Multivariate asset models using Lévy processes and applications

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December 7, 2010

Abstract

In this paper we propose a multivariate asset model based on Lévy processes for pricing of products written on more than one underlying asset. We investigate the properties of the model and introduce a multivariate generalization of some processes which are quite common in financial applications, such as time changed Brownian motions and jump diffusion processes. Finally, we explore the issue of model calibration for the proposed setting.

Keywords: Jump Diffusion process, Lévy processes, model calibration, multinames derivative contracts, time changed Brownian motions.

JEL Classification: C51, D52, G12, G13

1 Introduction

The aim of this paper is to introduce a simple, parsimonious and robust model for multivariate Lévy processes with dependence between components, which can be easily implemented for financial applications, such as the pricing of several types of basket options commonly used in the credit and the energy derivatives markets.

The proposed model generalizes the approaches existing in the literature to any class of multivariate Lévy process, from time changed Brownian motions to jump diffusion models. Further, the model has a simple and intuitive economic interpretation. Our construction, in fact, is based on a parsimonious two-factor linear representation of the assets (log)-returns, in the sense that it uses a linear combination of two independent Lévy processes representing respectively the systematic factor and the idiosyncratic shock. Consequently, dependence between assets in a given portfolio is originated by the common component of the overall risk. As Lévy processes are invariant under linear transformations, our approach allows to specify any one-dimensional model for each of the components. In order to guarantee that the model still allows a calibration procedure to market prices of traded vanilla options, which is parsimonious and independent of the factor structure adopted for the asset (log-)returns, in this paper we also study the conditions under which dependence can be separated from the behaviour of the margins. The empirical analysis presented in this paper shows that our approach is flexible enough to accommodate the full range of possible dependence, from negative to positive dependence, from complete dependence to independence, but, at the same time, it is relatively parsimonious in terms of number of parameters involved, as this grows linearly with the number of names in the basket.

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Currently, multivariate asset models are essentially based on Brownian motions, i.e. Gaussian distributions, due to their simple structure. However, the limitations of the Brownian motion for the purpose of financial modelling are well known since the early '60s, whilst the recent crisis in the financial markets has stressed even more the importance of capturing market shocks with more refined models. In the last few years, attention has been paid to the construction of multi-dimensional financial models based on Lévy processes, with applications especially in the credit derivatives area, where the proliferation of complex multivariate products, such as CDOs, has stressed the need of simple and parsimonious models describing both dependence among the underlying instruments, and shock-driven changes in their creditworthiness. For example, Baxter (2007) uses a linear combination of two processes, representing respectively the global factor and the idiosyncratic factor of the overall risk driver, and explore the application to CDO pricing. A similar construction has been adopted by Moosbrucker (2006.a, 2006.b), who also explores its application to both time changed Brownian motions, such as the Variance Gamma (VG) process and the Normal Inverse Gaussian process (NIG), and Jump Diffusion (JD) processes of the Merton (Merton, 1976) and the Kou (Kou, 2002) type. Although the formulation proposed in these models is very simple, the model can only capture positive correlation, due to restrictions on the corresponding parameters required to guarantee the existence of the components' characteristic functions. Further, all assets have the same pairwise correlation coefficient and the processes used in the multivariate construction are independent copies of the same driving motion. In this respect, our contribution is given by a richer correlation structure, which is also facilitated by the fact that the idiosyncratic shock and the systematic component can be chosen separately one of the other.

The idea of correlating Lévy processes using linear transformations has also been adopted by Madan and Yen (2008) in the context of developing asset allocation strategies which take into account higher moments in investment returns. Their application makes use of independent Variance Gamma (VG) processes which are combined using a mixing matrix obtained by independent component analysis. Luciano and Schoutens (2006), instead, develop a multivariate Variance Gamma process using a common subordinator and study the application of this model to the pricing of a number of products including CDS. It has been noted, though, that this model does not accommodate independence, and linear correlation cannot be fitted once the margins are fixed (see Luciano and Semeraro, 2010, for example). The same limitations are suffered by the model proposed by Cont and Tankov (2004) who assume that the Brownian motions are correlated. Differently from Luciano and Schoutens (2006), though, the model of Cont and Tankov (2004) allows for a non-zero correlation coefficient in the case of symmetric processes. Another alternative approach is the one proposed by Eberlein and Madan (2009), who introduce dependence by correlating the Brownian motions being time changed, whilst the subordinators are independent for each asset in the portfolio.

Semeraro (2008) improves the multivariate VG model of Luciano and Schoutens (2006) by means of a multivariate subordinator, incorporating both global and idiosyncratic components; Luciano and Semeraro (2010) extend this approach to other time changed Brownian motions, like the Normal Inverse Gaussian process and the CGMY process (Carr et al., 2002). The construction proposed in these papers (which is also the same adopted by Fiorani et al., 2009) recognizes that business time is related to trading activity with both global and idiosyncratic parts, as supported by the empirical evidence put forward by Lo and Wang (2000) for a two-factor representation of trading volume. Further, it captures the case of full independence which occurs when the subordinators are all independent; finally, linear correlation can be fitted by suitably choosing the parameters of the common component of the subordinator. However, the coefficient of linear correlation is still governed by the parameters of the margins, in the sense that it is zero in the case of symmetric processes, although they are still dependent. Further, the correlation coefficient is bounded above by the correlation of the subordinator, which affects the performance of the model when fitting high correlation values.

In order to overcome these drawbacks, Semeraro (2008) and Luciano and Semeraro (2010) introduce a more general version which makes use of correlated Brownian motions; the presence in the model of this additional parameters set allows to separate the correlation coefficient from the behaviour of the margins, so that the full range of possible dependence can be captured. As pointed out by the authors, though, this general formulation is no longer parsimonious in terms of parameters: the presence of a correlation matrix for the Brownian motion part of the components, in fact, implies that the number of parameters grows with the square of the number of assets included in the basket. The same dimensionality problem is suffered by the approach put forward by Kawai (2009), in which the number of the components is the same as the number of the names in the portfolio; further, the construction applies only to pure jump processes with the same variance. Although our model resembles in principle the more general version of the time changed Brownian motion proposed by Luciano and Semeraro (2010), it is shown in the following of this paper to be relatively less demanding in terms of the overall number of parameters involved which improves the tractability of the model and its calibration to market data; further, our construction extends from the pure jump setting to accommodate any type of Lévy process.

The remaining of the paper is organized as follows. In section 2, we introduce the construction for multivariate Lévy processes and investigate the properties of the resulting model. In sections 3 and 4 we show how, under certain conditions, the proposed formulation can be used to build multivariate time changed Brownian motions and JD processes. A financial application aimed at discussing the issue of model calibration and testing the robustness and the flexibility of the model is presented in section 5; section 6 concludes.

2 Multivariate Lévy process via linear transformation

Lévy processes are characterized by independent and stationary increments; they are fully described by their characteristic function which admits Lévy-Khintchine representation

$$\begin{aligned}\phi(u;t) &= e^{t\varphi(u)}, \qquad u \in \mathbb{C}\\ \varphi(u) &= \alpha u + \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \left(e^{ux} - 1 - ux1_{(|x|<1)}\right)\Pi(dx).\end{aligned}$$

The terms in the characteristic exponent, $\varphi(\cdot)$, i.e. (α, σ, Π) represent the characteristic triple of the Lévy process. The parameter $\alpha \in \mathbb{R}$ describes the drift of the process, $\sigma \in \mathbb{R}^{++}$ represents its diffusion part, whilst the jumps are fully characterized by the Lévy measure Π, i.e. a positive measure satisfying

$$\int_{\mathbb{R}} \left(1 \wedge |x|^2\right)\Pi(dx) < \infty.$$

To construct a multivariate Lévy process with dependent components, we use the property that these processes are invariant under linear transformations. The main result is given in the following.

Corollary 1 *Let*

$$\mathbf{Y}(t) = \begin{pmatrix} Y_1(t) \\ \vdots \\ Y_n(t) \end{pmatrix},$$

and $Z(t)$ be independent Lévy processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P}), \forall j = 1, ..., n,$ with characteristic functions $\phi_{Y_j}(u;t)$ and $\phi_Z(u;t)$ respectively. Then, for $a_j \in \mathbb{R}, j = 1, ..., n$

$$\mathbf{X}(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix} = \begin{pmatrix} Y_1(t) + a_1 Z(t) \\ \vdots \\ Y_n(t) + a_n Z(t) \end{pmatrix}$$

is a Lévy process on $\mathbb{R}^n$ with characteristic function

$$\phi_{\mathbf{X}}(\mathbf{u};t) = \phi_Z\left(\sum_{j=1}^{n} a_j u_j; t\right) \prod_{j=1}^{n} \phi_{Y_j}(u_j;t), \quad \mathbf{u} \in \mathbb{C}^n. \tag{1}$$

The proof follows from the properties of a Lévy process (see, for example, Theorem 4.1 in Cont and Tankov, 2004).

The cumulant process of the margins is given by their characteristic exponent, therefore the corresponding m^{th} cumulant, K_m, can be calculated as

$$\begin{aligned} K_m(X_j(t)) &= K_m(Y_j(t)) + a_j^m K_m(Z(t)) \\ &= t\left[K_m(Y_j(1)) + a_j^m K_m(Z(1))\right], \end{aligned} \tag{2}$$

where the last equality follows from the fact that Lévy processes have an infinitely divisible distribution.

The construction proposed in Corollary 1 offers a simple and intuitive economic interpretation as for each margin, X_j, the process Z can be considered as the systematic part of the risk inducing dependence between names, whilst the process Y_j can be seen as capturing the idiosyncratic shock. Hence, dependence in the model is induced by the dependence of each margin process on the common factor $Z(t)$; in fact,

$$\mathbb{C}ov(X_j(t), X_l(t)) = a_j a_l \mathbb{V}ar(Z(1))\, t,$$

which implies that the pairwise (linear) correlation coefficient is

$$\rho_{jl} = \mathbb{C}orr(X_j(t), X_l(t)) = \frac{a_j a_l \mathbb{V}ar(Z(1))}{\sqrt{\mathbb{V}ar(X_j(1))}\sqrt{\mathbb{V}ar(X_l(1))}}. \tag{3}$$

Thus, it follows from equation (3) that $\rho_{jl} = 0$ if and only if either $a_j a_l = 0$ or $\mathbb{V}ar(Z(1)) = 0$. Both cases imply that the pairwise correlation coefficient is zero if and only if the margins are independent (i.e. there is no common factor). On the other hand, full dependence is achieved if and only if there is no idiosyncratic factor in the margins. Moreover $sign(\rho_{jl}) = sign(a_j a_l)$ and therefore both positive and negative correlation can be accommodated.

