1 Appendix for Section 3

{sec:ApndxMTarget

1.1 Asymptotic Consumption Growth Factors

{subsec:AppxCgrow

Proof for Proposition 4. For consumption growth, as $m \to 0$ we have:

$$\begin{aligned}
\lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) \tilde{\mathcal{G}}_{t+1} \right] &> \lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\tilde{\mathcal{R}}_{t+1}\mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1})}{\bar{\kappa} m_t} \right) \tilde{\mathcal{G}}_{t+1} \right] \\
&= \wp \lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\tilde{\mathcal{R}}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \mathcal{G}_{t+1} \right] \qquad (1) \\
&\quad + (1-\wp) \lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\tilde{\mathcal{R}}_{t+1}\mathrm{a}(m_t) + \theta_{t+1}/(1-\wp))}{\bar{\kappa} m_t} \right) \tilde{\mathcal{G}}_{t+1} \right] \qquad (2) \\
&> (1-\wp) \lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/(1-\wp))}{\bar{\kappa} m_t} \right) \tilde{\mathcal{G}}_{t+1} \right] \\
&= \infty
\end{aligned}$$

{eq:consGrowth}

where the second-to-last line follows because $\lim_{m_t \to 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\tilde{\mathcal{R}}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \tilde{\mathcal{G}}_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.

Next we establish the limit of the expected consumption growth factor as $m_t \to \infty$:

$$\lim_{m_t \to \infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] = \lim_{m_t \to \infty} \mathbb{E}_t[\tilde{\mathcal{G}}_{t+1} c_{t+1}/c_t].$$

But

$$\mathbb{E}_t[\tilde{\mathcal{G}}_{t+1} \underline{c}_{t+1}/\bar{c}_t] \leq \mathbb{E}_t[\tilde{\mathcal{G}}_{t+1} c_{t+1}/c_t] \leq \mathbb{E}_t[\tilde{\mathcal{G}}_{t+1} \bar{c}_{t+1}/\underline{c}_t]$$

and

$$\lim_{m_t \to \infty} \tilde{\mathcal{G}}_{t+1} \underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t \to \infty} \tilde{\mathcal{G}}_{t+1} \bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t \to \infty} \tilde{\mathcal{G}}_{t+1} m_{t+1}/m_t,$$

while (for convenience defining $\mathrm{a}(m_t) = m_t - \mathrm{c}(m_t)$),

$$\begin{aligned}
\lim_{m_t \to \infty} \tilde{\mathcal{G}}_{t+1} m_{t+1}/m_t &= \lim_{m_t \to \infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \tilde{\mathcal{G}}_{t+1}\boldsymbol{\xi}_{t+1}}{m_t} \right) \qquad (3) \\
&= (\mathsf{R}\beta)^{1/\gamma} = \text{Þ}
\end{aligned}$$

{eq:xtp1toinfty}

because $\lim_{m_t \to \infty} \mathrm{a}'(m) = \text{Þ}/\mathsf{R}$[1] and $\tilde{\mathcal{G}}_{t+1}\boldsymbol{\xi}_{t+1}/m_t \leq (\mathcal{G}\bar{\psi}\bar{\theta}/(1-\wp))/m_t$ which goes to zero

[1] $\lim_{m_t \to \infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t \to \infty} \mathrm{c}(m_t)/m_t = 1 - \lim_{m_t \to \infty} \mathrm{c}'(m_t) = \text{Þ}/\mathsf{R}$.

as m_t goes to infinity. Hence we have:

$$\text{Þ} \leq \lim_{m_t \to \infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] \leq \text{Þ}$$

so as cash goes to infinity, consumption growth approaches its value **Þ** in the perfect foresight model. □

This appendix proves Theorems 3-4 and:

{lemma:orderingP

Lemma 1. *If $\check{m}$ and $\hat{m}$ both exist, then $\check{m} \leq \hat{m}$.*

