Fuzzy Sets and Fuzzy Logic Systems

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Genesis of Fuzzy Logic (FL)

- According to Aristotle every proposition must be either true or false
- Plato later proposed that a third region between true and false exists
- Lukasiewicz in the beginning of the XXth century presented a systematic alternative to the bi-valud logic of Aristotle
- On top of Lukasiewicz's possibility theory Zadeh introduced the concept of *fuzzy logic* as multi-valued logic mainly to capture imprecise, ambiguous and uncertain nature of the world
- Zadeh also proposed the idea of applying natural language terms in the realm of fuzzy logic

Introduction to Fuzzy Logic

- Fuzzy logic as a branch of fuzzy set theory – representation and inference from knowledge
- Imprecise and uncertain knowledge – based on the idea that physical reality is graded
- Fuzzy logic can also be synonymously used with fuzzy set theory
- Fuzzy logic is not logic that is fuzzy, but logic that is used to describe fuzziness. Fuzzy logic is the theory of fuzzy sets, sets that calibrate vagueness.

Motivation

- The importance of common sense and intuition when solving problems by experts
- Representing experts' knowledge that relies on vague ambiguous formulations is a challenging task
- Such imprecise knowledge however proves very effective in many applications where the complexity of problems prevents from quantitative understanding
- Fuzzy logic systems resemble human reasoning in how it exploits approximation and how it accounts for uncertainty (mathematical apparatus to do that.)

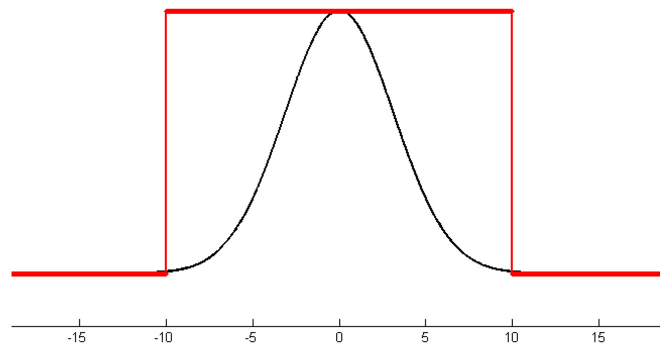
Examples of a wide range of applications

- automatic train control;
- tunnel digging machinery;
- washing machines;
- rice cookers;
- vacuum cleaners;
- air conditioners, etc

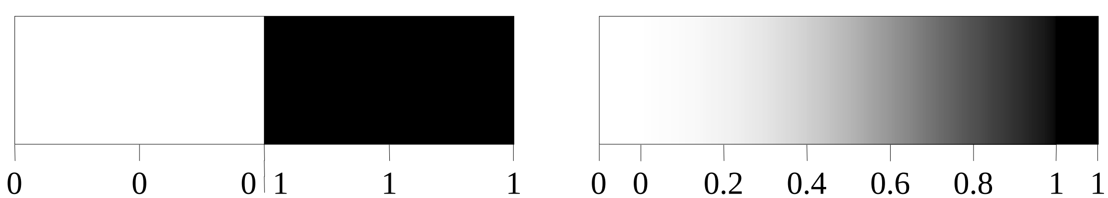
Fuzzy vs. crisp (conventional) set

$$A = \left\{ x, \mu_A(x) \mid x \in X \right\}$$

$$B = \left\{ x \mid x \in \left(-10, 10 \right] \right\}$$



Set A is fuzzy whereas set B has crisp characteristics.

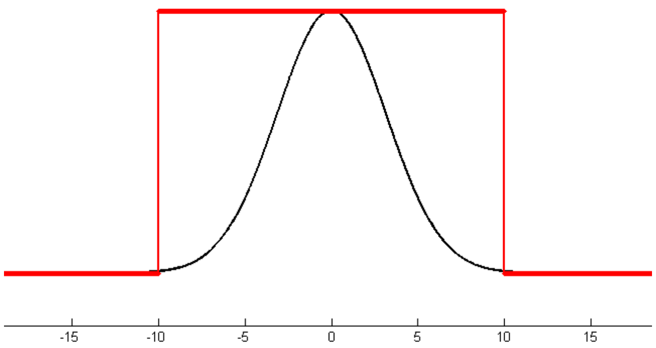


(*a*) Boolean Logic. (*b*) Multi-valued Logic.

Fuzzy vs. crisp (conventional) set

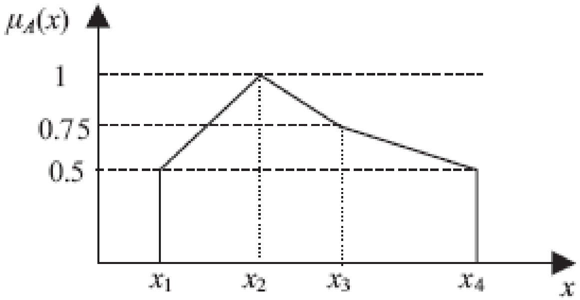
$$A = \{x, \mu_A(x) \mid x \in X\}$$

$$B = \{x \mid x \in (-10,10]\}$$



Set A is fuzzy whereas set B has crisp characteristics.

Both continuous and discrete quantities can be described using fuzzy sets



Fuzzy representation – membership function

- In the fuzzy theory, fuzzy set A of universe X is defined by function $\mu_A(x)$ called the *membership function* of set A

$\mu_A(x): X \rightarrow [0, 1]$, where

$$\mu_A(x) = 1 \text{ if } x \text{ is totally in } A;$$

$$\mu_A(x) = 0 \text{ if } x \text{ is not in } A;$$

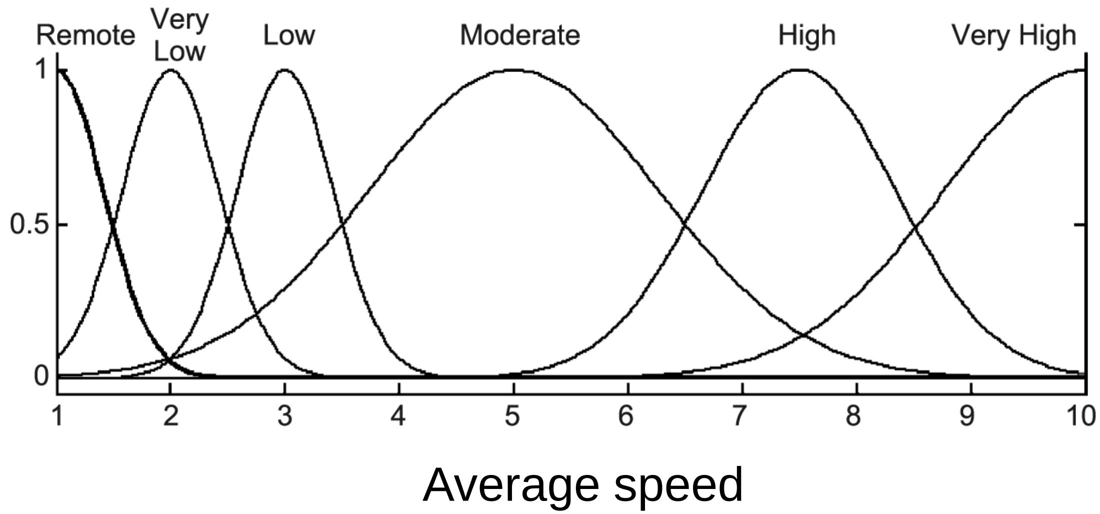
$$0 < \mu_A(x) < 1 \text{ if } x \text{ is partly in } A.$$

