POLITECNICO
MILANO 1863

THE FINITE ELEMENT METHOD (2a)

THE MATRIX ASSEMBLY

Milano, 18 May 2016

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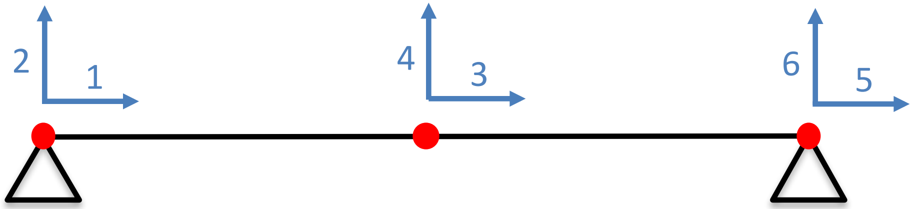
Matrix assembly

All the structures studied with the FE method are composed by several elements.

The structure is divided into **finite elements**. The DoFs of the structure are the **nodal displacements**.



Matrix assembly

Each element has it's own **mass and stiffness matrixes** depending on the type of the element (string, beam, shell, brick…)

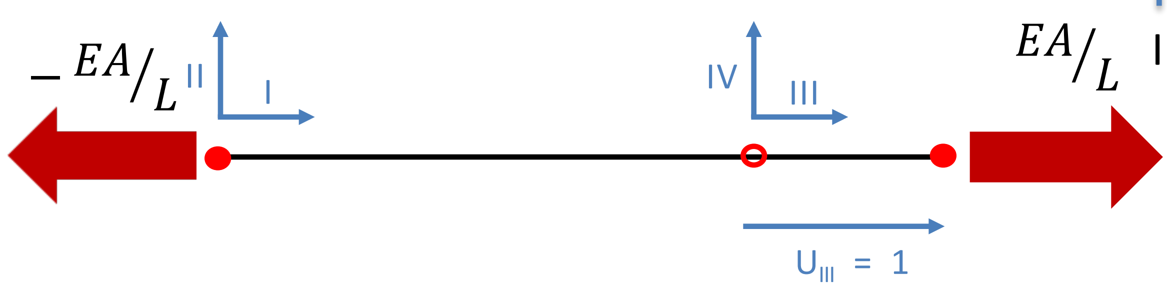
$$M = L\,m\begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{6} & 0 & \frac{1}{3} \end{bmatrix} \qquad K = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & \frac{S}{L} & 0 & -\frac{S}{L} \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & -\frac{S}{L} & 0 & \frac{S}{L} \end{bmatrix}$$

How to obtain the system matrixes?

Matrix assembly

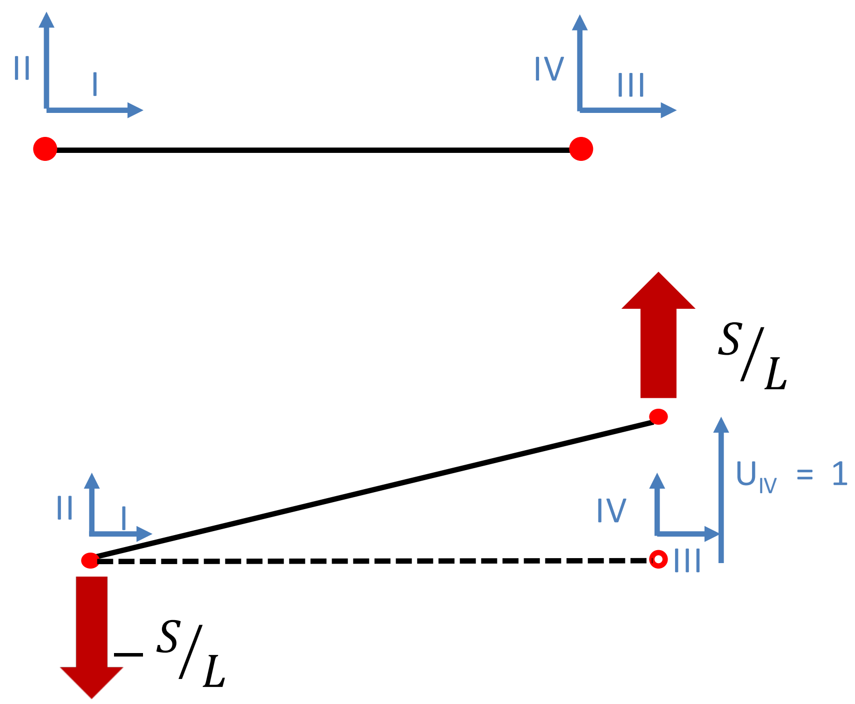
What does each element of the elemental matrix represent?

