# 1 Supporting Standard Results in Real Analysis

**Proposition 1.** *Let* $\mathrm{f} : \mathbb{R}_{++} \to \mathbb{R}_+$ *be a continuous function. Consider sequences* $x^n$ *in* $\mathbb{R}_{++}$ *and* $\mathrm{f}^n(x^n)$ *in* $\mathbb{R}_+$. *If* $\mathrm{f}^n(x^n) \to \mathrm{f}(x)$ *as* $n \to \infty$, *then* $x^n \to x$ *as* $n \to \infty$.

*Proof.* Given that f is continuous at $x$ (with $x \in \mathbb{R}_{++}$), for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $y$ in $\mathbb{R}_{++}$ with $|y - x| < \delta$, we have $|\mathrm{f}(y) - \mathrm{f}(x)| < \epsilon$.

Given $\mathrm{f}^n(x^n) \to \mathrm{f}(x)$, for the above $\epsilon$, there exists an $N$ such that for all $n > N$, $|\mathrm{f}^n(x^n) - \mathrm{f}(x)| < \epsilon$.

Assume for the sake of contradiction that $x^n$ doesn't converge to $x$. This implies that there exists a $\delta > 0$ such that for infinitely many terms of the sequence $x^n$, $|x^n - x| \geq \delta$.

By the continuity of f at $x$, if $|x^n - x| \geq \delta$ for infinitely many $n$, then $|\mathrm{f}^n(x^n) - \mathrm{f}(x)| \geq \epsilon$ for those $n$, contradicting our assumption that $\mathrm{f}^n(x^n) \to \mathrm{f}(x)$.

Therefore, our assumption for contradiction is false, and it follows that $x^n \to x$ as $n \to \infty$. $\square$

**Fact 1.** *Let* $\mathrm{g} : X \to \mathbb{R}_+$ *be a continuous function, where* $X \subseteq \mathbb{R}^n$ *is an open convex set. Define the weighted supremum norm* $\|\cdot\|_{\mathrm{g}}$ *of a real-valued function* $\mathrm{f} : X \to \mathbb{R}$ *by*

$$\|\mathrm{f}\|_{\mathrm{g}} := \sup_{x \in X} \frac{|\mathrm{f}(x)|}{\mathrm{g}(x)}. \tag{1}$$

*If* $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_{\mathrm{g}} = 0$, $\mathrm{f}_n$ *converges to* $\mathrm{f}^\star$ *uniformly on compact sets.*

*Proof.* Let $\tilde{X}$ be an arbitrary compact subset of $X$. Since $\tilde{X}$ is compact, there exists a positive lower bound for g on this subset, denoted as

$$\bar{\mathrm{g}} = \min_{x \in \tilde{X}} \mathrm{g}(x) > 0. \tag{2}$$

Hence, on $\tilde{X}$, if $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_{\mathrm{g}} = 0$, then $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_\infty = 0$ on $\tilde{X}$, where $\|\cdot\|_\infty$ denotes the supremum norm.

Now, let $K$ be a compact subset of $X$. Given the continuity of g, there exists a positive maximum value for g on $K$, denoted as $M_K$. Then, we have

$$\sup_{x \in K} |\mathrm{f}_n(x) - \mathrm{f}(x)| \leq M_K \sup_{x \in K} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)} \leq M_K \sup_{x \in X} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)}. \tag{3}$$

Thus, $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}\|_{=}0$ implies that $\mathrm{f}_n$ converges uniformly to f on the compact set $K$. It's also worth noting that the convexity and openness of $X$ aren't strictly necessary for this argument. $\square$

**Fact 2.** *Let* $\{\mathrm{f}_n\}$ *be a sequence of continuous functions defined on a subset of the real line and converging uniformly to a function*$\mathrm{f}$ *on compact sets. If*$\{x_n\}$ *is a convergent sequence of real numbers with limit* $x$, *then* $\mathrm{f}_n(x_n)$ *converges to* $\mathrm{f}(x)$.

*Proof.* Let $\tilde{X}$ be an arbitrary compact subset of $X$. Since $\tilde{X}$ is compact, there exists a positive lower bound for g on this subset, denoted as

$$\bar{\mathrm{g}} = \min_{x \in \tilde{X}} \mathrm{g}(x) > 0. \tag{4}$$

Hence, on $\tilde{X}$, if $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_{\mathrm{g}} = 0$, then $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}^\star\|_\infty = 0$ on $\tilde{X}$, where $\|\cdot\|_\infty$ denotes the supremum norm.