Further, the pairwise linear correlation between the margin processes, in fact, can be expressed in terms of the correlation between each margin and the systematic component as

$$\mathbb{C}orr\left(X_j\left(t\right), Z\left(t\right)\right) = a_j \sqrt{\frac{\mathbb{V}ar\left(Z\left(1\right)\right)}{\mathbb{V}ar\left(X_j\left(1\right)\right)}} \qquad \forall j = 1, ..., n, \tag{4}$$

implying that $\rho_{jl} = \mathbb{C}orr\left(X_j\left(t\right), Z\left(t\right)\right)\mathbb{C}orr\left(X_l\left(t\right), Z\left(t\right)\right)$. This has also an implication on the calibration of financial asset models built on the multivariate Lévy process $\mathbf{X}(t)$, in the sense that the calibration of the marginal distribution to observable market data can be independent of the fitting of the correlation matrix only if the linear combination $\mathbf{Y}\left(t\right)+\mathbf{a}Z\left(t\right)$ returns a known distribution for the process $\mathbf{X}\left(t\right)$. A way of guaranteeing this is to impose suitable convolution conditions on the processes X, Y and Z. Under this restriction, we note that, if the parameters of the margin processes are given (from calibration to univariate option prices, for example), and the correlation matrix is known, the fitting of the joint distribution requires $n(m+1)+m$ parameters, where m is the number of parameters describing the component processes Y_j and Z.

The multidimensional modelling approach put forward in this section is quite flexible as it allows to specify any univariate process for each of the components; further the margins do not have the same pairwise correlation coefficient. Consequently, the construction proposed by Baxter (2007), Moosbrucker (2006.a, 2006.b) and Kawai (2009) can be recovered as a particular case of the model proposed in this paper. Our model also generalizes Semeraro (2008) and Luciano and Semeraro (2010) in that it can be applied to any type of Lévy process from time changed Brownian motions to jump diffusion processes. Examples illustrating the construction of a multivariate version of both time changed Brownian motions and jump diffusion processes are discussed in the following sections, together with some further economic considerations on the roles of the idiosyncratic and systematic components.

3 Multivariate time changed Brownian motions

A time changed Brownian motion $X = \left(X\left(t\right) : t \geq 0\right)$ is a Lévy process obtained by observing a (arithmetic) Brownian motion on a time scale governed by an independent subordinator, i.e. an increasing, positive Lévy process. Hence $X(t)$ has general form

$$X\left(t\right) = \theta G\left(t\right) + \sigma W\left(G\left(t\right)\right), \qquad \theta \in \mathbb{R}, \sigma \in \mathbb{R}^{++}, \tag{5}$$

where $W = \left(W\left(t\right) : t \geq 0\right)$ is a Brownian motion and $G = \left(G\left(t\right) : t \geq 0\right)$ is a subordinator independent of W. The resulting characteristic function is

$$\phi_X\left(u; t\right) = e^{t\varphi_G\left(u\theta + u^2\frac{\sigma^2}{2}\right)}, \quad u \in \mathbb{C},$$

where $\varphi_G(\cdot)$ denotes the Lévy symbol of the subordinator. It follows that (see, for example, Ané and Geman, 2000)

$$\begin{aligned} \mathbb{E}X\left(t\right) &= \theta\mathbb{E}\left(G(t)\right); \\ \mathbb{V}ar\left(X\left(t\right)\right) &= \sigma^2\mathbb{E}\left(G(t)\right) + \theta^2\mathbb{V}ar\left(G(t)\right); \end{aligned}$$

the indices of skewness and excess kurtosis are respectively

$$\gamma_1(t) = \frac{3\theta\sigma^2 \mathbb{V}ar\left(G(t)\right) + \theta^3 K_3\left(G(t)\right)}{\left(\sigma^2 \mathbb{E}\left(G(t)\right) + \theta^2 \mathbb{V}ar\left(G(t)\right)\right)^{3/2}} \tag{6}$$

$$\gamma_2(t) = \frac{\theta^4 K_4\left(G(t)\right) + 6\theta^2\sigma^2 K_3\left(G(t)\right) + 3\sigma^4 \mathbb{V}ar\left(G(t)\right)}{\left(\sigma^2 \mathbb{E}\left(G(t)\right) + \theta^2 \mathbb{V}ar\left(G(t)\right)\right)^{2}}. \tag{7}$$

In general, the parameters of the distribution of the subordinator are chosen so that $\mathbb{E}G(t) = t$, in order to guarantee that the stochastic clock $G(t)$ is an unbiased reflection of calendar time (see, for example, Madan et al., 1998). If $k > 0$ is the variance rate of $G(t)$, we say that the process $X(t)$ has parameters (θ, σ, k); it follows from equations (6)-(7) that θ controls the skewness of $X(t)$, σ controls its volatility and k is the percentage excess kurtosis. The law of the increments of $G(t)$ allows us to characterize the resulting process. There are different methods for choosing a subordinator which is suitable for financial modelling; one class of such processes which proves to be quite popular due to its mathematical tractability is the family of tempered stable subordinators. These are Lévy processes with characteristic exponent

$$\varphi_G(u) = \frac{\alpha - 1}{\alpha k}\left[\left(1 - \frac{uk}{1-\alpha}\right)^{\alpha} - 1\right], \quad u \in \mathbb{C}, \tag{8}$$

where $k > 0$ is the variance rate as above and $\alpha \in [0, 1)$ is the index of stability. In particular, if $\alpha = 0$, $G(t)$ is a Gamma process and $X(t)$ is a VG process (Madan and Seneta, 1990, Madan and Milne, 1991, Madan et al., 1998); if, instead, $\alpha = 1/2$, the subordinator follows an Inverse Gaussian process and $X(t)$ is the NIG process introduced by Barndorff-Nielsen (1995). Other Lévy processes with known time changed Brownian motion representation are the CGMY and the Meixner process (Madan and Yor, 2008).

Constructing Lévy processes by subordination has particular economic appeal as, in first place, empirical evidence shows that stock log-returns are Gaussian but only under trade time, rather than standard calendar time (see, for example, Geman and Ané, 1996). Further, the time change construction recognizes that stock prices are largely driven by news, and the time between one piece of news and the next is random as is its impact. Finally, the representation of Lévy processes as time changed Brownian motions offers a high degree of mathematical tractability as, once we operate under business time, log-returns are once again Gaussian and therefore the results derived for the Black-Scholes model still hold.

To build the multivariate version of a time changed Brownian motion of the form (5), we follow Corollary 1 and let $Y_j(t)$ and $Z(t)$ be independent time changed Brownian motions chosen from the same family of distributions and with parameters $(\beta_j, \gamma_j, \nu_j)$, $\forall j = 1, ..., n$, and $(\beta_Z, \gamma_Z, \nu_Z)$ respectively. Then, $\mathbf{X}(t)$ is a multivariate time changed Brownian motion with margins of the same distribution's class as $Y_j(t)$ and $Z(t)$ if the following convolution condition is satisfied.

$$\varphi_{G_j}\left(u\theta_j + u^2\frac{\sigma_j^2}{2}\right) = \varphi_{G_{Y_j}}\left(u\beta_j + u^2\frac{\gamma_j^2}{2}\right) + \varphi_{G_Z}\left(ua_j\beta_Z + u^2\frac{a_j^2\gamma_Z^2}{2}\right) \quad \forall j = 1, ..., n, \tag{9}$$

where φ_{G_j}, $\varphi_{G_{Y_j}}$, φ_{G_Z} denote respectively the characteristic exponents of the subordinators of the margin process, the idiosyncratic part and the systematic component. The pairwise linear correlation coefficient is

$$\rho_{jl} = \frac{a_j a_l \left(\gamma_Z^2 + \beta_Z^2 \nu_Z\right)}{\sqrt{\sigma_j^2 + \theta_j^2 k_j}\sqrt{\sigma_l^2 + \theta_l^2 k_l}}. \tag{10}$$

We note that in this construction of the multivariate time changed Brownian motion, dependence stems from both the subordinator and the associated Wiener process. Equation (9), in particular, implies that

$$\begin{aligned} G_j(t) &= \frac{\beta_j G_{Yj}(t) + a_j \beta_Z G_Z(t)}{\theta_j} \\ &= \frac{\gamma_j^2 G_{Yj}(t) + a_j^2 \gamma_Z^2 G_Z(t)}{\sigma_j^2}, \end{aligned}$$

where both equalities are intended in distribution. This shows that the subordinator of the margin process can be decomposed into a stochastic clock which is common to all names in a given basket, and a time change that is specific to each asset, capturing the idiosyncratic part of the overall trading activity. This split between a common and an idiosyncratic component is supported also by the empirical analysis performed by Lo and Wang (2000).

Further, the convolution condition implies $\theta_j = \beta_j + a_j \beta_Z$ and $\sigma_j^2 = \gamma_j^2 + a_j^2 \gamma_Z^2$. In the case in which $G_{Yj}, \forall j = 1, ..., n$ and G_Z are tempered stable subordinators, equation (8) implies that the convolution condition (9) is satisfied if

$$\begin{cases} \nu_j \beta_j = \nu_Z a_j \beta_Z & j = 1, ..., n \\ \nu_j \gamma_j^2 = \nu_Z a_j^2 \gamma_Z^2 & j = 1, ..., n \end{cases}$$

and consequently $k_j = \nu_j \nu_Z / (\nu_j + \nu_Z)$. The subordinators, in fact, are assumed to have the same stability index in order to guarantee that $Y_j(t)$ and $Z(t)$ belong to the same family of distributions.