Fuzzy representation – membership function

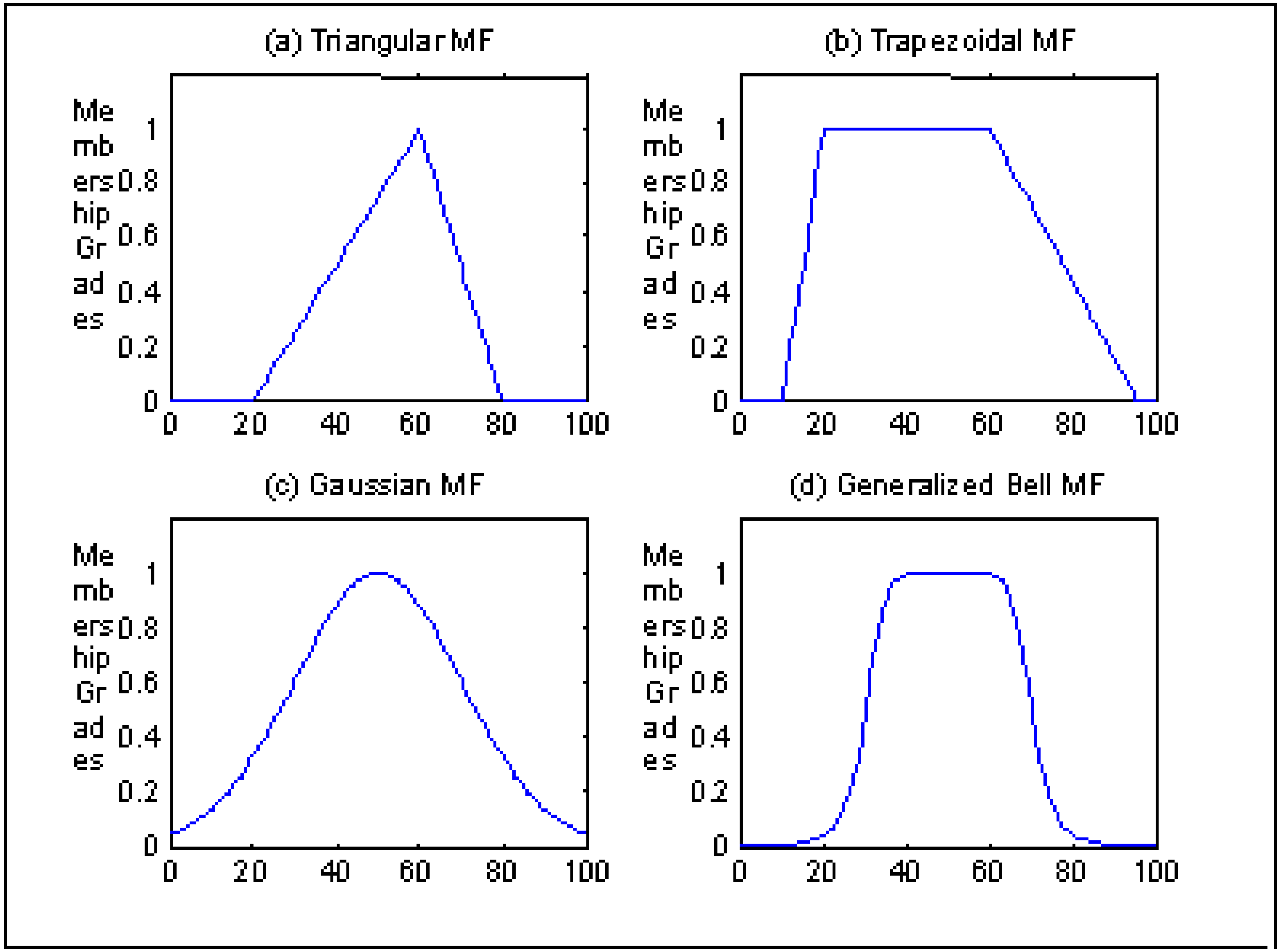
- In the fuzzy theory, fuzzy set *A* of universe *X* is defined by function $\mu_A(x)$ called the *membership function* of set *A*

- This set allows a continuum of possible choices. For any element *x* of universe *X*, membership function $\mu_A(x)$ equals the degree to which *x* is an element of set *A*. This degree, a value between 0 and 1, represents the **degree of membership**, also called **membership value**, of element *x* in set *A*.

Fuzzy representation



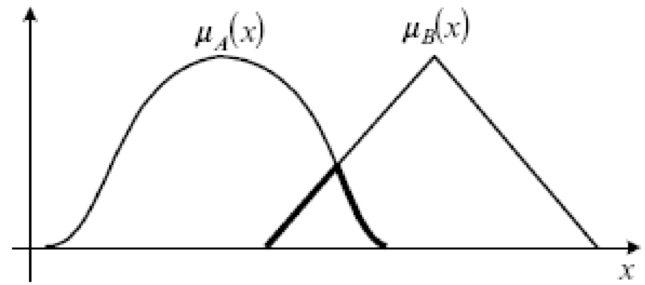
Family of membership functions



Fundamental operations on fuzzy sets

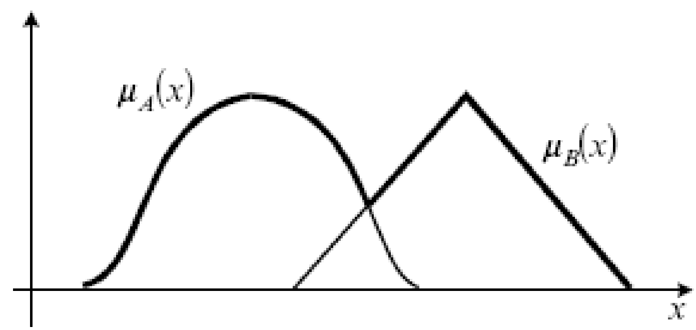
- Intersection

$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$$



- Union

$$\mu_{A\cap B}(x) = \max(\mu_A(x), \mu_B(x))$$



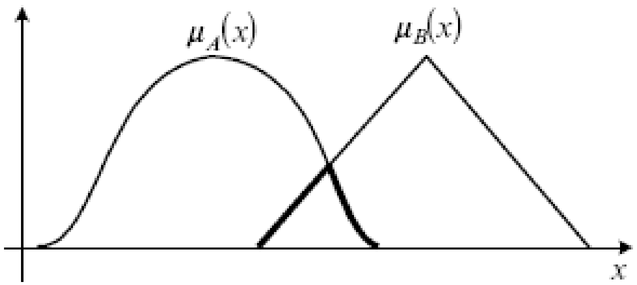
$$\mu_{A\cap B}(x) = \mu_A(x) * \mu_B(x)$$

