POLITECNICO
MILANO 1863

THE FINITE ELEMENT METHOD (3)

EXERCISE

Milano, 30 May 2017
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Exercise

DATA:

Beam Properties

Mass per unit length

$m = 19.5$ kg/m

Flexural rigidity

$EJ = 1.1 \cdot 10^5$ Nm^2

Axial stiffness

$EA = 5.15 \cdot 10^8$ N

Lumped mass

$m_c = 20$ kg

Springs

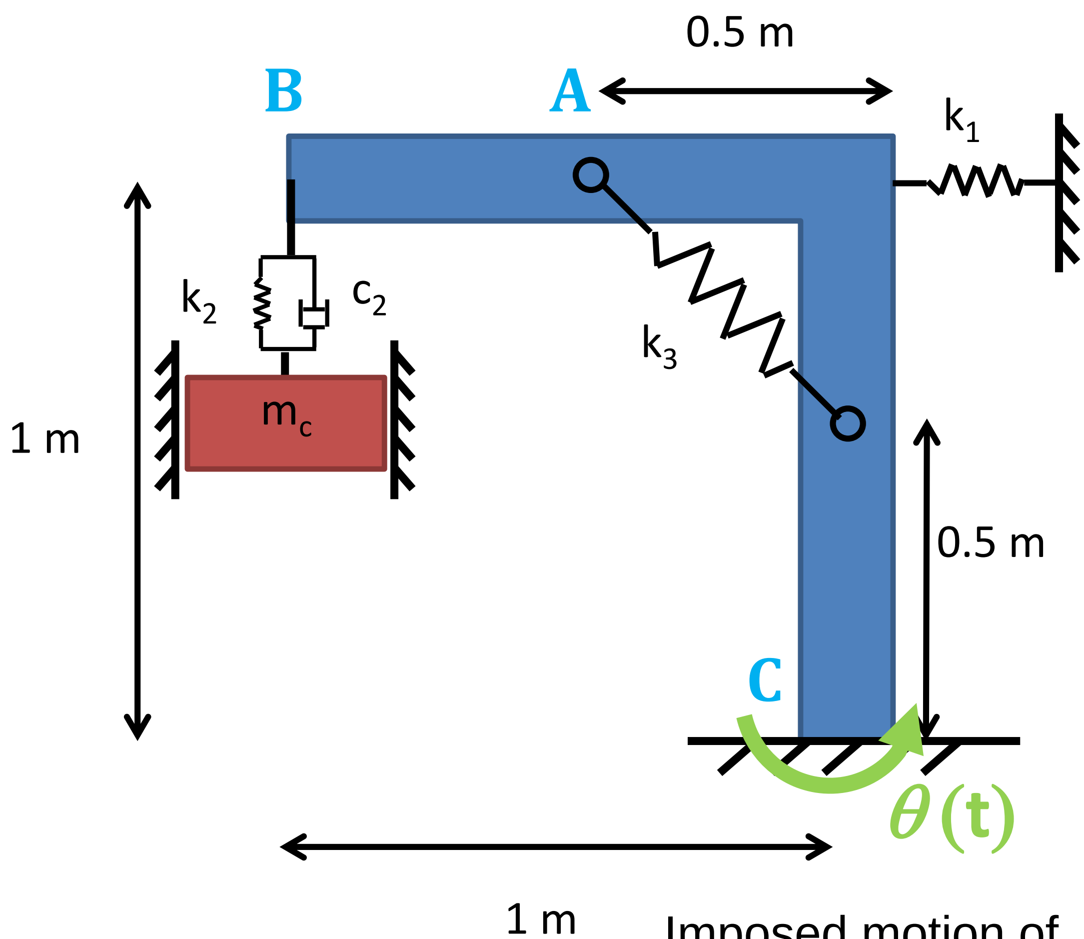
$k_1 = 2.0 \cdot 10^6$ N/m

$k_2 = 3.0 \cdot 10^6$ N/m

