KTH Royal Institute of Technology
Department of Mathematics

SF279X, Master's Thesis

Partizan Poset Games

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Abstract

This thesis analyzes a class of element-removal partizan games played on colored posets. In these games a player moves by removing an element of its color together with all greater elements in the poset. A player loses if it has no elements left to remove.
It is shown that all such games are numbers and that the dominating game options are to remove elements not lower than any other element of the same color.
In particular, the thesis concerns games played on posets that are chess-colored Young diagrams. It is shown that it is easy to compute the value for any such game with ≤ 3 rows by proving a proposed formula for computing the value.

Sammanfattning

I den här uppsatsen analyseras en klass av partiska spel som spelas på färgade pomängder. Spelen spelas i omgångar mellan två spelare där spelaren under sin tur väljer ut ett element i pomängden som är i spelarens färg och avlägsnar det elementet och alla större element i pomängden. En spelare förlorar om den inte längre har något element att avlägsna.
I uppsatsen visas det att alla sådana spel är tal och att de dominerande spelalternativen är att avlägsna element som inte är mindre än något annat element av samma färg.
I synnerhet fokuserar denna uppsats på spel som spelas på pomängder som är schackfärgade Young-diagram. Det visas att det är lätt att beräkna värdet på alla sådana spel med ≤ 3 rader genom att bevisa en föreslagen formel för att räkna ut värdet.

Acknowledgements

I would like express my gratitude to my supervisor at Royal Institute of Technology, Jonas Sjöstrand, for his valuable support and continuous interest and dedication throughout the process of this thesis and for keeping me inspired and devoted even after the worst of setbacks.

Contents

1 Introduction

In this section we will give an informal introduction of a partizan two-player game played on black-and-white-colored posets. The game is played by removing poset elements of your color, with the result that all greater elements in the poset are removed as well. The game ends when one of the players has no elements left to remove. This player then loses.
An example of such a game is provided below in Example 1.

Example 1. An example of a partizan element removal game played on a colored poset and an example of gameplay on this game.

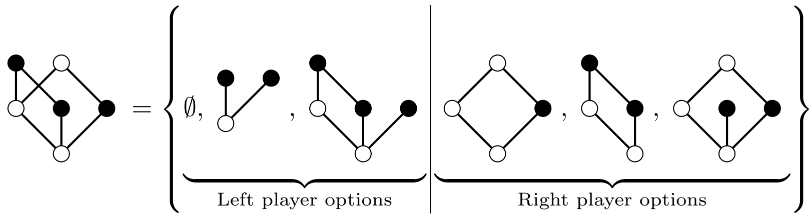

Figure 1: Example of a colored poset element removal game.

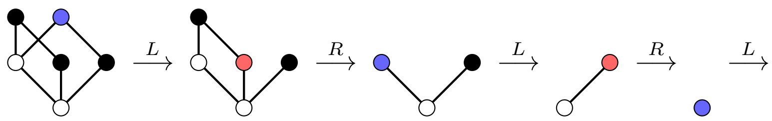

Figure 2: Example of gameplay on the game in Figure 1 where Left player wins. Play moves are highlighted with blue (Left) and red (Right).

In particular, this thesis will focus on games played on *chess-colored* posets, where the posets are in the form of *Young diagrams*. An example of a chess-colored Young diagram and a game played on this Young diagram is provided in Example 2.

Example 2. An example of a chess-colored Young diagram and an example of a gameplay on this Young diagram.

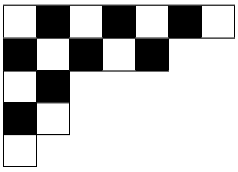

Figure 3: Example of a chess-colored Young diagram.

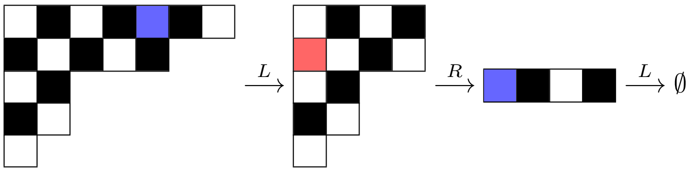

Figure 4: Example of gameplay on the chess-colored Young diagram of Figure 3 where Left player wins. Play moves are highlighted with blue (Left) and red (Right).

For these games in general, we will show that they are all surreal numbers, and that, given some properties, they always are valued between 0 and 1.
Finally, for games played on chess-colored Young diagrams with ≤ 3 rows, we will show that the value is easy to compute by proving that they can be computed with a given formula.

First a brief background of poset games will be covered.

1.1 Background

A poset game is an element-removal game played on a poset, where a player selects an element and removes this element and all greater elements. A poset game can be both impartial (if it is not colored) and partizan (if it is colored, i.e., each element has a color which specifies who can select and remove it). It is known that the problem of deciding the winner of an impartial uncolored poset game is PSPACE-complete [4].

The simplest possible impartial poset game is the one played on a collection of one-dimensional chain posets, also known as *Nim*. The game of Nim is played by removing a number of elements from one of multiple piles of elements. The player removing the last element wins the game. For a more intuitive understanding of Nim, an example of a gameplay on a game of Nim is provided in Example 3.

Example 3. An example gameplay on a game of Nim.

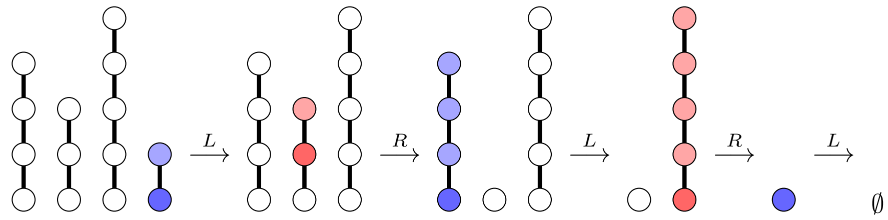

Figure 5: Example of gameplay on a game of Nim where Left player wins. Play moves are highlighted with blue (Left) and red (Right).

Another impartial poset game is the game *Chomp*, more thoroughly introduced in Section 2.2.3. A game of Chomp can be represented by a poset game with a two-dimensional $n \times m$ lattice poset, $n, m > 0$ integers, with the bottom element removed, as illustrated in Figure 7.

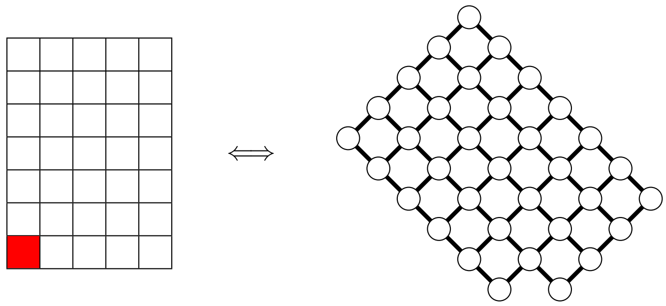

Figure 7: A game of Chomp is equivalent to an impartial poset game.

While the game of Nim is solved [2], i.e., there is a known optimal strategy, there is not so much known about the game of Chomp in general. This also points out how the difficulty can differ for two classes of impartial poset games. Therefore, it is interesting to investigate the properties of *partizan* poset games, i.e., games on colored posets.
In general, there have been very few studies on partizan poset games. One class of partizan poset games that have been studied are *pomax games*. A pomax game is played on a colored poset, where each player can remove only maximal elements of their own color. Examples of pomax games can be found in Figures 8 and 9.

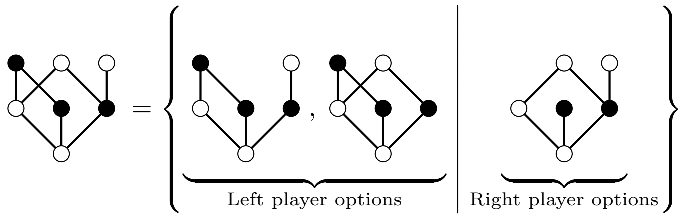

Figure 8: Example of an arbitrary pomax game.

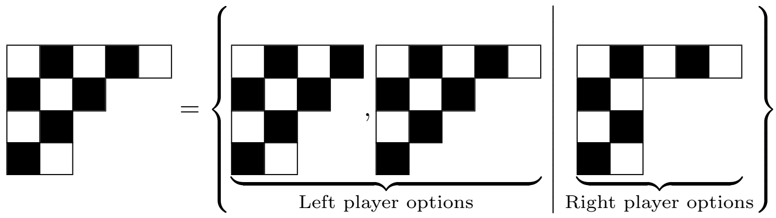

Figure 9: Example of a pomax game on a chess-colored Young diagram.

It has been shown that all pomax games are integer valued [5], that it is easy to determine the value of pomax games played on trees or on chess-colored Young diagrams [5] and that the problem of determining the winner of an arbitrary pomax game is PSPACE-complete [6].

This thesis will focus on partizan poset games without the constraint of only being able to remove maximal elements, i.e., more similar to the gameplay of the regular poset games, only played on a colored poset. An illustrative example of this type of game is provided in Example 1.

2 Preliminaries

In this section a background of the theory behind the combinatorial games studied for this thesis is provided. In addition to this, the notation introduced here is used throughout the thesis.

2.1 Partially Ordered Sets

This thesis is about a type of combinatorial games called *poset games*. In order to be able to introduce the theory of these games, we must first define what a poset is.

Definition 4 (Partially Ordered Sets [7, p. 278])**.** A *partially ordered set* (*poset*) $(P, \leq)$ is a set with a binary order relation $\leq$ satisfying the following three axioms:

1. For all $t \in P$, $t \leq t$ (reflexivity).
2. If $s \leq t$ and $t \leq s$,then $s = t$ (antisymmetry).
3. If $s \leq t$ and $t \leq u$,then $s \leq u$ (transitivity).

We use the obvious notation $t \geq s$ to mean $s \leq t$, $s < t$ to mean $s \leq t$ and $s \neq t$, and $t > s$ to mean $s < t$. We say that two elements s and t of P are *comparable* if $s \leq t$ or $t \leq s$, otherwise s and t are *incomparable*.
We define an *interval* $[p, q]$ of a poset to be $\{x \in P \mid p \leq x \leq q\}$. We say that v covers u if $[u, v] = \{u, v\}$, and we denote this by $u \lessdot v$.

In a partizan element removal game, every element must have a color.

Definition 5 (Colored Posets)**.** A *colored poset* is a poset where each element has a color of either black or white.

Example 6. An example of a regular and a colored poset.

