# A note on greedy algorithms for maximum weighted independent set problem

Version 2

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**Abstract**

In this paper, we consider three simple and natural greedy algorithms for the Maximum Weighted Independent Set problem. We show that two of them output an independent set of weight $\geq \sum_{v \in V(G)} \frac{W(v)}{d(v)+1}$ and the other outputs an independent set of weight $\geq \sum_{v \in V(G)} \frac{W(v)^2}{\sum_{u \in N_G^+(v)} W(u)}$. Those results are generalization of Turán's theorem.

## 1 Introduction

The Maximum (Weighted) Independent Set (MIS(MWIS)) is one of most important optimization problems [7, 8]. In several heuristic methods for optimization problems, greedy strategy is most natural and simplest one. For MIS, two simple greedy algorithms were investigated. One is called *MIN*, which selects a vertex of minimum degree, removes it and its neighbors from the graph, and iterates this process on the remaining graph until no vertex remains. The other is called *MAX*, which deletes a vertex of maximum degree until no edge remains.

Halldórsson and Radhakrishnan showed that for graphs $G$ with degree bounded by $\Delta$ *MIN* always outputs an independent set of size at least $\frac{3}{\Delta+2} \times \alpha(G)$ where $\alpha(G)$ is the size of an maximum independent set in $G$. They also demonstrated that the ratio is tight [6]. Griggs [5] and Chvátal and C. McDiarmid [3] proved independently that *MAX* outputs an independent set of size at least $\sum_{v \in V} \frac{1}{d(v)+1}$ for any graph $G$. This implies that *MAX* always outputs an independent set of size at least $\frac{1}{\Delta+1} \times \alpha(G)$ for graphs with degree bounded by $\Delta$. Halldórsson and Radhakrishnan showed that the ratio is at most $\frac{2}{\Delta+1}$ [6].

In this paper, we consider three simple greedy algorithms for MWIS. In section 2, we review terminology and concepts used throughout the paper. In section 3, first we give two simple algorithms *GWMIN* and *GWMAX* which are generalization of *MIN* and *MAX* respectively. Then we show that both greedy algorithms output an independent set of weight $\geq \sum_{v \in V(G)} \frac{W(v)}{d(v)+1}$. This can be considered as a natural extension of Turán's theorem. We also give another simple greedy algorithm, which outputs an independent set of weight $\geq \sum_{v \in V(G)} \frac{W(v)^2}{\sum_{u \in N_G^+(v)} W(u)}$. This can be also thought as an extension of Turán's theorem.

## 2 Definitions

Let $G = (V, E, W)$ be a weighted undirected graph without loops and multiple edges, where $V$ is the set of vertices, $E$ is the set of edges, and $W$ is the vertex weighting function such that $W : 2^V \rightarrow Z^+$

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(i.e. each vertex has a positive integral weight), $W(u) \in Z^+$ for all $u \in V$ and $W(S) = \sum_{u \in S} W(u)$ for $S \subseteq V$. We also use the notation $V(G)$ and $E(G)$ to denote the set of vertices and edges in $G$. For a subset $S \subseteq V$, $W(S)$ and $|S|$ are referred as the *weight* and *size* of $S$ respectively. $G$ is *unweighted* if $W(u) = 1$ for all $u \in V$. A subset $I \subseteq V$ is an *independent set* of $G$ if for any two vertices $u, v \in I$, $\{u, v\} \notin E$. An independent set $I$ of $G$ is *maximum* if there is no independent set $I'$ of $G$ such that $W(I) < W(I')$. We denote the weight of maximum independent set of $G$ by $\alpha(G)$ (i.e. $W(I) = \alpha(G)$ for a maximum independent set $I$ of $G$). Let $G[V']$ denote the subgraph of $G$ induced by $V'$, $d_G(u)$ the degree of vertex $u$, $\Delta_G$ the maximum degree of vertex in $G$, $\bar{d}_G$ the average degree of $G$, $N_G(v)$ the neighborhoods of $v$, and $N_G^+(v)$ $\{v\} \cup N_G(v)$. If $G$ is understood, then we often omit the inscription $G$ in $d_G(u)$, $\Delta_G$, $\bar{d}_G$, $N_G(v)$, and $N_G^+(v)$. For an independent set algorithm $A$, $A(G)$ is the weight of the solution obtained by $A$ on graph $G$. The *performance ratio* $\rho_A$ of $A$ is defined by $\rho_A = \inf_G \frac{A(G)}{\alpha(G)}$.

# 3 Greedy algorithms and extension of Turán's theorem

## 3.1 Known results

The following theorem is known as Turán's theorem [2].