$k_3 = 2.0 \cdot 10^6$ N/m

Damper

$C_2 = 310$ Ns/m

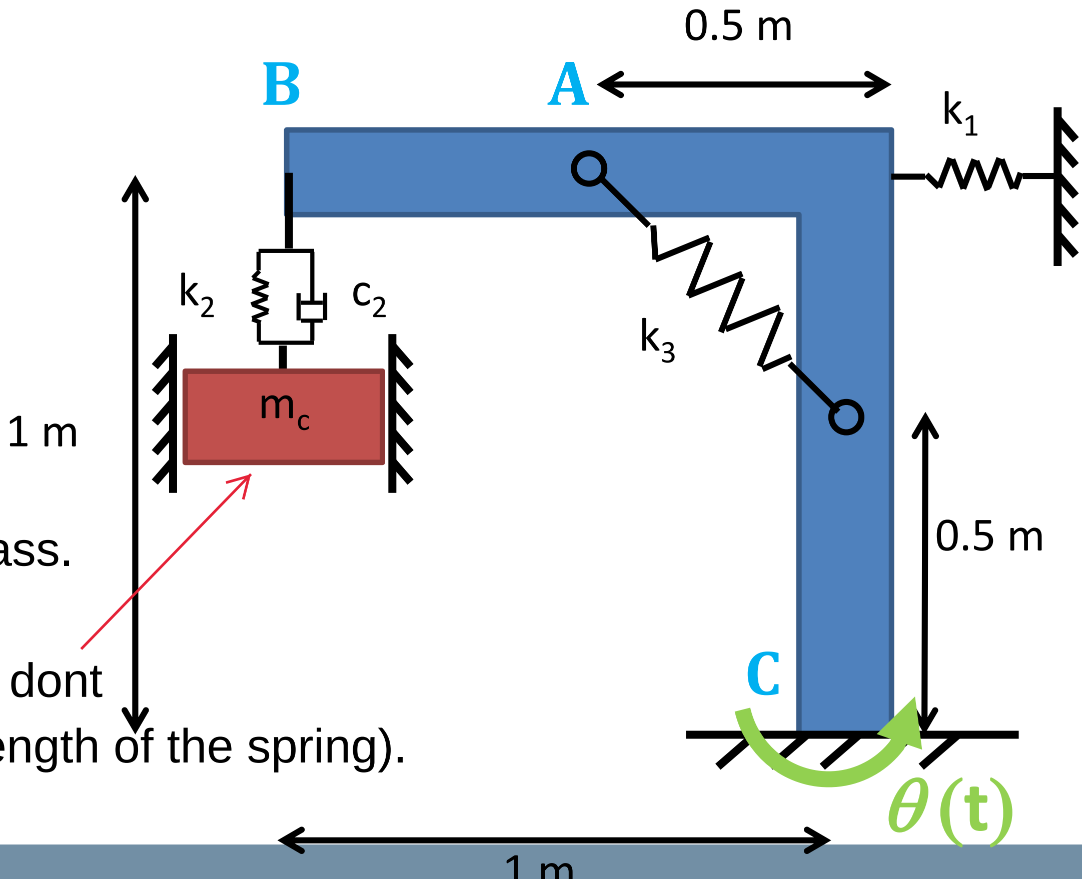


Imposed motion of theta

Exercise

We want to:

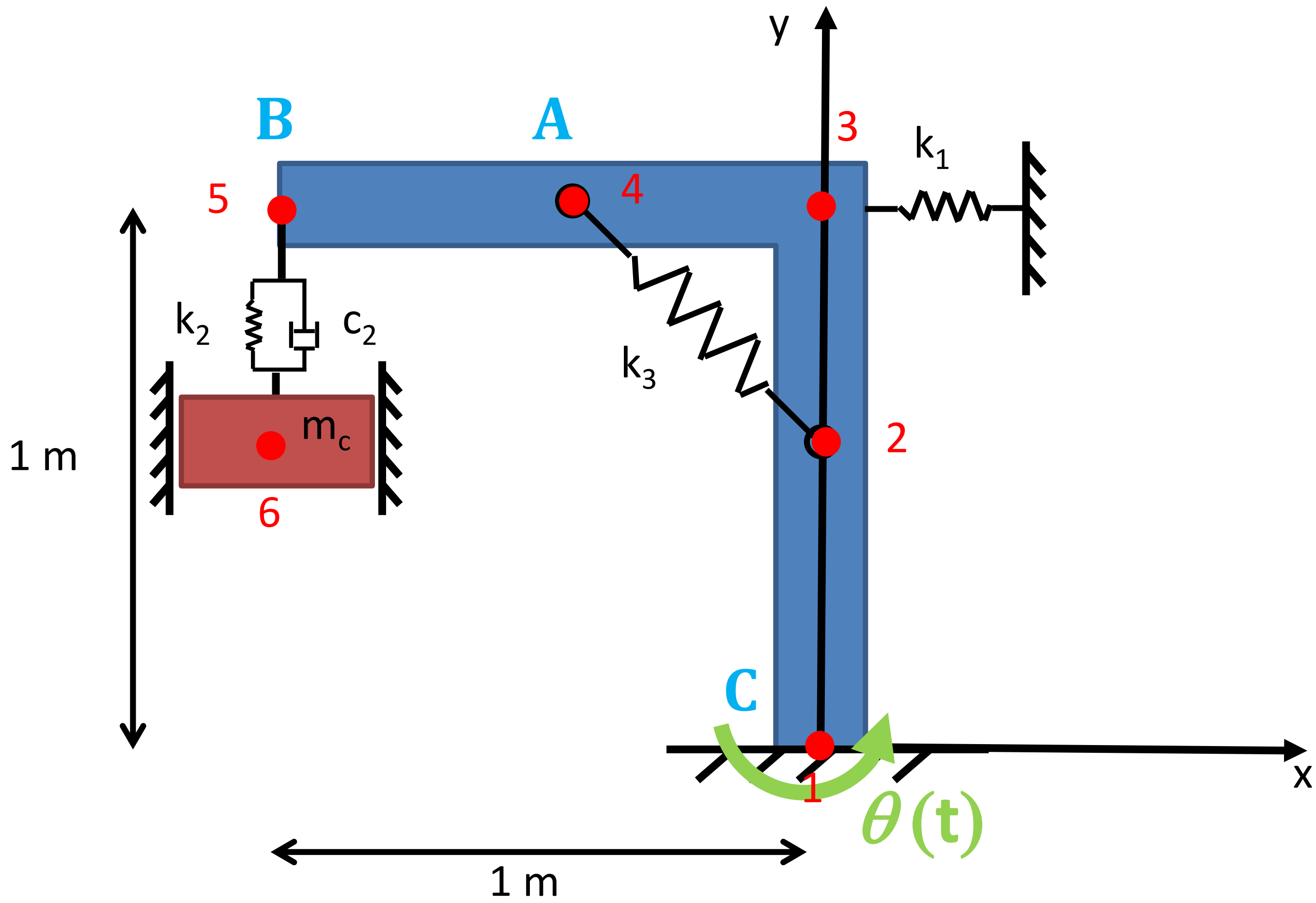
1. **Define mesh for the analysis up to 200 Hz;** fmax
2. **Calculate the first 3 natural frequencies and the related mode shapes;**
3. **Calculate the FRFs Y_A/θ and Y_B/θ, where: $\theta(t) = \theta_0 \cdot \cos(\Omega t)$**