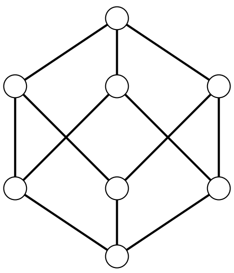

Figure 10: Example of a poset.

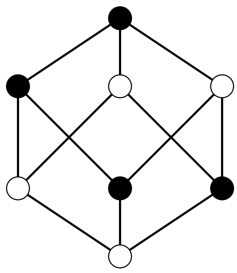

Figure 11: Example of a colored poset.

In particular, this thesis focuses on poset games with a specific coloring called chess-coloring.

Definition 7 (Chess-Colored Posets). A *chess-colored poset* is a colored poset such that no element covers an element of the same color. Equivalently we may regard $P = W \cup B$ as a bipartite graph, with white vertices W and black vertices B, with the cover relation as an edge relation.
To avoid confusion, we will assume that the least element is colored white when there is only one smallest element.

Example 8.

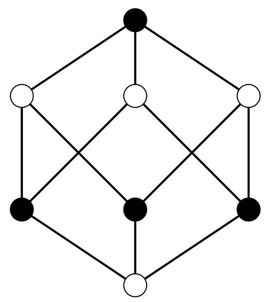

Figure 12: Example of a chess-colored poset.

2.1.1 Young Diagrams

This thesis mainly focuses on an object called *Young diagram*. We will formally define exactly what a Young diagram is in Definition 9, but before that we need the following definition:

Definition 9. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a weakly decreasing sequence of positive integers, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. We say that λ partitions n, denoted by $\lambda \vdash n$, if $\sum_{i=1}^{k} \lambda_i = n$.

Definition 10 (Young Diagrams). A *Young diagram* is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row. If the number of boxes is n and $\lambda \vdash n$, we say that λ generate a Young diagram with λ_1 boxes in the first row, λ_2 boxes in the second row,$\dots$, λ_k in the k'th row. Moreover, a Young diagram can always be represented as a poset.

This definition is best illustrated with an example.

Example 11. With $\lambda = (7, 5, 2, 2, 1)$ we have the Young diagram in Figure 13 and the corresponding representation as a poset in Figure 14.

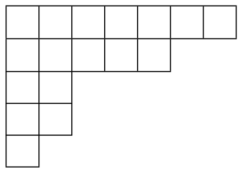

Figure 13: Young diagram generated by $\lambda = (7, 5, 2, 2, 1)$.

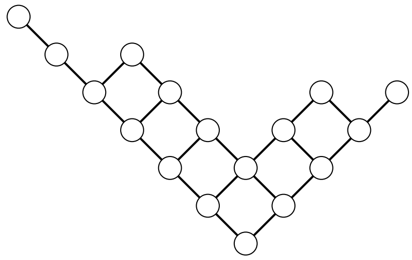

Figure 14: Poset of the Young diagram generated by $\lambda = (7, 5, 2, 2, 1)$.

In analogy with the previous definitions, a colored Young diagram is a Young diagram where each box has a color of either black or white, and a chess-colored Young diagram is a Young diagram where no adjacent boxes have the same color.

Example 12. With $\lambda = (7, 5, 2, 2, 1)$ as before, we have the chess-colored Young diagram and the corresponding chess-colored poset in Figures 15 and 16 respectively.

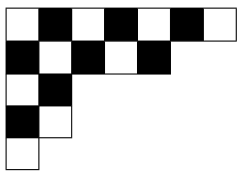

Figure 15: Chess-colored Young diagram generated by $\lambda = (7, 5, 2, 2, 1)$.

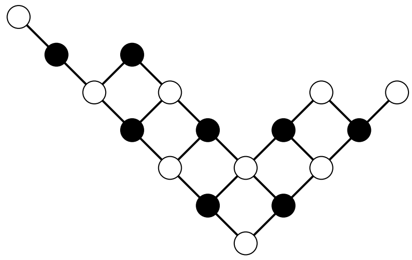

Figure 16: Chess-colored poset of the Young diagram in figure 15.

2.2 Combinatorial Game Theory

This thesis deals with combinatorial game theory, an area which studies sequential games with perfect information, that is, games where the players play in turns and where they have complete knowledge of the game, i.e., know all possible game options for all players.
In particular, the thesis will focus on two-player partizan combinatorial games, in which the game options of the two players can be different. Furthermore, we call the two players Left and Right (or White and Black or Blue and Red).
In general, a combinatorial game has *positions*, and at any given position every player has a set of *options* of moving to a new position. Under normal play convention a player loses if they have no options available at their turn to move.

Definition 13 (Partizan Game Position [3, p. 71])**.** A position in a partizan game is defined by its left and right options, and we denote it by $G = \{L|R\}$, where L and R are the sets of left and right options respectively.

Since we will be using the notation used by Conway [3], the notation above will not always be used, and we will instead often abuse it by writing $G = \{G_1, G_2, G_3|H_1, H_2\}$ as short for $G = \{\{G_1, G_2, G_3\}|\{H_1, H_2\}\}$, and in the general case $G = \{G^L|G^R\}$.
Following this we will introduce some notation for the games depending on the winner and who starts.

Definition 14 (Value Notation [3, p. 73])**.**

- $G > 0$ (G is *positive*) if there is a winning strategy for Left.
- $G < 0$ (G is *negative*) if there is a winning strategy for Right.
- $G = 0$ (G is *zero*) if there is a winning strategy for the second player to move.
- $G \parallel 0$ (G is *fuzzy* to zero) if there is a winning strategy for first player move.

This notation is easy to understand with help of some examples.

Example 15. Consider the simplest possible game, the game with no options for either player, i.e., $G_1 = \{|\}$. We obviously have $G_1 = 0$, since the first player has no options to play and therefore loses. This game is denoted by $0 := \{|\}$.
Now consider the game where Left has the option to move to 0, but Right still has no options, i.e., $G_2 = \{0|\}$. Here we have that $G_2 > 0$ since either Left starts and moves to 0, and then Right has no option and loses, or Right starts and has no options and therefore loses, i.e., Left has a winning strategy. This game is denoted by $1 := \{0|\}$.
Similarly we have that the game $-1 < 0$ where -1 is defined as $-1 := \{|0\}$.
Finally, consider the game where both players have the option to move to 0, i.e., $G_3 = \{0|0\}$. We now have $G_3 \parallel 0$ since both players have the option to move to 0, where the second player then will lose. This game is denoted as $* := \{0|0\}$.

The value notation of Definition 14 can be combined and extended in the following way.

Definition 16 (Extended Value Notation [3, p. 73])**.**

- If $G \geq 0$, then Left always wins if Left is the second player to move.
- If $G \leq 0$, then Right always wins if Right is the player second to move.
- If $G \rhd 0$, then Left always wins if Left is the first player to move.
- If $G \lhd 0$, then Right always wins if Right is the first player to move.

In addition to the value notations, it is also possible to add and subtract games.

Definition 17 (Addition and Negation [3, p. 73])**.**

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$$
$$-G = \{-G^R | -G^L\}$$

Informally, we can note that addition of two games is the same as playing in both games at the same time, and the negation of a game is the game with reversed roles of Left and Right. Combining these, it is possible to subtract games as $G - H = G + (-H)$.
Using this, we define the following relations between games:

Definition 18 (Game Relations [3, p. 78])**.**

- $G > H$ iff $G - H > 0$.
- $G < H$ iff $G - H < 0$.
- $G = H$ iff $G - H = 0$.
- $G \parallel H$ iff $G - H \parallel 0$.

Using these relations, which can be combined and extended as the extended notation of Definition 16, we can define what a *dominated* option is.

Definition 19 (Dominated Options [3, p. 110])**.** For a game we say that a left option G^{L_1} is dominated by another left option option G^{L_0} if $G^{L_1} \leq G^{L_0}$. Similarly, a right option G^{R_1} is dominated by another right option G^{R_0} if $G^{R_1} \geq G^{R_0}$.

In fact, an important property of a game is that you always can remove any dominated options [3, p. 110].

Theorem 20. *Let* $G = \{G^{L_0}, G^{L_1}, \ldots | G^{R_0}, G^{R_1} \ldots, \}$.

- *If* $G^{L_0} \leq G^{L_1}$*, then* $G = G'$*, where* $G' = \{G^{L_1}, \ldots | G^{R_0}, G^{R_1} \ldots, \}$.
- *If* $G^{R_0} \geq G^{R_1}$*, then* $G = G''$*, where* $G'' = \{G^{L_0}, G^{L_1}, \ldots | G^{R_1} \ldots, \}$.

A significant class of games are the *short* games.

Definition 21 (Short Games [1, p. 3] [3, p.97])**.** A game G is short if only a finite number of positions can be reached and a position may never be repeated.

Moreover, every short game G has a unique simplest form. This is called G's canonical form [1, p. 78]. It is possible to reduce any short game to its canonical form by just removing dominated and *reversible* options [3, p. 111].

A methodology that is extremely useful when proving properties of games is *Conway induction.*

Theorem 22 (Conway Induction [3, p. 5]). *Let P be a property which games might have, such that any game G has property P whenever all left and right options of G have this property. Then every game has property P.*

The methodology using Conway induction makes it possible to prove that a game has a property by assuming that all its options have this property, and from this proving that the game itself has it. This methodology is using that the definitions of games are inductive.

2.2.1 Numbers

Another important property and concept in combinatorial games is that of numbers, which is a class of games with some special characteristics.

Definition 23 (Numbers [1, p. 91]). A number is any game x such that all $x^L < x < x^R$ and x^L and x^R are numbers. For short games, we can, equivalently, for $j > 0$ and m odd, define a number as

$$\frac{m}{2^j} = \left\{ \frac{m-1}{2^j} \middle| \frac{m+1}{2^j} \right\}. \tag{1}$$

It should also be noted that all games are not numbers. For $* = \{0|0\}$ we have $G^L = 0 = G^R$, and hence $G^L \not< G^R$, i.e., $*$ is not a number.
Equation (1) can be generalized to recognize when G is a number even if it is not in canonical form. For this, we need to define what the *simplest number* is.

Definition 24 (Simplest Number [1, p. 93]). For $x^L < x^R$, the simplest number x between x^L and x^R is given by the following:

- If there are integer(s) n such that $x^L < n < x^R$, x is the one that is smallest in absolute value.
- Otherwise, x is the number of the form $\frac{i}{2^j}$ between x^L and x^R for which j is minimal.

