

# Machine Learning (in One Lecture)

## EC 607, Set 12

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Spring 2020

# Prologue

# Schedule

## Last time

Resampling methods

## Today

A one-lecture introduction to machine-learning methods

## Upcoming

The end is near. As is the final.

# Prediction: What's the goal?

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## What's different?

Machine-learning methods focus on **prediction**. What's different?

Up to this point, we've focused on causal **identification/inference** of  $\beta$ , i.e.,

$$\mathbf{Y}_i = \mathbf{X}_i\boldsymbol{\beta} + u_i$$

meaning we want an unbiased (consistent) and precise estimate  $\hat{\boldsymbol{\beta}}$ .

With **prediction**, we shift our focus to accurately estimating outcomes.

In other words, how can we best construct  $\hat{\mathbf{Y}}_i$ ?

# Prediction: What's the goal?

... so?

So we want "nice"-performing estimates  $\hat{y}$  instead of  $\hat{\beta}$ .

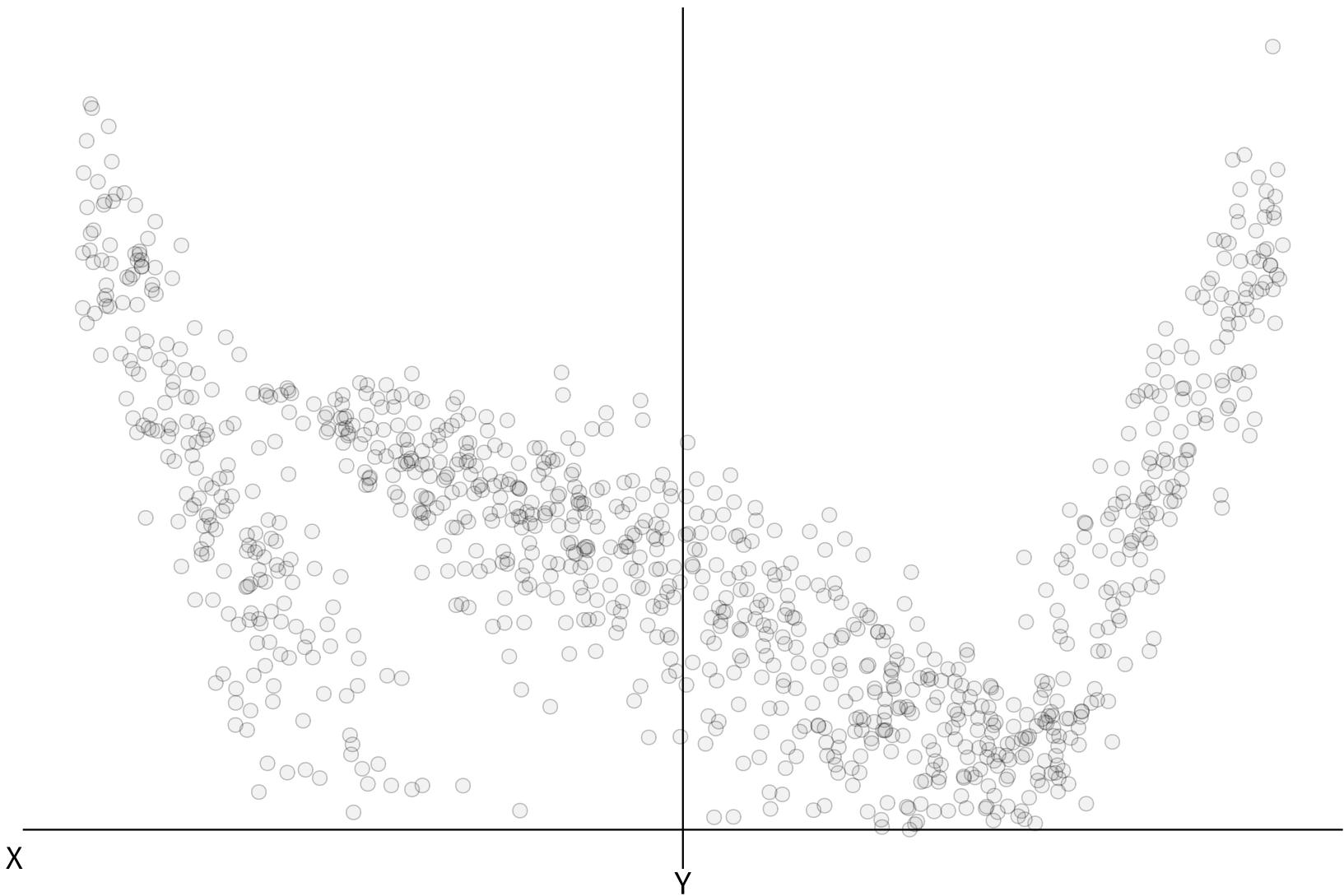
**Q** Can't we just use the same methods (*i.e.*, OLS)?

**A** It depends. How well does your **linear**-regression model approximate the underlying data? (And how do you plan to select your model?)

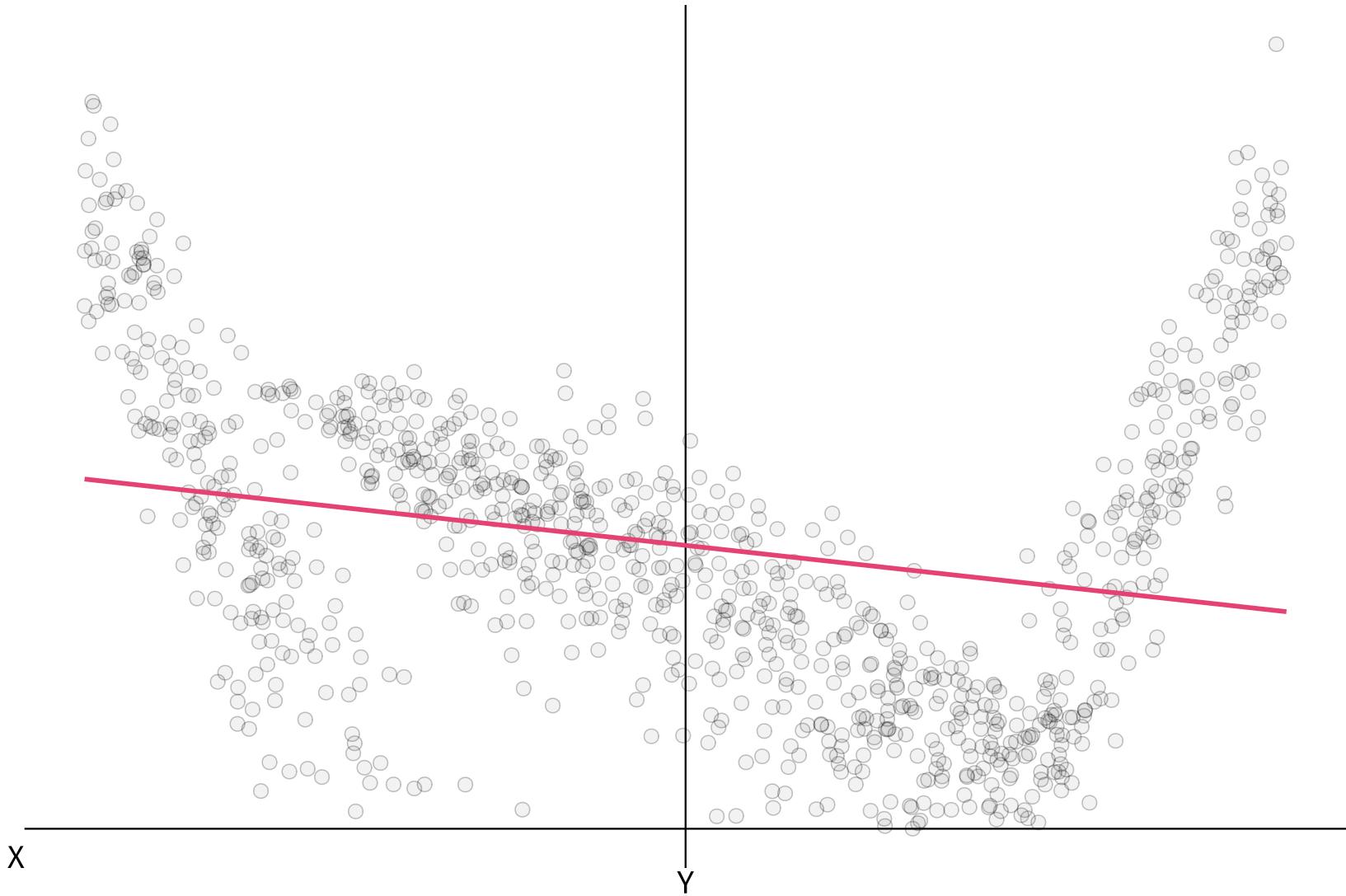
*Recall* Least-squares regression is a great **linear** estimator.

Data data be tricky<sup>†</sup>—as can understanding many relationships.

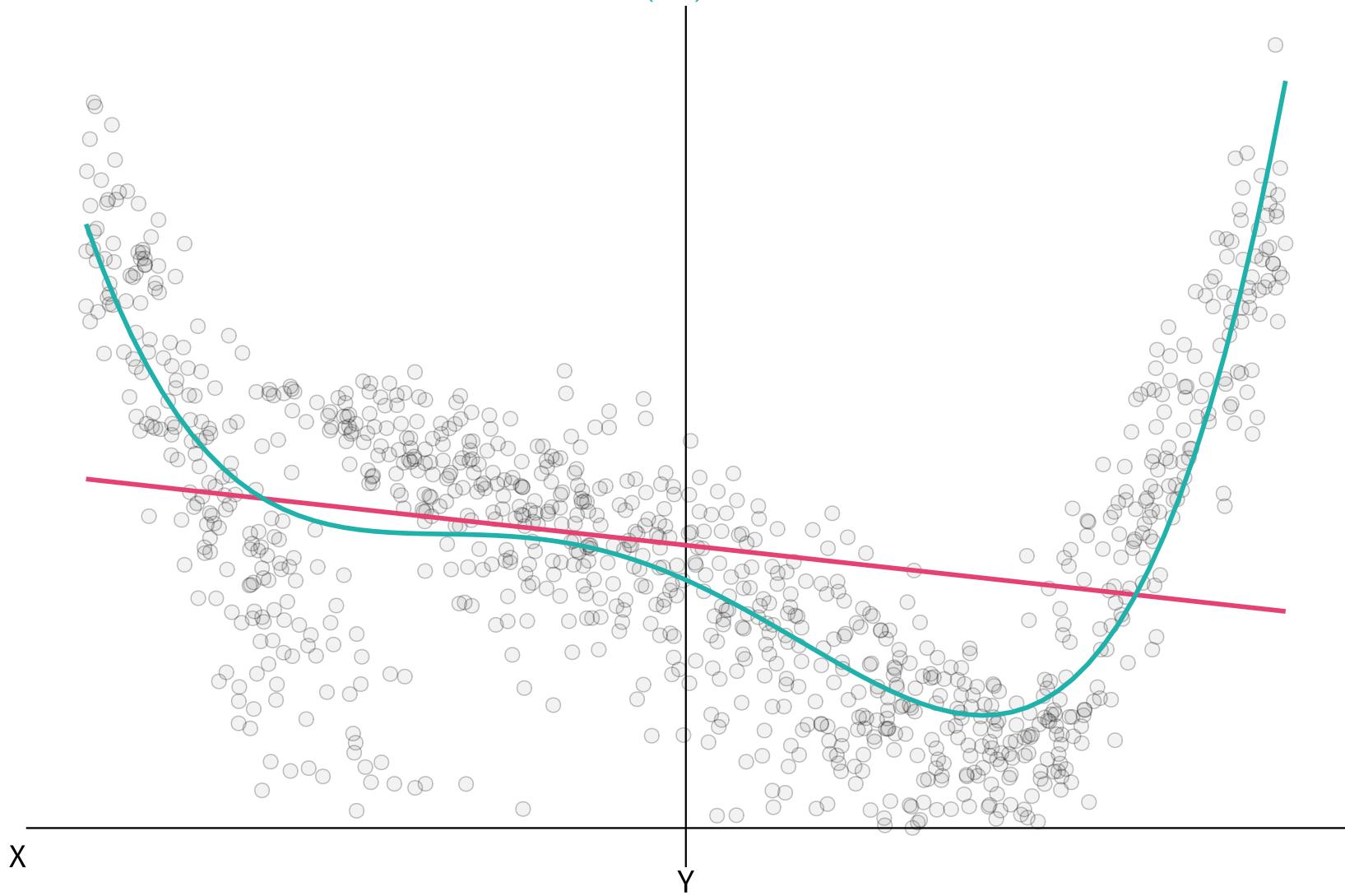
<sup>†</sup> "Tricky" might mean nonlinear... or many other things...