1.2 Existence of Buffer Stock Target

1.2.1 Existence of Individual Buffer Stock Target

{subsubsec:AppxInc

Proof of Theorem 3. First, observe that $\mathbb{E}_t[m_{t+1}/m_t] = \frac{\mathbb{E}_t\big((m_t - \mathrm{c}(m_t))\tilde{\mathcal{R}}_{t+1} + \boldsymbol{\xi}_{t+1}\big)}{m_t}$. Note that c is continuous since c is concave on $\mathbb{R}_{++}$ by Lemma 2. Thus for any convergent sequence $\left\{m_t^j\right\}_{j=0}^{\infty}$, with $m_t^j \in \mathbb{R}_{++}$, $(m_t^j - \mathrm{c}(m_t^j))\tilde{\mathcal{R}}_{t+1} + \boldsymbol{\xi}_{t+1}$ will be bounded above and below. It follows that, using the Dominated Convergence Theorem, $\mathbb{E}_t[m_{t+1}/m_t]$ will be continuous in m_t.

The remainder of the proof proceeds as follows. To establish Equation (25), we will show (i) that there exists a point $\check{m}_t$ where $\mathbb{E}_t[\check{m}^{\star}_{t+1}/\check{m}^{\star}_t] < 1$ and (ii) a point $\grave{m}$ where $\mathbb{E}_t[\grave{m}_{t+1}/\grave{m}_t] > 1$. By continuity of $\mathbb{E}[m_{t+1}/m_t]$ in m_t and the Intermediate Value Theorem, there will exist $\hat{m}$ such that $\mathbb{E}_t[\hat{m}_{t+1}/\hat{m}_t] = 1$. In turn, to establish that $\hat{m}$ is a point of stability, Equation (26), we will show that (iii) $\mathbb{E}_t[m_{t+1}] - m_t$ is decreasing.

Part (i). Existence of $\check{m}_t$ where $\mathbb{E}_t[\check{m}_{t+1}/\check{m}_t] < 1$.

To proceed, first suppose return impatience holds and take the steps analogous to those leading to Equation (94) in the proof of proof for Proposition 4, but dropping the $\mathcal{G}_{t+1}$ from the RHS:

$$\begin{aligned}
\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \to \infty} \mathbb{E}_t\left[\frac{\tilde{\mathcal{R}}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1}}{m_t}\right] \\
&= \mathbb{E}_t[(\mathsf{R}/\tilde{\mathcal{G}}_{t+1})\text{Þ}/\mathsf{R}] \\
&= \mathbb{E}_t[\text{Þ}/\tilde{\mathcal{G}}_{t+1}] \qquad (4) \\
&< 1,
\end{aligned}$$

{eq:emgro}

where the inequality follows from strong growth impatience. By continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ in m_t, there exists $\check{m}_t$ large enough such that $\mathbb{E}_t[\check{m}_{t+1}/\check{m}_t] < 1$.

Next, suppose return impatience fails. The fact that $\lim_{m_t \to \infty} \frac{\mathrm{c}(m_t)}{m_t} = 0$ (Lemma 2) means the limit of the RHS of (95) as $m_t \to \infty$ is $\bar{\bar{\mathcal{R}}} = \mathbb{E}_t[\tilde{\mathcal{R}}_{t+1}]$. Equations (99)-(100) below show that when strong growth impatience holds and return impatience fails $\bar{\bar{\mathcal{R}}} < 1$.

Thus, we have $\lim_{m \to \infty} \mathbb{E}[m_{t+1}/m_t] < 1$ whether the return impatience holds or fails.

Part (ii). Existence of $\grave{m}_t > 1$ where $\mathbb{E}_t[\grave{m}_{t+1}/\grave{m}_t] > 1$.

Analogous to Equation (92), the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \to 0$ because $\lim_{m_t \to 0} \mathbb{E}[m_{t+1}] > 0$. Thus, if $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous in m_t, and takes on values above and below one at $\grave{m}_t$ and $\breve{m}_t$, by the Intermediate Value Theorem, there must be at least one point at which it is equal to one.

