TOWARDS LARGE-EDDY SIMULATION OF COMPLEX GEOMETRY TURBULENT BOUNDARY LAYER FLOWS: EXTENSION OF FRACTIONAL STEP METHOD TO NONORTHOGONAL CURVILINEAR SYSTEMS

by
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ABSTRACT

This report documents the *details* of extension of the fractional step method to three-dimensional, time-dependent incompressible flows in non-orthogonal curvilinear coordinate systems. A formulation based on block-LU decomposition is combined with a mixed implicit/explicit treatment of the discretized equations. using local volume fluxes as dependent variables, the block-LU decomposition enables a unique definition of the sequential operations of the fractional step method for general coordinate systems. As a result of the LU decomposition, boundary information for the velocity field is carried directly from the discretized momentum and continuity equations; therefore no artificial boundary conditions are required. In this work a semi-direct scheme is developed for solution of the Poisson equation using series expansion along one coordinate direction which is discretized on a uniform, Cartesian grid. The relaxation parameter in the Poisson solver was optimized with respect to wavenumber through numerical experiments. The overall accuracy of the method developed in this work is demonstrated through comparison of numerical solutions of unsteady decaying vortices, polar lid-driven cavity flow, and skewed lid-driven cavity flow to analytical, experimental, and existing numerical solutions.

CONTENTS

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NOMENCLATURE

A	coefficient submatrix
$\mathbf{A}, \mathbf{B}, \mathbf{C}$	vector symbols
C	coefficients of discretized Poisson equation
$\mathbf{cbc}$	boundary condition vector for continuity equation
D	discrete divergence operator
$D_q, D_{q,ex}$	implicit and explicit of diffusion terms
div	continuous divergence operator
$\mathbf{e_q}$	natural local base in general system
$\mathbf{F}$	total flux through the cell
G	discrete gradient operator
$grad$	continuous gradient operator
H_q	convective term
I	identity matrix
k_m'	modified wave number
L	diffusive term
L_{im}, L_{ex}	implicit and explicit part of diffusive term
l_z	domain length in z direction
$\mathbf{mbc}$	boundary condition vector for momentum equation
m	index in Fourier transformation
N_z	total grid points in z direction
P	pressure
Q	right-hand-side of the Poisson equation
q, q'	coordinates of general system, ξ, η, z
R_q	pressure terms
$\mathbf{r}$	radius vector
$\mathbf{rhs}$	right-hand-side vector
r	radius
r_0	radius of polar cavity inner wall
Re	characteristic Reynolds number
$\mathbf{S}$	surface tensor bounding control volume V
$\mathbf{S}^q, \mathbf{S}_q$	reciprocal bases in general system
t	time
$\mathbf{u}, u_i, u_j, u_k$	Cartesian velocity vector
$\hat{\mathbf{u}}$	Cartesian intermediate velocity vector
$\mathbf{u^q}, u^\xi, u^\eta, u^z$	velocity flux vector in general system
$\hat{\mathbf{u}}, \hat{u}^\xi, \hat{u}^\eta, \hat{u}^z$	intermediate velocity flux vector in general system
$V, V^{\phi_{ijk}}$	control volume for general variable ϕ
x_1, x_2, x_3	Cartesian coordinates for Taylor-Green flow

Greek Letters

α	grid crossing angle
β	cavity skewing angle
ξ, η, z	non-orthogonal curvilinear coordinates
ρ	density
μ	molecular viscosity
ν	kinematic viscosity
Γ	boundary in physical space
ϕ	temporal pressure flux variable
Φ_{ijk}	general variables
θ	angular position in polar lid-driven cavity flow
δ	Kronecker delta
$\Delta\xi, \Delta\eta, \Delta z$	grid spacing in ξ, η, z direction

Subscripts

$cross$	cross-derivative terms
$comp$	computational solution
$exac$	exact solution
im, ex	implicit and explicit terms
max	maximum value
∞	undisturbed incoming flow
Γ	variables defined on boundary
q, ξ, η, z	covariant components
x, y, z	components along x, y, z directions

Superscripts

0	initial conditions
$n, n-1$	variables at old time step n and $n-1$
$n+1$	variables at new time step $n+1$
$*$	intermediate variable
q, ξ, η, z	contravariant components
T	transpose operation

CHAPTER 1
INTRODUCTION

A common approach adopted for solution of the unsteady, incompressible Navier-Stokes equations is the fractional step method [1], [2]. The method is comprised of three sequential operations: computation of a provisional velocity field using the non-linear and viscous terms; calculation of the pressure field by solving the Poisson equation, and finally projection of the intermediate velocity field onto a divergence-free space at the new time step using the pressure gradient. For calculations on staggered grids the fractional step method is a variant of the well known Marker-and-Cell method of Harlow & Welch [3].

Kim & Moin [4] developed a fractional step method based on a time splitting of the continuous equations. This approach has subsequently been used for calculation of a large number of incompressible flows, albeit mostly for simple geometries [5], [6]. While these and other applications have demonstrated the robustness of the method, the wide application of traditional time-splitting fractional step methods has also been accompanied by considerable controversy about the method itself, particularly surrounding the correct formulation of boundary conditions for the intermediate solution variables. Because time-splitting is performed before discretization, the only known boundary conditions are for the velocity field, those for the intermediate velocity and pressure fields are left unknown. Kim & Moin [4] found that unless the boundary conditions for the intermediate velocity field are chosen to be consistent with the governing equations, the solution may suffer from appreciable numerical errors. More recently, Perot [7] analyzed the traditional fractional step method by treating it as an approximate block-LU factorization of the fully discretized equations instead of a time splitting scheme. Perot showed that boundary conditions for the intermediate velocity and pressure are carried directly from the discretized governing equations in the LU decomposition and that no *ad hoc* boundary conditions are required.

In contrast to the numerous applications in simple Cartesian systems, relatively little effort has been devoted to application of the fractional step method in complex geometries. Notable is the scheme developed by Rosenfeld *et al.* [8] in which a formulation based on volume flux variables using the traditional time-splitting fractional step method was used to solve the unsteady, three-dimensional Navier-Stokes equation in general curvilinear systems. The sequential operations in the algorithm in Ref. [8] were derived through a series of special manipulations rather than from properties of the general fractional step scheme. Time-splitting was performed before spatial discretization, thus inevitably bringing in the issue of boundary condition for the intermediate velocity variable. Further, in the scheme developed in Ref. [8] all gradient and divergence operators are evaluated via their coordinate-free definitions $[\nabla(\cdots) = \int\int_S(\cdots)d\mathbf{S}/V]$. However, it is not clear how this could be implemented at boundaries where no closed bounding surface exists, unless some of the velocity components are defined outside the physical domain.

The objective of this study has been extension of the fractional step method to the unsteady three-dimensional Navier-Stokes equations for incompressible flows; the principal goal being construction of a time-accurate fractional step scheme in general curvilinear systems which is consistent with the LU decomposition developed in Ref. [7]. It is shown that by expanding the Cartesian velocities with respect to a new area-vector basis in a general curvilinear system, sequential operations in the method are uniquely defined using the LU decomposition and specification of boundary conditions for the provisional velocity field are not required. Spatial discretization of the momentum equations is performed on a staggered grid using the contravariant components (local volume fluxes in physical space) as dependent variables in the new coordinate system. The method developed in this study is applied to geometries which are complex in two dimensions, i.e., the third (e.g., spanwise) direction is treated using Cartesian coordinates and periodic boundary conditions. This in turn permits application of a semi-direct method for solution of the Poisson equation. Through Fourier transformation along the periodic direction the three-dimensional Poisson equation is reduced to a series of wavenumber-dependent two-dimensional problems which are solved using iterative methods.

Following in §2 is a summary of the governing equations. §3 presents the derivations of the fractional step method for generalized curvilinear systems, including thorough documentation of the spatial and temporal discretization procedures employed in this study. Optimization of solution to the Poisson equation for geometries which are curvilinear in two coordinate directions is also discussed in §3. Finally, results obtained using the method developed in this study are presented in §4 in which the accuracy and robustness of the numerical scheme is investigated through calculation of unsteady Taylor-Green decaying vortices in orthogonal as well as non-orthogonal systems, polar lid-driven cavity flow, and skewed lid-driven cavity flow.

CHAPTER 2
GOVERNING EQUATIONS

The unsteady incompressible Navier-Stokes equation written in tensor form is,

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \tag{2.1}$$

due to the following identities,

$$\frac{\partial}{\partial x_j}(u_i u_j) = u_i\frac{\partial u_j}{\partial x_j} + u_j\frac{\partial u_i}{\partial x_j} \qquad \frac{\partial u_j}{\partial x_j} = 0$$

and

$$\frac{\partial P}{\partial x_i} = \frac{\partial}{\partial x_j}(P\delta_{ij})$$

(2.1) is equivalent to

$$\rho\frac{\partial u_i}{\partial t} + \rho\frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial}{\partial x_j}(P\delta_{ij}) + \frac{\partial}{\partial x_j}\mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \tag{2.2}$$

Suppose that the characteristic length and velocity of the flow being concerned is L and u_∞, respectively. It is then possible to render (2.2) dimensionless by introducing

$$u_i = u_i/u_\infty, \qquad x = x/L, \qquad P = P/(\rho u_\infty^2), \qquad t = tL/u_\infty$$

thus, the non-dimensional form of (2.2) becomes

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial}{\partial x_j}(P\delta_{ij}) + \frac{\partial}{\partial x_j}\frac{\nu}{u_\infty L}[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}] \tag{2.3}$$

if $\nu = constant$, (2.3) reduces to the form that appears frequently in literature,

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial P}{\partial x_i} + \frac{\nu}{u_\infty L}\frac{\partial}{\partial x_j}(\frac{\partial u_i}{\partial x_j}) \tag{2.4}$$

However in this study (2.3) will be used for proceeding derivations due to the fact that the viscous terms of (2.3) are in a form that is easily replaceable by a variable kinematic viscosity coefficient (subgrid stress models). It is further noted that the term $\partial u_i/\partial x_j + \partial u_j/\partial x_i$ is just twice the rate of deformation tensor S_{ij}.

Integrate (2.3) over an arbitrary time-constant control volume V,

$$\begin{aligned}\int\int\int_V \frac{\partial u_i}{\partial t}dV &= \int\int\int_V \frac{\partial}{\partial x_j}(-u_i u_j)dV + \int\int\int_V \frac{\partial}{\partial x_j}(-P\delta_{ij})dV \\ &\quad + \int\int\int_V \frac{\partial}{\partial x_j}\frac{\nu}{u_\infty L}[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}]dV \end{aligned} \tag{2.5}$$

For simplicity, define a Reynolds number such as $Re = u_\infty L/\nu(\mathbf{r})$. With the use of divergence and gradient operator symbols, (2.5) becomes

$$\begin{aligned}\int\int\int_V \frac{\partial \mathbf{u}}{\partial t} dV &= \int\int\int_V \nabla \bullet (-\mathbf{u}\mathbf{u}) dV + \int\int\int_V \nabla \bullet (-P\overline{I}) dV \\ &+ \int\int\int_V \nabla \bullet \frac{1}{Re}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] dV \end{aligned} \tag{2.6}$$

Note: **uu** is a dyad,(second order tensor), there is no multiplication sign and it should not be confused with the scalar or vector product.

With the aid of Gauss' theorem $\int\int\int_V \nabla \bullet \mathbf{A} dV = \int\int_S \mathbf{A} \bullet d\mathbf{S}$, the equations governing the flow of constant density isothermal incompressible fluids in a time-constant control volume V with face S can finally be written as:

$$\int\int_S \mathbf{u} \bullet d\mathbf{S} = 0 \tag{2.7}$$

$$\int\int\int_V \frac{\partial \mathbf{u}}{\partial t} dV = \int\int_S \overline{T} \bullet d\mathbf{S} \tag{2.8}$$

where $\overline{T}$ is given by operator form

$$\overline{T} = -\mathbf{u}\mathbf{u} - P\overline{I} + \frac{1}{Re}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \tag{2.9}$$

or in tensor notation, $\overline{T}$ can be expressed as

$$\overline{T}_{ij} = -u_i u_j - P\delta_{ij} + \frac{1}{Re}[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}] \tag{2.10}$$

CHAPTER 3
NUMERICAL FORMULATIONS

3.1 Basic Geometric Quantities

The first step to solve (2.7) and (2.8) in a two-dimensionally complex geometry is to transform the radius vector $\mathbf{r}$ from the Cartesian system to a curvilinear non-orthogonal system as shown in Figure 3.1,

$$\mathbf{r}(x, y, z) \rightarrow \mathbf{r}(\xi, \eta, z)$$

note z direction is retained as Cartesian coordinate and usually flow is assumed to be statistically homogeneous along this direction to facilitate space averaging.

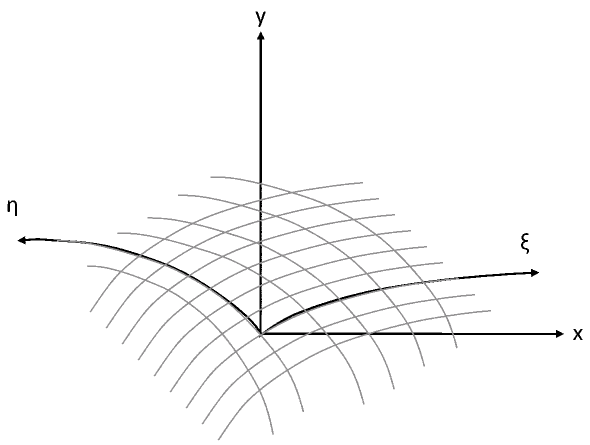

Figure 3.1: Curvilinear coordinate system.

According to Rosenfeld *et al* [8], at an arbitrary point (i, j, k) the area tensor is defined as

$$\mathbf{S} = (\mathbf{S}^{\xi}, \mathbf{S}^{\eta}, \mathbf{S}^{z})$$

where the vector components are defined as

$$\mathbf{S}^{\xi} = \frac{\partial \mathbf{r}}{\partial \eta} \times \frac{\partial \mathbf{r}}{\partial z}, \qquad \mathbf{S}^{\eta} = \frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial \xi}, \qquad \mathbf{S}^{z} = \frac{\partial \mathbf{r}}{\partial \xi} \times \frac{\partial \mathbf{r}}{\partial \eta}$$

or in a more compact form

$$\mathbf{S}^{q} = \frac{\partial \mathbf{r}}{\partial (q+1)} \times \frac{\partial \mathbf{r}}{\partial (q+2)} \tag{3.1}$$

where $q = \xi, \eta, z$ are in cyclic order. The vector quantity $\mathbf{S}^{q}$ has the magnitude of the area of the face and a direction normal to it.

The partial derivative of location vector $\mathbf{r}(x(\xi,\eta), y(\xi,\eta), z)$ with respect to ξ, η, z can be evaluated by using the central finite difference approximation which is easy to be done. For example, with reference to Figure 3.2,

$$\frac{\partial \mathbf{r}}{\partial \xi}_P = \frac{\mathbf{r}[x(\xi,\eta,z), y(\xi,eta), z]_e - [x(\xi,\eta,z), y(\xi,eta), z]_w}{\Delta\xi}$$

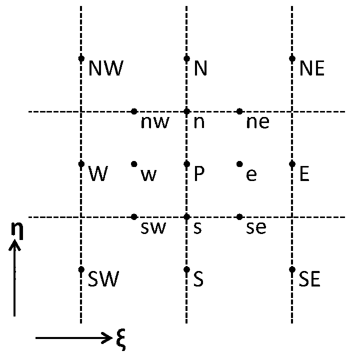

Figure 3.2: Staggered grid arrangement.

The evaluated area vectors $\mathbf{S}^q$ should satisfy the following restraint, (a cell is closed),

$$\int\int_S d\mathbf{S} = 0 \tag{3.2}$$

or, in discrete form

$$\sum_q \mathbf{S}^q = 0 \tag{3.3}$$

To satisfy (3.2) exactly in the computation, $\partial\mathbf{r}/\partial q$ should be carefully evaluated using grid point locations. In this work, only at pressure control volume are all the area vectors $\mathbf{S}_q$ and their reciprocal bases evaluated and stored. Area vectors for velocity control volumes are obtained via simple arithmetic averaging the primary pressure control volume quantities.

Also according to [8], the volume of the computational cell is evaluated from

$$V = \frac{\mathbf{S}^\xi_w + \mathbf{S}^\eta_s + \mathbf{S}^z_b}{3} \bullet (\mathbf{r}_{fne} - \mathbf{r}_{bsw}) \tag{3.4}$$

The evaluated cell volume should satisfy the following restraint

$$\sum_{cells} V = V_{domain} \tag{3.5}$$

(3.5) is not satisfied exactly in the computation, as a matter of fact it is satisfied asymptotically as the grid spacing decreases.

3.2 Discretization of Continuity Equation

Discretization of the integral continuity equation (2.7) over the primary control volume for pressure such as shown on Figure 3.2 yields,

$$\begin{aligned}(\mathbf{S}^\xi \bullet \mathbf{u})_e - (\mathbf{S}^\xi \bullet \mathbf{u})_w + (\mathbf{S}^\eta \bullet \mathbf{u})_n - (\mathbf{S}^\eta \bullet \mathbf{u})_s & \\ (\mathbf{S}^z \bullet \mathbf{u})_f - (\mathbf{S}^z \bullet \mathbf{u})_b &= 0\end{aligned}$$

Note that although (2.7) holds true for any field with $\nabla \mathbf{u} = 0$, the above equation is not true in general, again it is satisfied asymptotically as the grid spacing decreases.

A discretized mass conservation equation, which is identical to the form of the Cartesian case, can be derived if the volume fluxes over the faces of the computational cells are chosen as the unknowns instead of the Cartesian velocity components. Define

$$\begin{aligned}u^\xi &= \mathbf{S}^\xi \bullet \mathbf{u} = S^\xi_x u_i + S^\xi_y u_j + S^\xi_z u_k \\ u^\eta &= \mathbf{S}^\eta \bullet \mathbf{u} = S^\eta_x u_i + S^\eta_y u_j + S^\eta_z u_k \\ u^z &= \mathbf{S}^z \bullet \mathbf{u} = S^z_x u_i + S^z_y u_j + S^z_z u_k\end{aligned} \tag{3.6}$$

where u^ξ, u^η, u^z are the volume fluxes over the ξ, η, z faces of a primary control volume, and u_i, u_j, j_k are the Cartesian velocity components.

With the volume fluxes as new unknowns, we have the continuity equation

$$u^\xi_e - u^\xi_w + u^\eta_n - u^\eta_s + u^z_f - u^z_b = 0 \tag{3.7}$$

where $u^q = (u^\xi, u^\eta, u^z)$.

Summation over the whole computational domain yields

$$\sum_{\text{total cell faces}} u^q = 0$$

3.3 Momentum Equations In ξ, η, z Directions

Recall the integral momentum equation with Cartesian velocity components as unknown variables, (2.8),

$$\int\int\int_V \frac{\partial \mathbf{u}}{\partial t} dV = \int\int_S \overline{T} \bullet d\mathbf{S}$$

Approximate (2.8) over an arbitrary time-constant control volume V yields,

$$V\frac{\partial \mathbf{u}}{\partial t} = \sum_q \mathbf{S}^q \bullet \overline{T} = \mathbf{F} \tag{3.8}$$

where $\mathbf{F}$ represents the total flux through the computational cell.

Recall the definitions given in (3.6), the momentum equations along direction q with u^ξ, u^η, u^z as variables are,

$$\mathbf{S}^q_{\text{cell center plane}} \bullet (V\frac{\partial \mathbf{u}}{\partial t}) = V\frac{\partial u^q}{\partial t} = \mathbf{S}^q_{\text{cell center plane}} \bullet \mathbf{F} \tag{3.9}$$

where $q = \xi$ or η or z. These three equations should be further discretized over different staggered control volumes.

3.4 Temporal Discretization of Momentum Equations

Write (3.9) for ξ direction, with reference to Figure 3.2,

$$V_\xi \frac{\partial u^\xi}{\partial t} = \mathbf{S}^\xi_{\text{cell center plane}} \bullet \mathbf{F} = \mathbf{S}^\xi_{\text{cell center plane}} \bullet [\sum_q \mathbf{S}^q \bullet \overline{T}] = L_\xi \tag{3.10}$$

Split L_ξ into several parts,

$$L_\xi = H_\xi + R_\xi + D_\xi + D_{\xi,ex} \tag{3.11}$$

where H_ξ are nonlinear convection terms, R_ξ are pressure term. D_ξ are the implicit diffusion terms and $D_{\xi,ex}$ are the explicit diffusion terms. The reason to split the whole diffusion term into two parts is to facilitate the implementation of fractional step method. Basically, $D_\xi(u^\xi)$ contains terms similar to $\partial^2 u^\xi/\partial\xi^2 + \partial^2 u^\xi/\partial\eta^2 + \partial^2 u^\xi/\partial z^2$, and $D_{\xi,ex}$ contains the rest of the diffusion terms, mostly cross-derivative terms.

We use second order explicit Adams-Bashforth scheme for the nonlinear convection term

$$H_\xi(u^q) = \frac{3}{2}H_\xi(u^q)^n - \frac{1}{2}H_\xi(u^q)^{n-1}$$

we use Euler fully implicit treatment for Pressure terms

$$R_\xi(P) = R_\xi(P)^{n+1}$$

we use Crank-Nicolson scheme for the implicit diffusion terms,

$$D_\xi(u^\xi) = \frac{1}{2}D_\xi(u^\xi)^{n+1} + \frac{1}{2}D_\xi(u^\xi)^n$$

we use second order explicit Adams-Bashforth scheme for the explicit diffusion terms

$$D_{\xi,ex}(u^q) = \frac{3}{2}D_{\xi,ex}(u^q)^n - \frac{1}{2}D_{\xi,ex}(u^q)^{n-1}$$

Note: Rosenfeld *et al* [8] claims that the above scheme is not stable. However, while a seemingly complicated scheme was used, the treatment on explicit diffusion terms are still explicit.

On the other hand, the scheme of Adams-Bashforth/Crank-Nicolson was successfully used by us (so by Kim and Moin [4]).

Finally, the temporal discretization form of (3.10) can be written as

$$\begin{aligned} V_\xi \frac{(u^\xi)^{n+1} - (u^\xi)^n}{\Delta t} &= \frac{1}{2}[3H_\xi(u^q)^n - H_\xi(u^q)^{n-1}] + R_\xi(P^{n+1}) \\ &+ \frac{1}{2}[3D_{\xi,ex}(u^q)^n - D_{\xi,ex}(u^q)^{n-1}] + \frac{1}{2}[D_\xi(u^\xi)^{n+1} + D_\xi(u^\xi)^n] \end{aligned} \tag{3.12}$$

where the specific expressions of H, D, D_{ex}, R are to be derived in the spatial discretization section.