Example 1 (The VG process) Let G be a gamma process, i.e. a tempered stable process with scale parameter $\alpha = 0$; then $X(t)$ is a VG process with characteristic function

$$\phi_X(u;t) = \left(1 - u\theta k - u^2 \frac{\sigma^2}{2} k\right)^{-\frac{t}{k}}, \qquad u \in \mathbb{C},$$

which exists for

$$\frac{-\theta - \sqrt{\theta^2 + \frac{2\sigma^2}{k}}}{\sigma^2} < \Re(u) < \frac{-\theta + \sqrt{\theta^2 + \frac{2\sigma^2}{k}}}{\sigma^2}.$$

Therefore

$$\begin{aligned} \mathbb{E}X(t) &= \theta t, \\ \mathbb{V}ar(X(t)) &= \left(\sigma^2 + \theta^2 k\right) t; \end{aligned}$$

the indices of skewness and excess kurtosis are respectively

$$\begin{aligned} \gamma_1(t) &= \frac{\left(3\sigma^2 + 2\theta^2 k\right)\theta k}{\left(\sigma^2 + \theta^2 k\right)^{3/2}\sqrt{t}}, \\ \gamma_2(t) &= \frac{\left(3\sigma^4 + 12\sigma^2\theta^2 k + 6\theta^4 k^2\right)k}{\left(\sigma^2 + \theta^2 k\right)^2 t}. \end{aligned}$$

Under the restrictions imposed by the convolution condition (9), $\mathbf{X}(t)$ is a multivariate VG process with margins' parameters $(\theta_j, \sigma_j, k_j)$ constructed as above and characteristic function

$$\phi_{\mathbf{X}}(\mathbf{u};t) = \left(1 - \beta_Z \nu_Z \sum_{j=1}^{n} a_j u_j - \frac{\gamma_Z^2}{2} \nu_Z \left(\sum_{j=1}^{n} a_j u_j\right)^2\right)^{-\frac{t}{\nu_Z}} \prod_{j=1}^{n} \left(1 - u_j \beta_j \nu_j - u_j^2 \frac{\gamma_j^2}{2} \nu_j\right)^{-\frac{t}{\nu_j}}.$$

The corresponding pairwise correlation coefficient is given by equation (10).

Example 2 (The NIG process) In the case in which the tempered stable subordinator $G(t)$ has scale parameter $\alpha = 1/2$, i.e. is an Inverse Gaussian process, then $X(t)$ is a NIG process with characteristic function

$$\phi_X(u;t) = e^{\frac{t}{k}\left(1-\sqrt{1-2u\theta k - u^2\sigma^2 k}\right)}, \qquad u \in \mathbb{C},$$

which exists for

$$\frac{-\theta - \sqrt{\theta^2 + \frac{\sigma^2}{k}}}{\sigma^2} < \Re(u) < \frac{-\theta + \sqrt{\theta^2 + \frac{\sigma^2}{k}}}{\sigma^2}.$$

Further

$$\begin{aligned} \mathbb{E}X(t) &= \theta t, \\ \mathbb{V}ar(X(t)) &= \left(\sigma^2 + \theta^2 k\right) t; \end{aligned}$$

the indices of skewness and excess kurtosis are respectively

$$\begin{aligned} \gamma_1(t) &= \frac{3\theta k}{\sqrt{t\left(\sigma^2 + \theta^2 k\right)}}, \\ \gamma_2(t) &= \frac{3k\left(\sigma^4 + 6\sigma^2\theta^2 k + 5\theta^4 k^2\right)}{\left(\sigma^2 + \theta^2 k\right)^2 t}. \end{aligned}$$

Under the convolution condition (9), the margins $X_j(t)$ are NIG processes with parameters $(\theta_j, \sigma_j, k_j)$ as constructed as above. The resulting characteristic function of the multivariate NIG process is

$$\begin{aligned} \phi_{\mathbf{X}}(\mathbf{u};t) &= e^{t\varphi(\mathbf{u})} \\ \varphi(\mathbf{u}) &= \frac{1}{\nu_Z}\left(1 - \sqrt{1 - 2\beta_Z \nu_Z \sum_{j=1}^{n} a_j u_j - \gamma_Z^2 \nu_Z \left(\sum_{j=1}^{n} a_j u_j\right)^2}\right) \\ &\quad + \sum_{j=1}^{n} \frac{1}{\nu_j}\left(1 - \sqrt{1 - 2u_j\beta_j\nu_j - u_j^2\gamma_j^2\nu_j}\right). \end{aligned}$$

Equation (10) describes the pairwise correlation coefficient also in this case.

4 Multivariate jump-diffusion (JD) process

An alternative representation of Lévy processes quite common in financial applications relies on the observation that stock prices appear to have small continuous movements most of the time (due, for example, to a temporary imbalance between demand and supply); but sometimes they experience large jumps upon the arrival of important information with more than a marginal impact. By its very nature, important information arrives only at discrete points in time and the jumps it causes have finite activity. A motion portraying such a dynamic is a jump-diffusion process, which can be decomposed as the sum of a Brownian motion with drift and an independent compound Poisson process. Hence, a Lévy process in the JD class has form

$$X(t) = \mu t + \sigma W(t) + \sum_{k=1}^{N(t)} \xi(k), \quad \mu \in \mathbb{R}, \sigma \in \mathbb{R}^{++},$$

where $W = (W(t) : t \geq 0)$ is a Brownian motion, $N = (N(t) : t \geq 0)$ is a Poisson process counting the jumps of X and $\xi(k)$ are i.i.d. random variables capturing the jump sizes (severities). W, N and ξ are independent of each other. The Brownian motion captures the marginal changes in stock prices, whilst the compound Poisson process models the market shocks.

We assume that the rate of arrival of the Poisson process is $\lambda \in \mathbb{R}^{++}$. In this case, we say that the process $X(t)$ has parameters (μ, σ, λ) and jump sizes density f_ξ; the resulting characteristic function is

$$\begin{aligned} \phi_X(u;t) &= e^{t\left(u\mu + u^2\frac{\sigma^2}{2} + \lambda\left(\phi_\xi(u)-1\right)\right)}, \\ \phi_\xi(u) &= \mathbb{E}\left(e^{u\xi}\right), \qquad u \in \mathbb{C}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}X(t) &= (\mu + \lambda\mathbb{E}(\xi))\,t, \\ \mathbb{V}ar(X(t)) &= \left(\sigma^2 + \lambda\mathbb{E}\left(\xi^2\right)\right)t; \end{aligned}$$

whilst the indices of skewness and excess kurtosis are respectively

$$\begin{aligned} \gamma_1(t) &= \frac{\lambda\mathbb{E}\left(\xi^3\right)}{\left(\sigma^2 + \lambda\mathbb{E}\left(\xi^2\right)\right)^{3/2}\sqrt{t}}, \\ \gamma_2(t) &= \frac{\lambda\mathbb{E}\left(\xi^4\right)}{\left(\sigma^2 + \lambda\mathbb{E}\left(\xi^2\right)\right)^2 t}. \end{aligned}$$

Popular examples of JD processes used in finance are the so-called Merton process (Merton, 1976), for which the jump sizes are Gaussian, and the Kou process (Kou, 2002) in which case the jump sizes follow an asymmetric double exponential distribution.

In order to construct the multivariate version of the JD process, we follow the same steps as in the previous sections and set the idiosyncratic factor, Y_j, and the global factor, Z, to be two independent JD processes, respectively with parameters $(\beta_j, \gamma_j, \delta_j)$ and jump sizes' density f_{η_j}, and $(\beta_Z, \gamma_Z, \delta_Z)$ and jump sizes' density f_{η_Z}. Further, for the convolution property to hold, i.e.

for the process

$$\mathbf{X}(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix} = \begin{pmatrix} Y_1(t) + a_1 Z(t) \\ \vdots \\ Y_n(t) + a_n Z(t) \end{pmatrix}$$

to be a multivariate JD process, whose margins have parameters $(\mu_j, \sigma_j, \lambda_j)$ and jump sizes' density f_{ξ_j}, we require $\mu_j = \beta_j + a_j \beta_Z$, $\sigma_j^2 = \gamma_j^2 + a_j^2 \gamma_Z^2$ and

$$\lambda_j \left(\phi_{\xi_j}(u) - 1 \right) = \delta_j \left(\phi_{\eta_j}(u) - 1 \right) + \delta_Z \left(\phi_{\eta_Z}(a_j u) - 1 \right), \quad u \in \mathbb{C}. \tag{11}$$

The corresponding pairwise correlation coefficient is

$$\rho_{ij} = \frac{a_j a_l \left(\gamma_Z^2 + \delta_Z \mathbb{E}\left(\eta_Z^2 \right) \right)}{\sqrt{\sigma_j^2 + \lambda_j \mathbb{E}\left(\xi_j^2 \right)} \sqrt{\sigma_l^2 + \lambda_l \mathbb{E}\left(\xi_l^2 \right)}}. \tag{12}$$

Thus, we note that under the proposed construction, the compound Poisson process components are allowed to jump at different points in time. Further, the convolution conditions reported above show the decomposition of both the continuous part of the risk and the pure jump one into their corresponding asset specific part and the one common to the entire basket under consideration.

As the Poisson process is closed under convolution, we could further assume $\lambda_j = \delta_j + \delta_Z \ \forall j$, so that equation (11) implies the following convolution on the distribution of the jump sizes:

$$\phi_{\xi_j}(u) = \frac{\delta_j \phi_{\eta_j}(u) + \delta_Z \phi_{\eta_Z}(a_j u)}{\delta_j + \delta_Z}, \quad u \in \mathbb{C}. \tag{13}$$

The proposed construction of multivariate JD processes falls in the more general common Poisson shock framework, reviewed in Lindskog and McNeil (2003) and further extended by Brigo et al. (2007). In our case, we use only two different types of shock (systematic and idiosyncratic); however, the distribution of the jump sizes (severities) depends on the nature of the underlying shock.

We note that a trivial solution to equation (13) can be obtained by assuming that ξ_j, η_j and $a_j \eta_Z$ are identically distributed. This is the case discussed by Moosbrucker (2006.b). However, in the following we do not consider this alternative as it imposes the unrealistic restriction that the jump sizes of each margin and the ones of its idiosyncratic component are identically distributed. Therefore, we make use of the (numerical) solution of equation (13).

