1 Theoretical Foundations

This section formalizes our problem and derives formulae, for any period t earlier than the terminal period T, for the maximum and minimum MPCs as wealth approaches zero and infinity (these formulae are derived recursively backward from T). If the environment is that of an infinite-horizon 'income fluctuation problem,' our formulae yield the limiting upper and lower bounds of the nondegenerate stationary solution.

1.1 Setup

We start by stating the problem with permanent income growth in levels and then normalize by permanent income.

Our time index t can take on values in $\{T, T-1, T-2, \dots\}$. We assume that our consumer has a Constant Relative Risk Aversion (CRRA) per-period utility function, $\mathrm{u}(c) = \frac{c^{1-\gamma}}{1-\gamma}$, where $\gamma > 1$. β is the (strictly positive) discount factor. In each period, the consumer faces income shocks, with the permanent shock $\psi_t \in \mathbb{R}_{++}$ and the transitory shock by $\xi_t \in \mathbb{R}_+$.

In each t, value will be a function of 'market resources' $\boldsymbol{m}_t$ and permanent income $\mathbf{p}_t$, with $\boldsymbol{m}_t$ and $\mathbf{p}_t$ strictly positive real numbers ($\{\boldsymbol{m}_t, \mathbf{p}_t\} \in \mathbb{R}_{++} \times \mathbb{R}_{++}$).

Letting $\mathbf{v}_{T+1} = 0$, the finite-horizon value functions are recursively defined by:[1]

$$\mathbf{v}_t(\boldsymbol{m}_t, \mathbf{p}_t) = \max_{0 \le \boldsymbol{c}_t \le \boldsymbol{m}_t} \mathrm{u}(\boldsymbol{c}_t) + \beta\, \mathbb{E}_t\, \mathbf{v}_{t+1}(\boldsymbol{m}_{t+1}, \mathbf{p}_{t+1}) \tag{$\mathcal{P}_L$}$$

where $\boldsymbol{c}_t$ is the level of consumption at time t. We assume the consumer cannot die in debt:

$$\boldsymbol{c}_T \le \boldsymbol{m}_T. \tag{1}$$

For maximal clarity, we separately describe every step in the dynamic budget evolution that determines next period's $\boldsymbol{m}_{t+1}$ from this period's $\boldsymbol{m}_t$ and choice of $\boldsymbol{c}_t$:[2,3]

$$\begin{aligned}
\boldsymbol{a}_t &= \boldsymbol{m}_t - \boldsymbol{c}_t \\
\boldsymbol{k}_{t+1} &= \boldsymbol{a}_t \\
\mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{G\psi_{t+1}}_{:=\tilde{G}_{t+1}} \\
\boldsymbol{m}_{t+1} &= \underbrace{R\boldsymbol{k}_{t+1}}_{:=\boldsymbol{b}_{t+1}} + \underbrace{\mathbf{p}_{t+1}\xi_{t+1}}_{:=\boldsymbol{y}_{t+1}}.
\end{aligned}$$

[1] Notation throughout follows guidelines specified for the Econ-ARK toolkit; see Notation in the ARK for rationales and details. (Consequently, there is an exact mapping between objects in the paper and objects in the code.)

[2] The steps are broken down also so that the notation of the paper will correspond exactly to the variable names in the toolkit, because it is required for solving life cycle problems.

[3] Allowing a stochastic interest factor would complicate the notation but not affect the points we want to address; however, see Benhabib, Bisin, and Zhu (2015), Ma and Toda (2020), and Ma, Stachurski, and Toda (2020) for the implications of capital income risk for the distribution of wealth and other interesting questions not considered here.

The consumer's assets at the end of t, $\boldsymbol{a}_t$, translate one-for-one into capital $\boldsymbol{k}_{t+1}$ at the beginning of the next period. In turn, $\boldsymbol{k}_{t+1}$ is augmented by a fixed interest factor R to become the consumer's financial ('bank') balances $\boldsymbol{b}_{t+1} = R\boldsymbol{k}_{t+1}$.[4] 'Market resources,' $\boldsymbol{m}_{t+1}$, are the sum of financial wealth $R\boldsymbol{k}_{t+1}$ and noncapital income $\boldsymbol{y}_{t+1} = \mathbf{p}_{t+1}\xi_{t+1}$ (permanent noncapital income $\mathbf{p}_{t+1}$ multiplied by the transitory-income-shock factor ξ_{t+1} described below). Permanent noncapital income $\mathbf{p}_{t+1}$ is derived from $\mathbf{p}_t$ by application of a growth factor G,[5] modified by the permanent income shock ψ_{t+1}, and the resulting idiosyncratic growth factor for permanent income is compactly written as $\tilde{G}_{t+1}$.

The finite-horizon problems furnish a sequence of value functions $\{\mathbf{v}_T, \mathbf{v}_{T-1}, \ldots, \mathbf{v}_{T-n}\}$ and associated consumption functions $\{\mathbf{c}_T, \mathbf{c}_{T-1}, \ldots, \mathbf{c}_{T-n}\}$. We define the infinite-horizon solution as the (limiting) first-period solution to a sequence of finite-horizon problems as the first period becomes arbitrarily distant from the terminal period (that is, as $n \to \infty$). An infinite-horizon solution will be 'nondegenerate' (or, 'sensible') if the limiting consumption function, denoted by $\mathbf{c}(\boldsymbol{m}, \mathbf{p}) = \lim_{n\to\infty} \mathbf{c}_{T-n}(\boldsymbol{m}, \mathbf{p})$, is neither $\mathbf{c} = 0$ everywhere (for all $(\boldsymbol{m}, \mathbf{p})$) nor $\mathbf{c} = \infty$ everywhere.[6]

The following assumption defines the income process.

Assumption I.1. *(Friedman-Muth Income Process). The following holds for all t:*

1. *For $n > 0$, the permanent shocks ψ_t are independently and identically distributed (iid), with $\mathbb{E}_{t-n}[\psi_t] = 1$ and support $[\underline{\psi}, \overline{\psi}]$, where $0 < \underline{\psi} \leq 1$ and $1 \leq \overline{\psi} < \infty$.*

2. *The transitory shocks satisfy:*

$$\xi_t = \begin{cases} 0 & \text{with probability } q > 0 \\ \theta_t/(1-q) & \text{with probability } (1-q) \end{cases} \tag{2}$$

where θ_t is an iid random variable with $\mathbb{E}_{t-n}[\theta_t] = 1$ and $\underline{\theta} \leq \theta_t \leq \bar{\theta}$, where $\underline{\theta} > 0$ and $\underline{\theta} \leq 1 \leq \bar{\theta} < \infty$.

Following Zeldes (1989), the income process incorporates a small probability q that income will be zero (a 'zero-income event'). At date $T-1$, the (strictly positive) probability q of zero income in period T will prevent the consumer from spending all resources, because saving nothing would mean arriving in the following period with zero bank balances and thus facing the possibility of being required to consume 0, which would yield utility of $-\infty$. This logic holds recursively from $T-1$ back, so the consumer will never spend everything, giving rise to what Aiyagari (1994) dubbed a 'natural

[4] See below for a brief discussion of the case where returns are stochastic.

[5] A time-varying G has straightforward consequences for the analysis below; this is an option allowed for in the HARK toolkit.

[6] The traditional approach to study recursive problems defines an infinite-horizon maximization problem over stochastic recursive sequences (Sargent and Stachurski, 2023). Using the Bellman Principle of Optimality, the definition of degenerate solutions we use here can be shown to be equivalent to the optimal solution of a dynamic stochastic sequence problem (see Appendix A.1.3). The framing we use allows for a more direct link to life-cycle models (see Gourinchas and Parker (2002) for an instance where buffer stock saving is discussed in the context of a life-cycle model).

borrowing constraint.'[7] (Formally, this establishes that the upper-bound constraint on consumption in the problem ($\mathcal{P}_L$) will not bind.)

1.1.1 *Normalized Problem*

Let nonbold variables be the boldface counterpart normalized by $\mathbf{p}_t$, allowing us to reduce the number of states from two ($\boldsymbol{m}$ and $\mathbf{p}$) to one ($m = \boldsymbol{m}/\mathbf{p}$). Now, in a one-time deviation from the notational convention established in the last sentence, define nonbold 'normalized value' not as $\boldsymbol{v}_t/\mathbf{p}_t$ but as $v_t = \boldsymbol{v}_t/\mathbf{p}_t^{1-\gamma}$, because this allows us to write nonbold v_t to denote the 'normalized value function':

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{c_t \le m_t} \ \mathrm{u}(c_t) + \beta\,\mathbb{E}_t[\tilde{G}_{t+1}^{1-\gamma}\mathrm{v}_{t+1}(m_{t+1})], \qquad m_t \in \mathbb{R}_{++} \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
k_{t+1} &= a_t/\tilde{G}_{t+1} \\
b_{t+1} &= k_{t+1}R \\
m_{t+1} &= \ b_{t+1} + \xi_{t+1}.
\end{aligned}
\tag{$\mathcal{P}_N$}
$$

(Appendix A.1.1 explains how the solution to the original problem in levels can be recovered from the normalized problem.)

The time t normalized consumption *policy function* for the finite-horizon problem, c_t, is defined by:

$$
\mathrm{c}_t(m_t) = \operatorname*{arg\,max}_{c_t \le m_t} \ \mathrm{u}(c_t) + \beta\,\mathbb{E}_t[\tilde{G}_{t+1}^{1-\gamma}\mathrm{v}_{t+1}(m_{t+1})] \tag{3}
$$

The normalized problem's first order condition becomes:

$$
c_t^{-\gamma} = R\beta\,\mathbb{E}_t[\tilde{G}_{t+1}^{-\gamma}c_{t+1}^{-\gamma}]. \tag{4}
$$

We now formally define the limiting nondegenerate solution to the normalized problem, letting n index the planning horizon.

Definition 1. *Problem $\mathcal{P}_N$ has a nondegenerate solution if there exists* c*, with* $\mathrm{c}\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$*, such that* $\mathrm{c}(0) = 0$ *and:*

$$
\mathrm{c}(m) = \lim_{n\to\infty} \mathrm{c}_{T-n}(m), \qquad m \in \mathbb{R}_{++}.
$$

We will similarly use v without a subscript to refer to the pointwise limit of the value functions as the planning horizon recedes.

Having defined the limiting solution, we now explain why the standard dynamic programming approach for showing existence cannot be used here. Let $\mathbb{T}$ denote the mapping $\mathrm{v}_{t+1} \mapsto \mathrm{v}_t$ given by Problem $\mathcal{P}_N$:

$$
\mathbb{T}\mathrm{v}_{t+1}(m) = \max_{c\in(0,m]} \left\{\mathrm{u}(c) + \beta\,\mathbb{E}\,\tilde{G}^{1-\gamma}\mathrm{v}_{t+1}(\tilde{R}(m-c)+\xi)\right\}, \quad m \in \mathbb{R}_{++}. \tag{5}
$$

[7] We specify zero as the lowest-possible-income event without loss of generality, see for example the discussion by Aiyagari (1994).

If $\mathbb{T}$ can be shown to be a contraction mapping operator, our finite-horizon value functions will converge to a nondegenerate solution. However, $\mathbb{T}$ will not a be well defined operator on a space of continuous functions due to the naturally arising liquidity constraint.

Remark 1. *(Challenges with Standard Dynamic Programming). Standard dynamic programming works by showing that $\mathbb{T}$ is a well-defined contraction map on a Banach space, implying the sequence of value functions given by Problem $\mathcal{P}_N$ will converge to a fixed point of $\mathbb{T}$, a non-degenerate solution. In our case, there are two challenges (further details in Appendix A.1.2). First, utility is unbounded, so we must construct a suitable metric space in which the sequential value functions remain bounded. Second, we cannot show $\mathbb{T}$ is a contraction mapping because we cannot immediately assert that $\mathbb{T}$ is a well-defined mapping on a suitable function space. Even if we verified that a maximizer exists for arbitrary continuous function f, that is, $\mathrm{f} \mapsto \mathbb{T}\mathrm{f}$ is well-defined, we cannot then assert $\mathbb{T}$ maps continuous functions to continuous functions since the feasible set on the RHS is not compact-valued. (Without compact valued feasible sets, we cannot use Berge's Theorem and g, with $\mathrm{g} = \mathbb{T}\mathrm{f}$, may not be continuous.) Ma, Stachurski, and Toda (2022) show that the problem can be transformed to include zero consumption in the feasibility constraint; however, the operator continues to be ill-defined unless income has a strictly positive lower bound (or equivalently, there is an 'artificial' borrowing constraint is strictly tighter than the natural borrowing constraint).*[8]

Remark 2. *If we set $q = 0$, the normalized problem becomes a special case of the problem considered by Ma, Stachurski, and Toda (2020), with $\tilde{R}_{t+1} = R/\tilde{G}_{t+1}$ corresponding to the stochastic rate of return on capital and $\beta\tilde{G}_{t+1}^{1-\gamma}$ corresponding to the stochastic discount factor.*

Notwithstanding remark 2, there are important economic consequences of the fact that in our problem $\tilde{R}_{t+1} = \tilde{R}/\tilde{G}_{t+1}$ is tightly tied to 'normalized stochastic discount factor,' $\beta\tilde{G}_{t+1}^{1-\gamma}$; we discuss these below.

1.2 Patience Conditions

In order to have a central reference point for them, we now collect (without much explanation) conditions relating to growth factors and various different 'patience factors' that underpin results in the remainder of the paper. Assumptions L.1 - L.3 (finite value of autarky and return impatience) will be used to prove the existence of limiting solutions in Section 1.4, and Assumptions S.1 - S.2 (growth impatience and strong growth impatience) are required for existence of alternative definitions of a stable target buffer stock in Section 2.

[8] Feinberg, Kasyanov, and Zadoianchuk (2012) generalize the requirement of continuity of feasibility correspondences to K-Inf-Compactness of the Bellman operator, yielding a mapping from semi-continuous to semi-continuous functions. Shanker (2017) introduces a generalization, mild-Sup-compactness, which can be verified in the weak topology generated on the infinite dimensional product space of feasible random variables controlled by the consumers. Our approach, by contrast, has the advantage that it can be used to verify existence using more standard topological notions.

We start by generalizing the standard $\beta < 1$ condition to our setting with permanent income growth and uncertainty.[9] The updated condition requires that the expected net discounted value of utility from consumption is finite under our definition of 'autarky' – where consumption is always equal to permanent income. A finite value of autarky helps guarantee that as the horizon extends, discounted value remains finite along *any* consumption path the consumer might choose.