Similarly

$$\begin{aligned} V_\eta \frac{(u^\eta)^{n+1} - (u^\eta)^n}{\Delta t} &= \frac{1}{2}[3H_\eta(u^q)^n - H_\eta(u^q)^{n-1}] + R_\eta(P^{n+1}) \\ &+ \frac{1}{2}[3D_{\eta,ex}(u^q)^n - D_{\eta,ex}(u^q)^{n-1}] + \frac{1}{2}[D_\eta(u^\eta)^{n+1} + D_\eta(u^\eta)^n] \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} V_z \frac{(u^z)^{n+1} - (u^z)^n}{\Delta t} &= \frac{1}{2}[3H_z(u^q)^n - H_z(u^q)^{n-1}] + R_z(P^{n+1}) \\ &+ \frac{1}{2}[3D_{z,ex}(u^z)^n - D_{z,ex}(u^z)^{n-1}] + \frac{1}{2}[D_z(u^z)^{n+1} + D_z(u^z)^n] \end{aligned} \tag{3.14}$$

3.5 Equations of Intermediate Velocities

By introducing intermediate volume flux variables $\hat{u}^\xi, \hat{u}^\eta, \hat{u}^z$ and temporary pressure flux variable ϕ, we have the following split equations,

$$\begin{aligned} V_\xi \frac{\hat{u}^\xi - (u^\xi)^n}{\Delta t} &= \frac{1}{2}[3H_\xi(u^q)^n - H_\xi(u^q)^{n-1}] \\ &+ \frac{1}{2}[3D_{\xi,ex}(u^q)^n - D_{\xi,ex}(u^q)^{n-1}] + \frac{1}{2}[D_\xi(\hat{u}^\xi) + D_\xi(u^\xi)^n] \end{aligned} \tag{3.15}$$

$$V_\xi \frac{(u^\xi)^{n+1} - \hat{u}^\xi}{\Delta t} = R_\xi(\phi)^{n+1} \tag{3.16}$$

Similarly

$$\begin{aligned} V_\eta \frac{\hat{u}^\eta - (u^\eta)^n}{\Delta t} &= \frac{1}{2}[3H_\eta(u^q)^n - H_\eta(u^q)^{n-1}] \\ &+ \frac{1}{2}[3D_{\eta,ex}(u^q)^n - D_{\eta,ex}(u^q)^{n-1}] + \frac{1}{2}[D_\eta(\hat{u}^\eta) + D_\eta(u^\eta)^n] \end{aligned} \tag{3.17}$$

$$V_\eta \frac{(u^\eta)^{n+1} - \hat{u}^\eta}{\Delta t} = R_\eta(\phi)^{n+1} \tag{3.18}$$

and

$$V_z \frac{\hat{u}^z - (u^z)^n}{\Delta t} = \frac{1}{2}[3H_z(u^q)^n - H_z(u^q)^{n-1}] + \frac{1}{2}[3D_{z,ex}(u^z)^n - D_{z,ex}(u^z)^{n-1}] + \frac{1}{2}[D_z(\hat{u}^z)^{n+1} + D_z(u^z)^n] \tag{3.19}$$

$$V_z \frac{(u^z)^{n+1} - \hat{u}^z}{\Delta t} = R_z(\phi)^{n+1} \tag{3.20}$$

For the purpose of implementing fractional step method, we need to rewrite (3.15),(3.17) and (3.19) as follows,

$$[I - A_\xi][\hat{u}^\xi - (u^\xi)^n] = \frac{\Delta t}{2V_\xi}[3H_\xi(u^q)^n - H_\xi(u^q)^{n-1}] + \frac{\Delta t}{2V_\xi}[3D_{\xi,ex}(u^q)^n - D_{\xi,ex}(u^q)^{n-1}] + 2A_\xi(u^\xi)^n \tag{3.21}$$

where $A_\xi = [\Delta t/(2V_\xi)]D_\xi$.

$$[I - A_\eta][\hat{u}^\eta - (u^\eta)^n] = \frac{\Delta t}{2V_\eta}[3H_\eta(u^q)^n - H_\eta(u^q)^{n-1}] + \frac{\Delta t}{2V_\eta}[3D_{\eta,ex}(u^q)^n - D_{\eta,ex}(u^q)^{n-1}] + 2A_\eta(u^\eta)^n \tag{3.22}$$

where $A_\eta = [\Delta t/(2V_\eta)]D_\eta$.

$$[I - A_z][\hat{u}^z - (u^z)^n] = \frac{\Delta t}{2V_z}[3H_z(u^q)^n - H_z(u^q)^{n-1}] + \frac{\Delta t}{2V_\eta}[3D_{z,ex}(u^z)^n - D_{z,ex}(u^z)^{n-1}] + 2A_z(u^z)^n \tag{3.23}$$

where $A_z = [\Delta t/(2V_z)]D_z$.

It should be noted that the Laplacian operators A_ξ, A_η, A_z need be further split to facilitate the use of tridiagonal matrices inversion. This will be done after the implicit diffusion terms D_ξ, D_η, D_z are explicitly derived.

3.6 Derivation of Poisson Equation

Recall the discretized continuity equation (3.7), for the primary control volume, we have,

$$u_e^\xi - u_w^\xi + u_n^\eta - u_s^\eta + u_f^z - u_b^z = 0$$

where $u^q = (u^\xi, u^\eta, u^z)$.

From (3.16), (3.18) and (3.20), we have the velocity update equations as follows,

$$\begin{aligned}(u^\xi)^{n+1} &= \frac{\Delta t}{V_\xi} R_\xi(\phi)^{n+1} + \hat{u}^\xi \\ (u^\eta)^{n+1} &= \frac{\Delta t}{V_\eta} R_\eta(\phi)^{n+1} + \hat{u}^\eta \\ (u^z)^{n+1} &= \frac{\Delta t}{V_z} R_z(\phi)^{n+1} + \hat{u}^z\end{aligned}$$

With the help of the above equations, we have

$$\begin{aligned}&\frac{R_{\xi,e}(\phi)^{n+1}}{V_{\xi,e}} - \frac{R_{\xi,w}(\phi)^{n+1}}{V_{\xi,w}} + \frac{R_{\eta,n}(\phi)^{n+1}}{V_{\eta,n}} - \frac{R_{\eta,s}(\phi)^{n+1}}{V_{\eta,s}} + \frac{R_{z,f}(\phi)^{n+1}}{V_{z,f}} - \frac{R_{z,b}(\phi)^{n+1}}{V_{z,b}} \\ = &-\frac{\hat{u}^\xi_e - \hat{u}^\xi_w + \hat{u}^\eta_n - \hat{u}^\eta_s + \hat{u}^z_f - \hat{u}^z_b}{\Delta t} \qquad (3.24)\end{aligned}$$

This is the Poisson equation for the intermediate pressure flux variable ϕ. Note that (3.24) is not in a detailed form and in order to numerically solve the intermediate pressure flux variable it is necessary to obtain the expressions for the operator R and this will be done later.

3.7 More About Geometric Quantities

Recall (3.10)

$$\mathbf{S}^\xi_{\text{cell center plane}} \bullet [\sum_q \mathbf{S}^q \bullet \overline{T}] = L_\xi$$

where the tensor $\overline{T}$ is given by

$$\overline{T} = -\mathbf{uu} - P\overline{I} + \frac{1}{Re}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

To compute the fluxes over each face, the Cartesian velocity vector should be computed from the volume-flux unknowns by the identity

$$\mathbf{u} = \mathbf{S}_\xi u^\xi + \mathbf{S}_\eta u^\eta + \mathbf{S}_z u^z = \mathbf{S}_m u^m \qquad (3.25)$$

where $\mathbf{S}^q \bullet \mathbf{S}_m = \delta^q_m$ and $\mathbf{S}_m$ is the reciprocal base of $\mathbf{S}^q$. More specifically,

$$u_i = S_{\xi,x} u^\xi + S_{\eta,x} u^\eta, \qquad u_j = S_{\xi,y} u^\xi + S_{\eta,y} u^\eta, \qquad u_k = S_z u^z$$

According to Rosenfeld *et al* [8],

$$\mathbf{S}_m = \frac{\mathbf{S}^{m+1} \times \mathbf{S}^{m+2}}{\mathbf{S}^m \bullet (\mathbf{S}^{m+1} \times \mathbf{S}^{m+2})} \qquad (3.26)$$

here m is the cyclic permutation of (ξ, η, z). More explicitly,

$$\mathbf{S}_\xi = \frac{\mathbf{S}^\eta \times \mathbf{S}^z}{\mathbf{S}^\xi \bullet (\mathbf{S}^\eta \times \mathbf{S}^z)}, \qquad \mathbf{S}_\eta = \frac{\mathbf{S}^z \times \mathbf{S}^\xi}{\mathbf{S}^\eta \bullet (\mathbf{S}^z \times \mathbf{S}^\xi)}, \qquad \mathbf{S}_z = \frac{\mathbf{S}^\xi \times \mathbf{S}^\eta}{\mathbf{S}^z \bullet (\mathbf{S}^\xi \times \mathbf{S}^\eta)}, \tag{3.27}$$

3.8 Derivation of L_ξ

Recall the momentum equation for ξ direction (3.10)

$$V_\xi \frac{\partial u^\xi}{\partial t} = \mathbf{S}^\xi_{\text{cell center plane}} \bullet [\sum_q \mathbf{S}^q \bullet \overline{T}] = L_\xi$$

where $q = e, w, n, s, f, b$, i.e. summation over all the faces of the staggered u^ξ control volume shown in Figure 3.2; and the tensor $\overline{T}$ is given by

$$\overline{T} = -\mathbf{uu} - P\overline{I} + \frac{1}{Re}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

and also from (3.11),

$$L_\xi = H_\xi + R_\xi + D_\xi + D_{\xi,ex}$$

The task now is to derive the expressions for H_ξ, R_ξ, D_ξ and $D_{\xi,ex}$.

$$\begin{aligned} \sum_q \mathbf{S}^q \bullet \overline{T} &= \sum_q \mathbf{S}^q \bullet \{-\mathbf{uu} - P\overline{I} + \frac{1}{Re}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]\} \\ &= \sum_q \mathbf{S}^q \bullet (-\mathbf{uu}) + \sum_q \mathbf{S}^q \bullet (-P\overline{I}) + \sum_q \frac{\mathbf{S}^q}{Re} \bullet [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \qquad (3.28) \end{aligned}$$

Note: $\mathbf{uu}$ is a dyad, second order tensor, there has no multiplication sign and it should not be confused with the scalar and vector product.

Because $\mathbf{u} = \mathbf{S}_m u^m$, and $\mathbf{S}^q \bullet \mathbf{S}_m = \delta^q_m$, we have

$$\begin{aligned} \sum_q \mathbf{S}^q \bullet (-\mathbf{uu}) &= \sum_q \mathbf{S}^q \bullet (-\mathbf{S}_m u^m \mathbf{S}_m u^m) \\ &= \sum_q -u^q u^m \mathbf{S}_m \end{aligned}$$

where $q = e, w, n, s, b, f$, $m = \xi, \eta, z$ and $u^q(e, w) = u^\xi(e, w)$, $u^q(n, s) = u^\eta(n, s)$, and $u^q(f, b) = u^z(f, b)$.

Therefore, the convection term for u^ξ equation is

$$\begin{aligned} H_\xi(u^q) &= \mathbf{S}^\xi_P \bullet \sum_q -u^q u^m \mathbf{S}_m \qquad (3.29) \\ &= -\mathbf{S}^\xi_P \bullet [u^\xi_e u^m_e \mathbf{S}_{m,e} - u^\xi_w u^m_w \mathbf{S}_{m,w} + u^\eta_n u^m_n \mathbf{S}_{m,n} - u^\eta_s u^m_s \mathbf{S}_{m,s} + \\ &\quad u^z_f u^m_f \mathbf{S}_{m,f} - u^z_b u^m_b \mathbf{S}_{m,b}] \qquad (3.30) \end{aligned}$$

where the subscript P refers to the center of the staggered u_ξ control volume.

$$\sum_q \mathbf{S}^q \bullet (-P\overline{I}) = \sum_q -\mathbf{S}^q P$$

where $q = e, w, n, s, f, b$, and $\mathbf{S}^q(e,w) = \mathbf{S}^\xi(e,w), \mathbf{S}^q(n,s) = \mathbf{S}^\eta(n,s), \mathbf{S}^q(f,b) = \mathbf{S}^z(f,b)$.

Therefore, the pressure term for u^ξ equation is

$$\begin{aligned} R_\xi(P) &= \mathbf{S}_P^\xi \bullet \sum_q -\mathbf{S}^q P && (3.31)\\ &= -\mathbf{S}^\xi \bullet [\mathbf{S}_e^\xi P_e - \mathbf{S}_w^\xi P_w + \mathbf{S}_n^\eta P_n - \mathbf{S}_s^\eta P_s + \mathbf{S}_f^z P_f - \mathbf{S}_b^z P_b] && (3.32)\end{aligned}$$

$$\text{Diffusion} = \mathbf{S}_P^\xi \bullet \sum_q \frac{\mathbf{S}^q}{Re} \bullet [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

According to the coordinate-free definition of ∇ given by Borisenko and Tarapov [11],

$$\nabla(\cdots) = \lim_{V\to 0} \frac{1}{V} \int\int_S d\mathbf{S}(\cdots)$$

$$\nabla \mathbf{u} = \frac{\int\int_S d\mathbf{S}\mathbf{u}}{V}$$

$\nabla \mathbf{u}$ should be evaluated at every face l of the staggered u^ξ control volume. Let us evaluate $\nabla \mathbf{u}$ at an arbitrary point (i,j,k) which is the center of its own control volume $V_{i,j,k}$,

$$\begin{aligned}
\nabla \mathbf{u}_{ijk} &= \frac{1}{V_{ijk}}[\mathbf{S}^\xi_{i+1/2}\mathbf{u}_{i+1/2} - \mathbf{S}^\xi_{i-1/2}\mathbf{u}_{i-1/2} + \mathbf{S}^\eta_{j+1/2}\mathbf{u}_{j+1/2} - \\
&\quad \mathbf{S}^\eta_{j-1/2}\mathbf{u}_{j-1/2} + \mathbf{S}^z_{k+1/2}\mathbf{u}_{k+1/2} - \mathbf{S}^z_{k-1/2}\mathbf{u}_{k-1/2}] \\
&= \frac{1}{V_{ijk}}\{[\mathbf{S}^\xi_{i+1/2}\mathbf{S}_{m,i+1/2}u^m_{i+1/2} - \mathbf{S}^\xi_{i-1/2}\mathbf{S}_{m,i-1/2}u^m_{i-1/2}] \\
&\quad [\mathbf{S}^\eta_{j+1/2}\mathbf{S}_{m,j+1/2}u^m_{j+1/2} - \mathbf{S}^\eta_{j-1/2}\mathbf{S}_{m,j-1/2}u^m_{j-1/2}] \\
&\quad [\mathbf{S}^z_{k+1/2}\mathbf{S}_{m,k+1/2}u^m_{k+1/2} - \mathbf{S}^z_{k-1/2}\mathbf{S}_{m,k-1/2}u^m_{k-1/2}]\} \\
&= \frac{1}{V_{ijk}}\{[(\mathbf{S}^\xi_{i+1/2}\mathbf{S}_{\xi,i+1/2}u^\xi_{i+1/2} + \mathbf{S}^\xi_{i+1/2}\mathbf{S}_{\eta,i+1/2}u^\eta_{i+1/2} + \mathbf{S}^\xi_{i+1/2}\mathbf{S}_{z,i+1/2}u^z_{i+1/2}) - \\
&\quad (\mathbf{S}^\xi_{i-1/2}\mathbf{S}_{\xi,i-1/2}u^\xi_{i-1/2} + \mathbf{S}^\xi_{i-1/2}\mathbf{S}_{\eta,i-1/2}u^\eta_{i-1/2} + \mathbf{S}^\xi_{i-1/2}\mathbf{S}_{z,i-1/2}u^z_{i-1/2})] + \qquad (3.33) \\
&\quad [(\mathbf{S}^\eta_{j+1/2}\mathbf{S}_{\xi,j+1/2}u^\xi_{j+1/2} + \mathbf{S}^\eta_{j+1/2}\mathbf{S}_{\eta,j+1/2}u^\eta_{j+1/2} + \mathbf{S}^\eta_{j+1/2}\mathbf{S}_{z,j+1/2}u^z_{j+1/2}) - \\
&\quad (\mathbf{S}^\eta_{j-1/2}\mathbf{S}_{\xi,j-1/2}u^\xi_{j-1/2} + \mathbf{S}^\eta_{j-1/2}\mathbf{S}_{\eta,j-1/2}u^\eta_{j-1/2} + \mathbf{S}^\eta_{j-1/2}\mathbf{S}_{z,j-1/2}u^z_{j-1/2})] + \\
&\quad [(\mathbf{S}^z_{k+1/2}\mathbf{S}_{\xi,k+1/2}u^\xi_{k+1/2} + \mathbf{S}^z_{k+1/2}\mathbf{S}_{\eta,k+1/2}u^\eta_{k+1/2} + \mathbf{S}^z_{k+1/2}\mathbf{S}_{z,k+1/2}u^z_{k+1/2}) - \\
&\quad (\mathbf{S}^z_{k-1/2}\mathbf{S}_{\xi,k-1/2}u^\xi_{k-1/2} + \mathbf{S}^z_{k-1/2}\mathbf{S}_{\eta,k-1/2}u^\eta_{k-1/2} + \mathbf{S}^z_{k-1/2}\mathbf{S}_{z,k-1/2}u^z_{k-1/2})]\}
\end{aligned}$$

where $m = \xi, \eta, z$. Thus for a ∇u at one point, there are 18 terms, taking into account the transpose part, there are 36 terms. In one control volume, there are 6 faces, therefore, the total diffusion term number for one control volume is 216.

At this point, some tensor algebra is in order. First given two vectors $\mathbf{A} = a_i$ and $\mathbf{B} = b_j$, according to tensor algebra the dyad of $\mathbf{AB}$ is a second order tensor $\overline{C}$ with $C_{ij} = a_i b_j$.

$$\overline{C} = \mathbf{AB} = a_i b_j e^i e^j = C_{ij} e^i e^j$$

written in a matrix form, we have

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{pmatrix}$$

And the inner product of a vector $\mathbf{D} = D_k$ on tensor $\overline{C}$ gives a vector $\mathbf{E} = E_j$ as $E_j = D_k C_{kj}$ (contraction of a third order tensor),

$$\begin{aligned} (\, d_1 \quad d_2 \quad d_3) \bullet \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{pmatrix} \quad &= \quad (d_1c_1 + d_2c_4 + d_3c_7)\mathbf{i} \\ &\quad +(d_1c_2 + d_2c_5 + d_3c_8)\mathbf{j} \\ &\quad +(d_1c_3 + d_2c_6 + d_3c_9)\mathbf{k} \end{aligned}$$

For the control volume of u^ξ,

$$\begin{aligned} \sum_q \frac{\mathbf{S}^q}{Re} \bullet [\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \quad = \quad & \frac{\mathbf{S}_e^\xi}{Re_e} \bullet [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]_e - \frac{\mathbf{S}_w^\xi}{Re_w} \bullet [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]_w + \\ & \frac{\mathbf{S}_n^\eta}{Re_n} \bullet [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]_n - \frac{\mathbf{S}_s^\eta}{Re_s} \bullet [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]_s + \\ & \frac{\mathbf{S}_f^z}{Re_f} \bullet [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]_f - \frac{\mathbf{S}_b^z}{Re_b} \bullet [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]_b \end{aligned}$$

In order to use the matured fractional step method, we need to split the $\nabla\mathbf{u}$ to make the implicit diffusion terms look like $\partial^2 u^\xi/\partial\xi^2 + \partial^2 u^\xi/\partial\eta^2 + \partial^2 u^\xi/\partial z^2$, the rest will be lumped into explicit terms d_{ex}.

According to Borisenko and Tarapov [11], $(AB)^T = BA$,

$$\begin{aligned} \sum_q \frac{\mathbf{S}^q}{Re} \bullet [\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \quad = \quad & \frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_E^\xi \mathbf{S}_{\xi,E} + \mathbf{S}_{\xi,E}\mathbf{S}_E^\xi)u_E^\xi - (\mathbf{S}_P^\xi \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\xi)u_P^\xi + d_{\xi,e,ex}] - \\ & \frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\xi)u_P^\xi - (\mathbf{S}_W^\xi \mathbf{S}_{\xi,W} + \mathbf{S}_{\xi,W}\mathbf{S}_W^\xi)u_W^\xi + d_{\xi,w,ex}] + \\ & \frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{\xi,N} + \mathbf{S}_{\xi,N}\mathbf{S}_N^\eta)u_N^\xi - (\mathbf{S}_P^\eta \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\eta)u_P^\xi + d_{\xi,n,ex}] - \\ & \frac{\mathbf{S}_s^\eta}{Re_s V_s} \bullet [(\mathbf{S}_P^\eta \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\eta)u_P^\xi - (\mathbf{S}_S^\eta \mathbf{S}_{\xi,S} + \mathbf{S}_{\xi,S}\mathbf{S}_S^\eta)u_S^\xi + d_{\xi,s,ex}] + \\ & \frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{\xi,F} + \mathbf{S}_{\xi,F}\mathbf{S}_F^z)u_F^\xi - (\mathbf{S}_P^z \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^z)u_P^\xi + d_{\xi,f,ex}] - \\ & \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^z)u_P^\xi - (\mathbf{S}_B^z \mathbf{S}_{\xi,B} + \mathbf{S}_{\xi,B}\mathbf{S}_B^z)u_B^\xi + d_{\xi,b,ex}] \end{aligned}$$

Therefore, the implicit part of the diffusion terms for the u^ξ momentum equation are

$$\begin{aligned} D_\xi(u^\xi) \quad = \quad & \mathbf{S}_P^\xi \bullet \{ \frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_E^\xi \mathbf{S}_{\xi,E} + \mathbf{S}_{\xi,E} \mathbf{S}_E^\xi) u_E^\xi - (\mathbf{S}_P^\xi \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P} \mathbf{S}_P^\xi) u_P^\xi] - \\ & \frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P} \mathbf{S}_P^\xi) u_P^\xi - (\mathbf{S}_W^\xi \mathbf{S}_{\xi,W} + \mathbf{S}_{\xi,W} \mathbf{S}_W^\xi) u_W^\xi] + \\ & \frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{\xi,N} + \mathbf{S}_{\xi,N} \mathbf{S}_N^\eta) u_N^\xi - (\mathbf{S}_P^\eta \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P} \mathbf{S}_P^\eta) u_P^\xi] - \\ & \frac{\mathbf{S}_s^\eta}{Re_s V_s} \bullet [(\mathbf{S}_P^\eta \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P} \mathbf{S}_P^\eta) u_P^\xi - (\mathbf{S}_S^\eta \mathbf{S}_{\xi,S} + \mathbf{S}_{\xi,S} \mathbf{S}_S^\eta) u_S^\xi] + \\ & \frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{\xi,F} + \mathbf{S}_{\xi,F} \mathbf{S}_F^z) u_F^\xi - (\mathbf{S}_P^z \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P} \mathbf{S}_P^z) u_P^\xi] - \\ & \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P} \mathbf{S}_P^z) u_P^\xi - (\mathbf{S}_B^z \mathbf{S}_{\xi,B} + \mathbf{S}_{\xi,B} \mathbf{S}_B^z) u_B^\xi] \} \end{aligned} \tag{3.34}$$

And the explicit part of the diffusion terms for the u^ξ momentum equation are evaluated via

$$\begin{aligned} d_{\xi,e,ex} \quad = \quad & [(\mathbf{S}_E^\xi \mathbf{S}_{\eta,E} + \mathbf{S}_{\eta,E} \mathbf{S}_E^\xi) u_E^\eta - (\mathbf{S}_P^\xi \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P} \mathbf{S}_P^\xi) u_P^\eta] + \\ & [(\mathbf{S}_E^\xi \mathbf{S}_{z,E} + \mathbf{S}_{z,E} \mathbf{S}_E^\xi) u_E^z - (\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P} \mathbf{S}_P^\xi) u_P^z] + \\ & [(\mathbf{S}_{ne}^\eta \mathbf{S}_{\xi,ne} + \mathbf{S}_{\xi,ne} \mathbf{S}_{ne}^\eta) u_{ne}^\xi + (\mathbf{S}_{ne}^\eta \mathbf{S}_{\eta,ne} + \mathbf{S}_{\eta,ne} \mathbf{S}_{ne}^\eta) u_{ne}^\eta] - \\ & [(\mathbf{S}_{se}^\eta \mathbf{S}_{\xi,se} + \mathbf{S}_{\xi,se} \mathbf{S}_{se}^\eta) u_{se}^\xi + (\mathbf{S}_{se}^\eta \mathbf{S}_{\eta,se} + \mathbf{S}_{\eta,se} \mathbf{S}_{se}^\eta) u_{se}^\eta] + \\ & [(\mathbf{S}_{ne}^\eta \mathbf{S}_{z,ne} + \mathbf{S}_{z,ne} \mathbf{S}_{ne}^\eta) u_{ne}^z - (\mathbf{S}_{se}^\eta \mathbf{S}_{z,se} + \mathbf{S}_{z,se} \mathbf{S}_{se}^\eta) u_{se}^z] + \\ & [(\mathbf{S}_{fe}^z \mathbf{S}_{\xi,fe} + \mathbf{S}_{\xi,fe} \mathbf{S}_{fe}^z) u_{fe}^\xi + (\mathbf{S}_{fe}^z \mathbf{S}_{\eta,fe} + \mathbf{S}_{\eta,fe} \mathbf{S}_{fe}^z) u_{fe}^\eta] - \\ & [(\mathbf{S}_{be}^z \mathbf{S}_{\xi,be} + \mathbf{S}_{\xi,be} \mathbf{S}_{be}^z) u_{be}^\xi + (\mathbf{S}_{be}^z \mathbf{S}_{\eta,be} + \mathbf{S}_{\eta,be} \mathbf{S}_{be}^z) u_{be}^\eta] + \\ & [(\mathbf{S}_{fe}^z \mathbf{S}_{z,fe} + \mathbf{S}_{z,fe} \mathbf{S}_{fe}^z) u_{fe}^z - (\mathbf{S}_{be}^z \mathbf{S}_{z,be} + \mathbf{S}_{z,be} \mathbf{S}_{be}^z) u_{be}^z] \end{aligned} \tag{3.35}$$

