1 Convergence in Euclidian Space

1.1 Convergence of v_t

{sec:vEuclidian}

Boyd's theorem shows that $\mathbb{T}$ defines a contraction mapping in an φ-bounded space. We now show that $\mathbb{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathbb{T}$, since $\mathrm{v}^*(m) = \mathbb{T}\mathrm{v}^*(m)$,

$$\left\| \mathrm{v}_{T-n+1} - \mathrm{v}^* \right\|_{\varphi} \leq \alpha^{n-1} \left\| \mathrm{v}_T - \mathrm{v}^* \right\|_{\varphi}. \tag{1}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_{\varphi}(\mathcal{A}, \mathcal{B})$ and $\kappa = \left\| \mathrm{v}_T - \mathrm{v}^* \right\|_{\varphi} < \infty$ because v_T and v^* are in $\mathcal{C}_{\varphi}(\mathcal{A}, \mathcal{B})$. It follows that

$$\left| \mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m) \right| \leq \kappa\alpha^{n-1} \left| \varphi(m) \right|. \tag{2}$$

Then we obtain

$$\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{3}$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\gamma}}{1-\gamma}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\gamma}}{1-\gamma} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathbb{T}\mathrm{v}_{T-1} \leq \mathbb{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

1.2 Convergence of c_t

{subsec:cConverges}

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

Consider any convergent subsequence $\left\{\mathrm{c}_{T-n(i)}(m)\right\}$ of $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to c^*. By the definition of $\mathrm{c}_{T-n}(m)$, we have

$$\begin{aligned} \mathrm{u}(c_{T-n(i)}(m)) + &\beta\mathbb{E}_{T-n(i)}[\mathcal{G}_{T-n(i)+1}^{1-\gamma}\mathrm{v}_{T-n(i)+1}(m)] \\ &\geq \mathrm{u}(c_{T-n(i)}) + \beta\mathbb{E}_{T-n(i)}[\mathcal{G}_{T-n(i)+1}^{1-\gamma}\mathrm{v}_{T-n(i)+1}(m)], \end{aligned} \tag{4}$$

for any $c_{T-n(i)} \in [\underline{\underline{\kappa}}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $\mathrm{u}(c^*) + \beta\mathbb{E}_t[\mathcal{G}_t^{1-\gamma}\mathrm{v}(m)]$, and the right hand side converges to $\mathrm{u}(c_{T-n(i)}) + \beta\mathbb{E}_t[\mathcal{G}_t^{1-\gamma}\mathrm{v}(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$\mathrm{u}(c^*) + \beta\mathbb{E}_t[\mathcal{G}_{t+1}^{1-\gamma}\mathrm{v}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\mathbb{E}_t[\mathcal{G}_{t+1}^{1-\gamma}\mathrm{v}(m)]. \tag{5}$$

Hence, $c^* \in \underset{c_{T-n(i)} \in [\underline{\underline{\kappa}}m, \bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_{T-n(i)}) + \beta\mathbb{E}_t[\mathcal{G}_{t+1}^{1-\gamma}\mathrm{v}(m)]\right\}$. By the uniqueness of $\mathrm{c}(m)$, $c^* = \mathrm{c}(m)$.