POLITECNICO
MILANO 1863

THE FINITE ELEMENT METHOD (2b)

EXERCISE

Milano, 18 May 2016
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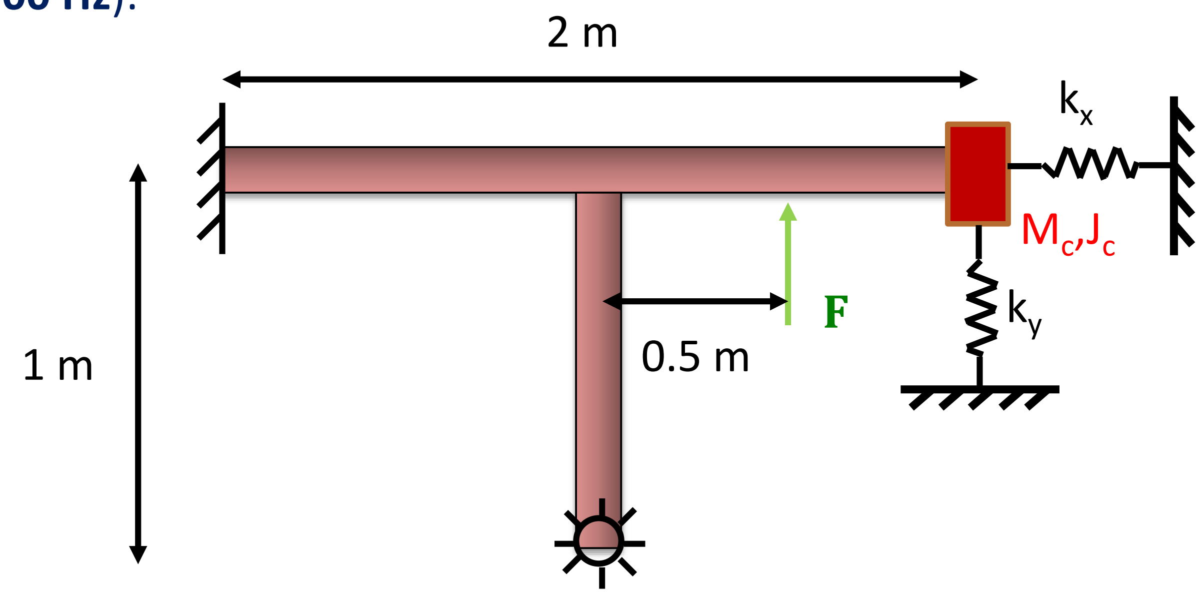
Exercise

DATA:

- Mass per unit length **$m = 9.75$ kg/m**
- Flexural rigidity **$EJ = 1.34 \cdot 10^4$ Nm^2**
- Axial stiffness **$EA = 2.57 \cdot 10^7$ N**
- Lumped mass **$M_c = 10$ kg; $J_c = 1$ kgm^2**
- Springs **$k_x = 2.0 \cdot 10^6$ N/m; $k_y = 3.0 \cdot 10^6$ N/m;**
- Structural Damping Matrix $[C_s] = \alpha[M]+\beta[K]$