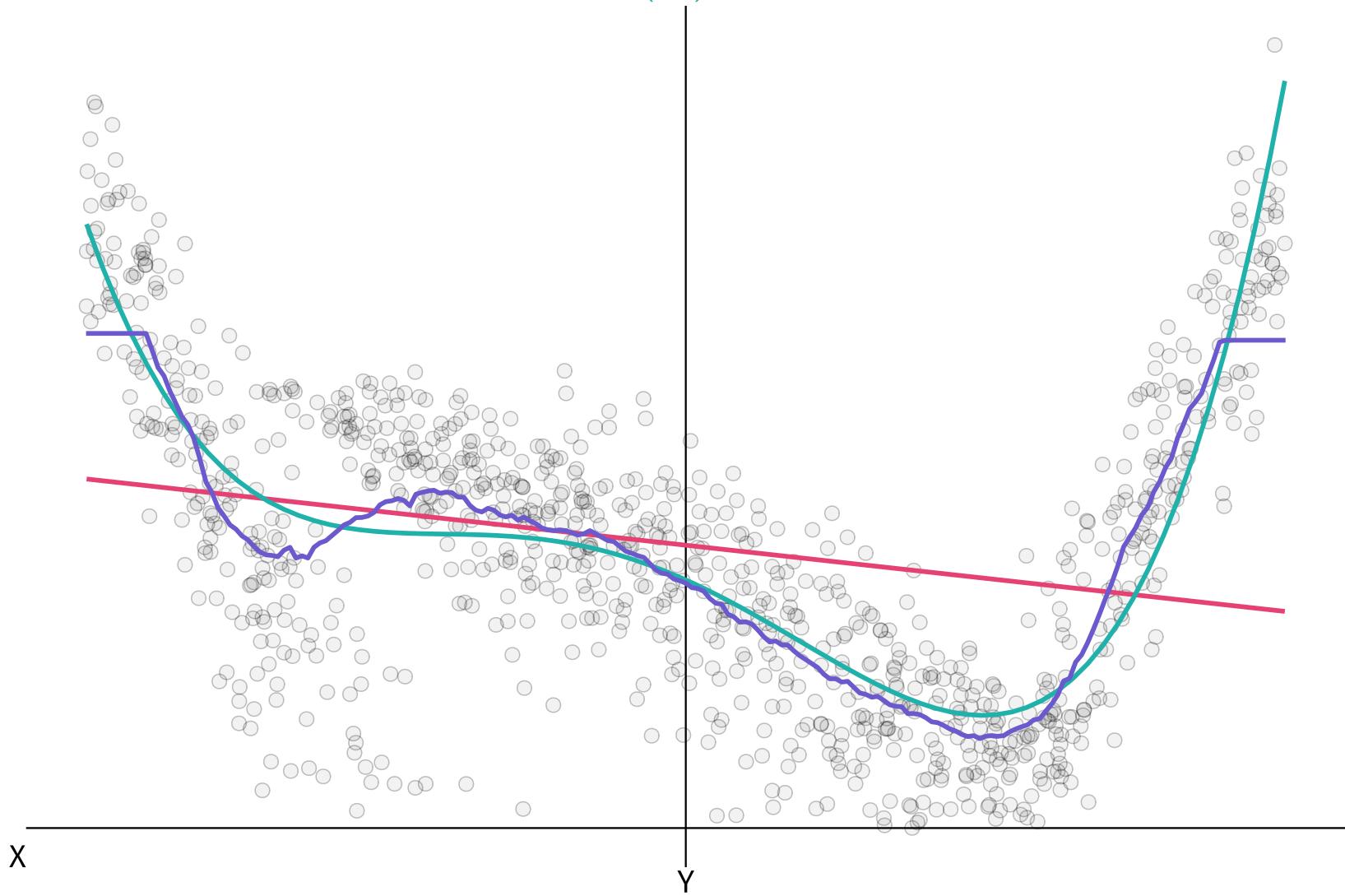
## Linear regression



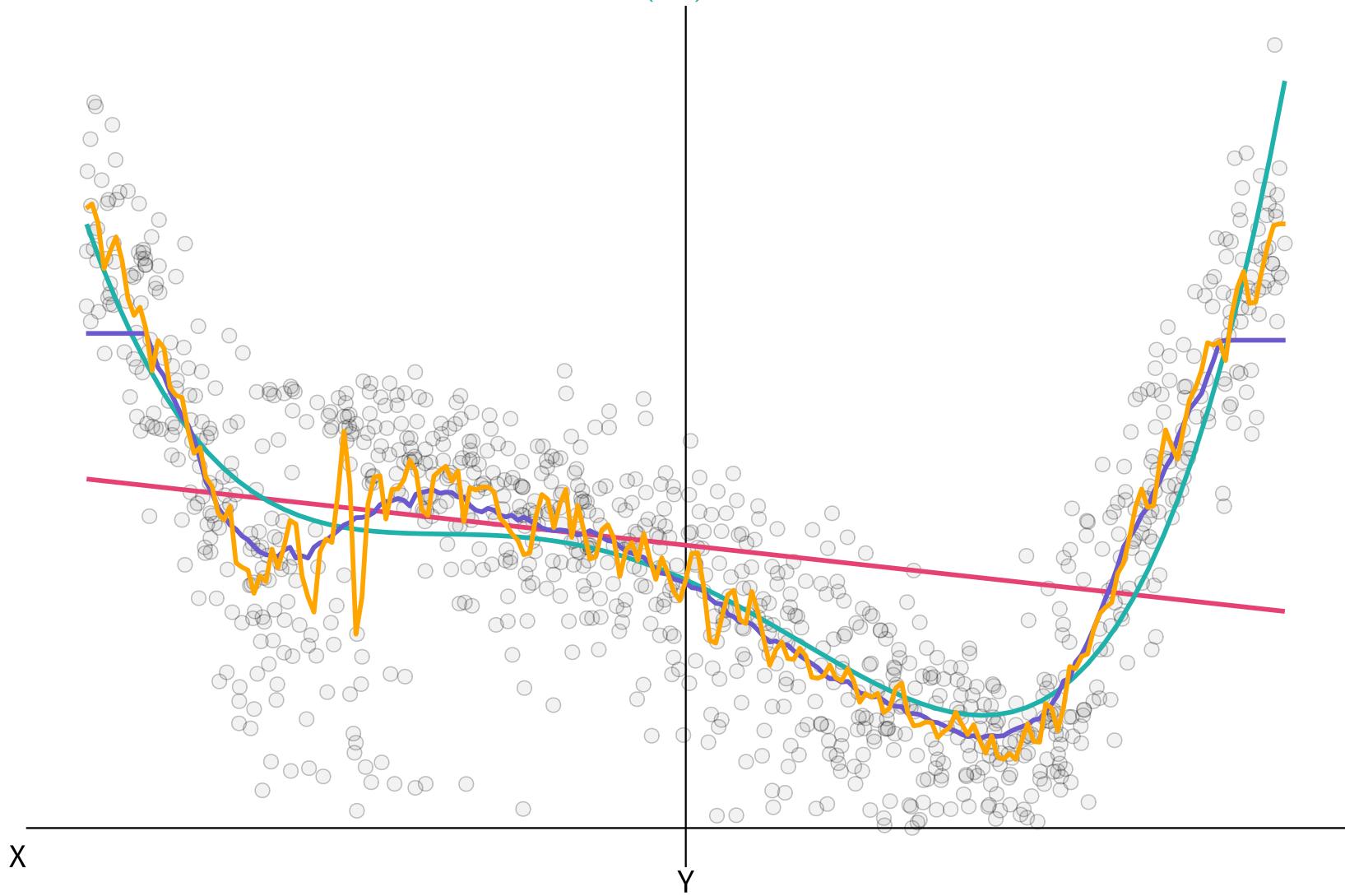
## Linear regression, linear regression ( $x^4$ )



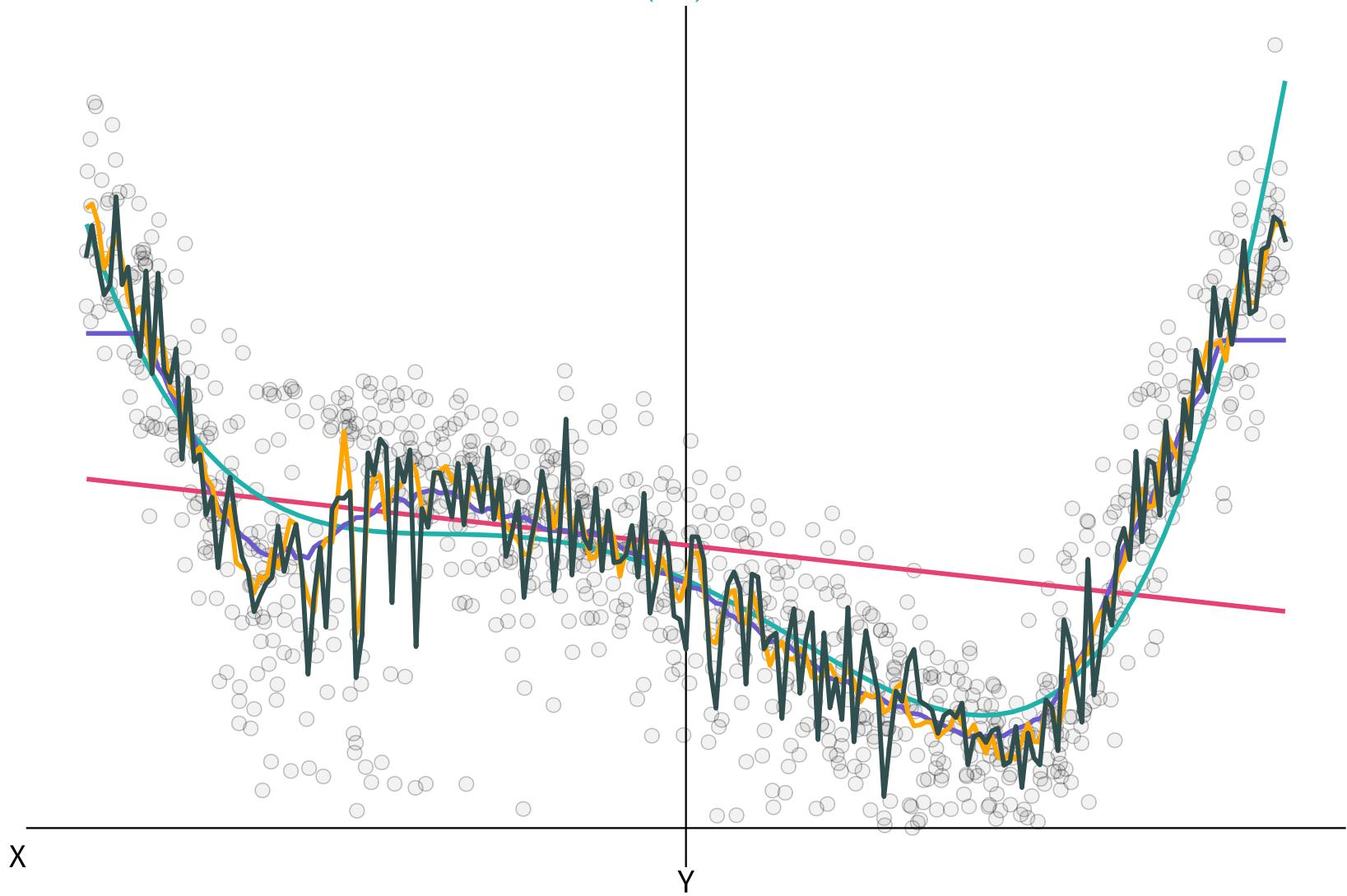
Linear regression, linear regression ( $x^4$ ), KNN (100)



Linear regression, linear regression ( $x^4$ ), KNN (100), KNN (10)



Linear regression, linear regression ( $x^4$ ), KNN (100), KNN (10), random forest



*Note* That example only had one predictor...

# What's the goal?

## Tradeoffs

In prediction, we constantly face many tradeoffs, *e.g.*,

- **flexibility** and **parametric structure** (and interpretability)
- performance in **training** and **test** samples
- **variance** and **bias**

As your economic training should have predicted, in each setting, we need to **balance the additional benefits and costs** of adjusting these tradeoffs.

Many machine-learning (ML) techniques/algorithms are crafted to optimize with these tradeoffs, but the practitioner (you) still needs to be careful.

# What's the goal?

There are many reasons to step outside the world of linear regression...

## **Multi-class** classification problems

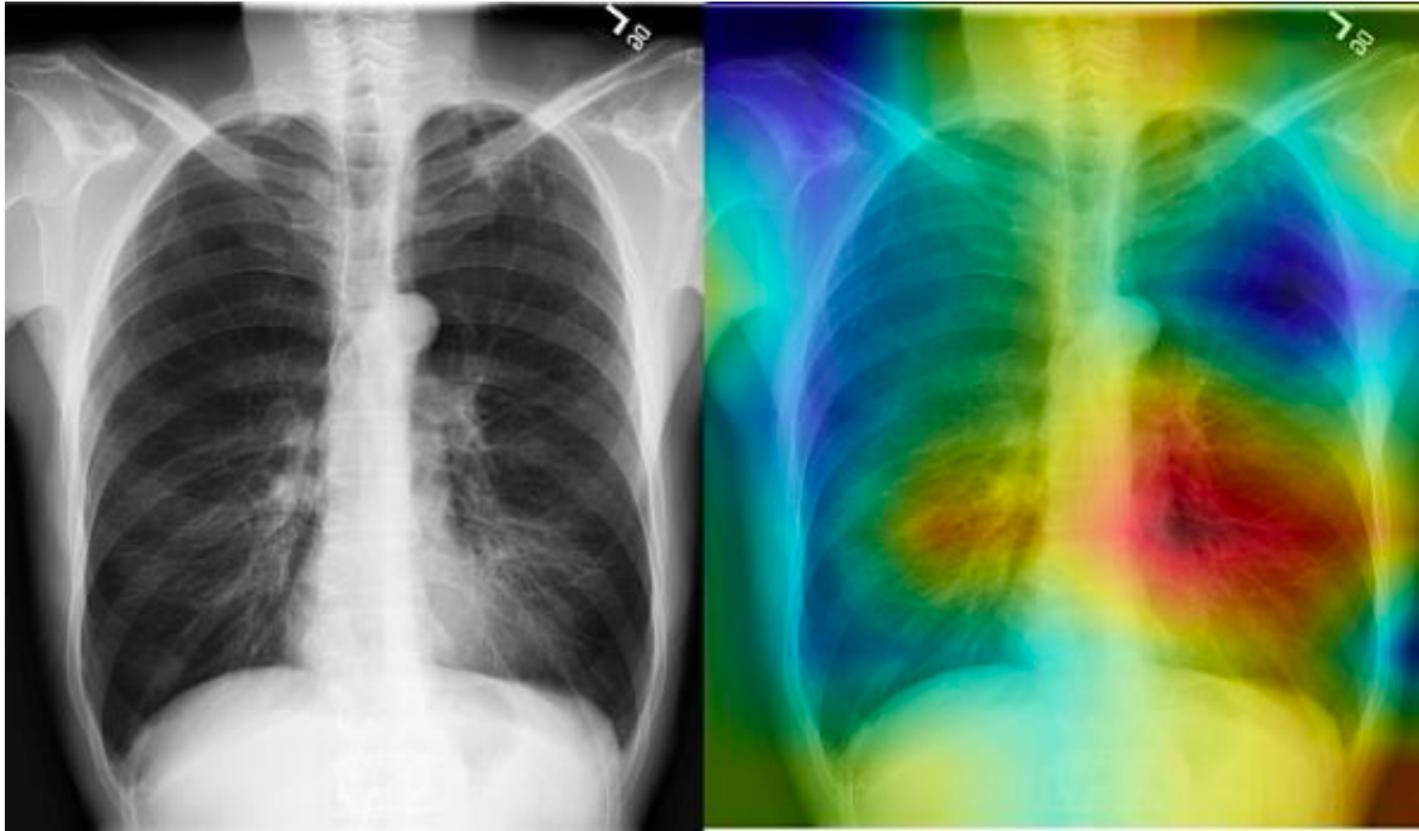
- Rather than {0,1}, we need to classify  $y_i$  into 1 of K classes
- *E.g.*, ER patients: {heart attack, drug overdose, stroke, nothing}

## **Text analysis** and **image recognition**

- Comb through sentences (pixels) to glean insights from relationships
- *E.g.*, detect sentiments in tweets or roof-top solar in satellite imagery

## **Unsupervised learning**

- You don't know groupings, but you think there are relevant groups
- *E.g.*, classify spatial data into groups



**Stanford University (Stanford, CA ) researchers have developed a deep-learning algorithm that can evaluate chest X-ray images for signs of disease at a level exceeding practicing radiologists.**



# Parking Lot Vehicle Detection Using Deep Learning

THE  
NEW YORKER

A REPORTER AT LARGE OCTOBER 14, 2019 ISSUE

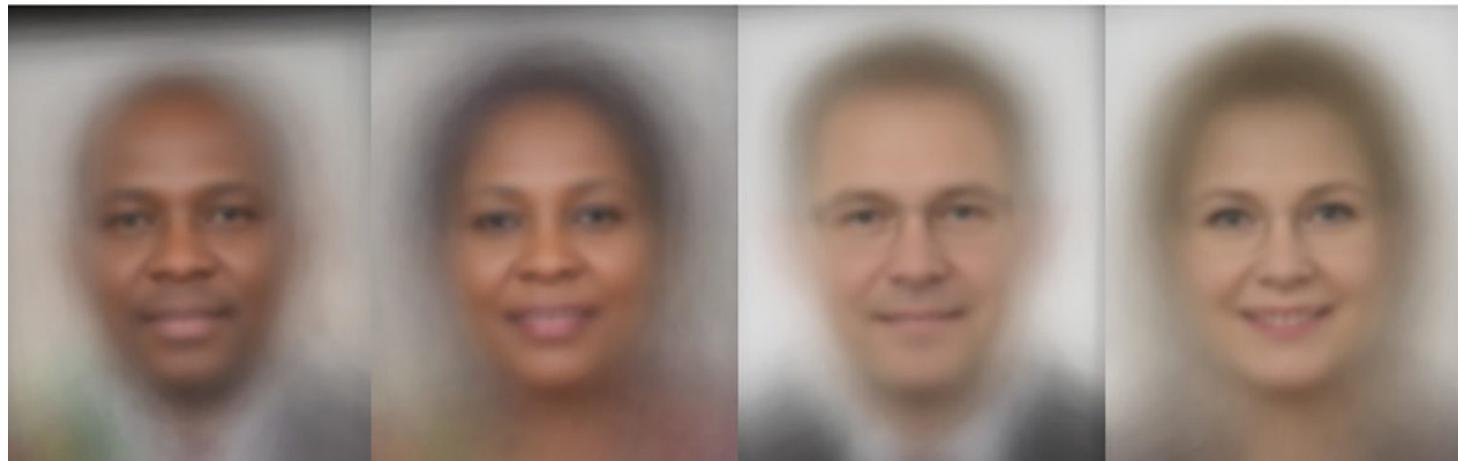
# The Next Word |

*Where will predictive text take us?*

Text by John Seabrook



Gender Classifier	Darker Male	Darker Female	Lighter Male	Lighter Female	Largest Gap
Microsoft	94.0%	79.2%	100%	98.3%	20.8%
FACE++	99.3%	65.5%	99.2%	94.0%	33.8%
IBM	88.0%	65.3%	99.7%	92.9%	34.4%



Flexibility is huge, but we still want to avoid overfitting.