$$K = \begin{array}{cc} \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & \frac{S}{L} & 0 & -\frac{S}{L} \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & -\frac{S}{L} & 0 & \frac{S}{L} \end{bmatrix} & \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array} \\ \begin{array}{cccc} \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} \end{array} & \end{array}$$



Matrix assembly

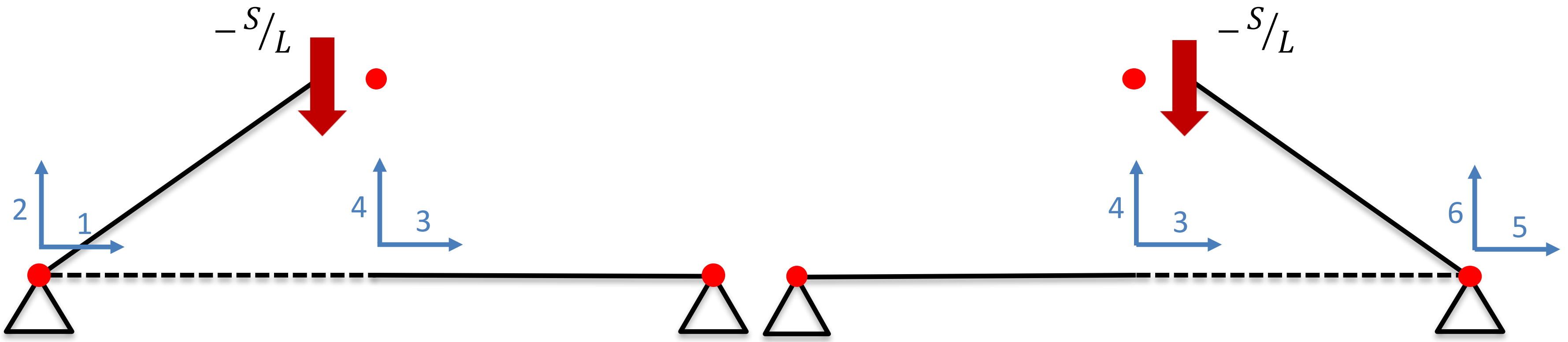
What does each element of the elemental matrix represent?



$$K = \begin{array}{c} \\ \end{array}\begin{array}{cc} \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & \frac{S}{L} & 0 & -\frac{S}{L} \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & -\frac{S}{L} & 0 & \frac{S}{L} \end{bmatrix} & \begin{array}{l} \leftarrow \mathrm{I} \\ \leftarrow \mathrm{II} \\ \leftarrow \mathrm{III} \\ \leftarrow \mathrm{IV} \end{array} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} \end{array} & \end{array}$$

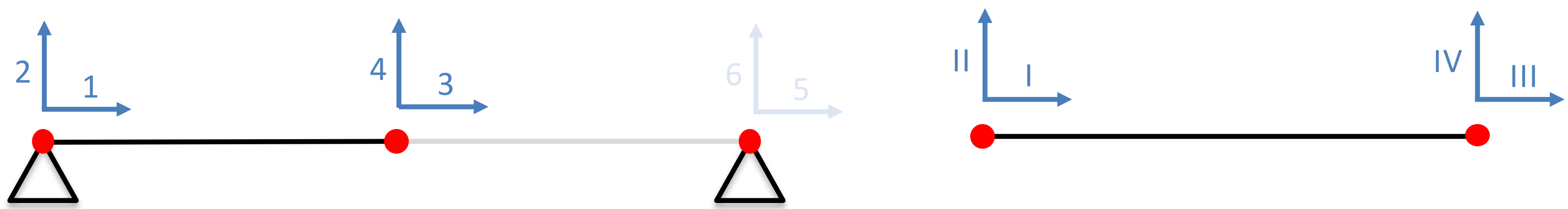
Matrix assembly

When I move a global dof I see the contribution of all the elements connected to that node



