POLITECNICO DI MILANO





THE FINITE ELEMENT METHOD

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DATA:

- Mass per unit length **$m = 9.75$ kg/m**
- Flexural rigidity **$EJ = 1.34 \cdot 10^4$ Nm^2**
- Axial stiffness **$EA = 2.57 \cdot 10^7$ N**
- Length **$L = 2$ m**
- Structural Damping Matrix $[C_s] = \alpha[M]+\beta[K]$ (α **$= 0.2$ s^{-1}; $\beta = 1 \cdot 10^{-4}$ s**)

We want to calculate:

- **Natural frequencies;**
- **Mode shapes.**

Definition of length of beam finite element

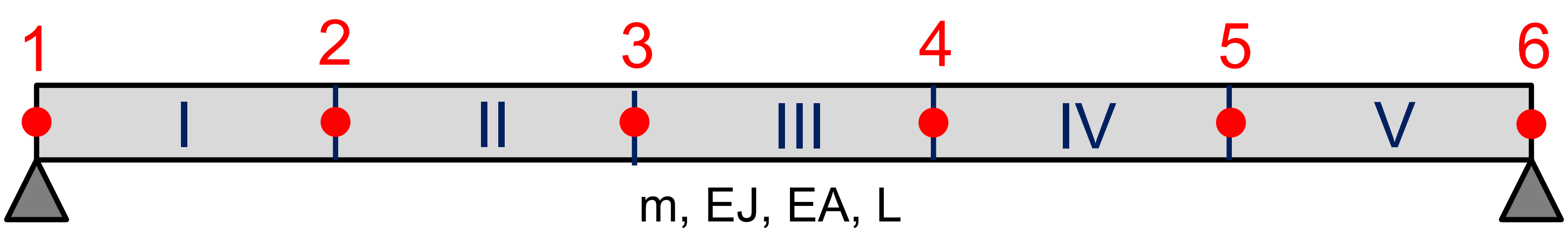
The length of the beam finite element influences the maximum frequency of the structure analysis.

The beam finite element must have a quasi-static behaviour (the considered shape functions are the static ones) during vibration of the structure.

$$\omega_{fe,i} = \left(\frac{\pi}{l_{fe,i}}\right)^2 \sqrt{\frac{EJ}{m}} >> \Omega_{max} \quad \Rightarrow \quad l_{fe,i,max} << \sqrt{\frac{\pi^2}{\Omega_{max}} \cdot \sqrt{\frac{EJ}{m}}}$$

For the considered structure:

f_{max} [Hz]	Ω_{max} [rad/s]	$l_{fe,i,max}$ [m]
5	31.4	3.41
10	62.8	2.41
20	125.7	1.71
50	314.2	1.08
100	628.3	0.76
200	1256.6	0.54
300	**1885.0**	**0.44**



Definition of input file (*.inp)


```
! FEM(1)
! 1st Exercise
!          -----------------------------------------
! list of nodes :
*NODES
! n. of node - constraint code (x,y,theta) - x coordinate- y coordinate.
1                   1 1 0              0.0      0.0
2                   0 0 0              0.4      0.0
3                   0 0 0              0.8      0.0
4                   0 0 0              1.2      0.0
5                   0 0 0              1.6      0.0
6                   1 1 0              2.0      0.0
! end card *NODES
*ENDNODES
!          -----------------------------------------
! list of elements :
*BEAMS
! n. of elem. - n. of input node - n. of output node - m, EA, EJ
1       1      2     9.75    2.57e7    1.34e4
2       2      3     9.75    2.57e7    1.34e4
3       3      4     9.75    2.57e7    1.34e4
4       4      5     9.75    2.57e7    1.34e4
5       5      6     9.75    2.57e7    1.34e4
*ENDBEAMS
```