Assumption L.1. *(Finite Value of Autarky).* $0 < \beta G^{1-\gamma}\mathbb{E}(\psi^{1-\gamma}) < 1$.

The discount factor β defines the consumer's 'pure' rate of time preference – the relative weight of utility across time. The term 'patience' does not have a similarly clear definition in the literature. Part of our objective in this paper is to provide a taxonomy for each of various useful definitions of patience.

We start with 'absolute (im)patience.' We will say that an unconstrained perfect foresight consumer exhibits absolute impatience if they optimally choose to spend so much today that their consumption must decline in the future. The growth factor for consumption implied by the Euler equation of a perfect foresight model is $\boldsymbol{c}_{t+1}/\boldsymbol{c}_t = (R\beta)^{1/\gamma}$,[10] which motivates our definition of an 'absolute patience factor' whose centrality (to everything that is to come later) justifies assigning to it a special symbol; we have settled on the archaic letter 'thorn':

$$\text{Þ} := (R\beta)^{1/\gamma}. \tag{6}$$

We will say that (in the perfect foresight problem) 'an absolutely impatient' consumer is one for whom $\text{Þ} < 1$; that is an absolutely impatient consumer prefers to consume more today than tomorrow (and vice versa for an 'absolutely patient' consumer, whose consumption will grow over time):

Assumption L.2. *(Absolute Impatience).* $\text{Þ} < 1$.

Remark 3. *A consumer who is absolutely impatient, $\text{Þ} < 1$, satisfies the standard impatience condition commonly used in the income fluctuation literature, $\beta R < 1$, which guarantees the existence of a stable distribution when there is no permanent income growth. However, as pointed out by Szeidl (2013) and Ma, Stachurski, and Toda (2022) (henceforth, 'MST'), $\beta R < 1$ is not necessary for an infinite-horizon solution. The shock process of the normalized problem here can be seen as a special case of MST (albeit with an artificial liquidity constraint). In their environment, MST show that a condition analogous to our finite value of autarky condition is sufficient for the existence of a nondegenerate solution.*

Recall now our earlier requirement that the limiting consumption function $\mathrm{c}(m)$ in our model must be 'sensible.' We will show below that for the perfect foresight unconstrained problem this requires

[9] In light of Remark 2, Ma, Stachurski, and Toda (2020) Assumption 2.1 is a generalization of this discount condition, albeit in a context with artificial liquidity constraints.

[10] See (10) below.

Assumption L.3. *(Return Impatience).* $\frac{\text{Þ}}{R} < 1$.

Return impatience can be best understood as the tension between the income effect of capital income and substitution effect. As we show below, in the perfect foresight model, it is straightforward to derive the MPC out of overall (human plus nonhuman) wealth that would result in next period's wealth being identical to the current period's wealth. The answer turns out to be an MPC ('κ') of $\kappa = (1 - \text{Þ}/R)$. The interesting point here is that κ depends both on our absolute patience factor **Þ** and on the return factor. This is the manifestation in this context of the interaction of the income effect (higher wealth yields higher interest income if $R > 1$) and the substitution effect (which we have already captured with **Þ**).

Next, consider the weaker condition of a consumer whose absolute patience factor is suitably adjusted to take account of the probability of zero income is less than the market return.

Assumption L.4. *(Weak Return Impatience).* $\underbrace{\frac{(qR\beta)^{1/\gamma}}{R}}_{=\frac{q^{1/\gamma}\text{Þ}}{R}} \leq 1$.

This condition is 'weak' (relative to the plain return impatience) because the probability of the zero income events q is strictly less than 1. The role of q in this equation is related to the fact that a consumer with zero end-of-period assets today has a probability q of having no income and no assets to finance consumption (and $m_{t+1} = 0$ would yield negative infinite utility). In the case with no artificial constraint, our main results below show weak return impatience and finite value of autarky are sufficient to guarantee a 'sensible' (nondegenerate) solution. Moreover, weak return impatience is also necessary and cannot be relaxed further without an artificial liquidity constraint.

Now that we have finished discussing the requirements for a nondegenerate solution, we turn to assumptions required for stability. We speak of a consumer whose absolute patience factor is less than the expected growth factor for their permanent income $G = \mathbb{E}[G\psi]$) as exhibiting 'growth impatience:'

Assumption S.1. *(Growth Impatience).* $\frac{\text{Þ}}{G} < 1$.

A final useful definition is 'strong growth impatience' which holds for a consumer for whom the expectation of the *ratio* of the absolute patience factor to the growth factor of permanent income is less than one,

Assumption S.2. *(Strong Growth Impatience).* $\mathbb{E}\left[\frac{\text{Þ}}{G\psi}\right] = \frac{\text{Þ}}{G}\,\mathbb{E}[\psi^{-1}] < 1$.

(The difference between growth impatience and strong growth impatience is that the first is the ratio of an expectation to an expectation, while the latter is the expectation of the ratio. With nondegenerate mean-one stochastic shocks to permanent income, the expectation of the ratio is strictly larger than the ratio of the expectations).

While neither growth impatience nor return impatience will by themselves be required for the existence of a limiting solution, the finite value of autarky condition stops individuals from becoming *both* growth and return patient.

Claim 1. *If growth impatience fails ($\frac{\text{Þ}}{G} > 1$) and return impatience fails ($\frac{\text{Þ}}{R} > 1$), then finite value of autarky fails ($\beta G^{1-\gamma}\mathbb{E}(\psi^{1-\gamma}) > 1$).*

Proof. Since $\frac{\text{Þ}}{R} > 1$, we have:

$$\frac{\text{Þ}}{R} = \frac{(R\beta)^{\frac{1}{\gamma}}}{R} > 1 \tag{7}$$

Next, multiplying both sides by $RG^{1-\gamma}$, we get

$$\beta G^{1-\gamma} R^{\frac{1}{\gamma}} \beta^{\frac{1-\gamma}{\gamma}} > RG^{1-\gamma} \Rightarrow \beta G^{1-\gamma} > \left(\frac{\text{Þ}}{G}\right)^{\gamma-1} \tag{8}$$

Finally, since $\gamma > 1$, applying $\frac{\text{Þ}}{G} > 1$ gives us the result.

□

We discuss further intuition for these conditions below when they are used in the formal results. The relationship between the conditions and their implications for consumption behaviour will also be be discussed in detail in Section 4.

1.3 Perfect Foresight Benchmarks

To understand the economic implications of the patience conditions, we begin with the perfect foresight case.

Assumption I.2. *(Perfect Foresight Income Process). The perfect foresight income process satisfies $q = 0$ and $\underline{\theta} = \bar{\theta} = \underline{\psi} = \overline{\psi} = 1$.*

Under perfect foresight, finite value of autarky reduces to a 'perfect foresight finite value of autarky' condition:

$$\beta G^{1-\gamma} < 1. \tag{9}$$

1.3.1 Perfect Foresight without Liquidity Constraints

Consider the familiar analytical solution to the perfect foresight model without liquidity constraints. In this case, the consumption Euler Equation always holds; with $\mathrm{u}'(\boldsymbol{c}) = \boldsymbol{c}^{-\gamma}$ and $\mathrm{u}'(\boldsymbol{c}_t) = R\beta\mathrm{u}'(\boldsymbol{c}_{t+1})$ we have:

$$\boldsymbol{c}_{t+1}/\boldsymbol{c}_t = (R\beta)^{1/\gamma}. \tag{10}$$

Defining $\bar{R} \colon = R/G$, 'human wealth' is the present discounted value of income:

$$\begin{aligned} \mathbf{h}_t &= \mathbf{p}_t + \bar{R}^{-1}\mathbf{p}_t + \bar{R}^{-2}\mathbf{p}_t + \cdots + \bar{R}^{t-T}\mathbf{p}_t \\ &= \underbrace{\left(\frac{1-\bar{R}^{-(T-t+1)}}{1-\bar{R}^{-1}}\right)}_{\equiv h_t}\mathbf{p}_t. \end{aligned} \tag{11}$$

For human wealth to have finite value, we must have:

Assumption I.3. *(Finite Human Wealth).*

$$\bar{R}^{-1} = G/R < 1. \tag{12}$$

If $\bar{R}^{-1}$ is less than one, human wealth will be finite in the limit as $T \uparrow \infty$ because (noncapital) income growth is smaller than the interest rate at which that income is being discounted.

Under these conditions we can define a normalized finite-horizon perfect foresight consumption function (see Appendix C.1 for details) as follows:

$$\bar{c}_{T-n}(m_{T-n}) = (\overbrace{m_{T-n} - 1}^{\equiv b_{T-n}} + h_{T-n})\underline{\kappa}_{t-n}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC) and satisfies:

$$\underline{\kappa}_{T-n}^{-1} = 1 + \left(\frac{\text{Þ}}{R}\right)\underline{\kappa}_{T-n+1}^{-1}. \tag{13}$$

Thus, for $\underline{\kappa}$ to be strictly positive as n goes to infinity, we must impose return impatience. The limiting consumption function then becomes:

$$\bar{c}(m) = (m + h - 1)\underline{\kappa}, \tag{14}$$

where (under return impatience)

$$\underline{\kappa} := 1 - \frac{\text{Þ}}{R}. \tag{15}$$

In order to rule out the degenerate limiting solution in which $\bar{c}(m) = \infty$, we also need to require (in the limit as the horizon extends to infinity) that human wealth remain bounded (that is, we require a limiting 'finite human wealth'). Thus, while return impatience prevents a consumer from saving everything in the limit, 'finite value of human wealth' prevents infinite borrowing (against infinite human wealth) in the limit.

Proposition 1. *Consider the normalized problem without liquidity constraints and with perfect foresight income (Assumption I.2). A nondegenerate limiting solution exists if and only if finite value of human wealth ($\bar{R}^{-1} < 1$) and return impatience (Assumption L.3) hold.*

The proof of the following claim is follows from straightforward algebra (see Appendix A.2).

Claim 2. *Assume finite limiting human wealth ($\bar{R}^{-1} < 1$). If growth impatience (Assumption S.1) holds, then finite value of autarky (Assumption L.1) holds. Moreover, if finite value of autarky (Assumption L.1) holds, then return impatience (Assumption L.3) holds.*

The claim implies that if we impose finite value of human wealth, then growth impatience is sufficient for nondegeneracy since finite value of autarky and return impatience follow. However, there are circumstances under which return impatience and finite value of human wealth can hold while the finite value of autarky fails. For example, if $G = 0$, the problem is a standard 'cake-eating' problem with a nondegenerate solution under return impatience.

1.3.2 Perfect Foresight with Liquidity Constraints

Our ultimate interest is in the unconstrained problem with uncertainty. Here, we show that the perfect foresight constrained solution defines a useful limit for the unconstrained problem with uncertainty.

If a liquidity constraint requiring $b \geq 0$ is ever to be relevant, it must be relevant at the lowest possible level of market resources, $m_t = 1$, defined by the lower bound, $b_t = 0$ (if the constraint were relevant at any higher m, it would certainly be relevant here, because $\mathrm{u}' > 0$). The constraint is 'relevant' if it prevents the choice that would otherwise be optimal; at $m_t = 1$ it is relevant if the marginal utility from spending all of today's resources $c_t = m_t = 1$, exceeds the marginal utility from doing the same thing next period, $c_{t+1} = 1$; that is, if such choices would violate the Euler equation, Equation (4):

$$1^{-\gamma} > R\beta G^{-\gamma}1^{-\gamma}, \tag{16}$$

which is just a restatement of growth impatience. So, the constraint is relevant if and only if growth impatience holds.

Proposition 2. *Consider the normalized problem with perfect foresight income (Assumption I.2) and assume $c_t \leq m_t$ for each t. If return impatience (Assumption L.3) holds, then a nondegenerate solution exists. Moreover, if return impatience does not hold, then a nondegenerate solution exists if and only if growth impatience (Assumption S.1) also holds.*

Importantly, if return impatience fails ($R < \text{\textbf{Þ}}$) and growth impatience holds ($\text{\textbf{Þ}} < G$), then finite value of human wealth also fails ($R < G$). Despite the unboundedness of human wealth as the horizon extends arbitrarily, for any finite horizon the relevant liquidity constraint prevents borrowing. Similarly, when uncertainty is present, the natural borrowing constraint plays an analogous role in permitting a finite limiting solution despite unbounded limiting human wealth – we discuss the various parametric cases in Section 4.

1.4 Main Results for Problem with Uncertainty

We are now ready to return to our primary interest, the model with permanent and transitory income shocks. Throughout this section, we assume the Friedman-Muth income process (Assumption I.1 holds).

1.4.1 Limiting MPCs

We first establish results regarding the shape of the consumption function.

Proposition 3. *For each t, c_t is increasing, twice continuously differentiable, strictly concave and $\mathrm{c}_t(0) = 0$.*

For a proof, see Appendix A.3.[11]

Next, we verify that the ratio of optimal consumption to market resources (c/m) is bounded by the minimal and maximal marginal propensities to consume (MPC). Recall that the MPCs answer the question 'if the consumer had an extra unit of resources, how much more spending would occur?' The minimal and maximal MPCs are the limits of the MPC as $m \to \infty$ and $m \to 0$, which we denote by $\underline{\kappa}_t$ and $\bar{\kappa}_t$ respectively. Since the consumer spends everything in the terminal period, $\underline{\kappa}_T = 1$ and $\bar{\kappa}_T = 1$. Furthermore, Proposition 3 will imply:[12]

$$\underline{\kappa}_t m_t \ \leq \mathrm{c}_t(m_t) \leq \ \bar{\kappa}_t m_t. \tag{17}$$

We define:

$$\underline{\kappa}: \ = \max\{0, 1 - \frac{\text{Þ}}{R}\}, \qquad \bar{\kappa}: \ = 1 - q^{1/\gamma}\frac{\text{Þ}}{R}, \tag{18}$$

as the 'limiting minimal and maximal MPCs.' The following result verifies that the consumption share is bounded each period by the minimal and maximal MPCs, that the consumption function is asymptotically linear and that the MPCs converge to the limiting MPCs as the planning horizon recedes.[13]

Lemma 1. *(Limiting MPCs). If weak return impatience (Assumption L.4) holds, then:*

(i) For each n:

$$\underline{\kappa}_{T-n}^{-1} = 1 + \left(\frac{\text{Þ}}{R}\right)\underline{\kappa}_{T-n+1}^{-1}. \tag{19}$$

Moreover, if return impatience (Assumption L.3) holds, then $\lim_{n\to\infty} \underline{\kappa}_{T-n} = \underline{\kappa} = 1 - \frac{\text{Þ}}{R}$ where $1 > \underline{\kappa} > 0$.

