Matching with Externalities: The Role of Prudence and Social Connectedness in Stability

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- Firms that consider the organizational structure of an industry before deciding to integrate;
- Markets where household formation can impact prices through its effects on aggregate demand;
- Students internalizing peer effects on preferences to choose among *ex-ante* identical courses.

A simple way to include externalities in matching problems is to assume that individual preferences are defined over the set of admissible matchings instead of the set of potential partners.

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These beliefs can be heterogeneous and may depend on the matching from which deviation occurs.

For instance, an extremely prudent agent considers all the possible reactions of other individuals before implementing a deviation, while a myopic agent disregards them.

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Two relevant pieces of information are incorporated in the beliefs of any individual:

- The share of potential reactions that she considers when evaluating a deviation, which is a measure of her prudence.
- The number of potential reactions that other individuals may implement, which is a measure of their social connectedness.

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Intuitively, a matching problem should be more likely to have a stable solution when these attributes are high.

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- Asymptotic stability—a property that never holds in the absence of externalities—is achieved even in situations where social connectedness is low and prudence vanishes.
- Those who ignore the reactions of other individuals have a destabilizing effect on society: the probability of a matching being stable converges to zero as the number of *myopic agents* increases, regardless of the percentage of the population they represent.

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- Those who ignore the reactions of other individuals have a destabilizing effect on society: the probability of a matching being stable converges to zero as the number of *myopic agents* increases, regardless of the percentage of the population they represent.

When preferences are correlated, some types of matchings might be always unstable, independently of the size of the population.

However, when the correlation of preferences is not too high, the probability of solvability still converges to one as the population grows.

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Each agent i has a strict, complete, and transitive preference relation $\succ^i$ defined over the set of *admissible matchings*

$$\mathcal{M} = \{\mu : I(n) \to I(n) : \ (j, \mu(j)) \in \mathcal{A} \ \wedge \ \mu(\mu(j)) = j, \ \forall j \in I(n)\},$$

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Let $\mathcal{M}(i,j) = \{\eta \in \mathcal{M} : \eta(i) = j\}$. If an agent i wants to deviate from μ to pair with j, then she believes that any matching in a set $\Theta^i(\mu, j) \subseteq \mathcal{M}(i,j)$ could be attained in the short-term.

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A matching μ is blocked by $(i,j) \in \mathcal{A}$ when $\mu \notin \mathcal{M}(i,j)$,

$$\eta \succ^i \mu, \quad \forall \eta \in \Theta^i(\mu, j), \qquad\qquad \rho \succ^j \mu, \quad \forall \rho \in \Theta^j(\mu, i).$$

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A matching $\mu \in \mathcal{M}$ is stable when it cannot be blocked by any pair $(i,j) \in \mathcal{A}$.

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where each X^i_μ has a continuous and increasing distribution F_i with convex support $K_i \subseteq \mathbb{R}$.

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The probability of $\mu \in \mathcal{M}$ being stable is given by

$$\int_K \prod_{\substack{i\in I(n)\\ \mu(i)\neq i}} \left(1 - F_i(x_i)^{\alpha^i(\mu,i)}\right) \times \prod_{\substack{(i,j)\in\mathcal{A}\\ i<j,\ \mu(i)\neq j}} \left(1 - F_i(x_i)^{\alpha^i(\mu,j)} F_j(x_j)^{\alpha^j(\mu,i)}\right) dF(x),$$

where $\alpha^i(\mu,j) = \#\Theta^i(\mu,j)$, $K = K_1 \times \cdots \times K_n$, and $F(x) = (F_1(x_1),\dots,F_n(x_n))$.

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The prudence of agent i at the moment of evaluating a deviation from μ to pair with j can be quantified by the proportion of matchings in $\mathcal{M}(i,j)$ that she considers as potential reactions of other agents:

$$\psi^i(\mu,j) = \#\Theta^i(\mu,j)/\#\mathcal{M}(i,j).$$

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The social connectedness of individuals in $I(n) \setminus \{i,j\}$ in the face of a deviation of (i,j) can be measured by the number of scenarios that could occur after their reactions:

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Since $\alpha^i(\mu,j) = \psi^i(\mu,j)\phi(i,j)$, it follows that prudence and social connectedness play a relevant role in stability when externalities are arbitrary.