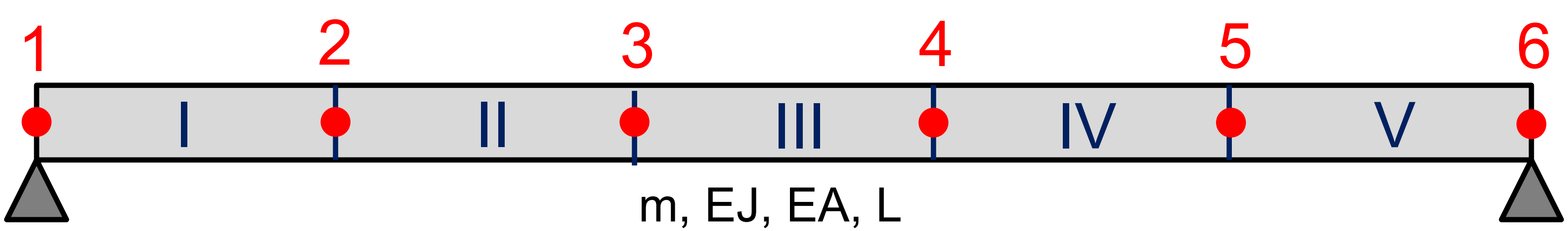
! = symbol for comment

*NODES = start definition of nodes

*ENDNODES = end definition of nodes

*BEAMS = start definition of beam finite element

*ENDBEAMS = end definition of beam finite element



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! end card *NODES
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!            ---------------------------------------------
! list of elements :
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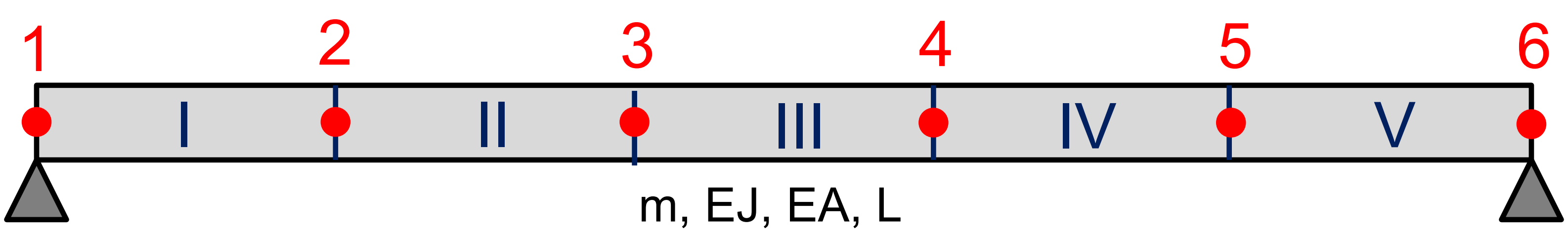
! n. of elem. - n. of input node - n. of output node - m, EA, EJ
1       1       2    9.75    2.57e7    1.34e4
2       2       3    9.75    2.57e7    1.34e4
3       3       4    9.75    2.57e7    1.34e4