Theorem 25 (Numbers [1, p. 93]). *If all options of a game G are numbers and all $G^L < G^R$, then G is the simplest number x satisfying $G^L < x < G^R$.*

This, together with Definition 19 and Theorem 20, yields that a game that is equal to a number is equal to the game consisting of only its greatest left options and smallest right options, i.e., $G \equiv \{G^{L_0}, G^{L_1}, \ldots | G^{R_0}, G^{R_1} \ldots, \} = \{G^{L_0} | G^{R_0}\} \equiv G'$ if $G^{L_0} \geq G^{L_i}$ and $G^{R_0} \leq G^{R_j}$ for $i, j > 0$.
It should also be noted that equation (1) is a number for integers $j > 0$ and m, regardless if m is odd or not.

2.2.2 Hackenbush

A game with properties similar to the ones studied in this thesis is *Blue-Red Hackenbush*. Hackenbush is a partizan two-player game that may be played on any configuration of colored line segments connected to one another by their endpoints and to a "ground" line. In the Blue-Red Hackenbush, the line segments are colored either blue or red. It is played by, in turns, removing a line segment of your color, by which all segments that are unconnected to the ground vanishes, until a player has no move left.

Example 26. An example of a game of Blue-Red Hackenbush and gameplay on that game.

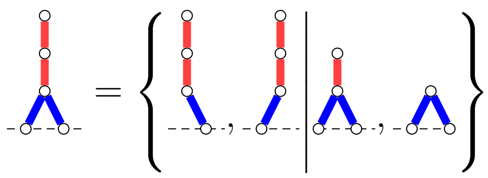

Figure 17: A simple game of Blue-Red Hackenbush.

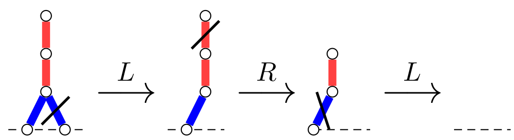

Figure 18: Example of gameplay on the Blue-Red Hackenbush in Figure 17 where Left player wins.

It is known that every game of Blue-Red Hackenbush is a surreal number, and in particular, any *finite* game of Blue-Red Hackenbush is a dyadic rational number, i.e., on the form $\frac{m}{2^j}$ where m and $j > 0$ are integers.

2.2.3 Chomp

Another game closely related to that of this thesis is the game of *Chomp*. Chomp is a impartial two-player game with the usual starting position consisting of a rectangle (possibly infinite) with one poison square in the lower-left corner. A move in Chomp is to choose a square and remove this and all other squares above or to the right of it, and a player loses if they has to choose the poison square.

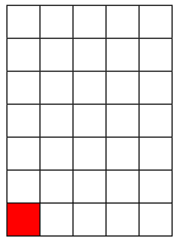

Figure 19: Example of a starting position of a game of Chomp.

Example 27. An example of gameplay on a game of Chomp with starting position as in Figure 19.

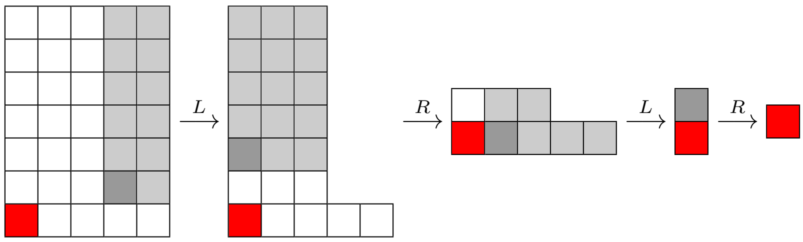

Figure 20: Example of gameplay in the game of Chomp with starting position as in Figure 19 where Right player wins.

3 Partizan Poset Games

Before the proofs that are the main results of this thesis we will start with a formal introduction of partizan poset games in general and partizan poset games played on chess-colored Young diagrams in particular.

Definition 28 (Partizan Poset Games)**.** Let P be any colored poset and let p_L, p_R denote arbitrary elements in P of Left and Right respectively. The partizan poset game $G_{\mathrm{PP}}(P)$ played on P is a partizan two-player game where the Left player has the option to select any white element p_L in P and then remove p_L together with all elements greater than p_L and equivalently for the Right player but with black elements in P. More formally we have:

$$
\begin{aligned}
&G_{\mathrm{PP}}(P) = \{L|R\} \\
&L = \{G_{\mathrm{PP}}(P \setminus S_{\mathrm{P}}(p_L)) : p_L \in P \text{ is white}\} \\
&R = \{G_{\mathrm{PP}}(P \setminus S_{\mathrm{P}}(p_R)) : p_R \in P \text{ is black}\} \\
&S_{\mathrm{P}}(p) = \{p' : p' \in P, p' \geq p\}
\end{aligned}
$$

More specifically, this thesis deals with partizan poset games played on posets that are chess-colored and in the form of Young diagrams. We will denote such a game, with $k \geq 1$ rows in the Young diagram, as A_λ, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is as in Definition 9 and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ denotes the lengths of the 1st, 2nd,$\ldots$, k'th rows respectively. In particular, when $k = 3$ or $k = 2$, we will use the notation $A_{x,y,z}$ and $A_{x,y}$ respectively.

3.1 All Poset Games Are Numbers

As it is, we have that all poset games are numbers. In particular, for a large number of partizan poset games, the value is bounded.

Theorem 29. *All poset games are numbers.*

Proof of Theorem 29. Assume that we have an arbitrary poset game G. Using *Conway Induction* (Theorem 22) it suffices to assume that G^L, G^R are numbers and then deduce that G is a number as well. By Theorem 25 it is then sufficient to show that $G^L < G^R$ for all options of G.
If we can show that $G^L < G$ and $G^R > G$ for any Left and Right options of G, then it also holds that $G^L < G^R$, and hence G must be a number.

Consider the scenario where Left moves to some option G^{L_1}, as illustrated in Figure 22. We have that $G^{L_1} < G$ since Right always wins in $G^{L_1} - G$.

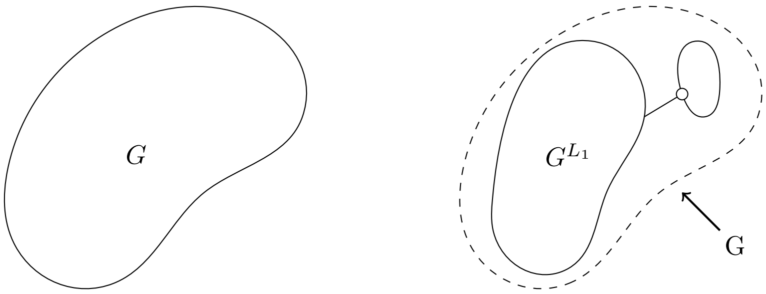

Figure 22: Arbitrary poset game G and G with an arbitrary Left option G^{L_1}.

This is easily understood from the following. In the game $G^{L_1} - G$ in Figure 23 the Right player wins if it plays first, since then Right can move to $-G^{L_1}$ in $-G$ and then mimic Left's moves until Left has no moves left. If Left starts and plays in the part of G that is not included in G^{L_1}, i.e., the small appendage of G in Figure 22, then Right can move to $-G^{L_1}$ in that component and copy Left in the same way as before. If Left starts and plays in the G^{L_1}-part of either component, then Right can copy Left until either Left has no moves or until Left plays in the appendage part of the $-G$ component, and then Right can just move to the option that removes that appendage, which makes the two components mirrored again, so Right can then copy Left until Left has no moves and loses.
Since Right wins in $G^{L_1} - G$ no matter if Right starts or not, then $G^{L_1} - G < 0$.

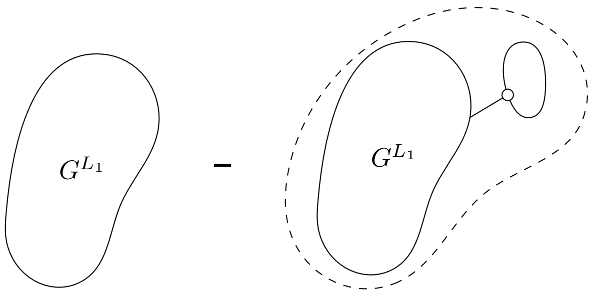

Figure 23: The poset game $G^{L_1} - G$.

In the exact same way, it is possible to show that Left always wins in $G^{R_1} - G$, and hence that $G^{R_1} - G > 0$. We therefore have $G^{L_1} < G^{R_1}$. Since G^{L_1} and G^{R_1} were arbitrary, this holds for any Left and Right options G^{L_1}, G^{R_1}, and therefore G must be a number. □

This lets us know that all poset games are numbers. But we can also bound the value of some partizan poset games, as will be seen in Theorem 30.

Theorem 30. *Any partizan poset game G with a single smallest element colored white, covered only by black elements, has a value $0 < G < 1$.*
An example game can be seen in Figure 24.

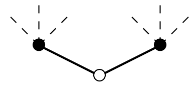

Figure 24

Proof of Theorem 30. Let G be any partizan poset game with a single smallest element colored white, covered only by black elements. Clearly $G > 0$ since Left can remove the smallest element in the poset, removing all elements, resulting in no options for Right, so Left wins.
Moreover, since the game with only the smallest (white) element left is either an option of Right, or an option of an option of Right, or an option of an option of an option of Right,..., etc., and all $G^R > G$, then $G < 1$. □

We may note that this theorem holds for all chess-colored partizan poset games with a single smallest elements, e.g., games played on chess-colored Young diagrams.

In addition to bounding the value of some games, it is also possible to determine that a player should try to play the option of removing as great elements as possible.

Theorem 31 (Play Strategy)**.** *A player should only play the options of removing elements not lower than any other element of the same color.*

Proof of Theorem 31. We will prove this theorem by showing that any option of removing an element that is lower than some other element of the same color is also a dominated option.
Let G be any partizan poset game, let $G^{L_{x_1}}, G^{L_{x_2}}$ be the Left options when removing the elements x_1 and x_2 respectively and let $x_1 > x_2$. The option $G^{L_{x_2}}$ must be an option of $G^{L_{x_1}}$. This is because $x_1 > x_2$, which yields that the option of removing x_2 also removes x_1, and therefore the option of removing x_1 does not remove any elements that are not removed when playing the option of removing x_2.
Since $G^{L_{x_2}}$ is an option of $G^{L_{x_1}}$ and Theorem 29 yields that G is a number, then $G^{L_{x_1}} > G^{L_{x_2}}$ and hence $G^{L_{x_2}}$ is dominated by $G^{L_{x_1}}$, i.e., Left should not play the option $G^{L_{x_2}}$.
Similarly, this holds for Right options as well. □

3.2 Chess-Colored Young Diagram Partizan Poset Games

For chess-colored Young diagrams, we have a very regular structure. This regularity makes it possible to reduce the games significantly, and make even stronger statements about the values of these games. A general result about how we can reduce games played on chess-colored Young diagrams is the following.