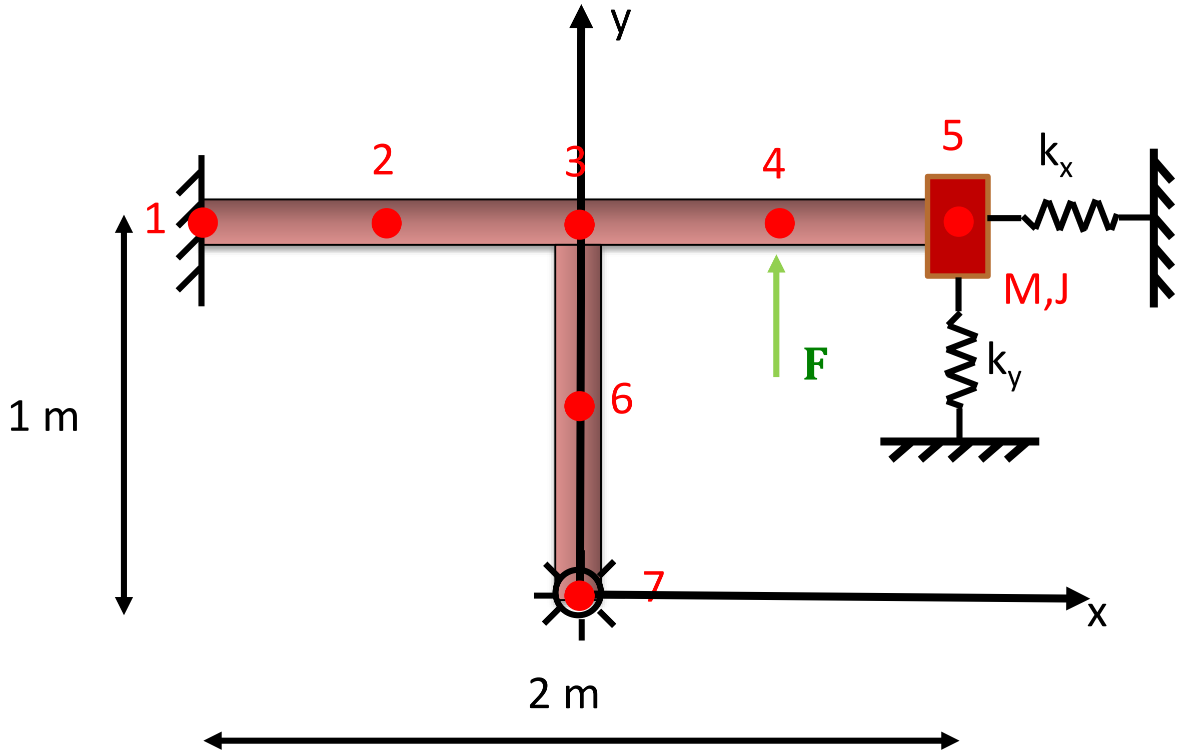
We want to calculate (**up to 100 Hz**):

- **Natural frequencies;**
- **Mode shapes.**



Definition of length of beam finite element

Length of beam finite element l_{fe} = 0.5 m



Assembly of Mass Matrix

$[M_{FF}]_{16\times16}$ $[M_{FC}]_{16\times5}$

$[M] =$

3,250	0	0	0,813	0	0	0	0	0	0	0	0	0	0	0	0	0,813	0	0	0	0
0	3,621	0	0	0,627	-0,075	0	0	0	0	0	0	0	0	0	0	0	0,627	0,075	0	0
0	0	0,023	0	0,075	-0,009	0	0	0	0	0	0	0	0	0	0	0	-0,075	-0,009	0	0
0,813	0	0	5,061	0	0,128	0,813	0	0	0	0	0	0,627	0	-0,075	0	0	0	0	0	0
0	0,627	0,075	0	5,246	0,000	0	0,627	-0,075	0,000	0	0	0	0,813	0	0	0	0	0	0	0
0	-0,075	-0,009	0,128	0	0,035	0	0,075	-0,009	0,000	0	0	0,075	0	-0,009	0	0	0	0	0	0
0	0	0	0,813	0	0	3,250	0	0,000	0,813	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0,627	0,075	0	3,621	0	0	0,627	-0,075	0	0	0	0	0	0	0	0	0
0	0	0	0	-0,075	-0,009	0	0	0,023	0	0,075	-0,009	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0,813	0	0	11,625	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0,627	0,075	0	11,811	-0,128	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-0,075	-0,009	0	-0,128	1,012	0	0	0	0	0	0	0	0	0
0	0	0	0,627	0	0,075	0	0	0	0	0	0	3,621	0	0	-0,075	0	0	0	0,627	0
0	0	0	0	0,813	0	0	0	0	0	0	0	0	3,250	0	0	0	0	0	0,000	0,813
0	0	0	-0,075	0	-0,009	0	0	0	0	0	0	0	0	0,023	-0,009	0	0	0	0,075	0
0	0	0	0	0	0	0	0	0	0	0	0	-0,075	0	-0,009	0,012	0	0	0	-0,128	0
0,813	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1,625	0	0	0	0
0	0,627	-0,075	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1,811	0,128	0	0
0	0,075	-0,009	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,128	0,012	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0,627	0	0,075	-0,128	0	0	0	1,811	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0,813	0	0	0	0	0	0	1,625