$$\mu_{A\cap B}(x) = \mu_A(x) + \mu_A(x) - \mu_A(x) * \mu_B(x)$$

Fundamental operations on fuzzy sets

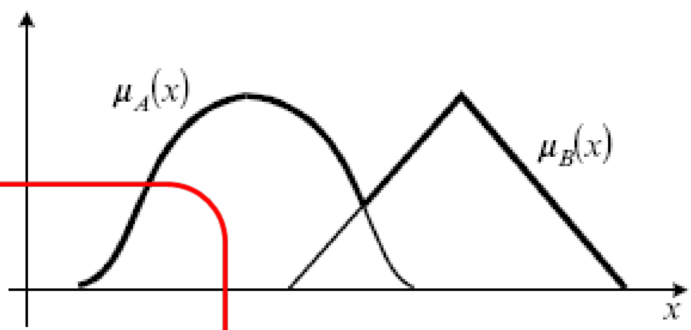
- Intersection

$$\mu_{A\cap B}(x)=\min(\mu_A(x),\mu_B(x))$$



- Union

$$\mu_{A\cap B}(x)=\max(\mu_A(x),\mu_B(x))$$



Alternatively,

$$\mu_{A\cap B}(x)=\mu_A(x)*\mu_B(x)$$

$$\mu_{A\cap B}(x)=\mu_A(x)+\mu_A(x)-\mu_A(x)*\mu_B(x)$$

T- and S-norms

T-Norms and S-Norms

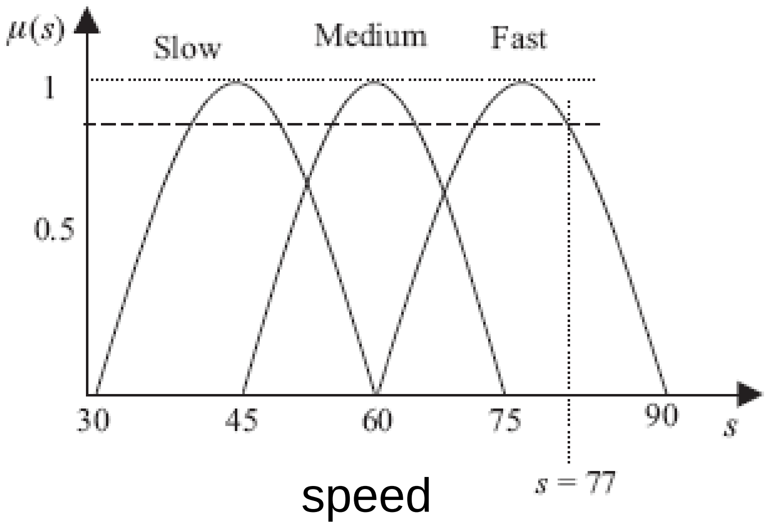
AND T-Norm $T(\mu_A(x), \mu_B(x))$	OR S-Norm $S(\mu_A(x), \mu_B(x))$
Minimum $\mathrm{MIN}(\mu_A(x), \mu_B(x))$	Maximum $\mathrm{MAX}(\mu_A(x), \mu_B(x))$
Algebraic product $\mu_A(x)\mu_B(x)$	Algebraic sum $\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$
Drastic product $\mathrm{MIN}(\mu_A(x), \mu_B(x))$ if $\mathrm{MAX}(\mu_A(x), \mu_B(x)) = 1$ 0 otherwise	Drastic sum $\mathrm{MAX}(\mu_A(x), \mu_B(x))$ if $\mathrm{MIN}(\mu_A(x), \mu_B(x)) = 0$ 1 otherwise
Lukasiewicz AND (Bounded Difference) $\mathrm{MAX}(0, \mu_A(x) + \mu_B(x) - 1)$	Lukasiewicz OR (Bounded Sum) $\mathrm{MIN}(1, \mu_A(x) + \mu_B(x))$
Einstein product $\mu_A(x)\mu_B(x)/(2 - (\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)))$	Einstein sum $(\mu_A(x) + \mu_B(x))/(1 + \mu_A(x)\mu_B(x))$
Hamacher product $\mu_A(x)\mu_B(x)/(\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x))$	Hamacher sum $(\mu_A(x) + \mu_B(x) - 2\mu_A(x)\mu_B(x))/(1 - \mu_A(x)\mu_B(x))$
Yager operator $1 - \mathrm{MIN}(1, ((1 - \mu_A(x))^b + (1 - \mu_B(x)^b)^{1/b}))$	Yager operator $\mathrm{MIN}(1, (\mu_A(x)^b + \mu_B(x)^b)^{1/b})$

Linguistic variables

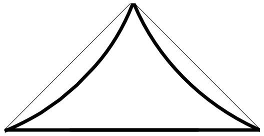
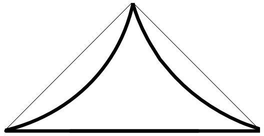
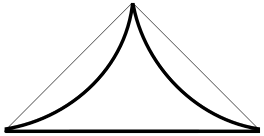
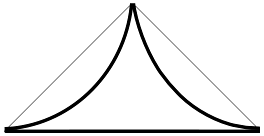
- At the core of fuzzy logic is the use of linguistic variables in fuzzy computations (inference)
- The range of possible values of a linguistic variable represents the universe of discourse of that variable.

IF *speed* is low THEN ....

IF *speed* is medium THEN ...