**Theorem 3.1** *For any unweighted graph $G$,*

$$\alpha(G) \geq \frac{n}{\bar{d}_G + 1}.$$

Erdös showed that for unweighted graphs *MIN* attains the above bound [4]. The following extension of Theorem 3.1 was proved first by Wei and later by Alon and Spencer in a different way from Wei.

**Theorem 3.2** *For any unweighted graph $G$,*

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v) + 1}.$$

Wei demonstrated that *MIN* outputs an independent set of at least $\sum_{v \in V} \frac{1}{d(v)+1}$ vertices [10]. Alon and Spencer gave an elegant probabilistic proof of Theorem 3.2 [1], and Selkow improved the probabilistic proof [9]. The probabilistic proof is nonconstructive, however. We can apply the probabilistic proof to the case where graphs are weighted. Thus we have the next theorem.

**Theorem 3.3** *For any weighted graph $G$,*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{W(v)}{d_G(v) + 1}.$$

## 3.2 An extension of *MIN*

Let us consider the following framework of *MIN* type algorithm.

**Algorithm** *WMIN*

INPUT : A weighted graph $G$
OUTPUT : A maximal independent set in $G$.

**begin**
  $I := \emptyset$; $i := 0$; $G_i := G$;
  **while** $V(G_i) \neq \emptyset$ **do**
    Choose a vertex, say $v_i$, in $G_i$;
    $I := I \cup \{v_i\}$;
    $G_{i+1} := G_i[V(G_i) - N^+_{G_i}(v_i)]$;
    $i := i + 1$;
  **od**
  Output $I$;
**end.**

**Theorem 3.4** *In WMIN, if each $v_i$ $(0 \leq i \leq |I|)$ satisfies $\sum_{u \in N^+_{G_i}(v_i)} \frac{W(u)}{d_{G_i}(u)+1} \leq W(v_i)$ (there exists such a node for any graph), then WMIN outputs an independent set of weight at least $\sum_{v \in V} \frac{W(v)}{d_G(v)+1}$.*

**Proof.**

$$\begin{aligned}
\sum_{i=1}^{|I|} W(v_i) &\geq \sum_{i=1}^{|I|} \left( \sum_{u \in N^+_{G_i}(v_i)} \frac{W(u)}{d_{G_i}(u)+1} \right) \\
&\geq \sum_{i=1}^{|I|} \left( \sum_{u \in N^+_{G_i}(v_i)} \frac{W(u)}{d_G(u)+1} \right) \\
&= \sum_{v \in V(G)} \frac{W(v)}{d_G(v)+1}.
\end{aligned}$$

□

We refer to a simple greedy algorithm (based on *WMIN*) in which a vertex $v$ maximizing $\frac{W(u)}{d_{G_i}(u)+1}$ over all $u \in V(G_i)$ is selected in each iteration as *GWMIN*.

**Corollary 3.5** *GWMIN outputs an independent set of weight at least $\sum_{v \in V} \frac{W(v)}{d_G(v)+1}$.*

**Theorem 3.6** $\rho_{GWMIN} = \frac{1}{\Delta}$.

**Proof.** First we show that $\frac{1}{\Delta} \leq \rho_{GWMIN}$ inductively. Let $G$ be a graph and $I$ be the output of *GWMIN* for $G$. It is clear that the theorem holds if $|V(G)| \leq 2$. Now suppose that the theorem holds if $|V(G)| \leq n-1$. Let us consider the case $|V(G)| = n$. Let $v$ be the node which *GWMIN* choose at the start. Then $\forall u \in V$, $\frac{W(u)}{d_G(u)+1} \leq \frac{W(v)}{k+1}$. Let $V_1 = N^+_G(v)$ and $V_2 = V(G) - V_1$. We need the following claim.

**Claim 1** $\alpha(G[V_1])/\Delta \leq W(v)$

**Proof.** We assume that $W(v) < \alpha(G[V_1])$, otherwise trivial. Then,

$$\alpha(G[V_1]) \leq \sum_{u \in V_1 \setminus \{v\}} W(u)$$

$$
\begin{aligned}
&= \sum_{u \in V_1 \setminus \{v\}} \frac{W(u)}{d_G(u)+1}(d_G(u)+1) \\
&\leq (\Delta+1) \sum_{u \in V_1 \setminus \{v\}} \frac{W(u)}{d_G(u)+1} \\
&\leq (\Delta+1) \sum_{u \in V_1 \setminus \{v\}} \frac{W(v)}{k+1} \\
&= (\Delta+1)k\frac{W(v)}{k+1}.
\end{aligned}
$$

Thus we have $\frac{\alpha(G[V_1])}{(\Delta+1)(k/k+1)} \leq W(v)$. Since $k \leq \Delta$, we have the claim. $\square$

It is clear that $\alpha(G) \leq \alpha(G[V_1]) + \alpha(G[V_2])$. From Claim 1 and inductive hypothesis, $\alpha(G)/\Delta \leq \alpha(G[V_1])/\Delta + \alpha(G[V_2])/\Delta \leq W(v) + W(I - \{v\}) = W(I)$

Next we show that $\rho_{GWMIN} \leq \frac{1}{\Delta}$. Let us consider the graph depicted in Fig. 1.

