POLITECNICO
MILANO 1863

THE FINITE ELEMENT METHOD

A NAÏVE INTRODUCTION

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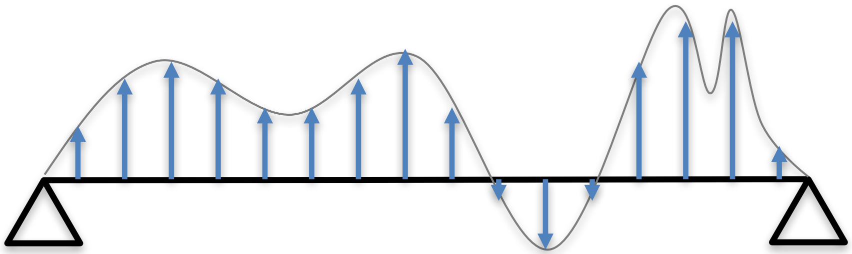
The Finite Element approach

In the ”continuum approach” the displacement of a system is described <u>in every point</u> i.e. it is a function of space.

This is difficult to “handle” and to automatise.

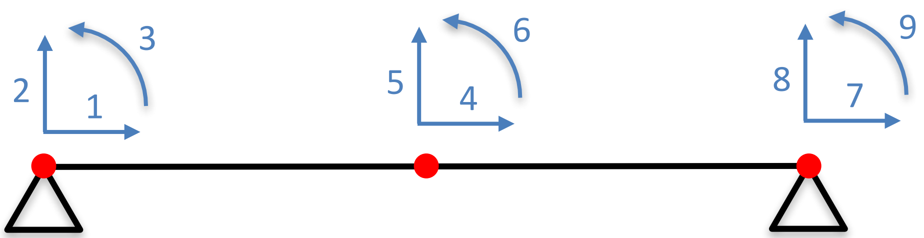


The Finite Element approach

When using the FE method the structure is divided into **finite elements**.

The displacement of the structure is controlled through the displacement of a finite number of **nodes**.

The DoFs of the structure are the **nodal displacements**.