# Statistical learning

## What is it good for?

A lot of things. We tend to break statistical-learning into two(-ish) classes:

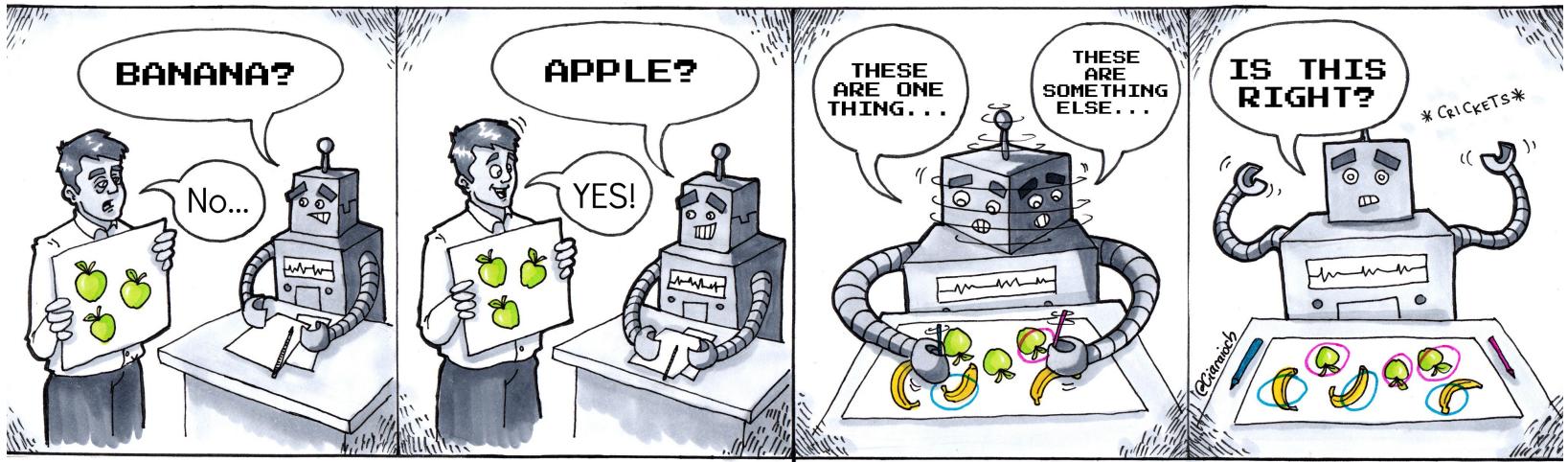
1. **Supervised learning** builds ("learns") a statistical model for predicting an **output ( $y$ )** given a set of **inputs ( $x_1, \dots, x_p$ )**, i.e., we want to build a model/function  $f$

$$y = f(x_1, \dots, x_p)$$

that accurately describes  $y$  given some values of  $x_1, \dots, x_p$ .

2. **Unsupervised learning** learns relationships and structure using only **inputs ( $x_1, \dots, x_p$ )** without any supervising output—letting the data "speak for itself."

**Semi-supervised learning** falls somewhere between these supervised and unsupervised learning—generally applied to supervised tasks when labeled **outputs** are incomplete.



## Supervised Learning

## Unsupervised Learning

Source

# Statistical learning

## Output

We tend to further break **supervised learning** into two groups, based upon the **output** (the **outcome** we want to predict):

1. **Classification tasks** for which the values of **y** are discrete categories  
*E.g., race, sex, loan default, hazard, disease, flight status*
2. **Regression tasks** in which **y** takes on continuous, numeric values.  
*E.g., price, arrival time, number of emails, temperature*

*Note<sub>1</sub>* The use of *regression* differs from our use of *linear regression*.

*Note<sub>2</sub>* Don't get tricked: Not all numbers represent continuous, numerical values—*e.g.*, zip codes, industry codes, social security numbers.<sup>†</sup>

<sup>†</sup> **Q** Where would you put responses to 5-item Likert scales?

# Statistical learning

## The goal

As defined before, we want to *learn* a model to understand our data.

1. Take our (numeric) output  $\mathbf{y}$ .
2. Imagine there is a function  $f$  that takes inputs  $\mathbf{X} = \mathbf{x}_1, \dots, \mathbf{x}_p$  and maps them, plus a random, mean-zero error term  $\varepsilon$ , to the output.

$$\mathbf{y} = f(\mathbf{X}) + \varepsilon$$

# Statistical learning

## Learning from $\hat{f}$

There are two main reasons we want to learn about  $f$

1. **Causal inference settings** How do changes in  $X$  affect  $y$ ?

What we've done all quarter.

2. **Prediction problems** Predict  $y$  using our estimated  $f$ , i.e.,

$$\hat{y} = \hat{f}(X)$$

our *black-box setting* where we care less about  $f$  than  $\hat{y}$ .<sup>†</sup>

Similarly, in causal-inference settings, we don't particularly care about  $\hat{y}$ .

<sup>†</sup> You shouldn't actually treat your prediction methods as total black boxes.

# Statistical learning

## Prediction errors

As tends to be the case in life, you will make errors in predicting  $\mathbf{y}$ .

The accuracy of  $\hat{\mathbf{y}}$  depends upon **two errors**:

1. **Reducible error** The error due to  $\hat{f}$  imperfectly estimating  $f$ .  
*Reducible* in the sense that we could improve  $\hat{f}$ .
2. **Irreducible error** The error component that is outside of the model  $f$ .  
*Irreducible* because we defined an error term  $\varepsilon$  unexplained by  $f$ .

*Note* As its name implies, you can't get rid of *irreducible* error—but we can try to get rid of *reducible* errors.

# Statistical learning

## Prediction errors

Why we're stuck with *irreducible* error

$$\begin{aligned} E[\{\mathbf{y} - \hat{\mathbf{y}}\}^2] &= E\left[\left\{\mathbf{f}(\mathbf{X}) + \varepsilon + \hat{\mathbf{f}}(\mathbf{X})\right\}^2\right] \\ &= \underbrace{\left[\mathbf{f}(\mathbf{X}) - \hat{\mathbf{f}}(\mathbf{X})\right]^2}_{\text{Reducible}} + \underbrace{\text{Var}(\varepsilon)}_{\text{Irreducible}} \end{aligned}$$

In less math:

- If  $\varepsilon$  exists, then  $\mathbf{X}$  cannot perfectly explain  $\mathbf{y}$ .
- So even if  $\hat{\mathbf{f}} = \mathbf{f}$ , we still have irreducible error.

Thus, to form our **best predictors**, we will **minimize reducible error**.

# Model accuracy

## MSE

Mean squared error (MSE) is the most common<sup>†</sup> way to measure model performance in a regression setting.

$$\text{MSE} = \frac{1}{n} \sum_{i=1}^n \left[ \textcolor{orange}{y}_i - \hat{\textcolor{teal}{f}}(\textcolor{blue}{x}_i) \right]^2$$

Recall:  $\textcolor{orange}{y}_i - \hat{\textcolor{teal}{f}}(\textcolor{blue}{x}_i) = \textcolor{orange}{y}_i - \hat{y}_i$  is our prediction error.

Two notes about MSE

1. MSE will be (relatively) very small when **prediction error** is nearly zero.
2. MSE **penalizes** big errors more than little errors (the squared part).

<sup>†</sup> Most common does not mean best—it just means lots of people use it.

# Model accuracy

## Training or testing?

Low MSE (accurate performance) on the data that trained the model isn't actually impressive—maybe the model is just overfitting our data.<sup>†</sup>

*What we want:* How well does the model perform **on data it has never seen?**

This introduces an important distinction:

1. **Training data:** The observations  $(y_i, x_i)$  used to **train** our model  $\hat{f}$ .
2. **Testing data:** The observations  $(y_0, x_0)$  that our model has yet to see—and which we can use to evaluate the performance of  $\hat{f}$ .

**Real goal: Low test-sample MSE** (not the training MSE from before).

<sup>†</sup> Recall the kNN performance for k=1.

# Model accuracy

## Regression and loss

For **regression settings**, the loss is our **prediction**'s distance from **truth**, i.e.,

$$\text{error}_i = \mathbf{y}_i - \hat{\mathbf{y}}_i \quad \text{loss}_i = |\mathbf{y}_i - \hat{\mathbf{y}}_i| = |\text{error}_i|$$

Depending upon our ultimate goal, we choose **loss/objective functions**.

$$\text{L1 loss} = \sum_i |\mathbf{y}_i - \hat{\mathbf{y}}_i|$$

$$\text{MAE} = \frac{1}{n} \sum_i |\mathbf{y}_i - \hat{\mathbf{y}}_i|$$

$$\text{L2 loss} = \sum_i (\mathbf{y}_i - \hat{\mathbf{y}}_i)^2$$

$$\text{MSE} = \frac{1}{n} \sum_i (\mathbf{y}_i - \hat{\mathbf{y}}_i)^2$$

Whatever we're using, we care about **test performance** (e.g., test MSE), rather than training performance.

# Model accuracy

## Classification

For **classification problems**, we often use the **test error rate**.

$$\frac{1}{n} \sum_{i=1}^n \mathbb{I}(y_i \neq \hat{y}_i)$$

The **Bayes classifier**

1. predicts class  $j$  when  $\Pr(y_0 = j | \mathbf{X} = \mathbf{x}_0)$  exceeds all other classes.
2. produces the **Bayes decision boundary**—the decision boundary with the lowest test error rate.
3. is unknown: we must predict  $\Pr(y_0 = j | \mathbf{X} = \mathbf{x}_0)$ .

# Flexibility

## The bias-variance tradeoff

Finding the optimal level of flexibility highlights the **bias-variance tradeoff**.

**Bias** The error that comes from inaccurately estimating  $\hat{f}$ .

- More flexible models are better equipped to recover complex relationships ( $f$ ), reducing bias. (Real life is seldom linear.)
- Simpler (less flexible) models typically increase bias.

**Variance** The amount  $\hat{f}$  would change with a different **training sample**

- If new **training sets** drastically change  $\hat{f}$ , then we have a lot of uncertainty about  $f$  (and, in general,  $\hat{f} \not\approx f$ ).
- More flexible models generally add variance to  $f$ .

# Flexibility

## The bias-variance tradeoff

The expected value<sup>†</sup> of the **test MSE** can be written

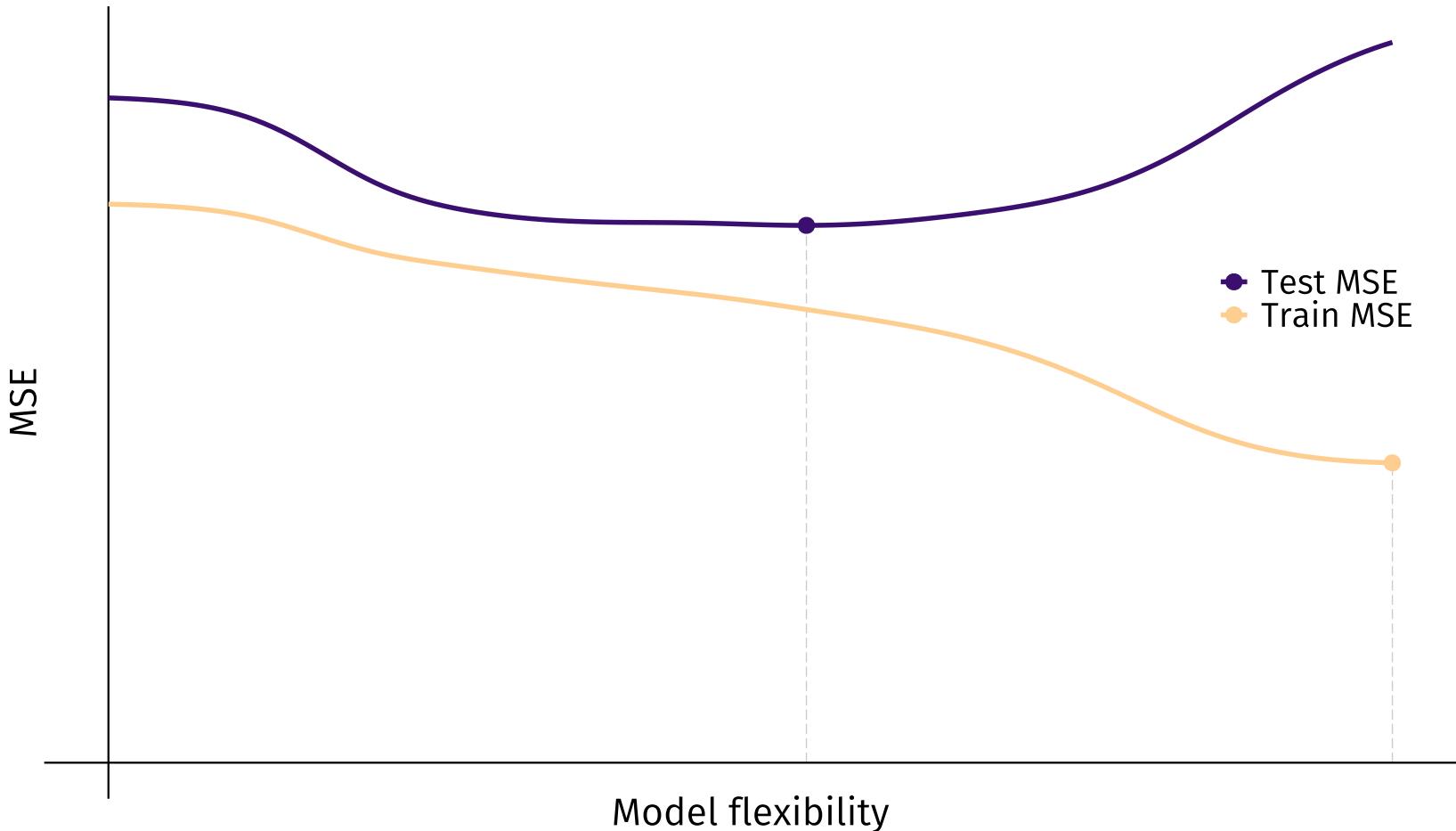
$$E\left[\left(\mathbf{y}_0 - \hat{\mathbf{f}}(\mathbf{X}_0)\right)^2\right] = \underbrace{\text{Var}\left(\hat{\mathbf{f}}(\mathbf{X}_0)\right)}_{\text{Variance}} + \underbrace{\left[\text{Bias}\left(\hat{\mathbf{f}}(\mathbf{X}_0)\right)\right]^2}_{\text{Bias}} + \underbrace{\text{Var}(\varepsilon)}_{\text{Irr. error}}$$

### The tradeoff in terms of model flexibility

- Increasing flexibility *from total inflexibility* generally **reduces bias more** than it increases variance (reducing test MSE).
- At some point, the marginal benefits of flexibility **equal** marginal costs.
- Past this point (optimal flexibility), we **increase variance more** than we reduce bias (increasing test MSE).