Lemma 32. *The dominating option of A_λ, with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, is always to remove the greatest element of your color in one of the rows.*

Lemma 32 follows from Theorem 31. For a better understanding of what the lemma yields, we will provide some examples of the concept.

Example 33. Let $k = 4$. With $\lambda_1 = 9, \lambda_2 = 7, \lambda_3 = 7, \lambda_4 = 2$, Lemma 32 gives us that:

$$A_{9,7,7,2} = \{A_{8,7,7,2}, A_{9,5,5,2}, A_{9,7,6,2}, A_{9,7,7,1} | A_{7,7,7,2}, A_{9,6,6,2}, A_{9,7,5,2}, A_{9,7,7,0}\}.$$

Example 34. For $x = y = 3, z = 1$ Lemma 32 gives us

$$A_{3,3,1} = \{A_{2,2,1}, A_{3,1,1}, A_{3,3,0} | A_{1,1,1}, A_{3,2,1}\},$$

as illustrated in Figure 25.

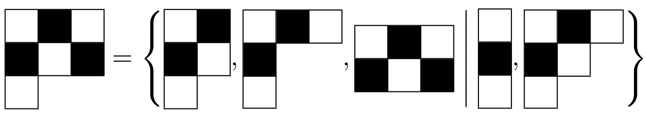

Figure 25: Concept of Lemma 32.

3.2.1 Two-Row Chess-Colored Young Diagrams

In this section we will prove that a partizan poset game played on a chess-colored two-row Young diagram is easy to compute by providing a formula to compute any such game $A_{x,y}$.

Theorem 35. *There is a formula, given by (2), to compute the value of partizan poset games played on chess-colored two-row Young diagrams.*

$$A_{x,y} = \frac{2}{5} + \frac{1}{15}2^{-(2y-2)}(-1)^y - \frac{1}{3}2^{-(x+y-1)}(-1)^x \tag{2}$$

This is proved by using Lemma 32 to reduce the game $A_{x,y}$ to two options per player, and, inductively, using the formula to determine which option is dominating to reduce the game to one option per player. Then, using Theorem 25, it is just a matter of showing that the formula in fact yields the simplest number between the two dominating options.

Proof of Theorem 35. Assume that Theorem 35 is true for all options of a game $A_{x,y}$. We will then show that the theorem then also holds for the game itself. By Conway Induction, this yields that Theorem 35 is true. From Lemma 32 we have that the dominating options of $A_{x,y}$ are to remove the greatest possible element in one of the rows.
Assume $x \geq y+2, y \geq 2$. This yields that $A_{x-2,y}, A_{x-1,y}, A_{x,y-1}, A_{x,y-2}$ are the dominating options of $A_{x,y}$. We want to find out when which options are dominating. We do that by comparing the differences between the values of the options using (2), to see which is greater and when.
If x and y are both even, then $A_{x,y} = \{A_{x-2,y}, A_{x,y-1} | A_{x-1,y}, A_{x,y-2}\}$. This yields the following differences between the option values:

$$\begin{aligned}
A_{x-2,y} - A_{x,y-1} &= 2^{-(2y-2)}(-1)^y \frac{1}{3}\Big(1 - (-2)^{-(x-y)}\Big) \\
&\geq 0 \text{ if } x \geq y \text{ and } y \text{ is even} \\
&\leq 0 \text{ if } x \geq y \text{ and } y \text{ is odd} \\
A_{x-1,y} - A_{x,y-2} &= 2^{-(2y-2)}(-1)^y \Big(-1 + (-2)^{-(x-y)}\Big) \\
&\geq 0 \text{ if } x \geq y \text{ and } y \text{ is odd} \\
&\leq 0 \text{ if } x \geq y \text{ and } y \text{ is even}
\end{aligned}$$

Since $x \geq y$, clearly $A_{x-2,y}$ dominates $A_{x,y-1}$ for Left and $A_{x-1,y}$ dominates $A_{x,y-2}$ for Right, and hence $A_{x,y} = \{A_{x-2,y} | A_{x-1,y}\}$ if x and y are even.
Similarly, if x and y are both odd, then $A_{x,y} = \{A_{x-1,y}, A_{x,y-2} | A_{x-2,y}, A_{x,y-1}\}$, and from the equations above, we have that $A_{x-1,y}$ dominates $A_{x,y-2}$ for Left and $A_{x-2,y}$ dominates $A_{x,y-1}$ for Right. Hence $A_{x,y} = \{A_{x-1,y} | A_{x-2,y}\}$ if x and y are odd.

For the other combination of parities of x and y, we have the following value differences:

$$\begin{aligned}
A_{x-2,y} - A_{x,y-2} &= (-4)^{-(y-1)} \\
&> 0 \text{ if } y \text{ is odd} \\
&< 0 \text{ if } y \text{ is even} \\
A_{x-1,y} - A_{x,y-1} &= 2^{-(2y-2)}(-1)^y \frac{1}{3}\left(1 + 2(-2)^{-(x-y)}\right) \\
&\geq 0 \text{ if } x \geq y \text{ and } y \text{ is even} \\
&\leq 0 \text{ if } x \geq y \text{ and } y \text{ is odd}
\end{aligned}$$

Clearly, $A_{x-2,y}, A_{x,y-1}$ are options of Left (Right) only if x, y are even (odd), and $A_{x-1,y}, A_{x,y-2}$ are options of Left (Right) only if x, y are odd (even). This yields that $A_{x-2,y}, A_{x-1,y}$ always dominate $A_{x,y-1}, A_{x,y-2}$, given that $x \geq y + 2$.
In other words, if $x \geq y + 2$,

$$A_{x,y} = \begin{cases} \{A_{x-2,y} | A_{x-1,y}\} & \text{if } x \text{ is even,} \\ \\ \{A_{x-1,y} | A_{x-2,y}\} & \text{if } x \text{ is odd.} \end{cases}$$

Using Theorem 25, we therefore only need to show that the value of $A_{x,y}$ given by (2) is the same as the simplest number in between the options above. If we compare the dominating options by examining the difference between their values, and the proposed value of the game, we get:

$$\begin{aligned}
A_{x-2,y} - A_{x-1,y} &= 2^{-(x+y-1)}\left(-2(-1)^x\right) \\
A_{x-2,y} - A_{x,y} &= 2^{-(x+y-1)}\left(-(-1)^x\right) \\
A_{x,y} - A_{x-1,y} &= 2^{-(x+y-1)}\left(-(-1)^x\right)
\end{aligned} \tag{*}$$

In other words we can see that the absolute difference in the numerator between the option values is 2, and that the proposed value is the only value with the same denominator in between the value of the options. From Definition 24 we can conclude that the proposed value for $A_{x,y}$ in fact is the simplest number between the options, and hence that Equation (2) holds for $x \geq y + 2, y \geq 2$.
Now, assume $x \geq y + 2, y = 1$. We can then reduce $A_{x,y}$ to:

$$A_{x,y} = \begin{cases} \{A_{x-2,1} | A_{x-1,1}, A_{x,0}\} & \text{if } x \text{ is even} \\ \\ \{A_{x-1,1} | A_{x-2,1}, A_{x,0}\} & \text{if } x \text{ is odd} \end{cases}$$

If we compare the value yields the value differences:

$$\begin{aligned}
A_{x-1,1} - A_{x,0} &= \frac{1}{3}\left(-1 + (-2)^{-(x-2)}\right) \\
&\leq 0 \text{ if } x \geq 2 \text{ is even} \\
A_{x-2,1} - A_{x,0} &= \frac{1}{3}\left(-1 + (-2)^{-(x-1)}\right) \\
&\leq 0 \text{ if } x \geq 1
\end{aligned}$$

As we can see, playing in the first row always dominates playing in the second row for Right if $x \geq y+2, y=1$. In other words, we have the game

$$A_{x,y} = \begin{cases} \{A_{x-2,1} | A_{x-1,1}\} & \text{if } x \text{ is even,} \\ \\ \{A_{x-1,1} | A_{x-2,1}\} & \text{if } x \text{ is odd.} \end{cases}$$

Comparing these options with the proposed game value yields the same results as in (*), but substituted with $y = 1$, and hence (2) holds for $x \geq y+2, y=1$ as well.
Now, assume $x \geq y+2, y=0$. Then

$$A_{x,y} = \begin{cases} \{A_{x-2,0} | A_{x-1,0}\} & \text{if } x \text{ is even,} \\ \\ \{A_{x-1,0} | A_{x-2,0}\} & \text{if } x \text{ is odd.} \end{cases}$$

As with $y = 1$, comparing the values of these options with the proposed game value yields the same results as in (*), but substituted with $y = 0$, and hence (2) holds for $x \geq y+2, y=0$ as well.
We can therefore conclude that (2) holds for $x \geq y+2, y \geq 0$.
Now, assume $x = y+1, y \geq 2$.We then have the following game:

$$A_{x,y} = A_{y+1,y} = \begin{cases} \{A_{y,y}, A_{y+1,y-1} | A_{y-1,y-1}, A_{y+1,y-2}\} & \text{if } y \text{ is even} \\ \\ \{A_{y-1,y-1}, A_{y+1,y-2} | A_{y,y}, A_{y+1,y-1}\} & \text{if } y \text{ is odd} \end{cases}$$

Comparing the options yields the differences:

$$\begin{aligned} A_{y,y} - A_{y+1,y-1} &= 0 \\ A_{y-1,y-1} - A_{y+1,y-2} &= (-4)^{-(y-1)} \\ &> 0 \text{ if } y \text{ is odd} \\ &< 0 \text{ if } y \text{ is even} \end{aligned}$$

As $A_{y-1,y-1}, A_{y+1,y-2}$ are options of Left (Right) only if y is odd (even), clearly $A_{y-1,y-1}$ dominates over $A_{y+1,y-2}$. Moreover, since $A_{y,y} = A_{y+1,y-1}$, then they dominate each other, so we can choose to always play in the first row here as well. This lets us reduce the game to

$$A_{x,y} = A_{y+1,y} = \begin{cases} \{A_{y,y} | A_{y-1,y-1}\} & \text{if } y \text{ is even,} \\ \\ \{A_{y-1,y-1} | A_{y,y}\} & \text{if } y \text{ is odd.} \end{cases}$$

If we compare the values of the dominating options and the proposed game value the same way as before, we have:

$$\begin{aligned} A_{y,y} - A_{y-1,y-1} &= 4^{-y}\left(-2(-1)^y\right) \\ A_{y,y} - A_{y+1,y} &= 4^{-y}\left(-(-1)^y\right) \\ A_{y+1,y} - A_{y-1,y-1} &= 4^{-y}\left(-(-1)^y\right) \end{aligned}$$

As before, this yields that the proposed value of the game is the simplest number between its dominating options, and hence (2) holds for $x = y+1, y \geq 2$.