One node for the mass. The vertical position does not matter (we dont consider the initial length of the spring).

Definition of length of beam finite element

Length of beam finite element l_{fe} = 0.5 m



Assembly of Mass Matrix

$[M_{FF}]_{13\times13}$ $[M_{FC}]_{13\times5}$

$[M] =$

7,24	0,00	0,00	1,25	0,00	0,15	0,00	0,00	0,00	0,00	0,00	0,00	0,00	1,25	0,00	-0,15	0,00	0,00
0,00	6,50	0,00	0,00	1,63	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	1,63	0,00	0,00	0,00
0,00	0,00	0,05	-0,15	0,00	-0,02	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,15	0,00	-0,02	0,00	0,00
1,25	0,00	-0,15	6,87	0,00	0,26	1,63	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
0,00	1,63	0,00	0,00	6,87	-0,26	0,00	1,25	0,15	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
0,15	0,00	-0,02	0,26	-0,26	0,05	0,00	-0,15	-0,02	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
0,00	0,00	0,00	1,63	0,00	0,00	6,50	0,00	0,00	1,63	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
0,00	0,00	0,00	0,00	1,25	-0,15	0,00	7,24	0,00	0,00	1,25	0,15	0,00	0,00	0,00	0,00	0,00	0,00
0,00	0,00	0,00	0,00	0,15	-0,02	0,00	0,00	0,05	0,00	-0,15	-0,02	0,00	0,00	0,00	0,00	0,00	0,00
0,00	0,00	0,00	0,00	0,00	0,00	1,63	0,00	0,00	3,25	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
0,00	0,00	0,00	0,00	0,00	0,00	0,00	1,25	-0,15	0,00	3,62	0,26	0,00	0,00	0,00	0,00	0,00	0,00
0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,15	-0,02	0,00	0,26	0,02	0,00	0,00	0,00	0,00	0,00	0,00
0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	20,00	0,00	0,00	0,00	0,00	0,00
1,25	0,00	0,15	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	3,62	0,00	-0,26	0,00	0,00
0,00	1,63	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	3,25	0,00	0,00	0,00
-0,15	0,00	-0,02	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	-0,26	0,00	0,02	0,00	0,00
0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00

$[M_{CF}]_{5\times13}$ $[M_{CC}]_{5\times5}$

Assembly of the Stiffness Matrix – Sping K1

$[K_{FF}]_{13\times13}$ $[K_{FC}]_{13\times5}$

$[K] =$

22120000	-1000000	0	-10560000	0	-2640000	-1000000	1000000	0	0	0	0	0	-10560000	0	2640000	0	0
-1000000	2061000000	0	0	-1030000000	0	1000000	-1000000	0	0	0	0	0	0	-1030000000	0	0	0
0	0	1760000	2640000	0	440000	0	0	0	0	0	0	0	-2640000	0	440000	0	0
-10560000	0	2640000	1042560000	0	2640000	-1030000000	0	0	0	0	0	0	0	0	0	0	0
0	-1030000000	0	0	1040560000	-2640000	0	-10560000	-2640000	0	0	0	0	0	0	0	0	0
-2640000	0	440000	2640000	-2640000	1760000	0	2640000	440000	0	0	0	0	0	0	0	0	0
-1000000	1000000	0	-1030000000	0	0	2061000000	-1000000	0	-1030000000	0	0	0	0	0	0	0	0
1000000	-1000000	0	0	-10560000	2640000	-1000000	22120000	0	0	-10560000	-2640000	0	0	0	0	0	0
0	0	0	0	-2640000	440000	0	0	1760000	0	2640000	440000	0	0	0	0	0	0
0	0	0	0	0	0	-1030000000	0	0	1030000000	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-10560000	2640000	0	13560000	2640000	-3000000	0	0	0	0	0
0	0	0	0	0	0	0	-2640000	440000	0	2640000	880000	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-3000000	0	3000000	0	0	0	0	0
-10560000	0	-2640000	0	0	0	0	0	0	0	0	0	0	10560000	0	-2640000	0	0
0	-1030000000	0	0	0	0	0	0	0	0	0	0	0	0	1030000000	0	0	0
2640000	0	440000	0	0	0	0	0	0	0	0	0	0	-2640000	0	880000	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