**U-shaped test MSE** with respect to model flexibility (KNN here).

Increases in variance eventually overcome reductions in (squared) bias.



# Resampling refresher

**Resampling methods** help understand uncertainty in statistical modeling.

The process behind the magic of resampling methods:

1. **Repeatedly draw samples** from the **training data**.
2. **Fit your model**(s) on each random sample.
3. **Compare** model performance (or estimates) **across samples**.
4. Infer the **variability/uncertainty in your model** from (3).

# Resampling

## Hold out

*Recall:* We want to find the model that **minimizes out-of-sample test error.**

If we have a large test dataset, we can use it (once).

**Q<sub>1</sub>** What if we don't have a test set?

**Q<sub>2</sub>** What if we need to select and train a model?

**Q<sub>3</sub>** How can we avoid overfitting our training<sup>†</sup> data during model selection?

**A<sub>1,2,3</sub>** **Hold-out methods** (e.g., cross validation) use training data to estimate test performance—**holding out** a mini "test" sample of the training data that we use to estimate the test error.

<sup>†</sup> Also relevant for *testing* data.

# Hold-out methods

## Option 1: The *validation set* approach

To estimate the **test error**, we can *hold out* a subset of our **training data** and then **validate** (evaluate) our model on this held out **validation set**.

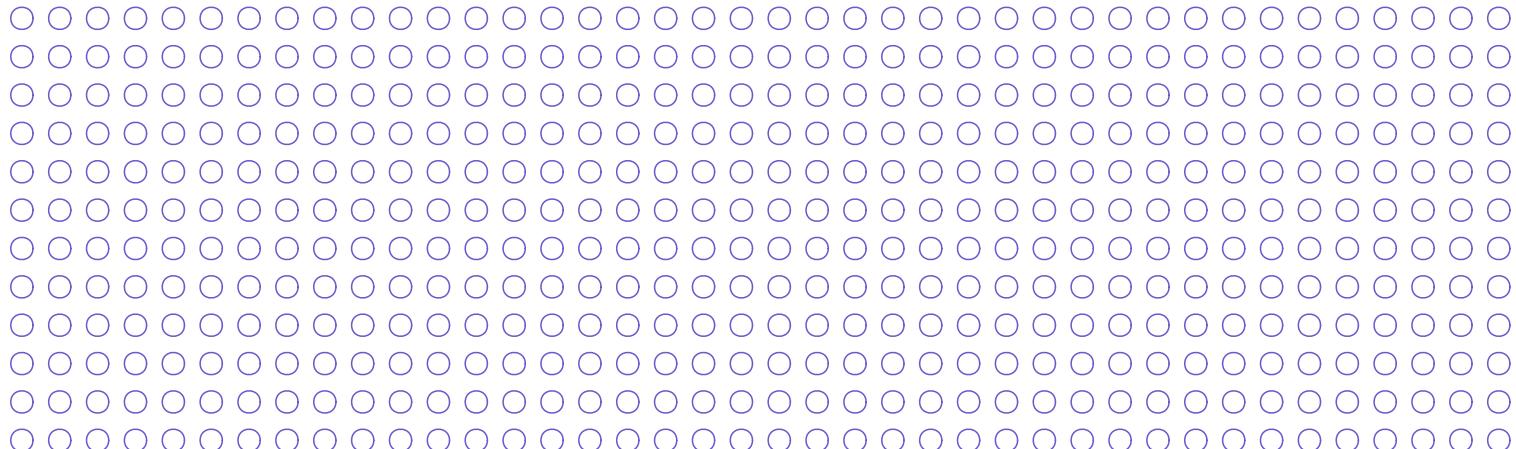
- The **validation error rate** estimates the **test error rate**
- The model only "sees" the non-validation subset of the **training data**.

# Hold-out methods

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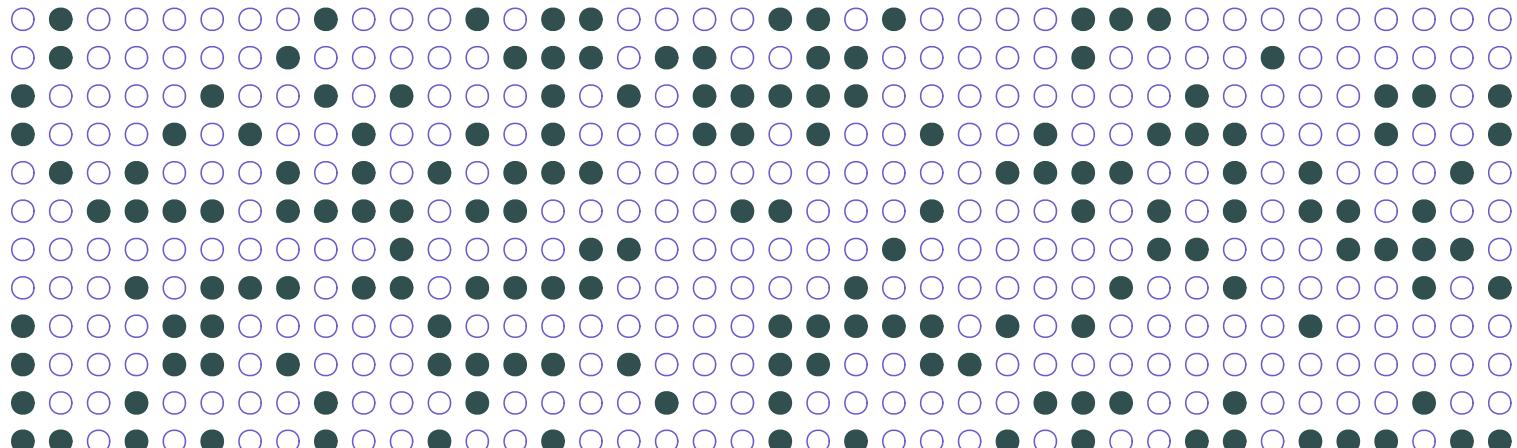
**Initial training set**

# Hold-out methods

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Validation (sub)set

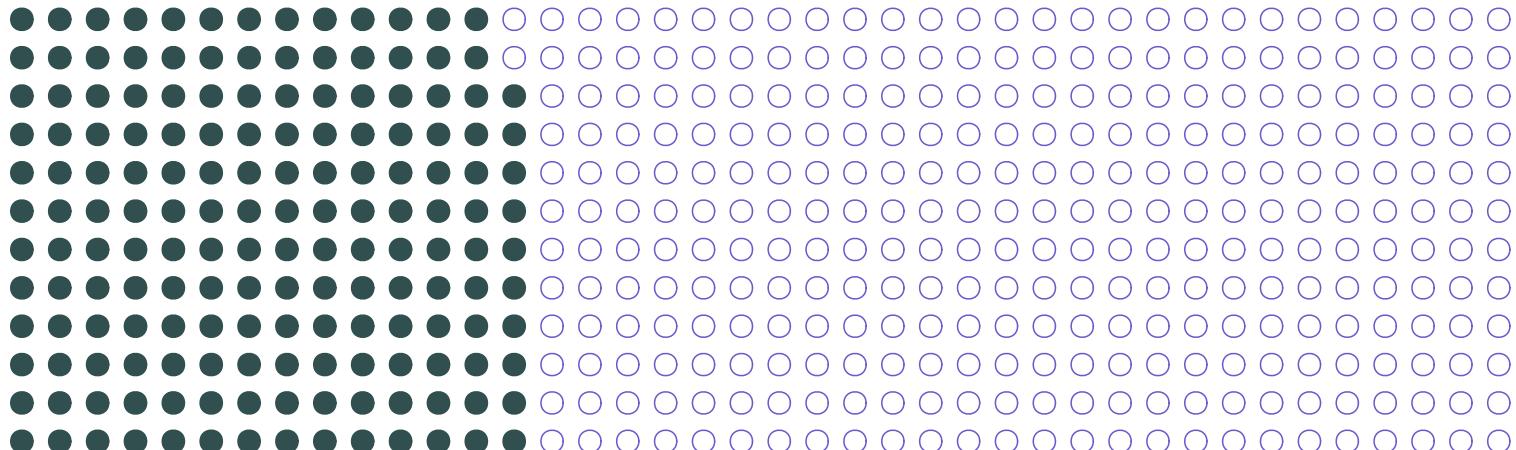
Training set: Model training

# Hold-out methods

## Option 1: The *validation set* approach

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**Validation (sub)set**

**Training set:** Model training

# Hold-out methods

## Option 1: The *validation set approach*

*Example* We could use the validation-set approach to help select the degree of a polynomial for a linear-regression model.

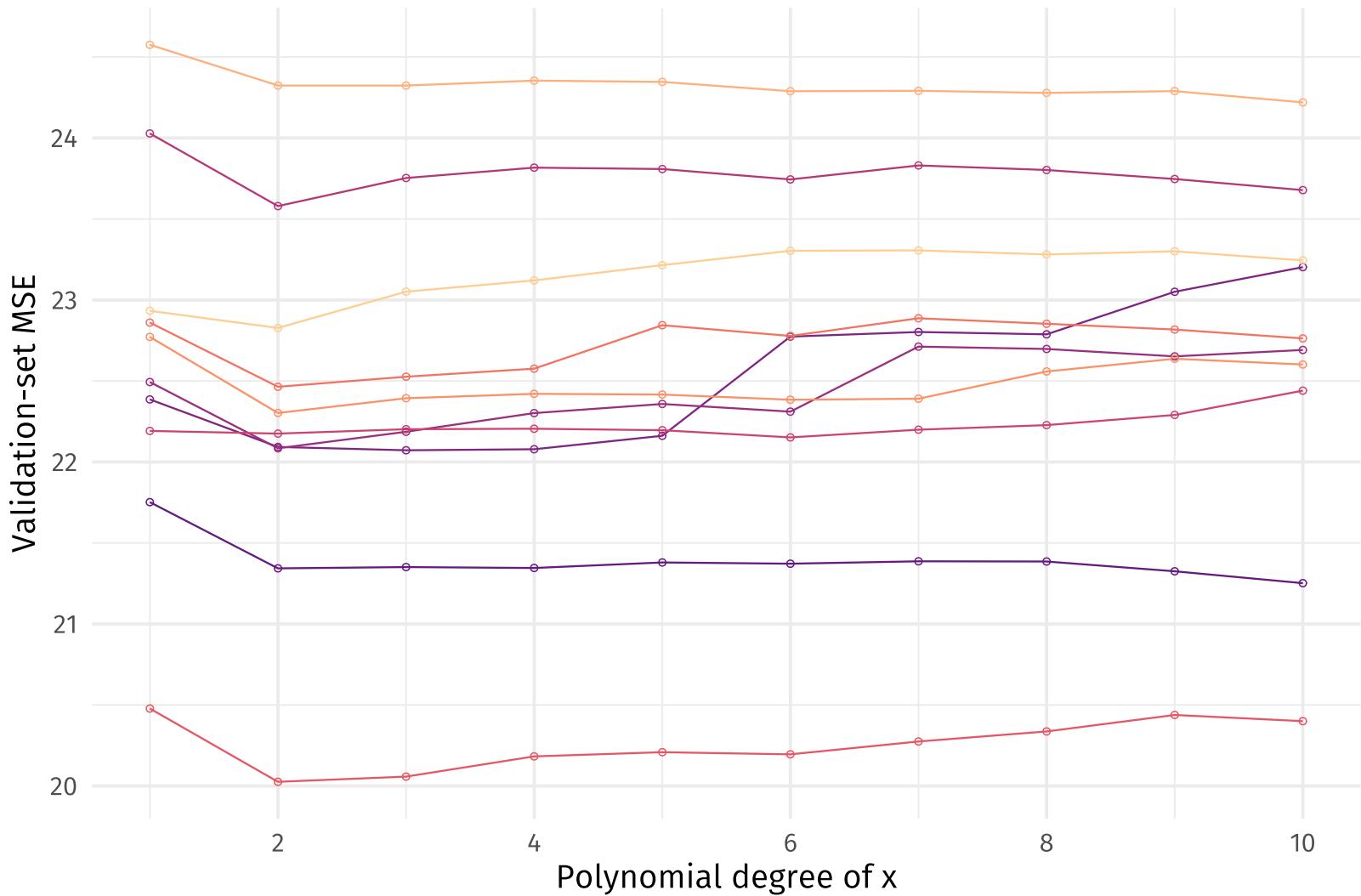
The goal of the validation set is to **estimate out-of-sample (test) error.**