$$\begin{aligned} d_{\xi,w,ex} \quad = \quad & [(\mathbf{S}_P^\xi \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P} \mathbf{S}_P^\xi) u_P^\eta - (\mathbf{S}_W^\xi \mathbf{S}_{\eta,W} + \mathbf{S}_{\eta,W} \mathbf{S}_W^\xi) u_W^\eta] + \\ & [(\mathbf{S}_{nw}^\eta \mathbf{S}_{\xi,nw} + \mathbf{S}_{\xi,nw} \mathbf{S}_{nw}^\eta) u_{nw}^\xi + (\mathbf{S}_{nw}^\eta \mathbf{S}_{\eta,nw} + \mathbf{S}_{\eta,nw} \mathbf{S}_{nw}^\eta) u_{nw}^\eta] - \\ & [(\mathbf{S}_{sw}^\eta \mathbf{S}_{\xi,sw} + \mathbf{S}_{\xi,sw} \mathbf{S}_{sw}^\eta) u_{sw}^\xi + (\mathbf{S}_{sw}^\eta \mathbf{S}_{\eta,sw} + \mathbf{S}_{\eta,sw} \mathbf{S}_{sw}^\eta) u_{sw}^\eta] + \\ & [(\mathbf{S}_{fw}^z \mathbf{S}_{\xi,fw} + \mathbf{S}_{\xi,fw} \mathbf{S}_{fw}^z) u_{fw}^\xi + (\mathbf{S}_{fw}^z \mathbf{S}_{\eta,fw} + \mathbf{S}_{\eta,fw} \mathbf{S}_{fw}^z) u_{fw}^\eta] - \\ & [(\mathbf{S}_{bw}^z \mathbf{S}_{\xi,bw} + \mathbf{S}_{\xi,bw} \mathbf{S}_{bw}^z) u_{bw}^\xi + (\mathbf{S}_{bw}^z \mathbf{S}_{\eta,bw} + \mathbf{S}_{\eta,bw} \mathbf{S}_{bw}^z) u_{bw}^\eta] + \\ & [(\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P} \mathbf{S}_P^\xi) u_P^z - (\mathbf{S}_W^\xi \mathbf{S}_{z,W} + \mathbf{S}_{z,W} \mathbf{S}_W^\xi) u_W^z] + \\ & [(\mathbf{S}_{nw}^\eta \mathbf{S}_{z,nw} + \mathbf{S}_{z,nw} \mathbf{S}_{nw}^\eta) u_{nw}^z - (\mathbf{S}_{sw}^\eta \mathbf{S}_{z,sw} + \mathbf{S}_{z,sw} \mathbf{S}_{sw}^\eta) u_{sw}^z] + \\ & [(\mathbf{S}_{fw}^z \mathbf{S}_{z,fw} + \mathbf{S}_{z,fw} \mathbf{S}_{fw}^z) u_{fw}^z - (\mathbf{S}_{bw}^z \mathbf{S}_{z,bw} + \mathbf{S}_{z,bw} \mathbf{S}_{bw}^z) u_{bw}^z] \end{aligned} \tag{3.36}$$

$$
\begin{aligned}
d_{\xi,n,ex} \quad = \quad & [(\mathbf{S}^{\eta}_{N}\mathbf{S}_{\eta,N} + \mathbf{S}_{\eta,N}\mathbf{S}^{\eta}_{N})u^{\eta}_{N} - (\mathbf{S}^{\eta}_{P}\mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^{\eta}_{P})u^{\eta}_{P}] + \\
& [(\mathbf{S}^{\xi}_{ne}\mathbf{S}_{\xi,ne} + \mathbf{S}_{\xi,ne}\mathbf{S}^{\xi}_{ne})u^{\xi}_{ne} + (\mathbf{S}^{\xi}_{ne}\mathbf{S}_{\eta,ne} + \mathbf{S}_{\eta,ne}\mathbf{S}^{\xi}_{ne})u^{\eta}_{ne}] - \\
& [(\mathbf{S}^{\xi}_{nw}\mathbf{S}_{\xi,nw} + \mathbf{S}_{\xi,nw}\mathbf{S}^{\xi}_{nw})u^{\xi}_{nw} + (\mathbf{S}^{\xi}_{nw}\mathbf{S}_{\eta,nw} + \mathbf{S}_{\eta,nw}\mathbf{S}^{\xi}_{nw})u^{\eta}_{nw}] + \\
& [(\mathbf{S}^{z}_{nf}\mathbf{S}_{\xi,nf} + \mathbf{S}_{\xi,nf}\mathbf{S}^{z}_{nf})u^{\xi}_{nf} + (\mathbf{S}^{z}_{nf}\mathbf{S}_{\eta,nf} + \mathbf{S}_{\eta,nf}\mathbf{S}^{z}_{nf})u^{\eta}_{nf}] - \\
& [(\mathbf{S}^{z}_{nb}\mathbf{S}_{\xi,nb} + \mathbf{S}_{\xi,nb}\mathbf{S}^{z}_{nb})u^{\xi}_{nb} + (\mathbf{S}^{z}_{nb}\mathbf{S}_{\eta,nb} + \mathbf{S}_{\eta,nb}\mathbf{S}^{z}_{nb})u^{\eta}_{nb}] + \\
& [(\mathbf{S}^{\eta}_{N}\mathbf{S}_{z,N} + \mathbf{S}_{z,N}\mathbf{S}^{\eta}_{N})u^{z}_{N} - (\mathbf{S}^{\eta}_{P}\mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^{\eta}_{P})u^{z}_{P}] + \\
& [(\mathbf{S}^{\xi}_{ne}\mathbf{S}_{z,ne} + \mathbf{S}_{z,ne}\mathbf{S}^{\xi}_{ne})u^{z}_{ne} - (\mathbf{S}^{\xi}_{nw}\mathbf{S}_{z,nw} + \mathbf{S}_{z,nw}\mathbf{S}^{\xi}_{nw})u^{z}_{nw}] + \\
& [(\mathbf{S}^{z}_{nf}\mathbf{S}_{z,nf} + \mathbf{S}_{z,nf}\mathbf{S}^{z}_{nf})u^{z}_{nf} - (\mathbf{S}^{z}_{nb}\mathbf{S}_{z,nb} + \mathbf{S}_{z,nb}\mathbf{S}^{z}_{nb})u^{z}_{nb}] \qquad (3.37)
\end{aligned}
$$

$$
\begin{aligned}
d_{\xi,s,ex} \quad = \quad & [(\mathbf{S}^{\eta}_{P}\mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^{\eta}_{P})u^{\eta}_{P} - (\mathbf{S}^{\eta}_{S}\mathbf{S}_{\eta,S} + \mathbf{S}_{\eta,S}\mathbf{S}^{\eta}_{S})u^{\eta}_{S}] + \\
& [(\mathbf{S}^{\xi}_{se}\mathbf{S}_{\xi,se} + \mathbf{S}_{\xi,se}\mathbf{S}^{\xi}_{se})u^{\xi}_{se} + (\mathbf{S}^{\xi}_{se}\mathbf{S}_{\eta,se} + \mathbf{S}_{\eta,se}\mathbf{S}^{\xi}_{se})u^{\eta}_{se}] - \\
& [(\mathbf{S}^{\xi}_{sw}\mathbf{S}_{\xi,sw} + \mathbf{S}_{\xi,sw}\mathbf{S}^{\xi}_{sw})u^{\xi}_{sw} + (\mathbf{S}^{\xi}_{sw}\mathbf{S}_{\eta,sw} + \mathbf{S}_{\eta,sw}\mathbf{S}^{\xi}_{sw})u^{\eta}_{sw}] + \\
& [(\mathbf{S}^{z}_{sf}\mathbf{S}_{\xi,sf} + \mathbf{S}_{\xi,sf}\mathbf{S}^{z}_{sf})u^{\xi}_{sf} + (\mathbf{S}^{z}_{sf}\mathbf{S}_{\eta,sf} + \mathbf{S}_{\eta,sf}\mathbf{S}^{z}_{sf})u^{\eta}_{sf}] - \\
& [(\mathbf{S}^{z}_{sb}\mathbf{S}_{\xi,sb} + \mathbf{S}_{\xi,sb}\mathbf{S}^{z}_{sb})u^{\xi}_{sb} + (\mathbf{S}^{z}_{sb}\mathbf{S}_{\eta,sb} + \mathbf{S}_{\eta,sb}\mathbf{S}^{z}_{sb})u^{\eta}_{sb}] + \\
& [(\mathbf{S}^{\eta}_{P}\mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^{\eta}_{P})u^{z}_{P} - (\mathbf{S}^{\eta}_{S}\mathbf{S}_{z,S} + \mathbf{S}_{z,S}\mathbf{S}^{\eta}_{S})u^{z}_{S}] + \\
& [(\mathbf{S}^{\xi}_{se}\mathbf{S}_{z,se} + \mathbf{S}_{z,se}\mathbf{S}^{\xi}_{se})u^{z}_{se} - (\mathbf{S}^{\xi}_{sw}\mathbf{S}_{z,sw} + \mathbf{S}_{z,sw}\mathbf{S}^{\xi}_{sw})u^{z}_{sw}] + \\
& [(\mathbf{S}^{z}_{sf}\mathbf{S}_{z,sf} + \mathbf{S}_{z,sf}\mathbf{S}^{z}_{sf})u^{z}_{sf} - (\mathbf{S}^{z}_{sb}\mathbf{S}_{z,sb} + \mathbf{S}_{z,sb}\mathbf{S}^{z}_{sb})u^{z}_{sb}] \qquad (3.38)
\end{aligned}
$$

$$
\begin{aligned}
d_{\xi,f,ex} \quad = \quad & [(\mathbf{S}^{z}_{F}\mathbf{S}_{\eta,F} + \mathbf{S}_{\eta,F}\mathbf{S}^{z}_{F})u^{\eta}_{F} - (\mathbf{S}^{z}_{P}\mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^{z}_{P})u^{\eta}_{P}] + \\
& [(\mathbf{S}^{\xi}_{fe}\mathbf{S}_{\xi,fe} + \mathbf{S}_{\xi,fe}\mathbf{S}^{\xi}_{fe})u^{\xi}_{fe} + (\mathbf{S}^{\xi}_{fe}\mathbf{S}_{\eta,fe} + \mathbf{S}_{\eta,fe}\mathbf{S}^{\xi}_{fe})u^{\eta}_{fe}] - \\
& [(\mathbf{S}^{\xi}_{fw}\mathbf{S}_{\xi,fw} + \mathbf{S}_{\xi,fw}\mathbf{S}^{\xi}_{fw})u^{\xi}_{fw} + (\mathbf{S}^{\xi}_{fw}\mathbf{S}_{\eta,fw} + \mathbf{S}_{\eta,fw}\mathbf{S}^{\xi}_{fw})u^{\eta}_{fw}] + \\
& [(\mathbf{S}^{\eta}_{fn}\mathbf{S}_{\xi,fn} + \mathbf{S}_{\xi,fn}\mathbf{S}^{\eta}_{fn})u^{\xi}_{fn} + (\mathbf{S}^{\eta}_{fn}\mathbf{S}_{\eta,fn} + \mathbf{S}_{\eta,fn}\mathbf{S}^{\eta}_{fn})u^{\eta}_{fn}] - \\
& [(\mathbf{S}^{\eta}_{fs}\mathbf{S}_{\xi,fs} + \mathbf{S}_{\xi,fs}\mathbf{S}^{\eta}_{fs})u^{\xi}_{fs} + (\mathbf{S}^{\eta}_{fs}\mathbf{S}_{\eta,fs} + \mathbf{S}_{\eta,fs}\mathbf{S}^{\eta}_{fs})u^{\eta}_{fs}] + \\
& [(\mathbf{S}^{z}_{F}\mathbf{S}_{z,F} + \mathbf{S}_{z,F}\mathbf{S}^{z}_{F})u^{z}_{F} - (\mathbf{S}^{z}_{P}\mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^{z}_{P})u^{z}_{P}] + \\
& [(\mathbf{S}^{\xi}_{fe}\mathbf{S}_{z,fe} + \mathbf{S}_{z,fe}\mathbf{S}^{\xi}_{fe})u^{z}_{fe} - (\mathbf{S}^{\xi}_{fw}\mathbf{S}_{z,fw} + \mathbf{S}_{z,fw}\mathbf{S}^{\xi}_{fw})u^{z}_{fw}] + \\
& [(\mathbf{S}^{\eta}_{fn}\mathbf{S}_{z,fn} + \mathbf{S}_{z,fn}\mathbf{S}^{\eta}_{fn})u^{z}_{fn} - (\mathbf{S}^{\eta}_{fs}\mathbf{S}_{z,fs} + \mathbf{S}_{z,fs}\mathbf{S}^{\eta}_{fs})u^{z}_{fs}] \qquad (3.39)
\end{aligned}
$$

$$
\begin{aligned}
d_{\xi,b,ex} \quad = \quad & [(\mathbf{S}^{z}_{P}\mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^{z}_{P})u^{\eta}_{P} - (\mathbf{S}^{z}_{B}\mathbf{S}_{\eta,B} + \mathbf{S}_{\eta,B}\mathbf{S}^{z}_{B})u^{\eta}_{B}] + \\
& [(\mathbf{S}^{\xi}_{be}\mathbf{S}_{\xi,be} + \mathbf{S}_{\xi,be}\mathbf{S}^{\xi}_{be})u^{\xi}_{be} + (\mathbf{S}^{\xi}_{be}\mathbf{S}_{\eta,be} + \mathbf{S}_{\eta,be}\mathbf{S}^{\xi}_{be})u^{\eta}_{be}] - \\
& [(\mathbf{S}^{\xi}_{bw}\mathbf{S}_{\xi,bw} + \mathbf{S}_{\xi,bw}\mathbf{S}^{\xi}_{bw})u^{\xi}_{bw} + (\mathbf{S}^{\xi}_{bw}\mathbf{S}_{\eta,bw} + \mathbf{S}_{\eta,bw}\mathbf{S}^{\xi}_{bw})u^{\eta}_{bw}] + \\
& [(\mathbf{S}^{\eta}_{bn}\mathbf{S}_{\xi,bn} + \mathbf{S}_{\xi,bn}\mathbf{S}^{\eta}_{bn})u^{\xi}_{bn} + (\mathbf{S}^{\eta}_{bn}\mathbf{S}_{\eta,bn} + \mathbf{S}_{\eta,bn}\mathbf{S}^{\eta}_{bn})u^{\eta}_{bn}] - \\
& [(\mathbf{S}^{\eta}_{bs}\mathbf{S}_{\xi,bs} + \mathbf{S}_{\xi,bs}\mathbf{S}^{\eta}_{bs})u^{\xi}_{bs} + (\mathbf{S}^{\eta}_{bs}\mathbf{S}_{\eta,bs} + \mathbf{S}_{\eta,bs}\mathbf{S}^{\eta}_{bs})u^{\eta}_{bs}] + \\
& [(\mathbf{S}^{z}_{P}\mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^{z}_{P})u^{z}_{P} - (\mathbf{S}^{z}_{B}\mathbf{S}_{z,B} + \mathbf{S}_{z,B}\mathbf{S}^{z}_{B})u^{z}_{B}] + \\
& [(\mathbf{S}^{\xi}_{be}\mathbf{S}_{z,be} + \mathbf{S}_{z,be}\mathbf{S}^{\xi}_{be})u^{z}_{be} - (\mathbf{S}^{\xi}_{bw}\mathbf{S}_{z,bw} + \mathbf{S}_{z,bw}\mathbf{S}^{\xi}_{bw})u^{z}_{bw}] + \\
& [(\mathbf{S}^{\eta}_{bn}\mathbf{S}_{z,bn} + \mathbf{S}_{z,bn}\mathbf{S}^{\eta}_{bn})u^{z}_{bn} - (\mathbf{S}^{\eta}_{bs}\mathbf{S}_{z,bs} + \mathbf{S}_{z,bs}\mathbf{S}^{\eta}_{bs})u^{z}_{bs}] \qquad (3.40)
\end{aligned}
$$

Therefore the explicit diffusion terms are

$$\begin{aligned} D_{\xi,ex} &= \mathbf{S}_P^{\xi} \bullet \{ \frac{\mathbf{S}_e^{\xi}}{Re_e V_e} \bullet d_{\xi,e,ex} - \frac{\mathbf{S}_w^{\xi}}{Re_w V_w} \bullet d_{\xi,w,ex} + \frac{\mathbf{S}_n^{\eta}}{Re_n V_n} \bullet d_{\xi,n,ex} - \\ & \frac{\mathbf{S}_s^{\eta}}{Re_s V_s} \bullet d_{\xi,s,ex} + \frac{\mathbf{S}_f^{z}}{Re_f V_f} \bullet d_{\xi,f,ex} - \frac{\mathbf{S}_b^{z}}{Re_b V_b} \bullet d_{\xi,b,ex} \} \end{aligned} \tag{3.41}$$

3.9 Derivation of L_η

Write the momentum equation for η direction from (3.9)

$$V_\eta \frac{\partial u^\eta}{\partial t} = \mathbf{S}^{\eta}_{\text{cell center plane}} \bullet [\sum_q \mathbf{S}^q \bullet \overline{T}] = L_\eta$$

where $q = e, w, n, s, f, b$, i.e. summation over all the faces of the staggered u^η control volume; and the tensor $\overline{T}$ is given by

$$\overline{T} = -\mathbf{uu} - P\overline{I} + \frac{1}{Re}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

and also similar to (3.11)

$$L_\eta = H_\eta + R_\eta + D_\eta + D_{\eta,ex}$$

$D_{\eta,ex}$.

$$\begin{aligned} \sum_q \mathbf{S}^q \bullet \overline{T} &= \sum_q \mathbf{S}^q \bullet \{-\mathbf{uu} - P\overline{I} + \frac{1}{Re}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]\} \\ &= \sum_q \mathbf{S}^q \bullet (-\mathbf{uu}) + \sum_q \mathbf{S}^q \bullet (-P\overline{I}) + \sum_q \frac{\mathbf{S}^q}{Re} \bullet [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \end{aligned} \tag{3.42}$$

Because $\mathbf{u} = \mathbf{S}_m u^m$, and $\mathbf{S}^q \bullet \mathbf{S}_m = \delta^l_m$, we have

$$\begin{aligned} \sum_q \mathbf{S}^q \bullet \mathbf{uu} &= \sum_q \mathbf{S}^q \bullet (-\mathbf{S}_m u^m \mathbf{S}_m u^m) \\ &= \sum_q -u^q u^m \mathbf{S}_m \end{aligned}$$

where $q = e, w, n, s, b, f$, $m = \xi, \eta, z$ and $u^q(e, w) = u^\xi(e, w)$, $u^q(n, s) = u^\eta(n, s)$, and $u^q(f, b) = u^z(f, b)$.

Therefore, the convection term for u^η equation is

$$H_\eta(u^q) = \mathbf{S}_P^{\eta} \bullet \sum_q -u^q u^m \mathbf{S}_m \tag{3.43}$$

$$\begin{aligned} &= -\mathbf{S}_P^{\eta} \bullet [u_e^{\xi} u_e^m \mathbf{S}_{m,e} - u_w^{\xi} u_w^m \mathbf{S}_{m,w} + u_n^{\eta} u_n^m \mathbf{S}_{m,n} - u_s^{\eta} u_s^m \mathbf{S}_{m,s} + \\ & u_f^z u_f^m \mathbf{S}_{m,f} - u_b^z u_b^m \mathbf{S}_{m,b}] \end{aligned} \tag{3.44}$$

where the subscript P refers to the center of the staggered u_η control volume.

$$\sum_q \mathbf{S}^q \bullet (-P\bar{I}) = \sum_q -\mathbf{S}^q P$$

where $q = e, w, n, s, f, b$, and $\mathbf{S}^q(e,w) = \mathbf{S}^\xi(e,w), \mathbf{S}^q(n,s) = \mathbf{S}^\eta(n,s), \mathbf{S}^q(f,b) = \mathbf{S}^z(f,b)$.

Therefore, the pressure term for u^η equation is

$$\begin{aligned} R_\eta(P) &= \mathbf{S}^\eta_P \bullet \sum_q -\mathbf{S}^q P && (3.45)\\ &= -\mathbf{S}^\eta \bullet [\mathbf{S}^\xi_e P_e - \mathbf{S}^\xi_w P_w + \mathbf{S}^\eta_n P_n - \mathbf{S}^\eta_s P_s + \mathbf{S}^z_f P_f - \mathbf{S}^z_b P_b] && (3.46)\end{aligned}$$

$$\text{Diffusion} = \mathbf{S}^\eta_P \bullet \sum_q \frac{\mathbf{S}^q}{Re} \bullet [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$$

For the control volume of u^η shown in Figure 3.2,

$$\begin{aligned} \sum_q \frac{\mathbf{S}^q}{Re} \bullet [\nabla\mathbf{u} + (\nabla\mathbf{u})^T] &= \frac{\mathbf{S}^\xi_e}{Re_e V_e} \bullet [(\mathbf{S}^\xi_E \mathbf{S}_{\eta,E} + \mathbf{S}_{\eta,E}\mathbf{S}^\xi_E)u^\eta_E - (\mathbf{S}^\xi_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\xi_P)u^\eta_P + d_{\eta,e,ex}] - \\ &\quad \frac{\mathbf{S}^\xi_w}{Re_w V_w} \bullet [(\mathbf{S}^\xi_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\xi_P)u^\eta_P - (\mathbf{S}^\xi_W \mathbf{S}_{\eta,W} + \mathbf{S}_{\eta,W}\mathbf{S}^\xi_W)u^\eta_W + d_{\eta,w,ex}] \\ &\quad + \frac{\mathbf{S}^\eta_n}{Re_n V_n} \bullet [(\mathbf{S}^\eta_N \mathbf{S}_{\eta,N} + \mathbf{S}_{\eta,N}\mathbf{S}^\eta_N)u^\eta_N - (\mathbf{S}^\eta_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\eta_P)u^\eta_P + d_{\eta,n,ex}] \\ &\quad - \frac{\mathbf{S}^\eta_s}{Re_s V_s} \bullet [(\mathbf{S}^\eta_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\eta_P)u^\eta_P - (\mathbf{S}^\eta_S \mathbf{S}_{\eta,S} + \mathbf{S}_{\eta,S}\mathbf{S}^\eta_S)u^\eta_S + d_{\eta,s,ex}] + \\ &\quad \frac{\mathbf{S}^z_f}{Re_f V_f} \bullet [(\mathbf{S}^z_F \mathbf{S}_{\eta,F} + \mathbf{S}_{\eta,F}\mathbf{S}^z_F)u^\eta_F - (\mathbf{S}^z_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^z_P)u^\eta_P + d_{\eta,f,ex}] - \\ &\quad \frac{\mathbf{S}^z_b}{Re_b V_b} \bullet [(\mathbf{S}^z_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^z_P)u^\eta_P - (\mathbf{S}^z_B \mathbf{S}_{\eta,B} + \mathbf{S}_{\eta,B}\mathbf{S}^z_B)u^\eta_B + d_{\eta,b,ex}] \end{aligned}$$

Therefore, the implicit part of the diffusion terms for the u^η momentum equation are

$$\begin{aligned} D_\eta(u^\eta) &= \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\xi_e}{Re_e V_e} \bullet [(\mathbf{S}^\xi_E \mathbf{S}_{\eta,E} + \mathbf{S}_{\eta,E}\mathbf{S}^\xi_E)u^\eta_E - (\mathbf{S}^\xi_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\xi_P)u^\eta_P] - \\ &\quad \frac{\mathbf{S}^\xi_w}{Re_w V_w} \bullet [(\mathbf{S}^\xi_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\xi_P)u^\eta_P - (\mathbf{S}^\xi_W \mathbf{S}_{\eta,W} + \mathbf{S}_{\eta,W}\mathbf{S}^\xi_W)u^\eta_W] + \\ &\quad \frac{\mathbf{S}^\eta_n}{Re_n V_n} \bullet [(\mathbf{S}^\eta_N \mathbf{S}_{\eta,N} + \mathbf{S}_{\eta,N}\mathbf{S}^\eta_N)u^\eta_N - (\mathbf{S}^\eta_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\eta_P)u^\eta_P] - \\ &\quad \frac{\mathbf{S}^\eta_s}{Re_s V_s} \bullet [(\mathbf{S}^\eta_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\eta_P)u^\eta_P - (\mathbf{S}^\eta_S \mathbf{S}_{\eta,S} + \mathbf{S}_{\eta,S}\mathbf{S}^\eta_S)u^\eta_S] + \\ &\quad \frac{\mathbf{S}^z_f}{Re_f V_f} \bullet [(\mathbf{S}^z_F \mathbf{S}_{\eta,F} + \mathbf{S}_{\eta,F}\mathbf{S}^z_F)u^\eta_F - (\mathbf{S}^z_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^z_P)u^\eta_P] - \\ &\quad \frac{\mathbf{S}^z_b}{Re_b V_b} \bullet [(\mathbf{S}^z_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^z_P)u^\eta_P - (\mathbf{S}^z_B \mathbf{S}_{\eta,B} + \mathbf{S}_{\eta,B}\mathbf{S}^z_B)u^\eta_B]\} && (3.47) \end{aligned}$$