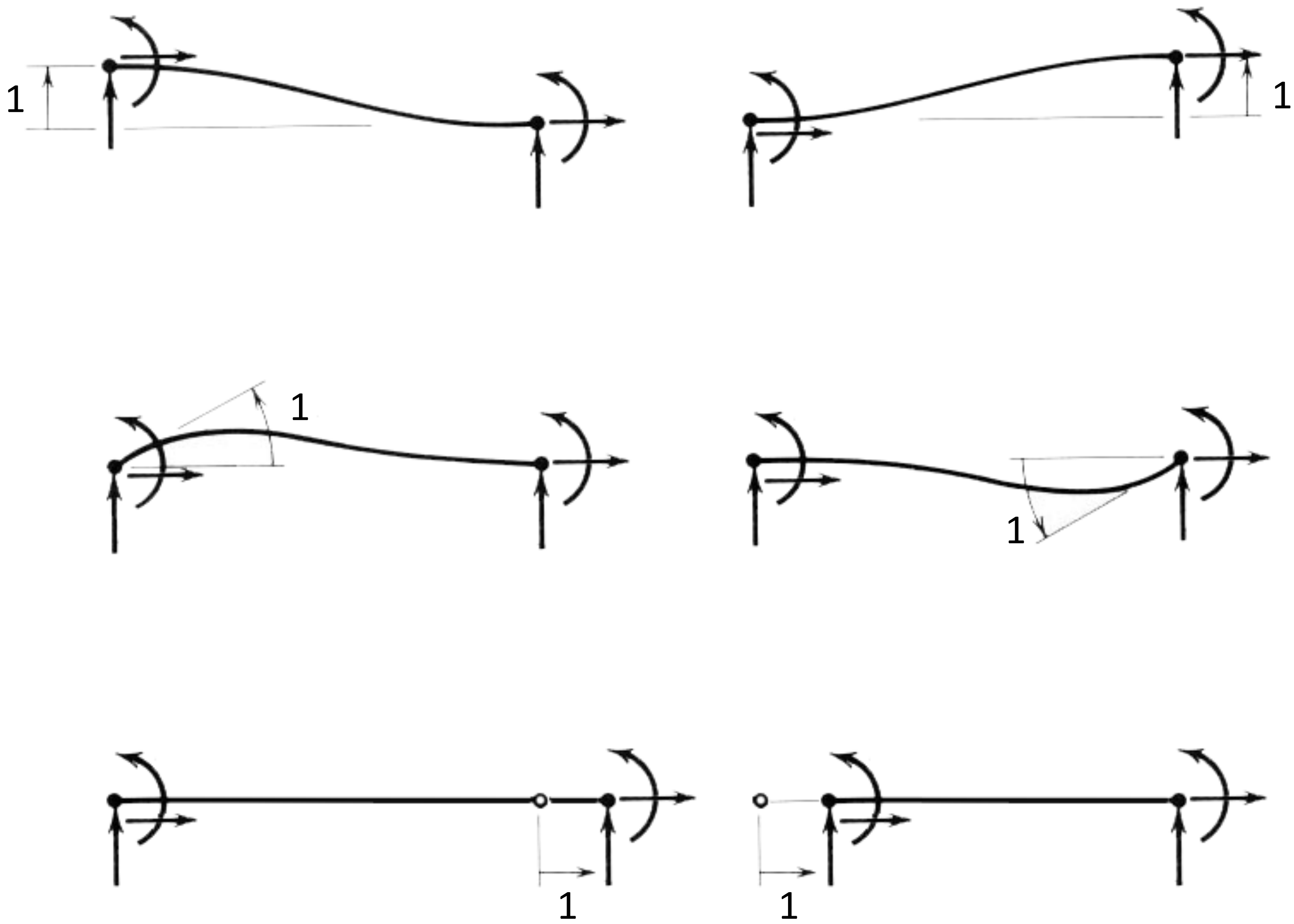
The shape functions

How can I relate the displacement of the **other points** of the structure to the displacement of the nodes?

I use the **shape functions**.

They correspond to the *static deformation* of an element due to the movement of one DoF.

The shape functions



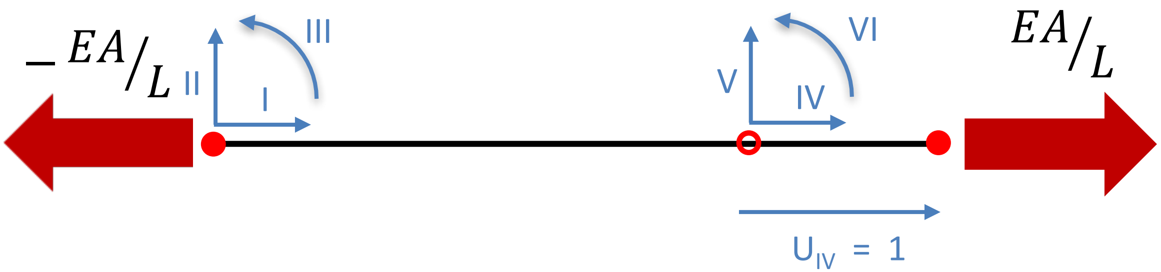
The stiffness matrix

How do we take into account the presence of the elements when we move a node?

I look at the **nodal forces** caused by that movement

The stiffness matrix

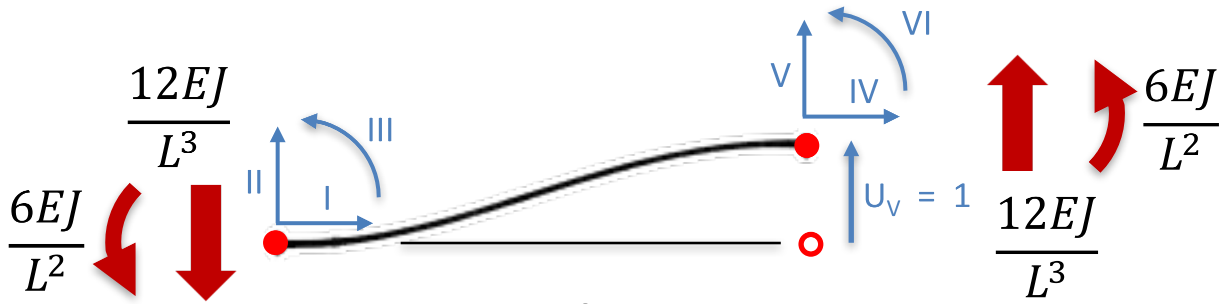
I impose a unitary displacement on one DoF and I observe the reaction forces appearing on the nodes



$$\begin{bmatrix} F_I \\ F_{II} \\ F_{III} \\ F_{IV} \\ F_V \\ F_{VI} \end{bmatrix} = \begin{bmatrix} -\dfrac{EA}{L} \\ 0 \\ 0 \\ \dfrac{EA}{L} \\ 0 \\ 0 \end{bmatrix} u_{IV}$$