Matrix assembly

We need to find the correspondence between each **element dof** and the **global dof.**



For the first element:

- I → 1
- II → 2
- III → 3
- IV → 4

Matrix assembly

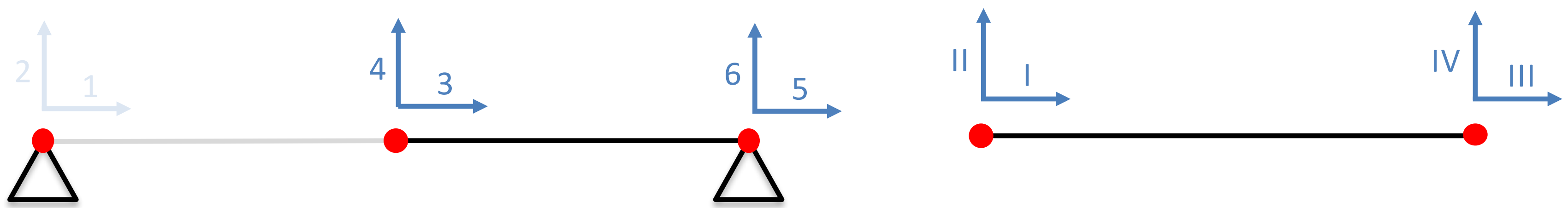
How the first element contributes to the overall stiffness?

$$K = \begin{array}{cc} \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & \frac{S}{L} & 0 & -\frac{S}{L} \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & -\frac{S}{L} & 0 & \frac{S}{L} \end{bmatrix} & \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array} \\ \begin{array}{cccc} \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} \end{array} & \end{array}$$

$$K_{tot} = \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Matrix assembly

We need to find the correspondence between each **element dof** and the **global dof.**



For the second element:

- I → 3
- II → 4
- III → 5
- IV → 6

Matrix assembly

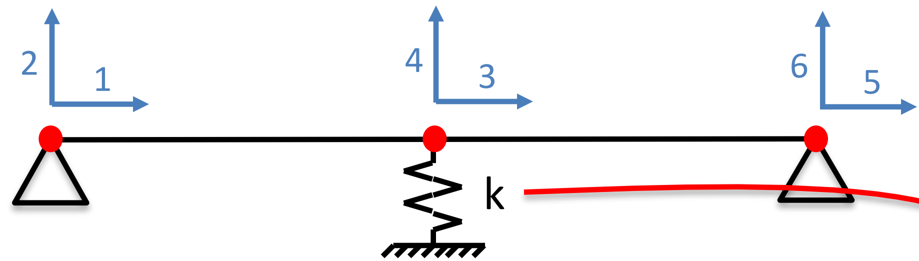
How the second element contributes to the overall stiffness?

$$
K = \begin{array}{cc}
\begin{bmatrix}
\frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\
0 & \frac{S}{L} & 0 & -\frac{S}{L} \\
-\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\
0 & -\frac{S}{L} & 0 & \frac{S}{L}
\end{bmatrix} & \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array} \\
\begin{array}{cccc} \quad\mathrm{I} & \quad\mathrm{II} & \quad\mathrm{III} & \quad\mathrm{IV} \end{array} &
\end{array}
$$

$$
K_{tot} = \begin{array}{cc}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\
\begin{bmatrix}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & *
\end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

Concentrated stiffness

How can I handle a concentrated spring?

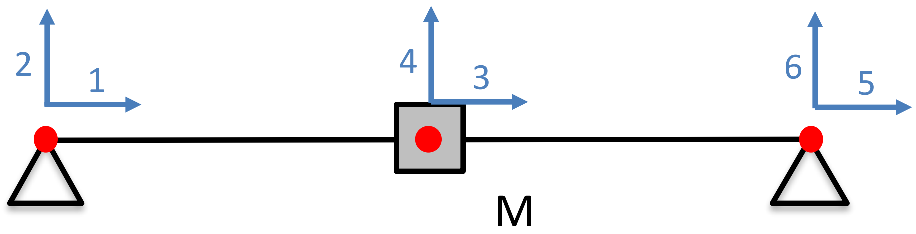


The spring "works" for the global dof number 4

$$K_{tot} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & \boldsymbol{k} & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Concentrated mass

And the mass? Exactly the same!