And the explicit part of the diffusion terms for the u^η momentum equation are evaluated via

$$
\begin{aligned}
d_{\eta,e,ex} \;=\;& [(\mathbf{S}^\xi_E \mathbf{S}_{\xi,E} + \mathbf{S}_{\xi,E}\mathbf{S}^\xi_E)u^\xi_E - (\mathbf{S}^\xi_P \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^\xi_P)u^\xi_P] + \\
& [(\mathbf{S}^\eta_{ne}\mathbf{S}_{\xi,ne} + \mathbf{S}_{\xi,ne}\mathbf{S}^\eta_{ne})u^\xi_{ne} + (\mathbf{S}^\eta_{ne}\mathbf{S}_{\eta,ne} + \mathbf{S}_{\eta,ne}\mathbf{S}^\eta_{ne})u^\eta_{ne}] - \\
& [(\mathbf{S}^\eta_{se}\mathbf{S}_{\xi,se} + \mathbf{S}_{\xi,se}\mathbf{S}^\eta_{se})u^\xi_{se} + (\mathbf{S}^\eta_{se}\mathbf{S}_{\eta,se} + \mathbf{S}_{\eta,se}\mathbf{S}^\eta_{se})u^\eta_{se}] + \\
& [(\mathbf{S}^z_{fe}\mathbf{S}_{\xi,fe} + \mathbf{S}_{\xi,fe}\mathbf{S}^z_{fe})u^\xi_{fe} + (\mathbf{S}^z_{fe}\mathbf{S}_{\eta,fe} + \mathbf{S}_{\eta,fe}\mathbf{S}^z_{fe})u^\eta_{fe}] - \\
& [(\mathbf{S}^z_{be}\mathbf{S}_{\xi,be} + \mathbf{S}_{\xi,be}\mathbf{S}^z_{be})u^\xi_{be} + (\mathbf{S}^z_{be}\mathbf{S}_{\eta,be} + \mathbf{S}_{\eta,be}\mathbf{S}^z_{be})u^\eta_{be}] + \\
& [(\mathbf{S}^\xi_E \mathbf{S}_{z,E} + \mathbf{S}_{z,E}\mathbf{S}^\xi_E)u^z_E - (\mathbf{S}^\xi_P \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^\xi_P)u^z_P] + \\
& [(\mathbf{S}^\eta_{ne}\mathbf{S}_{z,ne} + \mathbf{S}_{z,ne}\mathbf{S}^\eta_{ne})u^z_{ne} - (\mathbf{S}^\eta_{se}\mathbf{S}_{z,se} + \mathbf{S}_{z,se}\mathbf{S}^\eta_{se})u^z_{se}] + \\
& [(\mathbf{S}^z_{fe}\mathbf{S}_{z,fe} + \mathbf{S}_{z,fe}\mathbf{S}^z_{fe})u^z_{fe} - (\mathbf{S}^z_{be}\mathbf{S}_{z,be} + \mathbf{S}_{z,be}\mathbf{S}^z_{be})u^z_{be}] \qquad (3.48)
\end{aligned}
$$

$$
\begin{aligned}
d_{\eta,w,ex} \;=\;& [(\mathbf{S}^\xi_P \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^\xi_P)u^\xi_P - (\mathbf{S}^\xi_W \mathbf{S}_{\xi,W} + \mathbf{S}_{\xi,W}\mathbf{S}^\xi_W)u^\xi_W] + \\
& [(\mathbf{S}^\eta_{nw}\mathbf{S}_{\xi,nw} + \mathbf{S}_{\xi,nw}\mathbf{S}^\eta_{nw})u^\xi_{nw} + (\mathbf{S}^\eta_{nw}\mathbf{S}_{\eta,nw} + \mathbf{S}_{\eta,nw}\mathbf{S}^\eta_{nw})u^\eta_{nw}] - \\
& [(\mathbf{S}^\eta_{sw}\mathbf{S}_{\xi,sw} + \mathbf{S}_{\xi,sw}\mathbf{S}^\eta_{sw})u^\xi_{sw} + (\mathbf{S}^\eta_{sw}\mathbf{S}_{\eta,sw} + \mathbf{S}_{\eta,sw}\mathbf{S}^\eta_{sw})u^\eta_{sw}] + \\
& [(\mathbf{S}^z_{fw}\mathbf{S}_{\xi,fw} + \mathbf{S}_{\xi,fw}\mathbf{S}^z_{fw})u^\xi_{fw} + (\mathbf{S}^z_{fw}\mathbf{S}_{\eta,fw} + \mathbf{S}_{\eta,fw}\mathbf{S}^z_{fw})u^\eta_{fw}] - \\
& [(\mathbf{S}^z_{bw}\mathbf{S}_{\xi,bw} + \mathbf{S}_{\xi,bw}\mathbf{S}^z_{bw})u^\xi_{bw} + (\mathbf{S}^z_{bw}\mathbf{S}_{\eta,bw} + \mathbf{S}_{\eta,bw}\mathbf{S}^z_{bw})u^\eta_{bw}] + \\
& [(\mathbf{S}^\xi_P \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^\xi_P)u^z_P - (\mathbf{S}^\xi_W \mathbf{S}_{z,W} + \mathbf{S}_{z,W}\mathbf{S}^\xi_W)u^z_W] + \\
& [(\mathbf{S}^\eta_{nw}\mathbf{S}_{z,nw} + \mathbf{S}_{z,nw}\mathbf{S}^\eta_{nw})u^z_{nw} - (\mathbf{S}^\eta_{sw}\mathbf{S}_{z,sw} + \mathbf{S}_{z,sw}\mathbf{S}^\eta_{sw})u^z_{sw}] + \\
& [(\mathbf{S}^z_{fw}\mathbf{S}_{z,fw} + \mathbf{S}_{z,fw}\mathbf{S}^z_{fw})u^z_{fw} - (\mathbf{S}^z_{bw}\mathbf{S}_{z,bw} + \mathbf{S}_{z,bw}\mathbf{S}^z_{bw})u^z_{bw}] \qquad (3.49)
\end{aligned}
$$

$$
\begin{aligned}
d_{\eta,n,ex} \;=\;& [(\mathbf{S}^\eta_N \mathbf{S}_{\xi,N} + \mathbf{S}_{\xi,N}\mathbf{S}^\eta_N)u^\xi_N - (\mathbf{S}^\eta_P \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^\eta_P)u^\xi_P] + \\
& [(\mathbf{S}^\xi_{ne}\mathbf{S}_{\xi,ne} + \mathbf{S}_{\xi,ne}\mathbf{S}^\xi_{ne})u^\xi_{ne} + (\mathbf{S}^\xi_{ne}\mathbf{S}_{\eta,ne} + \mathbf{S}_{\eta,ne}\mathbf{S}^\xi_{ne})u^\eta_{ne}] - \\
& [(\mathbf{S}^\xi_{nw}\mathbf{S}_{\xi,nw} + \mathbf{S}_{\xi,nw}\mathbf{S}^\xi_{nw})u^\xi_{nw} + (\mathbf{S}^\xi_{nw}\mathbf{S}_{\eta,nw} + \mathbf{S}_{\eta,nw}\mathbf{S}^\xi_{nw})u^\eta_{nw}] + \\
& [(\mathbf{S}^z_{nf}\mathbf{S}_{\xi,nf} + \mathbf{S}_{\xi,nf}\mathbf{S}^z_{nf})u^\xi_{nf} + (\mathbf{S}^z_{nf}\mathbf{S}_{\eta,nf} + \mathbf{S}_{\eta,nf}\mathbf{S}^z_{nf})u^\eta_{nf}] - \\
& [(\mathbf{S}^z_{nb}\mathbf{S}_{\xi,nb} + \mathbf{S}_{\xi,nb}\mathbf{S}^z_{nb})u^\xi_{nb} + (\mathbf{S}^z_{nb}\mathbf{S}_{\eta,nb} + \mathbf{S}_{\eta,nb}\mathbf{S}^z_{nb})u^\eta_{nb}] + \\
& [(\mathbf{S}^\eta_N \mathbf{S}_{z,N} + \mathbf{S}_{z,N}\mathbf{S}^\eta_N)u^z_N - (\mathbf{S}^\eta_P \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^\eta_P)u^z_P] + \\
& [(\mathbf{S}^\xi_{ne}\mathbf{S}_{z,ne} + \mathbf{S}_{z,ne}\mathbf{S}^\xi_{ne})u^z_{ne} - (\mathbf{S}^\xi_{nw}\mathbf{S}_{z,nw} + \mathbf{S}_{z,nw}\mathbf{S}^\xi_{nw})u^z_{nw}] + \\
& [(\mathbf{S}^z_{nf}\mathbf{S}_{z,nf} + \mathbf{S}_{z,nf}\mathbf{S}^z_{nf})u^z_{nf} - (\mathbf{S}^z_{nb}\mathbf{S}_{z,nb} + \mathbf{S}_{z,nb}\mathbf{S}^z_{nb})u^z_{nb}] \qquad (3.50)
\end{aligned}
$$

$$
\begin{aligned}
d_{\eta,s,ex} \quad = \quad & [(\mathbf{S}^{\eta}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{\eta}_{P})u^{\xi}_{P} - (\mathbf{S}^{\eta}_{S}\mathbf{S}_{\xi,S} + \mathbf{S}_{\xi,S}\mathbf{S}^{\eta}_{S})u^{\xi}_{S}] + \\
& [(\mathbf{S}^{\xi}_{se}\mathbf{S}_{\xi,se} + \mathbf{S}_{\xi,se}\mathbf{S}^{\xi}_{se})u^{\xi}_{se} + (\mathbf{S}^{\xi}_{se}\mathbf{S}_{\eta,se} + \mathbf{S}_{\eta,se}\mathbf{S}^{\xi}_{se})u^{\eta}_{se}] - \\
& [(\mathbf{S}^{\xi}_{sw}\mathbf{S}_{\xi,sw} + \mathbf{S}_{\xi,sw}\mathbf{S}^{\xi}_{sw})u^{\xi}_{sw} + (\mathbf{S}^{\xi}_{sw}\mathbf{S}_{\eta,sw} + \mathbf{S}_{\eta,sw}\mathbf{S}^{\xi}_{sw})u^{\eta}_{sw}] + \\
& [(\mathbf{S}^{z}_{sf}\mathbf{S}_{\xi,sf} + \mathbf{S}_{\xi,sf}\mathbf{S}^{z}_{sf})u^{\xi}_{sf} + (\mathbf{S}^{z}_{sf}\mathbf{S}_{\eta,sf} + \mathbf{S}_{\eta,sf}\mathbf{S}^{z}_{sf})u^{\eta}_{sf}] - \\
& [(\mathbf{S}^{z}_{sb}\mathbf{S}_{\xi,sb} + \mathbf{S}_{\xi,sb}\mathbf{S}^{z}_{sb})u^{\xi}_{sb} + (\mathbf{S}^{z}_{sb}\mathbf{S}_{\eta,sb} + \mathbf{S}_{\eta,sb}\mathbf{S}^{z}_{sb})u^{\eta}_{sb}] + \\
& [(\mathbf{S}^{\eta}_{P}\mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^{\eta}_{P})u^{z}_{P} - (\mathbf{S}^{\eta}_{S}\mathbf{S}_{z,S} + \mathbf{S}_{z,S}\mathbf{S}^{\eta}_{S})u^{z}_{S}] + \\
& [(\mathbf{S}^{\xi}_{se}\mathbf{S}_{z,se} + \mathbf{S}_{z,se}\mathbf{S}^{\xi}_{se})u^{z}_{se} - (\mathbf{S}^{\xi}_{sw}\mathbf{S}_{z,sw} + \mathbf{S}_{z,sw}\mathbf{S}^{\xi}_{sw})u^{z}_{sw}] + \\
& [(\mathbf{S}^{z}_{sf}\mathbf{S}_{z,sf} + \mathbf{S}_{z,sf}\mathbf{S}^{z}_{sf})u^{z}_{sf} - (\mathbf{S}^{z}_{sb}\mathbf{S}_{z,sb} + \mathbf{S}_{z,sb}\mathbf{S}^{z}_{sb})u^{z}_{sb}] \qquad (3.51)
\end{aligned}
$$

$$
\begin{aligned}
d_{\eta,f,ex} \quad = \quad & [(\mathbf{S}^{z}_{F}\mathbf{S}_{\xi,F} + \mathbf{S}_{\xi,F}\mathbf{S}^{z}_{F})u^{\xi}_{F} - (\mathbf{S}^{z}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{z}_{P})u^{\xi}_{P}] + \\
& [(\mathbf{S}^{\xi}_{fe}\mathbf{S}_{\xi,fe} + \mathbf{S}_{\xi,fe}\mathbf{S}^{\xi}_{fe})u^{\xi}_{fe} + (\mathbf{S}^{\xi}_{fe}\mathbf{S}_{\eta,fe} + \mathbf{S}_{\eta,fe}\mathbf{S}^{\xi}_{fe})u^{\eta}_{fe}] - \\
& [(\mathbf{S}^{\xi}_{fw}\mathbf{S}_{\xi,fw} + \mathbf{S}_{\xi,fw}\mathbf{S}^{\xi}_{fw})u^{\xi}_{fw} + (\mathbf{S}^{\xi}_{fw}\mathbf{S}_{\eta,fw} + \mathbf{S}_{\eta,fw}\mathbf{S}^{\xi}_{fw})u^{\eta}_{fw}] + \\
& [(\mathbf{S}^{\eta}_{fn}\mathbf{S}_{\xi,fn} + \mathbf{S}_{\xi,fn}\mathbf{S}^{\eta}_{fn})u^{\xi}_{fn} + (\mathbf{S}^{\eta}_{fn}\mathbf{S}_{\eta,fn} + \mathbf{S}_{\eta,fn}\mathbf{S}^{\eta}_{fn})u^{\eta}_{fn}] - \\
& [(\mathbf{S}^{\eta}_{fs}\mathbf{S}_{\xi,fs} + \mathbf{S}_{\xi,fs}\mathbf{S}^{\eta}_{fs})u^{\xi}_{fs} + (\mathbf{S}^{\eta}_{fs}\mathbf{S}_{\eta,fs} + \mathbf{S}_{\eta,fs}\mathbf{S}^{\eta}_{fs})u^{\eta}_{fs}] + \\
& [(\mathbf{S}^{z}_{F}\mathbf{S}_{z,F} + \mathbf{S}_{z,F}\mathbf{S}^{z}_{F})u^{z}_{F} - (\mathbf{S}^{z}_{P}\mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^{z}_{P})u^{z}_{P}] + \\
& [(\mathbf{S}^{\xi}_{fe}\mathbf{S}_{z,fe} + \mathbf{S}_{z,fe}\mathbf{S}^{\xi}_{fe})u^{z}_{fe} - (\mathbf{S}^{\xi}_{fw}\mathbf{S}_{z,fw} + \mathbf{S}_{z,fw}\mathbf{S}^{\xi}_{fw})u^{z}_{fw}] + \\
& [(\mathbf{S}^{\eta}_{fn}\mathbf{S}_{z,fn} + \mathbf{S}_{z,fn}\mathbf{S}^{\eta}_{fn})u^{z}_{fn} - (\mathbf{S}^{\eta}_{fs}\mathbf{S}_{z,fs} + \mathbf{S}_{z,fs}\mathbf{S}^{\eta}_{fs})u^{z}_{fs}] \qquad (3.52)
\end{aligned}
$$

$$
\begin{aligned}
d_{\xi,b,ex} \quad = \quad & [(\mathbf{S}^{z}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{z}_{P})u^{\xi}_{P} - (\mathbf{S}^{z}_{B}\mathbf{S}_{\xi,B} + \mathbf{S}_{\xi,B}\mathbf{S}^{z}_{B})u^{\xi}_{B}] + \\
& [(\mathbf{S}^{\xi}_{be}\mathbf{S}_{\xi,be} + \mathbf{S}_{\xi,be}\mathbf{S}^{\xi}_{be})u^{\xi}_{be} + (\mathbf{S}^{\xi}_{be}\mathbf{S}_{\eta,be} + \mathbf{S}_{\eta,be}\mathbf{S}^{\xi}_{be})u^{\eta}_{be}] - \\
& [(\mathbf{S}^{\xi}_{bw}\mathbf{S}_{\xi,bw} + \mathbf{S}_{\xi,bw}\mathbf{S}^{\xi}_{bw})u^{\xi}_{bw} + (\mathbf{S}^{\xi}_{bw}\mathbf{S}_{\eta,bw} + \mathbf{S}_{\eta,bw}\mathbf{S}^{\xi}_{bw})u^{\eta}_{bw}] + \\
& [(\mathbf{S}^{\eta}_{bn}\mathbf{S}_{\xi,bn} + \mathbf{S}_{\xi,bn}\mathbf{S}^{\eta}_{bn})u^{\xi}_{bn} + (\mathbf{S}^{\eta}_{bn}\mathbf{S}_{\eta,bn} + \mathbf{S}_{\eta,bn}\mathbf{S}^{\eta}_{bn})u^{\eta}_{bn}] - \\
& [(\mathbf{S}^{\eta}_{bs}\mathbf{S}_{\xi,bs} + \mathbf{S}_{\xi,bs}\mathbf{S}^{\eta}_{bs})u^{\xi}_{bs} + (\mathbf{S}^{\eta}_{bs}\mathbf{S}_{\eta,bs} + \mathbf{S}_{\eta,bs}\mathbf{S}^{\eta}_{bs})u^{\eta}_{bs}] + \\
& [(\mathbf{S}^{z}_{P}\mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^{z}_{P})u^{z}_{P} - (\mathbf{S}^{z}_{B}\mathbf{S}_{z,B} + \mathbf{S}_{z,B}\mathbf{S}^{z}_{B})u^{z}_{B}] + \\
& [(\mathbf{S}^{\xi}_{be}\mathbf{S}_{z,be} + \mathbf{S}_{z,be}\mathbf{S}^{\xi}_{be})u^{z}_{be} - (\mathbf{S}^{\xi}_{bw}\mathbf{S}_{z,bw} + \mathbf{S}_{z,bw}\mathbf{S}^{\xi}_{bw})u^{z}_{bw}] + \\
& [(\mathbf{S}^{\eta}_{bn}\mathbf{S}_{z,bn} + \mathbf{S}_{z,bn}\mathbf{S}^{\eta}_{bn})u^{z}_{bn} - (\mathbf{S}^{\eta}_{bs}\mathbf{S}_{z,bs} + \mathbf{S}_{z,bs}\mathbf{S}^{\eta}_{bs})u^{z}_{bs}] \qquad (3.53)
\end{aligned}
$$

Therefore the explicit diffusion terms are

$$
\begin{aligned}
D_{\eta,ex} \quad = \quad & \mathbf{S}^{\eta}_{P} \bullet \{ \frac{\mathbf{S}^{\xi}_{e}}{Re_e V_e} \bullet d_{\eta,e,ex} - \frac{\mathbf{S}^{\xi}_{w}}{Re_w V_w} \bullet d_{\eta,w,ex} + \frac{\mathbf{S}^{\eta}_{n}}{Re_n V_n} \bullet d_{\eta,n,ex} - \\
& \frac{\mathbf{S}^{\eta}_{s}}{Re_s V_s} \bullet d_{\eta,s,ex} + \frac{\mathbf{S}^{z}_{f}}{Re_f V_f} \bullet d_{\eta,f,ex} - \frac{\mathbf{S}^{z}_{b}}{Re_b V_b} \bullet d_{\eta,b,ex} \} \qquad (3.54)
\end{aligned}
$$

3.10 Derivation of L_z

Write the momentum equation for η direction from (3.9)

$$V_z \frac{\partial u^z}{\partial t} = \mathbf{S}^z_{\text{cell center plane}} \bullet [\sum_q \mathbf{S}^q \bullet \overline{T}] = L_z$$

where $q = e, w, n, s, f, b$, i.e. summation over all the faces of the staggered u^z control volume; and the tensor $\overline{T}$ is given by

$$\overline{T} = -\mathbf{uu} - P\overline{I} + \frac{1}{Re}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

and also similar to (3.11)

$$L_z = H_z + R_z + D_{z,ex} + D_z$$

The task now is to derive the expressions for $H_z, R_z, D_{ex,z}$ and D_z.

$$\begin{aligned}
\sum_q \mathbf{S}^q \bullet \overline{T} &= \sum_q \mathbf{S}^q \bullet \{-\mathbf{uu} - P\overline{I} + \frac{1}{Re}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]\} \\
&= \sum_q \mathbf{S}^q \bullet (-\mathbf{uu}) + \sum_q \mathbf{S}^q \bullet (-P\overline{I}) + \sum_q \frac{\mathbf{S}^q}{Re} \bullet [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \qquad (3.55)
\end{aligned}$$

Because $\mathbf{u} = \mathbf{S}_m u^m$, and $\mathbf{S}^q \bullet \mathbf{S}_m = \delta^q_m$, we have

$$\begin{aligned}
\sum_q \mathbf{S}^q \bullet \mathbf{uu} &= \sum_q \mathbf{S}^q \bullet (-\mathbf{S}_m u^m \mathbf{S}_m u^m) \\
&= \sum_q -u^q u^m \mathbf{S}_m
\end{aligned}$$

where $q = e, w, n, s, b, f,\ m = \xi, \eta, z$ and $u^q(e, w) = u^\xi(e, w),\ u^q(n, s) = u^\eta(n, s)$, and $u^q(f, b) = u^z(f, b)$.

Therefore, the convection term for u^z equation is

$$\begin{aligned}
H_z(u^q) &= \mathbf{S}^z_P \bullet \sum_q -u^q u^m \mathbf{S}_m \qquad (3.56) \\
&= -\mathbf{S}^z_P \bullet [u^\xi_e u^m_e \mathbf{S}_{m,e} - u^\xi_w u^m_w \mathbf{S}_{m,w} + u^\eta_n u^m_n \mathbf{S}_{m,n} - u^\eta_s u^m_s \mathbf{S}_{m,s} + \\
&\quad u^z_f u^m_f \mathbf{S}_{m,f} - u^z_b u^m_b \mathbf{S}_{m,b}] \qquad (3.57)
\end{aligned}$$

where the subscript P refers to the center of the staggered u_z control volume.

$$\sum_q \mathbf{S}^q \bullet (-P\overline{I}) = \sum_q -\mathbf{S}^q P$$

where $q = e, w, n, s, f, b$, and $\mathbf{S}^q(e, w) = \mathbf{S}^\xi(e, w), \mathbf{S}^q(n, s) = \mathbf{S}^\eta(n, s), \mathbf{S}^q(f, b) = \mathbf{S}^z(f, b)$.