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Consider the following global measures of prudence and social connectedness:

$$\psi = \min_{(i,j)\in\mathcal{A}} \; \min_{\mu\in\mathcal{M}\setminus\mathcal{M}(i,j)} \psi^i(\mu, j), \qquad \phi = \min_{(i,j)\in\mathcal{A}} \phi(i,j).$$

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Theorem

In a matching problem with externalities $(I(n),\mathcal{A},(\succ^i,\Theta^i)_{i\in I(n)})$ both

$$\min_{\mu\in\mathcal{M}} P\left[\mu\in\mathcal{S}\right] \qquad \textit{and} \qquad P\left[\#\mathcal{S}\geq 1\right]$$

are bounded from below by $\left(1-1/n^4\right)^{n^2}\left(1/n^4\right)^{\frac{n}{\psi\phi}}$.

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$$\lim_{n\to+\infty} \min_{\mu\in\mathcal{M}_n} P[\mu \in \mathcal{S}_n] = 1. \qquad \textbf{[AS]}$$

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Corollary 1

Suppose that, for some $\delta \in (0, 1)$, in any problem $(I(n), \mathcal{A}_n, (\succ_n^i, \Theta_n^i)_{i \in I(n)})$ each agent has at least $[\delta n]$ feasible partners.

Then, the asymptotic property $[\mathrm{AS}]$ holds whenever $\{\psi_n\}_{n \in \mathbb{N}}$ is bounded away from zero or decays at most exponentially.

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Corollary 2

Suppose that, for some $\delta \in (0, 1]$ and $r > 0$, any $(I(n), \mathcal{A}_n, (\succ_n^i, \Theta_n^i)_{i \in I(n)})$ is a bilateral matching problem with m_n firms where the following conditions hold:

- *The firms/workers ratio satisfies $r^{-1} \leq m_n/(n - m_n) \leq r$.*
- *Each firm has at least $[\delta(n - m_n)]$ feasible partners.*
- *Each worker has at least $[\delta m_n]$ feasible partners.*

Then, [AS] holds whenever $\{\psi_n\}_{n \in \mathbb{N}}$ is bounded away from zero or decays at most exponentially.

Asymptotic solvability

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Corollary 3

For any sequence $\{(I(n), \mathcal{A}_n, (\succ_n^i, \Theta_n^i)_{i \in I(n)})\}_{n \in \mathbb{N}}$ *of roommate problems with externalities, the probability of solvability satisfies the following asymptotic property*

$$\lim_{n \to +\infty} P[\#\mathcal{S}_n \geq 1] = 1 \qquad \textbf{[SO]}$$

whenever $\{\psi_n \phi_n\}_{n \in \mathbb{N}}$ *grows at least exponentially.*

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$$\lim_{n\to+\infty} P[\#\mathcal{S}_n \geq 1] = 1 \qquad \textbf{[SO]}$$

whenever $\{\psi_n\phi_n\}_{n\in\mathbb{N}}$ grows at least exponentially.

Moreover, **[SO]** holds even in scenarios where individuals' prudence vanishes.

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Analogous results hold in the presence of externalities as long as the number of myopic agents increases without bound, regardless of the share of the population they represent.

Proposition

Let $\{(I(n), \mathcal{A}_n, (\succ^i_n, \Theta^i_n)_{i\in I(n)})\}_{n\in\mathbb{N}}$ be a sequence of matching problems with externalities and random preferences.