4       4       5    9.75    2.57e7    1.34e4
5       5       6    9.75    2.57e7    1.34e4
*ENDBEAMS
```

Constraint code:
0 = free d.o.f.
1 = constrained d.o.f.

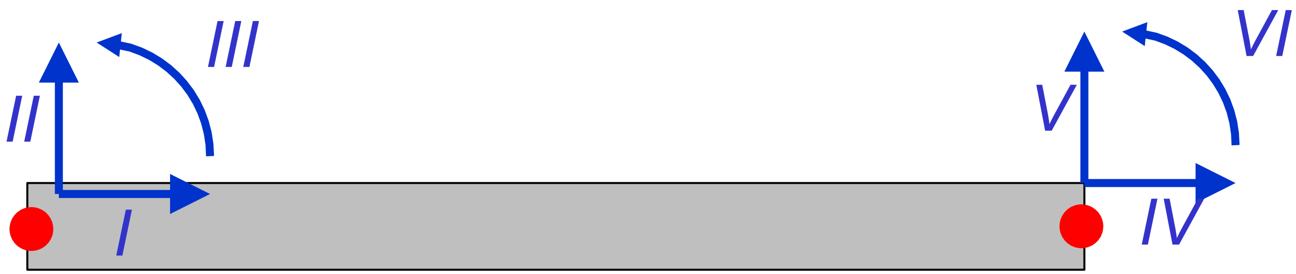
Coordinates of node:
x [m], y [m]

Left and right nodes of the beam finite element

Properties of the beam finite element:
m [kg/m], EA [N], EJ [Nm^2]



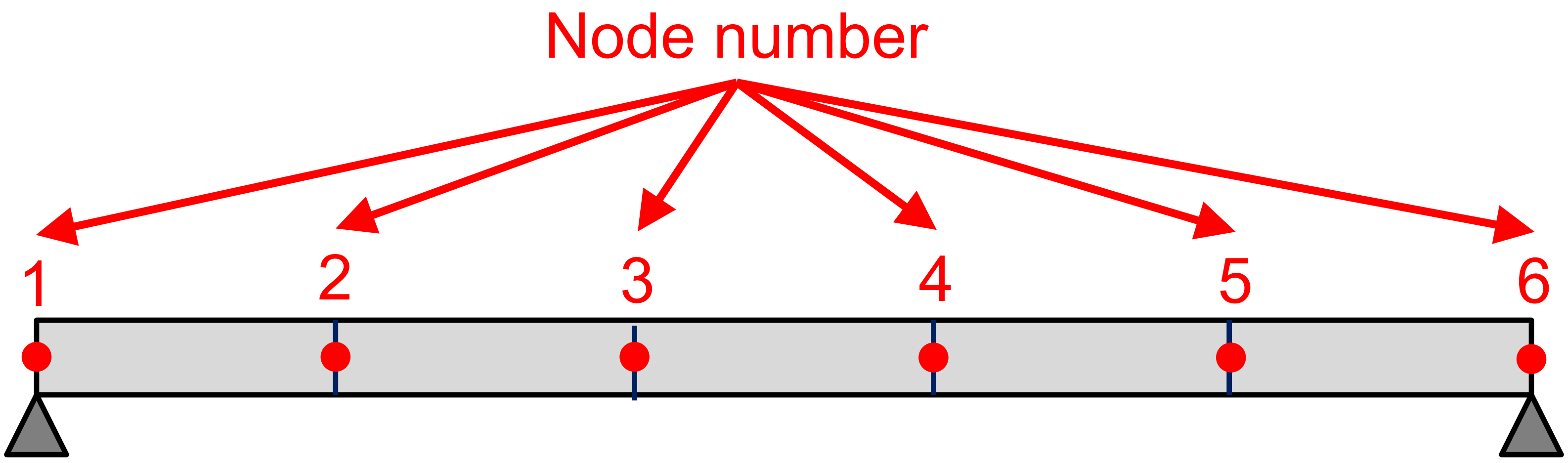
Each element has it’s own **mass and stiffness matrixes** depending on the type of the element (string, beam, shell, brick…)



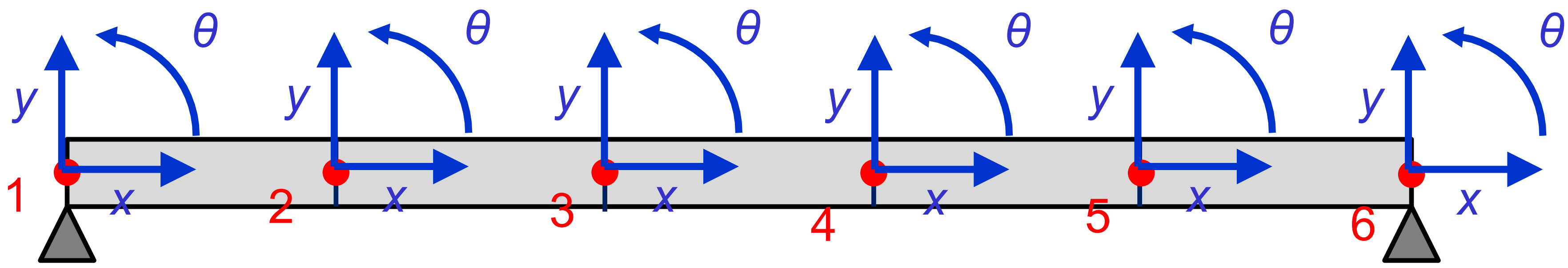
$$\boldsymbol{K}_{el}=\frac{EJ}{L^3}\begin{bmatrix}12 & 6\,L & -12 & 6\,L\\ 6\,L & 4\,L^2 & 6\,L & 2\,L^2\\ -12 & -6\,L & 12 & -6\,L\\ 6\,L & 2\,L^2 & -6\,L & 4\,L^2\end{bmatrix}\quad \boldsymbol{M}_{el}=mL\begin{bmatrix}\frac{12}{35} & -\frac{11}{210}L & \frac{9}{70} & \frac{13}{420}L\\ -\frac{11}{210}L & \frac{1}{105}L^2 & -\frac{13}{420}L & -\frac{1}{140}L^2\\ \frac{9}{70} & -\frac{13}{420}L & \frac{13}{35} & \frac{11}{210}L\\ \frac{13}{420}L & -\frac{1}{140}L^2 & \frac{11}{210}L & \frac{1}{105}L^2\end{bmatrix}$$

How to obtain the system matrixes?

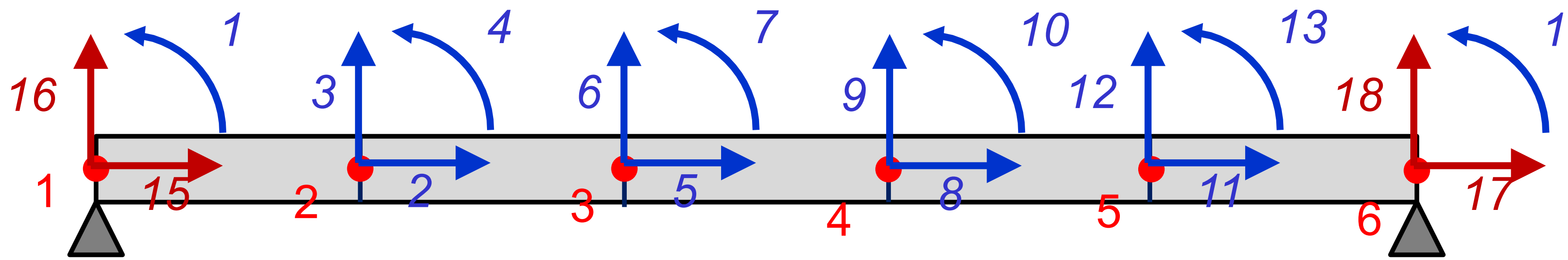
The matrix assembly



The matrix assembly

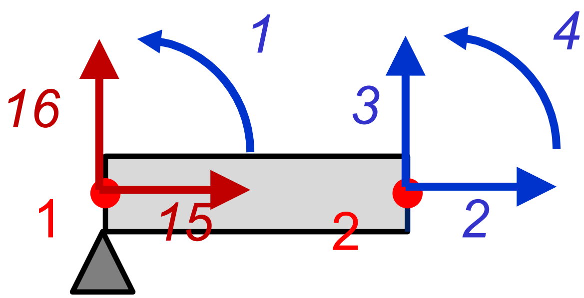


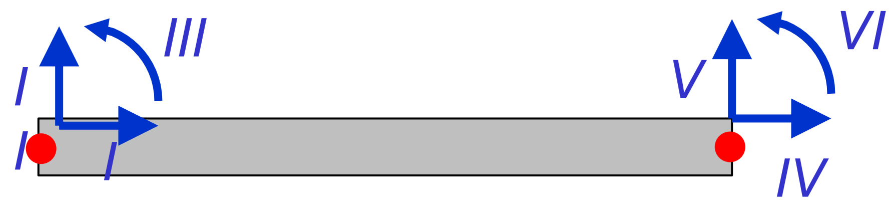
The matrix assembly



idf matrix

Node	X DoF	Y DoF	Rotation
1	15	16	1
2	2	3	4
3	5	6	7
4	8	9	10
5	11	12	13
6	17	18	14





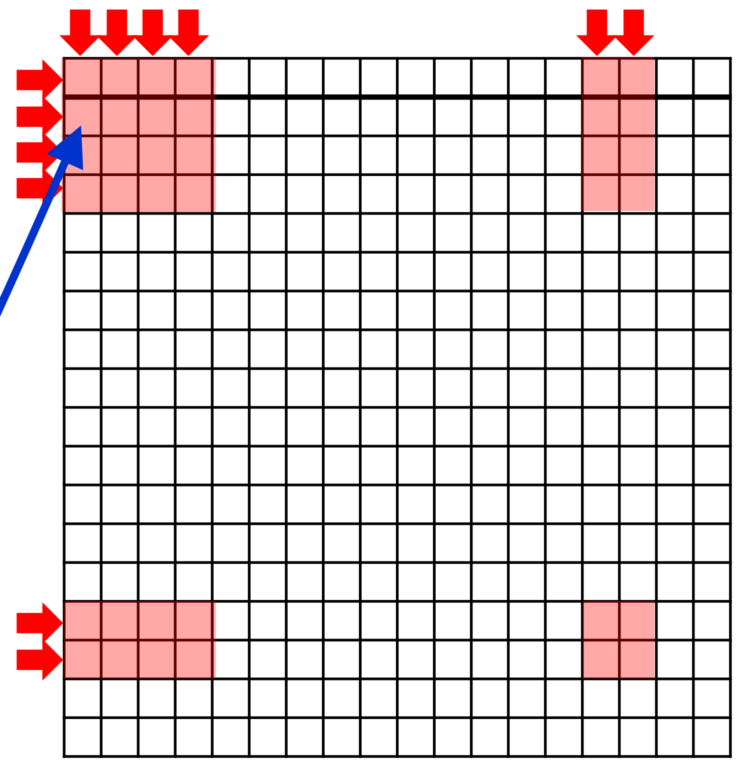
Incidence matrix

Element	I	II	III	IV	V	VI
1	15	16	1	2	3	4