Therefore, the pressure term for u^z equation is

$$\begin{aligned} R_z(P) &= \mathbf{S}_P^z \bullet \sum_q -\mathbf{S}^q P && (3.58) \\ &= -\mathbf{S}^z \bullet [\mathbf{S}_f^z P_f - \mathbf{S}_b^z P_b] && (3.59) \end{aligned}$$

where terms corresponding to $l = e, w, n, s$ vanish.

$$\text{Diffusion} = \mathbf{S}_P^z \bullet \sum_q \frac{\mathbf{S}^q}{Re} \bullet [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

$$\begin{aligned} \sum_q \frac{\mathbf{S}^q}{Re} \bullet [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \;=\; & \frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_E^\xi \mathbf{S}_{z,E} + \mathbf{S}_{z,E}\mathbf{S}_E^\xi) u_E^z - (\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\xi) u_P^z] - \\ & \frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\xi) u_P^z - (\mathbf{S}_W^\xi \mathbf{S}_{z,W} + \mathbf{S}_{z,W}\mathbf{S}_W^\xi) u_W^z] + \\ & \frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{z,N} + \mathbf{S}_{z,N}\mathbf{S}_N^\eta) u_N^z - (\mathbf{S}_P^\eta \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\eta) u_P^z] - \\ & \frac{\mathbf{S}_s^\eta}{Re_s V_s} \bullet [(\mathbf{S}_P^\eta \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\eta) u_P^z - (\mathbf{S}_S^\eta \mathbf{S}_{z,S} + \mathbf{S}_{z,S}\mathbf{S}_S^\eta) u_S^z] + \\ & \frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{z,F} + \mathbf{S}_{z,F}\mathbf{S}_F^z) u_F^z - (\mathbf{S}_P^z \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^z) u_P^z] - \\ & \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^z) u_P^z - (\mathbf{S}_B^z \mathbf{S}_{z,B} + \mathbf{S}_{z,B}\mathbf{S}_B^z) u_B^z] \end{aligned}$$

Therefore, the diffusion terms for the u^z momentum equation are

$$\begin{aligned} D_z(u^z) \;=\; & \mathbf{S}_P^z \bullet \{ \frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_E^\xi \mathbf{S}_{z,E} + \mathbf{S}_{z,E}\mathbf{S}_E^\xi) u_E^z - (\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\xi) u_P^z + d_{z,e,ex}] - \\ & \frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\xi) u_P^z - (\mathbf{S}_W^\xi \mathbf{S}_{z,W} + \mathbf{S}_{z,W}\mathbf{S}_W^\xi) u_W^z + d_{z,w,ex}] + \\ & \frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{z,N} + \mathbf{S}_{z,N}\mathbf{S}_N^\eta) u_N^z - (\mathbf{S}_P^\eta \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\eta) u_P^z + d_{z,n,ex}] - \\ & \frac{\mathbf{S}_s^\eta}{Re_s V_s} \bullet [(\mathbf{S}_P^\eta \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\eta) u_P^z - (\mathbf{S}_S^\eta \mathbf{S}_{z,S} + \mathbf{S}_{z,S}\mathbf{S}_S^\eta) u_S^z + d_{z,s,ex}] + \\ & \frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{z,F} + \mathbf{S}_{z,F}\mathbf{S}_F^z) u_F^z - (\mathbf{S}_P^z \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^z) u_P^z] - \\ & \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^z) u_P^z - (\mathbf{S}_B^z \mathbf{S}_{z,B} + \mathbf{S}_{z,B}\mathbf{S}_B^z) u_B^z] \} && (3.60) \end{aligned}$$

And the explicit part of the diffusion terms for the u^z momentum equation are evaluated via

$$
\begin{aligned}
d_{z,e,ex} &= [(\mathbf{S}^{\xi}_{E}\mathbf{S}_{\eta,E} + \mathbf{S}_{\eta,E}\mathbf{S}^{\xi}_{E})u^{\eta}_{E} - (\mathbf{S}^{\xi}_{P}\mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^{\xi}_{P})u^{\eta}_{P}] + \\
&\quad [(\mathbf{S}^{\xi}_{E}\mathbf{S}_{\xi,E} + \mathbf{S}_{\xi,E}\mathbf{S}^{\xi}_{E})u^{\xi}_{E} - (\mathbf{S}^{\xi}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{\xi}_{P})u^{\xi}_{P}] + \\
&\quad [(\mathbf{S}^{\eta}_{ne}\mathbf{S}_{z,ne} + \mathbf{S}_{z,ne}\mathbf{S}^{\eta}_{ne})u^{z}_{ne} - (\mathbf{S}^{\eta}_{se}\mathbf{S}_{z,se} + \mathbf{S}_{z,se}\mathbf{S}^{\eta}_{se})u^{z}_{se}] + \\
&\quad [(\mathbf{S}^{\eta}_{ne}\mathbf{S}_{\xi,ne} + \mathbf{S}_{\xi,ne}\mathbf{S}^{\eta}_{ne})u^{\xi}_{ne} - (\mathbf{S}^{\eta}_{se}\mathbf{S}_{\xi,se} + \mathbf{S}_{\xi,se}\mathbf{S}^{\eta}_{se})u^{\xi}_{se}] + \\
&\quad [(\mathbf{S}^{\eta}_{ne}\mathbf{S}_{\eta,ne} + \mathbf{S}_{\eta,ne}\mathbf{S}^{\eta}_{ne})u^{\eta}_{ne} - (\mathbf{S}^{\eta}_{se}\mathbf{S}_{\eta,se} + \mathbf{S}_{\eta,se}\mathbf{S}^{\eta}_{se})u^{\eta}_{se}] + \\
&\quad [(\mathbf{S}^{z}_{fe}\mathbf{S}_{z,fe} + \mathbf{S}_{z,fe}\mathbf{S}^{z}_{fe})u^{z}_{fe} - (\mathbf{S}^{z}_{be}\mathbf{S}_{z,be} + \mathbf{S}_{z,be}\mathbf{S}^{z}_{be})u^{z}_{be}] + \\
&\quad [(\mathbf{S}^{z}_{fe}\mathbf{S}_{\xi,fe} + \mathbf{S}_{\xi,fe}\mathbf{S}^{z}_{fe})u^{\xi}_{fe} - (\mathbf{S}^{z}_{be}\mathbf{S}_{\xi,be} + \mathbf{S}_{\xi,be}\mathbf{S}^{z}_{be})u^{\xi}_{be}] + \\
&\quad [(\mathbf{S}^{z}_{fe}\mathbf{S}_{\eta,fe} + \mathbf{S}_{\eta,fe}\mathbf{S}^{z}_{fe})u^{\eta}_{fe} - (\mathbf{S}^{z}_{be}\mathbf{S}_{\eta,be} + \mathbf{S}_{\eta,be}\mathbf{S}^{z}_{be})u^{\eta}_{be}]
\end{aligned}
\tag{3.61}
$$

$$
\begin{aligned}
d_{z,w,ex} &= [(\mathbf{S}^{\xi}_{P}\mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^{\xi}_{P})u^{\eta}_{P} - (\mathbf{S}^{\xi}_{W}\mathbf{S}_{\eta,W} + \mathbf{S}_{\eta,W}\mathbf{S}^{\xi}_{W})u^{\eta}_{W}] + \\
&\quad [(\mathbf{S}^{\xi}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{\xi}_{P})u^{\xi}_{P} - (\mathbf{S}^{\xi}_{W}\mathbf{S}_{\xi,W} + \mathbf{S}_{\xi,W}\mathbf{S}^{\xi}_{W})u^{\xi}_{W}] + \\
&\quad [(\mathbf{S}^{\eta}_{nw}\mathbf{S}_{z,nw} + \mathbf{S}_{z,nw}\mathbf{S}^{\eta}_{nw})u^{z}_{nw} - (\mathbf{S}^{\eta}_{sw}\mathbf{S}_{z,sw} + \mathbf{S}_{z,sw}\mathbf{S}^{\eta}_{sw})u^{z}_{sw}] + \\
&\quad [(\mathbf{S}^{\eta}_{nw}\mathbf{S}_{\xi,nw} + \mathbf{S}_{\xi,nw}\mathbf{S}^{\eta}_{nw})u^{\xi}_{nw} - (\mathbf{S}^{\eta}_{sw}\mathbf{S}_{\xi,sw} + \mathbf{S}_{\xi,sw}\mathbf{S}^{\eta}_{sw})u^{\xi}_{sw}] + \\
&\quad [(\mathbf{S}^{\eta}_{nw}\mathbf{S}_{\eta,nw} + \mathbf{S}_{\eta,nw}\mathbf{S}^{\eta}_{nw})u^{\eta}_{nw} - (\mathbf{S}^{\eta}_{sw}\mathbf{S}_{\eta,sw} + \mathbf{S}_{\eta,sw}\mathbf{S}^{\eta}_{sw})u^{\eta}_{sw}] + \\
&\quad [(\mathbf{S}^{z}_{fw}\mathbf{S}_{z,fw} + \mathbf{S}_{z,fw}\mathbf{S}^{z}_{fw})u^{z}_{fw} - (\mathbf{S}^{z}_{bw}\mathbf{S}_{z,bw} + \mathbf{S}_{z,bw}\mathbf{S}^{z}_{bw})u^{z}_{bw}] + \\
&\quad [(\mathbf{S}^{z}_{fw}\mathbf{S}_{\xi,fw} + \mathbf{S}_{\xi,fw}\mathbf{S}^{z}_{fw})u^{\xi}_{fw} - (\mathbf{S}^{z}_{bw}\mathbf{S}_{\xi,bw} + \mathbf{S}_{\xi,bw}\mathbf{S}^{z}_{bw})u^{\xi}_{bw}] + \\
&\quad [(\mathbf{S}^{z}_{fw}\mathbf{S}_{\eta,fw} + \mathbf{S}_{\eta,fw}\mathbf{S}^{z}_{fw})u^{\eta}_{fw} - (\mathbf{S}^{z}_{bw}\mathbf{S}_{\eta,bw} + \mathbf{S}_{\eta,bw}\mathbf{S}^{z}_{bw})u^{\eta}_{bw}]
\end{aligned}
\tag{3.62}
$$

$$
\begin{aligned}
d_{z,n,ex} &= [(\mathbf{S}^{\eta}_{N}\mathbf{S}_{\xi,N} + \mathbf{S}_{\xi,N}\mathbf{S}^{\eta}_{N})u^{\xi}_{N} - (\mathbf{S}^{\eta}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{\eta}_{P})u^{\xi}_{P}] + \\
&\quad [(\mathbf{S}^{\eta}_{N}\mathbf{S}_{\eta,N} + \mathbf{S}_{\eta,N}\mathbf{S}^{\eta}_{N})u^{\eta}_{N} - (\mathbf{S}^{\eta}_{P}\mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^{\eta}_{P})u^{\eta}_{P}] + \\
&\quad [(\mathbf{S}^{\xi}_{ne}\mathbf{S}_{z,ne} + \mathbf{S}_{z,ne}\mathbf{S}^{\xi}_{ne})u^{z}_{ne} - (\mathbf{S}^{\xi}_{nw}\mathbf{S}_{z,nw} + \mathbf{S}_{z,nw}\mathbf{S}^{\xi}_{nw})u^{z}_{nw}] + \\
&\quad [(\mathbf{S}^{\xi}_{ne}\mathbf{S}_{\xi,ne} + \mathbf{S}_{\xi,ne}\mathbf{S}^{\xi}_{ne})u^{\xi}_{ne} - (\mathbf{S}^{\xi}_{nw}\mathbf{S}_{\xi,nw} + \mathbf{S}_{\xi,nw}\mathbf{S}^{\xi}_{nw})u^{\xi}_{nw}] + \\
&\quad [(\mathbf{S}^{\xi}_{ne}\mathbf{S}_{\eta,ne} + \mathbf{S}_{\eta,ne}\mathbf{S}^{\xi}_{ne})u^{\eta}_{ne} - (\mathbf{S}^{\xi}_{nw}\mathbf{S}_{\eta,nw} + \mathbf{S}_{\xi,nw}\mathbf{S}^{\eta}_{nw})u^{\eta}_{nw}] + \\
&\quad [(\mathbf{S}^{z}_{nf}\mathbf{S}_{z,nf} + \mathbf{S}_{z,nf}\mathbf{S}^{z}_{nf})u^{z}_{nf} - (\mathbf{S}^{z}_{nb}\mathbf{S}_{z,nb} + \mathbf{S}_{z,nb}\mathbf{S}^{z}_{nb})u^{z}_{nb}] + \\
&\quad [(\mathbf{S}^{z}_{nf}\mathbf{S}_{\xi,nf} + \mathbf{S}_{\xi,nf}\mathbf{S}^{z}_{nf})u^{\xi}_{nf} - (\mathbf{S}^{z}_{nb}\mathbf{S}_{\xi,nb} + \mathbf{S}_{\xi,nb}\mathbf{S}^{z}_{nb})u^{\xi}_{nb}] + \\
&\quad [(\mathbf{S}^{z}_{nf}\mathbf{S}_{\eta,nf} + \mathbf{S}_{\eta,nf}\mathbf{S}^{z}_{nf})u^{\eta}_{nf} - (\mathbf{S}^{z}_{nb}\mathbf{S}_{\eta,nb} + \mathbf{S}_{\eta,nb}\mathbf{S}^{z}_{nb})u^{\eta}_{nb}]
\end{aligned}
\tag{3.63}
$$

$$
\begin{aligned}
d_{z,s,ex} &= (\mathbf{S}^{\eta}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{\eta}_{P})u^{\xi}_{P} - (\mathbf{S}^{\eta}_{S}\mathbf{S}_{\xi,S} + \mathbf{S}_{\xi,S}\mathbf{S}^{\eta}_{S})u^{\xi}_{S} + \\
&\quad (\mathbf{S}^{\eta}_{P}\mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^{\eta}_{P})u^{\eta}_{P} - (\mathbf{S}^{\eta}_{S}\mathbf{S}_{\eta,S} + \mathbf{S}_{\eta,S}\mathbf{S}^{\eta}_{S})u^{\eta}_{S} + \\
&\quad (\mathbf{S}^{\xi}_{se}\mathbf{S}_{z,se} + \mathbf{S}_{z,se}\mathbf{S}^{\xi}_{se})u^{z}_{se} - (\mathbf{S}^{\xi}_{sw}\mathbf{S}_{z,sw} + \mathbf{S}_{z,sw}\mathbf{S}^{\xi}_{sw})u^{z}_{sw} + \\
&\quad [(\mathbf{S}^{\xi}_{se}\mathbf{S}_{\xi,se} + \mathbf{S}_{\xi,se}\mathbf{S}^{\xi}_{se})u^{\xi}_{se} - (\mathbf{S}^{\xi}_{sw}\mathbf{S}_{\xi,sw} + \mathbf{S}_{\xi,sw}\mathbf{S}^{\xi}_{sw})u^{\xi}_{sw}] + \\
&\quad [(\mathbf{S}^{\xi}_{se}\mathbf{S}_{\eta,se} + \mathbf{S}_{\eta,se}\mathbf{S}^{\xi}_{se})u^{\eta}_{se} - (\mathbf{S}^{\xi}_{sw}\mathbf{S}_{\eta,sw} + \mathbf{S}_{\xi,sw}\mathbf{S}^{\eta}_{sw})u^{\eta}_{sw}] + \\
&\quad [(\mathbf{S}^{z}_{sf}\mathbf{S}_{z,sf} + \mathbf{S}_{z,sf}\mathbf{S}^{z}_{sf})u^{z}_{sf} - (\mathbf{S}^{z}_{sb}\mathbf{S}_{z,sb} + \mathbf{S}_{z,sb}\mathbf{S}^{z}_{sb})u^{z}_{sb}] + \\
&\quad [(\mathbf{S}^{z}_{sf}\mathbf{S}_{\xi,sf} + \mathbf{S}_{\xi,sf}\mathbf{S}^{z}_{sf})u^{\xi}_{sf} - (\mathbf{S}^{z}_{sb}\mathbf{S}_{\xi,sb} + \mathbf{S}_{\xi,sb}\mathbf{S}^{z}_{sb})u^{\xi}_{sb}] + \\
&\quad [(\mathbf{S}^{z}_{sf}\mathbf{S}_{\eta,sf} + \mathbf{S}_{\eta,sf}\mathbf{S}^{z}_{sf})u^{\eta}_{sf} - (\mathbf{S}^{z}_{sb}\mathbf{S}_{\eta,sb} + \mathbf{S}_{\eta,sb}\mathbf{S}^{z}_{sb})u^{\eta}_{sb}]
\end{aligned}
\tag{3.64}
$$

$$\begin{aligned}
d_{z,f,ex} &= [(\mathbf{S}^z_F\mathbf{S}_{\xi,F}+\mathbf{S}_{\xi,F}\mathbf{S}^z_F)u^\xi_F-(\mathbf{S}^z_P\mathbf{S}_{\xi,P}+\mathbf{S}_{\xi,P}\mathbf{S}^z_F)u^\xi_P]+\\
&[(\mathbf{S}^z_F\mathbf{S}_{\eta,F}+\mathbf{S}_{\eta,F}\mathbf{S}^z_F)u^\eta_F-(\mathbf{S}^z_P\mathbf{S}_{\eta,P}+\mathbf{S}_{\eta,P}\mathbf{S}^z_F)u^\eta_P]+\\
&[(\mathbf{S}^\xi_{fe}\mathbf{S}_{z,fe}+\mathbf{S}_{z,fe}\mathbf{S}^\xi_{fe})u^z_{fe}-(\mathbf{S}^\xi_{fw}\mathbf{S}_{z,fw}+\mathbf{S}_{z,fw}\mathbf{S}^\xi_{fw})u^z_{fw}]+\\
&[(\mathbf{S}^\xi_{fe}\mathbf{S}_{\xi,fe}+\mathbf{S}_{\xi,fe}\mathbf{S}^\xi_{fe})u^\xi_{fe}-(\mathbf{S}^\xi_{fw}\mathbf{S}_{\xi,fw}+\mathbf{S}_{\xi,fw}\mathbf{S}^\xi_{fw})u^\xi_{fw}]+\\
&[(\mathbf{S}^\xi_{fe}\mathbf{S}_{\eta,fe}+\mathbf{S}_{\eta,fe}\mathbf{S}^\xi_{fe})u^\eta_{fe}-(\mathbf{S}^\xi_{fw}\mathbf{S}_{\eta,fw}+\mathbf{S}_{\eta,fw}\mathbf{S}^\xi_{fw})u^\eta_{fw}]+\\
&[(\mathbf{S}^\eta_{fn}\mathbf{S}_{z,fn}+\mathbf{S}_{z,fn}\mathbf{S}^\eta_{fn})u^z_{fn}-(\mathbf{S}^\eta_{fs}\mathbf{S}_{z,fs}+\mathbf{S}_{z,fs}\mathbf{S}^\eta_{fs})u^z_{fs}]+\\
&[(\mathbf{S}^\eta_{fn}\mathbf{S}_{\xi,fn}+\mathbf{S}_{\xi,fn}\mathbf{S}^\eta_{fn})u^\xi_{fn}-(\mathbf{S}^\eta_{fs}\mathbf{S}_{\xi,fs}+\mathbf{S}_{\xi,fs}\mathbf{S}^\eta_{fs})u^\xi_{fs}]+\\
&[(\mathbf{S}^\eta_{fn}\mathbf{S}_{\eta,fn}+\mathbf{S}_{\eta,fn}\mathbf{S}^\eta_{fn})u^\eta_{fn}-(\mathbf{S}^\eta_{fs}\mathbf{S}_{\eta,fs}+\mathbf{S}_{\eta,fs}\mathbf{S}^\eta_{fs})u^\eta_{fs}] \qquad (3.65)\\
d_{z,b,ex} &= [(\mathbf{S}^z_P\mathbf{S}_{\xi,P}+\mathbf{S}_{\xi,P}\mathbf{S}^z_P)u^\xi_P-(\mathbf{S}^z_B\mathbf{S}_{\xi,B}+\mathbf{S}_{\xi,B}\mathbf{S}^z_P)u^\xi_B]+\\
&[(\mathbf{S}^z_P\mathbf{S}_{\eta,P}+\mathbf{S}_{\eta,P}\mathbf{S}^z_P)u^\eta_P-(\mathbf{S}^z_B\mathbf{S}_{\eta,B}+\mathbf{S}_{\eta,B}\mathbf{S}^z_P)u^\eta_B]+\\
&[(\mathbf{S}^\xi_{be}\mathbf{S}_{z,be}+\mathbf{S}_{z,be}\mathbf{S}^\xi_{be})u^z_{be}-(\mathbf{S}^\xi_{bw}\mathbf{S}_{z,bw}+\mathbf{S}_{z,bw}\mathbf{S}^\xi_{bw})u^z_{bw}]+\\
&[(\mathbf{S}^\xi_{be}\mathbf{S}_{\xi,be}+\mathbf{S}_{\xi,be}\mathbf{S}^\xi_{be})u^\xi_{be}-(\mathbf{S}^\xi_{bw}\mathbf{S}_{\xi,bw}+\mathbf{S}_{\xi,bw}\mathbf{S}^\xi_{bw})u^\xi_{bw}]+\\
&[(\mathbf{S}^\xi_{be}\mathbf{S}_{\eta,be}+\mathbf{S}_{\eta,be}\mathbf{S}^\xi_{be})u^\eta_{be}-(\mathbf{S}^\xi_{bw}\mathbf{S}_{\eta,bw}+\mathbf{S}_{\eta,bw}\mathbf{S}^\xi_{bw})u^\eta_{bw}]+\\
&[(\mathbf{S}^\eta_{bn}\mathbf{S}_{z,bn}+\mathbf{S}_{z,bn}\mathbf{S}^\eta_{bn})u^z_{bn}-(\mathbf{S}^\eta_{bs}\mathbf{S}_{z,bs}+\mathbf{S}_{z,bs}\mathbf{S}^\eta_{bs})u^z_{bs}]+\\
&[(\mathbf{S}^\eta_{bn}\mathbf{S}_{\xi,bn}+\mathbf{S}_{\xi,bn}\mathbf{S}^\eta_{bn})u^\xi_{bn}-(\mathbf{S}^\eta_{bs}\mathbf{S}_{\xi,bs}+\mathbf{S}_{\xi,bs}\mathbf{S}^\eta_{bs})u^\xi_{bs}]+\\
&[(\mathbf{S}^\eta_{bn}\mathbf{S}_{\eta,bn}+\mathbf{S}_{\eta,bn}\mathbf{S}^\eta_{bn})u^\eta_{bn}-(\mathbf{S}^\eta_{bs}\mathbf{S}_{\eta,bs}+\mathbf{S}_{\eta,bs}\mathbf{S}^\eta_{bs})u^\eta_{bs}] \qquad (3.66)
\end{aligned}$$

Therefore the explicit diffusion terms are

$$\begin{aligned}
D_{z,ex} &= \mathbf{S}^z_P \bullet \{\frac{\mathbf{S}^\xi_e}{Re_eV_e}\bullet d_{z,e,ex}-\frac{\mathbf{S}^\xi_w}{Re_wV_w}\bullet d_{z,w,ex}+\frac{\mathbf{S}^\eta_n}{Re_nV_n}\bullet d_{z,n,ex}-\\
&\frac{\mathbf{S}^\eta_s}{Re_sV_s}\bullet d_{z,s,ex}+\frac{\mathbf{S}^z_f}{Re_fV_f}\bullet d_{z,f,ex}-\frac{\mathbf{S}^z_b}{Re_bV_b}\bullet d_{z,b,ex}\} \qquad (3.67)
\end{aligned}$$

3.11 Details About Equations Of Intermediate Velocities

The equation for $\hat{u}^\xi$ (3.21) can be approximated as

$$\begin{aligned}
(I-A_{\xi,1})(I-A_{\xi,2})(I-A_{\xi,3})[\hat{u}^\xi-(u^\xi)^n] &= \frac{\Delta t}{2V_\xi}[3H_\xi(u^q)^n-H_\xi(u^q)^{n-1}] \qquad (3.68)\\
&+\frac{\Delta t}{2V_\xi}[3D_{\xi,ex}(u^q)^n-D_{\xi,ex}(u^q)^{n-1}]+2A_\xi(u^\xi)^n
\end{aligned}$$

where $A_\xi = \Delta t/(2V_\xi)D_\xi$, and according to the expression of D_ξ,

$$
\begin{aligned}
A_{\xi,1} &= \frac{\Delta t}{2V_\xi}\mathbf{S}_P^\xi \bullet \{\frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_E^\xi \mathbf{S}_{\xi,E} + \mathbf{S}_{\xi,E}\mathbf{S}_E^\xi)u_E^\xi - (\mathbf{S}_P^\xi \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\xi)u_P^\xi] - \\
&\quad \frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\xi)u_P^\xi - (\mathbf{S}_W^\xi \mathbf{S}_{\xi,W} + \mathbf{S}_{\xi,W}\mathbf{S}_W^\xi)u_W^\xi]\} \qquad (3.69) \\
A_{\xi,2} &= \frac{\Delta t}{2V_\xi}\mathbf{S}_P^\xi \bullet \{\frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{\xi,N} + \mathbf{S}_{\xi,N}\mathbf{S}_N^\eta)u_N^\xi - (\mathbf{S}_P^\eta \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\eta)u_P^\xi] - \\
&\quad \frac{\mathbf{S}_s^\eta}{Re_s V_s} \bullet [(\mathbf{S}_P^\eta \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\eta)u_P^\xi - (\mathbf{S}_S^\eta \mathbf{S}_{\xi,S} + \mathbf{S}_{\xi,S}\mathbf{S}_S^\eta)u_S^\xi]\} \qquad (3.70) \\
A_{\xi,3} &= \frac{\Delta t}{2V_\xi}\mathbf{S}_P^\xi \bullet \{\frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{\xi,F} + \mathbf{S}_{\xi,F}\mathbf{S}_F^z)u_F^\xi - (\mathbf{S}_P^z \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^z)u_P^\xi] - \\
&\quad \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^z)u_P^\xi - (\mathbf{S}_B^z \mathbf{S}_{\xi,B} + \mathbf{S}_{\xi,B}\mathbf{S}_B^z)u_B^\xi]\} \qquad (3.71)
\end{aligned}
$$