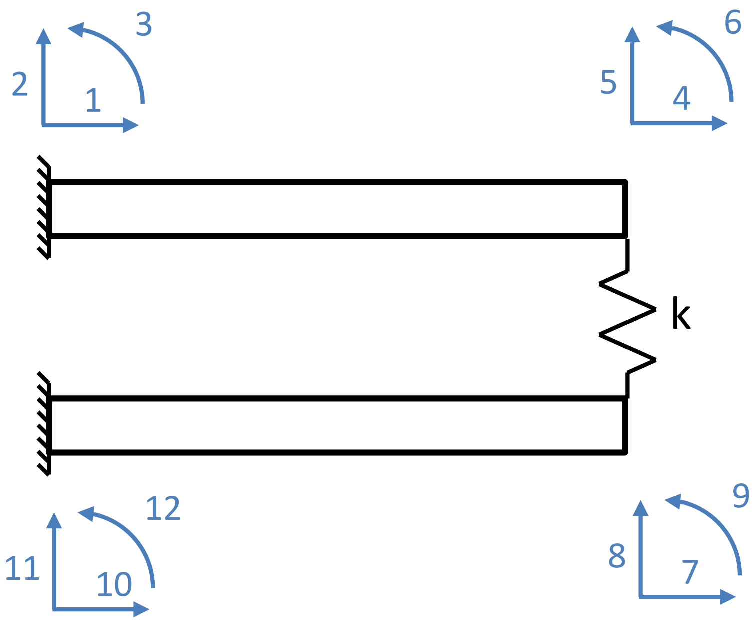
$[K_{CF}]_{5\times13}$ $[K_{CC}]_{5\times5}$

Contribution due to k_1 spring

Internal contrains

Like k2 and k3

How do I handle an internal spring?



The potential energy of the spring is:

$$V = \frac{1}{2}k(u_5 - u_8)^2$$

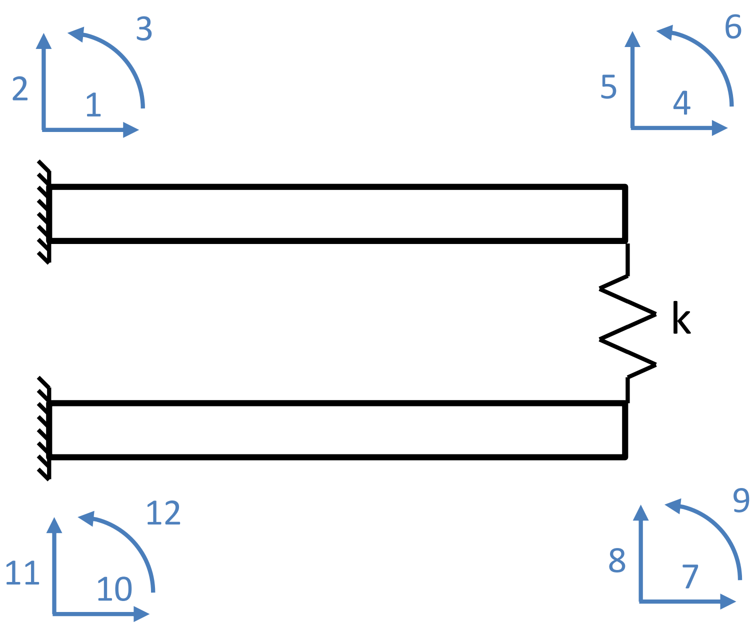
$$= \frac{1}{2}k\left(u_5^2 \boxed{- u_5 u_8 - u_8 u_5} + u_8^2\right)$$

2 positive contributions

2 negative contributions

Internal contrains

How do I handle an internal spring?



$$
\begin{array}{cc}
 & \begin{array}{ccccc} & 5 & & 8 & \end{array} \\
k_{tot} = & \left[\begin{array}{ccccc} \ddots & \vdots & & \vdots & \\ \cdots & k & \cdots & \boxed{-k} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & \boxed{-k} & \cdots & k & \cdots \\ & \vdots & & \vdots & \ddots \end{array}\right] \begin{array}{c} \\ 5 \\ \\ 8 \\ \\ \end{array}
\end{array}
$$