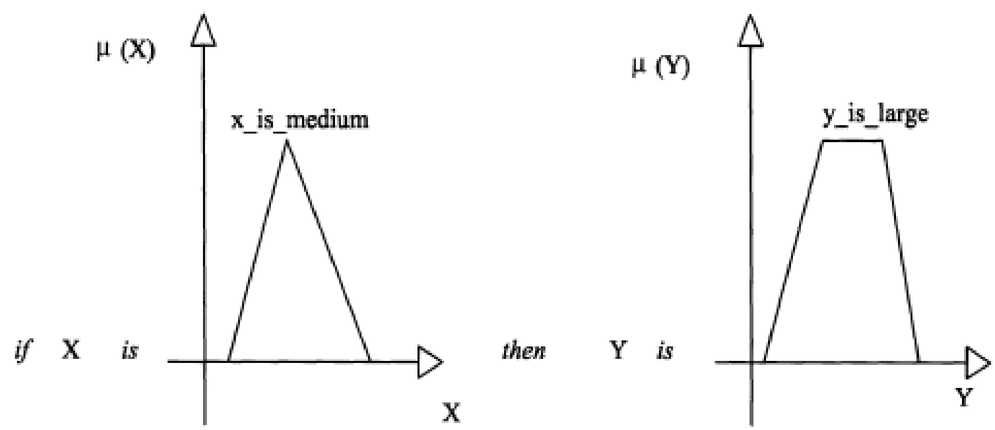


Linguistic variables - hedges

Hedge	*Mathematical Expression*	*Graphical Representation*
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^{2}$	
Extremely	$[\mu_A(x)]^{3}$	

Fuzzy implication – IF .. THEN rule

$$\underbrace{\text{IF } x \text{ is A}}_{\text{antecedent}} \; \overbrace{\text{THEN } y \text{ is B}}^{\text{consequent}}$$



Fuzzy inference boils down to <u>the composition of the antecedent set with the fuzzy relation</u> characterising the implication X->Y

Fuzzy relations and IF...THEN... rules

Fuzzy relation *R* is a fuzzy set itself

$$R = \{(x, y), \mu_R(x, y)\}$$

There is a number of different ways to define $\mu_R(x, y)$

product rule

$$\mu_{A-|B}(x, y) = \mu_R(x, y) = \mu_A(x) * \mu_B(y)$$

Mamdani-type rule

$$\mu_{A-|B}(x, y) = \mu_R(x, y) = \min(\mu_A(x), \mu_B(y))$$

Approximate resoning

- Modus Ponens

Premise:	**A**	*”Paul is a driver”*
Implication:	**A -> B**	
Conclusion:	**B**	*”Paul has a driving license”*

Approximate resoning

- **Generalized** Modus Ponens

Premise:	**A**	*”The car speed is high”*
Implication:	**A -> B**	”If the car speed is very high, then the noise level is high”
Conclusion:	**B**	*”The noise level in the car is medium hight”*

Approximate resoning

Generalized Modus Ponens

Premise: **A'**

Implication: **A -> B**

Conclusion: **B'**

Modus Ponens

Premise: A

Implication: A -> B

Conclusion: B

IF *x* is A' THEN *y is B'*

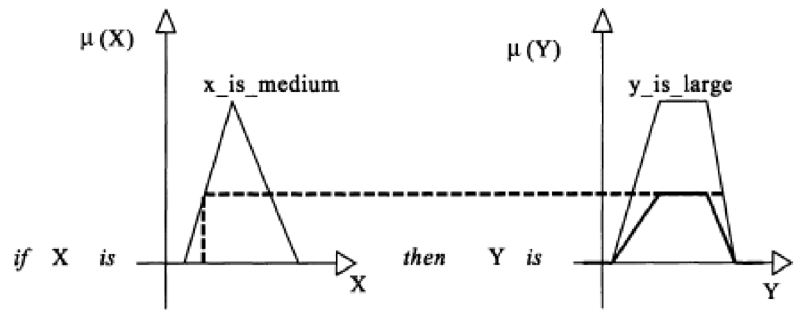
fuzzy sets, so we can talk about a degree of fulfilment

Fuzzy inference

Inference is a composition of the antecedent and the relation R.

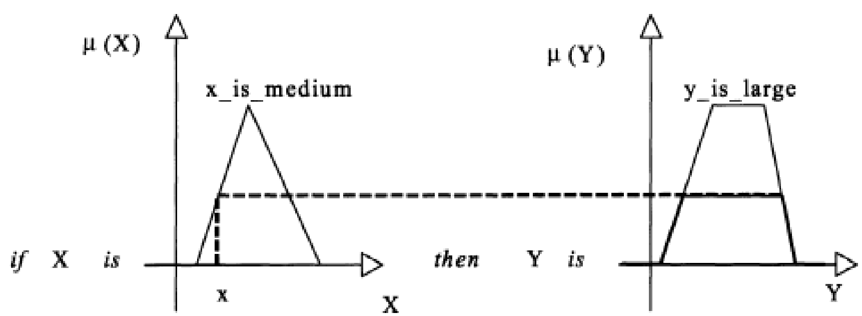
$$\mu_B(y) = \sup\{\mu_A(x) \overset{T}{*} \mu_R(x,y)\}$$