Now, assume $x = y, y \geq 2$. This yields the game

$$A_{x,y} = A_{y,y} = \begin{cases} \{A_{y-2,y-2}, A_{y,y-1} | A_{y-1,y-1}, A_{y,y-2}\} & \text{if } y \text{ is even,} \\ \{A_{y-1,y-1}, A_{y,y-2} | A_{y-2,y-2}, A_{y,y-1}\} & \text{if } y \text{ is odd.} \end{cases}$$

Comparing the values of these options yields the differences:

$$\begin{aligned} A_{y-2,y-2} - A_{y,y-1} &= (-4)^{-(y-1)} \\ &> 0 \text{ if } y \text{ is odd} \\ &< 0 \text{ if } y \text{ is even} \\ A_{y-1,y-1} - A_{y,y-2} &= 0 \end{aligned}$$

As $A_{y-2,y-2}, A_{y,y-1}$ are options of Left (Right) only if y is even (odd), then clearly $A_{y,y-1}$ dominates over $A_{y-2,y-2}$, i.e., it is dominating to play in the second row. Since $A_{y-1,y-1} = A_{y,y-2}$, we can choose to always play in the second row here too. This yields the reduced game

$$A_{x,y} = A_{y,y} = \begin{cases} \{A_{y,y-1} | A_{y,y-2}\} & \text{if } y \text{ is even,} \\ \{A_{y,y-2} | A_{y,y-1}\} & \text{if } y \text{ is odd.} \end{cases}$$

If we compare the values of these options and the proposed value of the game as before, we get:

$$\begin{aligned} A_{y,y-1} - A_{y,y-2} &= 4^{-y}\left(-4(-1)^y\right) \\ A_{y,y-1} - A_{y,y} &= 4^{-y}\left(-2(-1)^y\right) \\ A_{y,y} - A_{y,y-2} &= 4^{-y}\left(-2(-1)^y\right) \end{aligned}$$

In analogy with before, this yields that the proposed value of the game is the simplest number between its dominating options, and hence (2) holds for $x = y, y \geq 2$.
Now, the only cases left are when $x < y + 2, y < 2$, a finite number of cases. It is therefore sufficient to check by hand if (2) holds for these four cases. For these cases, i.e., $(x, y) \in \{(0, 0), (1, 1), (1, 0), (2, 1)\}$, we have, respectively,

$$\begin{aligned} A_{0,0} &= \{|\} = 0 \\ A_{1,1} &= \{A_{0,0} | A_{1,0}\} = \{0|1\} = \frac{1}{2}, \\ A_{1,0} &= \{A_{0,0}|\} = \{0|\} = 1, \\ A_{2,1} &= \{A_{0,0} | A_{1,1}, A_{2,0}\} = \left\{0 \middle| \frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{4}, \end{aligned}$$

which are all equal to the formula proposed values.
As we can see that (2) also holds for these for cases, we can conclude that it also holds for $x < y + 2, y < 2$, which also yields that it holds for any $x \geq y, y \geq 0$.
This completes the proof. $\square$

3.2.2 Three-Row Chess-Colored Young Diagrams

In this section we will prove that a partizan poset game played on a chess-colored three-row Young diagram is easy to compute by providing and proving correctness of a formula to compute the value of any such game $A_{x,y,z}$.

Theorem 36. *There is a formula, with equations given by (3), to compute the value of partizan poset games played on any chess-colored three-row Young diagrams.*

$$\begin{array}{lll}
\frac{237}{512} - 2^{-(2z+1)}\left(\frac{2^{z+1}}{3}\left((-1)^z - 2^{z-4}\right) - \frac{1}{5}\left((-1)^z - 4^{z-4}\right)\right) & & \\
\quad - 2^{-(y+z+1)}\left(2^{z+1}-1\right)(-1)^y\frac{1}{3}\left(1-(-2)^{y-z}\right) & x \geq y \geq z \geq 4 & \text{(3a)} \\
\quad - 2^{-x}(-1)^x\frac{1}{3}\left(1-(-2)^{x-y}\right) & & \\
\frac{119}{256} - 2^{-(y+4)}(-1)^y 5\left(1-(-2)^{y-4}\right) & & \\
\quad - 2^{-x}(-1)^x\frac{1}{3}\left(1-(-2)^{x-y}\right) & x \geq y \geq 4, z = 3 & \text{(3b)} \\
\frac{59}{128} - 2^{-(y+3)}(-1)^y\frac{7}{3}\left(1-(-2)^{y-4}\right) & & \\
\quad - 2^{-x}(-1)^x\frac{1}{3}\left(1-(-2)^{x-y}\right) & x \geq y \geq 4, z = 2 & \text{(3c)} \\
\frac{59}{128} - 2^{-x}(-1)^x\frac{1}{3}\left(1-(-2)^{x-4}\right) & x \geq 4, y = z = 3 & \text{(3d)} \\
\frac{29}{64} - 2^{-x}(-1)^x\frac{1}{3}\left(1-(-2)^{x-4}\right) & x \geq 4, y = 3, z = 2 & \text{(3e)} \\
\frac{15}{32} - 2^{-x}(-1)^x\frac{1}{3}\left(1-(-2)^{x-4}\right) & x \geq 4, y = z = 2 & \text{(3f)} \\
\frac{1}{2} - 2^{-(y+3)}(-1)^y\left(3-(-1)^y\right) & x = y, z = 1 & \text{(3g)} \\
\frac{1}{2} - 2^{-(y+3)}(-1)^y\left(-3-(-1)^y\right) & x = y+1, z = 1 & \text{(3h)} \\
\frac{1}{2} + \frac{1}{3}2^{-(y+1)} - \frac{1}{3}2^{-(x-1)}(-1)^x & x > y \geq z = 1 \text{ and } x \text{ is even} & \text{(3i)} \\
\frac{1}{2} - \frac{1}{3}2^{-(y+2)} - \frac{1}{3}2^{-x}(-1)^x & x > y \geq z = 1 \text{ and } x \text{ is odd} & \text{(3j)} \\
\frac{2}{5} + \frac{1}{15}4^{-(y-1)}(-1)^y - \frac{1}{3}2^{-(x+y-1)}(-1)^x & x \geq y \geq z = 0 & \text{(3k)} \\
\frac{35}{64} & x = y = z = 3 & \text{(3l)} \\
\frac{17}{32} & x = y = 3, z = 2 & \text{(3m)} \\
\frac{9}{16} & x = 3, y = z = 2 & \text{(3n)} \\
\frac{13}{32} & x = y = z = 2 & \text{(3o)}
\end{array}$$

We will prove the theorem by induction, assuming that it is true for any option of an arbitrary game $A_{x,y,z}$, and from this showing that it is then also true for the game itself. But first, we will prove a lemma that we will use in order to reduce the game by specifying when to play in which row.

Lemma 37. *Assuming Theorem 36 is true for all options of the game $A_{x,y,z}$, then the following is true.*

(i) Playing the option to remove the greatest element in the first row of your color is the dominating option if $x \geq y + z + 1$.

(ii) Playing the option to remove the greatest element in the second row of your color is the dominating option if $x < y + z + 1$ and $y > z$.

(iii) Playing the option to remove the greatest element in the third row of your color is the dominating option if $x < y + z + 1$ and $y = z$ and $z > 2$.

(iv) If $x < y + z + 1$ and $y = z = 1$, playing the option to remove the (only) element in the third row is the dominating option for Left and playing the option to remove the (only) element in the second row is the dominating option for Right.

(v) If $x < y + z + 1$ and $y = z = 2$, playing the option to remove the (only) white element in the second row is the dominating option for Left and playing the option to remove the (only) black element in the third row is the dominating option for Right.

We will provide a short example of the concept for a more intuitive understanding of Lemma 37:

Example 38.
With $x = 23, y = 10, z = 6$, Lemma 37 gives us that $A_{23,10,6} = \{A_{22,10,6} | A_{21,10,6}\}$, i.e., we play in the first row.
With $x = 11, y = 10, z = 6$, Lemma 37 gives us that $A_{11,10,6} = \{A_{11,8,6} | A_{11,9,6}\}$, i.e., we play in the second row.
With $x = 11, y = 6, z = 6$, Lemma 37 gives us that $A_{11,6,6} = \{A_{11,6,4} | A_{11,9,5}\}$, i.e., we play in the third row.
With $x = 5, y = 2, z = 2$, Lemma 37 gives us that $A_{5,2,2} = \{A_{4,2,2} | A_{3,2,2}\}$, i.e., we play in the first row.
With $x = 4, y = 2, z = 2$, Lemma 37 gives us that $A_{4,2,2} = \{A_{4,1,1} | A_{4,2,1}\}$, i.e., we play in the second row.