Assembly of the Stiffness Matrix – Sping K2

$[K_{FF}]_{13\times13}$ $[K_{FC}]_{13\times5}$

$[K] =$

22120000	-1000000	0	-10560000	0	-2640000	-1000000	1000000	0	0	0	0	0	-10560000	0	2640000	0	0
-1000000	2061000000	0	0	-1030000000	0	1000000	-1000000	0	0	0	0	0	0	-1030000000	0	0	0
0	0	1760000	2640000	0	440000	0	0	0	0	0	0	0	-2640000	0	440000	0	0
-10560000	0	2640000	1042560000	0	2640000	-1030000000	0	0	0	0	0	0	0	0	0	0	0
0	-1030000000	0	0	1040560000	-2640000	0	-10560000	-2640000	0	0	0	0	0	0	0	0	0
-2640000	0	440000	2640000	-2640000	1760000	0	2640000	440000	0	0	0	0	0	0	0	0	0
-1000000	1000000	0	-1030000000	0	0	2061000000	-1000000	0	-1030000000	0	0	0	0	0	0	0	0
1000000	-1000000	0	0	-10560000	2640000	-1000000	22120000	0	0	-10560000	-2640000	0	0	0	0	0	0
0	0	0	0	-2640000	440000	0	0	1760000	0	2640000	440000	0	0	0	0	0	0
0	0	0	0	0	0	-1030000000	0	0	1030000000	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-10560000	2640000	0	13560000	2640000	-3000000	0	0	0	0	0
0	0	0	0	0	0	0	-2640000	440000	0	2640000	880000	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-3000000	0	3000000	0	0	0	0	0
-10560000	0	-2640000	0	0	0	0	0	0	0	0	0	0	10560000	0	-2640000	0	0
0	-1030000000	0	0	0	0	0	0	0	0	0	0	0	0	1030000000	0	0	0
2640000	0	440000	0	0	0	0	0	0	0	0	0	0	-2640000	0	880000	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

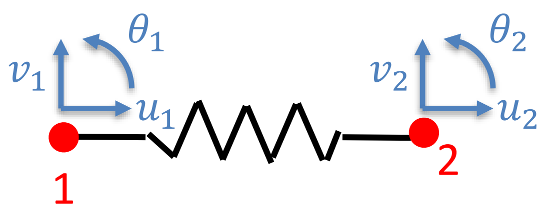
$[K_{CF}]_{5\times13}$ $[K_{CC}]_{5\times5}$

Contribution due to k_1 spring

Contribution due to k_2 spring

Assembly of Stiffness Matrix

Contribution due to the spring "k_3". We can consider the spring element as a 6 DoFs element (we don't have any dof corresponding to the direction ok k3)



$$\underline{X}_k^L = \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

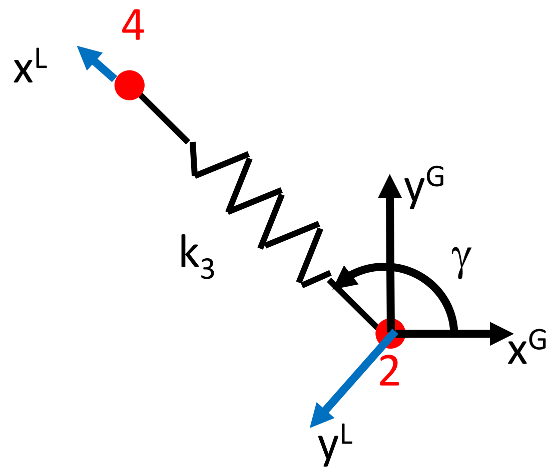
The stiffness matrix is:

A sprin element is a beam element with 0 stiffness on any direction exept for the axial one.

$$K_L = \begin{bmatrix} k_3 & 0 & 0 & -k_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Assembly of Stiffness Matrix

We can project in global coordinates using the same way we use for the FE elements:



$$[\Lambda_{k_3}] = \begin{bmatrix} [\lambda_{k_3}] & [0] \\ [0] & [\lambda_{k_3}] \end{bmatrix}$$

and

$$[\lambda_{k_3}] = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_G = [\Lambda_{k_3}]^T [K_L][\Lambda_{k_3}]$$

Assembly of the Stiffness Matrix – Sping K3

$[K_{FF}]_{13\times13}$ $[K_{FC}]_{13\times5}$

$[K] =$

22120000	-1000000	0	-10560000	0	-2640000	-1000000	1000000	0	0	0	0	0	-10560000	0	2640000	0	0
-1000000	2061000000	0	0	-1030000000	0	1000000	-1000000	0	0	0	0	0	0	-1030000000	0	0	0
0	0	1760000	2640000	0	440000	0	0	0	0	0	0	0	-2640000	0	440000	0	0
-10560000	0	2640000	1042560000	0	2640000	-1030000000	0	0	0	0	0	0	0	0	0	0	0
0	-1030000000	0	0	1040560000	-2640000	0	-10560000	-2640000	0	0	0	0	0	0	0	0	0
-2640000	0	440000	2640000	-2640000	1760000	0	2640000	440000	0	0	0	0	0	0	0	0	0
-1000000	1000000	0	-1030000000	0	0	2061000000	-1000000	0	-1030000000	0	0	0	0	0	0	0	0
1000000	-1000000	0	0	-10560000	2640000	-1000000	22120000	0	0	-10560000	-2640000	0	0	0	0	0	0
0	0	0	0	-2640000	440000	0	0	1760000	0	2640000	440000	0	0	0	0	0	0
0	0	0	0	0	0	-1030000000	0	0	1030000000	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-10560000	2640000	0	13560000	2640000	-3000000	0	0	0	0	0
0	0	0	0	0	0	0	-2640000	440000	0	2640000	880000	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-3000000	0	3000000	0	0	0	0	0
-10560000	0	-2640000	0	0	0	0	0	0	0	0	0	0	10560000	0	-2640000	0	0
0	-1030000000	0	0	0	0	0	0	0	0	0	0	0	0	1030000000	0	0	0
2640000	0	440000	0	0	0	0	0	0	0	0	0	0	-2640000	0	880000	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