2	2	3	4	5	6	7
3	5	6	7	8	9	10
4	8	9	10	11	12	13
5	11	12	13	17	18	14

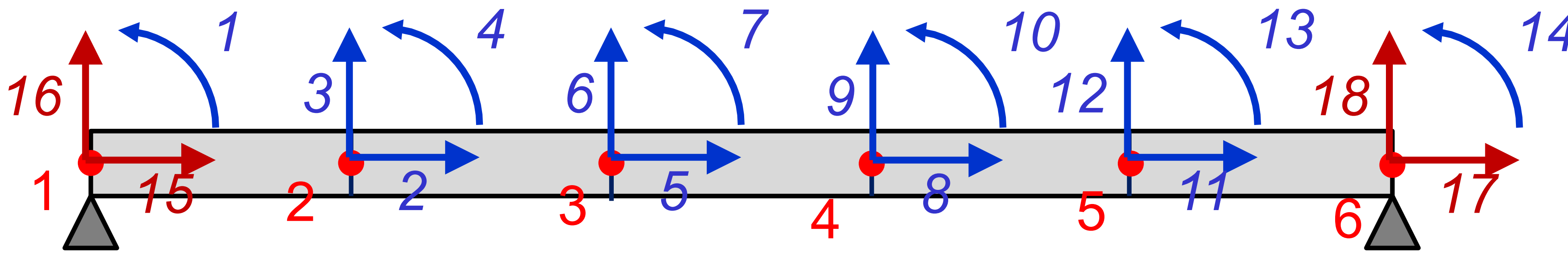
The matrix assembly

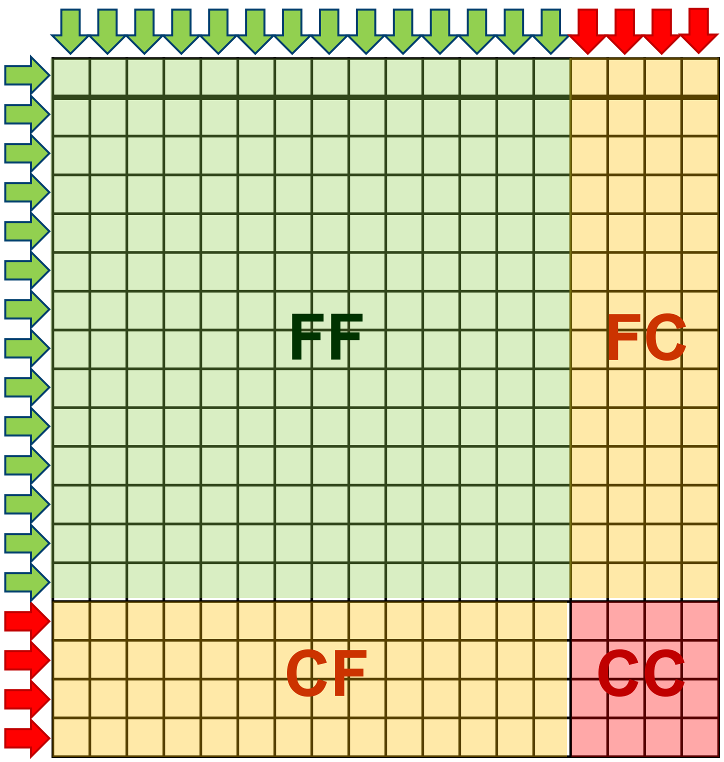
Element	I	II	III	IV	V	VI
1	15	16	1	2	3	4
2	2	3	4	5	6	7

$$\boldsymbol{M}_{el} = mL \begin{array}{c} \\ \\ \\ \\ \text{IV} \end{array} \begin{array}{cccc} & & \text{III} & \\ \left[\begin{matrix} \frac{12}{35} & -\frac{11}{210}L & \frac{9}{70} & \frac{13}{420}L \\ -\frac{11}{210}L & \frac{1}{105}L^2 & -\frac{13}{420}L & -\frac{1}{140}L^2 \\ \frac{9}{70} & -\frac{13}{420}L & \frac{13}{35} & \frac{11}{210}L \\ \frac{13}{420}L & -\frac{1}{140}L^2 & \frac{11}{210}L & \frac{1}{105}L^2 \end{matrix} \right] & & & \end{array}$$



The matrix partitioning



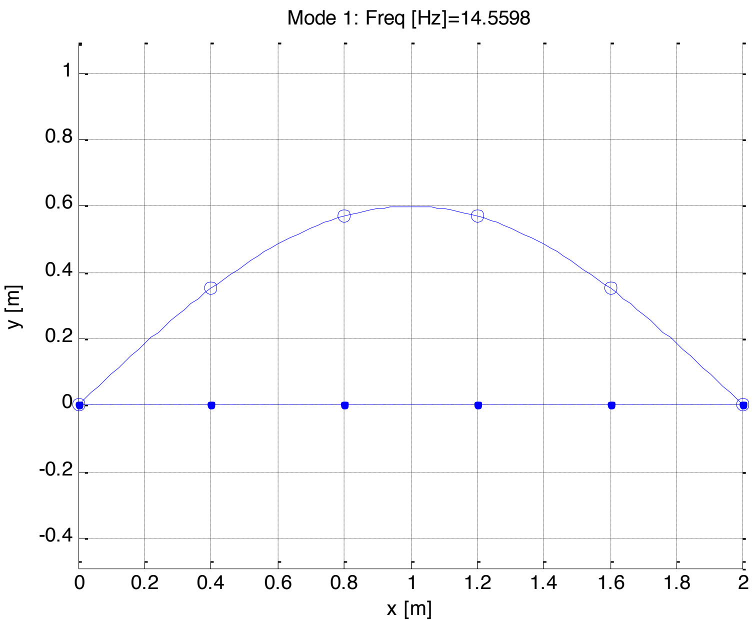