Part (iii). $\mathbb{E}_t[m_{t+1}] - m_t$ *is strictly decreasing.*

Finally to show $\mathbb{E}_t[m_{t+1}] - m_t$ is strictly decreasing m_t, define $\boldsymbol{\zeta}(m_t) \colon = \mathbb{E}_t[m_{t+1}] - m_t$ and note that:

$$\begin{aligned} \boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,, \end{aligned} \tag{5}$$

{eq:difNrmioEquiv}

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$. Let Δ_ϵ be the finite forward difference for spacing $\epsilon > 0$. Fixing $\epsilon > 0$, we will have:

$$\begin{aligned} \Delta_\epsilon \boldsymbol{\zeta}(m_t) &= \mathbb{E}_t \left[\Delta_\epsilon \left(\tilde{\mathcal{R}}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1} - m_t \right) \right] \\ &= \bar{\bar{\mathcal{R}}} \left(\epsilon - \Delta_\epsilon \mathrm{c}(m_t) \right) - \epsilon = \epsilon \left(\bar{\bar{\mathcal{R}}} \left[1 - \frac{\Delta_\epsilon \mathrm{c}(m_t)}{\epsilon} \right] - 1 \right). \end{aligned} \tag{6}$$

{eq:finiteDiff2}

Notice that $\frac{\Delta_\epsilon \mathrm{c}(m_t)}{\epsilon} \leq \frac{\mathrm{c}(m_t)}{m_t} < 1$ since $\frac{\mathrm{c}(m_t)}{m_t}$ is decreasing in m_t by Claim 7 in Appendix F. Consider the case when return impatience holds. Equation (15) and Lemma 2 indicate $0 < \underline{\kappa} \leq \frac{\mathrm{c}(m_t)}{m_t} < 1$. It follows that:

$$\begin{aligned} \bar{\bar{\mathcal{R}}} \left[1 - \frac{\Delta_\epsilon \mathrm{c}'(m_t)}{\epsilon} \right] - 1 &\leq \bar{\bar{\mathcal{R}}}(1 - \underbrace{(1 - \text{\textbf{Þ}}/\mathsf{R})}_{\underline{\kappa}}) - 1 \\ &= \bar{\bar{\mathcal{R}}}\text{\textbf{Þ}}/\mathsf{R} - 1 \\ &= \mathbb{E}_t \left[\frac{\mathsf{R}}{\mathcal{G}\psi_{t+1}} \frac{\text{\textbf{Þ}}}{\mathsf{R}} \right] - 1 \\ &= \underbrace{\mathbb{E}_t \left[\frac{\text{\textbf{Þ}}}{\mathcal{G}\psi_{t+1}} \right]}_{= \text{\textbf{Þ}}/\mathcal{G}\mathbb{E}[\psi^{-1}]} - 1 \end{aligned}$$

which is negative because the strong growth impatience says $\text{\textbf{Þ}}/\mathcal{G}\mathbb{E}[\psi^{-1}] < 1$. Conversely, when return impatience holds fails, recall $\lim_{m_t \to \infty} \frac{\mathrm{c}(m_t)}{m_t} = 0$. This means $\Delta_\epsilon \boldsymbol{\zeta}(m_t)$ from (97) is guaranteed to be negative if:

$$\bar{\bar{\mathcal{R}}} = \mathbb{E}_t \left[\frac{\mathsf{R}}{\mathcal{G}\psi_{t+1}} \right] < 1. \tag{7}$$

{eq:RbarBelowOne}

But the combination of the strong growth impatience holding and the return impatience failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\mathcal{G}\psi_{t+1}}\right]}^{\text{Þ}/\mathcal{G}\mathbb{E}[\psi^{-1}]} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}/\mathsf{R}}, \tag{8}$$ {eq:GICStrRICfails

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives:

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\mathcal{G}\psi_{t+1}}\right] < \mathsf{R}/\text{Þ} < 1, \tag{9}$$ {eq:GICStrRICfails

which satisfies our requirement in (98), thus completing the proof.

□

1.2.2 Existence of Pseudo-Steady-State

{subsubsec:AppxPs

Proof of Theorem 4. Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in B.2.1 that demonstrated existence and continuity of $\mathbb{E}[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}[\psi_{t+1}m_{t+1}/m_t]$.