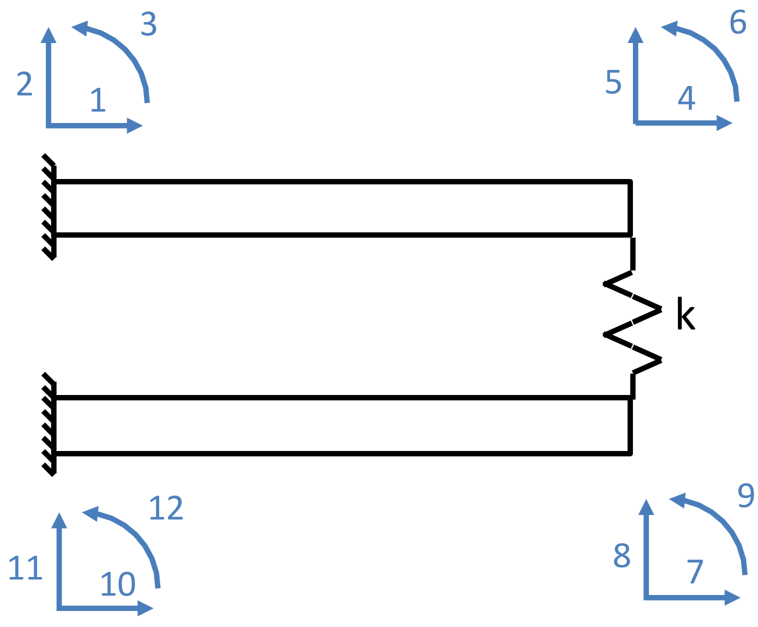


The mass "works" for the global dofs number 3 and 4

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
M_{tot} = & \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & \boldsymbol{M} & * & * & * \\ * & * & * & \boldsymbol{M} & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

Internal contrains

How do I handle an internal spring?

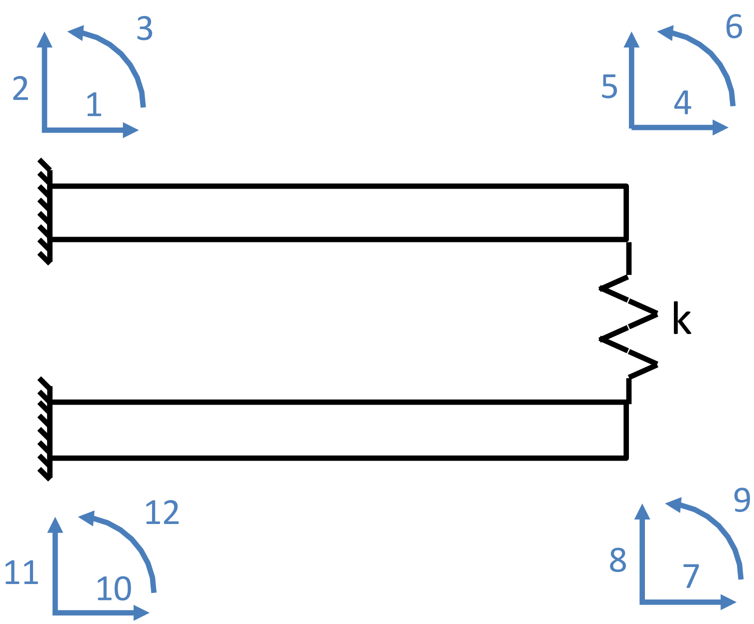


The potential energy of the spring is:

$$V = \frac{1}{2}k(u_5 - u_8)^2$$
$$= \frac{1}{2}k(u_5^2 - u_5 u_8 - u_8 u_5 + u_8^2)$$

