as the value of a corresponding to $m = \hat{m}$, we can write

$$a_{t+1,i} = \hat{a} + (m_{t+1,i} - \hat{m})\bar{\mathrm{a}}'(\overbrace{\tilde{R}a_{t,i} + \xi_{t+1,i}}^{m_{t+1,i}} - \hat{m})$$

so

$$\text{cov}_t(a_{t+1,i}, \boldsymbol{p}_{t+1,i}) = \text{cov}_t\left(\bar{\mathrm{a}}'(\tilde{R}a_{t,i} + \xi_{t+1,i} - \hat{m}), G\boldsymbol{p}_{t,i}\right).$$

But since $R^{-1}(qR\beta)^{1/\gamma} < \bar{\mathrm{a}}'(m) < \frac{\text{Þ}}{R}$,

$$|\text{cov}_t((qR\beta)^{1/\gamma}a_{t+1,i}, \boldsymbol{p}_{t+1,i})| < |\text{cov}_t(a_{t+1,i}, \boldsymbol{p}_{t+1,i})| < |\text{cov}_t(\text{Þ}a_{t+1,i}, \boldsymbol{p}_{t+1,i})|$$

and for the version of the model with no permanent shocks the GIC-Mod says that $\text{Þ} < G$, while the FHWC says that $G < R$; combining these facts we get:

$$|\text{cov}_t(a_{t+1,i}, \boldsymbol{p}_{t+1,i})| < G|\text{cov}_t(a_{t,i}, \boldsymbol{p}_{t,i})|.$$

This means that from any arbitrary starting value, the relative size of the covariance term shrinks to zero over time (compared to the $\check{\mathbf{A}}G^n$ term which is growing steadily by the factor G). Thus, $\lim_{n\to\infty} \mathbf{A}_{t+n+1}/\mathbf{A}_{t+n} = G$.

This logic unfortunately does not go through when there are permanent shocks, because the $\tilde{R}_{t+1,i}$ terms are not independent of the permanent income shocks.

To see the problem clearly, define $\check{\tilde{R}} = \mathbb{M}\left[\tilde{R}_{t+1,i}\right]$ and consider a first order Taylor expansion of $\bar{\mathrm{a}}'(m_{t+1,i})$ around $\hat{m}_{t+1,i} = \check{\tilde{R}}a_{t,i} + 1$,

$$\bar{\mathrm{a}}'_{t+1,i} \approx \bar{\mathrm{a}}'(\hat{m}_{t+1,i}) + \bar{\mathrm{a}}''(\hat{m}_{t+1,i})\left(m_{t+1,i} - \hat{m}_{t+1,i}\right).$$

The problem comes from the $\bar{\mathrm{a}}''$ term (which we implicitly define as the derivative of $\bar{\mathrm{a}}'$). The concavity of the consumption function implies convexity of the a function, so this term is strictly positive but we have no theory to place bounds on its size as we do for its level $\bar{\mathrm{a}}'$. We cannot rule out by theory that a positive shock to permanent income (which has a negative effect on $m_{t+1,i}$) could have a (locally) unboundedly positive effect on $\bar{\mathrm{a}}''$ (as for instance if it pushes the consumer arbitrarily close to the self-imposed liquidity constraint).

J Endogenous Gridpoints Solution Method

The model is solved using an extension of the method of endogenous gridpoints (?): A grid of possible values of end-of-period assets $\vec{a}$ is defined, and at these points, marginal end-of-period-t value is computed as the discounted next-period expected marginal utility of consumption (which the Envelope theorem says matches expected marginal value). The results are then used to identify the corresponding levels of consumption at the beginning of the period:[18]

$$
\begin{aligned}
\mathrm{u}'(\mathfrak{c}_t(\vec{a})) &= R\beta\,\mathbb{E}_t[\mathrm{u}'(G_{t+1}\mathrm{c}_{t+1}(\tilde{R}_{t+1}\vec{a}+\xi_{t+1}))] \\
\vec{c}_t \equiv \mathfrak{c}_t(\vec{a}) &= \left(R\beta\,\mathbb{E}_t\left[\left(G_{t+1}\mathrm{c}_{t+1}(\tilde{R}_{t+1}\vec{a}+\xi_{t+1})\right)^{-\gamma}\right]\right)^{-1/\gamma}.
\end{aligned}
$$

[18]The software can also solve a version of the model with explicit liquidity constraints, where the Envelope condition does not hold.

Table 1 Appendix: Perfect Foresight Liquidity Constrained Taxonomy

For constrained $\grave{c}$ and unconstrained $\bar{c}$ consumption functions

Main Condition Subcondition	Math			Outcome, Comments or Results
~~GIC~~		$1 <$	$\text{Þ}/G$	Constraint never binds for $m \geq 1$
and RIC	$\text{Þ}/R$	< 1		FHWC holds ($R > G$); $\grave{\mathrm{c}}(m) = \bar{\mathrm{c}}(m)$ for $m \geq 1$
and ~~RIC~~		$1 <$	$\text{Þ}/R$	$\grave{\mathrm{c}}(m)$ is degenerate: $\grave{\mathrm{c}}(m) = 0$
GIC	$\text{Þ}/G$	< 1		Constraint binds in finite time $\forall\ m$
and RIC	$\text{Þ}/R$	< 1		FHWC may or may not hold $\lim_{m\uparrow\infty} \bar{\mathrm{c}}(m) - \grave{\mathrm{c}}(m) = 0$ $\lim_{m\uparrow\infty} \grave{\kappa}(m) = \underline{\kappa}$
and ~~RIC~~		$1 <$	$\text{Þ}/R$	~~FHWC~~ $\lim_{m\uparrow\infty} \grave{\kappa}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~GIC~~ and RIC both hold, while the third row indicates that when the GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the GICholds, the constraint will bind in finite time.