Part (i). Existence of a stable point

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in Subsection B.2.1 that the ratio of $\mathbb{E}[m_{t+1}]$ to m_t is unbounded as $m_t \to 0$ implies that the ratio $\mathbb{E}[\psi_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \to 0$. The limit of the expected ratio as $m_t \to \infty$ goes to infinity is can be found as follows:

$$\begin{aligned}
\lim_{m_t\to\infty}\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t\to\infty}\mathbb{E}_t\left[\frac{\tilde{\mathcal{G}}_{t+1}\left((\mathsf{R}/\tilde{\mathcal{G}}_{t+1})\mathrm{a}(m_t)+\boldsymbol{\xi}_{t+1}\right)/\mathcal{G}}{m_t}\right] \\
&= \lim_{m_t\to\infty}\mathbb{E}_t\left[\frac{(\mathsf{R}/\mathcal{G})\mathrm{a}(m_t)+\psi_{t+1}\boldsymbol{\xi}_{t+1}}{m_t}\right] \\
&= \lim_{m_t\to\infty}\left[\frac{(\mathsf{R}/\mathcal{G})\mathrm{a}(m_t)+1}{m_t}\right] \\
&= (\mathsf{R}/\mathcal{G})\text{Þ}/\mathsf{R} \qquad (10) \\
&= \text{Þ}/\mathcal{G} \\
&< 1,
\end{aligned}$$ {eq:emgro2}

where the last two lines are merely a restatement of growth impatience.

To conclude Part (i) of the proof, the Intermediate Value Theorem says that if $\mathbb{E}[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

Part (ii). $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ is monotonically decreasing.

Define $\boldsymbol{\zeta}(m_t) := \mathbb{E}_t[\psi_{t+1} m_{t+1}] - m_t$ and note that:

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] > 1,
\end{aligned} \tag{11}$$

{eq:difLvlEquiv}

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$. Letting Δ_ϵ be the forward difference operator, we have:

$$\begin{aligned}
\Delta_\epsilon \boldsymbol{\zeta}(m_t) &= \mathbb{E}\left[\Delta_\epsilon \left(\frac{\mathsf{R}}{\mathcal{G}}(m_t - \mathrm{c}(m_t)) + \psi_{t+1}\boldsymbol{\xi}_{t+1} - m_t\right)\right] \\
&= \frac{\mathsf{R}}{\mathcal{G}}\left(\epsilon - \Delta_\epsilon \mathrm{c}'(m_t)\right) - \epsilon = \epsilon\left(\frac{\mathsf{R}}{\mathcal{G}}\left[1 - \frac{\Delta_\epsilon \mathrm{c}(m_t)}{\epsilon}\right] - 1\right).
\end{aligned} \tag{12}$$

{eq:finiteDiff}

for any given $\epsilon > 0$. Notice that $\frac{\Delta_\epsilon \mathrm{c}'(m_t)}{\epsilon} \leq \frac{\mathrm{c}(m_t)}{m_t} < 1$ since $\frac{\mathrm{c}(m_t)}{m_t}$ is decreasing in m_t by Claim 7 in Appendix. Now, we show that $\boldsymbol{\zeta}(m)$ is decreasing when return impatience holds and when return impatience fails. When return impatience holds, Equation (15) and Lemma 2 indicate that $\underline{\kappa} > 0$ and $0 < \underline{\kappa} \leq \frac{\mathrm{c}(m_t)}{m_t} < 1$. It follows that:

$$\begin{aligned}
\frac{\mathsf{R}}{\mathcal{G}}\left(1 - \mathrm{c}'(m_t)\right) - 1 &< \frac{\mathsf{R}}{\mathcal{G}}(1 - \underbrace{(1 - \text{Þ}/\mathsf{R})}_{\underline{\kappa}}) - 1 \\
&= (\mathsf{R}/\mathcal{G})\text{Þ}/\mathsf{R} - 1,
\end{aligned}$$

which is negative because growth impatience says $\text{Þ}/\mathcal{G} < 1$. Conversely, when return impatience holds fails, recall $\lim_{m_t \to \infty} \frac{\mathrm{c}(m_t)}{m_t} = 0$. In turn, this means $\Delta_\epsilon \boldsymbol{\zeta}(m_t)$ from (103) is guaranteed to be negative if:

$$(\mathsf{R}/\mathcal{G}) < 1. \tag{13}$$

{eq:FHWCFails}

But we showed in Section 2.3.1, Equation (44), that the only circumstances under which the problem has a non-degenerate solution while return impatience fails were ones where the finite limiting human wealth also fails. Thus, $(\mathsf{R}/\mathcal{G}) < 1$, completing the proof.

□