(ii) For each n:

$$\bar{\kappa}_{T-n}^{-1} = 1 + \left(q^{1/\gamma}\frac{\text{Þ}}{R}\right)\bar{\kappa}_{T-n+1}^{-1}. \tag{20}$$

Moreover, $\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa}$, where $1 \geq \bar{\kappa} > 0$.

For the proof, see Appendix A.3.

The MPC bound as market resources approach infinity is easy to understand. Recall that $\bar{\mathrm{c}}$ from the perfect foresight case will be an upper bound in the problem with uncertainty; analogously, $\underline{\kappa}$ becomes the MPC's lower bound. As the *proportion* of consumption that will be financed out of human wealth approaches zero, the proportional difference between the solution to the model with uncertainty and the perfect foresight model shrinks to zero.

[11] Carroll and Kimball (1996) proved concavity but not continuous differentiability.

[12] Note c_t' is positive, bounded above by 1 and decreasing, then apply L'Hôpital's Rule.

[13] Benhabib, Bisin, and Zhu (2015) show that the consumption function becomes linear as wealth approaches infinity in a model with capital income risk and liquidity constraints; Ma and Toda (2020) show that these results generalize to the limits derived here if capital income is added to the model.

To understand the maximal limiting MPC, the essence of the argument is that as market resources approach zero, the overriding consideration that limits consumption is the (recursive) fear of the zero-income events — this is why the probability of the zero income event q appears in the expression for the maximal MPC. Weak return impatience is too weak to guarantee a lower bound on the share of consumption to market resources; it merely prevents the upper bound on the share of consumption to market resources from approaching zero. Weak return impatience thereby prevents a situation where *everyone* consumes an arbitrarily small share of current market resources as the planning horizon recedes. This insight plays a key role in the proof for the existence of a non-degenerate solution in what follows.

1.4.2 Existence of Limiting Nondegenerate Solution

To address the challenges of unbounded state-spaces, Boyd (1990) provided a weighted contraction mapping theorem. We use this approach to first show that while the stationary operator $\mathbb{T}$ may be undefined, each period's Bellman operator will be a contraction. We then show the value function generated by the Bellman iteration given by Problem ($\mathcal{P}_N$) generates a Cauchy sequence in a complete metric space; that is, the sequence of value functions converges to a nondegenerate solution in $\mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$.

Let $\mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$ be the space of continuous functions from $\mathbb{R}_{++}$ to $\mathbb{R}$.

Definition 2. *Fix* f *such that* $\mathrm{f} \in \mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$ *and let* φ *be a function such that* $\varphi \in \mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$ *and* $\varphi > 0$*. The function* f *will be* φ*-bounded if the* φ*-norm of* f*:*

$$\|\mathrm{f}\|_{\varphi} = \sup_{s \in \mathbb{R}_{++}} \left[\frac{|\mathrm{f}(s)|}{\varphi(s)} \right], \tag{21}$$

is finite. We will call $\mathcal{C}_{\varphi}(\mathbb{R}_{++}, \mathbb{R})$ *the subspace of functions in* $\mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$ *that are* φ*-bounded.*

We define the weighting function as:

$$\varphi(x) = \zeta + x^{1-\gamma}, \tag{22}$$

where $\zeta \in \mathbb{R}_{++}$ is a constant.

Next, for any lower bound $\underline{\nu}$ and upper-bound $\overline{\nu}$ on the share of consumption to market resources, define the Bellman operator $\mathbb{T}^{\underline{\nu},\overline{\nu}}$, with $\mathbb{T}^{\underline{\nu},\overline{\nu}} : \mathcal{C}_{\varphi}(S, Y) \to \mathcal{C}_{\varphi}(S, Y)$, as:

$$\begin{aligned} &\mathbb{T}^{\underline{\nu},\overline{\nu}}\mathrm{f}(m) \\ &\quad = \max_{c \in [\underline{\nu}m, \overline{\nu}m]} \left\{ \mathrm{u}(c) + \beta\, \mathbb{E}\, \tilde{G}^{1-\gamma} \mathrm{f}(\tilde{R}(m-c) + \xi) \right\},\ m \in \mathbb{R}_{++}, \mathrm{f} \in \mathcal{C}_{\varphi}(\mathbb{R}_{++}, Y). \end{aligned} \tag{23}$$

Even if in the limit the minimal marginal propensity to consume approaches zero, the finite-horizon value functions defined by the normalized recursive problem, Problem ($\mathcal{P}_N$), will satisfy $\mathrm{v}_t = \mathbb{T}^{\underline{\kappa}_t,\bar{\kappa}_t}\mathrm{v}_{t+1}$ since consumption shares are bounded by the minimal and maximal MPCs (Lemma 3 and Equation (17)). We now show this implies that the operator $\mathbb{T}^{\underline{\nu},\overline{\nu}}$ is a contraction on $\mathcal{C}_{\varphi}(\mathbb{R}_{++}, Y)$ for a suitably narrow interval $[\underline{\nu}, \overline{\nu}]$.

Theorem 1. *(Contraction Mapping Under Consumption Bounds). If weak return impatience (Assumption L.4) and finite value of autarky (Assumption L.1) hold, then there exists k large enough and $\alpha \in (0,1)$ such that for all $\overline{\nu}$ with $\overline{\nu} \leq \bar{\kappa}_{T-k}$ and $\underline{\nu}$ with $\overline{\nu} > \underline{\nu} > 0$, the Bellman operator $\mathbb{T}^{\underline{\nu},\overline{\nu}}$ is a contraction with modulus α.*

An implication of the theorem is that eventually, the maximal MPCs will be small enough such that each of the Bellman operators generating the sequence of value functions as the terminal time T recedes (that is, as the horizon expands) will be contraction maps. We can now relate the sequence of contraction maps to a limiting solution defined in Section 1.1.1.

Theorem 2. *(Existence of Nondegenerate Solution). If weak return impatience (Assumption L.4) and finite value of autarky (Assumption L.1) hold, then:*

(i) There exists k such that for all t with $t < k$ and $\underline{\nu}$, with $\bar{\kappa}_t > \underline{\nu} > 0$, $\mathbb{T}^{\underline{\nu},\bar{\kappa}_t}$ is a contraction with modulus α, where $\alpha < 1$ and the sequence $\{\mathrm{v}_{T-n}\}_{n=0}^{\infty}$ converges point-wise to v, with $\mathrm{v} \in \mathcal{C}_{\varphi}(\mathbb{R}_{++}, \mathbb{R})$.

(ii) The function v is a fixed point of $\mathbb{T}$ and there exists a measurable limiting policy function, c, such that $\mathrm{c}\colon \mathbb{R}_{+} \to \mathbb{R}$ and:

$$\mathbb{T}\mathrm{v}(m) = \mathrm{u}(\mathrm{c}(m)) + \beta\,\mathbb{E}\,\tilde{G}^{1-\gamma}\mathrm{v}(\tilde{R}(m - \mathrm{c}(m)) + \xi), \qquad m \in \mathbb{R}_{++}. \tag{24}$$

(iii) The sequence $\{\mathrm{c}_{T-n}\}_{n=0}^{\infty}$ converges point-wise to c and c is a limiting nondegenerate solution.

Proof. The first claim of Item (i) follows from Theorem 1, since $\underline{\nu} > 0$ and for each t, $\bar{\kappa}_t < \bar{\kappa}_k$ by Lemma 3. We now prove $\{\mathrm{v}_{T-n}\}_{n=0}^{\infty}$ converges point-wise to a limiting nondegenerate solution v. In the proof, to streamline the notation, we define an index t_n for the sequence t that depends on n. Specifically, $t_n = T - n$. Now, for all $n > k+2$, $\mathrm{v}_{t_n} = \mathbb{T}^{\underline{\kappa}_{t_n},\bar{\kappa}_{t_n}}\mathrm{v}_{t_{n-1}}$ holds by definition of the value function given by the Bellman Equation ($\mathcal{P}_N$). Moreover, since $\bar{\kappa}_{t_{n-1}} \geq \bar{\kappa}_{t_n}$ by Lemma 3, we have:

$$\mathrm{v}_{t_n} = \mathbb{T}^{\underline{\kappa}_{t_n},\bar{\kappa}_{t_{n-1}}}\mathrm{v}_{t_{n-1}}$$

and since $\underline{\kappa}_{t_n} \leq \underline{\kappa}_{t_{n-1}}$, we have:

$$\begin{aligned}\mathrm{v}_{t_{n-1}} &= \mathbb{T}^{\underline{\kappa}_{t_{n-1}},\bar{\kappa}_{t_{n-1}}}\mathrm{v}_{n-2} \\ &= \mathbb{T}^{\underline{\kappa}_{t_n},\bar{\kappa}_{t_{n-1}}}\mathrm{v}_{t_{n-2}}\end{aligned}$$

Next, take the φ-norm distance between v_{t_n} and $\mathrm{v}_{t_{n-1}}$, and note $\mathbb{T}^{\underline{\nu},\bar{\kappa}_t}$ is a contraction. We have:

$$\|\mathrm{v}_{t_n} - \mathrm{v}_{t_{n-1}}\|_{\varphi} = \|\mathbb{T}^{\underline{\kappa}_{t_n},\bar{\kappa}_{t_{n-1}}}\mathrm{v}_{t_{n-1}} - \mathbb{T}^{\underline{\kappa}_{t_n},\bar{\kappa}_{t_{n-1}}}\mathrm{v}_{t_{n-2}}\|_{\varphi} \leq \alpha\|\mathrm{v}_{t_{n-1}} - \mathrm{v}_{t_{n-2}}\|_{\varphi}.$$

As such, $\|\mathrm{v}_{t_n} - \mathrm{v}_{t_{n-1}}\|_{\varphi} \leq \alpha\|\mathrm{v}_{t_{n-1}} - \mathrm{v}_{t_{n-2}}\|_{\varphi}$; because n is arbitrary and α holds for all n by Theorem 1, this is a sufficient condition for $\{\mathrm{v}_{T-n}\}_{n=k+2}^{\infty}$ to be Cauchy. Since $\mathcal{C}_{\varphi}(\mathbb{R}_{++}, Y)$ is a complete metric space, and $\mathrm{v}_{t_{n-2}} \in \mathcal{C}_{\varphi}(\mathbb{R}_{++}, Y)$ for each n, v_{t_n} converges to v, with $v \in \mathcal{C}_{\varphi}(\mathbb{R}_{++}, Y)$. (Proof continued in Appendix A.4.) $\square$

Without the stronger assumption holding, $\underline{\kappa} = 0$ and $\mathbb{T}^{0,\kappa_k}$ will not be a well-defined operator from $\mathcal{C}_\varphi(\mathbb{R}_{++}, Y)$ to $\mathcal{C}_\varphi(\mathbb{R}_{++}, Y)$, even for k large enough (recall our discussion below Equation (5)). Nonetheless, the sequence of value functions produced by the composition of the per-period Bellman operators $\mathbb{T}^{\underline{\kappa}_t,\bar{\kappa}_t}$ will be a Cauchy sequence converging to the limiting solution. Due to weak return impatience, the upper bound on consumption converges to a strictly positive share of market resources, preventing consumption to limit to zero. The remainder of this proof for Item (ii) and Item (iii) in the appendix shows that the limiting value functions is a fixed point to the operator $\mathbb{T}$ and that the sequence of consumption functions converge.

Finite value of autarky is the second key assumption required to show existence of limiting solutions and guarantees the value is finite (in levels) for a consumer who spent exactly their permanent income every period (see Section 4.2). In the normalized problem ($\mathcal{P}_N$), finite value of autarky ensures the expected value of the random discount factor is less than one.[14] The intuition for the finite value of autarky condition is that, with an infinite-horizon, with any strictly positive initial amount of bank balances b_0, in the limit your value can always be made greater than you would get by consuming exactly the sustainable amount (say, by consuming $(\mathsf{r}/R)b_0 - \epsilon$ for some arbitrarily small $\epsilon > 0$).

Finally, we verify that the converged nondegenerate consumption functions satisfies the same consumption bounds as the per-period consumption functions.

Claim 3. *If weak return impatience (Assumption L.4) holds, then $\underline{\kappa}m \leq \mathrm{c}(m) \leq \bar{\kappa}m$, ii) $\lim_{m\to\infty} \mathrm{c}(m)/m = \underline{\kappa}m$ and iii) $\lim_{m\to 0} \mathrm{c}(m)/m = \bar{\kappa}m$.*

1.4.3 The Liquidity Constrained Solution as a Limit

Recall the common assumption (Deaton, 1991; Aiyagari, 1994; Li and Stachurski, 2014; Ma, Stachurski, and Toda, 2020) of a strictly positive minimum value of income and a non-trivial artificial liquidity constraint, namely $a_t \geq 0$. We will refer to the set-up from Section 1.1, with Assumption 2 modified so $q = 0$ as the "liquidity constrained problem." We now show a finite-horizon solution to the liquidity constrained problem is the limit of the problems as the probability q of the zero-income event approaches zero. Let $\mathrm{c}_t(\bullet; q)$ be the consumption function for a problem where Assumption I.1 holds for a given fixed q, with $q > 0$. Moreover, let $\grave{\mathrm{c}}_t$ be the limiting consumption function for the liquidity constrained problem (note that the liquidity constraint $c_t \leq m_t$, or $a_t \geq 0$, becomes relevant only when $q = 0$).

Proposition 4. *Assume the setting of Theorem 2. We have $\lim_{q\to 0} \mathrm{c}_t(m; q) = \grave{\mathrm{c}}_t(m)$ for each t and $m \in \mathbb{R}_{++}$.*

Intuitively, if we impose the artificial constraint without changing q and maintain $q > 0$, it would not affect behavior. This is because the possibility of earning zero income

[14] Assumption 2.1 by Ma, Stachurski, and Toda (2020) specializes to finite value of autarky in our case. Assumption 2.2 by Ma, Stachurski, and Toda (2020) requires specializes $\beta R < 1$. Our proofs do not require $\beta R < 1$.

over the remaining horizon already prevents the consumer from ending the period with zero assets. For precautionary reasons, the consumer will save something. However, the *extent* to which the consumer feels the need to make this precautionary provision depends on the *probability* that it will turn out to matter. As $q \to 0$, the precautionary saving induced by the zero-income events approaches zero, and "zero" is the amount of precautionary saving that would be induced by a zero-probability event by the impatient liquidity constrained consumer. See Appendix A.5 for the formal proof.