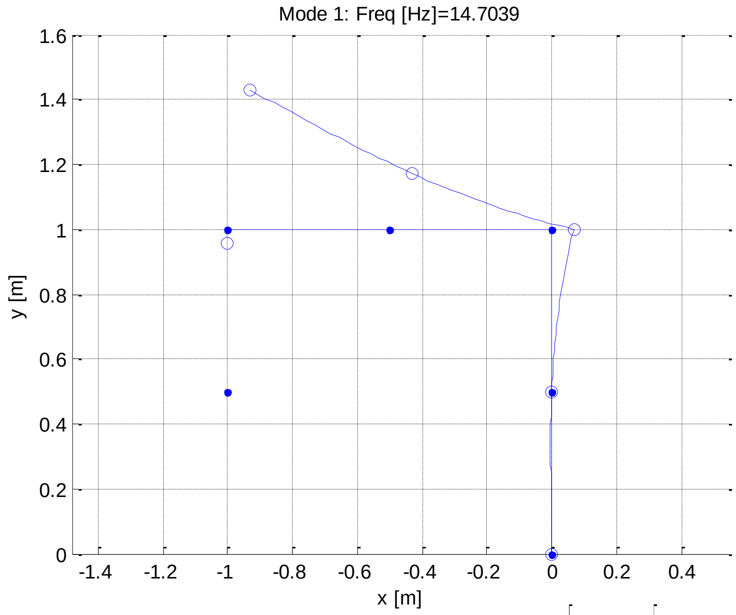
$[K_{CF}]_{5\times13}$ $[K_{CC}]_{5\times5}$