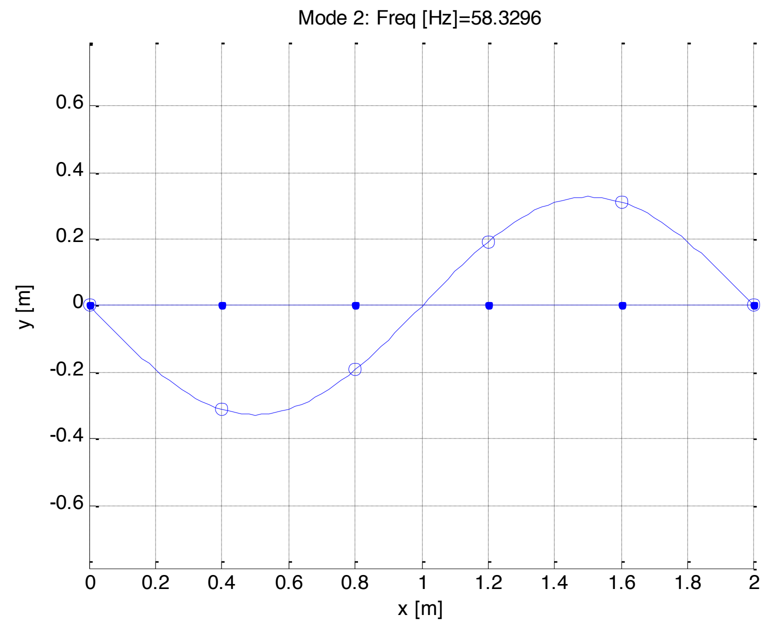
If we divide the structure in 5 beam finite elements (6 nodes – 3 d.o.f. for each node = 18 d.o.f.):

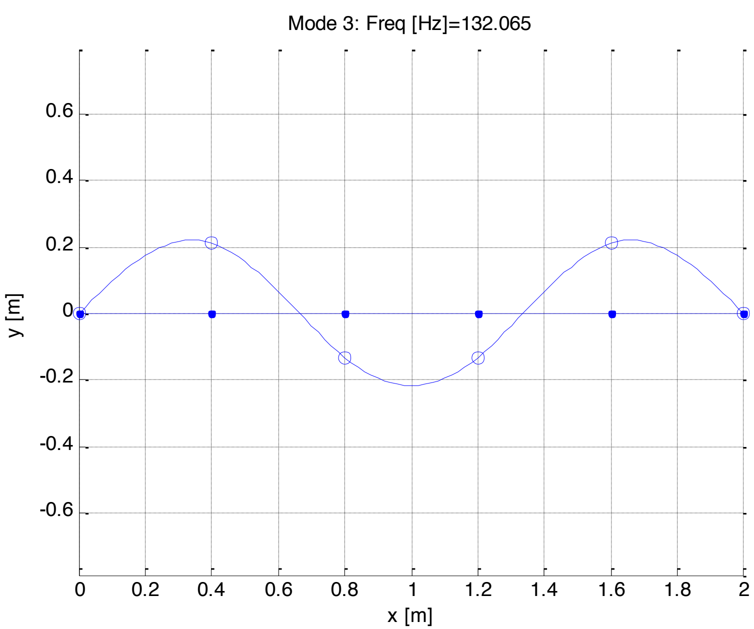
$$[\mathbf{M}]_{18\times 18} = \begin{bmatrix} [\mathbf{M_{FF}}]_{14\times 14} & [\mathbf{M_{FC}}]_{14\times 4} \\ [\mathbf{M_{CF}}]_{4\times 14} & [\mathbf{M_{CC}}]_{4\times 4} \end{bmatrix}$$

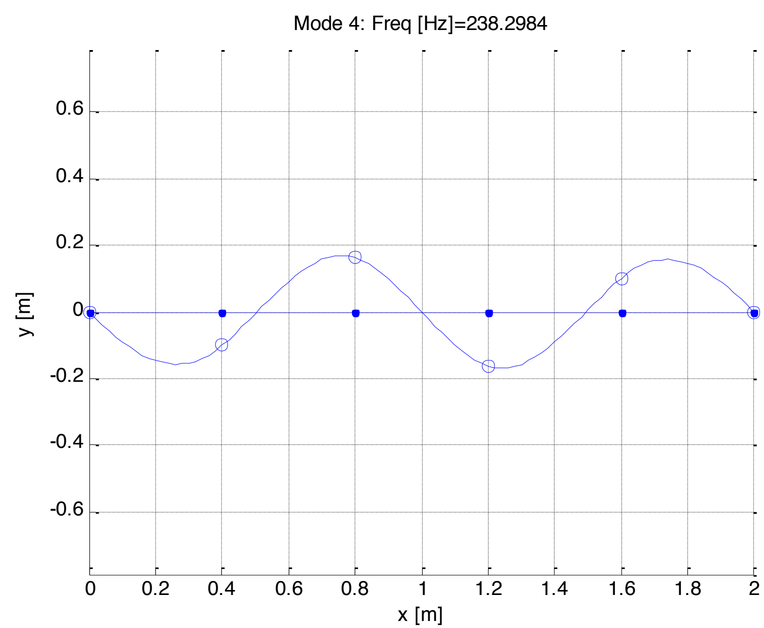
$$[\mathbf{K}]_{18\times 18} = \begin{bmatrix} [\mathbf{K_{FF}}]_{14\times 14} & [\mathbf{K_{FC}}]_{14\times 4} \\ [\mathbf{K_{CF}}]_{4\times 14} & [\mathbf{K_{CC}}]_{4\times 4} \end{bmatrix}$$

Natural Frequencies and Mode Shapes









Effect of beam finite element length

Ridurre il numero di elementi aumenta la rigidezza "apparente" del sistema