Example 3 (The Merton process) Assume that the distribution of the jump sizes is Gaussian. Then, if $\eta_j \sim N\left(\vartheta_{Yj}, \upsilon_{Yj}^2 \right)$ and $\eta_Z \sim N\left(\vartheta_Z, \upsilon_Z^2 \right)$, the process $X_j(t) = Y_j(t) + a_j Z(t)$ is a Merton JD process with parameters $(\mu_j, \sigma_j, \lambda_j)$ as defined above, and jump sizes $\xi_j \sim N\left(\vartheta_j, \upsilon_j^2 \right)$, where ϑ_j and υ_j are the solutions of

$$e^{u\vartheta_j + u^2 \frac{\upsilon_j^2}{2}} = \frac{\delta_j e^{u\vartheta_{Yj} + u^2 \frac{\upsilon_{Yj}^2}{2}} + \delta_Z e^{u a_j \vartheta_Z + u^2 \frac{a_j^2 \upsilon_Z^2}{2}}}{\delta_j + \delta_Z}, \qquad u \in \mathbb{C}. \tag{14}$$

The above implies

$$\vartheta_j = \frac{\delta_j \vartheta_{Yj} + \delta_Z a_j \vartheta_Z}{\delta_j + \delta_Z},$$
$$\upsilon_j^2 = \frac{\delta_j(\vartheta_{Yj}^2 + \upsilon_{Yj}^2) + \delta_Z a_j^2(\vartheta_Z^2 + \upsilon_Z^2)}{\delta_j + \delta_Z} - \vartheta_j^2.$$

The coefficient of correlation is given by equation (12), with

$$\mathbb{E}\left(\xi_j^2\right) = \vartheta_j^2 + \upsilon_j^2, \qquad \forall j = 1, ..., n$$
$$\mathbb{E}\left(\eta_Z^2\right) = \vartheta_Z^2 + \upsilon_Z^2.$$

Example 4 (The Kou process) In the case of the Kou process, the jump sizes follow a double exponential distribution with parameters (p, α^+, α^-), i.e. their density function is given by

$$p\alpha^+ e^{-\alpha^+ y} 1_{(y\geq 0)} + (1-p)\,\alpha^- e^{\alpha^- y} 1_{(y<0)}, \quad \alpha^+, \alpha^- \in \mathbb{R}^{++}, p \in [0,1].$$

Thus, if η_j, η_Z, ξ_j have a double exponential distribution respectively with parameters $(p_{Yj}, \alpha_{Yj}^+, \alpha_{Yj}^-)$, $(p_Z, \alpha_Z^+, \alpha_Z^-)$, $(p_j, \alpha_j^+, \alpha_j^-)$, then, for the convolution condition (13) to hold, these parameters must satisfy the following

$$p_j \frac{\alpha_j^+}{\alpha_j^+ - u} + (1-p_j)\,\frac{\alpha_j^-}{\alpha_j^- + u} =$$
$$\frac{1}{\delta_j + \delta_Z}\left[p_{Yj}\frac{\delta_j \alpha_{Yj}^+}{\alpha_{Yj}^+ - u} + (1-p_{Yj})\,\frac{\delta_j \alpha_{Yj}^-}{\alpha_{Yj}^- + u} + p_Z \frac{\delta_Z \alpha_Z^+}{\alpha_Z^+ - a_j u} + (1-p_Z)\,\frac{\delta_Z \alpha_Z^-}{\alpha_Z^- + a_j u}\right]. \quad (15)$$

The correlation coefficient is obtained from equation (12) for

$$\mathbb{E}\left(\xi_j^2\right) = 2\left(\frac{p_j}{\left(\alpha_j^+\right)^2} + \frac{1-p_j}{\left(\alpha_j^-\right)^2}\right), \qquad \forall j = 1, ..., n$$
$$\mathbb{E}\left(\eta_Z^2\right) = 2\left(\frac{p_Z}{\left(\alpha_Z^+\right)^2} + \frac{1-p_Z}{\left(\alpha_Z^-\right)^2}\right).$$

Finally, we note that for the multivariate Kou model, the reconstruction of the margin parameters (p, α^+, α^-) from the components parameters can only be performed numerically.

In the following, we consider applications of our multivariate approach to option pricing problems; therefore, without loss of generality, we consider the case of a JD process with no drift, i.e. we set $\mu_j = \beta_j = \beta_Z = 0\ \forall j = 1, ..., n$.

5 Multivariate asset modelling: calibration and derivative pricing

In this section we discuss how the multivariate Lévy process constructed in the previous part of this paper can be used to set up a model for the financial market. In particular, we analyze the calibration of the model to market data in view of applications to the problem of pricing multinames products.

To this purpose, we consider a frictionless market in which asset log-returns are modelled by the multivariate Lévy process defined in Corollary 1, so that under the risk neutral martingale measure $\mathbb{P}$ asset prices are given by

$$S_j(t) = S_j(0)e^{(r-q_j-\varpi_j)t+X_j(t)}, \quad j = 1, ..., n$$

where $r > 0$ is the risk free rate of interest, $S_j(0)$ and q_j denote respectively the spot price and the dividend yield of the j^{th} asset, $X_j(t)$ is the j^{th} component of the multivariate Lévy process, and ϖ_j denotes the exponential compensator of $X_j(t)$.

As in general the given market is incomplete, there are infinitely many risk neutral martingale measures; the availability of market prices for European vanilla options, though, allows us to "complete" the market and extract the pricing measure by calibration. We note that a full calibration procedure aimed at extracting the correlation coefficient as well should also make use of derivatives written on all the assets, like for example a basket option. However, as these products are only traded on the OTC market, the implied correlation cannot be estimated at this stage. For this reason, in this application we use as a proxy the historical pairwise correlation coefficient ρ_{jl} between asset log-returns; we leave the issue of its calibration to further research.

We test the flexibility of the model by calibrating it to option prices on Ford Motor Company, Abbott Laboratories and Baxter International Inc. We use Bloomberg quotes at three different valuation dates, September 30, 2008, February 27, 2009 and September 30, 2009, in order to explore the behaviour of the proposed model when fitting different correlation values. A synopsis of the three assets is reported in Table 1. The risk free rate of interest is taken from Bloomberg as well in correspondence of the relevant dates. Historical correlation between assets has been estimated on a time window of 125 days up to (and including) the valuation date.

The three assets considered in this analysis are constituents of the S&P100 index, and represent three different industries: automotive, drug manufacturers and medical instruments and supplies respectively. Abbott Laboratories and Baxter International Inc. are part of the same healthcare sector. A principal component analysis performed on daily share prices over a 20 month time period preceding each valuation date shows that the first two components explain between 94% and 97% of the total variance of the log-returns of the three assets, hence supporting the idea of a two factor model for the driving process.

Further, from Table 1 we observe that in September 2008 the three assets exhibit positive correlation, at a level which is fairly similar between Ford and the remaining two assets, whilst it is significantly higher between Abbott and Baxter. This date, in fact, coincides with the peak of the financial crisis which led to the collapse of Lehman Brothers; the car industry was also experiencing a particularly difficult period following the General Motors liquidity crisis and the sales fall also reported by its main competitors. Correlation values further increase in February 2009, when the effects of the credit crisis are fully captured by the estimation procedure used in this analysis. These observations lead us to expect the common component $Z(t)$ to play a significant role in the prices of Abbott Lab. and Baxter, whilst we expect it to have a smaller impact of Ford prices. The same consideration holds especially for the September 2009 valuation date, when Ford exhibits negative correlation with the other two assets considered in this analysis.

The calibration of the proposed multivariate asset model is performed in steps. In first place, we extract the parameters of the margin processes, $\mathbf{X}(t)$, using the bid and ask implied volatilities of European call options written on each asset; in more details, we minimize a weighted root mean squared error, where the error is defined as

$$\begin{cases} \bar{C}_{bid} - C_m & \text{if } C_m < \bar{C}_{bid} \\ 0 & \text{if } \bar{C}_{bid} < C_m < \bar{C}_{ask} \\ C_m - \bar{C}_{ask} & \text{if } C_m > \bar{C}_{ask}. \end{cases}$$

$\bar{C}$ denotes the market quote; the weights are obtained using a procedure based on Huang and Wu (2004). Model prices, denoted by C_m, are computed using the FFT procedures of Carr and Madan (1999); out-of-the-money options are dealt with the time value approach. The second step consists in the calibration of the parameters of the idiosyncratic process, $\mathbf{Y}(t)$, and the systematic component, $Z(t)$, by fitting the correlation coefficient using least squares, and imposing the relevant convolution conditions. In more details, the joint distribution is obtained by fitting the correlation coefficient given in equation (10) for the case of the VG and NIG models, and in equation (12) for the case of Merton and Kou jump diffusion models. The relevant convolution conditions on the corresponding parameters are summarized in Table 2. Finally, we note that the VG and the NIG are 3-parameter processes, therefore the total number of parameters required for the joint fit is $4n + 3$; the Merton JD process, instead, is fully described by 4 parameters implying that the joint fit requires the calibration of $5n + 4$ parameters. This number increases to $6n + 5$ in the case of the Kou JD process.

The calibrated parameters of the margins, the idiosyncratic components and the systematic process are reported in Tables 3-6 for all the valuation dates considered and each model analysed in this paper. We note that the convolution conditions can only be solved numerically which, potentially, could introduce an approximation error. In the attempt of quantifying it, in the Tables we also report the difference between the moments of the distribution of the processes $\mathbf{X}(t)$, calculated using the components parameters in conjunction with equation (2), and the same moments calculated instead using the margin parameters and the exact formulae reported in sections 3 and 4. Further, Figures 1-2 show the QQ plots of the (simulated) samples of the margin process obtained by direct calibration to European vanilla options and the same process obtained, instead, by linear combination of the idiosyncratic process and the systematic process. In particular, in these plots we consider the case of the multivariate VG model and Merton jump diffusion model (similar results can be obtained for the other models presented in this paper and are available from the authors). These results prove the goodness of the fit provided by the multivariate model proposed in this paper, although the accuracy of the approximation tends to deteriorate at the very far end of the tails. Tables 3-6 also report the error produced by the multivariate construction in fitting the given correlation matrices. In particular, the full range of correlation values provided by the data is captured with a satisfactory degree of accuracy.