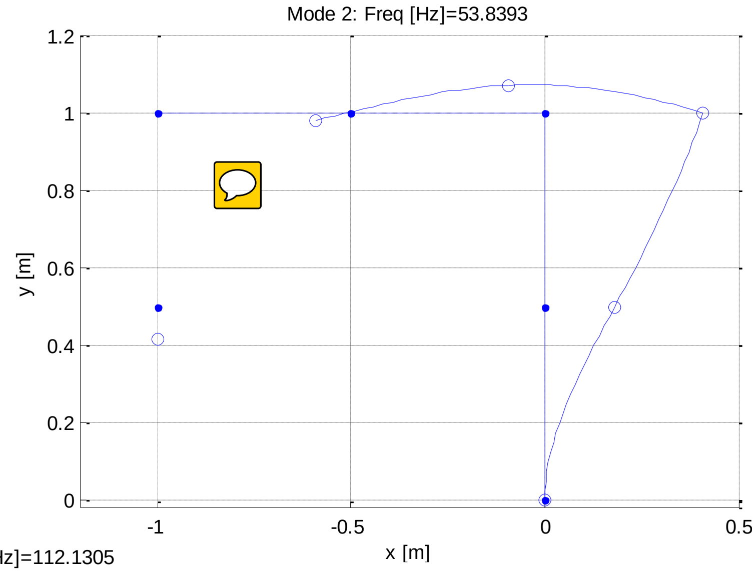
Contribution due to k_1 spring

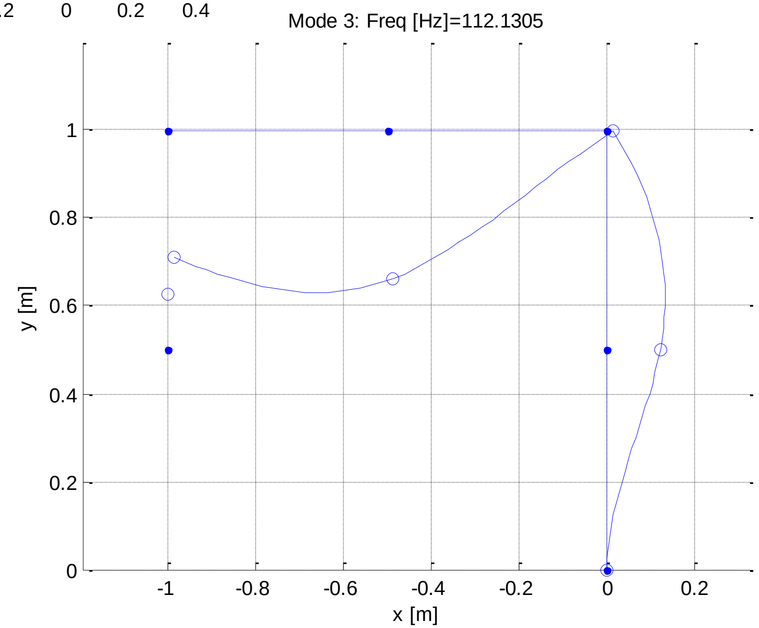
Contribution due to k_2 spring

Contribution due to k_3 spring

Natural Frequencies and Mode Shapes



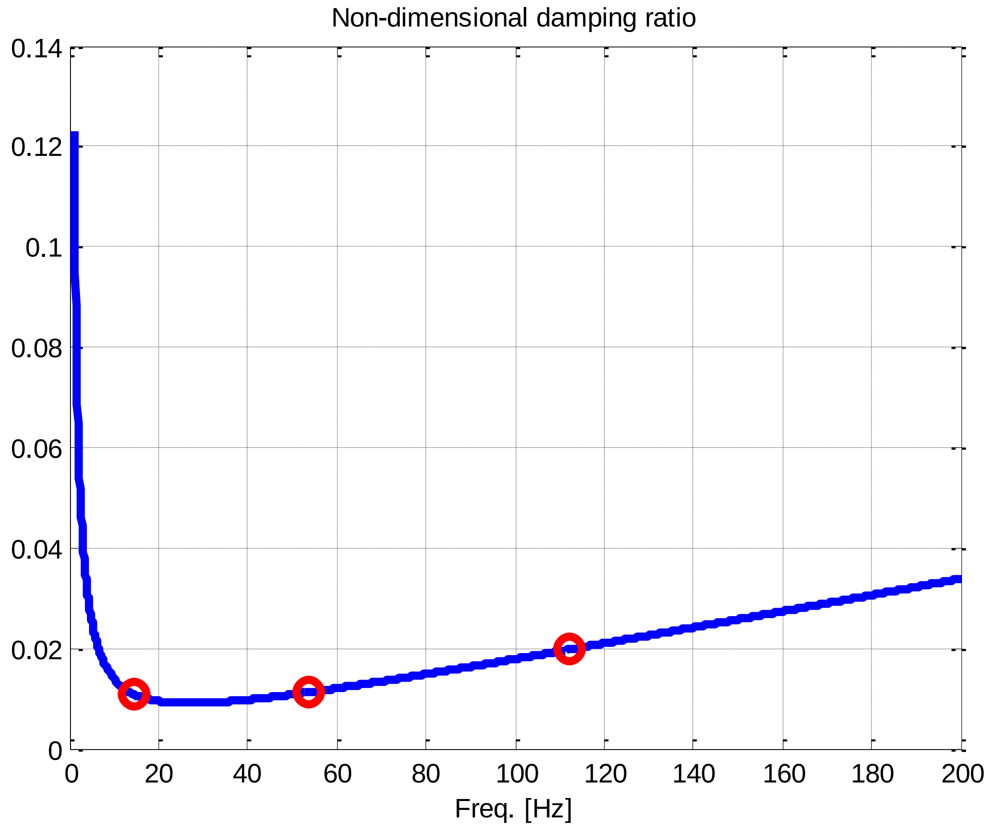




Structural Damping Matrix

As an example, from experimental analysis $h_1 = 0.01$, $h_2 = 0.015$ and $h_3 = 0.018$. We want to estimate α and β parameters by using the method of least square.

$$h_i = \frac{c_i}{2m_i\omega_{n,i}} = \frac{\alpha}{2\omega_{n,i}} + \frac{\beta\omega_{n,i}}{2}$$



$$\alpha = 1.55\ s^{-1}$$

$$\beta = 5.28 \cdot 10^{-5}\ s$$

$$[C_s] = \alpha[\mathrm{M}] + \beta[\mathrm{K}]$$

Overcontrained system of equations -> not exact solution -> we can compute alpha and beta to have a damping as close as possible to h1, h2 and h3 on the first 3 modes. Matlab does it automatically,

Assembly of Damping Matrix

$[C_{FF}]_{13\times13}$ $[C_{FC}]_{13\times5}$

$[C] =$

1180	-53	0	-556	0	-139	-53	53	0	0	0	0	0	-556	0	139	0	0
-53	108897	0	0	-54415	0	53	-53	0	0	0	0	0	0	-54415	0	0	0
0	0	93	139	0	23	0	0	0	0	0	0	0	-139	0	23	0	0
-556	0	139	55091	0	140	-54415	0	0	0	0	0	0	0	0	0	0	0
0	-54415	0	0	54986	-140	0	-556	-139	0	0	0	0	0	0	0	0	0
-139	0	23	140	-140	93	0	139	23	0	0	0	0	0	0	0	0	0
-53	53	0	-54415	0	0	108897	-53	0	-54415	0	0	0	0	0	0	0	0
53	-53	0	0	-556	139	-53	1180	0	0	-556	-139	0	0	0	0	0	0
0	0	0	0	-139	23	0	0	93	0	139	23	0	0	0	0	0	0
0	0	0	0	0	0	-54415	0	0	54422	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-556	139	0	1032	140	-468	0	0	0	0	0
0	0	0	0	0	0	0	-139	23	0	140	47	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-468	0	499	0	0	0	0	0
-556	0	-139	0	0	0	0	0	0	0	0	0	0	564	0	-140	0	0
0	-54415	0	0	0	0	0	0	0	0	0	0	0	0	54422	0	0	0
139	0	23	0	0	0	0	0	0	0	0	0	0	-140	0	47	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$[C_{CF}]_{5\times13}$ $[C_{CC}]_{5\times5}$

Reaction force

Remember the equation of motion for the system

$$\begin{bmatrix} M_{FF} & M_{FC} \\ M_{CF} & M_{CC} \end{bmatrix} \begin{Bmatrix} \ddot{x}_F \\ \ddot{x}_C \end{Bmatrix} + \begin{bmatrix} C_{FF} & C_{FC} \\ C_{CF} & C_{CC} \end{bmatrix} \begin{Bmatrix} \dot{x}_F \\ \dot{x}_C \end{Bmatrix} + \begin{bmatrix} K_{FF} & K_{FC} \\ K_{CF} & K_{CC} \end{bmatrix} \begin{Bmatrix} x_F \\ x_C \end{Bmatrix} = \begin{Bmatrix} F \\ R \end{Bmatrix}$$

known unknown

In this problem $x_C = \left[0 \; \dots \theta_0 e^{i\Omega t} \dots 0\right]$ therefore

$$M_{FF}\ddot{x}_F + M_{FC}\ddot{x}_C + C_{FF}\dot{x}_F + C_{FC}\dot{x}_C + K_{FF}x_F + K_{FC}x_C = F$$

