# 1 When Is Consumption Growth Declining in $m$?

Figure 1 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\boldsymbol{\Upsilon}(m_t) \equiv \mathcal{G}_{t+1}\mathrm{c}(\tilde{\mathcal{R}}_{t+1}\mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1})/\mathrm{c}(m_t) = \boldsymbol{c}_{t+1}/\boldsymbol{c}_t$$

and the proposition in which we are interested is

$$(d/dm_t)\mathbb{E}_t[\underbrace{\boldsymbol{\Upsilon}(m_t)}_{\equiv \boldsymbol{\Upsilon}_{t+1}}] < 0$$

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t\left[\mathcal{G}_{t+1}\left(\frac{\mathrm{c}'(m_{t+1})\tilde{\mathcal{R}}_{t+1}\mathrm{a}'(m_t)\mathrm{c}(m_t) - \mathrm{c}(m_{t+1})\mathrm{c}'(m_t)}{\mathrm{c}(m_t)^2}\right)\right] < 0. \tag{1}$$

{eq:kappaPrimeLT0

Henceforth indicating appropriate arguments by the corresponding subscript (e.g. $\mathrm{c}'_{t+1} \equiv \mathrm{c}'(m_{t+1})$), since $\mathcal{G}_{t+1}\tilde{\mathcal{R}}_{t+1} = \mathsf{R}$, the portion of the LHS of equation (1) in brackets can be manipulated to yield

$$\begin{aligned}
\mathrm{c}_t\boldsymbol{\Upsilon}'_{t+1} &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\mathcal{G}_{t+1}\mathrm{c}_{t+1}/\mathrm{c}_t \\
&= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}.
\end{aligned}$$

Now differentiate the Euler equation with respect to $m_t$:

$$\begin{aligned}
1 &= \mathsf{R}\beta\mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\gamma}] \\
0 &= \mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\gamma-1}\boldsymbol{\Upsilon}'_{t+1}] \\
&= \mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\gamma-1}]\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] + \mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\gamma-1}, \boldsymbol{\Upsilon}'_{t+1}) \\
\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] &= -\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\gamma-1}, \boldsymbol{\Upsilon}'_{t+1})/\mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\gamma-1}]
\end{aligned} \tag{2}$$

{eq:covgen}

but since $\boldsymbol{\Upsilon}_{t+1} > 0$ we can see from (2) that (1) is equivalent to

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\gamma-1}, \boldsymbol{\Upsilon}'_{t+1}) > 0$$

which, using (2), will be true if

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\gamma-1}, \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}) > 0$$

which in turn will be true if both

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\gamma-1}, \mathrm{c}'_{t+1}) > 0$$

and

$$\text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\gamma-1}, \mathbf{\Upsilon}_{t+1}) < 0.$$

The latter proposition is obviously true under our assumption $\gamma > 1$. The former will be true if

$$\text{cov}_t\left((\mathcal{G}\psi_{t+1}\text{c}(m_{t+1}))^{-\gamma-1}, \text{c}'(m_{t+1})\right) > 0.$$

The two shocks cause two kinds of variation in $m_{t+1}$. Variations due to $\boldsymbol{\xi}_{t+1}$ satisfy the proposition, since a higher draw of $\boldsymbol{\xi}$ both reduces $c_{t+1}^{-\gamma-1}$ and reduces the marginal propensity to consume. However, permanent shocks have conflicting effects. On the one hand, a higher draw of $\psi_{t+1}$ will reduce $m_{t+1}$, thus increasing both $c_{t+1}^{-\gamma-1}$ and $c'_{t+1}$. On the other hand, the $c_{t+1}^{-\gamma-1}$ term is multiplied by $\mathcal{G}\psi_{t+1}$, so the effect of a higher $\psi_{t+1}$ could be to decrease the first term in the covariance, leading to a negative covariance with the second term. (Analogously, a lower permanent shock $\psi_{t+1}$ can also lead a negative correlation.)