$[M_{CF}]_{5\times16}$ $[M_{CC}]_{5\times5}$

Assembly of Stiffness Matrix

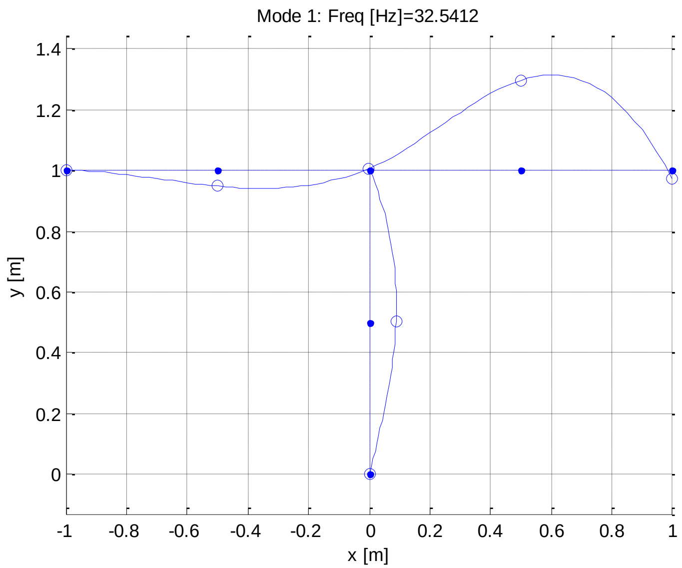
$[K_{FF}]_{16\times16}$ $[K_{FC}]_{16\times5}$