**Q** So what?

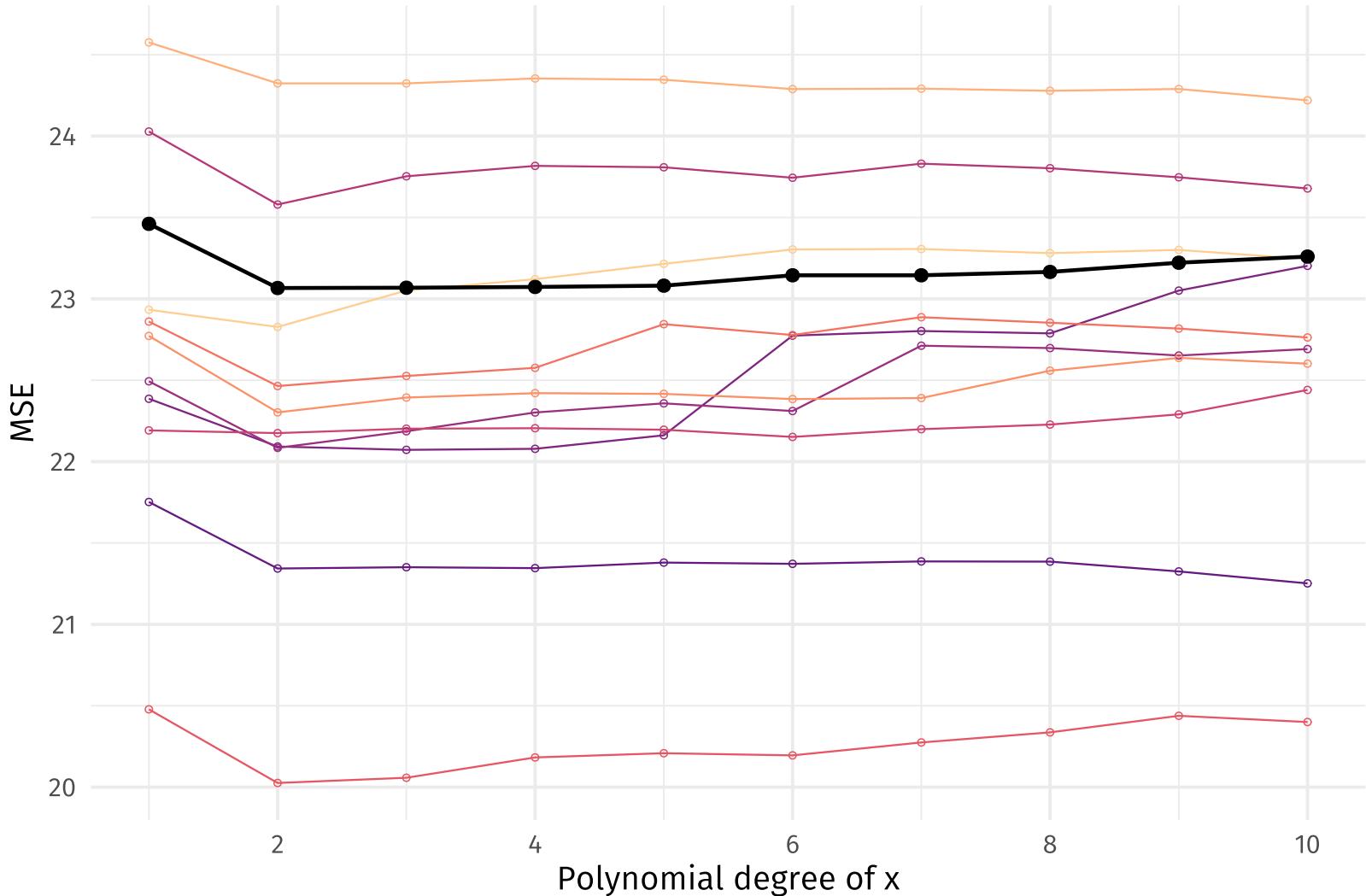
- Estimates come with **uncertainty**—varying from sample to sample.
- Variability (standard errors) is larger with **smaller samples**.

**Problem** This estimated error is often based upon a fairly small sample (<30% of our training data). So its variance can be large.

## Validation MSE for 10 different validation samples



## True test MSE compared to validation-set estimates



# Hold-out methods

## Option 1: The *validation set* approach

Put differently: The validation-set approach has ( $\geq$ ) two major drawbacks:

1. **High variability** Which observations are included in the validation set can greatly affect the validation MSE.
2. **Inefficiency in training our model** We're essentially throwing away the validation data when training the model—"wasting" observations.  
(2)  $\implies$  validation MSE may overestimate test MSE.

Even if the validation-set approach provides an unbiased estimator for test error, it is likely a pretty noisy estimator.

# Hold-out methods

## Option 2: Leave-one-out cross validation

**Cross validation** solves the validation-set method's main problems.

- Use more (= all) of the data for training (lower variability; less bias).
- Still maintains separation between training and validation subsets.

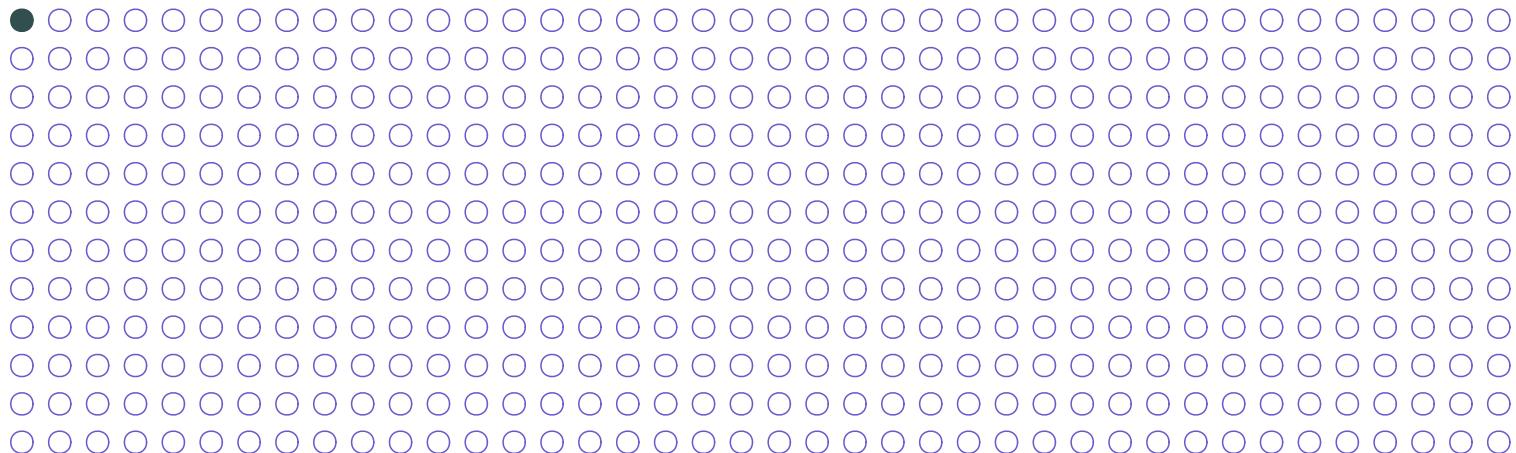
**Leave-one-out cross validation** (LOOCV) is perhaps the cross-validation method most similar to the validation-set approach.

- Your validation set is exactly one observation.
- *New* You repeat the validation exercise for every observation.
- *New* Estimate MSE as the mean across all observations.

# Hold-out methods

## Option 2: Leave-one-out cross validation

Each observation takes a turn as the **validation set**,  
while the other  $n-1$  observations get to **train the model**.

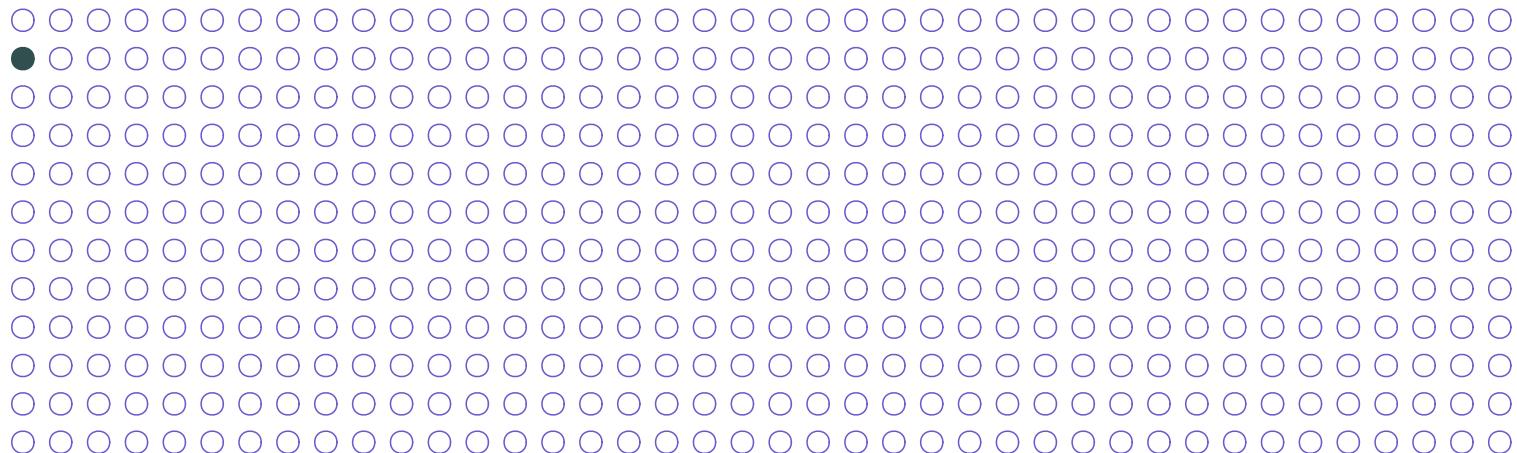


Observation 1's turn for validation produces  $MSE_1$ .

# Hold-out methods

## Option 2: Leave-one-out cross validation

Each observation takes a turn as the **validation set**,  
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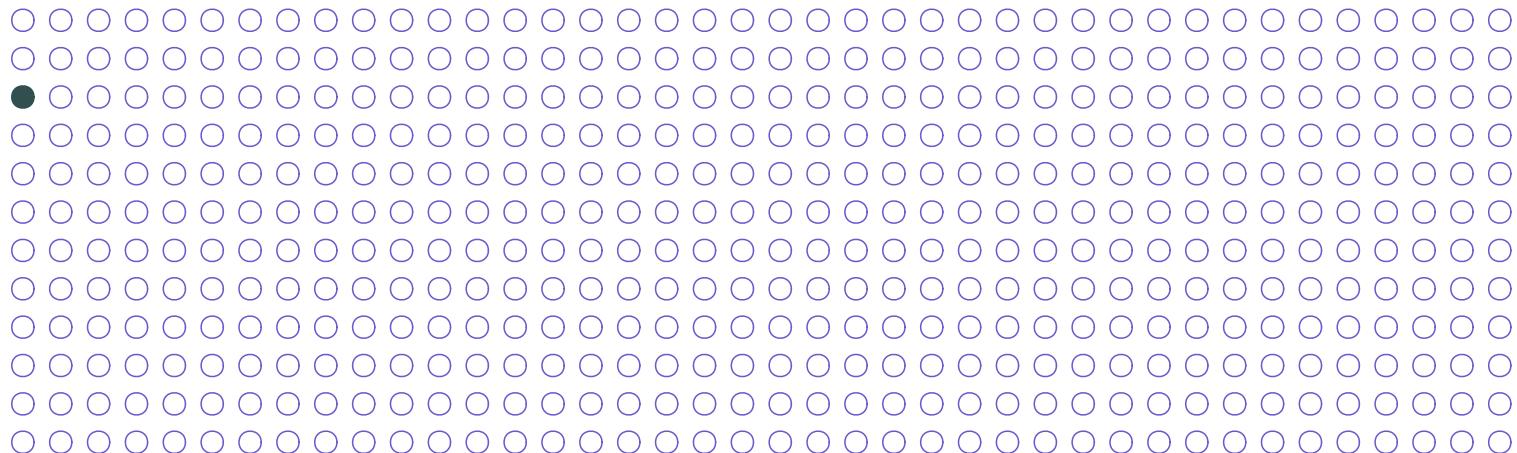


Observation 2's turn for validation produces  $MSE_2$ .

# Hold-out methods

## Option 2: Leave-one-out cross validation

Each observation takes a turn as the **validation set**,  
while the other n-1 observations get to **train the model**.

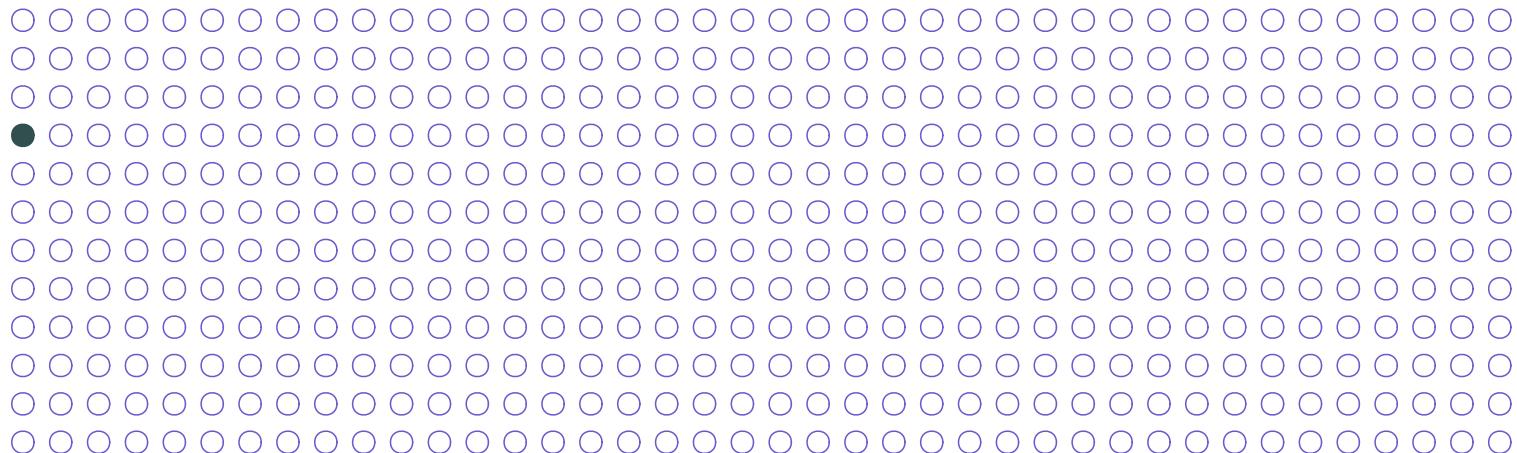


Observation 3's turn for validation produces  $MSE_3$ .

# Hold-out methods

## Option 2: Leave-one-out cross validation

Each observation takes a turn as the **validation set**,  
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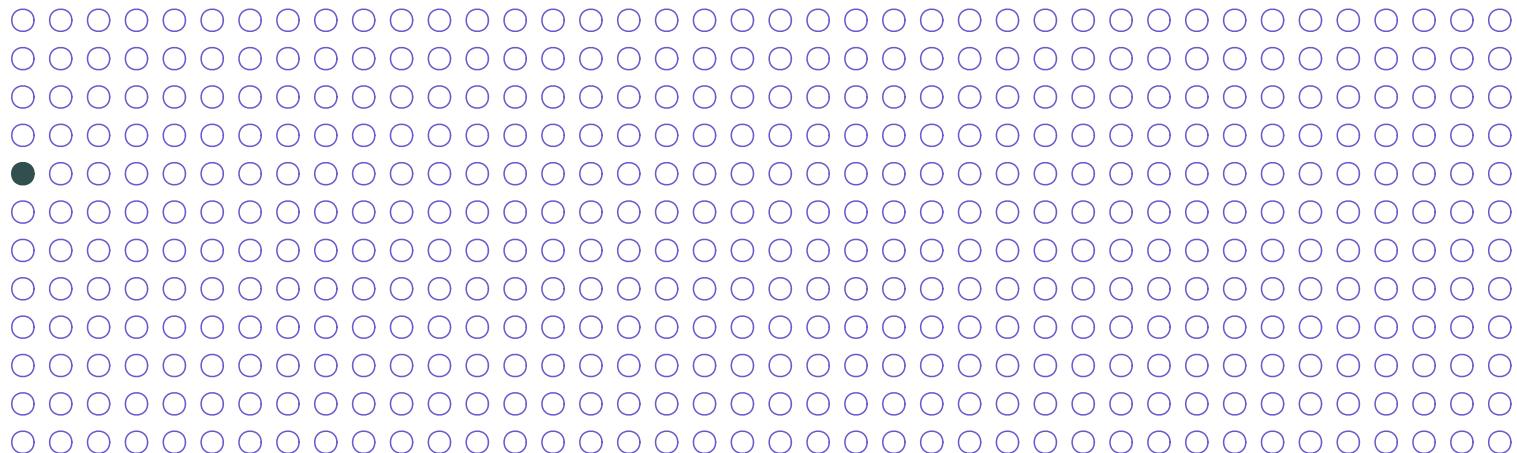


Observation 4's turn for validation produces  $MSE_4$ .