Proof of Lemma 37. Assume Theorem 36 holds for all options of a game $A_{x,y,z}$. By Lemma 32, we only need to look at the options when removing maximal elements. We will begin to show when it is better to play in the second and third row, independently of the first row.
Assume $y = z$, we then have the following dominating options when playing in the

second and third row:

$$A_{x,z-1,z-1}, A_{x,z-2,z-2}, A_{x,z,z-2}, A_{x,z,z-1}$$

We will determine the dominance of the third row over the second by examining the differences between the values of the options, to determine when an option has a value with greater value. Since we have different formulas for different z's, we have to check a combination of the equations on (3). But they all give the same result, namely the following:

$$\begin{aligned} A_{x,z,z-2} - A_{x,z-1,z-1} &= 0 \text{ if } z \geq 3 \\ A_{x,z,z-1} - A_{x,z-2,z-2} &= \begin{cases} -(-4)^{-z} & \text{if } z \geq 3 \text{ and } x \geq 4 \\ 2^{-5} & \text{if } z = 3 \text{ and } x = 3 \end{cases} \\ &> 0 \text{ if } z \text{ odd} \\ &< 0 \text{ if } z \text{ even} \end{aligned}$$

Clearly $A_{x,z-2,z-2}, A_{x,z,z-1}$ are options of Left (Right) only if z is odd (even), so $A_{x,z-2,z-2}$ is dominated by $A_{x,z,z-1}$. Moreover, since $A_{x,z-1,z-1} = A_{x,z,z-2}$, then $A_{x,z-1,z-1}$ is dominated by $A_{x,z,z-2}$ (and vice versa). These results are evidently independent of x, so we have that playing in the second row is always dominated by playing in the third row if $y = z \geq 3$.
If $y = z = 2$ (3) yields:

$$\begin{aligned} A_{x,2,0} - A_{x,1,1} &= 2^{-3}\frac{1}{3}\left(-1 + (-2)^{-(x-2)}\right) \\ &\leq 0 \text{ if } x \geq 2 \\ A_{x,2,1} - A_{x,0,0} &= \begin{cases} -2^{-3} & \text{if } x \geq 3 \\ -2^{-4} & \text{if } x = 2 \end{cases} \\ &< 0 \end{aligned}$$

As we can see, this is consistent with Lemma 37 (v). Moreover, if $y = z = 1$, then Left only has the option to remove the element in the third row, and Right only has the option to remove the element in the second row, which is consistent with Lemma 37 (iv).
If we instead have $y > z$, we have the following dominating options when playing in the second or third row:

$$\begin{cases} A_{x,y-1,z}, A_{x,y-2,z}, A_{x,y,z-2}, A_{x,y,z-1} & \text{if } y \geq z+2 \\ \\ A_{x,z-1,z-1}, A_{x,z,z}, A_{x,z+1,z-2}, A_{x,z+1,z-1} & \text{if } y = z+1 \end{cases}$$

If $y \geq z+2$ and $z \geq 3$ we have the following possible value differences between the options:

$$\begin{aligned} A_{x,y-1,z} - A_{x,y,z-2} &= \begin{cases} \frac{1}{3}(-2)^{-x} - \frac{5}{3}(-2)^{-(y+3)} - 2^{-6} & \text{if } x > y, z = 3 \text{ and } y \text{ is even} \\ -2^{-(y+3)} - 2^{-6} & \text{if } x = y, z = 3 \text{ and } x \text{ is odd} \\ 2^{-z}\left(-(-2)^{-y} + (-2)^{-z}\right) & \text{othwerwise} \end{cases} \\ &> 0 \text{ if } y > z \text{ and } z \text{ is even} \\ &< 0 \text{ if } y > z \text{ and } z \text{ is odd} \end{aligned}$$

$$
\begin{aligned}
A_{x,y-1,z} - A_{x,y,z-1} &= 2^{-z}\frac{1}{3}\left(-2(-2)^{-y} - (-2)^{-z}\right)\\
&\geq 0 \text{ if } y > z \text{ and } z \text{ is odd}\\
&\leq 0 \text{ if } y > z \text{ and } z \text{ is even}\\
A_{x,y-2,z} - A_{x,y,z-2} &= \begin{cases} \frac{1}{3}(-2)^{-x} - \frac{1}{3}(-2)^{-(y+2)} - 2^{-6} & \text{if } x > y, z = 3 \text{ and } y \text{ is even}\\ -2^{-(y+2)} - 2^{-6} & \text{if } x = y, z = 3 \text{ and } x \text{ is odd}\\ (-4)^{-z} & \text{othwerwise}\end{cases}\\
&> 0 \text{ if } z \text{ even}\\
&< 0 \text{ if } z \text{ odd}\\
A_{x,y-2,z} - A_{x,y,z-1} &= 2^{-z}\frac{1}{3}\left((-2)^{-y} - (-2)^{-z}\right)\\
&> 0 \text{ if } y > z \text{ and } z \text{ odd}\\
&< 0 \text{ if } y > z \text{ and } z \text{ even}
\end{aligned}
$$

Similarly, for $z < 3$ we also have

$$
\begin{aligned}
A_{x,y-1,2} - A_{x,y,0} &> 0\\
A_{x,y-2,2} - A_{x,y,0} &> 0\\
A_{x,y-2,2} - A_{x,y,1} &\leq 0\\
A_{x,y-1,2} - A_{x,y,1} &< 0\\
A_{x,y-1,1} - A_{x,y,0} &> 0\\
A_{x,y-2,1} - A_{x,y,0} &> 0
\end{aligned}
$$

As $A_{x,y-1,z}, A_{x,y,z-2}$ are options of Left (Right) only if y, z are even (odd) and $A_{x,y-2,z}, A_{x,y,z-1}$ are options of Left (Right) only if y, z are odd (even), then clearly $A_{x,y,z-2}, A_{x,y,z-1}$ are dominated by $A_{x,y-1,z}, A_{x,y-2,z}$. Again, these results where independent of x.
Moreover, if $y = z + 1$ and $z \geq 2$, we have the following differences:

$$
\begin{aligned}
A_{x,z-1,z-1} - A_{x,z+1,z-2} &= \begin{cases} (-4)^{-z} & \text{if } z \geq 4\\ \frac{1}{3}\left((-2)^{-x} - 2^{-4}\right) & \text{if } z = 3 \text{ and } x > y\\ -2^{-6} & \text{if } z = 3 \text{ and } x = y\\ -2^{-2}\frac{1}{3}\left(-7(-2)^{-x} + 9\cdot 2^{-2}\right) & \text{if } z = 2\end{cases}\\
&> 0 \text{ if } z \text{ even}\\
&< 0 \text{ if } z \text{ odd}\\
A_{x,z,z} - A_{x,z+1,z-1} &= 0 \text{ if } z \geq 2
\end{aligned}
$$

If $z = 1$ and $y = z + 1$, we have the value differences

$$
\begin{aligned}
A_{x,y-1,z} - A_{x,y,z-2} = A_{x,1,1} - A_{x,2,0} &= 2^{-1}\frac{1}{3}\left(-(-2)^{-x} + 2^{-2}\right)\\
&\geq 0
\end{aligned}
$$

Again, we can see that $A_{x,z-1,z-1}, A_{x,z+1,z-2}$ are options of Left (Right) only if z is even (odd) and $A_{x,z,z}, A_{x,z+1,z-1}$ are options of Left (Right) only if z is odd (even),

so clearly $A_{x,z+1,z-2}, A_{x,z+1,z-1}$ are dominated by $A_{x,z-1,z-1}, A_{x,z,z}$.
Lastly, if $z = 0$, there are no option to play in the third row, so playing in the second row will always be a better strategy.
Since these scenarios are independent of x, we can conclude that playing in the second row is always dominated by playing in the third row if $y > z$ and $z \geq 0$, which is consistent with the lemma.

Now we only need to deduce when to play in the first row, and when to play in the second or third row. Assume that $x \geq y + 2$. For $y \geq 3, z \geq 2$ we then have the following dominating options:

$$\begin{aligned}
&A_{x-2,y,z}, A_{x-1,y,z}, A_{x,y-1,z}, A_{x,y-2,z} && \text{if } y \geq z+2 \\
&A_{x-2,y,y-1}, A_{x-1,y,y-1}, A_{x,y-1,y-1}, A_{x,y-2,y-2} && \text{if } y = z+1 \\
&A_{x-2,y,y}, A_{x-1,y,y}, A_{x,y,y-2}, A_{x,y,y-1} && \text{if } y = z
\end{aligned}$$

If we compare the values of the options for the three scenarios, it turns out that they have the same differences:

$$\begin{aligned}
A_{x-2,y,z} - A_{x,y-1,z} &= A_{x-2,y,y-1} - A_{x,y-1,y-1} = A_{x-2,y,y} - A_{x,y,y-2} \\
&= -2^{-x}(-1)^x + 2^{-(y+z+1)}(-1)^y \\
&\geq 0 \text{ if } \begin{cases} x \geq y+z+1 \text{ and } y \text{ even} \\ x \leq y+z+1 \text{ and } x \text{ odd} \end{cases} \\
&\leq 0 \text{ if } \begin{cases} x \geq y+z+1 \text{ and } y \text{ odd} \\ x \leq y+z+1 \text{ and } x \text{ even} \end{cases} \\
A_{x-1,y,z} - A_{x,y-2,z} &= A_{x-1,y,y-1} - A_{x,y-2,y-2} = A_{x-1,y,y} - A_{x,y,y-1} \\
&= 2^{-x}(-1)^x - 2^{-(y+z+1)}(-1)^y \\
&\geq 0 \text{ if } \begin{cases} x \geq y+z+1 \text{ and } y \text{ odd} \\ x \leq y+z+1 \text{ and } x \text{ odd} \end{cases} \\
&\leq 0 \text{ if } \begin{cases} x \geq y+z+1 \text{ and } y \text{ even} \\ x \leq y+z+1 \text{ and } x \text{ even} \end{cases} \\
A_{x-2,y,z} - A_{x,y-2,z} &= A_{x-2,y,y-1} - A_{x,y-2,y-2} = A_{x-2,y,y} - A_{x,y,y-1} \\
&= -2^{-x}(-1)^x - 2^{-(y+z+1)}(-1)^y \\
&\geq 0 \text{ if } \begin{cases} x \geq y+z+1 \text{ and } y \text{ even} \\ x \leq y+z+1 \text{ and } x \text{ even} \end{cases} \\
&\leq 0 \text{ if } \begin{cases} x \geq y+z+1 \text{ and } y \text{ odd} \\ x \leq y+z+1 \text{ and } x \text{ odd} \end{cases} \\
A_{x-1,y,z} - A_{x,y-1,z} &= A_{x-1,y,y-1} - A_{x,y-1,y-1} = A_{x-1,y,y} - A_{x,y,y-2} \\
&= 2^{-x}(-1)^x + 2^{-(y+z+1)}(-1)^y \\
&\geq 0 \text{ if } \begin{cases} x \geq y+z+1 \text{ and } y \text{ odd} \\ x \leq y+z+1 \text{ and } x \text{ even} \end{cases} \\
&\leq 0 \text{ if } \begin{cases} x \geq y+z+1 \text{ and } y \text{ even} \\ x \leq y+z+1 \text{ and } x \text{ odd} \end{cases}
\end{aligned}$$

Obviously $A_{x-2,y,z}, A_{x,y-1,z}, A_{x,y-1,y-1}, A_{x,,y,y-2}$ are options of Left (Right) only if

x, y are even (odd) and $A_{x-1,y,z}, A_{x,y-2,z}, A_{x,y-2,y-2}, A_{x,y,y-1}$ are options of Left (Right) only if x, y are odd (even). We can therefore clearly see that $A_{x-2,y,z}, A_{x-1,y,z}$ dominate over $A_{x,y-1,z}, A_{x,y-2,z}, A_{x,y-1,y-1}, A_{x,y-2,y-2}, A_{x,y,y-2}, A_{x,y,y-1}$ if $x \geq y + z + 1$ and reversely if $x \leq y + z + 1$. In other words, when $x \geq y + 2, y \geq 3, z \geq 2$, playing in the first row is the dominating option if $x \geq y + z + 1$ and playing in the second or third row is the dominating option if $x < y + z + 1$, which is consistent with the lemma.