The equation for $\hat{u}^\eta$ (3.22) can be approximated as

$$
\begin{aligned}
(I - A_{\eta,1})(I - A_{\eta,2})(I - A_{\eta,3})[\hat{u}^\eta - (u^\eta)^n] &= \frac{\Delta t}{2V_\eta}[3H_\eta(u^q)^n - H_\eta(u^q)^{n-1}] \qquad (3.72) \\
&\quad + \frac{\Delta t}{2V_\eta}[3D_{\eta,ex}(u^q)^n - D_{\eta,ex}(u^q)^{n-1}] + 2A_\eta(u^\eta)^n
\end{aligned}
$$

where $A_\eta = \Delta t/(2V_\eta)D_\eta$, and according to the expression of D_η,

$$
\begin{aligned}
A_{\eta,1} &= \frac{\Delta t}{2V_\eta}\mathbf{S}_P^\eta \bullet \{\frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_E^\xi \mathbf{S}_{\eta,E} + \mathbf{S}_{\eta,E}\mathbf{S}_E^\xi)u_E^\eta - (\mathbf{S}_P^\xi \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}_P^\xi)u_P^\eta] - \\
&\quad \frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}_P^\xi)u_P^\eta - (\mathbf{S}_W^\xi \mathbf{S}_{\eta,W} + \mathbf{S}_{\eta,W}\mathbf{S}_W^\xi)u_W^\eta]\} \qquad (3.73) \\
A_{\eta,2} &= \frac{\Delta t}{2V_\eta}\mathbf{S}_P^\eta \bullet \{\frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{\eta,N} + \mathbf{S}_{\eta,N}\mathbf{S}_N^\eta)u_N^\eta - (\mathbf{S}_P^\eta \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}_P^\eta)u_P^\eta] - \\
&\quad \frac{\mathbf{S}_s^\eta}{Re_s V_s} \bullet [(\mathbf{S}_P^\eta \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}_P^\eta)u_P^\eta - (\mathbf{S}_S^\eta \mathbf{S}_{\eta,S} + \mathbf{S}_{\eta,S}\mathbf{S}_S^\eta)u_S^\eta]\} \qquad (3.74) \\
A_{\eta,3} &= \frac{\Delta t}{2V_\eta}\mathbf{S}_P^\eta \bullet \{\frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{\eta,F} + \mathbf{S}_{\eta,F}\mathbf{S}_F^z)u_F^\eta - (\mathbf{S}_P^z \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}_P^z)u_P^\eta] - \\
&\quad \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}_P^z)u_P^\eta - (\mathbf{S}_B^z \mathbf{S}_{\eta,B} + \mathbf{S}_{\eta,B}\mathbf{S}_B^z)u_B^\eta]\} \qquad (3.75)
\end{aligned}
$$

$$
\begin{aligned}
(I - A_{z,1})(I - A_{z,2})(I - A_{z,3})[\hat{u}^z - (u^z)^n] &= \frac{\Delta t}{2V_z}[3H_z(u^q)^n - H_z(u^q)^{n-1}] \qquad (3.76) \\
&\quad + \frac{\Delta t}{2V_z}[3D_{z,ex}(u^z)^n - D_{z,ex}(u^z)^{n-1}] + 2A_z(u^z)^n
\end{aligned}
$$

where $A_z = \Delta t/(2V_z)D_z$, and according to the expression of D_z,

$$\begin{aligned}
A_{z,1} &= \frac{\Delta t}{2V_z}\mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_E^\xi \mathbf{S}_{z,E} + \mathbf{S}_{z,E}\mathbf{S}_E^\xi)u_E^z - (\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\xi)u_P^z] - \\
&\quad \frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\xi)u_P^z - (\mathbf{S}_W^\xi \mathbf{S}_{z,W} + \mathbf{S}_{z,W}\mathbf{S}_W^\xi)u_W^z]\} \qquad (3.77) \\
A_{z,2} &= \frac{\Delta t}{2V_z}\mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{z,N} + \mathbf{S}_{z,N}\mathbf{S}_N^\eta)u_N^\eta - (\mathbf{S}_P^\eta \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\eta)u_P^z] - \\
&\quad \frac{\mathbf{S}_s^\eta}{Re_s V_s} \bullet [(\mathbf{S}_P^\eta \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\eta)u_P^z - (\mathbf{S}_S^\eta \mathbf{S}_{z,S} + \mathbf{S}_{z,S}\mathbf{S}_S^\eta)u_S^z]\} \qquad (3.78) \\
A_{z,3} &= \frac{\Delta t}{2V_z}\mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{z,F} + \mathbf{S}_{z,F}\mathbf{S}_F^z)u_F^z - (\mathbf{S}_P^z \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^z)u_P^z] - \\
&\quad \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^z)u_P^z - (\mathbf{S}_B^z \mathbf{S}_{z,B} + \mathbf{S}_{z,B}\mathbf{S}_B^z)u_B^z]\} \qquad (3.79)
\end{aligned}$$

3.12 Inversion of $\hat{u}^\xi$ Equation

The equation for $\hat{u}^\xi$ (3.68) can be written as

$$(I - A_{\xi,1})(I - A_{\xi,2})(I - A_{\xi,3})[\hat{u}^\xi - (u^\xi)^n] = Rhs \qquad (3.80)$$

where

$$\begin{aligned}
A_{\xi,1} &= \frac{\Delta t}{2V_\xi}\mathbf{S}_P^\xi \bullet \{\frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_E^\xi \mathbf{S}_{\xi,E} + \mathbf{S}_{\xi,E}\mathbf{S}_E^\xi)u_E^\xi - (\mathbf{S}_P^\xi \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\xi)u_P^\xi] - \\
&\quad \frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\xi)u_P^\xi - (\mathbf{S}_W^\xi \mathbf{S}_{\xi,W} + \mathbf{S}_{\xi,W}\mathbf{S}_W^\xi)u_W^\xi]\} \qquad (3.81) \\
A_{\xi,2} &= \frac{\Delta t}{2V_\xi}\mathbf{S}_P^\xi \bullet \{\frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{\xi,N} + \mathbf{S}_{\xi,N}\mathbf{S}_N^\eta)u_N^\xi - (\mathbf{S}_P^\eta \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\eta)u_P^\xi] - \\
&\quad \frac{\mathbf{S}_s^\eta}{Re_s V_s} \bullet [(\mathbf{S}_P^\eta \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^\eta)u_P^\xi - (\mathbf{S}_S^\eta \mathbf{S}_{\xi,S} + \mathbf{S}_{\xi,S}\mathbf{S}_S^\eta)u_S^\xi]\} \qquad (3.82) \\
A_{\xi,3} &= \frac{\Delta t}{2V_\xi}\mathbf{S}_P^\xi \bullet \{\frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{\xi,F} + \mathbf{S}_{\xi,F}\mathbf{S}_F^z)u_F^\xi - (\mathbf{S}_P^z \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^z)u_P^\xi] - \\
&\quad \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}_P^z)u_P^\xi - (\mathbf{S}_B^z \mathbf{S}_{\xi,B} + \mathbf{S}_{\xi,B}\mathbf{S}_B^z)u_B^\xi]\} \qquad (3.83)
\end{aligned}$$

Define

$$(I - A_{\xi,2})(I - A_{\xi,3})[\hat{u}^\xi - (u^\xi)^n] = \Delta u^{'} \qquad (3.84)$$

we have

$$(I - A_{\xi,1})\Delta u^{'} = Rhs \qquad (3.85)$$

let

$$\beta = -\frac{\Delta t}{2V_\xi} \qquad (3.86)$$

we have

$$c\Delta u^{'}_{i+1} + b\Delta u^{'}_{i} + a\Delta u^{'}_{i-1} = Rhs \tag{3.87}$$

where

$$\begin{aligned}
c &= \beta \mathbf{S}^{\xi}_{P} \bullet \{\frac{\mathbf{S}^{\xi}_{e}}{Re_e V_e} \bullet [(\mathbf{S}^{\xi}_{E}\mathbf{S}_{\xi,E} + \mathbf{S}_{\xi,E}\mathbf{S}^{\xi}_{E})]\} && (3.88)\\
b &= 1 - \beta \mathbf{S}^{\xi}_{P} \bullet \{\frac{\mathbf{S}^{\xi}_{e}}{Re_e V_e} \bullet [(\mathbf{S}^{\xi}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{\xi}_{P})] + \\
&\quad \frac{\mathbf{S}^{\xi}_{w}}{Re_w V_w} \bullet [(\mathbf{S}^{\xi}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{\xi}_{P})]\} && (3.89)\\
a &= \beta \mathbf{S}^{\xi}_{P} \bullet \{\frac{\mathbf{S}^{\xi}_{w}}{Re_w V_w} \bullet [(\mathbf{S}^{\xi}_{W}\mathbf{S}_{\xi,W} + \mathbf{S}_{\xi,W}\mathbf{S}^{\xi}_{W})]\} && (3.90)
\end{aligned}$$

If $i = 2$

$$a = 0$$

$$Rhs^{'} = Rhs - \beta \mathbf{S}^{\xi}_{P} \bullet \{\frac{\mathbf{S}^{\xi}_{w}}{Re_w V_w} \bullet [(\mathbf{S}^{\xi}_{W}\mathbf{S}_{\xi,W} + \mathbf{S}_{\xi,W}\mathbf{S}^{\xi}_{W})]\}\Delta u^{'}_{i=1} \tag{3.91}$$

where

$$\Delta u^{'}_{i=1} = (I - A_{\xi,2})(I - A_{\xi,3})[\hat{u}^{\xi} - (u^{\xi})^{n}]_{1jk} \tag{3.92}$$

Because $\hat{u}^{\xi}$ is not defined at $i = 1$, we use the boundary condition for u^{ξ} at $n + 1$ time step here to substitute it,

$$\begin{aligned}
\Delta u^{'}_{i=1} &= (I - A_{\xi,2})(I - A_{\xi,3})[u^{\xi}_{bcw} - (u^{\xi})^{n}_{1jk}] \\
&= (A_{\xi,2})[(u^{\xi}_{bcw} - (u^{\xi})^{n}_{1jk}) - A_{\xi,3}(u^{\xi}_{bcw} - (u^{\xi})^{n}_{1jk})] \\
&= [(u^{\xi}_{bcw} - (u^{\xi})^{n}_{1jk}) - A_{\xi,3}(u^{\xi}_{bcw} - (u^{\xi})^{n}_{1jk})] \\
&\quad - A_{\xi,2}\{[(u^{\xi}_{bcw} - (u^{\xi})^{n}_{1jk}) - A_{\xi,3}(u^{\xi}_{bcw} - (u^{\xi})^{n}_{1jk})]\} && (3.93)
\end{aligned}$$

Similarly, at $i = nx - 1$,

$$c = 0$$

$$Rhs^{'} = Rhs - \beta \mathbf{S}^{\xi}_{P} \bullet \{\frac{\mathbf{S}^{\xi}_{e}}{Re_e V_e} \bullet [(\mathbf{S}^{\xi}_{E}\mathbf{S}_{\xi,E} + \mathbf{S}_{\xi,E}\mathbf{S}^{\xi}_{E})]\}\Delta u^{'}_{i=nx} \tag{3.94}$$

where

$$\Delta u^{'}_{i=nx} = (I - A_{\xi,2})(I - A_{\xi,3})[\hat{u}^{\xi} - (u^{\xi})^{n}]_{nxjk} \tag{3.95}$$

Because $\hat{u}^{\xi}$ is not defined at $i = nx$, we use the boundary condition for u^{ξ} here to substitute it,

$$\begin{aligned}
\Delta u^{'}_{i=nx} &= (I - A_{\xi,2})(I - A_{\xi,3})[u^{\xi}_{bce} - (u^{\xi})^{n}_{nxjk}] \\
&= (I - A_{\xi,2})[(u^{\xi}_{bce} - (u^{\xi})^{n}_{nxjk}) - A_{\xi,3}(u^{\xi}_{bce} - (u^{\xi})^{n}_{nxjk})] \\
&= [(u^{\xi}_{bce} - (u^{\xi})^{n}_{nxjk}) - A_{\xi,3}(u^{\xi}_{bce} - (u^{\xi})^{n}_{nxjk})] \\
&\quad - A_{\xi,2}\{[(u^{\xi}_{bce} - (u^{\xi})^{n}_{nxjk}) - A_{\xi,3}(u^{\xi}_{bce} - (u^{\xi})^{n}_{nxjk})]\} && (3.96)
\end{aligned}$$

Next step, let,

$$(I - A_{\xi,3})[\hat{u}^{\xi} - (u^{\xi})^n] = \Delta u^{''} \tag{3.97}$$

we have

$$(I - A_{\xi,2})\Delta u^{''} = \Delta u^{'} = Rhs(2) \tag{3.98}$$

let

$$\beta = -\frac{\Delta t}{2V_{\xi}} \tag{3.99}$$

we have

$$c\Delta u^{''}_{j+1} + b\Delta u^{''}_{j} + a\Delta u^{''}_{j-1} = Rhs(2) \tag{3.100}$$

where

$$\begin{aligned}
c &= \beta \mathbf{S}^{\xi}_{P} \bullet \{\frac{\mathbf{S}^{\eta}_{n}}{Re_n V_n} \bullet [(\mathbf{S}^{\eta}_{N}\mathbf{S}_{\xi,N} + \mathbf{S}_{\xi,N}\mathbf{S}^{\eta}_{N})] & (3.101)\\
b &= 1 - \beta \mathbf{S}^{\xi}_{P} \bullet \{\frac{\mathbf{S}^{\eta}_{n}}{Re_n V_n} \bullet [(\mathbf{S}^{\eta}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{\eta}_{P})] + \\
&\quad \frac{\mathbf{S}^{\eta}_{s}}{Re_s V_s} \bullet [(\mathbf{S}^{\eta}_{P}\mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^{\eta}_{P})]\} & (3.102)\\
a &= \beta \mathbf{S}^{\xi}_{P} \bullet \{\frac{\mathbf{S}^{\eta}_{s}}{Re_s V_s} \bullet [(\mathbf{S}^{\eta}_{S}\mathbf{S}_{\xi,S} + \mathbf{S}_{\xi,S}\mathbf{S}^{\eta}_{S})]\} & (3.103)
\end{aligned}$$

If $j = 1$

$$a = 0$$

$$Rhs^{'} = Rhs(2) - \beta \mathbf{S}^{\xi}_{P} \bullet \{\frac{\mathbf{S}^{\eta}_{s}}{Re_s V_s} \bullet [(\mathbf{S}^{\eta}_{S}\mathbf{S}_{\xi,S} + \mathbf{S}_{\xi,S}\mathbf{S}^{\eta}_{S})]\}\Delta u^{''}_{j=0} \tag{3.104}$$

where

$$\Delta u^{''}_{j=0} = (I - A_{\xi,3})[\hat{u}^{\xi} - (u^{\xi})^n]_{i0k} \tag{3.105}$$

Because $\hat{u}^{\xi}$ is not defined at $j = 0$, we use the boundary condition for u^{ξ} here to substitute it,

$$\begin{aligned}
\Delta u^{''}_{j=0} &= (I - A_{\xi,3})[u^{\xi}_{bcs} - (u^{\xi})^n_{i0k}] \\
&= [(u^{\xi}_{bcs} - (u^{\xi})^n_{i0k}) - A_{\xi,3}(u^{\xi}_{bcs} - (u^{\xi})^n_{i0k})] & (3.106)
\end{aligned}$$

Similarly, at $j = ny - 1$,

$$c = 0$$

$$Rhs^{'} = Rhs(2) - \beta \mathbf{S}^{\xi}_{P} \bullet \{\frac{\mathbf{S}^{\eta}_{n}}{Re_n V_n} \bullet [(\mathbf{S}^{\eta}_{N}\mathbf{S}_{\xi,N} + \mathbf{S}_{\xi,N}\mathbf{S}^{\eta}_{N})]\}\Delta u^{''}_{j=ny} \tag{3.107}$$

where

$$\Delta u^{''}_{j=ny} = (I - A_{\xi,3})[\hat{u}^{\xi} - (u^{\xi})^n]_{inyk} \tag{3.108}$$

Because $\hat{u}^\xi$ is not defined at $j = ny$, we use the boundary condition for u^ξ here to substitute it,

$$\begin{aligned}\Delta u''_{j=ny} &= (I - A_{\xi,3})[u^\xi_{bcn} - (u^\xi)^n_{inyk}] \\ &= [(u^\xi_{bcn} - (u^\xi)^n_{inyk}) - A_{\xi,3}(u^\xi_{bcn} - (u^\xi)^n_{inyk})] \qquad (3.109)\end{aligned}$$

Next step

$$(I - A_{\xi,3})[\hat{u}^\xi - (u^\xi)^n] = \Delta u'' = Rhs(3) \qquad (3.110)$$

let

$$\Delta u''' = \hat{u}^\xi - (u^\xi)^n$$

we have

$$c\Delta u'''_{k+1} + b\Delta u'''_k + a\Delta u'''_{k-1} = Rhs(3) \qquad (3.111)$$

where

$$\begin{aligned}c &= \beta \mathbf{S}^\xi_P \bullet \{\frac{\mathbf{S}^z_f}{Re_f V_f} \bullet [(\mathbf{S}^z_F \mathbf{S}_{\xi,F} + \mathbf{S}_{\xi,F}\mathbf{S}^z_F)] \qquad (3.112)\\ b &= 1 - \beta \mathbf{S}^\xi_P \bullet \{\frac{\mathbf{S}^z_f}{Re_f V_f} \bullet [(\mathbf{S}^z_P \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^z_P)] + \\ &\quad \frac{\mathbf{S}^z_b}{Re_b V_b} \bullet [(\mathbf{S}^z_P \mathbf{S}_{\xi,P} + \mathbf{S}_{\xi,P}\mathbf{S}^z_P)]\} \qquad (3.113)\\ a &= \beta \mathbf{S}^\xi_P \bullet \{\frac{\mathbf{S}^z_b}{Re_b V_b} \bullet [(\mathbf{S}^z_B \mathbf{S}_{\xi,B} + \mathbf{S}_{\xi,B}\mathbf{S}^z_B)]\} \qquad (3.114)\end{aligned}$$

3.13 Inversion of $\hat{u}^\eta$ Equation

The equation for $\hat{u}^\eta$ (3.72) can be written as

$$(I - A_{\eta,1})(I - A_{\eta,2})(I - A_{\eta,3})[\hat{u}^\eta - (u^\eta)^n] = Rhs \qquad (3.115)$$

where

$$\begin{aligned}A_{\eta,1} &= \frac{\Delta t}{2V_\eta}\mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\xi_e}{Re_e V_e} \bullet [(\mathbf{S}^\xi_E \mathbf{S}_{\eta,E} + \mathbf{S}_{\eta,E}\mathbf{S}^\xi_E)u^\eta_E - (\mathbf{S}^\xi_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\xi_P)u^\eta_P] - \\ &\quad \frac{\mathbf{S}^\xi_w}{Re_w V_w} \bullet [(\mathbf{S}^\xi_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\xi_P)u^\eta_P - (\mathbf{S}^\xi_W \mathbf{S}_{\eta,W} + \mathbf{S}_{\eta,W}\mathbf{S}^\xi_W)u^\eta_W]\} \qquad (3.116)\\ A_{\eta,2} &= \frac{\Delta t}{2V_\eta}\mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\eta_n}{Re_n V_n} \bullet [(\mathbf{S}^\eta_N \mathbf{S}_{\eta,N} + \mathbf{S}_{\eta,N}\mathbf{S}^\eta_N)u^\eta_N - (\mathbf{S}^\eta_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\eta_P)u^\eta_P] - \\ &\quad \frac{\mathbf{S}^\eta_s}{Re_s V_s} \bullet [(\mathbf{S}^\eta_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\eta_P)u^\eta_P - (\mathbf{S}^\eta_S \mathbf{S}_{\eta,S} + \mathbf{S}_{\eta,S}\mathbf{S}^\eta_S)u^\eta_S]\} \qquad (3.117)\\ A_{\eta,3} &= \frac{\Delta t}{2V_\eta}\mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^z_f}{Re_f V_f} \bullet [(\mathbf{S}^z_F \mathbf{S}_{\eta,F} + \mathbf{S}_{\eta,F}\mathbf{S}^z_F)u^\eta_F - (\mathbf{S}^z_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^z_P)u^\eta_P] - \\ &\quad \frac{\mathbf{S}^z_b}{Re_b V_b} \bullet [(\mathbf{S}^z_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^z_P)u^\eta_P - (\mathbf{S}^z_B \mathbf{S}_{\eta,B} + \mathbf{S}_{\eta,B}\mathbf{S}^z_B)u^\eta_B]\} \qquad (3.118)\end{aligned}$$

Define

$$(I - A_{\eta,2})(I - A_{\eta,3})[\hat{u}^\eta - (u^\eta)^n] = \Delta u^{'} \tag{3.119}$$

we have

$$(I - A_{\eta,1})\Delta u^{'} = Rhs \tag{3.120}$$

let

$$\beta = -\frac{\Delta t}{2V_\eta} \tag{3.121}$$

we have

$$c\Delta u^{'}_{i+1} + b\Delta u^{'}_i + a\Delta u^{'}_{i-1} = Rhs \tag{3.122}$$

where

$$\begin{aligned}
c &= \beta \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\xi_e}{Re_e V_e} \bullet [(\mathbf{S}^\xi_E \mathbf{S}_{\eta,E} + \mathbf{S}_{\eta,E}\mathbf{S}^\xi_E)]\} && (3.123)\\
b &= 1 - \beta \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\xi_e}{Re_e V_e} \bullet [(\mathbf{S}^\xi_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\xi_P)] + \\
&\quad \frac{\mathbf{S}^\xi_w}{Re_w V_w} \bullet [(\mathbf{S}^\xi_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\xi_P)]\} && (3.124)\\
a &= \beta \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\xi_w}{Re_w V_w} \bullet [(\mathbf{S}^\xi_W \mathbf{S}_{\eta,W} + \mathbf{S}_{\eta,W}\mathbf{S}^\xi_W)]\} && (3.125)
\end{aligned}$$

If $i = 1$

$$a = 0$$

$$Rhs^{'} = Rhs - \beta \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\xi_w}{Re_w V_w} \bullet [(\mathbf{S}^\xi_W \mathbf{S}_{\eta,W} + \mathbf{S}_{\eta,W}\mathbf{S}^\xi_W)]\}\Delta u^{'}_{i=0} \tag{3.126}$$

where

$$\Delta u^{'}_{i=0} = (I - A_{\eta,2})(I - A_{\eta,3})[\hat{u}^\eta - (u^\eta)^n]_{0jk} \tag{3.127}$$

Because $\hat{u}^\eta$ is not defined at $i = 0$, we use the boundary condition for u^η at $n + 1$ time step here to substitute it,

$$\begin{aligned}
\Delta u^{'}_{i=0} &= (I - A_{\eta,2})(I - A_{\eta,3})[u^\eta_{bcw} - (u^\eta)^n_{0jk}]\\
&= (I - A_{\eta,2})[(u^\eta_{bcw} - (u^\eta)^n_{0jk}) - A_{\eta,3}(u^\eta_{bcw} - (u^\eta)^n_{0jk})]\\
&= [(u^\eta_{bcw} - (u^\eta)^n_{0jk}) - A_{\eta,3}(u^\eta_{bcw} - (u^\eta)^n_{0jk})]\\
&\quad -A_{\eta,2}\{[(u^\eta_{bcw} - (u^\eta)^n_{0jk}) - A_{\eta,3}(u^\eta_{bcw} - (u^\eta)^n_{0jk})]\} && (3.128)
\end{aligned}$$

Similarly, at $i = nx - 1$,

$$c = 0$$

$$Rhs^{'} = Rhs - \beta \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\eta_e}{Re_e V_e} \bullet [(\mathbf{S}^\xi_E \mathbf{S}_{\eta,E} + \mathbf{S}_{\eta,E}\mathbf{S}^\xi_E)]\}\Delta u^{'}_{i=nx} \tag{3.129}$$

where

$$\Delta u'_{i=nx} = (I - A_{\eta,2})(I - A_{\eta,3})[\hat{u}^\eta - (u^\eta)^n]_{nxjk} \tag{3.130}$$

Because $\hat{u}^\eta$ is not defined at $i = nx$, we use the boundary condition for u^η here to substitute it,

$$\begin{aligned}\Delta u'_{i=nx} &= (I - A_{\eta,2})(I - A_{\eta,3})[u^\eta_{bce} - (u^\eta)^n_{nxjk}] \\ &= (I - A_{\eta,2})[(u^\eta_{bce} - (u^\eta)^n_{nxjk}) - A_{\eta,3}(u^\eta_{bce} - (u^\eta)^n_{nxjk})] \\ &= [(u^\eta_{bce} - (u^\eta)^n_{nxjk}) - A_{\eta,3}(u^\eta_{bce} - (u^\eta)^n_{nxjk})] \\ &\quad -A_{\eta,2}\{[(u^\eta_{bce} - (u^\eta)^n_{nxjk}) - A_{\eta,3}(u^\eta_{bce} - (u^\eta)^n_{nxjk})]\end{aligned} \tag{3.131}$$

Next step, let,

$$(I - A_{\eta,3})[\hat{u}^\eta - (u^\eta)^n] = \Delta u'' \tag{3.132}$$

we have

$$(I - A_{\eta,2})\Delta u'' = \Delta u' = Rhs(2) \tag{3.133}$$

let

$$\beta = -\frac{\Delta t}{2V_\eta} \tag{3.134}$$

we have

$$c\Delta u''_{j+1} + b\Delta u''_j + a\Delta u''_{j-1} = Rhs(2) \tag{3.135}$$

where

$$c = \beta \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\eta_n}{Re_n V_n} \bullet [(\mathbf{S}^\eta_N \mathbf{S}_{\eta,N} + \mathbf{S}_{\eta,N}\mathbf{S}^\eta_N)] \tag{3.136}$$

$$\begin{aligned}b &= 1 - \beta \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\eta_n}{Re_n V_n} \bullet [(\mathbf{S}^\eta_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\eta_P)] + \\ &\quad \frac{\mathbf{S}^\eta_s}{Re_s V_s} \bullet [(\mathbf{S}^\eta_P \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P}\mathbf{S}^\eta_P)]\}\end{aligned} \tag{3.137}$$

$$a = \beta \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\eta_s}{Re_s V_s} \bullet [(\mathbf{S}^\eta_S \mathbf{S}_{\eta,S} + \mathbf{S}_{\eta,S}\mathbf{S}^\eta_S)]\} \tag{3.138}$$

If $j = 2$

$$a = 0$$

$$Rhs' = Rhs(2) - \beta \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\eta_s}{Re_s V_s} \bullet [(\mathbf{S}^\eta_S \mathbf{S}_{\eta,S} + \mathbf{S}_{\eta,S}\mathbf{S}^\eta_S)]\}\Delta u''_{j=1} \tag{3.139}$$

where

$$\Delta u''_{j=1} = (I - A_{\eta,3})[\hat{u}^\eta - (u^\eta)^n]_{i1k} \tag{3.140}$$

Because $\hat{u}^\eta$ is not defined at $j = 1$, we use the boundary condition for u^η here to substitute it,

$$\begin{aligned}\Delta u''_{j=1} &= (I - A_{\eta,3})[u^\eta_{bcs} - (u^\eta)^n_{i1k}] \\ &= [(u^\eta_{bcs} - (u^\eta)^n_{i1k}) - A_{\eta,3}(u^\eta_{bcs} - (u^\eta)^n_{i1k})]\end{aligned} \tag{3.141}$$