# Hold-out methods

## Option 2: Leave-one-out cross validation

Each observation takes a turn as the **validation set**,  
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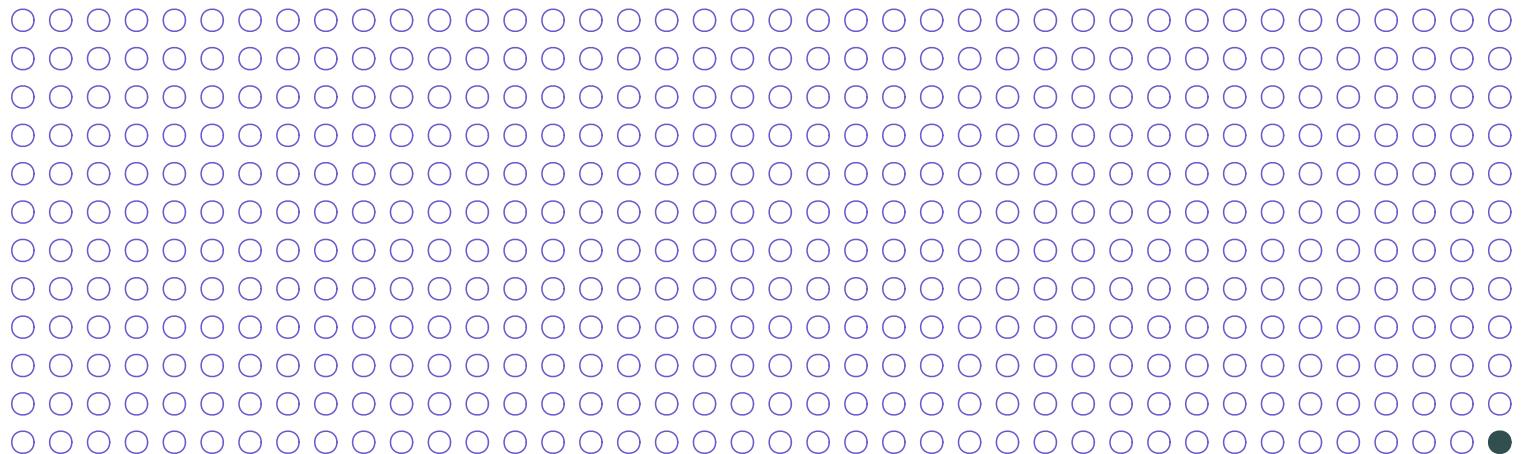


Observation 5's turn for validation produces  $MSE_5$ .

# Hold-out methods

## Option 2: Leave-one-out cross validation

Each observation takes a turn as the **validation set**,  
while the other  $n-1$  observations get to **train the model**.



Observation  $n$ 's turn for validation produces  $MSE_n$ .

# Hold-out methods

## Option 2: Leave-one-out cross validation

Because **LOOCV uses n-1 observations** to train the model,<sup>†</sup>  $\text{MSE}_i$  (validation MSE from observation i) is approximately unbiased for test MSE.

**Problem**  $\text{MSE}_i$  is a terribly noisy estimator for test MSE (albeit  $\approx$ unbiased).

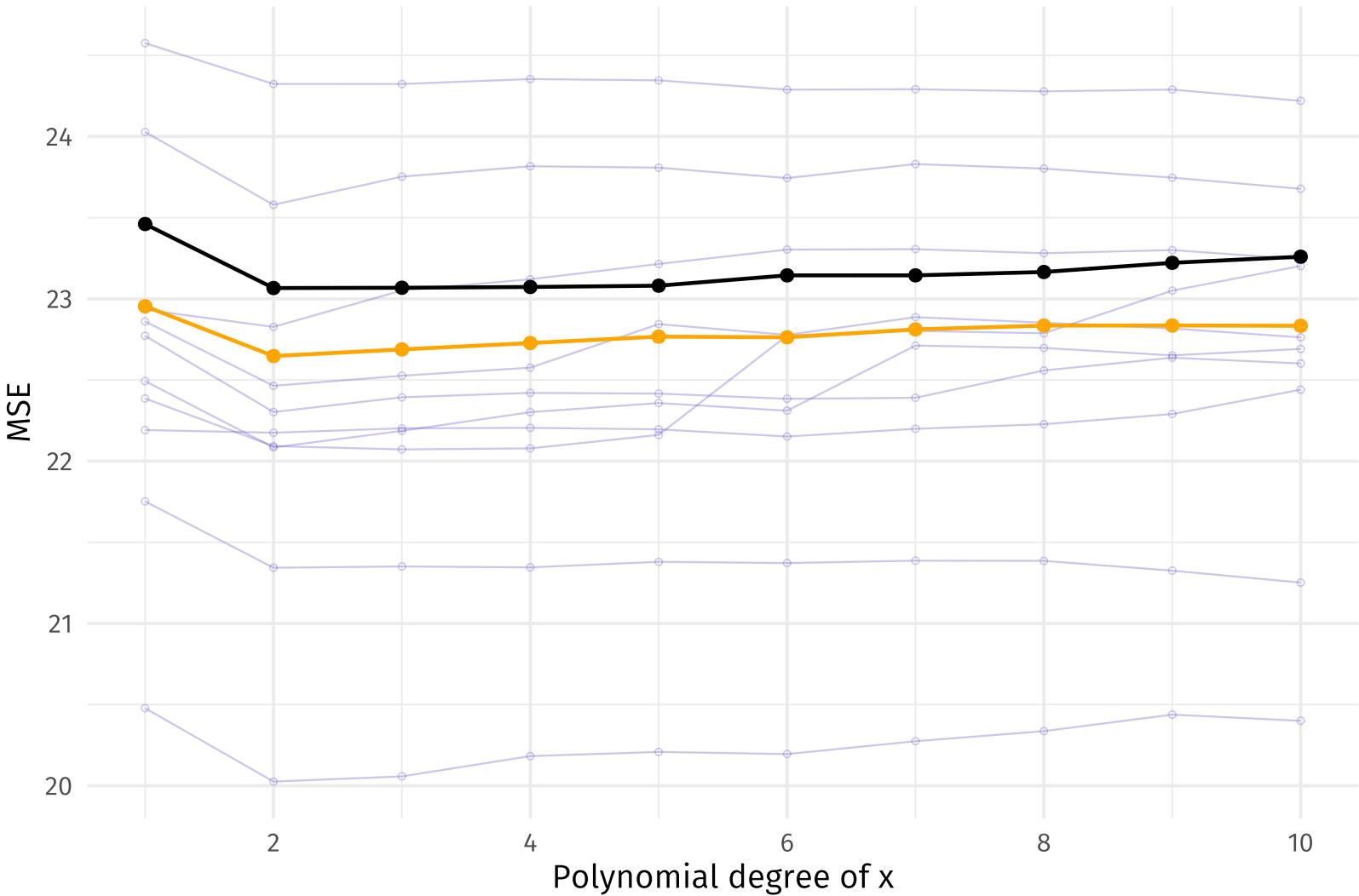
**Solution** Take the mean!

$$\text{CV}_{(n)} = \frac{1}{n} \sum_{i=1}^n \text{MSE}_i$$

1. LOOCV **reduces bias** by using n-1 (almost all) observations for training.
2. LOOCV **resolves variance**: it makes all possible comparison (no dependence upon which validation-test split you make).

<sup>†</sup> And because often  $n-1 \approx n$ .

## True test MSE and LOOCV MSE compared to validation-set estimates



# Hold-out methods

## Option 3: k-fold cross validation

Leave-one-out cross validation is a special case of a broader strategy:  
**k-fold cross validation**.

1. **Divide** the training data into  $k$  equally sized groups (folds).
2. **Iterate** over the  $k$  folds, treating each as a validation set once (training the model on the other  $k - 1$  folds).
3. **Average** the folds' MSEs to estimate test MSE.

Benefits?

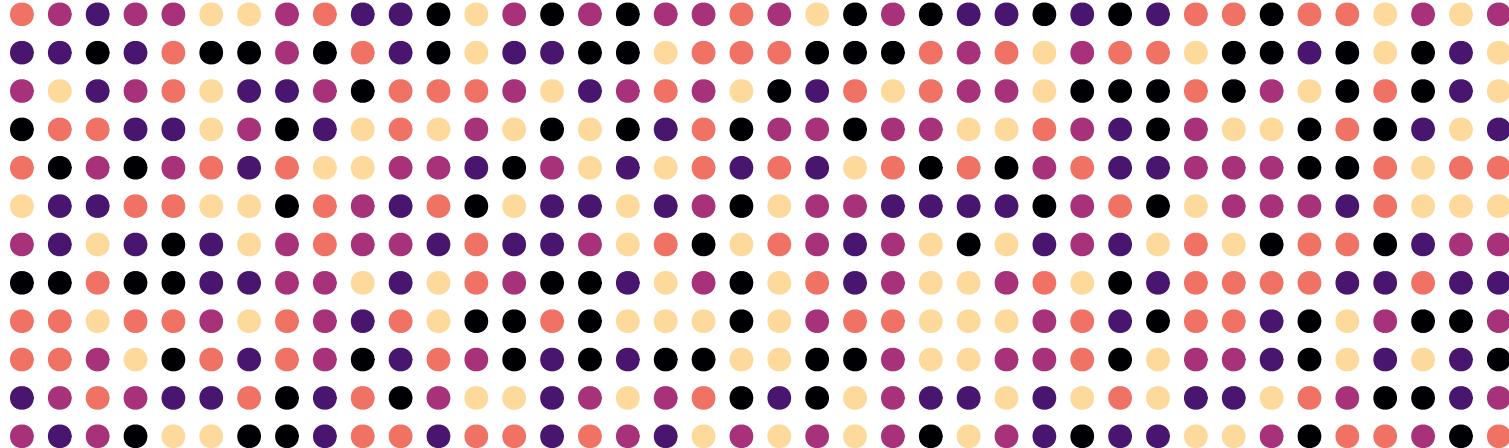
1. **Less computationally demanding** (fit model  $k = 5$  or 10 times; not  $n$ ).
2. **Greater accuracy** (in general) due to bias-variance tradeoff!
  - Somewhat higher bias, relative to LOOCV:  $n - 1$  vs.  $(k - 1)/k$ .
  - Lower variance due to high-degree of correlation in LOOCV  $\text{MSE}_i$ . 

# Hold-out methods

## Option 3: k-fold cross validation

With  $k$ -fold cross validation, we estimate test MSE as

$$\text{CV}_{(k)} = \frac{1}{k} \sum_{i=1}^k \text{MSE}_i$$



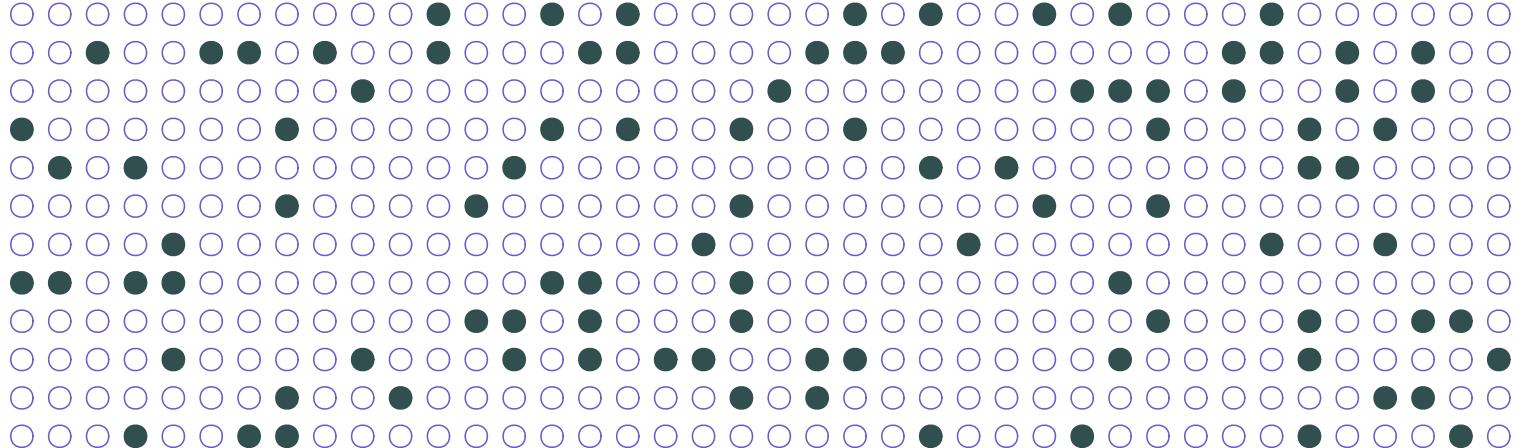
Our  $k = 5$  folds.

# Hold-out methods

## Option 3: k-fold cross validation

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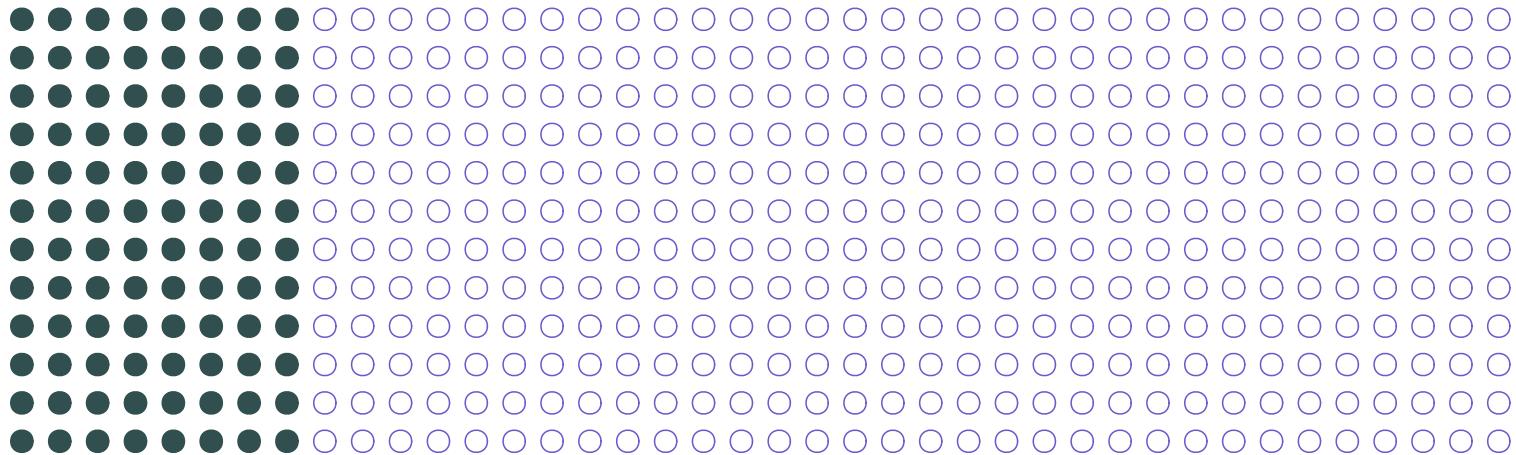
Each fold takes a turn at **validation**. The other  $k - 1$  folds **train**.