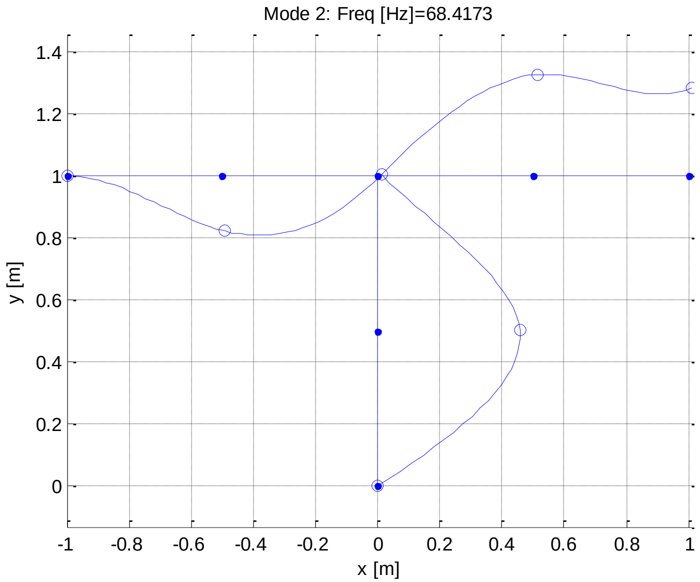
$[K] =$

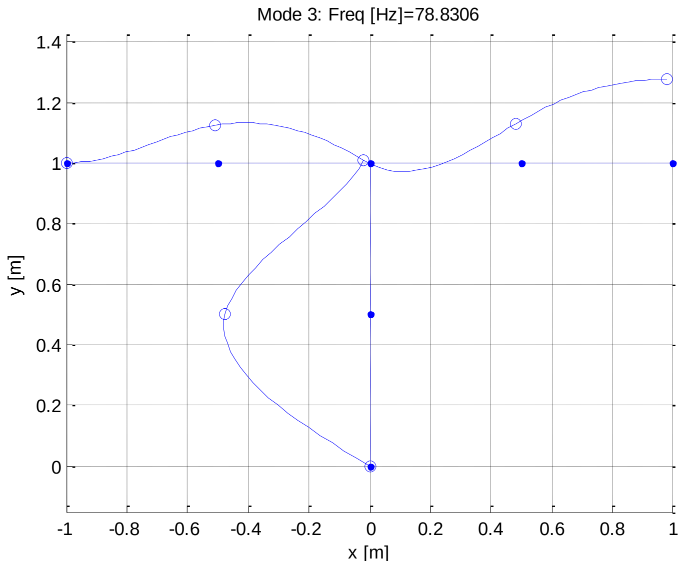
102800000	0	0	-51400000	0	0	0	0	0	0	0	0	0	0	0	0	-51400000	0	0	0	0
0	2572800	0	0	-1286400	321600	0	0	0	0	0	0	0	0	0	0	0	-1286400	-321600	0	0
0	0	214400	0	-321600	53600	0	0	0	0	0	0	0	0	0	0	0	321600	53600	0	0
-51400000	0	0	104086400	0	321600	-51400000	0	0	0	0	0	-1286400	0	321600	0	0	0	0	0	0
0	-1286400	-321600	0	53972800	0	0	-1286400	321600	0	0	0	0	-51400000	0	0	0	0	0	0	0
0	321600	53600	321600	0	321600	0	-321600	53600	0	0	0	-321600	0	53600	0	0	0	0	0	0
0	0	0	-51400000	0	0	102800000	0	0	-51400000	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-1286400	-321600	0	2572800	0	0	-1286400	321600	0	0	0	0	0	0	0	0	0
0	0	0	0	321600	53600	0	0	214400	0	-321600	53600	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-51400000	0	0	53400000	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-1286400	-321600	0	4286400	-321600	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	321600	53600	0	-321600	107200	0	0	0	0	0	0	0	0	0
0	0	0	-1286400	0	-321600	0	0	0	0	0	0	2572800	0	0	321600	0	0	0	-1286400	0
0	0	0	0	-51400000	0	0	0	0	0	0	0	0	102800000	0	0	0	0	0	0	-51400000
0	0	0	321600	0	53600	0	0	0	0	0	0	0	0	214400	53600	0	0	0	-321600	0
0	0	0	0	0	0	0	0	0	0	0	0	321600	0	53600	107200	0	0	0	-321600	0
-51400000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	51400000	0	0	0	0
0	-1286400	321600	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1286400	321600	0	0
0	-321600	53600	0	0	0	0	0	0	0	0	0	0	0	0	0	0	321600	107200	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-1286400	0	-321600	-321600	0	0	0	1286400	0
0	0	0	0	0	0	0	0	0	0	0	0	0	-51400000	0	0	0	0	0	0	51400000