Similarly, at $j = ny - 1$,

$$c = 0$$

$$Rhs^{'} = Rhs(2) - \beta \mathbf{S}_P^\eta \bullet \{ \frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{\eta,N} + \mathbf{S}_{\eta,N} \mathbf{S}_N^\eta)] \} \Delta u^{''}_{j=ny} \tag{3.142}$$

where

$$\Delta u^{''}_{j=ny} = (I - A_{\eta,3})[\hat{u}^\eta - (u^\eta)^n]_{inyk} \tag{3.143}$$

Because $\hat{u}^\eta$ is not defined at $j = ny$, we use the boundary condition for u^η here to substitute it,

$$\begin{aligned} \Delta u^{''}_{j=ny} &= (I - A_{\eta,3})[u^\eta_{bcn} - (u^\eta)^n_{inyk}] \\ &= [(u^\eta_{bcn} - (u^\eta)^n_{inyk}) - A_{\eta,3}(u^\eta_{bcn} - (u^\eta)^n_{inyk})] \end{aligned} \tag{3.144}$$

Next step

$$(I - A_{\eta,3})[\hat{u}^\eta - (u^\eta)^n] = \Delta u^{''} = Rhs(3) \tag{3.145}$$

let

$$\Delta u^{'''} = \hat{u}^\eta - (u^\eta)^n$$

we have

$$c\Delta u^{'''}_{k+1} + b\Delta u^{'''}_k + a\Delta u^{'''}_{k-1} = Rhs(3) \tag{3.146}$$

where

$$c = \beta \mathbf{S}_P^\eta \bullet \{ \frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{\eta,F} + \mathbf{S}_{\eta,F} \mathbf{S}_F^z)] \tag{3.147}$$

$$\begin{aligned} b = {} & 1 - \beta \mathbf{S}_P^\eta \bullet \{ \frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_P^z \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P} \mathbf{S}_P^z)] + \\ & \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{\eta,P} + \mathbf{S}_{\eta,P} \mathbf{S}_P^z)] \} \end{aligned} \tag{3.148}$$

$$a = \beta \mathbf{S}_P^\eta \bullet \{ \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_B^z \mathbf{S}_{\eta,B} + \mathbf{S}_{\eta,B} \mathbf{S}_B^z)] \} \tag{3.149}$$

3.14 Inversion of $\hat{u}^z$ Equation

The equation for $\hat{u}^z$ (3.76) can be written as

$$(I - A_{z,1})(I - A_{z,2})(I - A_{z,3})[\hat{u}^z - (u^z)^n] = Rhs \tag{3.150}$$

where

$$\begin{aligned} A_{z,1} &= \frac{\Delta t}{2V_z}\mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_E^\xi \mathbf{S}_{z,E} + \mathbf{S}_{z,E}\mathbf{S}_E^\xi)u_E^z - (\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\xi)u_P^z] - \\ &\quad \frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\xi)u_P^z - (\mathbf{S}_W^\xi \mathbf{S}_{z,W} + \mathbf{S}_{z,W}\mathbf{S}_W^\xi)u_W^z]\} \qquad (3.151) \\ A_{z,2} &= \frac{\Delta t}{2V_z}\mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{z,N} + \mathbf{S}_{z,N}\mathbf{S}_N^\eta)u_N^z - (\mathbf{S}_P^\eta \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\eta)u_P^z] - \\ &\quad \frac{\mathbf{S}_s^\eta}{Re_s V_s} \bullet [(\mathbf{S}_P^\eta \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\eta)u_P^z - (\mathbf{S}_S^\eta \mathbf{S}_{z,S} + \mathbf{S}_{z,S}\mathbf{S}_S^\eta)u_S^z]\} \qquad (3.152) \\ A_{z,3} &= \frac{\Delta t}{2V_z}\mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{z,F} + \mathbf{S}_{z,F}\mathbf{S}_F^z)u_F^z - (\mathbf{S}_P^z \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^z)u_P^z] - \\ &\quad \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^z)u_P^z - (\mathbf{S}_B^z \mathbf{S}_{z,B} + \mathbf{S}_{z,B}\mathbf{S}_B^z)u_B^z]\} \qquad (3.153) \end{aligned}$$

Define

$$(I - A_{z,2})(I - A_{z,3})[\hat{u}^z - (u^z)^n] = \Delta u^{'} \qquad (3.154)$$

we have

$$(I - A_{z,1})\Delta u^{'} = Rhs \qquad (3.155)$$

let

$$\beta = -\frac{\Delta t}{2V_z} \qquad (3.156)$$

we have

$$c\Delta u^{'}_{i+1} + b\Delta u^{'}_i + a\Delta u^{'}_{i-1} = Rhs \qquad (3.157)$$

where

$$\begin{aligned} c &= \beta \mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_E^\xi \mathbf{S}_{z,E} + \mathbf{S}_{z,E}\mathbf{S}_E^\xi)]\} \qquad (3.158) \\ b &= 1 - \beta \mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_e^\xi}{Re_e V_e} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\xi)] + \\ &\quad \frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_P^\xi \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}_P^\xi)]\} \qquad (3.159) \\ a &= \beta \mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_W^\xi \mathbf{S}_{z,W} + \mathbf{S}_{z,W}\mathbf{S}_W^\xi)]\} \qquad (3.160) \end{aligned}$$

If $i = 1$

$$a = 0$$

$$Rhs^{'} = Rhs - \beta \mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_w^\xi}{Re_w V_w} \bullet [(\mathbf{S}_W^\xi \mathbf{S}_{z,W} + \mathbf{S}_{z,W}\mathbf{S}_W^\xi)]\}\Delta u^{'}_{i=0} \qquad (3.161)$$

where

$$\Delta u^{'}_{i=0} = (I - A_{z,2})(I - A_{z,3})[\hat{u}^z - (u^z)^n]_{0jk} \qquad (3.162)$$

Because $\hat{u}^z$ is not defined at $i = 0$, we use the boundary condition for u^z at $n+1$ time step here to substitute it,

$$\begin{aligned} \Delta u'_{i=0} &= (I - A_{z,2})(I - A_{z,3})[u^z_{bcw} - (u^z)^n_{0jk}] \\ &= (I - A_{z,2})[(u^z_{bcw} - (u^z)^n_{0jk}) - A_{z,3}(u^z_{bcw} - (u^z)^n_{0jk})] \\ &= [(u^z_{bcw} - (u^z)^n_{0jk}) - A_{z,3}(u^z_{bcw} - (u^z)^n_{0jk})] \\ &\quad -A_{z,2}\{[(u^z_{bcw} - (u^z)^n_{0jk}) - A_{z,3}(u^z_{bcw} - (u^z)^n_{0jk})]\} \end{aligned} \tag{3.163}$$

Similarly, at $i = nx - 1$,

$$c = 0$$

$$Rhs' = Rhs - \beta \mathbf{S}^z_P \bullet \{\frac{\mathbf{S}^\eta_e}{Re_e V_e} \bullet [(\mathbf{S}^\xi_E \mathbf{S}_{z,E} + \mathbf{S}_{z,E}\mathbf{S}^\xi_E)]\}\Delta u'_{i=nx} \tag{3.164}$$

where

$$\Delta u'_{i=nx} = (I - A_{z,2})(I - A_{z,3})[\hat{u}^z - (u^z)^n]_{nxjk} \tag{3.165}$$

Because $\hat{u}^z$ is not defined at $i = nx$, we use the boundary condition for u^z here to substitute it,

$$\begin{aligned} \Delta u'_{i=nx} &= (I - A_{z,2})(I - A_{z,3})[u^z_{bce} - (u^z)^n_{nxjk}] \\ &= (I - A_{z,2})[(u^z_{bce} - (u^z)^n_{nxjk}) - A_{z,3}(u^z_{bce} - (u^z)^n_{nxjk})] \\ &= [(u^z_{bce} - (u^z)^n_{nxjk}) - A_{z,3}(u^z_{bce} - (u^z)^n_{nxjk})] \\ &\quad -A_{z,2}\{[(u^z_{bce} - (u^z)^n_{nxjk}) - A_{z,3}(u^z_{bce} - (u^z)^n_{nxjk})]\} \end{aligned} \tag{3.166}$$

Next step ,let,

$$(I - A_{z,3})[\hat{u}^z - (u^z)^n] = \Delta u'' \tag{3.167}$$

we have

$$(I - A_{z,2})\Delta u'' = \Delta u' = Rhs(2) \tag{3.168}$$

let

$$\beta = -\frac{\Delta t}{2V_z} \tag{3.169}$$

we have

$$c\Delta u''_{j+1} + b\Delta u''_j + a\Delta u''_{j-1} = Rhs(2) \tag{3.170}$$

where

$$c = \beta \mathbf{S}^z_P \bullet \{\frac{\mathbf{S}^\eta_n}{Re_n V_n} \bullet [(\mathbf{S}^\eta_N \mathbf{S}_{z,N} + \mathbf{S}_{z,N}\mathbf{S}^\eta_N)] \tag{3.171}$$

$$\begin{aligned} b &= 1 - \beta \mathbf{S}^z_P \bullet \{\frac{\mathbf{S}^\eta_n}{Re_n V_n} \bullet [(\mathbf{S}^\eta_P \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^\eta_P)] + \\ &\quad \frac{\mathbf{S}^\eta_s}{Re_s V_s} \bullet [(\mathbf{S}^\eta_P \mathbf{S}_{z,P} + \mathbf{S}_{z,P}\mathbf{S}^\eta_P)]\} \end{aligned} \tag{3.172}$$

$$a = \beta \mathbf{S}^\eta_P \bullet \{\frac{\mathbf{S}^\eta_s}{Re_s V_s} \bullet [(\mathbf{S}^\eta_S \mathbf{S}_{z,S} + \mathbf{S}_{z,S}\mathbf{S}^\eta_S)]\} \tag{3.173}$$

If $j = 1$

$$a = 0$$

$$Rhs^{'} = Rhs(2) - \beta \mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_s^\eta}{Re_s V_s} \bullet [(\mathbf{S}_S^\eta \mathbf{S}_{z,S} + \mathbf{S}_{z,S} \mathbf{S}_S^\eta)]\} \Delta u_{j=0}^{''} \tag{3.174}$$

where

$$\Delta u_{j=0}^{''} = (I - A_{z,3})[\hat{u}^z - (u^z)^n]_{i0k} \tag{3.175}$$

Because $\hat{u}^z$ is not defined at $j = 0$, we use the boundary condition for u^z here to substitute it,

$$\begin{aligned} \Delta u_{j=0}^{''} &= (I - A_{z,3})[u_{bcs}^z - (u^z)_{i0k}^n] \\ &= [(u_{bcs}^z - (u^z)_{i0k}^n) - A_{z,3}(u_{bcs}^z - (u^z)_{i0k}^n)] \end{aligned} \tag{3.176}$$

Similarly, at $j = ny - 1$,

$$c = 0$$

$$Rhs^{'} = Rhs(2) - \beta \mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_n^\eta}{Re_n V_n} \bullet [(\mathbf{S}_N^\eta \mathbf{S}_{z,N} + \mathbf{S}_{z,N} \mathbf{S}_N^\eta)]\} \Delta u_{j=ny}^{''} \tag{3.177}$$

where

$$\Delta u_{j=ny}^{''} = (I - A_{z,3})[\hat{u}^z - (u^z)^n]_{inyk} \tag{3.178}$$

Because $\hat{u}^z$ is not defined at $j = ny$, we use the boundary condition for u^z here to substitute it,

$$\begin{aligned} \Delta u_{j=ny}^{''} &= (I - A_{z,3})[u_{bcn}^z - (u^z)_{inyk}^n] \\ &= [(u_{bcn}^z - (u^z)_{inyk}^n) - A_{z,3}(u_{bcn}^z - (u^z)_{inyk}^n)] \end{aligned} \tag{3.179}$$

Next step

$$(I - A_{z,3})[\hat{u}^z - (u^z)^n] = \Delta u^{''} = Rhs(3) \tag{3.180}$$

let

$$\Delta u^{'''} = \hat{u}^z - (u^z)^n$$

we have

$$c \Delta u_{k+1}^{'''} + b \Delta u_k^{'''} + a \Delta u_{k-1}^{'''} = Rhs(3) \tag{3.181}$$

where

$$c = \beta \mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_F^z \mathbf{S}_{z,F} + \mathbf{S}_{z,F} \mathbf{S}_F^z)] \tag{3.182}$$

$$\begin{aligned} b = \; & 1 - \beta \mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_f^z}{Re_f V_f} \bullet [(\mathbf{S}_P^z \mathbf{S}_{z,P} + \mathbf{S}_{z,P} \mathbf{S}_P^z)] + \\ & \frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_P^z \mathbf{S}_{z,P} + \mathbf{S}_{z,P} \mathbf{S}_P^z)]\} \end{aligned} \tag{3.183}$$

$$a = \beta \mathbf{S}_P^z \bullet \{\frac{\mathbf{S}_b^z}{Re_b V_b} \bullet [(\mathbf{S}_B^z \mathbf{S}_{z,B} + \mathbf{S}_{z,B} \mathbf{S}_B^z)]\} \tag{3.184}$$

3.15 Details of Poisson Equation

Recall the discretized continuity equation for the primary control volume such as shown in Figure 3.2, we have,

$$u_e^\xi - u_w^\xi + u_n^\eta - u_s^\eta + u_f^z - u_b^z = 0$$

where $u^q = (u^\xi, u^\eta, u^z)$ and D_{iv} is the summation operator. All the subscripts e, w, n, s, f, b are associated with primary pressure control volume such as those shown in Figure 3.2.

The velocity update equations are as follows,

$$\begin{aligned}
(u^\xi)^{n+1} &= \frac{\Delta t}{V_\xi} R_\xi(\phi)^{n+1} + \hat{u}^\xi \\
(u^\eta)^{n+1} &= \frac{\Delta t}{V_\eta} R_\eta(\phi)^{n+1} + \hat{u}^\eta \\
(u^z)^{n+1} &= \frac{\Delta t}{V_z} R_z(\phi)^{n+1} + \hat{u}^z
\end{aligned}$$

therefore, all the flux variables at the pressure control volume faces at n+1 time step can be expressed as,

$$\begin{aligned}
(u_e^\xi)^{n+1} &= \frac{\Delta t}{V_{\xi,e}} R_{\xi,e}(\phi)^{n+1} + \hat{u}_e^\xi \\
(u_w^\xi)^{n+1} &= \frac{\Delta t}{V_{\xi,w}} R_{\xi,w}(\phi)^{n+1} + \hat{u}_w^\xi \\
(u_n^\eta)^{n+1} &= \frac{\Delta t}{V_{\eta,n}} R_{\eta,n}(\phi)^{n+1} + \hat{u}_n^\eta \\
(u_s^\eta)^{n+1} &= \frac{\Delta t}{V_{\eta,s}} R_{\eta,s}(\phi)^{n+1} + \hat{u}_s^\eta \\
(u_f^z)^{n+1} &= \frac{\Delta t}{V_{z,f}} R_{z,f}(\phi)^{n+1} + \hat{u}_f^z \\
(u_b^z)^{n+1} &= \frac{\Delta t}{V_{z,b}} R_{z,b}(\phi)^{n+1} + \hat{u}_b^z
\end{aligned}$$

Substitute these expressions into the discretized continuity equation at time step at n+1, we have the Poisson equation written in operator form

$$\begin{aligned}
& \frac{R_{\xi,e}(\phi)^{n+1}}{V_{\xi,e}} - \frac{R_{\xi,w}(\phi)^{n+1}}{V_{\xi,w}} + \frac{R_{\eta,n}(\phi)^{n+1}}{V_{\eta,n}} - \frac{R_{\eta,s}(\phi)^{n+1}}{V_{\eta,s}} + \frac{R_{z,f}(\phi)^{n+1}}{V_{z,f}} - \frac{R_{z,b}(\phi)^{n+1}}{V_{z,b}} \\
= & -\frac{\hat{u}_e^\xi - \hat{u}_w^\xi + \hat{u}_n^\eta - \hat{u}_s^\eta + \hat{u}_f^z - \hat{u}_b^z}{\Delta t} \qquad (3.185)
\end{aligned}$$

again, all the subscripts e, w, n, s, f, b are associated with the pressure control volume shown in Figure 3.2.

The operators R_ξ, R_η, R_z are given by (3.32), (3.46) and (3.59), respectively. We recapitulate them in the following for subsequent use. The pressure term for u^ξ equation is

$$
\begin{aligned}
R_\xi(P) &= \mathbf{S}^\xi_P \bullet \sum_q -\mathbf{S}^q P && (3.186)\\
&= -\mathbf{S}^\xi \bullet [\mathbf{S}^\xi_e P_e - \mathbf{S}^\xi_w P_w + \mathbf{S}^\eta_n P_n - \mathbf{S}^\eta_s P_s] && (3.187)
\end{aligned}
$$

note all the subscripts e, w, n, s, P are associated with the u^ξ control volume shown in Fig.1.

The pressure term for u^η equation is

$$
\begin{aligned}
R_\eta(P) &= \mathbf{S}^\eta_P \bullet \sum_q -\mathbf{S}^q P && (3.188)\\
&= -\mathbf{S}^\eta \bullet [\mathbf{S}^\xi_e P_e - \mathbf{S}^\xi_w P_w + \mathbf{S}^\eta_n P_n - \mathbf{S}^\eta_s P_s] && (3.189)
\end{aligned}
$$

note all the subscripts e, w, n, s, P are associated with the u^η control volume shown in Figure 3.2.

The pressure term for u^z equation is

$$
\begin{aligned}
R_z(P) &= \mathbf{S}^z_P \bullet \sum_q -\mathbf{S}^q P && (3.190)\\
&= -\mathbf{S}^z \bullet [\mathbf{S}^z_f P_f - \mathbf{S}^z_b P_b] && (3.191)
\end{aligned}
$$

note all the subscripts f, b, P are associated with the u^z control volume shown in Figure 3.2.

Therefore, with reference with the primary pressure control volume shown in Fig.1, we can evaluate all the operators in the Poisson equation as follows,

$$
R_{\xi,e}(\phi)^{n+1} = -\mathbf{S}^\xi_e \bullet [\mathbf{S}^\xi_E \phi_E - \mathbf{S}^\xi_P \phi_P + \mathbf{S}^\eta_{ne}\phi_{ne} - \mathbf{S}^\eta_{se}\phi_{se}] \qquad (3.192)
$$

because

$$
\phi_{ne} = \frac{\phi_N + \phi_P + \phi_E + \phi_{NE}}{4}
$$

$$
\phi_{se} = \frac{\phi_S + \phi_P + \phi_E + \phi_{SE}}{4}
$$

we have

$$
\begin{aligned}
R_{\xi,e}(\phi)^{n+1} &= -\mathbf{S}^\xi_e \bullet [\mathbf{S}^\xi_E \phi_E - \mathbf{S}^\xi_P \phi_P + \mathbf{S}^\eta_{ne}\frac{\phi_N + \phi_P + \phi_E + \phi_{NE}}{4}\\
&\quad -\mathbf{S}^\eta_{se}\frac{\phi_S + \phi_P + \phi_E + \phi_{SE}}{4}] && (3.193)
\end{aligned}
$$

$$
R_{\xi,w}(\phi)^{n+1} = -\mathbf{S}^\xi_w \bullet [\mathbf{S}^\xi_P \phi_P - \mathbf{S}^\xi_W \phi_W + \mathbf{S}^\eta_{nw}\phi_{nw} - \mathbf{S}^\eta_{sw}\phi_{sw}] \qquad (3.194)
$$

because

$$
\phi_{nw} = \frac{\phi_N + \phi_P + \phi_W + \phi_{NW}}{4}
$$

$$\phi_{sw} = \frac{\phi_S + \phi_P + \phi_W + \phi_{SW}}{4}$$

we have

$$\begin{aligned} R_{\xi,w}(\phi)^{n+1} &= -\mathbf{S}^{\xi}_{w} \bullet [\mathbf{S}^{\xi}_{P}\phi_P - \mathbf{S}^{\xi}_{W}\phi_W + \mathbf{S}^{\eta}_{nw}\frac{\phi_N + \phi_P + \phi_W + \phi_{NW}}{4} \\ &\quad -\mathbf{S}^{\eta}_{sw}\frac{\phi_S + \phi_P + \phi_W + \phi_{SW}}{4}] \end{aligned} \tag{3.195}$$

$$R_{\eta,n}(\phi)^{n+1} = -\mathbf{S}^{\eta}_{n} \bullet [\mathbf{S}^{\eta}_{N}\phi_N - \mathbf{S}^{\eta}_{P}\phi_P + \mathbf{S}^{\xi}_{ne}\phi_{ne} - \mathbf{S}^{\xi}_{nw}\phi_{nw}] \tag{3.196}$$

because

$$\phi_{ne} = \frac{\phi_N + \phi_P + \phi_E + \phi_{NE}}{4}$$

$$\phi_{nw} = \frac{\phi_N + \phi_P + \phi_W + \phi_{NW}}{4}$$

we have

$$\begin{aligned} R_{\eta,n}(\phi)^{n+1} &= -\mathbf{S}^{\eta}_{n} \bullet [\mathbf{S}^{\eta}_{N}\phi_N - \mathbf{S}^{\eta}_{P}\phi_P + \mathbf{S}^{\xi}_{ne}\frac{\phi_N + \phi_P + \phi_E + \phi_{NE}}{4} \\ &\quad -\mathbf{S}^{\xi}_{nw}\frac{\phi_N + \phi_P + \phi_W + \phi_{NW}}{4}] \end{aligned} \tag{3.197}$$

$$R_{\eta,s}(\phi)^{n+1} = -\mathbf{S}^{\eta}_{s} \bullet [\mathbf{S}^{\eta}_{P}\phi_P - \mathbf{S}^{\eta}_{S}\phi_S + \mathbf{S}^{\xi}_{se}\phi_{se} - \mathbf{S}^{\xi}_{sw}\phi_{sw}] \tag{3.198}$$

because

$$\phi_{se} = \frac{\phi_S + \phi_P + \phi_E + \phi_{SE}}{4}$$

$$\phi_{sw} = \frac{\phi_S + \phi_P + \phi_W + \phi_{SW}}{4}$$

we have

$$\begin{aligned} R_{\eta,s}(\phi)^{n+1} &= -\mathbf{S}^{\eta}_{s} \bullet [\mathbf{S}^{\eta}_{P}\phi_P - \mathbf{S}^{\eta}_{S}\phi_S + \mathbf{S}^{\xi}_{se}\frac{\phi_S + \phi_P + \phi_E + \phi_{SE}}{4} \\ &\quad -\mathbf{S}^{\xi}_{sw}\frac{\phi_S + \phi_P + \phi_W + \phi_{SW}}{4} \end{aligned} \tag{3.199}$$

$$R_{z,f}(\phi)^{n+1} = -\mathbf{S}^{z}_{f} \bullet [\mathbf{S}^{z}_{F}\phi_F - \mathbf{S}^{z}_{P}\phi_P] \tag{3.200}$$

$$R_{z,b}(\phi)^{n+1} = -\mathbf{S}^{z}_{b} \bullet [\mathbf{S}^{z}_{P}\phi_P - \mathbf{S}^{z}_{B}\phi_B] \tag{3.201}$$

Substitute these operators into the Poisson equation in operator form yields the implementable Poisson equation for pressure point P_{ijk}

$$
\begin{aligned}
& \{\frac{1}{V_{\xi,e}}\mathbf{S}_e^\xi \bullet [\mathbf{S}_E^\xi\phi_E - \mathbf{S}_P^\xi\phi_P + \mathbf{S}_{ne}^\eta\frac{\phi_N+\phi_P+\phi_E+\phi_{NE}}{4} - \mathbf{S}_{se}^\eta\frac{\phi_S+\phi_P+\phi_E+\phi_{SE}}{4}]\} - \\
& \{\frac{1}{V_{\xi,w}}\mathbf{S}_w^\xi \bullet [\mathbf{S}_P^\xi\phi_P - \mathbf{S}_W^\xi\phi_W + \mathbf{S}_{nw}^\eta\frac{\phi_N+\phi_P+\phi_W+\phi_{NW}}{4} - \mathbf{S}_{sw}^\eta\frac{\phi_S+\phi_P+\phi_W+\phi_{SW}}{4}]\} + \\
& \{\frac{1}{V_{\eta,n}}\mathbf{S}_n^\eta \bullet [\mathbf{S}_N^\eta\phi_N - \mathbf{S}_P^\eta\phi_P + \mathbf{S}_{ne}^\xi\frac{\phi_N+\phi_P+\phi_E+\phi_{NE}}{4} - \mathbf{S}_{nw}^\xi\frac{\phi_N+\phi_P+\phi_W+\phi_{NW}}{4}]\} - \\
& \{\frac{1}{V_{\eta,s}}\mathbf{S}_s^\eta \bullet [\mathbf{S}_P^\eta\phi_P - \mathbf{S}_S^\eta\phi_S + \mathbf{S}_{se}^\xi\frac{\phi_S+\phi_P+\phi_E+\phi_{SE}}{4} - \mathbf{S}_{sw}^\xi\frac{\phi_S+\phi_P+\phi_W+\phi_{SW}}{4}]\} + \\
& \{\frac{1}{V_{z,f}}\mathbf{S}_f^z \bullet [\mathbf{S}_F^z\phi_F - \mathbf{S}_P^z\phi_P]\} - \{\frac{1}{V_{z,b}}\mathbf{S}_b^z \bullet [\mathbf{S}_P^z\phi_P - \mathbf{S}_B^z\phi_B]\} \\
= \; & \frac{\hat{u}_e^\xi - \hat{u}_w^\xi + \hat{u}_n^\eta - \hat{u}_s^\eta + \hat{u}_f^z - \hat{u}_b^z}{\Delta t}
\end{aligned} \tag{3.202}
$$