What if $x < y + 2$? Since any game option $A_{x',y,z}$ where $x' < y + 2$ is an option of any option $A_{x,y,z}$ where $x \geq y + 2$, then $A_{x',y,z}$ must be dominated by $A_{x,y,z}$. With $z \geq 2$, all $A_{x',y,z}$ such that $x' < y + 2$ are options of some $A_{x,y,z}, x \geq y + 2$. Therefore the above is also valid for $x < y + 2$. That is, when $y \geq 3, z \geq 2$, playing in the first row dominates by the option of playing in the second or third row if $x \geq y + z + 1$ and playing in the second or third row dominates playing in the first row if $x < y + z + 1$.

Now, assume $y = z = 2$. The game is then given by

$$A_{x,y,z} = A_{x,2,2} = \begin{cases} \{A_{x-2,2,2}, A_{x,1,1} | A_{x-1,2,2}, A_{x,2,1}\} & \text{if } x \geq y + 2 \text{ and } x \text{ is even} \\ \{A_{x-1,2,2}, A_{x,1,1} | A_{x-2,2,2}, A_{x,2,1}\} & \text{if } x \geq y + 2 \text{ and } x \text{ is odd} \\ \{A_{2,2,2}, A_{3,1,1} | A_{1,1,1}, A_{3,2,1}\} & \text{if } x = 3 \\ \{A_{0,0,0}, A_{2,1,1} | A_{1,1,1}, A_{2,2,1}\} & \text{if } x = 2 \end{cases}$$

Clearly, the first game is only valid if $x \geq 4$ and the second only if $x \geq 5$. We therefore only need to examine these options for these values. The possible value differences between the dominating options are then:

$$\begin{aligned} A_{x-2,2,2} - A_{x,1,1} &= -2^{-x}(-1)^x + 2^{-5} \\ &\geq 0 \text{ if } x \geq 5 = y + z + 1 \\ &\leq 0 \text{ if } x = 4 \\ A_{x-1,2,2} - A_{x,2,1} &= 2^{-5}\frac{1}{3}\left(-5 - (-2)^{7-x}\right) \\ &> 0 \text{ if } x = 4 \\ &< 0 \text{ if } x \geq 5 = y + z + 1 \\ A_{x-2,2,2} - A_{x,2,1} &= 2^{-5}\frac{1}{3}\left(-5 - (-2)^{6-x}\right) \\ &< 0 \text{ if } x \geq 5 = y + z + 1 \\ A_{x-1,2,2} - A_{x,1,1} &= 2^{-x}(-1)^x + 2^{-5} \\ &\geq 0 \text{ if } x \geq 5 = y + z + 1 \\ A_{2,2,2} - A_{3,1,1} &= -\frac{3}{32} \\ &< 0 \\ A_{1,1,1} - A_{3,2,1} &= \frac{1}{8} \\ &> 0 \\ A_{0,0,0} - A_{2,1,1} &= -\frac{3}{8} \\ &< 0 \end{aligned}$$

$$
\begin{aligned}
A_{1,1,1} - A_{2,2,1} &= \frac{5}{16} \\
&> 0
\end{aligned}
$$

These differences yields that playing in the first row is the dominating option if $x \geq y + z + 1$ and playing in the third row is the dominating option if $x < y + z + 1$, which is consistent with the lemma.
Similarly, for $z = 1$, this comparison methodology yields that playing in the first row is the dominating option if $x \geq y + z + 1$ and it is dominated by playing in the second or third row if $x < y + z + 1$.
Moreover, from the proof of Theorem 35, we can conclude that playing in the first row is the dominating option if $x > y$, and playing in the second row is the dominating strategy if $x = y$. Since here, $z = 0$, this is the same as saying that playing in the first row is the dominating strategy if $x \geq y + z + 1$ and playing in the second row is dominating if $x < y + z + 1$.
All this yields that, for any x, y, z, the dominating strategy is to play in the first row if $x \geq y + z + 1$ and in the second or third row if $x < y + z + 1$.
This completes the proof. $\square$

Now we can use Lemma 37 to prove Theorem 36. This will be done with a proof using Conway Induction (see Theorem 22). We do this by assuming that Theorem 36 holds for all options of an arbitrary game $A_{x,y,z}$ and then showing that it holds for the game $A_{x,y,z}$ itself.

Proof of Theorem 36. Assume that Theorem 36 holds for all options of the game $A_{x,y,z}$. We can then use the result of Lemma 37 to limit the number of options to one for each player. Using Theorem 25, we then just need to show for all of these that the value of $A_{x,y,z}$ computed with (3) of Theorem 36 is the same as the *simplest number* in between the values of the deduced dominating game options of the game.

Assume $x \geq y + z + 1$ and $x \geq y + 2$. We then have

$$
A_{x,y,z} = \begin{cases} \{A_{x-2,y,z} | A_{x-1,y,z}\} & \text{if } x \text{ is even,} \\ \\ \{A_{x-1,y,z} | A_{x-2,y,z}\} & \text{if } x \text{ is odd.} \end{cases}
$$

This yields that $A_{x,y,z}$ must be the simplest number between $A_{x-2,y,z}$ and $A_{x-1,y,z}$. We can compare these options and the game by comparing their computed values. For $z \geq 2$, these options have the value differences

$$
\begin{aligned}
A_{x-2,y,z} - A_{x-1,y,z} &= 2^{-x} \left(-2(-1)^x\right), \\
A_{x-2,y,z} - A_{x,y,z} &= 2^{-x} \left(-(-1)^x\right), \\
A_{x,y,z} - A_{x-1,y,z} &= 2^{-x} \left(-(-1)^x\right).
\end{aligned}
$$

Clearly the proposed value for $A_{x,y,z}$ is the only number with a numerator between those of $A_{x-2,y,z}$ and $A_{x-1,y,z}$ over the same denominator. Definition 24 yields that this then must be the simplest number.

Now, assume $x < y + z + 1$ and $y > z$. The game is then

$$A_{x,y,z} = \begin{cases} \left.\begin{array}{ll} \{A_{x,y-1,z}|A_{x,y-2,z}\} & \text{if } y \text{ is even} \\ \{A_{x,y-2,z}|A_{x,y-1,z}\} & \text{if } y \text{ is odd} \end{array}\right\} & \text{if } y \geq z+2, \\ \\ \left.\begin{array}{ll} \{A_{x,y-1,y-1}|A_{x,y-2,y-2}\} & \text{if } y \text{ is even} \\ \{A_{x,y-2,y-2}|A_{x,y-1,y-1}\} & \text{if } y \text{ is odd} \end{array}\right\} & \text{if } y = z+1. \end{cases}$$

If we compare the values of these games when $z \geq 2$ and $x \geq 4$ we have the differences

$$\begin{aligned} A_{x,y-1,z} - A_{x,y-2,z} &= A_{x,y-1,y-1} - A_{x,y-2,y-2} = \\ &= 2^{-(y+z+1)}\left(-2(-1)^y\right) \\ A_{x,y-1,z} - A_{x,y,z} &= A_{x,y-1,y-1} - A_{x,y,y-1} = \\ &= 2^{-(y+z+1)}\left(-(-1)^y\right) \\ A_{x,y,z} - A_{x,y-2,z} &= A_{x,y,y-1} - A_{x,y-2,y-2} = \\ &= 2^{-(y+z+1)}\left(-(-1)^y\right) \end{aligned}$$

Similarly, if $z \geq 2$ and $x < 4$, i.e., if $A_{x,y,z} = A_{3,3,2}$, we have

$$\begin{aligned} A_{3,2,2} - A_{3,1,1} &= \frac{2}{32} \\ A_{3,2,2} - A_{3,3,2} &= \frac{1}{32} \\ A_{3,3,2} - A_{3,1,1} &= \frac{1}{32} \end{aligned}$$

Again, we can see that the proposed value for $A_{x,y,z}$ is the simplest number between $A_{x,y-1,z}$ and $A_{x,y-2,z}$ and $A_{x,y-1,y-1}$ and $A_{x,y-2,y-2}$. Now, assume $y = z$. Assuming $z \geq 3$ we then have the game

$$A_{x,y,z} = \begin{cases} \{A_{x,y,y-2}|A_{x,y,y-1}\} & \text{if } y \text{ is even,} \\ \\ \{A_{x,y,y-1}|A_{x,y,y-2}\} & \text{if } y \text{ is odd.} \end{cases}$$

Similarly, these options gives us the value differences

$$\begin{aligned} A_{x,y,y-2} - A_{x,y,y-1} &= 2^{-(y+z+1)}\left(-2(-1)^y\right), \\ A_{x,y,y-2} - A_{x,y,z} &= 2^{-(y+z+1)}\left(-(-1)^y\right), \\ A_{x,y,z} - A_{x,y,y-1} &= 2^{-(y+z+1)}\left(-(-1)^y\right). \end{aligned}$$

As before, this clearly yields that the proposed value of $A_{x,y,z}$ is the simplest value between its dominating options if $x < y + z + 1, y = z$ and $z \geq 3$.
If $y = z = 2$ and $x < y + z + 1$, then $x \in \{2, 3, 4\}$. Using Lemma 37, we have:

$$A_{x,y,z} = \begin{cases} \{A_{4,1,1}|A_{4,2,1}\} = \left\{\frac{7}{16}\middle|\frac{1}{2}\right\} & = \frac{15}{32} & \text{if } x = 4 \\ \{A_{3,1,1}|A_{3,2,1}\} = \left\{\frac{1}{2}\middle|\frac{5}{8}\right\} & = \frac{9}{16} & \text{if } x = 3 \\ \{A_{2,1,1}|A_{2,2,1}\} = \left\{\frac{3}{8}\middle|\frac{7}{16}\right\} & = \frac{13}{32} & \text{if } x = 2 \end{cases}$$