Fig. 1

In the graph, the maximum degree $\Delta$ is equals to $l$ and $b >> l$. Note that the right side of the graph is the maximum independent set and $GWMIN$ outputs the left side of the graph as a maximum independent set. So we have $\rho_{GWMIN} \leq \frac{1}{\Delta}$. $\square$

It seems to be worth noting that we cannot guarantee the performance if we pick up a vertex maximizing $\frac{W(v)}{d_{G_i}(v)}$. The graph depicted in Fig.2 is a counterexample. In the graph, $\sum_{v \in V} \frac{W(v)}{d(v)+1} = 14$. If we choose the vertex $v_2$ (which maximizes $\frac{W(v)}{d(v)+1}$), we get the independent set $\{v_2\}$, and the total weight is 30. On the other hand, if we choose the vertex $v_1$ (which maximizes $\frac{W(v)}{d(v)}$), we get the independent set $\{v_1, v_3, v_4\}$, and the total weight is 13.

Fig. 2

### 3.3 An extension of $MAX$

Let us consider the following framework of $MAX$ type algorithm.

**Algorithm** $WMAX$

INPUT : A weighted graph $G$
OUTPUT : A maximal independent set in $G$.

**begin**
  $I := \emptyset$; $i := 0$; $G_i := G$;
  **while** $E(G_i) \neq \emptyset$ **do**
    Choose a vertex, say $v_i$, in $G_i$;
    $G_{i+1} := G_i[V(G_i) - \{v_i\}]$;
    $i := i + 1$;
  **od**
  $I := V(G)$;
  Output $I$;
**end.**

The next theorem is a generalization of the result of Griggs and Chvátal and C. McDiarmid. The essence of the proof is the same as Griggs's proof.

**Theorem 3.7** *In WMAX, if each $v_i$ ($0 \leq i < |V(G) - I|$) satisfies $\sum_{u \in N_{G_i}(v)} \frac{W(u)}{d_{G_i}(u)(d_{G_i}(u)+1)} \geq \frac{W(v)}{d_{G_i}(v)+1}$ and $d_{G_i}(v_i) \neq 0$ (there exists such a node for any graph $G$ such that $E(G) \neq \emptyset$), then WMAX outputs an independent set of weight at least $\sum_{v \in V} \frac{W(v)}{d_G(v)+1}$.*

**Proof.** For each $0 \leq i < |V(G) - I|$, the following inequality holds.

$$\begin{aligned}
\sum_{u \in V(G_{i+1})} \frac{W(u)}{d_{G_{i+1}}(u) + 1} &= \sum_{u \in V(G_i)} \frac{W(u)}{d_{G_i}(u) + 1} - \frac{W(v_i)}{d_{G_i}(v_i) + 1} + \sum_{u \in N_{G_i}(v)} \frac{W(u)}{d_{G_i}(u)(d_{G_i}(u) + 1)} \\
&\geq \sum_{u \in V(G_i)} \frac{W(u)}{d_{G_i}(u) + 1}.
\end{aligned}$$

Thus, $W(I) \geq \sum_{u \in V(G)} \frac{W(u)}{d_G(u)+1}$. $\square$

We refer to a simple greedy algorithm (based on $WMAX$) in which a vertex $v$ minimizing $\frac{W(u)}{d_{G_i}(u)(d_{G_i}(u)+1)}$ for all $u \in V(G_i)$ is selected in each iteration as $GWMAX$.

**Corollary 3.8** *$GWMAX$ outputs an independent set of weight at least* $\sum_{v \in V} \frac{W(v)}{d_G(v)+1}$.

**Theorem 3.9** $\frac{1}{\Delta+1} \leq \rho_{GWMAX} \leq \frac{1}{\Delta}$.

**Proof.** It is clear that $\frac{1}{\Delta+1} \leq \rho_{GWMAX}$ because $\sum_{v \in V} \frac{W(v)}{d_G(v)+1} \geq \sum_{v \in V} \frac{W(v)}{\Delta+1} \geq \frac{\alpha(G)}{\Delta+1}$. Let us consider the complete bipartite graph depicted in Fig. 3.