Compositional rule of inference



a

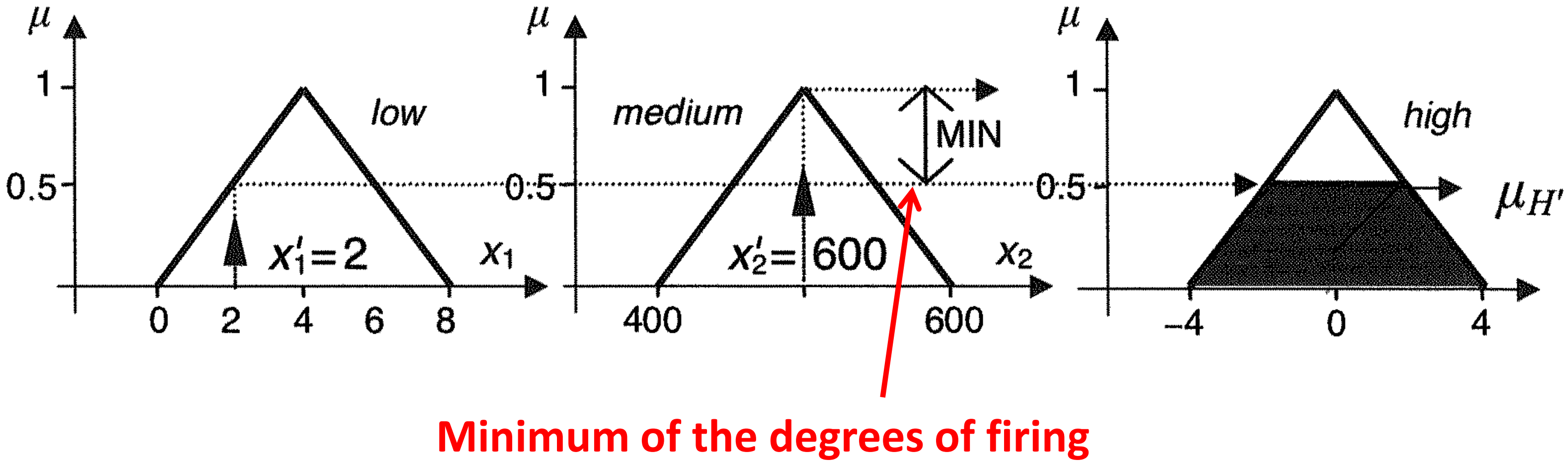
product rule



Mamdani-type rule

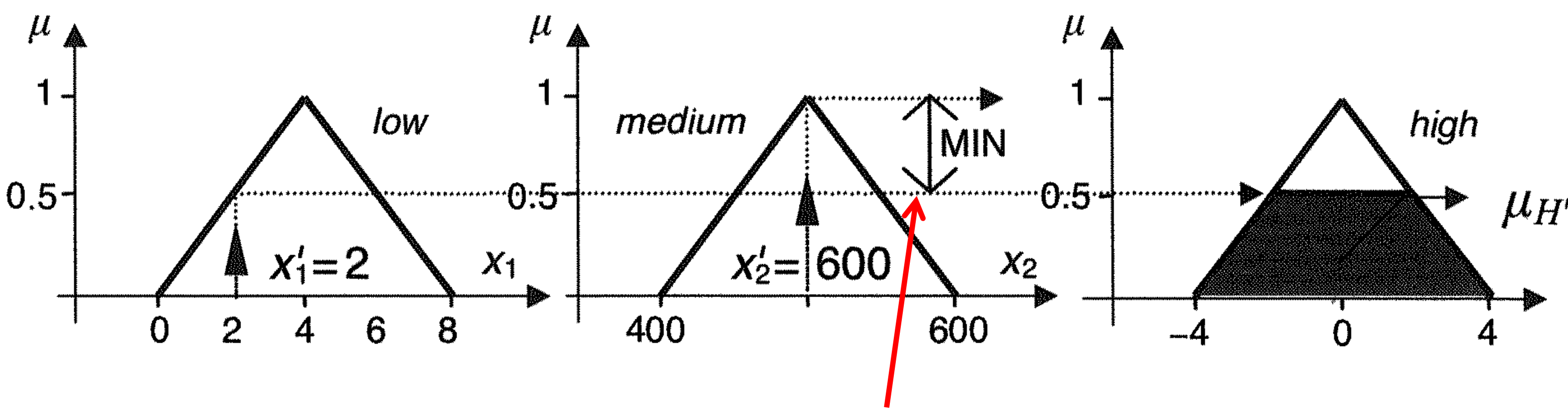
Fuzzy inference in general cases

- Multiple antecedents (combined with AND)



Fuzzy inference in general cases

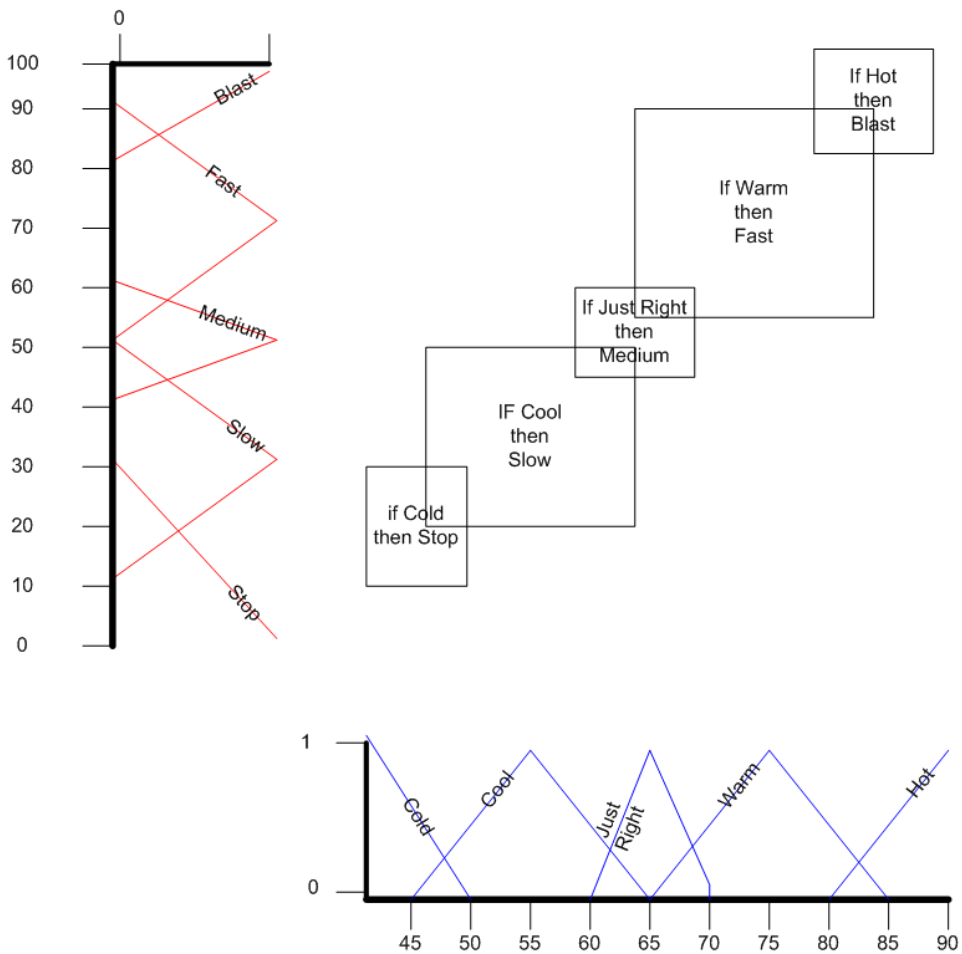
- Multiple antecedents (combined with AND)



Minimum of the degrees of firing

- Multiple rules
  - The outputs of inference have to be aggregated across all rules
  - Max operator is often used to aggregate the resultant consequents