2 Individual Buffer Stock Stability

In this section we analyse two notions of stability which will be useful for studying either an individual or a population of individuals who behave according to the converged consumption rule. Consider a individual who at time t holds normalized and non-normalized market resources m_t and $\boldsymbol{m}_t$ and follows the converged decision function c. The time-t consumption for the consumer will be $c_t = \mathrm{c}(m_t)$ and the time $t+1$ market resources will be a random variable $m_{t+1} = \tilde{R}_{t+1}(m_t - \mathrm{c}(m)) + \xi_{t+1}$. At the individual level, we are interested in whether the current level of market resources is above or below a 'target' level such that the magnitude of the precautionary motive (which causes a consumer to save) exactly balances the impatience motive (which makes them want to dissave). At the individual 'target,' the expected market resources ratio in the next period, *conditioned on current market resources*, will be the same as the ratio in the current period. The intensifying strength of the precautionary motive with decreasing market resources can ensure stability of the target. Below the target, the urgency to save due to the precautionary motive leads to an expected rise in market resources. Conversely, above the target, impatience prevails, leading to an expected reduction of market resources. In this way, the 'target' essentially defines the desired 'buffer stock' of resources for the consumer.

To help motivate the theoretical results concerning existence of a target level of market resources, Figure 1 shows the expected growth factors for consumption, the level of market resources, and the market resources to permanent income ratio, $\mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t]$, $\mathbb{E}_t[\boldsymbol{m}_{t+1}/\boldsymbol{m}_t]$, and $\mathbb{E}_t[m_{t+1}/m_t]$. The figure is generated using parameters discussed in Section 4, Table 2. First, the figure shows how as $m_t \to \infty$ the expected consumption growth factor goes to Þ, indicated by the lower bound in Figure 1. Moreover, as m_t approaches zero the consumption growth factor approaches ∞. The following proposition establishes the asymptotic growth factors formally (See Appendix B for a proof).

Proposition 5. *We have* $\lim_{m_t \to \infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] = \text{Þ}$ *and* $\lim_{m_t \to 0} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] = \infty$.

Next, consider the implications of Figure 1 for individual stability. The figure shows a value of the market resources ratio, $m_t = \check{m}$, at which point the expected growth factor of the level of market resources $\boldsymbol{m}$ matches the expected growth factor of permanent income G. A distinct and larger target ratio, $\hat{m}$, also exists. At this ratio, $\mathbb{E}[m_{t+1}/m_t] = 1$, and the expected growth factor of consumption is less than G. Importantly, at the individual level, this model does not have a single m at which $\mathbf{p}$, $\boldsymbol{m}$ and $\boldsymbol{c}$ are all expected to grow

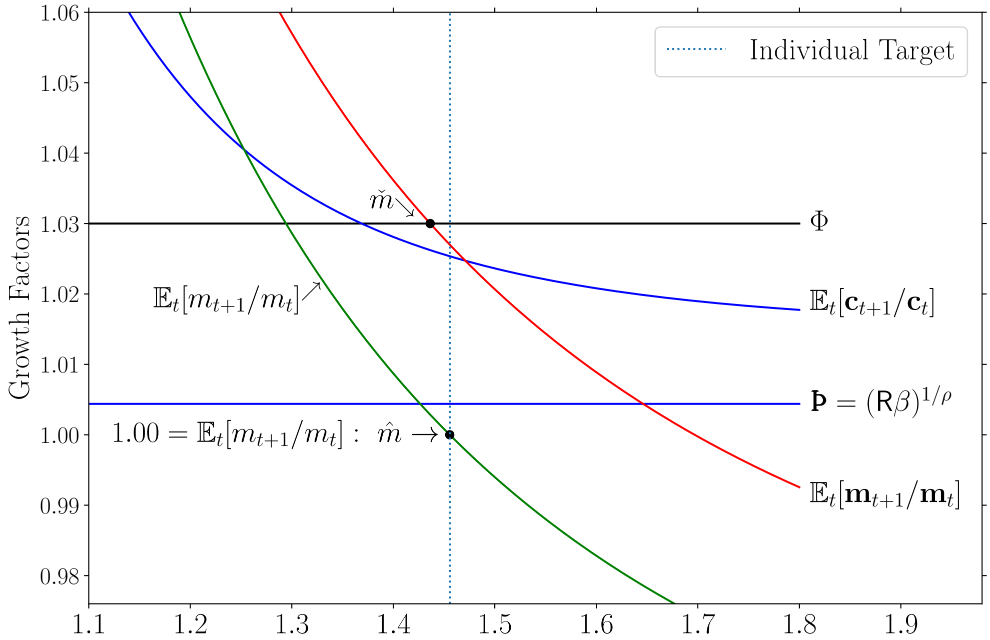

Figure 1 'Stable' (Target; Balanced Growth) m Values

at the same rate. Yet, when we aggregate across individuals, balanced growth paths can exist, even if there does not exist a target ratio where $\mathbb{E}[m_{t+1}/m_t] = 1$. Before we discuss aggregates further, we'll first set the conditions required for the existence of individual targets.

2.1 Unique 'Stable' Points

One kind of 'stable' point is a 'target' value $\hat{m}$ such that if $m_t = \hat{m}$, then $\mathbb{E}_t[m_{t+1}] = m_t$. Existence of such a target turns out to require the strong growth impatience condition.

Theorem 3. *(Individual Market-Resources-to-Permanent-Income Ratio Target). Consider the problem defined in Section 1.1. If weak return impatience (Assumption L.4), finite value of autarky (Assumption L.1) and strong growth impatience (Assumption S.2) hold, then there exists a unique market resources to permanent income ratio, $\hat{m}$, with $\hat{m} > 0$, such that:*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \textit{ if } m_t = \hat{m}. \tag{25}$$

Moreover, $\hat{m}$ is a point of 'stability' in the sense that:

$$\begin{aligned} \forall m_t \in (0, \hat{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\ \forall m_t \in (\hat{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t. \end{aligned} \tag{26}$$

Since $m_{t+1} = \tilde{R}_{t+1}(m_t - \mathrm{c}(m_t))\tilde{R}_{t+1} + \xi_{t+1}$, the implicit equation for $\hat{m}$ becomes:

$$
\begin{aligned}
\mathbb{E}_t[(\hat{m} - \mathrm{c}(\hat{m}))\tilde{R}_{t+1} + \xi_{t+1}] &= \hat{m} \\
(\hat{m} - \mathrm{c}(\hat{m}))\underbrace{\tilde{R}\,\mathbb{E}_t[\psi^{-1}]}_{\equiv\tilde{\tilde{R}}} + 1 &= \hat{m}.
\end{aligned} \tag{27}
$$

The market-resources-to-permanent-income ratio target is the most restrictive among several competing definitions of stability. Our least restrictive definition of 'stability' derives from a traditional aggregate question in macro models: whether or not there is a 'balanced growth' equilibrium in which aggregate variables (income, consumption, market resources) all grow by the same factor G. In particular, if growth impatience holds, the problem will exhibit a balanced-growth 'pseudo-steady-state' point, by which we mean that there is some $\check{m}$ such that if: $m_t > \check{m}$, then $\mathbb{E}_t[\boldsymbol{m}_{t+1}/\boldsymbol{m}_t] < G$. Conversely if $m_t < \check{m}$ then $\mathbb{E}_t[\boldsymbol{m}_{t+1}/\boldsymbol{m}_t] > G$. The target $\check{m}$ will be such that $\boldsymbol{m}$ growth matches G, allowing us to write the implicit equation for $\check{m}$ as follows:

$$
\begin{aligned}
\mathbb{E}_t[\boldsymbol{m}_{t+1}]/\boldsymbol{m}_t &= \mathbb{E}_t[\mathbf{p}_{t+1}]/\mathbf{p}_t \\
\mathbb{E}_t[m_{t+1}G\psi_{t+1}\mathbf{p}_t]/(m_t\mathbf{p}_t) &= \mathbb{E}_t[\mathbf{p}_tG\psi_{t+1}]/\mathbf{p}_t \\
\mathbb{E}_t\left[\psi_{t+1}\underbrace{((m_t - \mathrm{c}(m_t)R/(G\psi_{t+1})) + \xi_{t+1})}_{m_{t+1}}\right]/m_t &= 1 \\
\mathbb{E}_t\left[(\check{m} - \mathrm{c}(\check{m}))\overbrace{R/G}^{\tilde{R}} + \psi_{t+1}\xi_{t+1}\right] &= \check{m} \\
(\check{m} - \mathrm{c}(\check{m}))\tilde{R} + 1 &= \check{m}.
\end{aligned} \tag{28}
$$

The only difference between (28) and (27) is the substitution of $\tilde{R}$ for $\tilde{\tilde{R}}$.[15,16]

Theorem 4. *(Individual Balanced-Growth 'Pseudo Steady State'). Consider the problem defined in Section 1.1. If weak return impatience (Assumption L.4), finite value of autarky (Assumption L.1) and growth impatience (Assumption S.1) hold, then there*

[15] A third 'stable point' is the $\tilde{m}$ where $\mathbb{E}_t[\log \boldsymbol{m}_{t+1}] = \log G\boldsymbol{m}_t$; this can be conveniently rewritten as $\mathbb{E}_t\left[\log\left((\tilde{m} - \mathrm{c}(\tilde{m}))\tilde{R} + \psi_{t+1}\xi_{t+1}\right)\right] = \log \tilde{m}_t$. Because the expectation of the log of a stochastic variable is less than the log of the expectation, if a solution for $\tilde{m}$ exists it will satisfy $\tilde{m} > \check{m}$; in turn, if $\hat{m}$ exists, $\hat{m} > \tilde{m}$. The target $\tilde{m}$ is guaranteed to exist when the log growth impatience condition is satisfied (see below). For our purposes, little would be gained by an analysis of this point parallel to those of the other points of stability; but to accommodate potential practical uses, the Econ-ARK toolkit computes the value of this point (when it exists) as `mBalLog`.

[16] Our choice to call to this the individual problem's 'individual balanced-growth pseudo-steady-state' $\check{m}$ is motivated by what happens in the case where all draws of all future shocks just happen to take on their expected value of 1.0. (They unexpectedly always take on their expected values). In that infinitely improbable case, the economy *would* exhibit balanced growth:

$$
\mathbb{E}_t[m_{t+1}/m_t|\psi_{t+1} = \xi_{t+1} = 1] = G\left(\check{m} - \mathrm{c}(\check{m})\tilde{R} + 1\right)/m = G.
$$

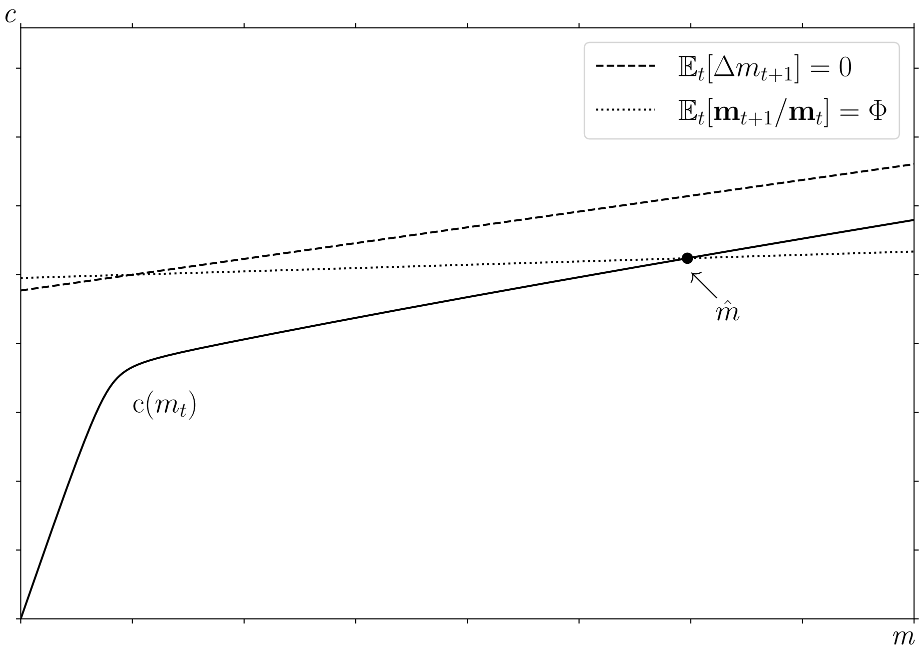

Figure 2 {FVAC,GIC,~~GIC-Mod~~}: No Target Exists But SS Does

exists a unique pseudo-steady-state market resources to permanent income ratio $\check{m} > 0$ *such that:*

$$\mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{29}$$

Moreover, $\check{m}$ *is a point of stability in the sense that:*

$$\begin{aligned} \forall m_t \in (0, \check{m}),\ \mathbb{E}_t[\boldsymbol{m}_{t+1}]/\boldsymbol{m}_t > G \\ \forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[\boldsymbol{m}_{t+1}]/\boldsymbol{m}_t < G. \end{aligned} \tag{30}$$

2.2 Example With Balanced-Growth $\check{m}$ But No Target $\hat{m}$

Because the equations defining target and pseudo-steady-state m, (27) and (28), differ only by substitution of $\tilde{R}$ for $\bar{\tilde{R}} = \tilde{R}\,\mathbb{E}[\psi^{-1}]$, if there are no permanent shocks ($\psi \equiv 1$), the conditions are identical. For many parameterizations (e.g., under the baseline parameter values used for constructing figure 1), $\hat{m}$ and $\check{m}$ will not differ much.