$[K_{CF}]_{5\times16}$ $[K_{CC}]_{5\times5}$

Natural Frequencies and Mode Shapes



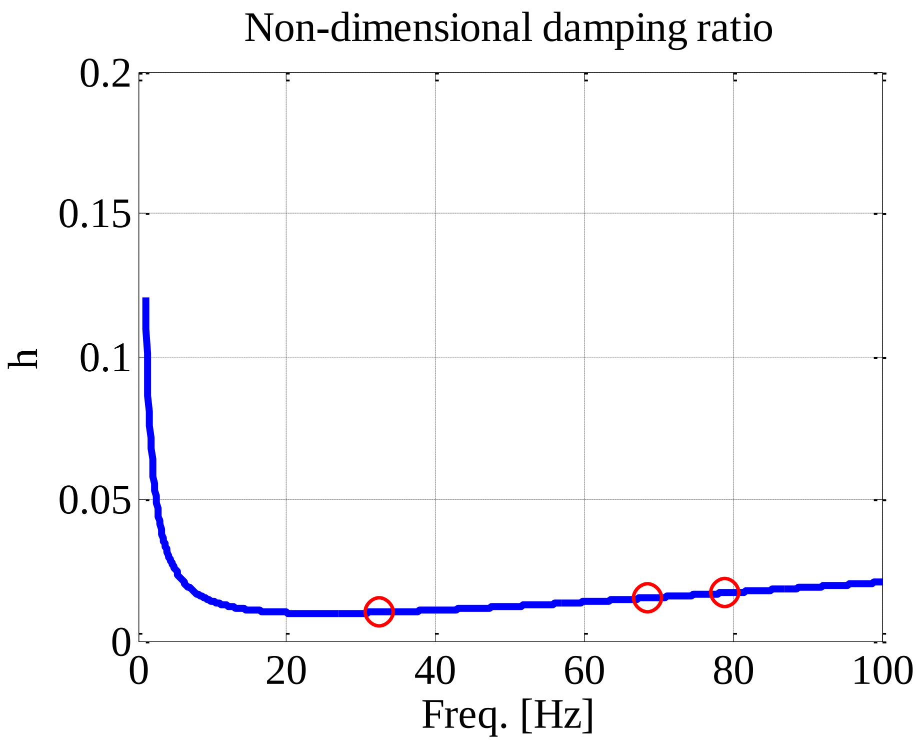




Structural Damping Matrix

As an example, from experimental analysis $h_1 = 0.01$ and $h_2 = 0.015$. We want to estimate α and β parameters.

$$h_i = \frac{c_i}{2m_i\omega_{n,i}} = \frac{\alpha}{2\omega_{n,i}} + \frac{\beta\omega_{n,i}}{2}$$



$$\begin{bmatrix} \frac{1}{2\omega_1} & \frac{\omega_1}{2} \\ \frac{1}{2\omega_2} & \frac{\omega_2}{2} \\ \cdots & \cdots \\ \frac{1}{2\omega_n} & \frac{\omega_n}{2} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \ldots \\ h_n \end{bmatrix}$$