As a further test, we re-calculate the prices of the European vanilla options using the joint characteristic function and quantify the error against the corresponding market data. Results are reported in Table 7; in particular, we note that the (weighted) root mean squared errors are very close to the ones generated by direct calibration of the marginal distribution, which shows that any potential approximation error introduced by the joint fitting procedure is relatively negligible for this type of application. This is confirmed in Figures 3-4 which show the actual fitting of the multivariate VG model and Merton jump diffusion model in correspondence of all available strikes and maturities (similar results can be obtained for the other models discussed

VALUATION DATE	ASSET	$S(0)$	q	125-day correlation F (Ford)	ABT (Abbott Lab.)	BAX (Baxter)
30/09/2008	F	5.20	0.0%	100%		
	ABT	57.58	2.8%	25%	100%	
	BAX	65.67	1.5%	30%	64%	100%
27/02/2009	F	2.00	0.0%	100%		
	ABT	47.34	3.0%	37%	100%	
	BAX	50.91	1.8%	34%	83%	100%
30/09/2009	F	7.21	0.0%	100%		
	ABT	49.47	3.0%	-22%	100%	
	BAX	57.02	1.7%	-15%	45%	100%

Table 1: Synopsis of market data for Ford Motor Company, Abbott Laboratories and Baxter International Inc. The correlation matrix is estimated using the historical log-returns of the three assets over a 125-day time window, up to (and including) the valuation date. Source: Bloomberg.

in this paper and are available from the authors).

We note at this stage that the higher the number of parameters in the joint distribution, the less flexible the fitting of the multivariate model, and an initial guess which is quite close to the optimal solution is required for the fitting procedure to converge efficiently. This explains the relatively poor (compared to the other models) performance of the multivariate Kou model in fitting the moments of the marginal distribution, as shown in Table 6. Table 7, however, shows that the errors generated by the calibration to market prices of both the univariate and multivariate Kou models are quite close, in spite of the lack of analytical expressions linking the parameters of the marginal jump sizes and the parameters of the components, as discussed in Example 4.

Finally, Table 8 reports the correlation between the process driving the assets log-returns and both the systematic risk component and the idiosyncratic risk process. Thus, we observe the very strong impact of the systematic process on the correlation between the log-returns of Abbott Lab. and Baxter; the role of the common factor though is not so relevant in the case of Ford, confirming the economic considerations offered above. Further evidence is provided by the parameters reported in Tables 3-6; for example, in the case of the VG model specification, the systematic component explains only 13% of the total variance of Ford log-returns in September 2008, against the 60% of the total variance of Abbot Lab. and 67% of Baxter. This increases in February 2009 to 14% for Ford, 90% for Abbott Lab. and 62% for Baxter. In September 2009, the contribution of Z accounts for 10% of the total variation of Ford, and 49%-41% for Abbot Lab. and Baxter respectively. Similar considerations hold for the other models analyzed in this paper.

6 Conclusions

In this note we studied an alternative construction of multivariate Lévy processes which keeps the appealing properties of the approaches existing in the literature and, at the same time, addresses their limitations. The relevance of this topic arises from the increasing need of more

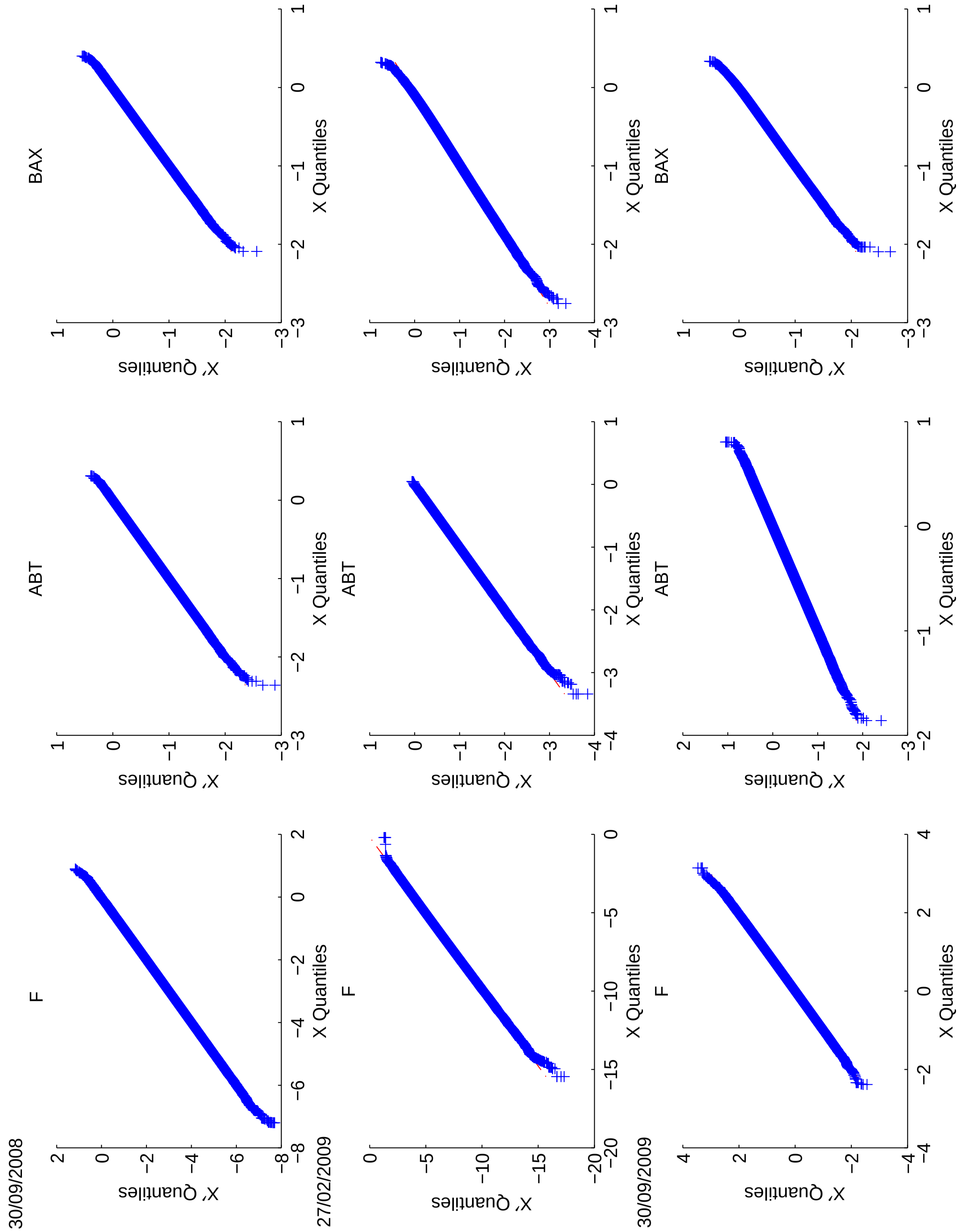

Figure 1: QQ plots of a Monte Carlo sample of the margin VG process, $X(t)$, and the linear transformation process, $X'(t) = Y(t) + aZ(t)$, for Ford, Abbott Lab. and Baxter at September 30, 2008, February 27, 2009 and September 30, 2009. Monte Carlo simulation based on 1,000,000 iterations.