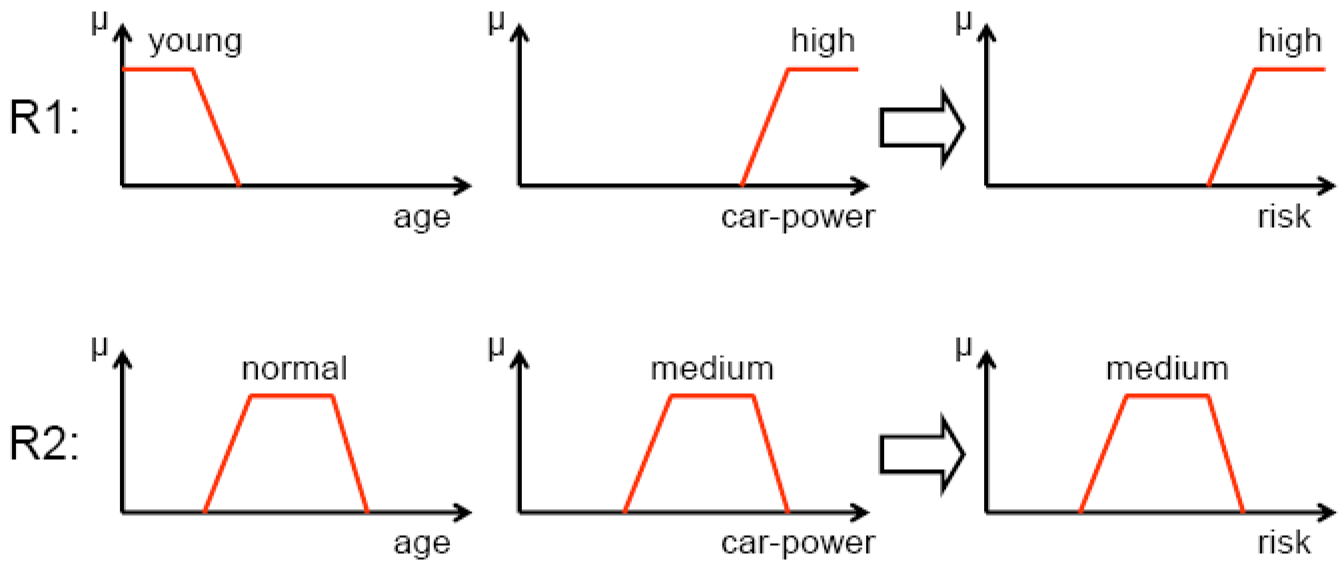
Example



0
100
90
80
70
60
50
40
30
20
10
0
Blast
Fast
Medium
Slow
Stop
If Hot then Blast
If Warm then Fast
If Just Right then Medium
IF Cool then Slow
if Cold then Stop
1
0
Cold
Cool
Just Right
Warm
Hot
45
50
55
60
65
70
75
80
85
90

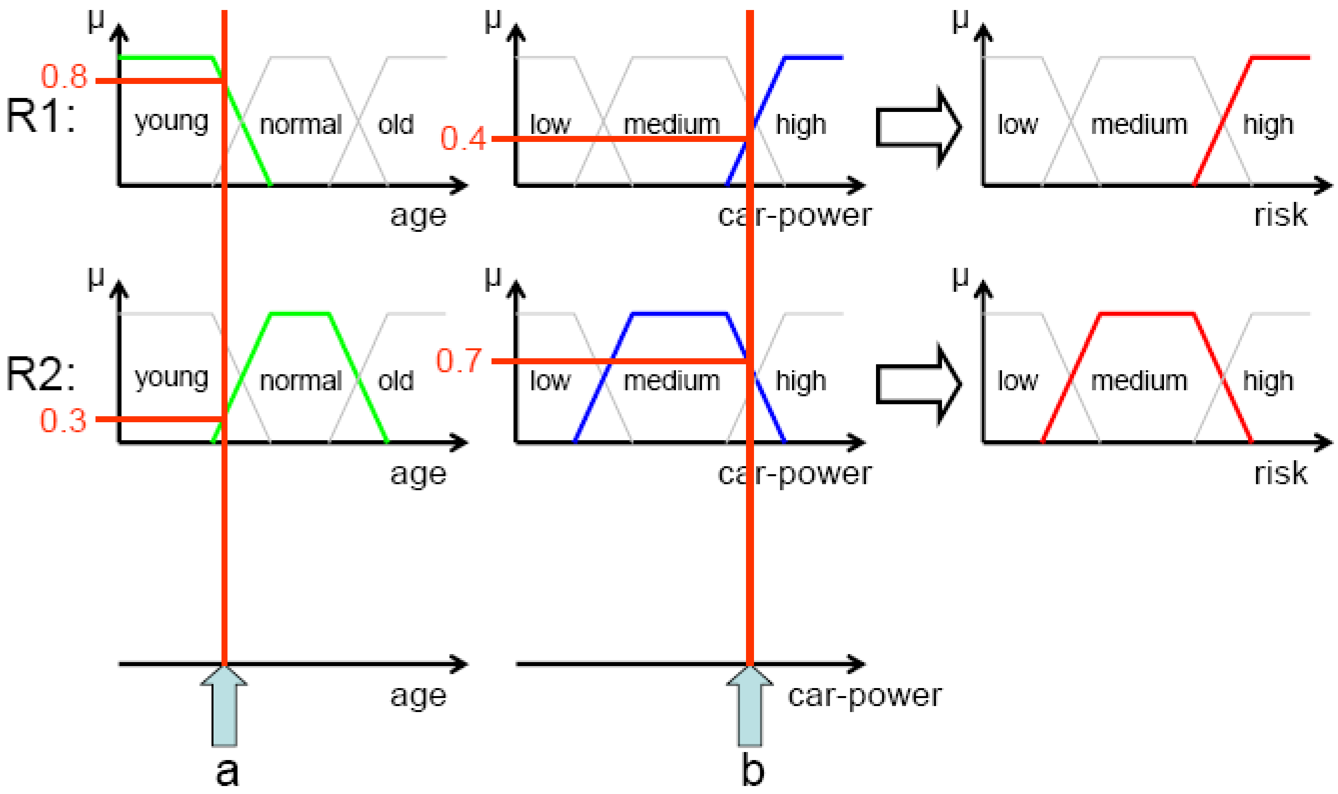
Example with multiple antecedents and multiple rules

R1: IF age IS young AND car-power IS high THEN risk IS high
R2: IF age IS normal AND car-power IS medium THEN risk IS medium