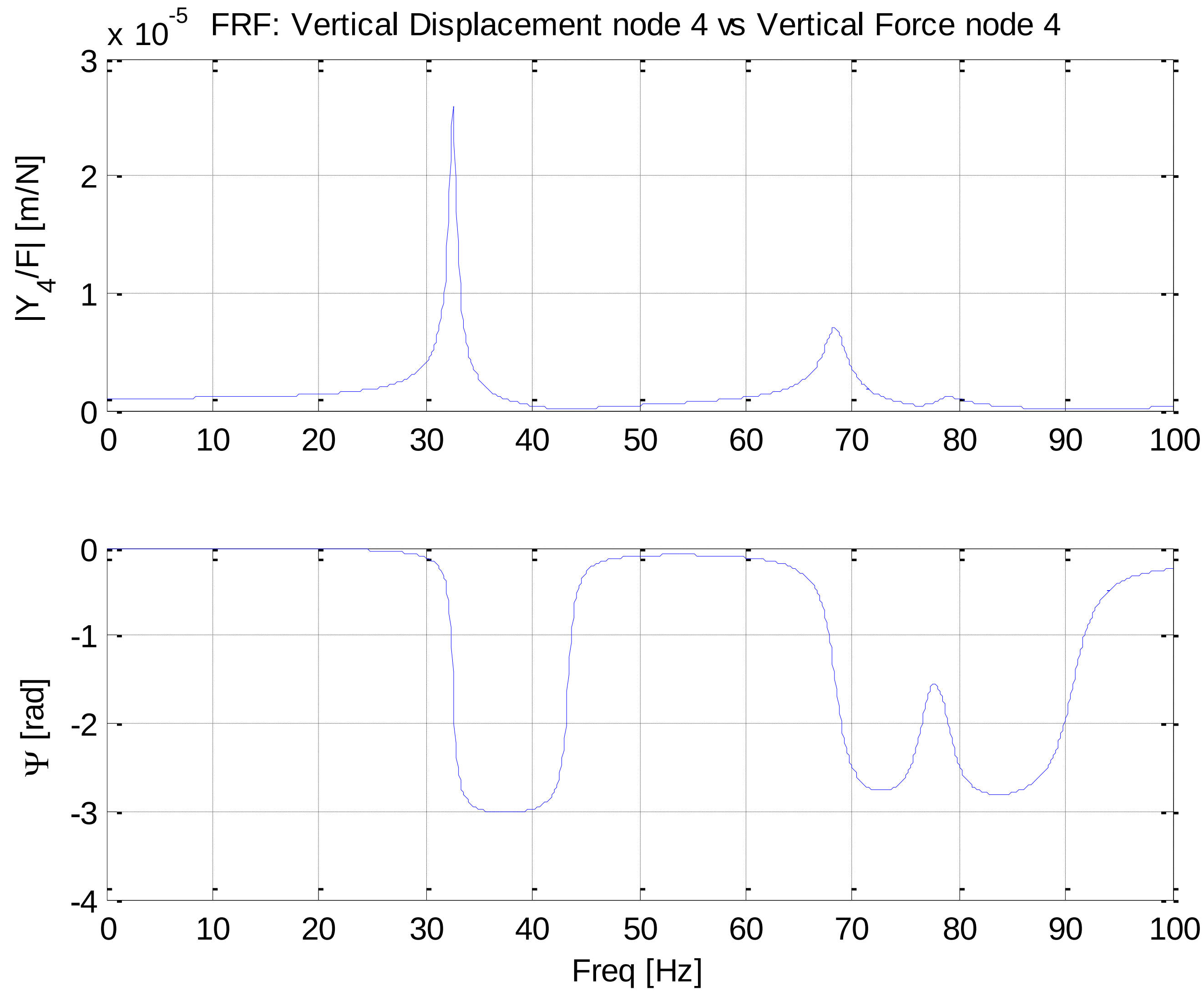
$$\alpha = 1.51\ s^{-1}$$

$$\beta = 6.16 \cdot 10^{-5}\ s$$

$$[C_s] = \alpha[\mathrm{M}] + \beta[\mathrm{K}]$$

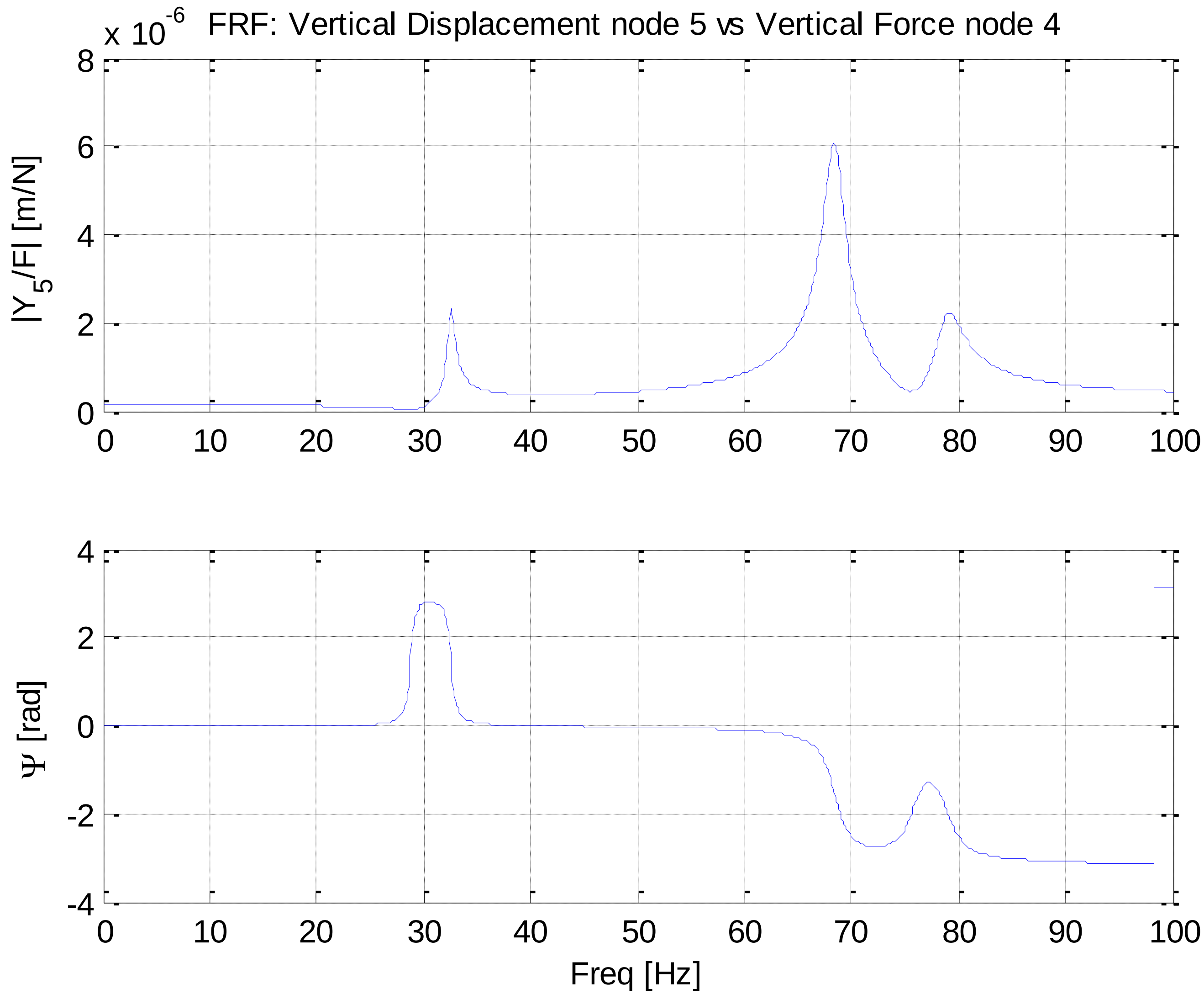
Frequency Response Function

FRF Y_4/F



Frequency Response Function

FRF Y_5/F



Reaction force

Remember the equation of motion for the system

$$\begin{bmatrix} M_{FF} & M_{FC} \\ M_{CF} & M_{CC} \end{bmatrix} \begin{Bmatrix} \ddot{x}_F \\ \ddot{x}_C \end{Bmatrix} + \begin{bmatrix} C_{FF} & C_{FC} \\ C_{CF} & C_{CC} \end{bmatrix} \begin{Bmatrix} \dot{x}_F \\ \dot{x}_C \end{Bmatrix} + \begin{bmatrix} K_{FF} & K_{FC} \\ K_{CF} & K_{CC} \end{bmatrix} \begin{Bmatrix} x_F \\ x_C \end{Bmatrix} = \begin{Bmatrix} F \\ R \end{Bmatrix}$$

In this problem $x_C = \underline{0}$ therefore

$$R = M_{CF}\ddot{x}_F + C_{CF}\dot{x}_F + K_{CF}x_F$$

Reaction force

Assuming

$$x_F = x_0 e^{i\Omega t}$$

In this problem $x_C = \underline{0}$ therefore

$$x_F = (-\Omega^2 M_{FF} + i\Omega C_{FF} \dot{x_F} + K_{FF})^{-1} F$$

And the reactions:

$$R = (-\Omega^2 M_{CF} + i\Omega C_{CF} \dot{x_F} + K_{CF})\, x_F$$

Frequency Response Function

FRF M_1/F