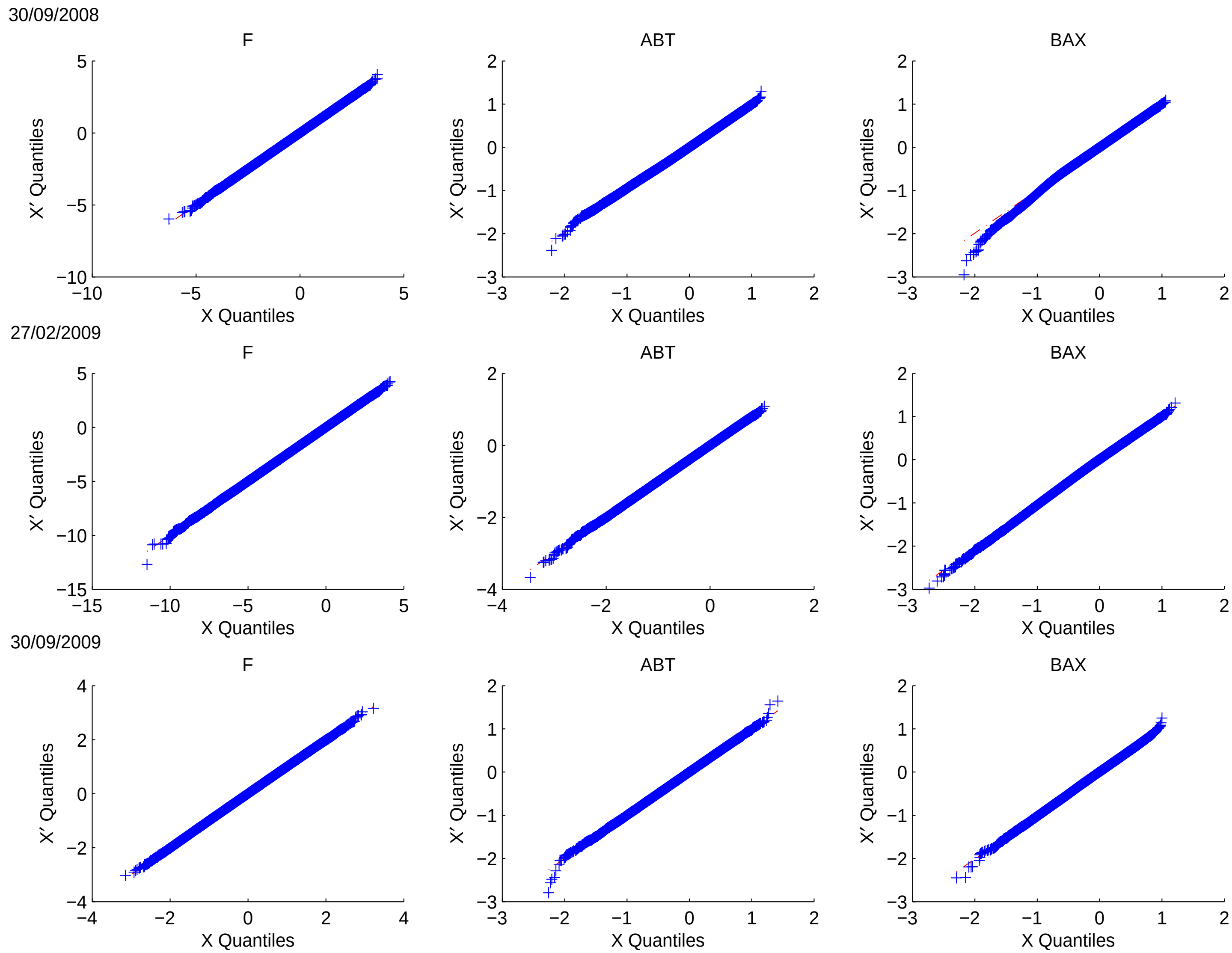

Figure 2: QQ plots of a Monte Carlo sample of the margin Merton process, $X(t)$, and the linear transformation process, $X'(t) = Y(t) + aZ(t)$, for Ford, Abbott Lab. and Baxter at September 30, 2008, February 27, 2009 and September 30, 2009. Monte Carlo simulation based on 1,000,000 iterations.

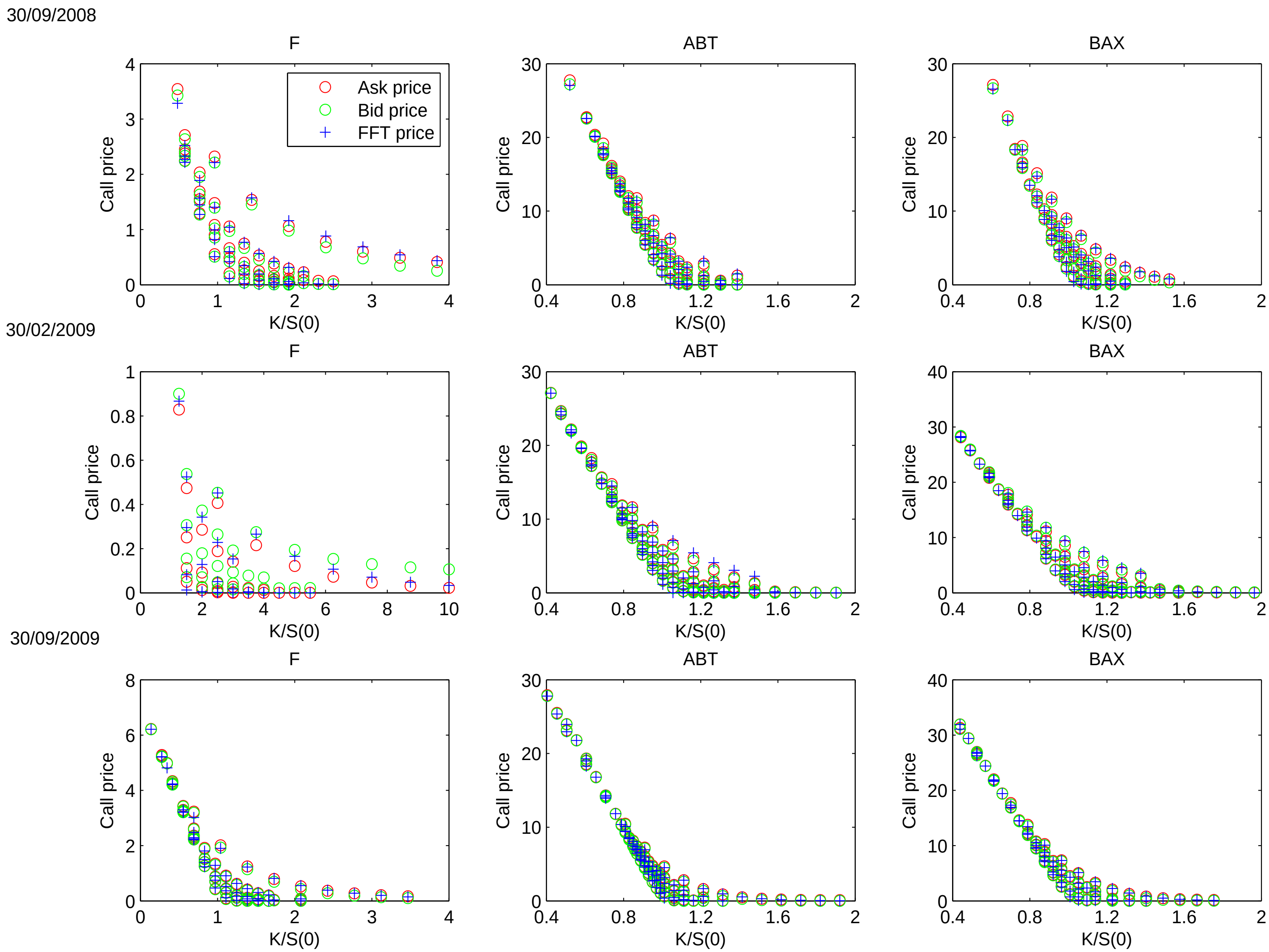

Figure 3: Calibration to market data using the joint distribution. The case of the VG model. In this case, the model option prices have been computed using the joint characteristic function and the parameters of the processes Y and Z reported in Tables 3.

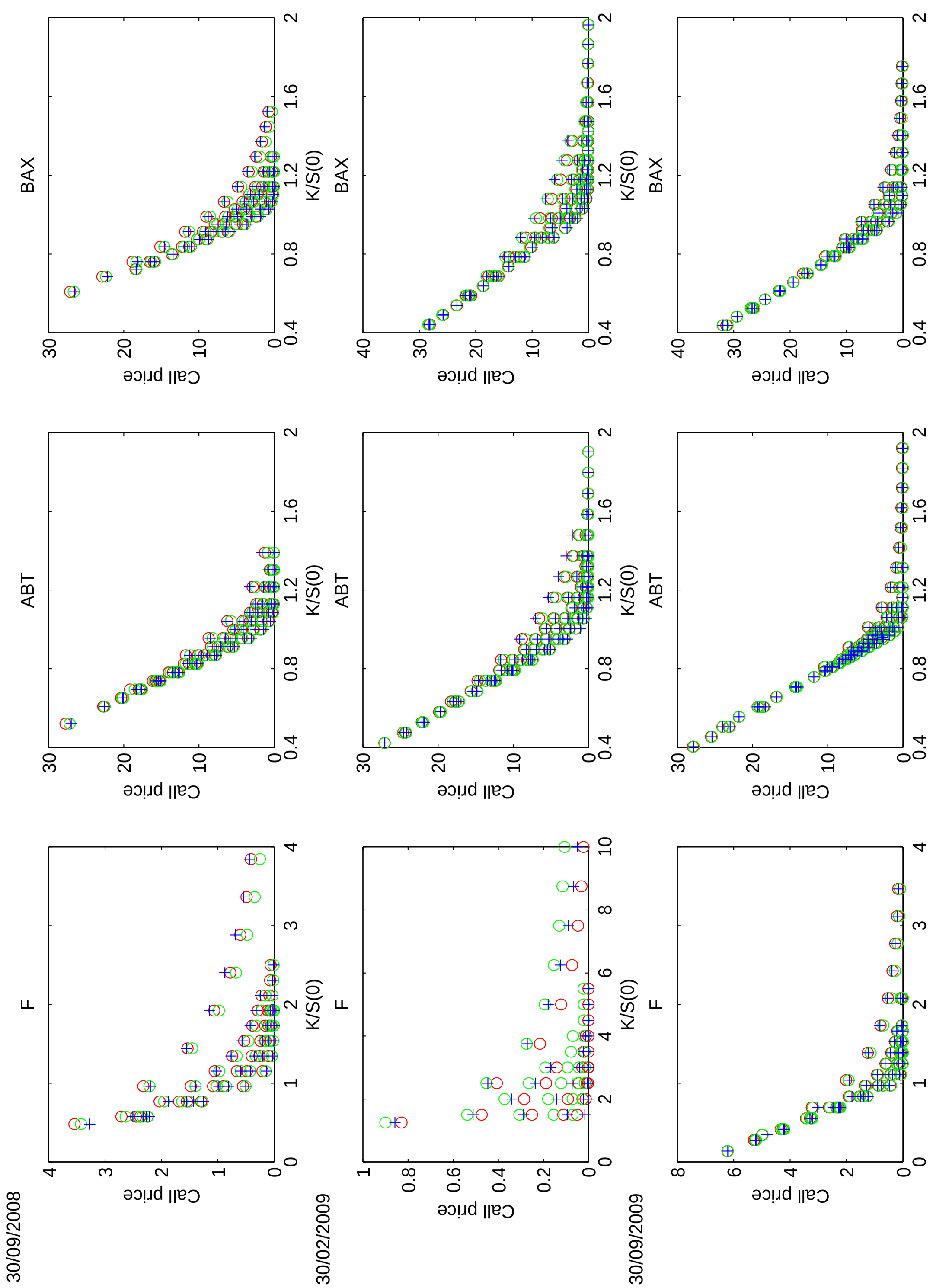

Figure 4: Calibration to market data using the joint distribution. The case of the Merton jump diffusion model. In this case, the model option prices have been computed using the joint characteristic function and the parameters of the processes Y and Z reported in Tables 5.