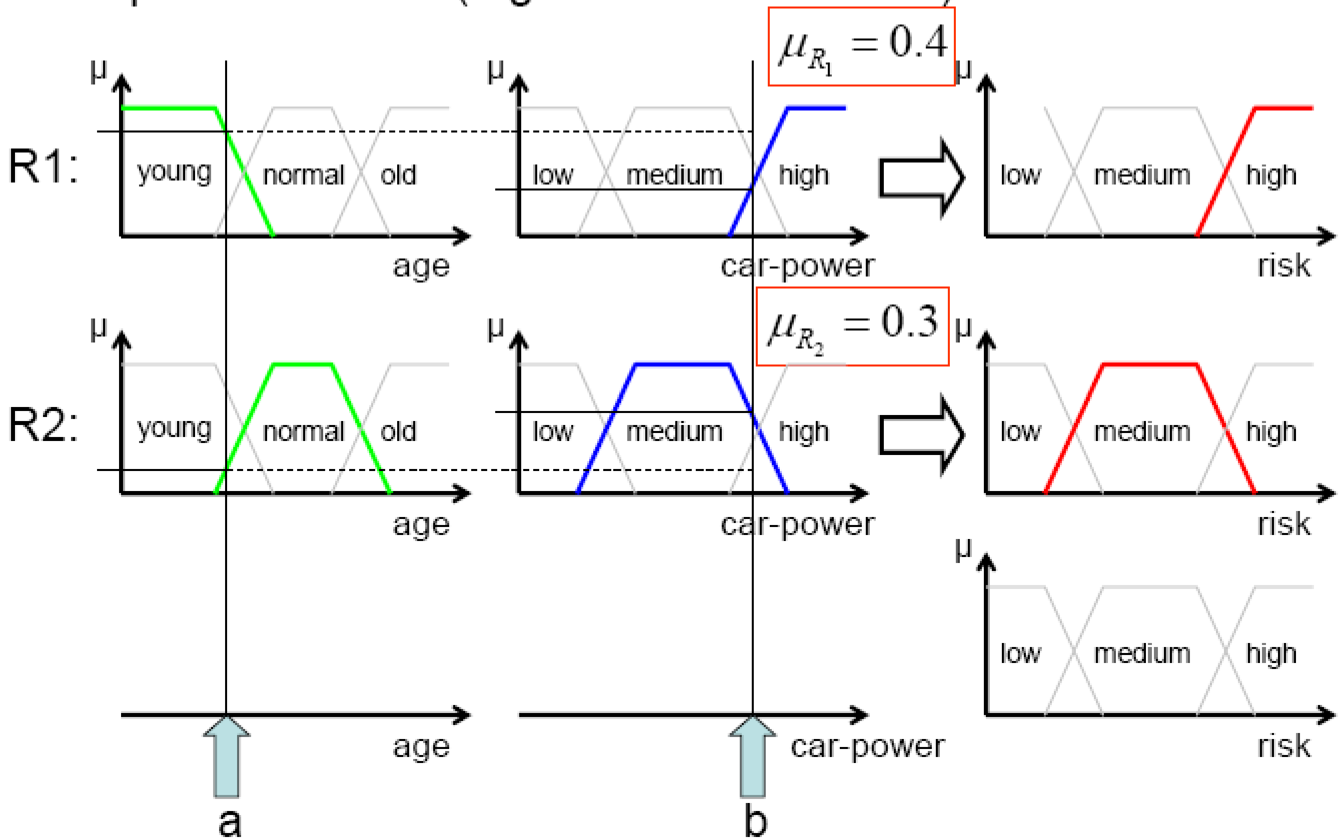
Example – fuzzification

- Step 1: Fuzzification of crisp inputs



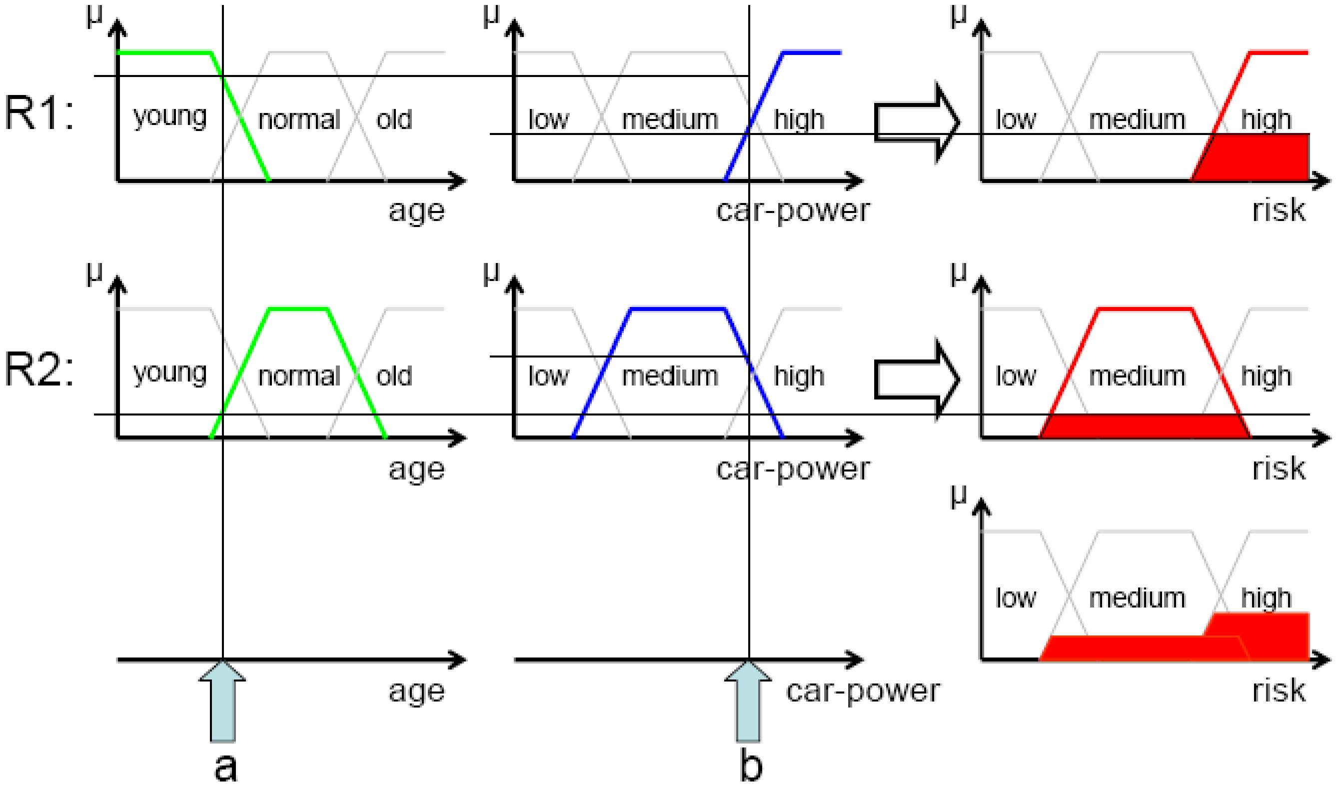
Example – inference

- Step 2a: Inference (e.g. via Min/Max-Norm)