Fig. 3

In the graph, the maximum degree $\Delta$ is equals to $l$. Obviously, the left side of the graph is the maximum independent set. Thus,

$$\begin{aligned}
\alpha(G) &= \frac{l(l+1)}{1 \times 2}b + \frac{l(l+1)}{2 \times 3}b + \cdots + \frac{l(l+1)}{l(l+1)}b \\
&= l(l+1)b\left\{\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{l} - \frac{1}{l+1}\right)\right\} \\
&= l(l+1)b\left(1 - \frac{1}{l+1}\right) \\
&= bl^2.
\end{aligned}$$

On the other hand, $GWMAX$ can output the right side of the graph. Hence $\frac{bl^2}{bl} = l = \Delta$. Therefore, $\rho_{GWMAX} \leq \frac{1}{\Delta}$. □

### 3.4 The other greedy algorithm

**Theorem 3.10** *A simple greedy algorithm (based on WMIN) in which a vertex $v$ maximizing* $\frac{W(u)}{\sum_{w \in N^+_{G_i}(u)} W(w)}$ *over all $u \in V(G_i)$, is selected in each iteration, outputs an independent set of weight at least* $\sum_{v \in V(G)} \frac{W(v)^2}{\sum_{u \in N^+_G(v)} W(u)}$.

**Proof.** Let $I = \{v_1, v_2, \ldots, v_t\}$ be the independent set obtained by the algorithm. Let $f_G(v) = \frac{W(v)}{\sum_{u \in N^+_G(v)} W(u)}$.

$$\begin{aligned}
\sum_{i=1}^{t} W(v_i) &= \sum_{i=1}^{t} \left( f_{G_i}(v_i) \times \sum_{u \in N^+_{G_i}(v_i)} W(u) \right) \\
&\geq \sum_{i=1}^{t} \left( \sum_{u \in N^+_{G_i}(v_i)} f_{G_i}(u)\ W(u) \right) && (\text{from} \quad f_{G_i}(v_i) \geq f_{G_i}(u)\ \forall u \in V(G_i)) \\
&\geq \sum_{v \in V(G)} f_G(v) W(v) && (\text{from} \quad f_{G_i}(u) \geq f_G(u)\ \forall u \in V(G)) \\
&= \sum_{v \in V(G)} \frac{W(v)^2}{\sum_{u \in N^+_G(v)} W(u)}
\end{aligned}$$

□

If $W(v) = 1$ for all $v \in V(G)$ (i.e. unweighted case), then $\sum_{v \in V(G)} \frac{W(v)^2}{\sum_{u \in N^+_G(v)} W(u)}$ is equal to $\sum_{v \in V(G)} \frac{1}{d(v)+1}$ which is the bound of Turán's theorem.

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## Figures

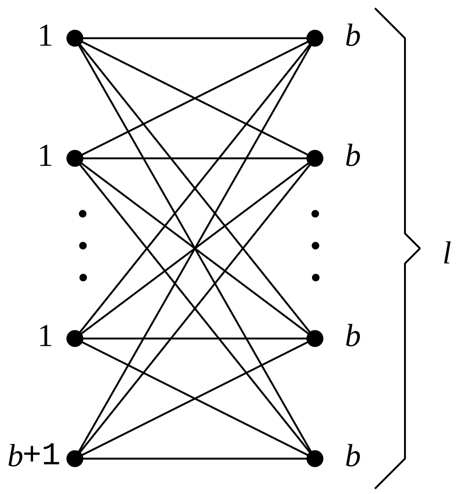


Figure 1: An example for *GWMIN*.

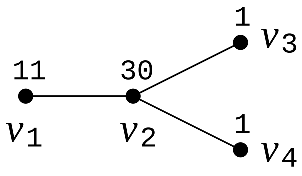


Figure 2: A counterexample.

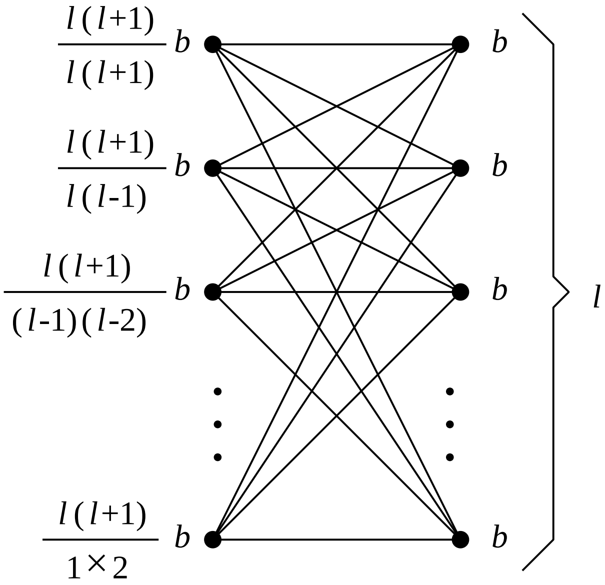

Figure 3: An example for $GWMAX$.