Normal tempered stable process (VG - NIG processes)	**Jump diffusion process** (Merton - Kou models)
$\theta_j = \beta_j + a_j\beta_Z$ $\sigma_j^2 = \gamma_j^2 + a_j^2\gamma_Z^2$ $k_j = \frac{\nu_j\nu_Z}{\nu_j+\nu_Z}$ $\nu_j\beta_j = \nu_Z a_j\beta_Z$ $\nu_j\gamma_j^2 = \nu_Z a_j^2\gamma_Z^2$	$\sigma_j^2 = \gamma_j^2 + a_j^2\gamma_Z^2$ $\lambda_j = \delta_j + \delta_Z$ $\phi_{\xi_j}(u) = \frac{\delta_j\phi_{\eta_j}(u)+\delta_Z\phi_{\eta_Z}(a_j u)}{\delta_j+\delta_Z}, \quad u \in \mathbb{C}$

Table 2: Convolution conditions for the joint fit of the multivariate time changed Brownian motion and JD processes. Full derivation is provided in sections 3 and 4.

sophisticated models as highlighted by the recent financial crisis. The model can be used for the pricing of multinames products; to this purpose, we discuss its calibration to market data, for a given correlation matrix. This has to be carried out in two steps, consisting of, firstly, the calibration of the margin processes using single names option prices, and secondly the (possibly numerical) solution of the correlation equation (3) and the relevant convolution conditions in order to extract the parameters of the idiosyncratic and systematic components. The proposed model could find further applications in extracting implied correlation surfaces from market data wherever quotes for multinames derivatives are available. Finally, we note that in the applications discussed in this paper, we assume that the margins and the overall multivariate process follow the same distribution; this is necessary in order to maintain the calibration of the margin processes separated from the fitting of the correlation structure. However, Corollary 1 holds for any Lévy process and therefore it allows to relax this assumption.

Acknowledgements

The authors would like to thank Elisa Luciano for providing valuable comments on an earlier version of this paper. Further thanks go to Frederik Ulmer for helpful discussions and suggestions on model calibration issues. The paper in its present format was presented at the IRMC 2010 Conference. We thank the participants for their interesting comments, criticisms, further references and suggestions. The views expressed in this paper are those of the authors; usual caveat applies.

VG MODEL									
	VALUATION DATE								
	30/09/2008			**27/02/2009**			**30/09/2009**		
MARGINS	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
θ	-2.6871	-0.6373	-0.5286	-6.3009	-0.8664	-0.7969	0.4058	-0.2283	-0.5425
σ	0.8537	0.2259	0.2296	0.5354	0.1509	0.2613	0.6040	0.2352	0.2129
k	0.0264	0.0928	0.0897	0.0588	0.1555	0.0805	0.0104	0.2339	0.0944
RMSE	3.75E-02	1.23E-01	1.49E-01	4.54E-02	2.72E-01	3.60E-01	4.78E-02	7.15E-02	1.01E-01
(w)RMSE	2.21E-03	9.94E-03	1.18E-02	4.64E-03	1.51E-02	1.61E-02	1.86E-03	4.09E-03	3.89E-03
IDIOSYNCRATIC PART									
β	-2.1117	-0.2552	-0.1467	-4.9115	-0.0838	-0.1316	0.2888	-0.1168	-0.4356
γ	0.8120	0.1429	0.1488	0.4710	0.0469	0.2311	0.5788	0.1682	0.1431
ν	0.0318	0.2316	0.2137	0.0892	1.6068	0.1512	0.0106	0.4570	0.1176
a	1.1564	0.7678	0.7675	1.4550	0.8197	0.6969	-0.9348	0.8903	0.8541
SYSTEMIC PART									
β_Z	-0.4976			-0.9547			-0.1252		
γ_Z	0.2278			0.1750			0.1846		
ν_Z	0.1547			0.1721			0.4790		
MOMENT MATCHING ERROR									
$\mathbb{E}X$ (1)	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
$stdX$ (1)	-1.34E-03	0.00E+00	-3.72E-03	-4.62E-02	-1.08E-14	-3.80E-02	-4.72E-03	-2.36E-14	-1.33E-14
γ_1 (1)	-3.61E-03	2.11E-15	7.77E-03	3.41E-02	6.64E-14	-1.91E-02	-1.80E-02	1.11E-12	-1.58E-07
γ_2 (1)	-5.32E-05	-5.27E-15	-2.25E-02	-6.16E-02	-2.08E-13	-5.84E-02	-1.28E-02	-4.05E-12	-8.48E-02
CORRELATION ERROR									
F	-			-			-		
ABT	3.05E-02	-		5.43E-08	-		4.60E-07	-	
BAX	-5.84E-04	5.47E-08	-	1.39E-07	1.22E-07	-	-5.28E-02	-9.41E-07	-

Table 3: Calibration of the multivariate VG model. The parameters of the margins, the systemic part and the idiosyncratic components as at 30/09/2009, 27/02/2009 and 30/09/2009. The parameters of the marginal distributions $(\theta_j, \sigma_j, k_j)$ are obtained by direct calibration to market prices. The parameters governing the idiosyncratic risk process, $(\beta_j, \gamma_j, \nu_j, a_j)$, and the systematic risk process, $(\beta_Z, \gamma_Z, \nu_Z)$, are obtained by fitting the correlation matrix and then solving the parameters conditions given in Example 1. The moment matching error is obtained as the difference between the exact moments provided in Example 1 (calculated using the parameters of the marginal process) and the moments reconstructed using equation (2). The final three rows report the difference between the model and the sample correlation.

NIG MODEL									
	VALUATION DATE								
	30/09/2008			**27/02/2009**			**30/09/2009**		
MARGINS	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
θ	-2.0985	-0.3917	-0.3879	-6.2583	-0.8635	-0.8041	0.5358	-0.2567	-0.5414
σ	0.8082	0.2206	0.2141	0.9382	0.2350	0.2570	0.5968	0.2303	0.2167
k	0.0175	0.0698	0.0559	0.0397	0.1140	0.0881	0.0196	0.2536	0.0937
RMSE	3.63E-02	1.20E-01	1.40E-01	4.25E-02	2.72E-01	3.60E-01	5.00E-02	7.15E-02	9.79E-02
(w)RMSE	2.11E-03	9.70E-03	1.13E-02	4.44E-03	1.50E-02	1.60E-02	1.94E-03	4.09E-03	3.76E-03
IDIOSYNCRATIC PART									
β	-1.7346	-0.1828	-0.1483	-4.9265	-0.0836	-0.1328	0.4072	-0.0783	-0.4024
γ	0.7579	0.1507	0.1081	0.8572	0.0731	0.1706	0.5806	0.1271	0.1569
ν	0.0201	0.1495	0.0976	0.0580	1.1777	0.2917	0.0207	0.8316	0.1260
a	1.1480	0.6591	0.7559	1.3965	0.8178	0.7039	-0.6866	0.9523	0.7420
SYSTEMIC PART									
β_Z		-0.3170			-0.9537			-0.1874	
γ_Z		0.2445			0.2731			0.2016	
ν_Z		0.1308			0.1262			0.3648	
MOMENT MATCHING ERROR									
$\mathbb{E}X$ (1)	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
$stdX$ (1)	-6.22E-04	0.00E+00	-2.66E-03	-2.39E-02	-6.48E-13	-7.14E-03	-3.18E-03	-3.35E-13	-8.07E-14
γ_1 (1)	-1.81E-04	4.44E-16	6.64E-02	3.02E-02	6.29E-12	1.23E-03	-2.61E-03	1.54E-11	-2.24E-07
γ_2 (1)	-2.40E-04	-1.39E-15	-1.39E-01	-6.35E-02	-2.36E-11	-4.23E-02	-3.76E-03	-7.30E-11	-2.83E-02
CORRELATION ERROR									
F	-			-			-		
ABT	1.50E-02	-		6.43E-07	-		9.88E-07	-	
BAX	1.80E-02	-5.48E-08	-	1.80E-07	4.38E-07	-	-1.59E-02	7.42E-02	-

Table 4: Calibration of the multivariate NIG model. The parameters of the margins, the systemic part and the idiosyncratic components as at 30/09/2009, 27/02/2009 and 30/09/2009. The parameters of the marginal distributions $(\theta_j, \sigma_j, k_j)$ are obtained by direct calibration to market prices. The parameters governing the idiosyncratic risk process, $(\beta_j, \gamma_j, \nu_j, a_j)$, and the systematic risk process, $(\beta_Z, \gamma_Z, \nu_Z)$, are obtained by fitting the correlation matrix and then solving the parameters conditions given in Example 2. The moment matching error is obtained as the difference between the exact moments provided in Example 2 (calculated using the parameters of the marginal process) and the moments reconstructed using equation (2). The final three rows report the difference between the model and the sample correlation.

MERTON MODEL									
	VALUATION DATE								
MARGINS	**30/09/2008**			**27/02/2009**			**30/09/2009**		
	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
σ	0.8232	0.2553	0.2353	0.9462	0.2193	0.2551	0.5858	0.2085	0.2219
λ	0.6969	0.1974	0.2017	2.3781	0.5619	0.5674	0.2708	0.2055	0.2186
ϑ	-0.4738	-0.2600	-0.2837	-0.6313	-0.3574	-0.2644	0.0177	-0.2099	-0.3031
υ	0.2770	0.1998	0.1867	0.4650	0.2251	0.1800	0.3120	0.2772	0.1500
RMSE	5.03E-02	1.53E-01	1.46E-01	3.69E-02	2.74E-01	3.60E-01	4.67E-02	6.97E-02	9.79E-02
(w)RMSE	3.37E-03	1.23E-02	1.20E-02	3.87E-03	1.51E-02	1.60E-02	1.82E-03	3.10E-03	3.88E-03
IDIOSYNCRATIC PART									
γ	0.7748	0.1751	0.1018	0.8909	0.1000	0.1960	0.5682	0.1000	0.1791
δ	0.5183	0.0187	0.0231	1.9165	0.1003	0.1057	0.1683	0.1030	0.1161
ϑ_Y	-0.5267	-0.5807	-0.3502	-0.6257	0.1445	0.0491	-0.0578	-0.2320	-0.4288
υ_Y	0.2736	0.1252	0.2651	0.4881	0.1469	0.0100	0.3352	0.2792	0.0784
a	0.6467	0.4320	0.4936	0.8863	0.5428	0.4544	-0.5606	0.7188	0.5148
SYSTEMIC PART									
γ_Z	0.4299			0.3595			0.2546		
δ_Z	0.1787			0.4617			0.1025		
ϑ_Z	-0.5073			-0.7437			-0.2695		
υ_Z	0.3062			0.3900			0.3738		
MOMENT MATCHING ERROR									
$\mathbb{E}X(1)$	1.40E-03	-1.30E-03	-4.42E-03	2.09E-03	-3.37E-06	8.19E-04	-9.76E-04	6.04E-04	-2.27E-03
$stdX(1)$	5.89E-04	4.99E-03	6.38E-03	-1.74E-04	1.26E-04	-1.32E-02	1.09E-04	6.30E-05	-5.12E-03
$\gamma_1(1)$	4.16E-04	-4.23E-02	–6.42E-02	-1.41E-03	9.06E-03	9.32E-02	1.03E-02	4.41E-03	3.51E-02
$\gamma_2(1)$	1.83E-03	7.07E-02	9.96E-02	-2.24E-04	-1.98E-02	-1.76E-01	-2.02E-03	5.89E-03	-5.94E-02
CORRELATION ERROR									
F	-			-			-		
ABT	-2.08E-04	-		-3.87E-07	-		5.03E-04	-	
BAX	-9.11E-04	3.56E-04	-	-3.56E-07	-8.68E-07	-	-7.91E-04	-3.47E-04	-

