Spacetime examples in **INLA**

Elias T. Krainski

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Outline

1. Separable space-time models
2. Infant mortality in Paraná
3. PM-10 concentration in Piemonte, Italy

Building spacetime models: simple way

- $\boldsymbol{x} = \{x_{11}, ..., x_{n1}, x_{12}, ..., x_{nT}\}$

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$$\pi(\boldsymbol{x}) \propto (|\boldsymbol{Q1} \otimes \boldsymbol{Q2}|^*)^{1/2} \exp\left(-\frac{1}{2}\boldsymbol{x}^T\{\boldsymbol{Q1} \otimes \boldsymbol{Q2}\}\boldsymbol{x}\right)$$

  where $|.|^*$ may be the generalized determinant when needed
- example:

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- 1st order dynamic models

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- E_i: expected number of death (under some suposition)
  - overal ratio

$$r_0 = \frac{\sum_{it} y_{it}}{\sum_{it} borns_{it}}$$

  - $E_{it} = r_0 borns_{it}$
  - E_{it}: expected deaths if the ratio is the same (over space and time)
  - observed relative risk

$$SMR_{it} = \frac{y_{it}}{E_{it}}$$

Model structure

- linear predictor evolution over time

$$x_{it} = \rho x_{i,t-1} + s_{it}$$

- s_{it} at each time $\rightarrow$ spatially correlated

$$s_{it}|s_{-i,t} \sim N(\sum_{j \sim i} s_{j,t}/n_i, \sigma_s^2/n_i)$$

- space-time precision matrix implied: $\boldsymbol{Q} = \boldsymbol{Q_T} \otimes \boldsymbol{Q_S}$
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- both smooth over time and space (if ρ is near 1)
- the full model

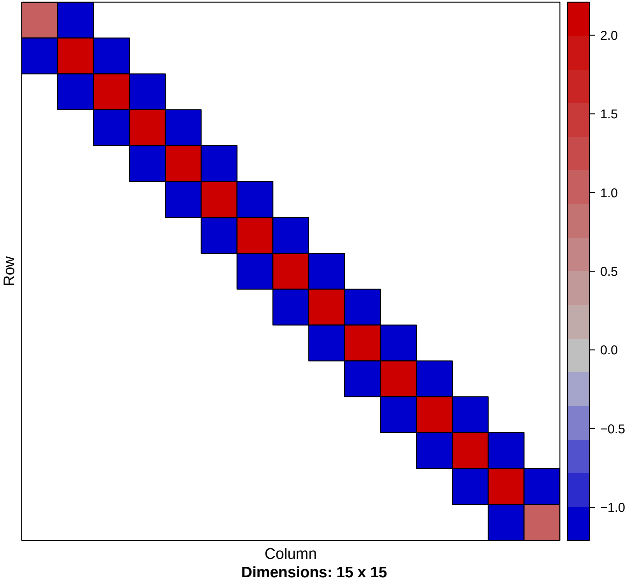
$$\eta_{it} = \alpha_0 + e_t + u_i + v_t + s_i + x_{it}$$

where

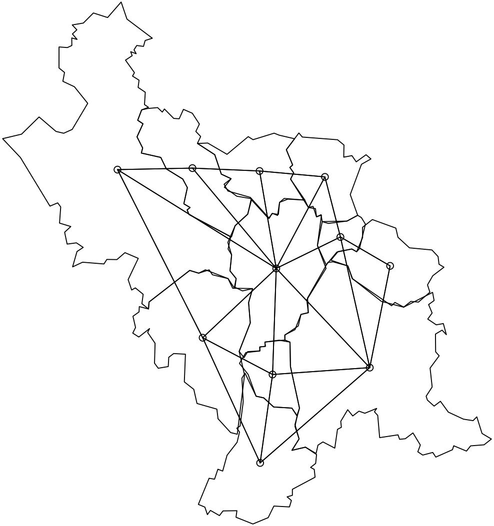
- α_0 is the intercept
- e_t is a unstructured temporal random effect
- u_i is a unstructured spatial random effect
- v_t is a structured temporal random effect
- s_t is a structured temporal random effect
- x_{it} is a space-time random effect

On space-time random effect

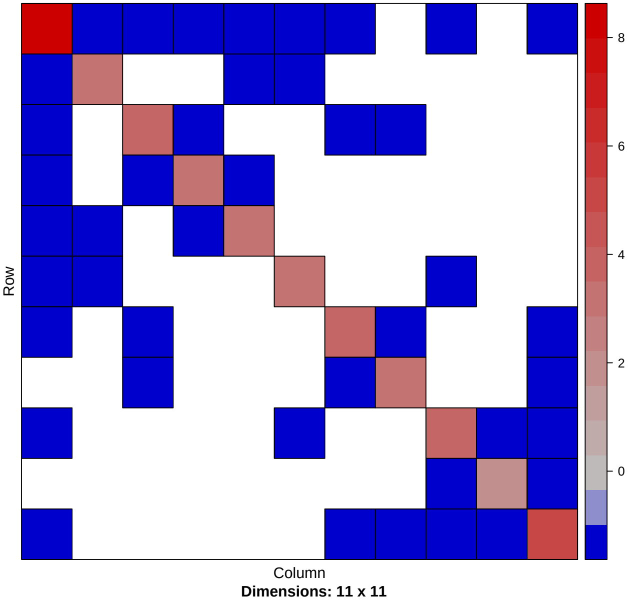
- it can be one of the four type interaction models
- dynamic model using the `besagproper2` model for space
    - $\lambda = 0$: no spatial structure
    - $\lambda = 1$: equals the intrinsic Besag
    - $\rho = 0$: no temporal structure
    - $\rho = 1$: equals RW1
    - $\rightarrow$ includes all the four interaction types