The stiffness matrix

I impose a unitary displacement on one DoF and I observe the reaction forces appearing on the nodes



$$\begin{bmatrix} F_I \\ F_{II} \\ F_{III} \\ F_{IV} \\ F_V \\ F_{VI} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{12EJ}{L^3} \\ \frac{6EJ}{L^2} \\ 0 \\ \frac{12EJ}{L^3} \\ \frac{6EJ}{L^2} \end{bmatrix} u_V$$

The stiffness matrix

Repeating for each DoF of the beam we obtain the stiffness matrix of the beam element

$$\boldsymbol{K}_{el} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EJ}{L^3} & \frac{6EJ}{L^2} & 0 & -\frac{12EJ}{L^3} & \frac{6EJ}{L^2} \\ 0 & \frac{6EJ}{L^2} & \frac{4EJ}{L} & 0 & \frac{6EJ}{L^2} & \frac{2EJ}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EJ}{L^3} & -\frac{6EJ}{L^2} & 0 & \frac{12EJ}{L^3} & -\frac{6EJ}{L^2} \\ 0 & \frac{6EJ}{L^2} & \frac{2EJ}{L} & 0 & -\frac{6EJ}{L^2} & \frac{4EJ}{L} \end{bmatrix}$$

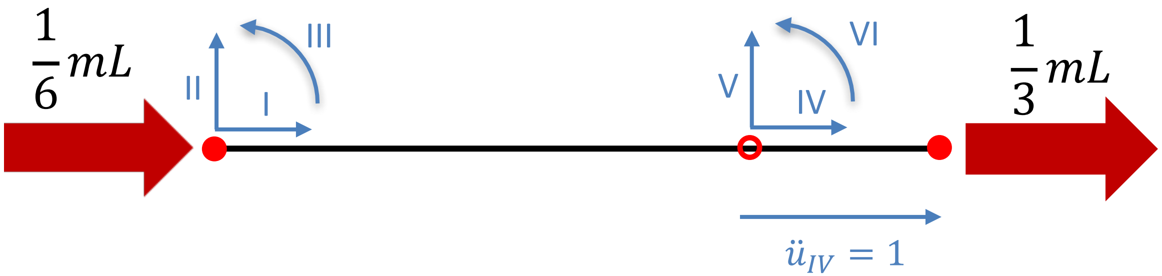
The stiffness matrix

The nodal forces due to the elastic potential of the beam can be computed as

$$F_{el} = Kx$$

The mass matrix

I impose a unitary **acceleration** on one DoF and I observe the reaction forces appearing on the nodes



$$\begin{bmatrix} F_I \\ F_{II} \\ F_{III} \\ F_{IV} \\ F_V \\ F_{VI} \end{bmatrix} = \begin{bmatrix} \frac{1}{6}mL \\ 0 \\ 0 \\ \frac{1}{3}mL \\ 0 \\ 0 \end{bmatrix} \ddot{u}_{IV}$$

The mass matrix

Repeating for each DoF of the beam we obtain the mass matrix of the beam element

$$\boldsymbol{M}_{el} = \begin{bmatrix} \frac{140mL}{420} & 0 & 0 & \frac{70mL}{420} & 0 & 0 \\ 0 & \frac{156mL}{420} & \frac{22mL^2}{420} & 0 & \frac{54mL}{420} & -\frac{13mL^2}{420} \\ 0 & \frac{22mL}{420} & \frac{4mL^3}{420} & 0 & \frac{13mL^2}{420} & -\frac{3mL^3}{420} \\ \frac{70mL}{420} & 0 & 0 & \frac{140mL}{420} & 0 & 0 \\ 0 & \frac{54mL}{420} & \frac{13mL^2}{420} & 0 & \frac{156mL}{420} & -\frac{22mL^2}{420} \\ 0 & -\frac{13mL^2}{420} & -\frac{3mL^3}{420} & 0 & -\frac{22mL^2}{420} & \frac{4mL^3}{420} \end{bmatrix}$$