	$l_{fe} = 1m$	$l_{fe} = 0.5m$	$l_{fe} = 0.4m$	$l_{fe} = 0.1m$
	$\omega_n = 58$ Hz	$\omega_n = 233$	$\omega_n = 364$	$\omega_n = 5823$
f_{real} [Hz]	f_{fem} [Hz]	f_{fem} [Hz]	f_{fem} [Hz]	f_{fem} [Hz]
14.56	14.62	14.56	14.56	14.56
58.23	64.63	58.46	58.33	58.23
131.02	162.46	133.42	132.06	131.03
232.93	296.19	258.54	238.3	232.96

$$f_{real,k} = \frac{1}{2\pi}\left(\frac{k\pi}{L}\right)^2 \sqrt{\frac{EJ}{m}}$$

Frequency response functions

The equation of motion for the system is

$$M\ddot{x} + R\dot{x} + Kx = F_0 e^{i\Omega t}$$

To solve the ODE we assume

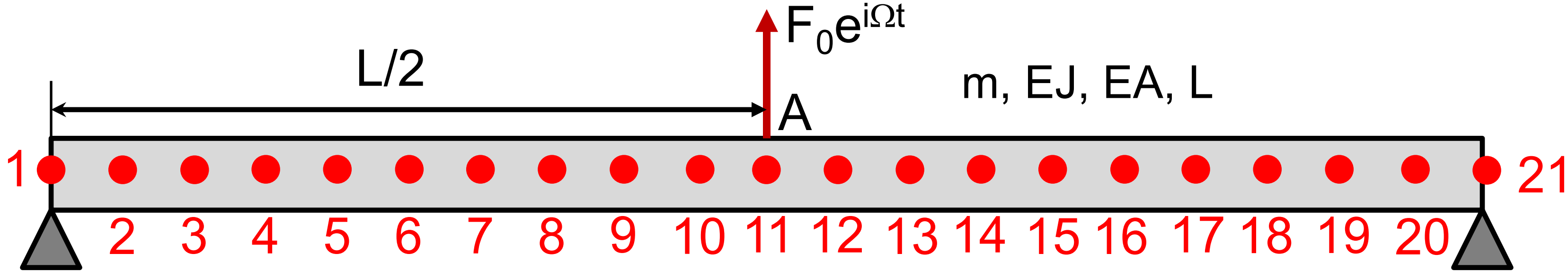
$$x = x_0 e^{i\Omega t}$$

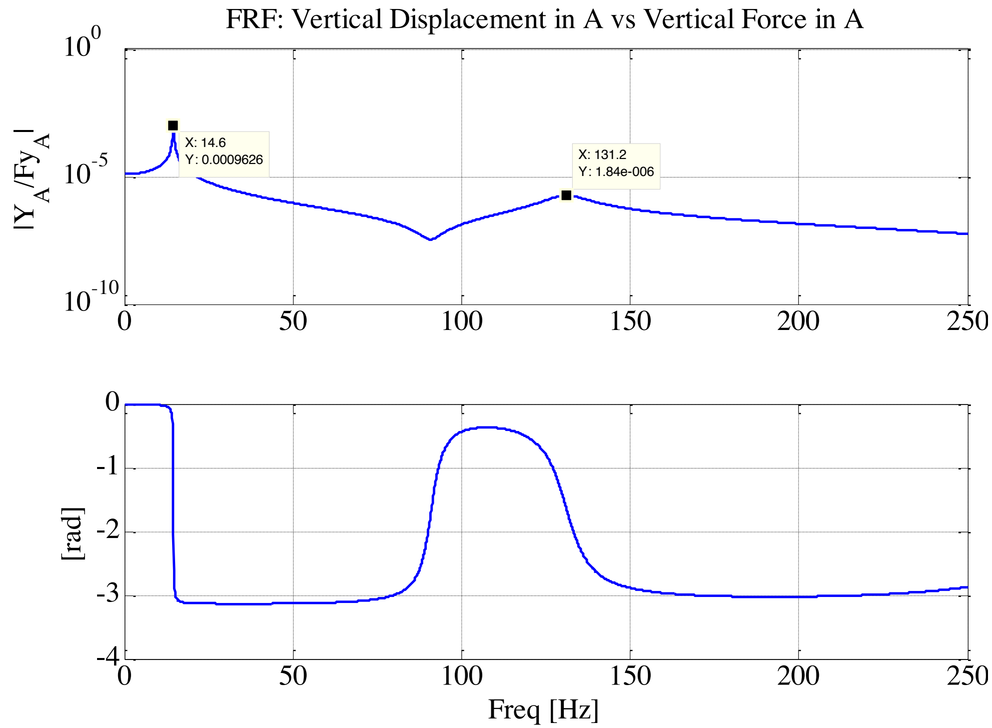
Therefore

$$\left(-\Omega^2 M + i\Omega R + Kx\right) x_0 = F_0$$