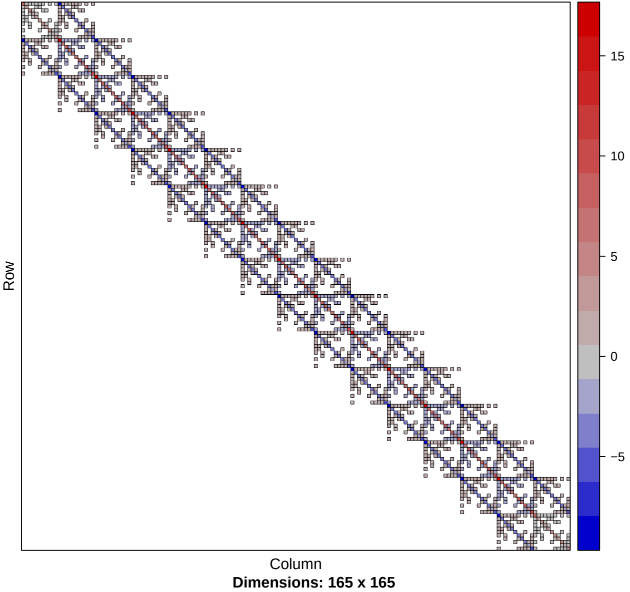
Temporal precision structure (for $\boldsymbol{Q}_T$)



Map example and the neighbourhood



Spatial precision structure (for $\boldsymbol{Q}_S$)



Spatio temporal precision structure (for $\boldsymbol{Q}$)

Five models for x_{it}

m_0 :	x_0	same ratio over space and time
m_1 :	$x_0 + x_{0,t}$	different ratio over time
m_2 :	$x_0 + x_{i,0}$	differet ratio over space
m_3 :	$x_0 + x_{0,t} + x_{i,0}$	common time trend + common sp. surface
m_4 :	$x_0 + x_{it}$	variation over space and time

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```
f0 <- y ~ 1
f1 <- y ~ 1 + f(t, model="ar1")
f2 <- y ~ 1 + f(i, model="besag", graph="...")
f3 <- y ~ 1 + f(t, model="ar1") +
    f(i, model="besag", graph="...")
f4 <- y ~ 1 + f(i, model="besag", graph="...",
          group=t, control.group=list(model="ar1"))
```

Outline

Space-time dynamic intercept

- The (linear) measurement equation

$$\boldsymbol{y}_{it} = F_{it}^{'}\beta + \boldsymbol{A}_{i(t)}\boldsymbol{x}_t + \epsilon_{it}$$

  - $\boldsymbol{F}_t$ is a matrix of covariates
  - $\boldsymbol{\beta}$ are the fixed effects
  - $\boldsymbol{A}_{(t)}$ picks out the appropriate values of $\boldsymbol{x}_t$
  - $\boldsymbol{\epsilon}_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2 \boldsymbol{I})$

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- vector AR(1) process for $\boldsymbol{x}$

$$\boldsymbol{x}_t = \rho\boldsymbol{x}_{t-1} + \boldsymbol{\omega}_t$$

  - $\boldsymbol{\omega_t}$: spatial SPDE model

$$\boldsymbol{\omega}_t \overset{\text{i.i.d.}}{\sim} N(\boldsymbol{0}, \boldsymbol{Q}^{-1}),$$

  - ρ is the time correlation

PM-10 concentration in Piemonte, Italy

Cameletti *et al.* (2011), on `r-inla.org`

- 24 monitoring stations
- Daily data from 10/05 to 03/06

Space model part

- Make the mesh

```
mesh <- inla.mesh.2d(points =NULL,
                     points.domain=borders,
                     offset=c(10, 140),
                     max.edge=c(40,1000))
```

- Make the latent model

```
spde = inla.create.spde(mesh,model="matern")
```

Using the group feature

- Construct a kronecker product model using the group feature

```
formula = y ~ -1 + intercept + WS + HMIX + ... +
   f(field, model=spde,
      group =time,
      control.group=list(model="ar1")
     )
```

- This tells INLA that the observations are grouped in a certain way.
- `control.group` contains the grouping model (`ar1`, `exchangable`, `rw1`, and others) as well as their prior specifications.

Make an **A** matrix

- Use the `group` argument

```
LocationMatrix = inla.spde.make.A(mesh = mesh,
          loc =dataLoc, group=time, n.group=nT)
```

- data locations in all `group=time` level
- builds an **A** matrix in an appropriate way

Organising the data

Covariates at the data points, but the latent field only defined their through the A matrix

*We need to make sure that **A** only applies to the random effect.*

```
idx.set <- inla.spde.make.index("mesh.idx",n.field=nmesh,
                  n.group=T)
stack = inla.stack( data = dat,
       A = list(1, LocationMatrix),
       effects = list( list(WS = cov$WS,...),
           c(idx.set,
                       list(intercept=rep(1,mesh$n*nT)))
       )
    )
```