An illuminating exception is exhibited in Figure 2, which modifies the baseline parameter values by quadrupling the variance of the permanent shocks, enough to cause failure of strong growth impatience; now there is no target level of market resources $\hat{m}$. Nonetheless, the pseudo-steady-state still exists because it turns off realizations of the permanent shock. It is tempting to conclude that the reason target $\hat{m}$ does not exist is that the increase in the size of the shocks induces a precautionary motive that increases the consumer's effective patience. The interpretation is not correct because as market resources approach infinity, precautionary saving against noncapital income risk becomes negligible (as the proportion of consumption financed out of such income approaches zero). The correct explanation is more prosaic: The increase in uncertainty boosts the expected uncertainty-modified rate of return factor from $\tilde{R}$ to $\tilde{\tilde{R}} > \tilde{R}$ which reflects the fact that in the presence of uncertainty the expectation of the inverse of the growth factor increases: $\underline{G} > G$. That is, in the limit as $m \to \infty$ the increase in effective impatience reflected in $\frac{\text{Þ}}{G}\,\mathbb{E}[\psi^{-1}] < \frac{\text{Þ}}{G}$ is entirely due to the certainty-equivalence growth adjustment, not to a (limiting) change in precaution. In fact, the next section will show that an aggregate balanced growth equilibrium will exist even when realizations of the permanent shock are not turned off: The required condition for aggregate balanced growth is the regular growth impatience, which ignores the magnitude of permanent shocks, not strong growth impatience.[17]

Before we get to the formal arguments, the key insight can be understood by considering an economy that starts, at date t, with the entire population at $m_t = \check{m}$, but then evolves according to the model's assumed dynamics between t and $t + 1$. Equation (28) will still hold, so for this first period, at least, the economy will exhibit balanced growth: the growth factor for aggregate $\boldsymbol{m}$ will match the growth factor for permanent income G. It is true that there will be people for whom financial balances, b_{t+1}, where $b_{t+1} = k_{t+1}R/(G\psi_{t+1})$, are boosted by a small draw of ψ_{t+1}. However, their contribution to the *level* of the aggregate variable is given by $\boldsymbol{b}_{t+1} = b_{t+1}\psi_{t+1}$, so their b_{t+1} is reweighted by an amount that exactly unwinds that divisor-boosting. This means that it is possible for the consumption-to-permanent-income ratio for every consumer to be small enough that their market resources ratio is expected to rise, and yet for the economy as a whole to exhibit a balanced growth equilibrium with a finite aggregate balanced growth steady state $\check{M}$ (this is not numerically the same as the individual pseudo-steady-state ratio $\check{m}$ because the problem's nonlinearities have consequences when aggregated).[18]

3 Aggregate Invariant Relationships

In this section, we move from characterizing the individual decision rule to properties of a distribution of individuals following the converged nondegenerate consumption rule c. Assume a continuum of *ex ante* identical buffer-stock households, with constant total mass normalized to one and indexed by i. Szeidl (2013) proved that such a population, following the consumption rule c, will be characterized by invariant distributions of m,

[17]Szeidl (2013)'s impatience condition, discussed below, also tightens as uncertainty increases, but this is also not a consequence of a precaution-induced increase in patience – it represents an increase in the tightness of the requirements of the 'mixing condition' used in his proof.

[18]Still, the pseudo-steady-state can be calculated from the policy function without any simulation, and therefore serves as a low-cost starting point for the numerical simulation process; see Harmenberg-Aggregation for an example.

c, and a under the log growth impatience condition:[19]

$$\log \frac{\text{\thorn}}{G} < \mathbb{E}[\log \psi] \tag{31}$$

which is stronger than our growth impatience ($\frac{\text{\thorn}}{G} < 1$), but weaker than our strong growth impatience ($\frac{\text{\thorn}}{G}\, \mathbb{E}[\psi^{-1}] < 1$).[20]

Harmenberg (2021) substitutes a clever change of probability-measure into Szeidl's proof, with the implication that under growth impatience, invariant *permanent-income-weighted* distributions of m and c exist. In particular, let $\mathcal{F}_{m_t,\mathbf{p}_t}$ be the joint CDF of normalized market resources and permanent income at time t.[21] The permanent-income-weighted CDF of m_t, $\bar{\mathcal{F}}_{m_t}$, will be:

$$\bar{\mathcal{F}}_{m_t}(x) = G^{-t} \int_0^x \int_0^\infty \mathbf{p} \mathcal{F}_{m_t,\mathbf{p}_t}(dm, d\mathbf{p}) \tag{32}$$

Simply put, the permanent-income-weighted CDF shows how the total 'mass' of permanent income is distributed along normalized market resources.[22] The change of variables allows Harmenberg (2021) to prove a conjecture from an earlier draft of this paper (Carroll (2019, Submitted)) that under growth impatience, aggregate consumption grows at the same rate G as aggregate noncapital income in the long run (with the corollary that aggregate assets and market resources grow at that same rate). Harmenberg (2021) also shows how the reformulation can reduce costs of calculation by over a factor of 100.[23] The remainder of this section draws out the implications of these points for aggregate balanced growth factors.

---

[19] Szeidl (2013)'s equation (9), in our notation, is:

$$\begin{aligned} \mathbb{E} \log R(1-\kappa) &< \mathbb{E} \log G\psi \\ \mathbb{E} \log R\frac{\text{\thorn}}{R} &< \mathbb{E} \log G\psi \\ \log \frac{\text{\thorn}}{G} &< \mathbb{E} \log \psi \end{aligned}$$

which, exponentiated, yields (31).

[20] Under our default (though not required) assumption that $\log \psi \sim \mathcal{N}(-\sigma_\psi^2/2, \sigma_\psi^2)$; strong growth impatience in this case, is $\frac{\text{\thorn}}{G} < \exp(-\sigma^2)$, so if strong growth impatience holds then Szeidl's condition will hold.

[21] In the notation in Harmenberg (2021), the *permanent-income-weighted* measures are denoted as $\tilde{\psi}^m$.

[22] The change of variables is analogous to weighting the mass of objects by coordinates and integrating to calculate the center of gravity. Wolf and Shanker (2021) also use a similar approach to compare the relative dependence on labour and capital income across the wealth distribution.

[23] The Harmenberg method is implemented in the Econ-ARK; see the last part of `test_Harmenbergs_method.sh`. Confirming the computational advantage of Harmenberg's method, this notebook finds that the Harmenberg method reduces the simulation size required for a given degree of accuracy by two orders of magnitude under the baseline parameter values defined above.

3.1 Aggregate Balanced Growth of Income, Consumption, and Wealth

Define $\mathbb{M}$ to yield the expected value operator with respect to the empirical distribution of a variable across the population (as distinct from the operator $\mathbb{E}$ which represents beliefs about the future for a given individual).[24] Using boldface capitals for aggregates, the growth factor for aggregate noncapital income becomes:

$$\mathbf{Y}_{t+1}/\mathbf{Y}_t = \mathbb{M}\left[\xi_{t+1}G\psi_{t+1}\mathbf{p}_t\right]/\mathbb{M}\left[\mathbf{p}_t\xi_t\right] = G$$

because of the independence assumptions we have made about the shocks ξ and ψ.

Consider an economy that satisfies the Szeidl impatience condition (31) and has existed for long enough by date t that we can consider it as Szeidl-converged. In such an economy a microeconomist with a population-representative panel dataset could calculate the growth factor of consumption for each individual household, and take the average:

$$\begin{aligned}\mathbb{M}\left[\Delta\log \boldsymbol{c}_{t+1}\right] &= \mathbb{M}\left[\log c_{t+1}\mathbf{p}_{t+1} - \log c_t\mathbf{p}_t\right] \\ &= \mathbb{M}\left[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t\right] + \mathbb{M}\left[\log c_{t+1} - \log c_t\right].\end{aligned} \tag{33}$$

Because this economy is Szeidl-converged, distributions of c_t and c_{t+1} will be identical, so that the second term in (33) disappears; hence, mean cross-sectional growth factors of consumption and permanent income are the same:

$$\mathbb{M}\left[\Delta\log \boldsymbol{c}_{t+1}\right] = \mathbb{M}\left[\Delta\log \mathbf{p}_{t+1}\right] = \log G. \tag{34}$$

In a Harmenberg-invariant economy (and therefore also any Szeidl-invariant economy), a similar proposition holds in the cross-section as a direct implication of the fact that a constant proportion of total permanent income is accounted for by the successive sets of consumers with any particular m (recall Equation (32)). This fact is one way of interpreting Harmenberg's definition of the density of the permanent-income-weighted invariant distribution of m; call this density $\bar{f}$. To understand $\bar{f}$, we can see how total aggregate market resources held by people with given m will be:

$$\boldsymbol{M}_t = \boldsymbol{P}_t\bar{f}(m)m \tag{35}$$

By implication of Theorem 4, $\boldsymbol{M}_t$ grows at a rate G. We will now use this property of $\bar{f}$ to show that aggregate consumption also grows at rate G. Call $\boldsymbol{C}_t(m)$ the total amount of consumption at date t by persons with market resources m, and note that in the invariant economy this is given by the converged consumption function $\mathrm{c}(m)$ multiplied by the amount of permanent income accruing to such people $\bar{f}(m)\boldsymbol{P}_t$. Since $\bar{f}(m)$ is invariant and aggregate permanent income grows according to $\boldsymbol{P}_{t+1} = G\boldsymbol{P}_t$, for any m,

[24] Formally, fix an individual i and let $\{\tilde{c}^i_t\}_{t=0}^{\infty}$ and $\{\tilde{m}^i_t\}_{t=0}^{\infty}$ be a stochastic recursive sequence generated by the converged consumption rule as follows, $\tilde{c}^i_t = \mathrm{c}(\tilde{m}^i_t)$ and $\tilde{m}^i_{t+1} = \tilde{R}^i_{t+1}(\tilde{m}^i_t - \mathrm{c}(\tilde{m}^i_t)) + \xi^i_{t+1}$, where the sequence of exogenous shocks are each defined on a *theoretical probability space* $(\Omega, \Sigma, \mathbb{P})$. Integration with respect to the measure $\mathbb{P}$ in the expected value operator $\mathbb{E}$ will be equivalent to *empirical* integration $\mathbb{M}$ with respect to a suitable measure of agents on a nonatomic agent space. In particular, for all j, $\mathbb{E}\,\mathrm{g}(\tilde{c}^j_t) = \int \tilde{c}_t\, d\mathbb{P} = \mathbb{M}\mathrm{g}(\tilde{c}_t) :\, = \int \mathrm{g}(\tilde{c}^i_t)\lambda(di)$, where λ is the measure of agents and for any measurable function g. For technical steps required to assert this claim, see Shanker (2017), which utilizes relatively recent results by Sun and Zhang (2009) and also the detailed construction by Cao (2020).

the following characterizes the growth of total consumption:

$$\begin{aligned}\log \boldsymbol{C}_{t+1}(m) - \log \boldsymbol{C}_t(m) &= \log \mathrm{c}(m)\bar{f}(m)\boldsymbol{P}_{t+1} - \log \mathrm{c}(m)\bar{f}(m)\boldsymbol{P}_t \\ &= \log G.\end{aligned}$$

3.2 Aggregate Balanced Growth and Idiosyncratic Covariances

Harmenberg shows that the covariance between the individual consumption ratio c and the idiosyncratic component of permanent income $\mathbf{p}$ does not shrink to zero; thus, covariances are another potential measurement for construction of microfoundations.

Consider a date-t Harmenberg-converged economy, and define the mean value of the consumption ratio as $\mathfrak{c}_{t+n} \equiv \mathbb{M}[c_{t+n}]$. Normalizing period-$t$ aggregate permanent income to $\boldsymbol{P}_t = 1$, total consumption at $t+1$ and $t+2$ are

$$\begin{aligned}\mathbf{C}_{t+1} = \mathbb{M}[c_{t+1}\mathbf{p}_{t+1}] = \mathfrak{c}_{t+1}G^1 + \mathrm{cov}_{t+1}(c_{t+1}, \mathbf{p}_{t+1}) \\ \mathbf{C}_{t+2} = \mathbb{M}[c_{t+2}\mathbf{p}_{t+2}] = \mathfrak{c}_{t+2}G^2 + \mathrm{cov}_{t+2}(c_{t+2}, \mathbf{p}_{t+2})\end{aligned} \tag{36}$$

and Harmenberg's proof that $\mathbf{C}_{t+2} - G\mathbf{C}_{t+1} = 0$ allows us to obtain:

$$\left(\mathfrak{c}_{t+2} - \mathfrak{c}_{t+1}\right) G^2 = G\mathrm{cov}_{t+1} - \mathrm{cov}_{t+2}. \tag{37}$$

In a Szeidl-invariant economy, $\mathfrak{c}_{t+2} = \mathfrak{c}_{t+1}$, so the economy exhibits balanced growth in the covariance:

$$\mathrm{cov}_{t+2} = G\mathrm{cov}_{t+1}. \tag{38}$$

The more interesting case is when the economy is Harmenberg- but not Szeidl-invariant. In that case, if the cov and the $\mathfrak{c}$ terms have constant growth factors Ω_{cov} and $\Omega_{\mathfrak{c}}$,[25] an equation corresponding to (37) will hold in $t+n$:

$$\begin{aligned}(\overbrace{\Omega^n_{\mathfrak{c}}\mathfrak{c}_t}^{\mathfrak{c}_{t+n}} - \Omega^{n-1}_{\mathfrak{c}}\mathfrak{c}_t)G^n &= \left(G\Omega^{n-1}_{\mathrm{cov}} - \Omega^n_{\mathrm{cov}}\right)\mathrm{cov}_t \\ (\Omega_{\mathfrak{c}}G)^{n-1}(\Omega_{\mathfrak{c}} - 1)\mathfrak{c}_t G &= \Omega^{n-1}_{\mathrm{cov}}(G - \Omega_{\mathrm{cov}})\mathrm{cov}_t\end{aligned} \tag{39}$$

so for the LHS and RHS to grow at the same rates we need

$$\Omega_{\mathrm{cov}} = \Omega_{\mathfrak{c}}G. \tag{40}$$

This is intuitive: In the Szeidl-invariant economy, it just reproduces our result above that the covariance exhibits balanced growth because $\Omega_{\mathfrak{c}} = 1$. The revised result just says that in the Harmenberg case where the mean value $\mathfrak{c}$ of the consumption ratio c can grow, the covariance must rise in proportion to any ongoing expansion of $\mathfrak{c}$ (as well as in proportion to the growth in $\mathbf{p}$).

[25]This 'if' is a conjecture, not something proven by Harmenberg (or anyone else). But see appendix F for an example of a Harmenberg-invariant economy in which simulations suggest this proposition holds.

3.3 Implications for Microfoundations

Thus we have microeconomic propositions, for both growth factors and for covariances of observable variables,[26] that can be tested in either cross-section or panel microdata to judge (and calibrate) the microfoundations that should hold for any macroeconomic analysis that requires balanced growth for its conclusions.