Table 5: Calibration of the multivariate Merton jump diffusion model. The parameters of the margins, the systemic part and the idiosyncratic components as at 30/09/2009, 27/02/2009 and 30/09/2009. The parameters of the marginal distributions $(\sigma_j, \lambda_j, \vartheta_j, \upsilon_j)$ are obtained by direct calibration to market prices. The parameters governing the idiosyncratic risk process, $(\gamma_j, \delta_j, \vartheta_{Yj}, \upsilon_{Yj}, a_j)$, and the systematic risk process, $(\gamma_Z, \delta_Z, \vartheta_Z, \upsilon_Z)$, are obtained by fitting the correlation matrix and then solving the parameters conditions given in Example 3. The moment matching error is obtained as the difference between the exact moments provided in Example 3 (calculated using the parameters of the marginal process) and the moments reconstructed using equation (2). The final three rows report the difference between the model and the sample correlation.

KOU MODEL									
	VALUATION DATE								
MARGINS	30/09/2008			27/02/2009			30/09/2009		
	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
σ	0.8116	0.2549	0.2500	1.1776	0.2234	0.2794	0.5828	0.2300	0.2432
λ	0.2549	0.2000	0.2170	0.9873	0.9939	0.5105	0.3624	0.2033	0.2115
p	0.0254	0.3000	0.6184	0.0579	0.0772	0.0658	0.2385	0.4600	0.2300
α^+	21.3413	22.1345	26.9924	5.5704	19.7977	11.7134	3.5667	8.0000	22.4224
α^-	2.4050	3.5001	2.7864	2.0662	4.1458	4.5614	7.3701	4.3438	4.8337
RMSE	5.11E-02	1.63E-01	1.55E-01	3.77E-02	2.71E-01	3.56E-01	4.81E-02	1.25E-01	1.25E-01
(w)RMSE	3.57E-03	1.31E-02	1.29E-02	4.51E-03	1.49E-02	1.59E-02	1.87E-03	6.66E-03	5.17E-03
IDIOSYNCRATIC PART									
γ	0.7667	0.1639	0.1169	1.1446	0.1042	0.2278	0.5629	0.1216	0.2000
δ	0.0668	0.0119	0.0289	0.4967	0.5033	0.0198	0.2424	0.0833	0.0915
p_Y	0.0115	0.0568	0.4937	0.0100	0.9000	0.1000	0.2681	0.2185	0.2031
α_Y^+	6.3406	6.4452	41.0497	1.9113	4.7301	38.6608	3.4307	4.5397	8.8803
α_Y^-	1.7582	1.9443	1.8969	2.0154	5.0739	87.2172	6.3361	4.5398	4.1503
a	0.5455	0.4003	0.4530	0.9538	0.6813	0.5581	-0.5226	0.6754	0.4788
SYSTEMIC PART									
γ_Z	0.4878			0.2900			0.2891		
δ_Z	0.1881			0.4906			0.1200		
p_Z	0.1908			0.0204			0.3709		
α_Z^+	2.3883			6.7138			22.9985		
α_Z^-	2.3863			2.0884			3.2841		
MOMENT MATCHING ERROR									
$\mathbb{E}X(1)$	-3.90E-02	-1.21E-02	4.70E-03	1.96E-02	-1.47E-01	2.61E-02	-1.52E-02	1.10E-02	-5.97E-03
$stdX(1)$	1.37E-02	1.13E-02	-1.41E-06	-2.34E-03	-4.74E-02	-3.05E-02	0.00E+00	2.55E-05	3.62E-03
$\gamma_1(1)$	-3.41E-02	-2.14E-01	-2.16E-01	-1.32E-03	-3.62E-01	2.93E-01	0.00E+00	1.20E-04	-4.84E-02
$\gamma_2(1)$	-2.13E-02	-4.83E-01	1.12E-04	2.53E-03	-9.59E-01	-1.02E+00	0.00E+00	-6.71E-04	-3.02E-04
CORRELATION ERROR									
F	-			-			-		
ABT	9.16E-03	-		1.06E-04	-		-2.57E-04	-	
BAX	0.00E+00	2.89E-08	-	6.25E-04	-2.44E-04	-	1.90E-04	-5.49E-04	-

Table 6: Calibration of the multivariate Kou jump diffusion model. The parameters of the margins, the systemic part and the idiosyncratic components as at 30/09/2009, 27/02/2009 and 30/09/2009. The parameters of the marginal distributions $(\sigma_j, \lambda_j, p_j, \alpha_j^+, \alpha_j^-)$ are obtained by direct calibration to market prices. The parameters governing the idiosyncratic risk process, $(\gamma_j, \delta_j, p_{Yj}, \alpha_{Yj}^+, \alpha_{Yj}^-, a_j)$, and the systematic risk process, $(\gamma_Z, \delta_Z, p_Z, \alpha_Z^+, \alpha_Z^-)$, are obtained by fitting the correlation matrix and then solving the parameters conditions given in Example 4. The moment matching error is obtained as the difference between the exact moments provided in Example 4 (calculated using the parameters of the marginal process) and the moments reconstructed using equation (2). The final three rows report the difference between the model and the sample correlation.

VALUATION DATE									
	30/09/2008			**27/02/2009**			**30/09/2009**		
MODEL	VG PROCESS								
	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
RMSE	9.49E-03	1.70E-07	1.18E-07	9.71E-09	-5.14E-08	-1.30E-08	3.69E-09	0.00E+00	1.11E-09
(w)RMSE	8.81E-04	-1.17E-08	2.45E-09	1.17E-09	-3.61E-10	-7.81E-10	9.43E-11	0.00E+00	5.45E-11
	NIG PROCESS								
	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
RMSE	3.82E-02	2.86E-01	2.95E-01	-1.85E-09	4.90E-09	-3.91E-09	4.16E-09	0.00E+00	-4.86E-11
(w)RMSE	3.00E-03	2.27E-02	3.04E-02	-2.35E-10	1.19E-11	9.51E-10	9.68E-11	0.00E+00	3.12E-10
	MERTON PROCESS								
	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
RMSE	3.92E-05	6.05E-03	6.69E-04	-9.62E-06	-7.44E-05	2.63E-02	5.80E-05	-6.14E-04	-1.11E-03
(w)RMSE	3.92E-06	5.20E-04	5.97E-04	-1.80E-06	-4.99E-07	8.55E-04	2.05E-06	-3.40E-05	-8.55E-05
	KOU PROCESS								
	F	ABT	BAX	F	ABT	BAX	F	ABT	BAX
RMSE	2.56E-02	5.37E-02	6.71E-02	-6.49E-04	4.68E-01	6.07E-02	-1.35E-12	-5.20E-05	8.08E-03
(w)RMSE	1.72E-03	4.32E-03	5.75E-03	-7.79E-05	2.35E-02	2.07E-03	2.12E-12	-3.44E-06	3.83E-04

Table 7: Calibration errors. The table reports the difference between the errors produced by the calibration to market option prices of the margin processes, $\mathbf{X}(t)$, and the linear transformation $\mathbf{Y}(t) + \mathbf{a}Z(t)$.

VALUATION DATE												
	30/09/2008				**27/02/2009**				**30/09/2009**			
	VG MODEL		NIG MODEL		VG MODEL		NIG MODEL		VG MODEL		NIG MODEL	
	Z	$\mathbf{Y}$	Z	$\mathbf{Y}$	Z	$\mathbf{Y}$	Z	$\mathbf{Y}$	Z	$\mathbf{Y}$	Z	$\mathbf{Y}$
F	0.3622	0.9336	0.3628	0.9326	0.3893	0.9519	0.3893	0.9377	-0.3148	0.9573	-0.2639	0.9700
ABT	0.7713	0.6328	0.7303	0.6831	0.9504	0.3111	0.9504	0.3111	0.6987	0.7154	0.8337	0.5522
BAX	0.8265	0.5862	0.8764	0.5050	0.8733	0.6850	0.8733	0.5276	0.6440	0.7650	0.6287	0.7776
	JD Merton		JD Kou		JD Merton		JD Kou		JD Merton		JD Kou	
	Z	$\mathbf{Y}$	Z	$\mathbf{Y}$	Z	$\mathbf{Y}$	Z	$\mathbf{Y}$	Z	$\mathbf{Y}$	Z	$\mathbf{Y}$
F	0.3416	0.9392	0.3485	0.9203	0.3893	0.9212	0.3899	0.9228	-0.2713	0.9623	-0.2709	0.9626
ABT	0.7313	0.6569	0.7436	0.6105	0.9504	0.3100	0.9495	0.5932	0.8090	0.5874	0.8129	0.5822
BAX	0.8763	0.4339	0.8607	0.5091	0.8733	0.5604	0.8739	0.6458	0.5558	0.8538	0.5529	0.8171

Table 8: Correlation between the asset log-returns and the common component, Z, and their idiosyncratic part, $\mathbf{Y}$. These values have been obtained using equation (4) and the parameters of the components.

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