# Hold-out methods

## Option 3: k-fold cross validation

With  $k$ -fold cross validation, we estimate test MSE as

$$\text{CV}_{(k)} = \frac{1}{k} \sum_{i=1}^k \text{MSE}_i$$



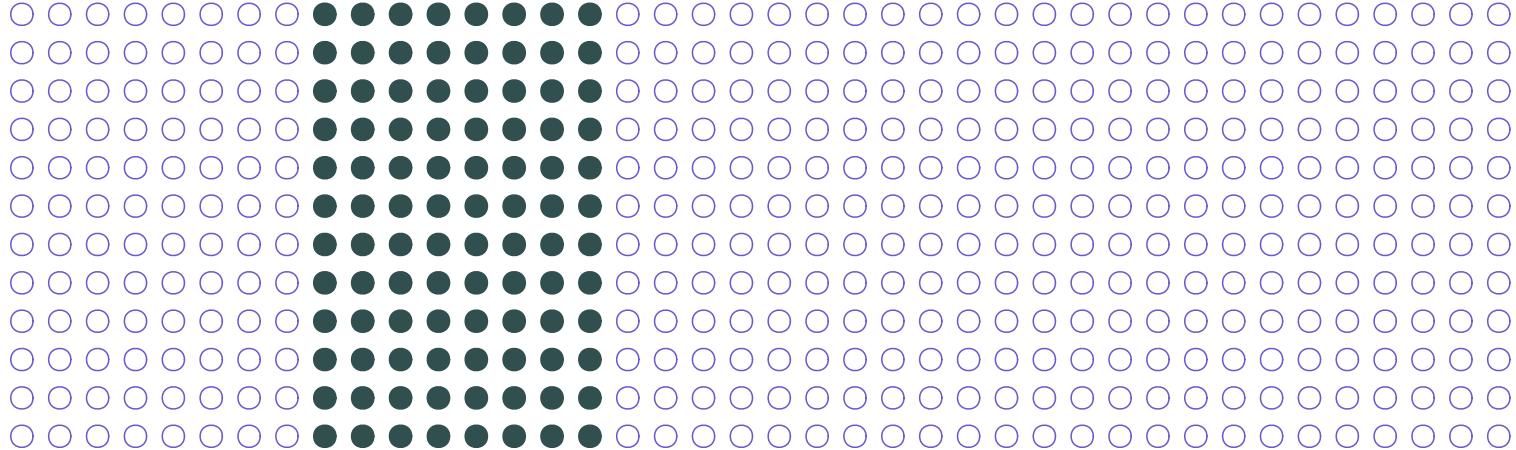
For  $k = 5$ , fold number 1 as the **validation set** produces  $\text{MSE}_{k=1}$ .

# Hold-out methods

## Option 3: k-fold cross validation

With  $k$ -fold cross validation, we estimate test MSE as

$$\text{CV}_{(k)} = \frac{1}{k} \sum_{i=1}^k \text{MSE}_i$$



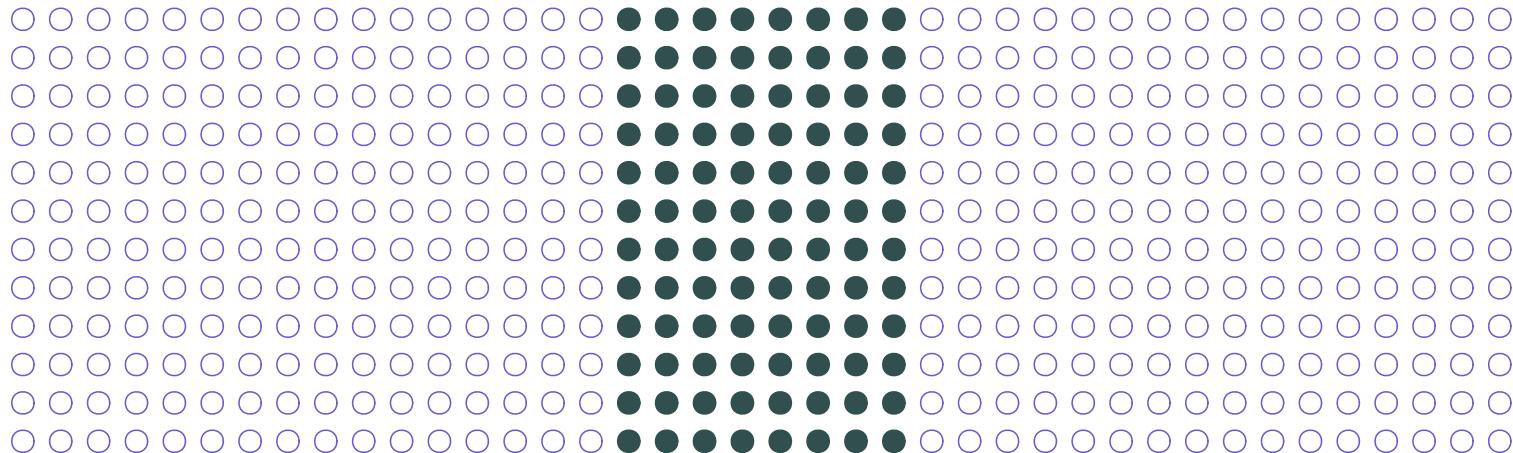
For  $k = 5$ , fold number 2 as the **validation set** produces  $\text{MSE}_{k=2}$ .

# Hold-out methods

## Option 3: k-fold cross validation

With  $k$ -fold cross validation, we estimate test MSE as

$$\text{CV}_{(k)} = \frac{1}{k} \sum_{i=1}^k \text{MSE}_i$$



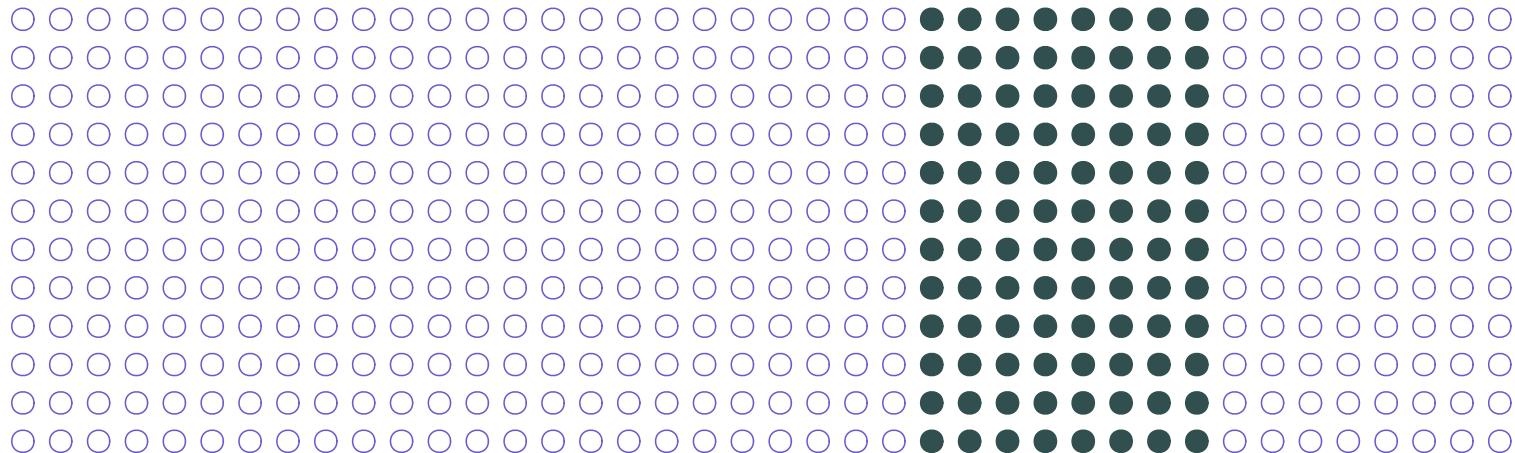
For  $k = 5$ , fold number 3 as the **validation set** produces  $\text{MSE}_{k=3}$ .

# Hold-out methods

## Option 3: k-fold cross validation

With  $k$ -fold cross validation, we estimate test MSE as

$$\text{CV}_{(k)} = \frac{1}{k} \sum_{i=1}^k \text{MSE}_i$$



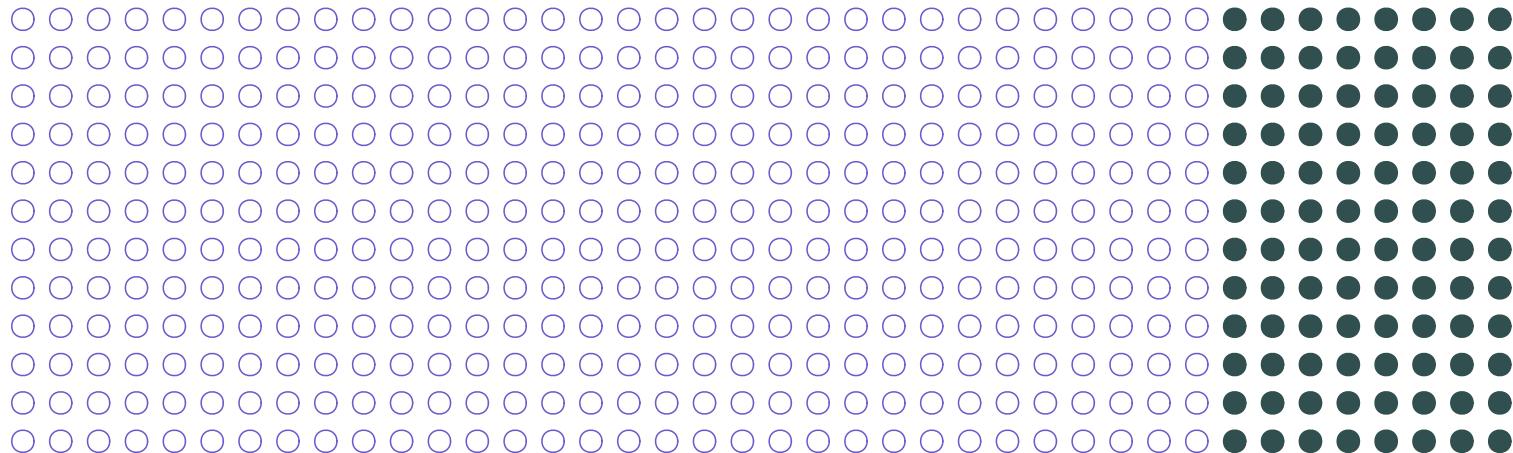
For  $k = 5$ , fold number 4 as the **validation set** produces  $\text{MSE}_{k=4}$ .

# Hold-out methods

## Option 3: k-fold cross validation

With  $k$ -fold cross validation, we estimate test MSE as

$$\text{CV}_{(k)} = \frac{1}{k} \sum_{i=1}^k \text{MSE}_i$$



For  $k = 5$ , fold number 5 as the **validation set** produces  $\text{MSE}_{k=5}$ .

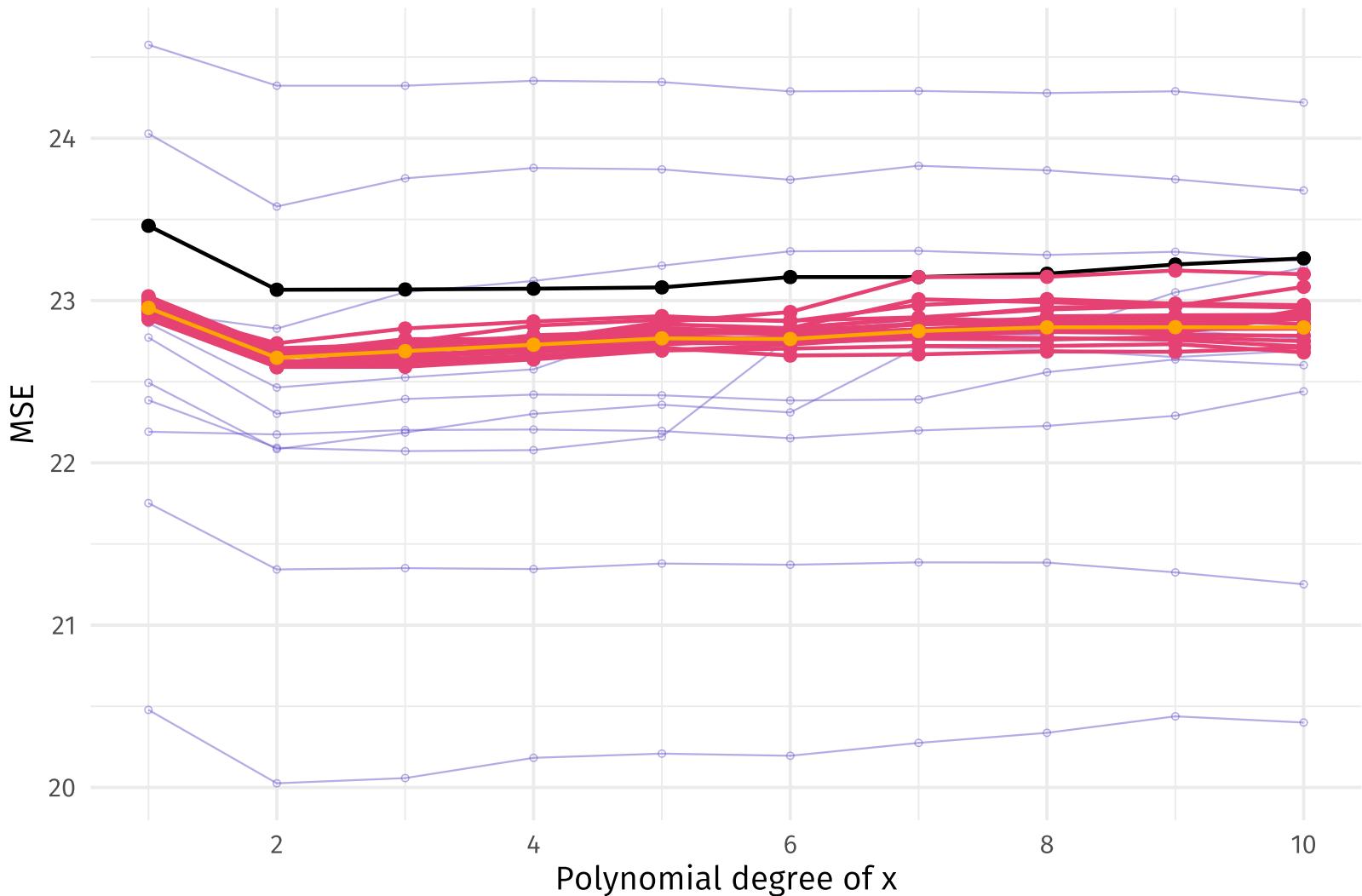
# Hold-out methods

## Option 3: k-fold cross validation

With  $k$ -fold cross validation, we estimate test MSE as

$$\text{CV}_{(k)} = \frac{1}{k} \sum_{i=1}^k \text{MSE}_i$$

## Test MSE vs. estimates: LOOCV, 5-fold CV (20x), and validation set (10x)



Note: Each of these methods extends to classification settings, e.g., LOOCV

$$\text{CV}_{(n)} = \frac{1}{n} \sum_{i=1}^n \mathbb{I}(\textcolor{orange}{y_i} \neq \hat{y}_i)$$

# Hold-out methods

## Caveat

So far, we've treated each observation as separate/independent from each other observation.