At first blush, these points are reassuring; one of the most persuasive arguments for the agenda of building microfoundations of macroeconomics is that newly available 'big data' allow us to measure cross-sectional covariances with great precision, so that we can use microeconomic natural experiments to disentangle questions that are hopelessly entangled in aggregate time-series data. Knowing that such covariances ought to be a stable feature of a stably growing economy is therefore encouraging.

But this discussion also highlights an uncomfortable point: In the model as specified, permanent income does not have a limiting distribution; it becomes ever more dispersed as the economy with infinite-horizon consumers continues to exist indefinitely.

A few microeconomic data sources attempt direct measurement of 'permanent income'; Carroll, Slacalek, Tokuoka, and White (2017), for example, show that their assumptions about the magnitude of permanent shocks (and mortality; see below) yield a simulated distribution of permanent income that roughly matches answers in the U.S. *Survey of Consumer Finances* ('SCF') to a question designed to elicit a direct measure of respondents' permanent income. They use those results to calibrate a model to match empirical facts about the distribution of permanent income and wealth, showing that the model also does fits empirical facts about the marginal propensity to consume. The quantitative credibility of the argument depends on the model's match to the distribution of permanent income inequality, which would not be possible in a model without a nondegenerate steady-state distribution of permanent income.

For macroeconomists who want to build microfoundations by comparing the microeconomic implications of their models to micro data (directly – not in ratios to difficult-to-meaure 'permanent income'), it would be something of a challenge to determine how to construct empirical-data-comparable simulated results from a model with no limiting distribution of permanent income.

Death can solve this problem.

3.4 Mortality Yields Invariance

Most heterogeneous-agent models incorporate a constant positive probability of death, following Blanchard (1985) and Yaari (1965). In the Blanchardian model, if the probability of death exceeds a threshold that depends on the size of the permanent shocks, Carroll, Slacalek, Tokuoka, and White (2017) show that the limiting distribution of permanent income has a finite variance. Blanchard (1985) assumes a universal annuitization scheme in which estates of dying consumers are redistributed to survivors in proportion to survivors' wealth, giving the recipients a higher effective rate of return. This treatment has considerable analytical advantages, most notably that the effect of mortality on the

[26] Parallel results to those for consumption can be obtained for other measures like market assets.

time preference factor is the exact inverse of its effect on the (effective) interest factor. That is, if the 'pure' time preference factor is β and probability of remaining alive (not dead) is $\mathcal{L}$, then the assumption that no utility accrues after death makes the effective discount factor $\underline{\beta} = \beta\mathcal{L}$ while the enhancement to the rate of return from the annuity scheme yields an effective interest factor $\bar{R} = R/\mathcal{L}$ (recall that because of white-noise mortality, the average wealth of the two groups is identical). Combining these, the effective patience factor in the new economy $\underline{\beta}\bar{R}$ is unchanged from its value in the infinite-horizon model:

$$\underline{\beta}\bar{R} = (\beta\mathcal{L}R/\mathcal{L})^{1/\gamma} = (R\beta)^{1/\gamma} = \text{Þ}. \tag{41}$$

The only adjustments this requires to the analysis above are therefore to the few elements that involve a role for the interest factor distinct from its contribution to **Þ** (principally, the RIC, which becomes $\text{Þ}/\bar{R}$).

Blanchard (1985)'s innovation was valuable not only for the insight it provided but also because when he wrote, the principal alternative, the Life Cycle model of Modigliani (1966), was computationally challenging given then-available technologies. Despite its (considerable) conceptual value, Blanchard's analytical solution is now rarely used because essentially all modern modeling incorporates uncertainty, constraints, and other features that rule out analytical solutions anyway.

The simplest alternative to Blanchard is to follow Modigliani in constructing a realistic description of income over the life cycle and assuming that any wealth remaining at death occurs accidentally (not implausible, given the robust finding that for the great majority of households, bequests amount to less than 2 percent of lifetime earnings, Hendricks (2001, 2016)).

Even if bequests are accidental, a macroeconomic model must make some assumption about how they are disposed of: As windfalls to heirs, estate tax proceeds, etc. We again consider the simplest choice, because it represents something of a polar alternative to Blanchard. Without a bequest motive, there are no behavioral effects of a 100 percent estate tax; we assume such a tax is imposed and that the revenues are effectively thrown in the ocean: The estate-related wealth effectively vanishes from the economy.

The chief appeal of this approach is the simplicity of the change it makes in the condition required for the economy to exhibit a balanced growth equilibrium (for consumers without a life cycle income profile). If $\mathcal{L}$ is the probability of remaining alive, the condition changes from the plain growth impatience to a looser mortality-adjusted version of growth impatience:

$$\mathcal{L}\text{Þ}_G < 1. \tag{42}$$

With no income growth, what is required to prohibit unbounded growth in aggregate wealth is the condition that prevents the per-capita wealth-to-permanent-income ratio of surviving consumers from growing faster than the rate at which mortality diminishes their collective population. With income growth, the aggregate wealth-to-income ratio will head to infinity only if a cohort of consumers is patient enough to make the desired rate of growth of wealth fast enough to counteract combined erosive forces of mortality and productivity.

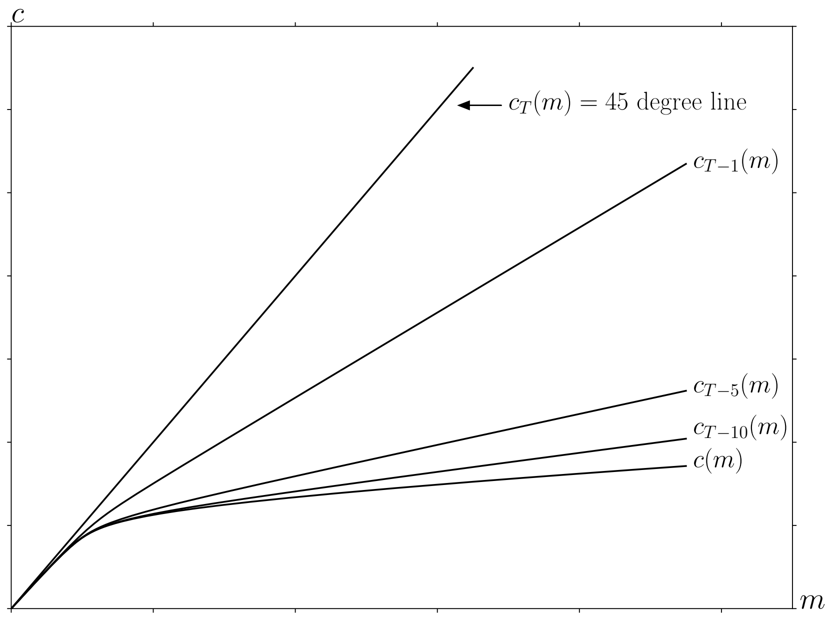

Figure 3 Convergence of the Consumption Rules

4 Patience and Limiting Consumer Behavior

Having established our formal results, we turn to how the variety of patience conditions influence the limiting consumption function. To fix ideas, we start with a quantitative example using the familiar benchmark case under both return impatience, growth impatience and finite human wealth, shown by Figure 3. The figure depicts the successive consumption rules that apply in the last period of life (c_T), the second-to-last period, and earlier periods under parameter values listed in Table 2 below. (The 45 degree line is $\mathrm{c}_T(m) = m$ because in the last period of life it is optimal to spend all remaining resources.)

Figures 4–5 capture the theoretical bounds and MPCs of the converged consumption rule when weak return impatience and strong growth impatience both hold (under the parameter values in Table 2). In Figure 4, as m rises, the marginal propensity to consume approaches $\underline{\kappa} = (1 - \frac{\text{Þ}}{R})$ as $m \to \infty$, the same as the perfect foresight MPC. Moreover, as m approaches zero, the MPC approaches $\bar{\kappa} = (1 - q^{1/\gamma}\frac{\text{Þ}}{R})$.

While neither return impatience nor growth impatience is necessary for nondegeneracy of $\mathrm{c}(m)$ in the presence of a constraint, a key argument of this section is that if both return impatience and growth impatience jointly fail, the consumption function *will* be degenerate (limiting either to $\mathrm{c}(m) = 0$ or $c(m) = \infty$ as the horizon recedes). So, for a useful solution, at least one of these conditions must hold (recall Claim 1). The case with growth impatience but return *patience* is particularly surprising, because it

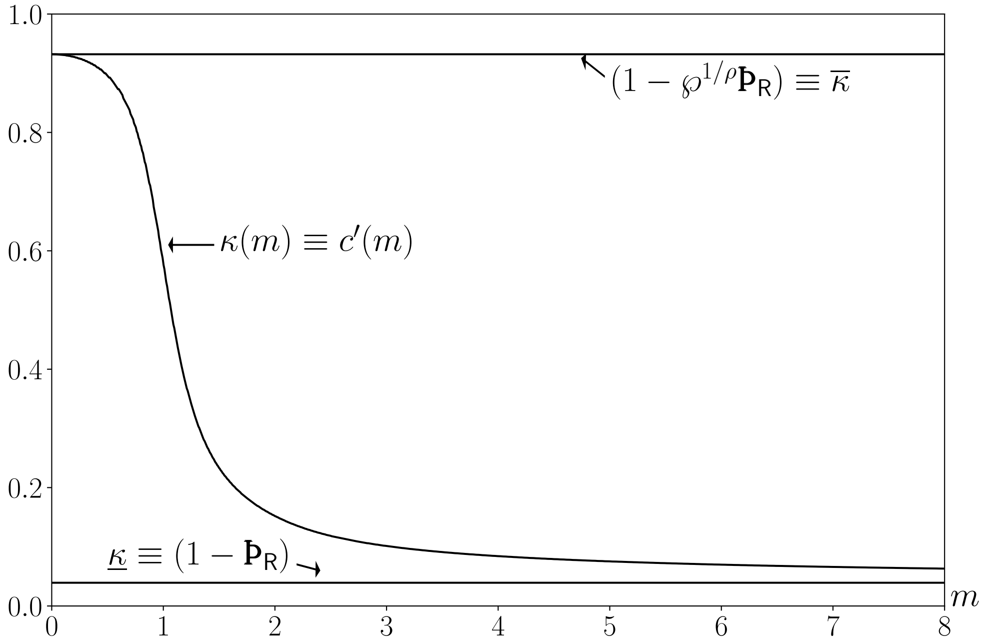

Figure 4 Limiting MPC's

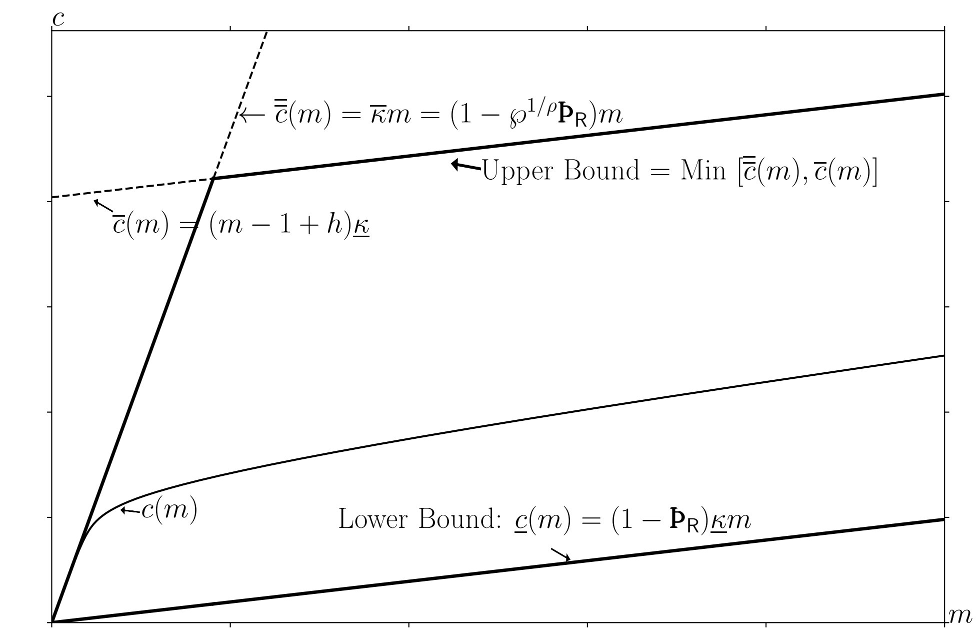

Figure 5 Upper and Lower Bounds on the Consumption Function

is not immediately clear what prevents our earlier conclusion that return patience in other circumstances leads $c(m)$ to asymptote to zero. The trick is to note that if return patience holds, $R > \text{Þ}$ while failure of growth impatience means $G < \text{Þ}$, which together let us conclude that (limiting) human wealth is infinite.[27] But, if human wealth is unbounded, what prevents c from asymptoting to $c(m) = \infty$? This is where the natural borrowing constraint comes in. It turns out that growth impatience is sufficient, at any fixed m, to guarantee an upper bound to $c(m)$. In the absence of return impatience in the presence of a borrowing constraint (either natural or artificial). The insight is best understood by first abstracting from uncertainty and studying the perfect foresight case (with and without constraints).

4.1 Model with Perfect Foresight

Recall Claims 1-2, which established the relationship between the finite value of autarky, return impatience and growth impatience in the context of a model with uncertainty. The easiest way to grasp the relations among these conditions is by studying Figure 6. Each node represents a quantity defined above. The arrow associated with each inequality imposes that condition. For example, one way we wrote the finite value of autarky (under perfect foresight) in Equation (9) is $\text{Þ} < R^{1/\gamma}G^{1-1/\gamma}$, so imposition of finite value of autarky is captured by the diagonal arrow connecting Þ and $R^{1/\gamma}G^{1-1/\gamma}$. Traversing the boundary of the diagram clockwise starting at Þ involves imposing first growth impatience then the finite human wealth, and the consequent arrival at the bottom right node tells us that these two conditions jointly imply perfect foresight finite value of autarky. Reversal of a condition reverses the arrow's direction; so, for example, the bottom-most arrow going to $R^{1/\gamma}G^{1-1/\gamma}$ implies finite human wealth fails; but we can cancel the cancellation and reverse the arrow. This would allow us to traverse the diagram clockwise from Þ through G to $R^{1/\gamma}G^{1-1/\gamma}$ to R, revealing that imposition of growth impatience and finite human wealth (and, redundantly, finite human wealth again) let us conclude that return impatience holds because the starting point is Þ and the endpoint is R. (Consult Appendix E for an exposition of diagrams of this type, which are a simple application of Category Theory (Riehl (2017).)