Now, let $K$ be a compact subset of $X$. Given the continuity of g, there exists a positive maximum value for g on $K$, denoted as $M_K$. Then, we have

$$\sup_{x\in K} |\mathrm{f}_n(x) - \mathrm{f}(x)| \le M_K \sup_{x\in K} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)} \le M_K \sup_{x\in X} \frac{|\mathrm{f}_n(x) - \mathrm{f}(x)|}{\mathrm{g}(x)}. \tag{5}$$

Thus, $\lim_{n\to\infty} \|\mathrm{f}_n - \mathrm{f}\|_{=}0$ implies that $\mathrm{f}_n$ converges uniformly to f on the compact set $K$. It's also worth noting that the convexity and openness of $X$ aren't strictly necessary for this argument. □

**Fact 3.** *Let $\{\mathrm{f}_n\}$ be a sequence of continuous functions defined on a subset of the real line and converging uniformly to a function* f *on compact sets. If $\{x_n\}$ is a convergent sequence of real numbers with limit $x$, then $\mathrm{f}_n(x_n)$ converges to $\mathrm{f}(x)$.*

*Proof.* Since $x_n$ converges to $x$, the sequence $\{x_n\}$ is bounded. Therefore, there exists a compact set $K$ (specifically, a closed interval in the real line) that contains all the $x_n$ as well as $x$.

Given the uniform convergence of $\mathrm{f}_n$ to f on $K$, for every $\epsilon > 0$, there exists an $N$ such that for all $n \ge N$ and for all $y \in K$, we have

$$|\mathrm{f}_n(y) - \mathrm{f}(y)| < \epsilon.$$

In particular, for $y = x_n$, we have

$$|\mathrm{f}_n(x_n) - \mathrm{f}(x_n)| < \epsilon$$

for all $n \ge N$.

Now, each $\mathrm{f}_n$ being continuous and $x_n$ converging to $x$ implies that $\mathrm{f}(x_n)$ converges to $\mathrm{f}(x)$. Thus, for sufficiently large $n$, $\mathrm{f}(x_n)$ can be made arbitrarily close to $\mathrm{f}(x)$.

Combining the two inequalities and taking $n$ large enough, we deduce that $|\mathrm{f}_n(x_n) - \mathrm{f}(x)|$ can be made smaller than any given $\epsilon$. Hence, $\mathrm{f}_n(x_n)$ converges to $\mathrm{f}(x)$. □

**Claim 1.** *If* f *is convex and* $\mathrm{f} < 0$ *on* $(0, \lambda)$*, then* $\frac{\mathrm{f}(s)}{s}$ *is increasing on* $(0, \lambda)$.

*Proof.* Let f be convex on $(0, \lambda)$ and $\mathrm{f} < 0$ on $(0, \lambda)$. Let $x_1$ and $x_2$ be two points in $(0, \lambda)$. Choose $0 < \alpha < x_1$. Then, any point (in particular, $x_1$) in $(\alpha, x_2)$ can be written as $x_1 = t\alpha + (1-t)x_2$ for some $0 < t < 1$.

Now, define a new function $F$ on $[\alpha, x_2]$ as:

$$F(x) = \mathrm{f}(x) - \mathrm{f}(\alpha).$$

Since f is convex, $F(x)$ is also convex on $[\alpha, x_2]$. To see this, observe that:

$$F(t\alpha+(1-t)x_2) = \mathrm{f}(t\alpha+(1-t)x_2)-\mathrm{f}(\alpha) \leq t\mathrm{f}(\alpha)+(1-t)\mathrm{f}(x_2)-\mathrm{f}(\alpha) = tF(\alpha)+(1-t)F(x_2).$$

Since $F(\alpha) = 0$, the inequality simplifies to $F(x_1) \leq (1-t)F(x_2)$. This implies that $\frac{F(s)}{s}$ is increasing. And thus, if $y_1 \leq y_2$, then:

$$\frac{F(y_1)}{y_1} \leq \frac{F(y_2)}{y_2}.$$

Now, using the definition of $F(x)$, we have:

$$\frac{\mathrm{f}(y_1)}{y_1} = \frac{F(y_1)}{y_1} + \frac{\mathrm{f}(\alpha)}{y_1}.$$

Similarly, for $y_2$:

$$\frac{\mathrm{f}(y_2)}{y_2} = \frac{F(y_2)}{y_2} + \frac{\mathrm{f}(\alpha)}{y_2}.$$

Since $\frac{F(y_1)}{y_1} \leq \frac{F(y_2)}{y_2}$ and $\mathrm{f}(\alpha) < 0$, we conclude that:

$$\frac{\mathrm{f}(y_1)}{y_1} \leq \frac{\mathrm{f}(y_2)}{y_2}.$$

Thus, $\frac{\mathrm{f}(s)}{s}$ is increasing on $(0, \lambda)$, completing the proof. $\square$