Rearranging the above Poisson equation, we have the final Poisson equation for internal pressure point P_{ijk},

$$
\begin{aligned}
& C_E\phi_E + C_{NE}\phi_{NE} + C_{SE}\phi_{SE} + C_N\phi_N + C_P\phi_P + C_S\phi_S + C_{NW}\phi_{NW} + C_W\phi_W + \\
& C_{SW}\phi_{SW} + C_F\phi_F + C_B\phi_B \\
= \; & \frac{\hat{u}_e^\xi - \hat{u}_w^\xi + \hat{u}_n^\eta - \hat{u}_s^\eta + \hat{u}_f^z - \hat{u}_b^z}{\Delta t}
\end{aligned} \tag{3.203}
$$

And,

$$
\begin{aligned}
C_E &= \frac{1}{V_{\xi,e}}\mathbf{S}_e^{\xi} \bullet [\mathbf{S}_E^{\xi} + \frac{\mathbf{S}_{ne}^{\eta}}{4} - \frac{\mathbf{S}_{se}^{\eta}}{4}] + \frac{1}{V_{\eta,n}}\mathbf{S}_n^{\eta} \bullet [\frac{\mathbf{S}_{ne}^{\xi}}{4}] - \frac{1}{V_{\eta,s}}\mathbf{S}_s^{\eta} \bullet [\frac{\mathbf{S}_{se}^{\xi}}{4}] && (3.204)\\
C_{NE} &= \frac{1}{V_{\xi,e}}\mathbf{S}_e^{\xi} \bullet [\frac{\mathbf{S}_{ne}^{\eta}}{4}] + \frac{1}{V_{\eta,n}}\mathbf{S}_n^{\eta} \bullet [\frac{\mathbf{S}_{ne}^{\xi}}{4}] && (3.205)\\
C_{SE} &= \frac{1}{V_{\xi,e}}\mathbf{S}_e^{\xi} \bullet [-\frac{\mathbf{S}_{se}^{\eta}}{4}] - \frac{1}{V_{\eta,s}}\mathbf{S}_s^{\eta} \bullet [\frac{\mathbf{S}_{se}^{\xi}}{4}] && (3.206)\\
C_N &= \frac{1}{V_{\xi,e}}\mathbf{S}_e^{\xi} \bullet [\frac{\mathbf{S}_{ne}^{\eta}}{4}] - \frac{1}{V_{\xi,w}}\mathbf{S}_w^{\xi} \bullet [\frac{\mathbf{S}_{nw}^{\eta}}{4}] + \frac{1}{V_{\eta,n}}\mathbf{S}_n^{\eta} \bullet [\mathbf{S}_N^{\eta} \\
&\quad + \frac{\mathbf{S}_{ne}^{\xi}}{4} - \frac{\mathbf{S}_{nw}^{\xi}}{4}] && (3.207)\\
C_P &= \frac{1}{V_{\xi,e}}\mathbf{S}_e^{\xi} \bullet [-\mathbf{S}_P^{\xi} + \frac{\mathbf{S}_{ne}^{\eta}}{4} - \frac{\mathbf{S}_{se}^{\eta}}{4}] - \frac{1}{V_{\xi,w}}\mathbf{S}_w^{\xi} \bullet [\mathbf{S}_P^{\xi} + \frac{\mathbf{S}_{nw}^{\eta}}{4} - \frac{\mathbf{S}_{sw}^{\eta}}{4}] \\
&\quad + \frac{1}{V_{\eta,n}}\mathbf{S}_n^{\eta} \bullet [-\mathbf{S}_P^{\eta} + \frac{\mathbf{S}_{ne}^{\xi}}{4} - \frac{\mathbf{S}_{nw}^{\xi}}{4}] - \frac{1}{V_{\eta,s}}\mathbf{S}_s^{\eta} \bullet [\mathbf{S}_P^{\eta} + \frac{\mathbf{S}_{se}^{\xi}}{4} - \frac{\mathbf{S}_{sw}^{\xi}}{4}] \\
&\quad + \frac{1}{V_{z,f}}\mathbf{S}_f^{z} \bullet [-\mathbf{S}_P^{z}] - \frac{1}{V_{z,b}}\mathbf{S}_b^{z} \bullet [\mathbf{S}_P^{z}] && (3.208)\\
C_S &= \frac{1}{V_{\xi,e}}\mathbf{S}_e^{\xi} \bullet [-\frac{\mathbf{S}_{se}^{\eta}}{4}] - \frac{1}{V_{\xi,w}}\mathbf{S}_w^{\xi} \bullet [-\frac{\mathbf{S}_{sw}^{\eta}}{4}] - \frac{1}{V_{\eta,s}}\mathbf{S}_s^{\eta} \bullet [-\mathbf{S}_S^{\eta} \\
&\quad + \frac{\mathbf{S}_{se}^{\xi}}{4} - \frac{\mathbf{S}_{sw}^{\xi}}{4}] && (3.209)\\
C_{NW} &= -\frac{1}{V_{\xi,w}}\mathbf{S}_w^{\xi} \bullet [\frac{\mathbf{S}_{nw}^{\eta}}{4}] + \frac{1}{V_{\eta,n}}\mathbf{S}_n^{\eta} \bullet [-\frac{\mathbf{S}_{nw}^{\xi}}{4}] && (3.210)\\
C_W &= -\frac{1}{V_{\xi,w}}\mathbf{S}_w^{\xi} \bullet [-\mathbf{S}_W^{\xi} + \frac{\mathbf{S}_{nw}^{\eta}}{4} - \frac{\mathbf{S}_{sw}^{\eta}}{4}] + \frac{1}{V_{\eta,n}}\mathbf{S}_n^{\eta} \bullet [-\frac{\mathbf{S}_{nw}^{\xi}}{4}] \\
&\quad - \frac{1}{V_{\eta,s}}\mathbf{S}_s^{\eta} \bullet [-\frac{\mathbf{S}_{sw}^{\xi}}{4}] && (3.211)\\
C_{SW} &= -\frac{1}{V_{\xi,w}}\mathbf{S}_w^{\xi} \bullet [-\frac{\mathbf{S}_{sw}^{\eta}}{4}] - \frac{1}{V_{\eta,s}}\mathbf{S}_s^{\eta} \bullet [-\frac{\mathbf{S}_{sw}^{\xi}}{4}] && (3.212)\\
C_F &= \frac{1}{V_{z,f}}\mathbf{S}_f^{z} \bullet [\mathbf{S}_F^{z}] && (3.213)\\
C_B &= -\frac{1}{V_{z,b}}\mathbf{S}_b^{z} \bullet [-\mathbf{S}_B^{z}] && (3.214)
\end{aligned}
$$

3.16 Summary of Numerical Scheme

Now after every aspect of the numerical scheme has been addressed, it is appropriate to summarize these derivations by putting them into the context of general fractional step method.

In short, the fractional method is comprised of three sequential operations: computation of a provisional velocity field using the non-linear and viscous terms; calculation of the pressure field by solving the Poisson equation, and finally projection of the intermediate velocity field onto a divergence-free space at the new time step using the pressure gradient. Written in a compact form, the series of operations are

$$\begin{aligned} A\hat{\mathbf{u}} &= \mathbf{rhs} + \mathbf{mbc} && (3.215)\\ \Delta t D G \phi^{n+1} &= D\hat{\mathbf{u}} - \mathbf{cbc} && (3.216)\\ \mathbf{u}^{n+1} &= \hat{\mathbf{u}} - \Delta t G \phi^{n+1} && (3.217) \end{aligned}$$

where submatrix G is the discrete gradient operator and D is the discrete divergence operator. The exact expressions for coefficient submatrix A and right-hand-side submatrix $\mathbf{r}$ are dependent on the specific temporal and spatial discretizations. The unknown discrete velocity and pressure vectors are denoted $\mathbf{u}^{n+1}$ and p^{n+1}. Boundary condition vectors $\mathbf{mbc}$ and $\mathbf{cbc}$ are such that when combined with operators G and D the original system (2.7)(2.8) with appropriate boundary conditions can be fully recovered.

Consider a coordinate transformation from Cartesian system $\mathbf{r}(x, y, z)$ to the curvilinear non-orthogonal system $\mathbf{r}(\xi, \eta, z)$, where $\mathbf{r}$ is the radius vector and the spanwise coordinate z is not transformed. The natural local base of the transformed curvilinear coordinate system (ξ, η, z) is denoted $\mathbf{e}_q = \partial \mathbf{r}/\partial q,\ q = \xi, \eta, z$. The transformation between the two systems is achieved by defining a new base in terms of the vector product of the natural local base $\mathbf{e}_q$, $\mathbf{S}^q = \mathbf{e}_{q+1} \times \mathbf{e}_{q+2}$ (q, cyclic permutation). Note that the vector $\mathbf{S}^q$ has a magnitude equal to the area of the parallelogram spanned by natural local basis $\mathbf{e}_{q+1}, \mathbf{e}_{q+2}$ and is perpendicular to the plane of $\mathbf{e}_{q+1}, \mathbf{e}_{q+2}$. Using the orthogonality between the new base $\mathbf{S}^q$ and its reciprocal base $\mathbf{S}_q$, the series operations in generalized curvilinear system with volume flux as dependent variables be expressed as follows,

$$\begin{aligned} A\hat{\mathbf{u}}^{\mathbf{q}} &= (\mathbf{S}^\xi, \mathbf{S}^\eta, \mathbf{S}^z)^T \bullet \mathbf{rhs} + (\mathbf{S}^\xi, \mathbf{S}^\eta, \mathbf{S}^z)^T \bullet \mathbf{mbc} && (3.218)\\ (\mathbf{S}^\xi, \mathbf{S}^\eta, \mathbf{S}^z)^T \bullet \Delta t D G \phi^{n+1} &= D\hat{\mathbf{u}}^{\mathbf{q}} - (\mathbf{S}^\xi, \mathbf{S}^\eta, \mathbf{S}^z)^T \bullet \mathbf{cbc} && (3.219)\\ \mathbf{u}^{\mathbf{q}n+1} &= \hat{\mathbf{u}}^{\mathbf{q}} - (\mathbf{S}^\xi, \mathbf{S}^\eta, \mathbf{S}^z)^T \bullet \Delta t G \phi^{n+1} && (3.220) \end{aligned}$$

where superscript T denotes the transpose. The vector $\mathbf{u}^{\mathbf{q}}$ is the contravariant component (volume flux in physical space) of $\mathbf{u}$ in the coordinate system defined by $\mathbf{S}^q$ and $\mathbf{S}_q$.

3.17 Optimization and Discussions of Poisson Solver

The fully discretized Poisson equation can be written as

$$
\begin{aligned}
& \left\{ \frac{1}{V_e^{U^\xi}} \mathbf{S}_e^\xi \bullet \left[\mathbf{S}_E^\xi \phi_E - \mathbf{S}_P^\xi \phi_P + \mathbf{S}_{ne}^\eta \phi_{ne} - \mathbf{S}_{se}^\eta \phi_{se} \right] \right\} - \\
& \left\{ \frac{1}{V_w^{U^\xi}} \mathbf{S}_w^\xi \bullet \left[\mathbf{S}_P^\xi \phi_P - \mathbf{S}_W^\xi \phi_W + \mathbf{S}_{nw}^\eta \phi_{nw} - \mathbf{S}_{sw}^\eta \phi_{sw} \right] \right\} + \\
& \left\{ \frac{1}{V_n^{U^\eta}} \mathbf{S}_n^\eta \bullet \left[\mathbf{S}_N^\eta \phi_N - \mathbf{S}_P^\eta \phi_P + \mathbf{S}_{ne}^\xi \phi_{ne} - \mathbf{S}_{nw}^\xi \phi_{nw} \right] \right\} - \\
& \left\{ \frac{1}{V_s^{U^\eta}} \mathbf{S}_s^\eta \bullet \left[\mathbf{S}_P^\eta \phi_P - \mathbf{S}_S^\eta \phi_S + \mathbf{S}_{se}^\xi \phi_{se} - \mathbf{S}_{sw}^\xi \phi_{sw} \right] \right\} + \\
& \left\{ \frac{1}{V_t^{U^z}} \mathbf{S}_t^z \bullet \left[\mathbf{S}_T^z \phi_T - \mathbf{S}_P^z \phi_P \right] \right\} - \left\{ \frac{1}{V_b^{U^z}} \mathbf{S}_b^z \bullet \left[\mathbf{S}_P^z \phi_P - \mathbf{S}_B^z \phi_B \right] \right\} \\
= \; & \frac{\hat{u}_e^\xi - \hat{u}_w^\xi + \hat{u}_n^\eta - \hat{u}_s^\eta + \hat{u}_t^z - \hat{u}_b^z}{\Delta t} \equiv Q
\end{aligned}
\tag{3.221}
$$

(see Figure 3.2 for arrangement of indices.). Numerical solution of the Poisson equation can be more easily facilitated by rewriting the above equation as

$$
C_{\xi,\eta}(\phi) - V^{U^z} \frac{\partial^2 \phi}{\partial z^2} = Q \tag{3.222}
$$

where $C_{\xi,\eta}(\phi)$ contains all the terms in the (ξ, η) directions, the z coordinate denotes the third spatial dimension aligned with the uniform Cartesian coordinate, and $V^{U^z} = V_t^{U^z} = V_b^{U^z}$. Fast transform techniques may now be used to decouple solution of the Poisson equation along the z direction [4]. In particular, for a uniform grid distribution along the z direction with grid spacing Δz and grid points N_z, Fourier transformation of (3.222) yields

$$
C_{\xi,\eta} \left[\widehat{\phi}(\xi, \eta, k_m^{'}) \right] - V^{U^z} k_m^{'} \widehat{\phi}(\xi, \eta, k_m^{'}) = \widehat{Q}(\xi, \eta, k_m^{'}) \tag{3.223}
$$

where $k_m^{'} = 2[1 - \cos(\pi m / N_z)]/\Delta z^2$, $(m = \pm 0, \pm 1, \cdots, \pm N_z - 1)$ is the modified wavenumber for central differences. For each wavenumber (3.223) is solved using standard iterative techniques on two dimensional ξ-η planes (SLOR in the present work). Significant reductions in computational time can be achieved using the semi-direct approach, especially for problems with a large number of grid points in the spanwise direction. In performing the series expansion used in the Poisson equation, , it is also important to make the series compatible with the boundary conditions in the z direction. For example, a periodic trigonometric expansion (Fourier) should be used for problems having periodic boundary conditions in the Cartesian coordinate direction; likewise, a cosine expansion is suitable for Neumann boundary conditions and a sine expansion for Dirichlet boundary conditions.

In this work a semi-direct Poisson solver was developed based on the procedure outlined above. The algorithm includes all terms in the discretized Poisson equation, including the cross derivative terms arising from transformation of the governing equations to curvilinear systems. The relaxation parameter Ω used in the two dimensional SLOR method was optimized to minimize the CPU consumption. It was found that if a constant Ω is used for different wavenumber, the CPU consumption by $k_0^{'}$ mode is much larger than those by other wavenumbers. It was also found that an optimal combination can be achieved by using a larger $\Omega(k_0) \approx 1.6$ and smaller $\Omega(k_m, m \neq 0) \approx 1.1$. A set of typical optimization results are presented in Figure 12. Further discussions about the role of cross-derivative terms in the Poisson equation can be found in Wu *et al* [12].

CHAPTER 4
NUMERICAL RESULTS

In this section numerical results are presented from applications of the numerical method described in §3 to three problems, viz., unsteady Taylor-Green decaying vortices, polar lid-driven cavity flow, and skewed lid-driven cavity flow. Temporal accuracy of the method is investigated by computing the transient decaying vortex flow using orthogonal as well as non-orthogonal grids. Effects of curvature and grid non-orthogonality are further examined using the polar and skewed lid-driven cavity flows.

4.1 Taylor-Green decaying vortices

The Taylor-Green solution to the incompressible Navier-Stokes equations provides an excellent check of the implementation of the numerical method. The solution is two dimensional and represents an infinite array of vortices. For an arbitrary x_1-x_2 plane the solution is

$$\begin{aligned} u_1(x_1,x_2,t) &= -\cos x_1 \sin x_2 e^{-2t} \\ u_2(x_1,x_2,t) &= \sin x_1 \cos x_2 e^{-2t} \\ p(x_1,x_2,t) &= -\frac{1}{4}\left(\cos 2x_1 + \cos 2x_2\right) e^{-4t}\,. \end{aligned} \tag{4.1}$$

The Taylor-Green flow has been extensively used to test numerical methods for incompressible flows ([4], [6]) since it permits application of a numerical scheme to a transient laminar flow which encompasses all possible boundary conditions (Dirichlet, Neumann, and periodic). The interest in this work is testing the numerical method using both skewed and orthogonal grids. The non-orthogonal curvilinear fractional stop method outlined in §3 was thoroughly tested against the solution given by (4.1) in a domain of volume $2\pi \times 2\pi \times 2\pi$. Note that since the Taylor-Green flow is two-dimensional, the length of the third dimension is arbitrary. The Taylor-Green flow was calculated in each of the three possible orientations, i.e., using the x-y, y-z, and x-z planes as the primary x_1-x_2 plane in (4.1), in order to test the three dimensional properties of the method. In the x and y directions both Dirichlet and Neumann boundary conditions were applied while periodic boundary conditions were used in the z direction.

Each test was performed using both Cartesian and non-orthogonal grids. Skewed grids were obtained by applying a uniform grid spacing along three sides of a square in the primary x_1-x_2 plane. Along the fourth side of the square hyperbolic stretching was used to distort the grid, resulting in a maximum skewing angle of approximately 5^o. These tests were used not only to investigate the accuracy of the method on uniform and non-uniform grids but also to examine the effect on the velocity field of solving a simplified Poisson equation. For all calculations the time step was maintained at 10^{-3} and computations were performed for up to 120 time steps using a mesh with a resolution of $81 \times 81 \times 4$ in the (x_1, x_2, x_3) directions, respectively. Shown in Figure 3 is the time history of the maximum relative and absolute error in velocity. It is evident from the Figure that the error in the numerical solution is $\mathcal{O}(\Delta t)$ and that the error is smaller for calculations in which periodic boundary conditions were employed (in the z direction). Calculations in the x-z plane were also approximately 20 times faster than the calculations in the other planes. This decrease in cpu time for calculations in the x-z plane is directly related to the use of Fourier transformation along the periodic direction to simplify solution of the Poisson equation.

As also shown in Figure 3, the relative error is larger for computations performed on non-orthogonal grids. The increase in the relative error can be attributed to a number of sources, e.g., increased truncation errors from the cross-derivative terms which are non-zero on non-orthogonal grids as well as time discretization errors associated with explicit treatment of the cross-derivatives. For a second-order accurate spatial discretization as used in this study the truncation errors can be reduced by increasing the resolution; time discretization errors can be reduced by either refining the time step or through the use of more accurate time-advance schemes (e.g., higher order Runge-Kutta methods). In any case, such improvements are straightforward to incorporate into the fractional step method. Comparison between the results in Figure 3 for cases on the non-orthogonal grids confirm the analysis in Wu *et al* [12] that the simplified Poisson equation (i.e., neglecting cross-derivative terms) will not lead to the same velocity field as obtained from the full Poisson equation. However, it is also important to note that the larger relative error arising from use of the simplified Poisson equation does not substantially degrade the overall accuracy of the velocity field for relatively small skewing angles.

4.2 Lid-driven flow in a polar cavity

To test the curvilinear character of the numerical scheme, computations of steady flow in a polar lid-driven cavity were performed. The two-dimensional configuration illustrated schematically in Figure 4 is the same as the cross-section studied experimentally by Fuchs & Tillmark [13]. The length of the cavity in the radial direction equals the radius of the inner wall. No slip boundary conditions are applied on all the cavity walls except at the inner wall where a negative unit tangential velocity is specified. Periodic boundary conditions were applied in the spanwise (z) direction. Hyperbolic stretching was used to cluster the $87 \times 87 \times 4$ orthogonal polar mesh to the four bounding surfaces. For all the steady-state computations presented in this paper, time history of velocity was used to determine when steady state conditions had been attained.

Shown in Figure 5 is the time variation of velocity for the polar cavity flow; the time step in this computation was fixed at 0.003. The time history shown in Figure 5 is for the velocity in the polar lid-driven cavity flow at the grid point immediately adjacent to the outer corner of $\theta = 1/2$ where the secondary vortex occurs. Fuchs & Tillmark [13] measured cavity velocities using laser Doppler velocimetry and performed flow visualization at two Reynolds numbers, $Re = 60$ and 350 (based on the radius and the surface velocity of the rotating inner cylinder). It was reported in Ref. [13] that at $Re = 350$ the maximum measurement errors are roughly 5% and some three-dimensional effects are present in the apparatus. As shown in Figure 4, the computed velocity vectors at $Re = 350$ qualitatively agree with the flow visualization reported in Ref. [13]. Comparison between computation and experimental measurements of tangential and radial velocities along the radial traverse at $\theta = 0$ is shown in Figure 6. Good agreement between simulation results and the experimental data is evident.

4.3 Lid-driven flow in a skewed cavity

Steady, laminar lid-driven flow in a skewed cavity is an excellent test of the "non-orthogonal" properties of the method, e.g., testing of cross-derivative terms. Skewed cavity flow examined in previous studies ([9], [14]) also provides a suitable basis for comparison of simulation results in the present work to those obtained by other investigators. Computations of the skewed cavity flow shown in Figure 7a were performed at a Reynolds number of 100 (based on the sliding velocity and the length of the top lid). Two skewing angles were used in the calculations, $\beta = \pi/4, \pi/6$ (see Figure 7a). Note that the height of the cavity, h, is specified in terms of the length of the driving lid, a, i.e., $h = a \tan \beta$. Thus, one of the main diagonals of the cavity is perpendicular to the horizontal lid. A hyperbolically stretched mesh ($61 \times 61 \times 4$) was used in the simulations and the time step maintained at 0.005.

Comparison of the flow pattern predicted in this study to that obtained in Ref. [9] is shown in Figure 8 for $\beta = \pi/4$. Good agreement between the simulation results in this study to those obtained in Ref. [9] is apparent. Velocity profiles along vertical and horizontal traverses from the present computations are compared to those obtained in Ref. [9] in Figures 9a and 9b. It may be observed from the Figures that the quantitative agreement between the present simulations and Ref. [9] is good. Also shown in Figure 9 are the results obtained using the simplified Poisson equation. Analogous to the results obtained for the Taylor-Green flow, the velocity field obtained using the simplified Poisson equation differs from that obtained from the full Poisson equation. It can be seen from Figure 9 that the greatest differences between the simplified and full Poisson equation occurs at $x/a = 5/4$ in the horizontal velocity u_x

and at $y/h = 3/5$ for the vertical velocity u_y. Shown in Figures 10 and 11 are the flow pattern and velocity profiles from the skewed cavities with $\beta = \pi/6$. As was observed in Figures 8 and 9, good agreement is obtained between the present simulations and the results in Ref. [9] for the higher skewing angle.

CHAPTER 5
CONCLUSION

The extension of the fractional step method to non-orthogonal curvilinear coordinate systems has been performed using matrix block-LU decomposition. The sequential fractional step operations for general curvilinear systems are defined by expanding the Cartesian velocity with respect to a new area vector base in which the new dependent variables are volume fluxes in physical space. The advantage of this formulation is that the issue of boundary conditions for intermediate solution variables is clearly defined. In particular, since boundary conditions are incorporated into the discretized equations before the LU decomposition, there is in fact no need to specify boundary conditions for the provisional velocity field in curvilinear coordinate systems.

In the method presented in this study, the third spatial dimension is treated using Cartesian coordinates on a uniform grid. A semi-direct scheme is developed for solution of the Poisson equation by using Fourier transformation along the uniform coordinate direction. The proper choice of the series eigenfunctions in the semi-direct solution of Poisson equation is also specified in accordance with the different types of boundary conditions employed in the Cartesian coordinate direction.

The accuracy and robustness of the numerical scheme developed in the present work was established through comparison of simulation results to analytical, numerical, and experimental measurements of the Taylor-Green decaying vortex flow, polar lid-driven cavity flow, and skewed lid-driven cavity flow. In all cases good agreement was obtained between the present numerical scheme and existing solutions.

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