$$\boldsymbol{x_0(\Omega) = \left(-\Omega^2 M + i\Omega R + K\right)^{-1} F_0}$$

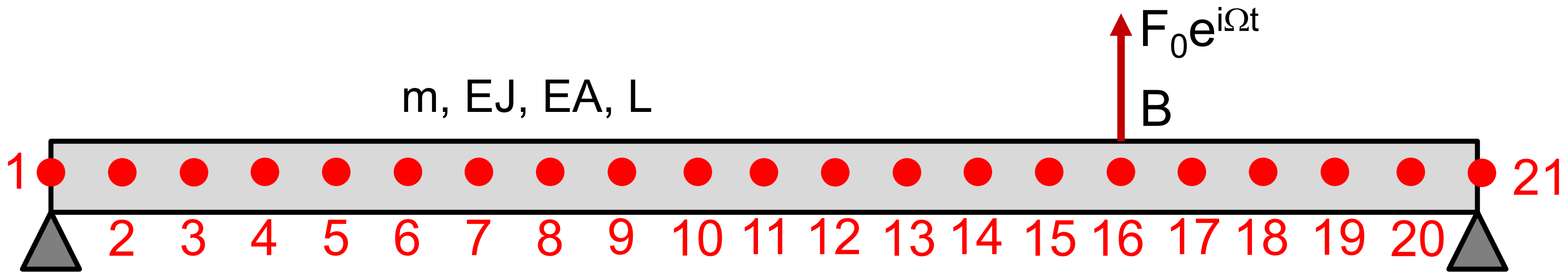
A harmonic force $F=F_0e^{i\Omega t}$ is applied in A: we want to calculate the FRF of the system (l_{fe} = 0.1m).

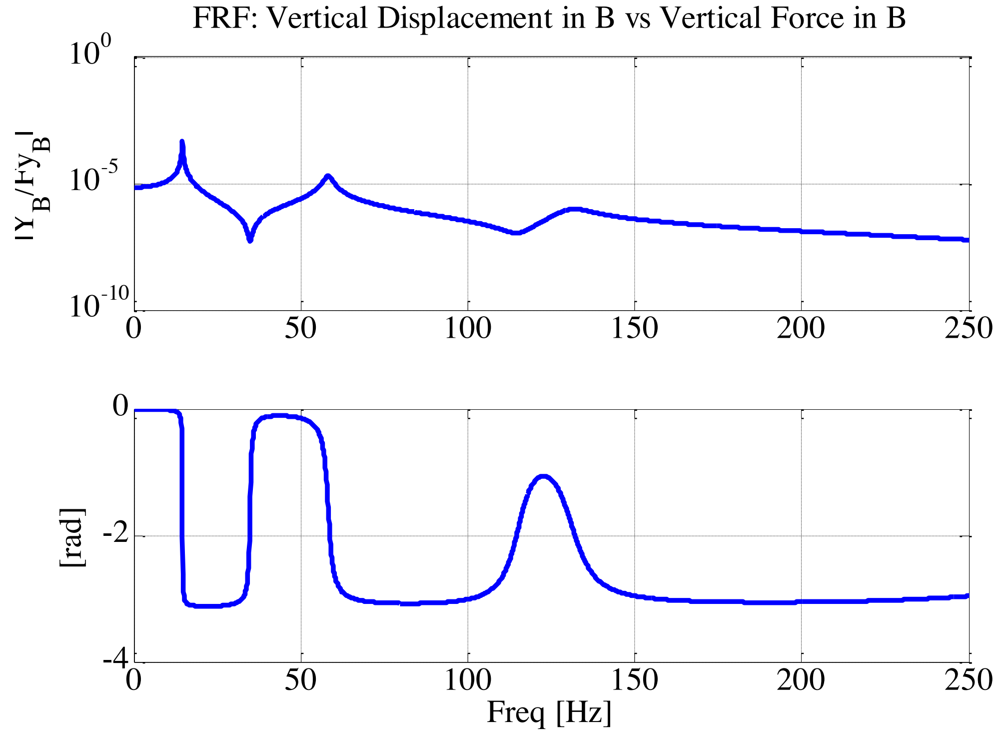




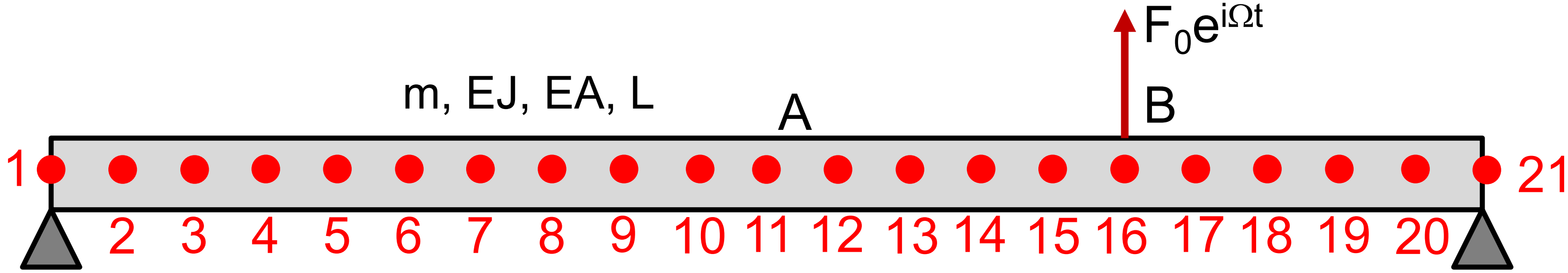
Now it's your turn…

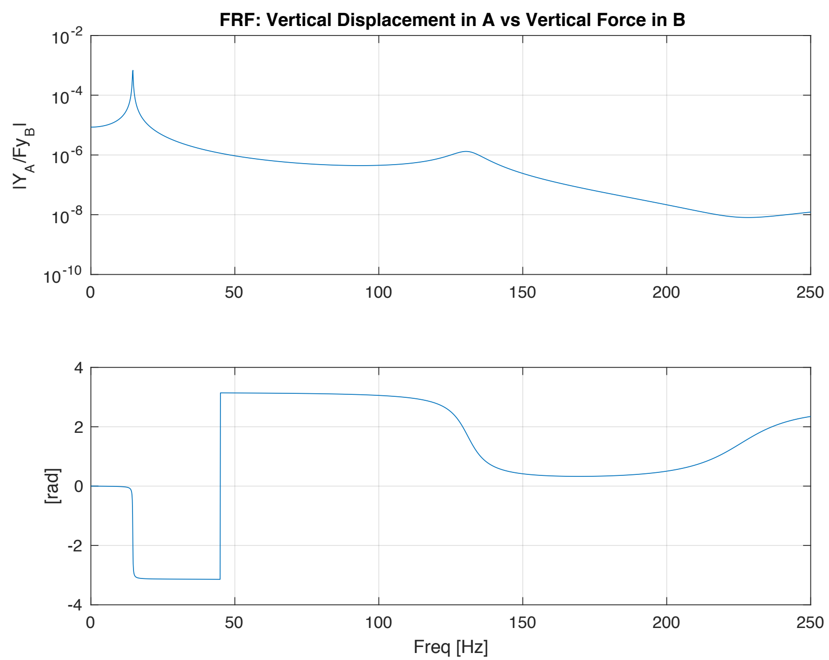
A harmonic force $F=F_0e^{i\Omega t}$ is applied in B: we want to calculate the FRF of the system for the displacement in B (l_{fe} = 0.1m).





A harmonic force $F=F_0e^{i\Omega t}$ is applied in B: we want to calculate the FRF of the system for the displacement in A (i.e. l_{fe} = 0.1m).





Modal shapes and Frequency Response Function