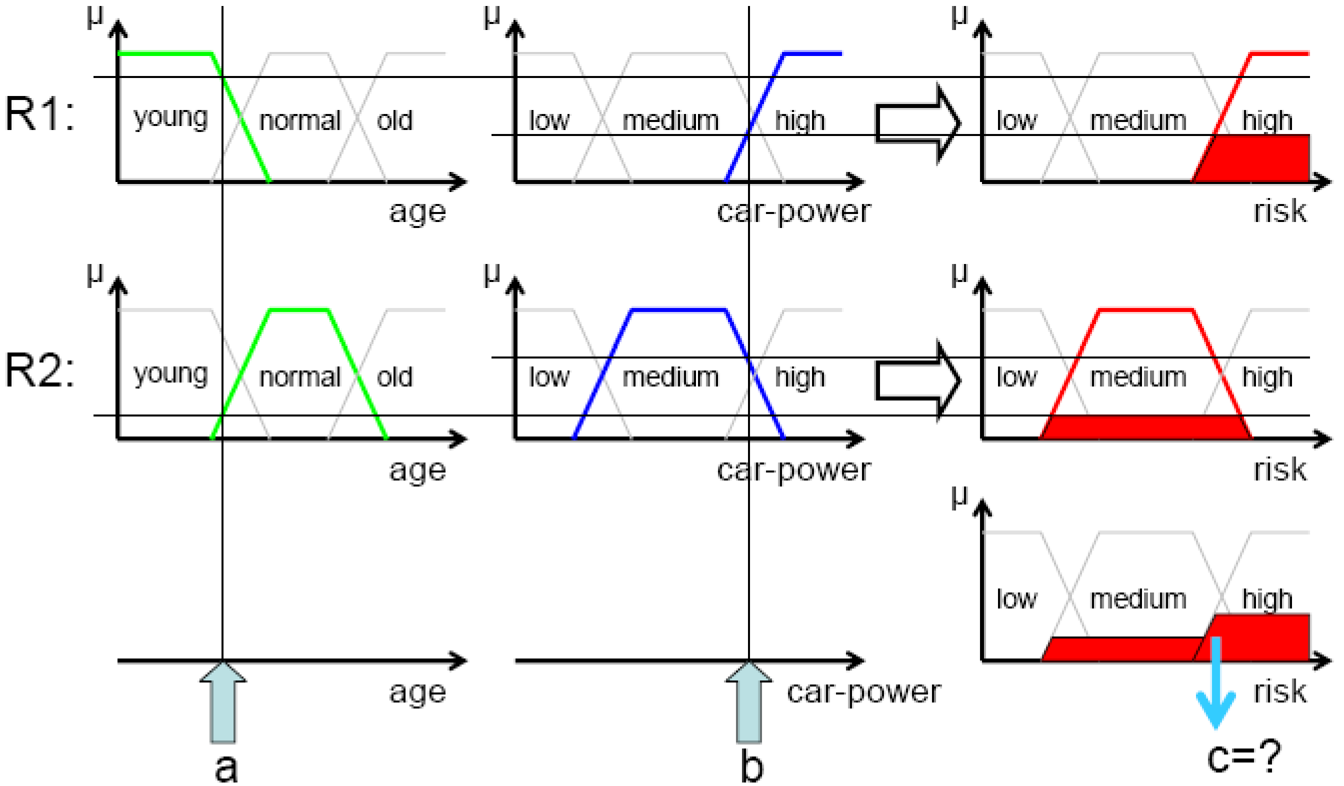
Example – aggregation

- Step 2b: Inference (e.g. via Maximum-Norm)



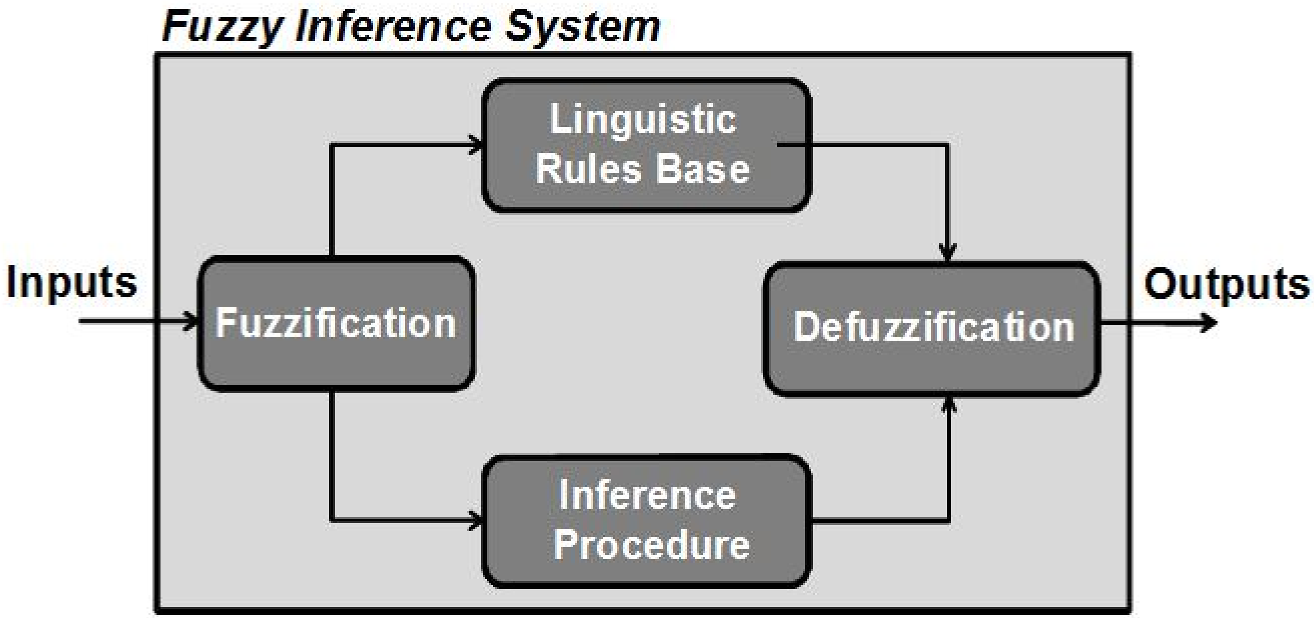
Example - defuzzification

- Step 3: Defuzzification



Fuzzy inference system

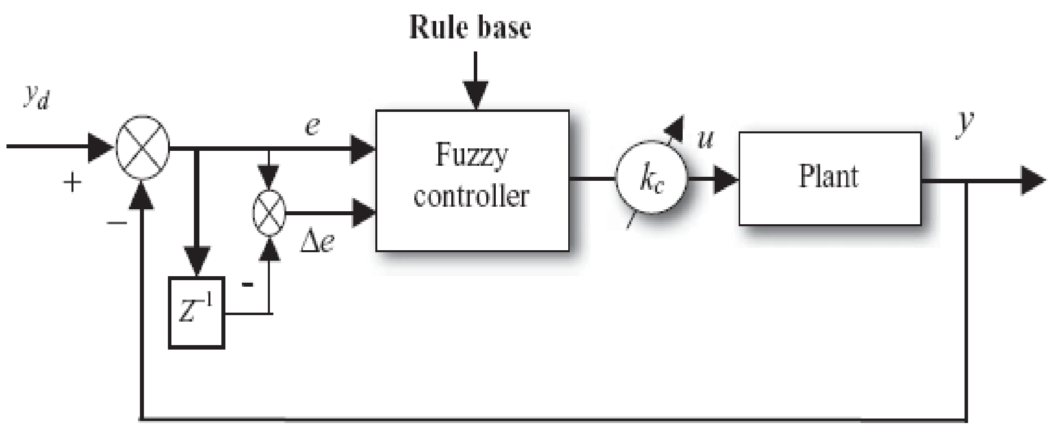
- Mapping input-output relationship with crisp data requires *fuzzification* and *defuzzification* steps



How to obtain rules?

Fuzzy Logic Control

PD-type controller



Rule base