In the unconstrained case, we saw how finite human wealth was necessary since, without constraints, only this condition could prevent infinite borrowing in the limit (recall Proposition 1 preceding discussion). Looking at Figure 6, following the diagonal from Þ to the bottom-right corner corresponds to the direct of imposition of the finite value of autarky, which implies that the existence of a non-degenerate solution *requires* return impatience to hold. To see why, if return impatience failed, proceeding clock-wise from R would lead to $R > R^{1/\gamma}G^{1-1/\gamma}$, (equivalently $(G/R)^{1-1/\gamma} < 1$) which corresponds to failure of finite human wealth (see also Case 3 in Section 4.2.1).

We can alternatively understand how failure of finite human wealth leads to infinite borrowing from the point of view of growth impatience. From Figure 6, let finite value

[27]This logic holds even if both R and G are less than one – in this case, because the agent can *borrow* at a negative interest rate and always repay with income that shrinks more slowly than their debt.

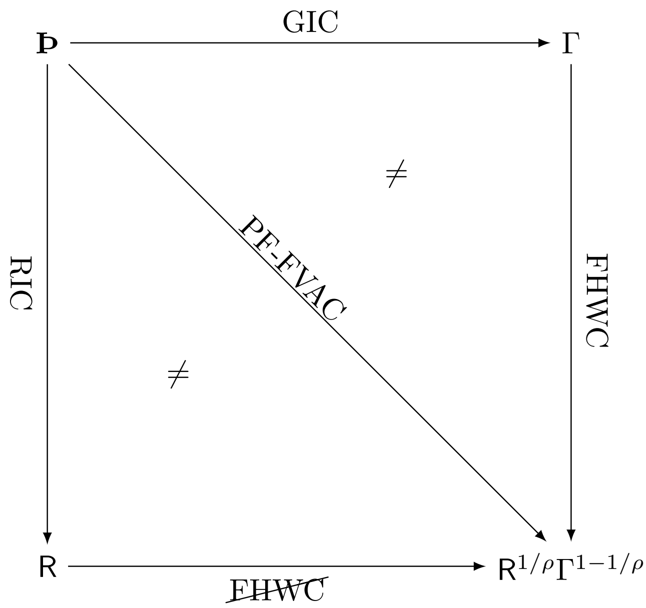

Figure 6 Perfect Foresight Relation of GIC, FHWC, RIC, and PFFVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\text{Þ} < R^{1/\gamma}G^{1-1/\gamma}$, which is one way of writing the PF-FVAC, equation (9)

of autarky hold (traverse the diagonal from **Þ**) and then reverse the downward arrow from G, signifying the failure of finite human wealth. The resulting chain of inequalities then gives us growth impatience ($\text{Þ} > G$). But, under growth impatience, as the horizon extends and income grows faster than the rate at which it is discounted, there is no upper bound to the present discounted value of future income (cf. Equation (16)). And therefore, at any fixed level of market resources, there is no upper bound to how much the consumer can and would wish to borrow as the horizon recedes.

Thus, return impatience is the only condition at our disposal that can prevent consumption from limiting to zero as the planning horizon recedes. However, when we impose a liquidity constraint, the range of admissible parameters becomes more interesting.

4.1.1 Perfect Foresight Constrained Solution

We now sketch the perfect foresight constrained solution and demonstrate a solution can exist either under return impatience or without return impatience but with growth impatience (Proposition 2). Our discussion proceeds by examining implications of possible configurations of the patience conditions. (Tables 3 and 4 codify.)

Case 1: Growth impatience fails and return impatience holds. If growth impatience fails but return impatience holds, Appendix C shows that, for some $m_{\#}$, with $0 < m_{\#} < 1$, an unconstrained consumer behaving according to the perfect foresight solution (14) would choose $c < m$ for all $m > m_{\#}$. In this case the solution to the constrained consumer's problem is simple; for any $m \geq m_{\#}$ the constraint does not bind (and will never bind in the future); for such m the constrained consumption function is identical to the unconstrained one. If the consumer were somehow[28] to arrive at an $m_{\#}$ such that $m < m_{\#} < 1$ the constraint would bind and the consumer would consume $c = m$. Using $\grave{\mathrm{c}}$ for the perfect foresight consumption function in the presence of constraints (and analogously for all other functions):

$$\grave{\mathrm{c}}(m) = \begin{cases} m & \text{if } m < m_{\#} \\ \bar{\mathrm{c}}(m) & \text{if } m \geq m_{\#} \end{cases}$$

where $\bar{\mathrm{c}}(m)$ is the unconstrained perfect foresight solution.

Case 2: Growth impatience holds and return impatience holds. When return impatience and growth impatience both hold, Appendix C shows that the limiting constrained consumption function is piecewise linear, with $\grave{\mathrm{c}}(m) = m$ up to a first 'kink point' at $m_{\#}^0 > 1$, and with discrete declines in the MPC at a set of kink points $\{m_{\#}^1, m_{\#}^2, \ldots\}$. As $m \to \infty$ the constrained consumption function $\grave{\mathrm{c}}(m)$ becomes arbitrarily close to the unconstrained $\bar{\mathrm{c}}(m)$, and the marginal propensity to consume, $\grave{\mathrm{c}}'(m)$, limits to $\underline{\kappa}$.[29] Similarly, the value function $\grave{\mathrm{v}}(m)$ is nondegenerate and limits to the value function of the unconstrained consumer.

This logic holds even when finite human wealth fails, because the constraint prevents the (limiting) consumer[30] from borrowing against unbounded human wealth to finance unbounded current consumption. Under these circumstances, the consumer who starts with any $b_t > 0$ will, over time, run those resources down so that after some finite number of periods τ the consumer will reach $b_{t+\tau} = 0$, and thereafter will set $\boldsymbol{c} = \mathbf{p}$ for eternity (which finite value of autarky says yields finite value). Using the same steps as for Equation (96), value of the interim program is also finite:

$$\boldsymbol{v}_{t+\tau} = G^{\tau(1-\gamma)}\mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta G^{1-\gamma})^{T-(t+\tau)+1}}{1-\beta G^{1-\gamma}}\right).$$

So, even when finite human wealth fails, the limiting consumer's value for any finite m will be the sum of two finite numbers: One due to the unconstrained choice made over the finite-horizon leading up to $b_{t+\tau} = 0$, and one reflecting the value of consuming $\mathbf{p}_{t+\tau}$ thereafter.

Case 3: Growth impatience holds and return impatience fails. The most peculiar possibility occurs when return impatience fails. As already discussed above, this possibility is unavailable to us without a constraint. Under this case finite human wealth must also fail (Appendix C), and the constrained consumption function is nondegenerate. (See appendix Figure 8 for a numerical example). Even though human wealth is unbounded

[28]"Somehow" because $m < 1$ could only be obtained by entering the period with $b < 0$ which the constraint forbids.

[29]See Carroll, Holm, and Kimball (2019) for details.

[30]That is, one obeying $\mathrm{c}(m) = \lim_{n \to \infty} \mathrm{c}_{t-n}(m)$.

Table 1 Microeconomic Model Calibration

Calibrated Parameters			
Description	Parameter	Value	Source
Permanent Income Growth Factor	G	1.03	PSID: Carroll (1992)
Interest Factor	R	1.04	Conventional
Time Preference Factor	β	0.96	Conventional
Coefficient of Relative Risk Aversion	γ	2	Conventional
Probability of Zero Income	q	0.005	PSID: Carroll (1992)
Std Dev of Log Permanent Shock	σ_ψ	0.1	PSID: Carroll (1992)
Std Dev of Log Transitory Shock	σ_θ	0.1	PSID: Carroll (1992)

at any given level of m, since borrowing is ruled out, consumption cannot become unbounded in the limit as the horizon recedes. However, due to failure of return impatience, $\lim_{m\to\infty} \grave{c}'(m) = 0$. Nevertheless the limiting constrained consumption function $\grave{c}(m)$ is finite, strictly positive, and strictly increasing in m. This result reconciles the conflicting intuitions from the unconstrained case, where failure of return impatience would suggest a degenerate limit of $\grave{c}(m) = 0$ while failure of finite human wealth would suggest a degenerate limit of $\grave{c}(m) = \infty$.

Regarding the case when both return impatience and growth impatience fail, finite value of autarky no longer holds. Even when there is a borrowing constraint, Appendix C.1 demonstrates how a non-degenerate solution cannot exist since human wealth become infinite and at the same time the marginal propensity to consume limits to zero. In this case, the limiting degenerate solution depends on the balance of the 'speeds' at which these limits converge.

4.2 Model with Uncertainty

We now examine the case with uncertainty but without constraints, which we argued was close parallel to the model with constraints but without uncertainty (recall Section 1.4.3).

Tables 1 and 2 present calibrations and values of model conditions in the case with uncertainty, where return impatience, growth impatience and finite value of autarky all hold. The full relationship among conditions is represented in Figure 7. Though the diagram looks complex, it is merely a modified version of the earlier simple diagram (Figure 6) with further (mostly intermediate) inequalities inserted. (Arrows with a "because" now label relations that always hold under the model's assumptions.)[31]

The 'weakness' of the additional condition sufficient for contraction beyond finite value of autarky, weak return impatience, can be seen by asking 'under what circumstances would the finite value of autarky hold but the weak return impatience fail?' Algebraically,

[31] Again, readers unfamiliar with such diagrams should see Appendix E for a more detailed exposition.

Table 2 Model Characteristics Calculated from Parameters

Description	Symbol and Formula			Approximate Calculated Value
Finite Human Wealth Factor	$\tilde{R}^{-1}$	$\equiv$	G/R	0.990
PF Value of Autarky Factor	$\daleth$	$\equiv$	$\beta G^{1-\gamma}$	0.932
Growth Compensated Permanent Shock	$\underline{\boldsymbol{\Psi}}$	$\equiv$	$(\mathbb{E}[\boldsymbol{\Psi}^{-1}])^{-1}$	0.990
Uncertainty-Adjusted Growth	$\underline{G}$	$\equiv$	$G\underline{\boldsymbol{\Psi}}$	1.020
Utility Compensated Permanent Shock	$\underline{\underline{\boldsymbol{\Psi}}}$	$\equiv$	$(\mathbb{E}[\psi^{1-\gamma}])^{1/(1-\gamma)}$	0.990
Utility Compensated Growth	$\underline{\underline{G}}$	$\equiv$	$G\underline{\underline{\boldsymbol{\Psi}}}$	1.020
Absolute Patience Factor	Þ	$\equiv$	$(R\beta)^{1/\gamma}$	0.999
Return Patience Factor	$\frac{\text{Þ}}{R}$	$\equiv$	$\text{Þ}/R$	0.961
Growth Patience Factor	$\frac{\text{Þ}}{G}$	$\equiv$	$\text{Þ}/G$	0.970
Modified Growth Patience Factor	$\frac{\text{Þ}}{G}\,\mathbb{E}[\psi^{-1}]$	$\equiv$	$\text{Þ}/\underline{G}$	0.980
Value of Autarky Factor	$\underline{\underline{\daleth}}$	$\equiv$	$\beta G^{1-\gamma}\underline{\underline{\boldsymbol{\Psi}}}^{1-\gamma}$	0.941
Weak Return Impatience Factor	$q^{1/\gamma}\text{Þ}$	$\equiv$	$(q\beta R)^{1/\gamma}$	0.071

the requirement becomes:

$$\beta G^{1-\gamma}\underline{\underline{\boldsymbol{\Psi}}}^{1-\gamma} < \ 1 \ < (q\beta)^{1/\gamma}/R^{1-1/\gamma}. \tag{43}$$

where $\underline{\underline{\boldsymbol{\Psi}}}: \ = (\mathbb{E}[\psi^{1-\gamma}])^{1/(1-\gamma)}$. If we require $R \geq 1$, the weak return impatience is 'redundant' because now $\beta < 1 < R^{\gamma-1}$, so that (with $\gamma > 1$ and $\beta < 1$) the return impatience (and weak return impatience) must hold. But neither theory nor evidence demand that $R \geq 1$. We can therefore approach the question of the relevance of weak return impatience by asking just how low R must be for the condition to be relevant. Suppose for illustration that $\gamma = 2$, $\underline{\underline{\boldsymbol{\Psi}}}^{1-\gamma} = 1.01$, $G^{1-\gamma} = 1.01^{-1}$ and $q = 0.10$. In that case (43) reduces to:

$$\beta < 1 < (0.1\beta/R)^{1/2},$$

but since $\beta < 1$ by assumption, the binding requirement becomes:

$$R < \beta/10,$$

so that for example if $\beta = 0.96$ we would need $R < 0.096$ (that is, a perpetual riskfree rate of return of worse than -90 percent a year) in order for weak return impatience to be relevant.