As these all correspond to the formula proposed values, this concludes that the formula works for $x \geq y+z+1$ if $x \geq y+2$ and $z \geq 2$, and for $x < y+z+1$ if $z \geq 2$. What if $x < y+2$?
If $z \geq 2$, then $y+z+1 > y+2$. So if $x < y+2$, then $x < y+z+1$ and hence, the dominating strategy is not to play in the first row. The above therefore holds for $x < y+2$ as well, and we can conclude that the formula works for all $A_{x,y,z}$ with $z \geq 2$.
What if $z < 2$, i.e., $z = 1$ or $z = 0$? The case when $z = 0$ follows from Theorem 35. Let us therefore assume that $z = 1$.
Assume $x \geq y+2$. Then $x \geq y+2 = y+z+1$, so the dominating strategy will be moving in the first row. This yields the game

$$A_{x,y,z} = \begin{cases} \{A_{x-2,y,1} | A_{x-1,y,1}\} & \text{if } x \text{ is even,} \\ \\ \{A_{x-1,y,1} | A_{x-2,y,1}\} & \text{if } x \text{ is odd.} \end{cases}$$

If we again compare these options, this yields the differences:

$$A_{x-2,y,1} - A_{x-1,y,1} = \begin{cases} 2^{-(x-1)}\left(-2(-1)^x\right) & \text{if } x \geq y+3 \text{ and } y \text{ is even} \\ 2^{-x}\left(-2(-1)^x\right) & \text{if } x \geq y+3 \text{ and } y \text{ is odd} \\ 2^{-x}\left(-3(-1)^x\right) & \text{if } x = y+2 \end{cases}$$

$$A_{x-2,y,1} - A_{x,y,1} = \begin{cases} 2^{-(x-1)}\left(-(-1)^x\right) & \text{if } x \geq y+3 \text{ and } y \text{ is even} \\ 2^{-x}\left(-(-1)^x\right) & \text{if } x \geq y+3 \text{ and } y \text{ is odd} \\ 2^{-x}\left(-\frac{3-(-1)^x}{2}(-1)^x\right) & \text{if } x = y+2 \end{cases}$$

$$A_{x-1,y,1} - A_{x-2,y,1} = \begin{cases} 2^{-(x-1)}\left(-(-1)^x\right) & \text{if } x \geq y+3 \text{ and } y \text{ is even} \\ 2^{-x}\left(-(-1)^x\right) & \text{if } x \geq y+3 \text{ and } y \text{ is odd} \\ 2^{-x}\left(-\frac{3+(-1)^x}{2}(-1)^x\right) & \text{if } x = y+2 \end{cases}$$

Again, we can clearly see that $A_{x,y,z}$ is the simplest number in between $A_{x-2,y,1}$ and $A_{x-1,y,1}$, if $z = 1$ and $x \geq y+3$. If $x = y+2$, from above, the proposed value of $A_{x,y,z} = A_{x,x-2,1}$ is between the two dominating options. Since the proposed value $A_{x,x-2,1} = \frac{1}{2}$ is the simplest possible number between these options, then the proposed value for $A_{x,y,z}$ is the simplest number between its dominating options if $x \geq y+2$ and $z = 1$.
If $x < y+2$, then $x < y+2 = y+z+1$. The dominating strategy will then to play in the second or third row. Assuming $y \geq z+2$, we have the game:

$$A_{x,y,z} = \begin{cases} \left.\begin{array}{ll} \{A_{x,x-3,1} | A_{x,x-2,1}\} & \text{if } x \text{ is even} \\ \\ \{A_{x,x-2,1} | A_{x,x-3,1}\} & \text{if } x \text{ is odd} \end{array}\right\} \text{if } x = y+1 \\ \\ \left.\begin{array}{ll} \{A_{x,x-1,1} | A_{x,x-2,1}\} & \text{if } x \text{ is even} \\ \\ \{A_{x,x-2,1} | A_{x,x-1,1}\} & \text{if } x \text{ is odd} \end{array}\right\} \text{if } x = y \end{cases}$$

Comparing these options as before yields:

$$A_{x,x-3,1} - A_{x,x-2,1} = \begin{cases} 2^{-(x+1)}\left(-2(-1)^x\right) & \text{if } x \text{ is even} \\ 2^{-x}\left(-2(-1)^x\right) & \text{if } x \text{ is odd} \end{cases}$$

$$A_{x,x-3,1} - A_{x,x-1,1} = \begin{cases} 2^{-(x+1)}\left(-(-1)^x\right) & \text{if } x \text{ is even} \\ 2^{-x}\left(-(-1)^x\right) & \text{if } x \text{ is odd} \end{cases}$$

$$A_{x,x-1,1} - A_{x,x-2,1} = \begin{cases} 2^{-(x+1)}\left(-(-1)^x\right) & \text{if } x \text{ is even} \\ 2^{-x}\left(-(-1)^x\right) & \text{if } x \text{ is odd} \end{cases}$$

$$A_{x,x-1,1} - A_{x,x-2,1} = \begin{cases} 2^{-(x+2)}\left(-2(-1)^x\right) & \text{if } x \text{ is even} \\ 2^{-(x+1)}\left(-2(-1)^x\right) & \text{if } x \text{ is odd} \end{cases}$$

$$A_{x,x-1,1} - A_{x,x,1} = \begin{cases} 2^{-(x+2)}\left(-(-1)^x\right) & \text{if } x \text{ is even} \\ 2^{-(x+1)}\left(-(-1)^x\right) & \text{if } x \text{ is odd} \end{cases}$$

$$A_{x,x,1} - A_{x,x-2,1} = \begin{cases} 2^{-(x+2)}\left(-(-1)^x\right) & \text{if } x \text{ is even} \\ 2^{-(x+1)}\left(-(-1)^x\right) & \text{if } x \text{ is odd} \end{cases}$$

Just as before, this yields that the proposed value of $A_{x,y,z}$ in fact is the simplest number between its dominated options when $z = 1, y \geq z + 2$ and $x < y + 2$.
Finally, we have the cases when $z = 1, y < z + 2$ and $x < y + 2$, i.e.,

$$A_{x,y,z} : (x, y, z) \in \{(3, 2, 1), (2, 1, 1), (2, 2, 1), (1, 1, 1)\}.$$

Again, since $x < y + 2 = y + z + 1$, the dominating strategy will be to move in the second or third row. This yields:

$$A_{3,2,1} = \{A_{3,1,1}|A_{3,0,0}\} = \left\{\frac{1}{2}\middle|\frac{3}{4}\right\} = \frac{5}{8}$$

$$A_{2,1,1} = \{A_{2,1,0}|A_{2,0,0}\} = \left\{\frac{1}{4}\middle|\frac{1}{2}\right\} = \frac{3}{8}$$

$$A_{2,2,1} = \{A_{2,1,1}|A_{2,0,0}\} = \left\{\frac{3}{8}\middle|\frac{1}{2}\right\} = \frac{7}{16}$$

$$A_{1,1,1} = \{A_{1,1,0}|A_{1,0,0}\} = \left\{\frac{1}{2}\middle|1\right\} = \frac{3}{4}$$

Since all of these are equal to the formula proposed values, then this yields that the formula is valid for any $x, y, z \geq 0$. This completes the proof of Theorem 36. □

4 Conclusions and Open Questions

With Theorem 35 we have that it is easy to compute the value of any partizan poset game played on a chess-colored two-row Young diagram, for which a formula was provided. With Theorem 36 we have proved that it is easy to compute the value of any partizan poset game played on a chess-colored three-row Young diagram.

The first question one might ask is why the formula to compute any three-row game is so complex?

When analyzing the three-row games in order to find a formula to compute their value, it was discovered that the value of the games were somewhat "chaotic" when the length of the third row of the games was below 4. Although no clear evidence for why have been found, one plausible explanation to this would be that games of lengths not exceeding 3 is the same if flipped, and it can be played from two different directions (from the right as usual, and from below). A short example is provided below to illustrate this.

Example 39.

The game $A_{3,3,1}$ is the same as the game $A_{3,2,2}$, and can therefore be seen to be played both from the right as usual, and from below, as illustrated in Figure 26.

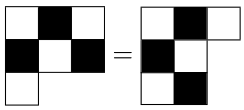

Figure 26

One thing that follows from the formulas of Theorems 35 and 36 is that the value of the games goes asymptotically toward a quotient when the length of the rows grows. In particular,

$$\lim_{n\to\infty} A_{n,0} = \frac{2}{3}, \qquad \lim_{n\to\infty} A_{n,n} = \frac{2}{5}, \qquad \lim_{n\to\infty} A_{n,n,n} = \frac{29}{60}.$$

This can be seen as a result of that the option to remove the greatest elements decreases as they are further from the root of the diagram. What one might ask is if this is still true for games with more than three rows? Intuitively, it seems very plausible for the above to be true, but it remains to be proved.

Another question concerning games with more than three rows is if it is possible to find formulas for these kind of games in general, for any number of rows. This is also something that seems very plausible, largely because of the regularity of the chess-coloring, but it is also something that remains to be proven. A follow-up question to this is also if such a formula will have the same issues as with the three-row-formula, that is, if the value of the games of more than three rows also will have some chaotic behavior when they are small enough?

Something other that would be interesting to investigate is also how the games are affected by other colorings, or with skew Young diagrams (a skew Young diagram is a Young diagram obtained by removing a smaller Young diagram from a larger one that contains the smaller one, see Figure 27 below). For games played on Young diagrams it is clearly very easy to determine the winner of the game, since it will always be the one with its color in the upper-left corner. The more interesting question is therefore if it is possible to say anything about the value of a game with some different coloring, for exampling a coloring with a more random nature.

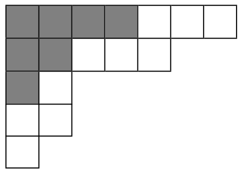

Figure 27: The non-gray boxes constitute a skew Young diagram.

Lastly, following the results of Theorem 31, that a player wants to play the options of removing an element as great as possible, we may note that this is very similar to the only allowed moves in pomax games when, with the difference of also being able to remove non-maximal elements as long as they are only smaller than elements of the opposite color. From these similarities, an interesting question is how much of the analysis of the pomax games that can be transferred to the regular partizan poset games.

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