If each $(I(n), \mathcal{A}_n, (\succ^i_n, \Theta^i_n))$ is a roommate problem with m_n myopic agents and $\{m_n\}_{n\in\mathbb{N}}$ increases without bound, then

$$\lim_{n\to+\infty} \max_{\mu\in\mathcal{M}_n} P[\mu \in \mathcal{S}_n] = 0. \qquad \textbf{[AI]}$$

Moreover, **[AI]** *holds when each $(I(n), \mathcal{A}_n, (\succ^i_n, \Theta^i_n)_{i\in I(n)})$ is a marriage market with f_n myopic firms and w_n myopic workers, $f_n + w_n \leq n$, such that $\lim_{n\to+\infty} \min\{f_n, w_n\} = +\infty$.*

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Example. Consider a roommate problem with an even number of agents such that $\mu \succ^i \eta$ if and only if $Y^i_\mu < Y^i_\eta$, where $\{Y^i_\mu\}_{i\in I(n),\mu\in\mathcal{M}}$ are independent random variables.

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Assume that each Y^i_μ is uniformly distributed on

$$K^i_\mu = \begin{cases} [1,2]\,, & \text{when } \mu(i) = 1,\ i \neq 1, \\ \vdots & \vdots \\ [m, m+1]\,, & \text{when } \mu(i) = m,\ i \neq m, \\ [m+1, m+2]\,, & \text{when } \mu(i) > m,\ \mu(i) \neq i, \\ [m+2, m+3]\,, & \text{when } \mu(i) = i, \end{cases}$$

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where $m \leq (n-2)$ is an even number.

It follows that, if μ is stable, then $\mu(2i) = 2i - 1$ for all $i \in \{1, \ldots, m/2\}$ and no agent is alone.

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Considering the following measures of prudence and social connectedness:

$$\psi^* = \min_{(i,j)\in\mathcal{A}^*} \ \min_{\mu\in\mathcal{M}^*\setminus\mathcal{M}^*(i,j)} \frac{\#\Theta^i(\mu,j)}{\#\mathcal{M}^*(i,j)}, \qquad \phi^* = \min_{(i,j)\in\mathcal{A}^*} \#\mathcal{M}^*(i,j),$$

it follows that

$$P[\#\mathcal{S} \geq 1] \geq \left(1 - \frac{1}{n^4}\right)^{(n-m)^2} \left(\frac{1}{n^4}\right)^{\frac{n-m}{2\psi^*\phi^*}}.$$

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Since $\phi^* = (n-m-3)!!$ and $\psi^* \in (0,1]$, if the correlation of preferences is very high, then $\psi^*\phi^*$ remains bounded as n increases.

Correlated preferences: an example

Let $\mathcal{M}^*$ be the set of matching satisfying these properties.
If $\mu \in \mathcal{M}^*$ is unstable, its blocking pairs belongs to $\mathcal{A}^* = (I(n) \setminus I(m)) \times (I(n) \setminus I(m))$.

Considering the following measures of prudence and social connectedness:

$$\psi^* = \min_{(i,j)\in\mathcal{A}^*} \min_{\mu\in\mathcal{M}^*\setminus\mathcal{M}^*(i,j)} \frac{\#\Theta^i(\mu,j)}{\#\mathcal{M}^*(i,j)}, \qquad \phi^* = \min_{(i,j)\in\mathcal{A}^*} \#\mathcal{M}^*(i,j),$$

it follows that

$$P[\#\mathcal{S} \geq 1] \geq \left(1 - \frac{1}{n^4}\right)^{(n-m)^2} \left(\frac{1}{n^4}\right)^{\frac{n-m}{2\psi^*\phi^*}}.$$

Since $\phi^* = (n-m-3)!!$ and $\psi^* \in (0,1]$, if the correlation of preferences is very high, then $\psi^*\phi^*$ remains bounded as n increases.

However, if m/n remains bounded from above for some $\kappa \in (0,1)$ as n increases, then **[SO]** holds whenever ψ^* remains bounded away from zero or decays at most exponentially.

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Indeed, the integral formula for the probability of stability was tractable as a consequence of two assumptions: the independence of random variables determining preferences, $\{X^i_\mu\}_{i\in I(n),\mu\in\mathcal{M}}$, and the common distribution of $\{X^i_\mu\}_{\mu\in\mathcal{M}}$ for each agent i.

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These hypotheses implied that externalities arbitrary.

Therefore, any attempt to extend the results of this paper to more general contexts, in which the correlation of preferences is allowed, should consider the difficulties to quantify the probability of a matching being stable.

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