Perhaps the best way of thinking about this is to note that the space of parameter values for which the weak return impatience remains relevant shrinks out of existence as $q \to 0$, which Section 1.4.3 showed was the precise limiting condition under which behavior becomes arbitrarily close to the liquidity constrained solution (in the absence of other risks). On the other hand, when $q = 1$, the consumer has no noncapital income (so finite human wealth holds) and with $q = 1$ weak return impatience is identical to the

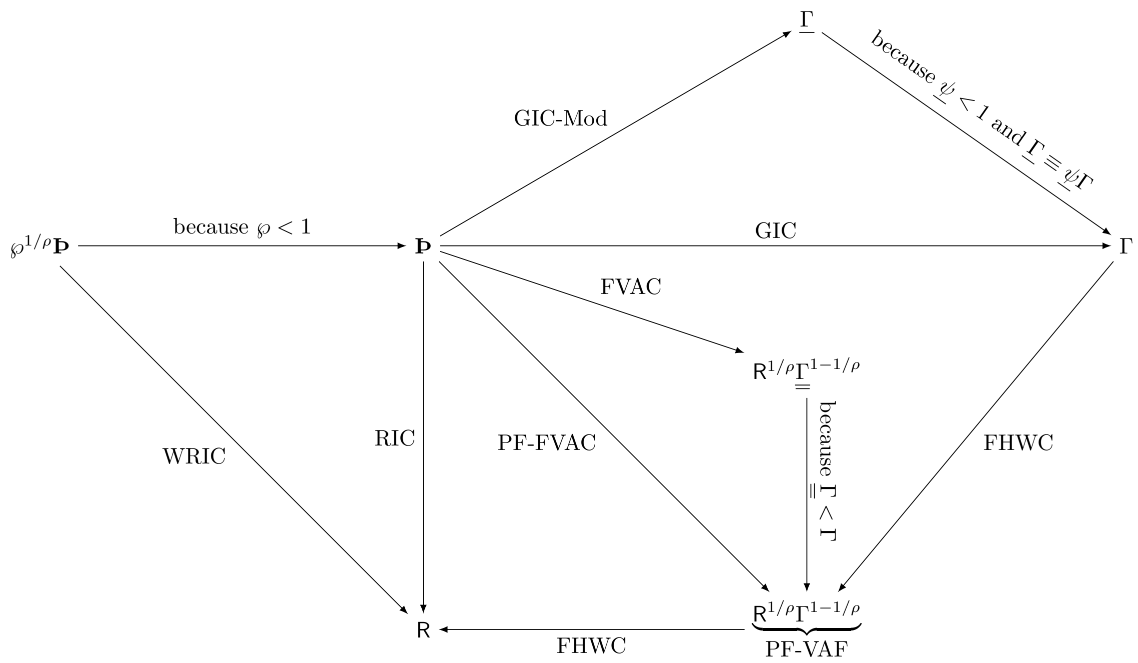

Figure 7 Relation of All Inequality Conditions
See Table 2 for Numerical Values of Nodes Under Baseline Parameters

weak return impatience. However, weak return impatience is the only condition required for a solution to exist for a perfect foresight consumer with no noncapital income. Thus weak return impatience forms a sort of 'bridge' between the liquidity constrained and the unconstrained problems as q moves from 0 to 1.

4.2.1 Behavior Under Cases of Conditions

Case 1: Return impatience fails and growth impatience holds In the unconstrained perfect foresight problem (Section 4.1), the return impatience was necessary for existence of a nondegenerate solution. It is surprising, therefore, that in the presence of uncertainty, the much weaker weak return impatience is sufficient for nondegeneracy (assuming that the finite value of autarky holds). Given finite value of autarky, we can derive the features the problem must exhibit for return impatience to fail (that is, $R < (R\beta)^{1/\gamma}$) as follows:

$$\begin{aligned} R < (R\beta)^{1/\gamma} \ &< \ \left(R(G\underline{\underline{\mathbf{\Psi}}})^{\gamma-1}\right)^{1/\gamma} \\ \Rightarrow R &< (R/G)^{1/\gamma} G\underline{\underline{\mathbf{\Psi}}}^{1-1/\gamma} \\ \Rightarrow R/G &< \underline{\underline{\mathbf{\Psi}}} \end{aligned} \tag{44}$$

but since $\underline{\underline{\Psi}} < 1$ (for $\gamma > 1$ and nondegenerate ψ), this requires $R/G < 1$. Thus, given finite value of autarky, return impatience can fail only if human wealth is unbounded and growth impatience holds.[32]

As in the perfect foresight constrained problem, unbounded limiting human wealth here does not lead to a degenerate limiting consumption function (finite human wealth is not required for Theorem 2). But, from equation (19) and the discussion surrounding it, an implication of the failure of return impatience is that $\lim_{m\to\infty} \mathrm{c}'(m) = 0$. Thus, interestingly, in this case (unavailable in the perfect foresight unconstrained model) the presence of uncertainty both permits unlimited human wealth (in the $n \to \infty$ limit) and at the same time prevents unlimited human wealth from resulting in (limiting) infinite consumption (at any finite m). Intuitively, the natural constraint that arises from the possibility of a zero income event prevents infinite borrowing and at the same time allows infinite human wealth to discipline patience, preventing $\mathrm{c}(m) = 0$ as the planning horizon recedes. Thus, in presence of uncertainty (zero income event?), pathological patience (which in the perfect foresight model results in a limiting consumption function of $\mathrm{c}(m) = 0$) plus unbounded human wealth (which the perfect foresight model prohibits because it leads to a limiting consumption function $\mathrm{c}(m) = \infty$ for any finite m) combine to yield a unique finite limiting (as $n \to \infty$) level of consumption and MPC for any finite value of m.

Note the close parallel to the conclusion in the perfect foresight liquidity constrained model in the case where return impatience fails (Case 3 in Section 4.1.1). There, too, the tension between infinite human wealth and pathological patience was resolved with a nondegenerate consumption function whose limiting MPC was zero.[33]

Case 2: Return impatience holds and growth impatience holds with finite human wealth This is the benchmark case we presented at the start of the Section. If return impatience and finite human wealth both hold, a perfect foresight solution exists (Section 4.1). As $m \to \infty$ the limiting c and v functions become arbitrarily close to those in the perfect foresight model, because human wealth pays for a vanishingly small portion of spending (Section **??**).

Case 3: Return impatience holds and growth impatience holds with infinite human wealth The more exotic case is where finite human wealth fails but both growth impatience and return impatience also hold. In the unconstrained perfect foresight model, this is the degenerate case with limiting $\bar{\mathrm{c}}(m) = \infty$. Here,infinite human wealth and finite value of autarky implies that (perfect foresight) finite value of autarky holds and that $\text{Þ} < G$. To see why, traverse Figure 7 clockwise from Þ by imposing finite value

[32] This algebraically complicated conclusion could be easily reached diagrammatically in Figure 7 by starting at the R node and imposing the failure of return impatience, which reverses the return impatience arrow and lets us traverse the diagram along any clockwise path to the perfect foresight finite value of autarky node at which point we realize that we *cannot* impose finite human wealth because that would let us conclude $R > R$.

[33] Ma and Toda (2020) derive conditions under which the limiting MPC is zero in an even more general case where there is also capital income risk.

of autarky and continue to the perfect foresight finite value of autarky node. Reversing the arrow connecting the R and perfect foresight finite value of autarky nodes implies that:

$$\text{Þ} < (R/G)^{1/\gamma} G \Rightarrow \text{Þ} < G$$

where the transition from the first to the second lines is justified because failure of finite human wealth implies $\Rightarrow (R/G)^{1/\gamma} < 1$. So, under return impatience and finite human wealth, we must have growth impatience.

However, we are not entitled to conclude that the strong growth impatience holds: $\text{Þ} < G$ does not imply $\text{Þ} < \underline{\mathbf{\Psi}} G$ where $\underline{\mathbf{\Psi}} < 1$.

We have now established the principal points of comparison between the perfect foresight solutions and the solutions under uncertainty; these are codified in the remaining parts of Tables 3 and 4.

Table 3 Definitions and Comparisons of Conditions

Perfect Foresight Versions	Uncertainty Versions
Finite Human Wealth Condition (FHWC)	
$G/R < 1$	$G/R < 1$
The growth factor for permanent income G must be smaller than the discounting factor R for human wealth to be finite.	The model's risks are mean-preserving spreads, so the PDV of future income is unchanged by their introduction.
Absolute Impatience Condition (AIC)	
$\text{Þ} < 1$	$\text{Þ} < 1$
The unconstrained consumer is sufficiently impatient that the level of consumption will be declining over time:	*If wealth is large enough,* the *expectation* of consumption next period will be smaller than this period's consumption:
$\boldsymbol{c}_{t+1} < \boldsymbol{c}_t$	$\lim_{m_t \to \infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}] < \boldsymbol{c}_t$
Return Impatience Conditions	
Return Impatience Condition (RIC)	Weak RIC (WRIC)
$\text{Þ}/R < 1$	$q^{1/\gamma}\text{Þ}/R < 1$
The growth factor for consumption Þ must be smaller than the discounting factor R, so that the PDV of current and future consumption will be finite:	If the probability of the zero-income event is $q = 1$ then income is always zero and the condition becomes identical to the RIC. Otherwise, weaker.
$\mathrm{c}'(m) = 1 - \text{Þ}/R < 1$	$\mathrm{c}'(m) < 1 - q^{1/\gamma}\text{Þ}/R < 1$
Growth Impatience Conditions	
GIC	GIC-Mod
$\text{Þ}/G < 1$	$\text{Þ}\,\mathbb{E}[\psi^{-1}]/G < 1$
For an unconstrained PF consumer, the ratio of $\boldsymbol{c}$ to $\mathbf{p}$ will fall over time. For constrained, guarantees the constraint eventually binds. Guarantees $\lim_{m_t \to \infty} m_{t+1}/m_t = \frac{\text{Þ}}{G}$	By Jensen's inequality stronger than GIC. Ensures consumers will not expect to accumulate m unboundedly. $\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \frac{\text{Þ}}{G}\mathbb{E}[\psi^{-1}]$
Finite Value of Autarky Conditions	
PF-FVAC	FVAC
$\beta G^{1-\gamma} < 1$ equivalently $\text{Þ} < R^{1/\gamma}G^{1-1/\gamma}$	$\beta G^{1-\gamma}\,\mathbb{E}[\psi^{1-\gamma}] < 1$
The discounted utility of constrained consumers who spend their permanent income each period should be finite.	By Jensen's inequality, stronger than the PF-FVAC because for $\gamma > 1$ and nondegenerate ψ, $\mathbb{E}[\psi^{1-\gamma}] > 1$.

Table 4 Sufficient Conditions for Nondegenerate[‡] Solution

Consumption Model(s)	Conditions	Comments
$\bar{\text{c}}(m)$: PF Unconstrained	RIC, FHWC°	RIC⇒ $\lvert\text{v}(m)\rvert < \infty$; FHWC⇒ $0 < \lvert\text{v}(m)\rvert$
$\underline{\text{c}}(m) = \underline{\kappa} m$		PF model with no human wealth ($h = 0$)
Section 1.3.1:		RIC prevents $\bar{\text{c}}(m) = \underline{\text{c}}(m) = 0$
Section 1.3.1:		FHWC prevents $\bar{\text{c}}(m) = \infty$
Eq (47):		PF-FVAC+FHWC ⇒ RIC
Eq (46):		GIC+FHWC ⇒ PF-FVAC
$\grave{\text{c}}(m)$: PF Constrained	~~GIC~~, RIC	FHWC holds ($G < \text{Þ} < R \Rightarrow G < R$)
Section 4.1.1:		$\grave{\text{c}}(m) = \bar{\text{c}}(m)$ for $m > m_{\#} < 1$
		(~~RIC~~ would yield $m_{\#} = 0$ so $\grave{\text{c}}(m) = 0$)
Appendix C:	GIC,RIC	$\lim_{m\to\infty} \grave{\text{c}}(m) = \bar{c}(m)$, $\lim_{m\to\infty} \grave{\kappa}(m) = \underline{\kappa}$
		kinks where horizon to $b = 0$ changes*
Appendix C:	GIC,~~RIC~~	$\lim_{m\to\infty} \grave{\kappa}(m) = 0$
		kinks where horizon to $b = 0$ changes*
$\text{c}(m)$: Friedman/Muth	Section 1.4.1 & 1.4.2 ,	$\underline{\text{c}}(m) < \text{c}(m) < \bar{\text{c}}(m)$
	Section ??	$\underline{\text{v}}(m) < \text{v}(m) < \bar{\text{v}}(m)$
Section ??:	FVAC, WRIC	Sufficient for Contraction
Section ??:		WRIC is weaker than RIC
Figure 7:		FVAC is stronger than PF-FVAC
Section ??: Case 3		~~FHWC~~+RIC ⇒GIC, $\lim_{m\to\infty} \kappa(m) = \underline{\kappa}$
Section ??: Case 1		~~RIC~~ ⇒~~FHWC~~, $\lim_{m\to\infty} \kappa(m) = 0$
Section 2.1:		"Buffer Stock Saving" Conditions
Theorem 3:		GIC ⇒ ∃ $\check{m}$ s.t. $0 < \check{m} < \infty$
Theorem 4:		GIC-Mod ⇒ ∃ $\hat{m}$ s.t. $0 < \hat{m} < \infty$

[‡]For feasible m satisfying $0 < m < \infty$, a nondegenerate limiting consumption function defines a unique optimal value of c satisfying $0 < c(m) < \infty$; a nondegenerate limiting value function defines a corresponding unique value of $-\infty < \text{v}(m) < 0$.

°RIC, FHWC are necessary as well as sufficient for the perfect foresight case. *That is, the first kink point in $c(m)$ is $m_{\#}$ s.t. for $m < m_{\#}$ the constraint will bind now, while for $m > m_{\#}$ the constraint will bind one period in the future. The second kink point corresponds to the m where the constraint will bind two periods in the future, etc.

**In the Friedman/Muth model, the RIC+FHWC are sufficient, but *not* necessary for nondegeneracy

5 Conclusions

Numerical solutions to optimal consumption problems, in both life cycle and infinite-horizon contexts, have become standard tools since the first reasonably realistic models were constructed in the late 1980s. One contribution of this paper is to show that finite-horizon ('life cycle') versions of the simplest such models, with assumptions about income shocks (transitory and permanent) dating back to Friedman (1957) and standard specifications of preferences — and without plausible (but computationally and mathematically inconvenient) complications like liquidity constraints — have attractive properties (like continuous differentiability of the consumption function, and analytical limiting MPC's as resources approach their minimum and maximum possible values).

The main focus of the paper, though, is on the limiting solution of the finite-horizon model as the time horizon approaches infinity. This simple model has other appealing features: A 'Finite Value of Autarky' condition guarantees convergence of the consumption function, under the mild additional requirement of a 'Weak Return Impatience Condition' that will never bind for plausible parameterizations, but provides intuition for the bridge between this model and models with explicit liquidity constraints. The paper also provides a roadmap for the model's relationships to the perfect foresight model without and with constraints. The constrained perfect foresight model provides an upper bound to the consumption function (and value function) for the model with uncertainty, which explains why the conditions for the model to have a nondegenerate solution closely parallel those required for the perfect foresight constrained model to have a nondegenerate solution.

The main use of infinite-horizon versions of such models is in heterogeneous-agent macroeconomics. The paper articulates intuitive 'Growth Impatience Conditions' under which populations of such agents, with Blanchardian (tighter) or Modiglianian (looser) mortality will exhibit balanced growth. Finally, the paper provides the analytical basis for many results about buffer-stock saving models that are so well understood that even without analytical foundations researchers uncontroversially use them as explanations of real-world phenomena like the cross-sectional pattern of consumption dynamics in the Great Recession.

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