The stiffness matrix

The nodal forces due to the inertia of the beam can be computed as

$$F_{in} = M\ddot{x}$$

The equation of motion

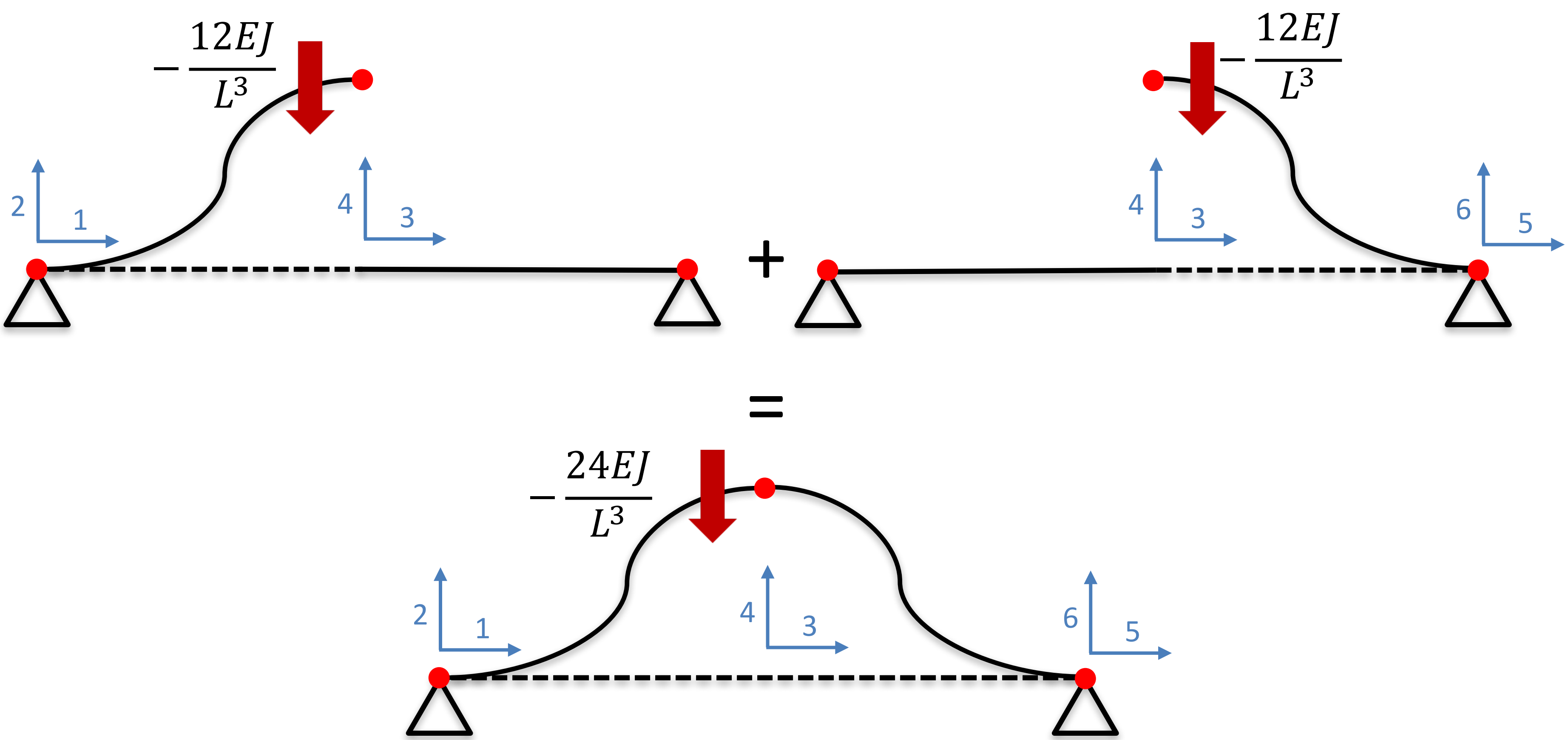
Imposing the equilibrium on each DoF we obtain the equation of motion for the system:

$$F_{el} + F_{in} = F_{ext}$$

$$M\ddot{x} + Kx = F_{ext}$$

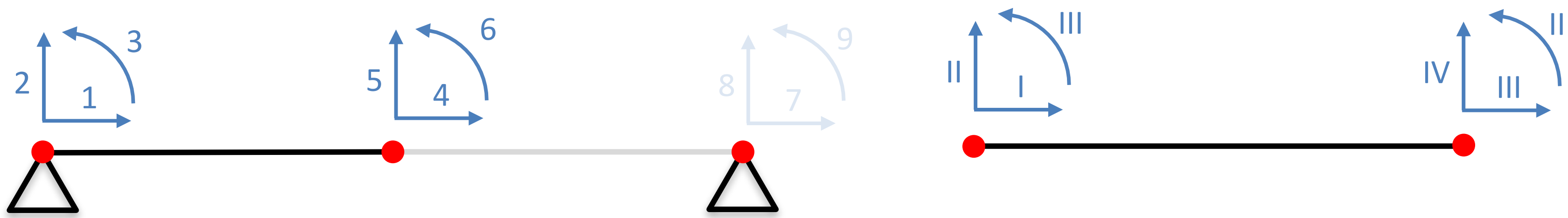
Matrix assembly

When I move a global dof I see the contribution of all the elements connected to that node



Matrix assembly

We need to find the correspondence between each **local dof** and the corresponding **global dof.**



For the first element:

- I → 1
- II → 2
- III → 3
- IV → 4
- V → 5
- VI → 6

Matrix assembly

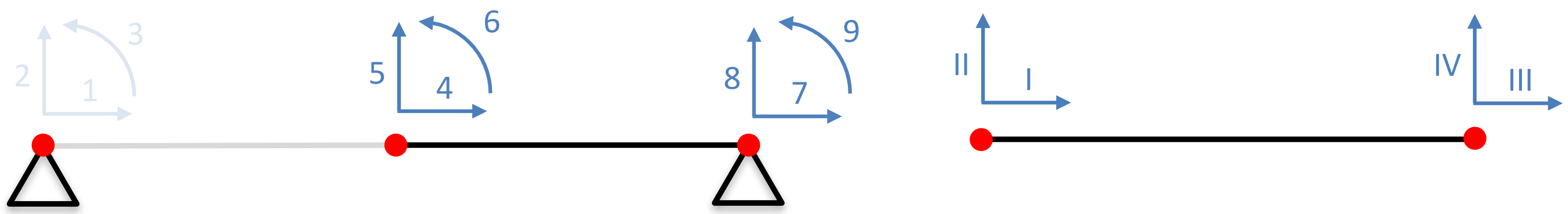
How the first element contributes to the overall stiffness?

$$
K = \begin{array}{c c}
\begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & \frac{S}{L} & 0 & -\frac{S}{L} \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & -\frac{S}{L} & 0 & \frac{S}{L} \end{bmatrix} & \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array} \\
\begin{array}{cccc} \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} \end{array} &
\end{array}
$$

$$
K_{tot} = \begin{array}{c c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\
\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

Matrix assembly

We need to find the correspondence between each **element dof** and the **global dof.**



For the second element:

- I → 4
- II → 5
- III → 6
- IV → 7
- V → 8
- VI → 9

Matrix assembly

How the second element contributes to the overall stiffness?

$$K = \begin{array}{c} \\ \left[\begin{array}{cccc} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & \frac{S}{L} & 0 & -\frac{S}{L} \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & -\frac{S}{L} & 0 & \frac{S}{L} \end{array}\right] \\ \begin{array}{cccc} \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} \end{array} \end{array} \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \\ \\ \end{array}$$

$$K_{tot} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & \boxed{*} & \boxed{*} & \boxed{*} & \boxed{*} \\ * & * & \boxed{*} & \boxed{*} & \boxed{*} & \boxed{*} \\ * & * & \boxed{*} & \boxed{*} & \boxed{*} & \boxed{*} \\ * & * & \boxed{*} & \boxed{*} & \boxed{*} & \boxed{*} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$