Internal contrains

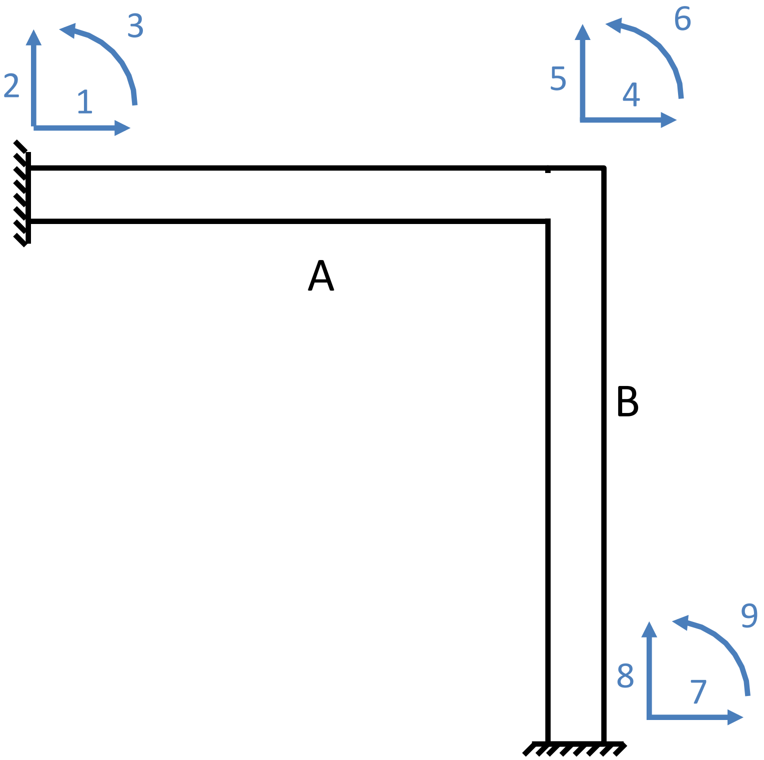
How do I handle an internal spring?



$$k_{tot} = \begin{array}{c} \begin{array}{ccccc} & 5 & & 8 & \end{array} \\ \begin{bmatrix} \ddots & \vdots & & \vdots & \\ \cdots & k & \cdots & -k & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & -k & \cdots & k & \cdots \\ & \vdots & & \vdots & \ddots \end{bmatrix} \end{array} \begin{array}{c} \\ \\ 5 \\ \\ 8 \\ \\ \end{array}$$

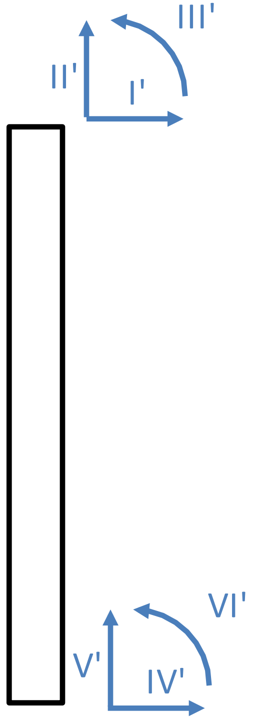
Rotation

What if an element is rotated?



Rotation

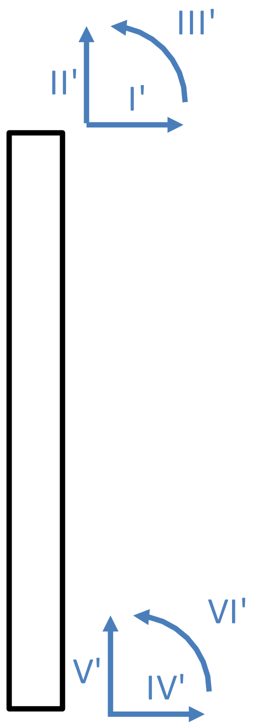
We compute the stiffness matrix and the mass matrix in the rotated reference frame:



$$\begin{bmatrix} I' \\ II' \\ III' \\ IV' \\ V' \\ VI' \end{bmatrix} = \underbrace{\left\{ \begin{matrix} \cos\gamma & \sin\gamma & 0 & & & \\ -\sin\gamma & \cos\gamma & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & \cos\gamma & \sin\gamma & 0 \\ & 0 & & -\sin\gamma & \cos\gamma & 0 \\ & & & 0 & 0 & 1 \end{matrix} \right\}}_{\Lambda} \begin{bmatrix} I \\ II \\ III \\ IV \\ V \\ VI \end{bmatrix}$$

Rotation

We compute the stiffness matrix and the mass matrix in the rotated reference frame:



$$K'_{el} = \Lambda^{-1} K_{el} \Lambda$$

$$M'_{el} = \Lambda^{-1} M_{el} \Lambda$$