The methods that we've defined so far actually need this independence.

# Hold-out methods

## Goals and alternatives

You can use CV for either of two important **modeling tasks**:

- **Model selection** Choosing and tuning a model
- **Model assessment** Evaluating a model's accuracy

*Alternative approach:* **Shrinkage methods**

- fit a model that contains *all  $p$  predictors*
- simultaneously: *shrink<sup>†</sup> coefficients* toward zero

*Idea:* Penalize the model for coefficients as they move away from zero.

<sup>†</sup> Synonyms for *shrink*: constrain or regularize

# Shrinkage

## Why?

**Q** How could shrinking coefficients toward zero help our predictions?

**A** Remember we're generally facing a tradeoff between bias and variance.

- Shrinking our coefficients toward zero **reduces the model's variance**.<sup>†</sup>
- **Penalizing** our model for **larger coefficients** shrinks them toward zero.
- The **optimal penalty** will balance reduced variance with increased bias.

Now you understand shrinkage methods.

- **Ridge regression**
- **Lasso**
- **Elasticnet**

<sup>†</sup> Imagine the extreme case: a model whose coefficients are all zeros has no variance.

# Ridge regression

# Ridge regression

## Back to least squares (again)

Recall Least-squares regression gets  $\hat{\beta}_j$ 's by minimizing RSS, i.e.,

$$\min_{\hat{\beta}} \text{RSS} = \min_{\hat{\beta}} \sum_{i=1}^n e_i^2 = \min_{\hat{\beta}} \sum_{i=1}^n \left( \textcolor{orange}{y_i} - \underbrace{\left[ \hat{\beta}_0 + \hat{\beta}_1 x_{i,1} + \cdots + \hat{\beta}_p x_{i,p} \right]}_{=\hat{y}_i} \right)^2$$

**Ridge regression** makes a small change

- adds a **shrinkage penalty** = the sum of squared coefficients  $(\lambda \sum_j \beta_j^2)$
- minimizes the (weighted) sum of RSS and the shrinkage penalty

$$\min_{\hat{\beta}^R} \sum_{i=1}^n \left( \textcolor{orange}{y_i} - \hat{y}_i \right)^2 + \lambda \sum_{j=1}^p \beta_j^2$$

# Ridge regression

## Ridge regression

$$\min_{\hat{\beta}^R} \sum_{i=1}^n \left( \mathbf{y}_i - \hat{\mathbf{y}}_i \right)^2 + \lambda \sum_{j=1}^p \beta_j^2$$

## Least squares

$$\min_{\hat{\beta}} \sum_{i=1}^n \left( \mathbf{y}_i - \hat{\mathbf{y}}_i \right)^2$$

$\lambda$  ( $\geq 0$ ) is a tuning parameter for the harshness of the penalty.

$\lambda = 0$  implies no penalty: we are back to least squares.

Each value of  $\lambda$  produces a new set of coefficients.

Ridge's approach to the bias-variance tradeoff: Balance

- reducing **RSS**, i.e.,  $\sum_i (\mathbf{y}_i - \hat{\mathbf{y}}_i)^2$
- reducing **coefficients** (ignoring the intercept)

$\lambda$  determines how much ridge "cares about" these two quantities.<sup>†</sup>

<sup>†</sup> With  $\lambda = 0$ , least-squares regression only "cares about" RSS.

# Ridge regression

## $\lambda$ and penalization

Choosing a *good* value for  $\lambda$  is key.

- If  $\lambda$  is too small, then our model is essentially back to OLS.
- If  $\lambda$  is too large, then we shrink all of our coefficients too close to zero.

**Q** So what do we do?

**A** Cross validate!

(You saw that coming, right?)

# Ridge regression

## Penalization and standardization

**Important** Predictors' **units** can drastically **affect ridge regression results**.

**Why?** Because  $\mathbf{x}_j$ 's units affect  $\beta_j$ , and ridge is very sensitive to  $\beta_j$ .

*Example* Let  $x_1$  denote distance.

### Least-squares regression

If  $x_1$  is *meters* and  $\beta_1 = 3$ , then when  $x_1$  is *km*,  $\beta_1 = 3,000$ .

The scale/units of predictors do not affect least squares' estimates.

**Ridge regression** pays a much larger penalty for  $\beta_1 = 3,000$  than  $\beta_1 = 3$ .

You will not get the same (scaled) estimates when you change units.

*Solution* Standardize your variables, i.e.,  $x_{\text{stnd}} = (x - \text{mean}(x))/\text{sd}(x)$ .

# Lasso

# Lasso

## Intro

Lasso simply replaces ridge's *squared* coefficients with absolute values.

## Ridge regression

$$\min_{\hat{\beta}^R} \sum_{i=1}^n (\textcolor{orange}{y_i} - \hat{y}_i)^2 + \lambda \sum_{j=1}^p \beta_j^2$$

## Lasso

$$\min_{\hat{\beta}^L} \sum_{i=1}^n (\textcolor{orange}{y_i} - \hat{y}_i)^2 + \lambda \sum_{j=1}^p |\beta_j|$$

Everything else will be the same—except one aspect...

# Lasso

## Shrinkage

Unlike ridge, lasso's penalty does not increase with the size of  $\beta_j$ .

You always pay  $\lambda$  to increase  $|\beta_j|$  by one unit.

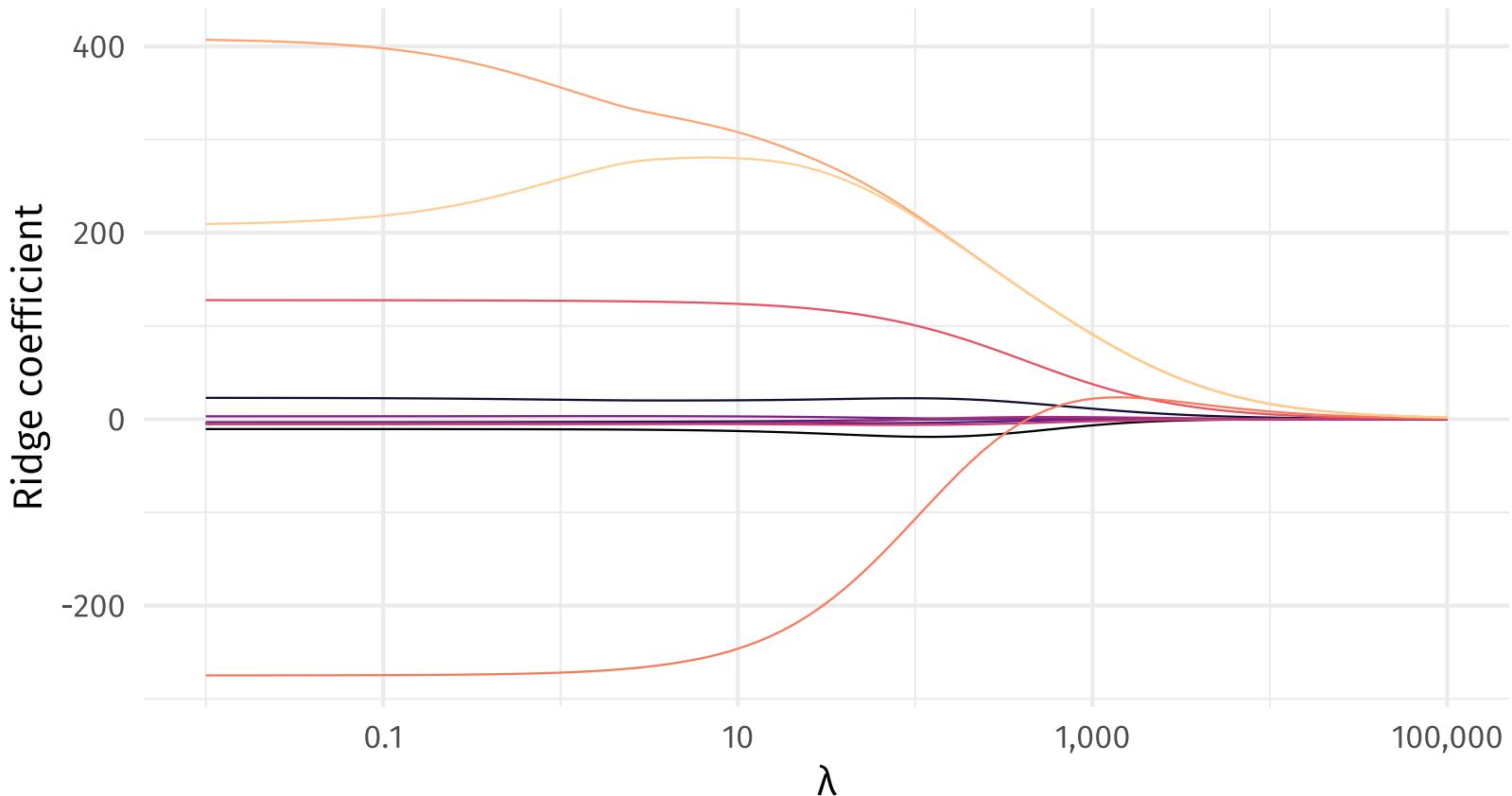
The only way to avoid lasso's penalty is to **set coefficients to zero**.

This feature has two **benefits**

1. Some coefficients will be **set to zero**—we get "sparse" models.
2. Lasso can be used for subset/feature **selection**.

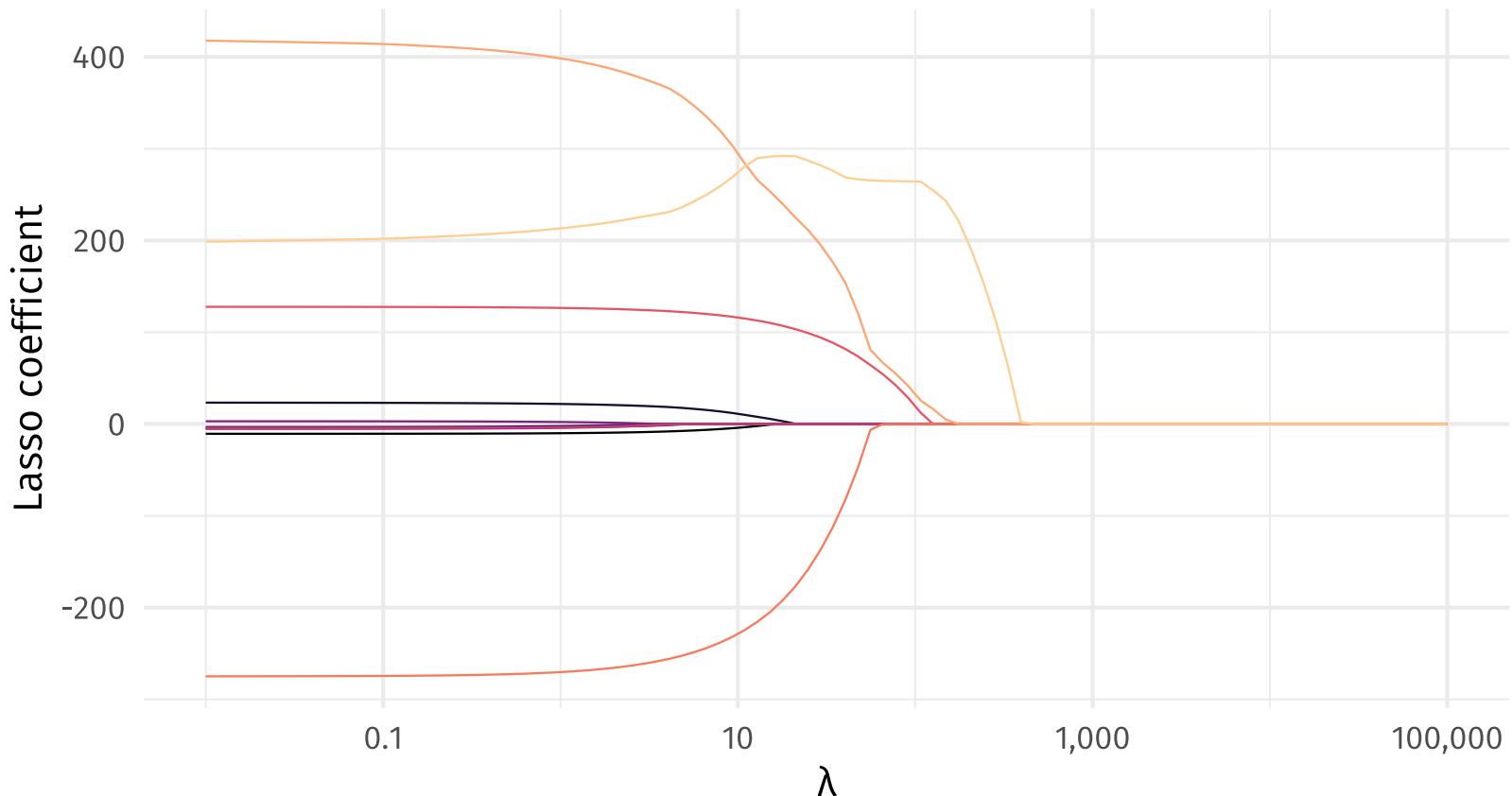
We will still need to carefully select  $\lambda$ .

## Ridge regression coefficients for $\lambda$ between 0.01 and 100,000



Predictor	— age	— cards	— education	— i_african_american	— i_asian_american	— i_female	— i_married	— i_student	— income	— limit	— rating
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## Lasso coefficients for $\lambda$ between 0.01 and 100,000



Predictor	age	i_african_american	i_married	limit
	—	—	—	—
cards	—	—	—	—
education	—	—	—	—
i_asian_american	—	—	—	—
i_female	—	—	—	—
i_student	—	—	—	—
income	—	—	—	—
rating	—	—	—	—

# Machine learning

## Wrap up

Now you understand the basic tenants of machine learning:

- How **prediction** differs from causal inference
- **Bias-variance tradeoff** (the benefits and costs of flexibility)
- **Cross validation**: Performance and tuning
- In- vs. out-of-sample **performance**

# Sources

Sources (articles) of images

- Deep learning and radiology
- Parking lot detection
- *New Yorker* writing
- Gender Shades

I pulled the comic from [Twitter](#).