Reaction force

Assuming

$$x_F = x_0 e^{i\Omega t}$$

And

$$x_C = x_{C0} e^{i\Omega t}$$

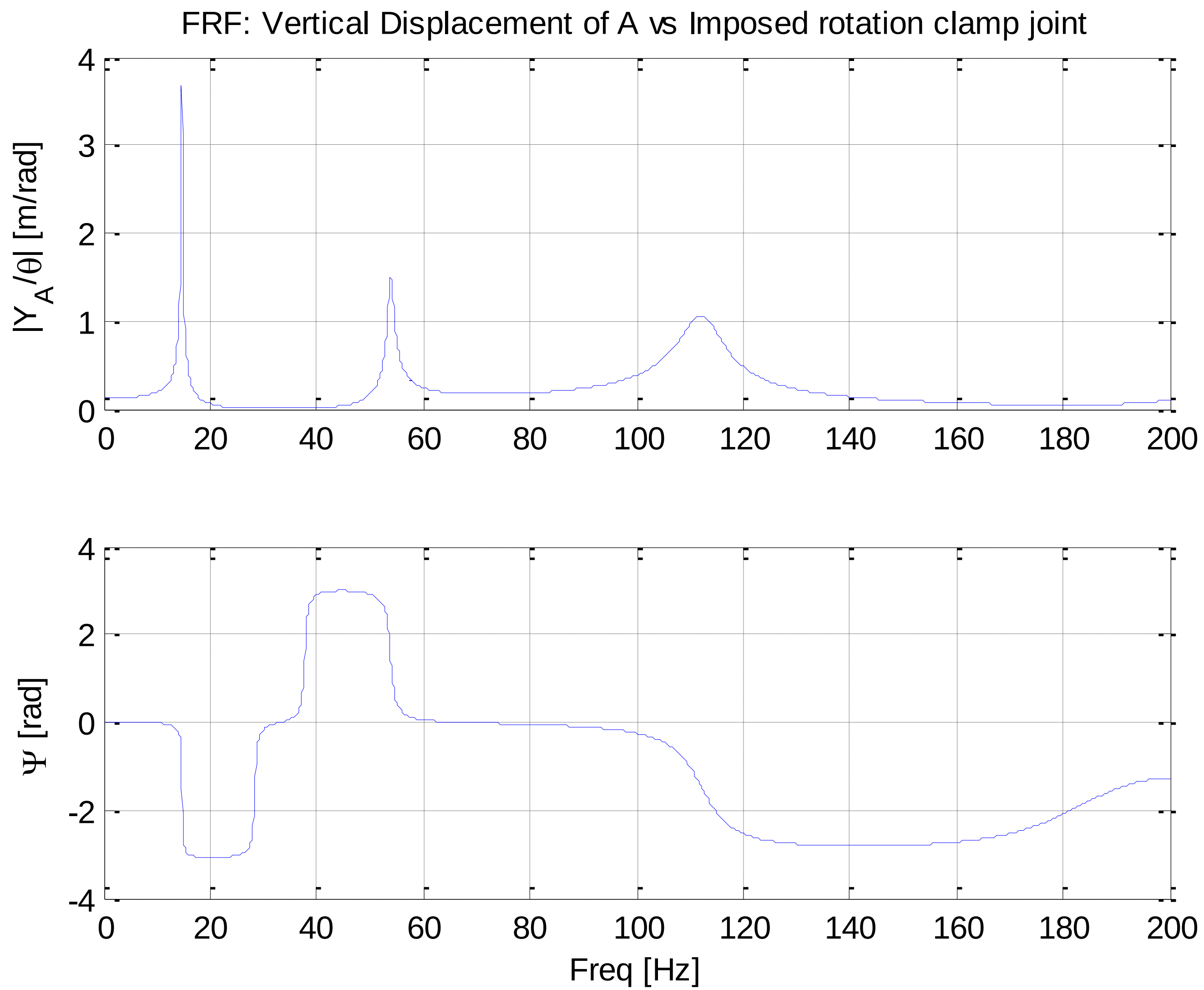
In this problem $F = \underline{0}$ therefore

$$(-\Omega^2 M_{FF} + i\Omega C_{FF} \dot{x}_F + K_{FF}) x_F e^{i\Omega t} = -(-\Omega^2 M_{FC} + i\Omega C_{FC} \dot{x}_F + K_{FC}) x_{C0} e^{i\Omega t} \cancel{+} F$$

$$x_F = (-\Omega^2 M_{FF} + i\Omega C_{FF} \boxed{} + K_{FF})^{-1} (-\Omega^2 M_{FC} + i\Omega C_{FC} \boxed{} + K_{FC}) x_C$$

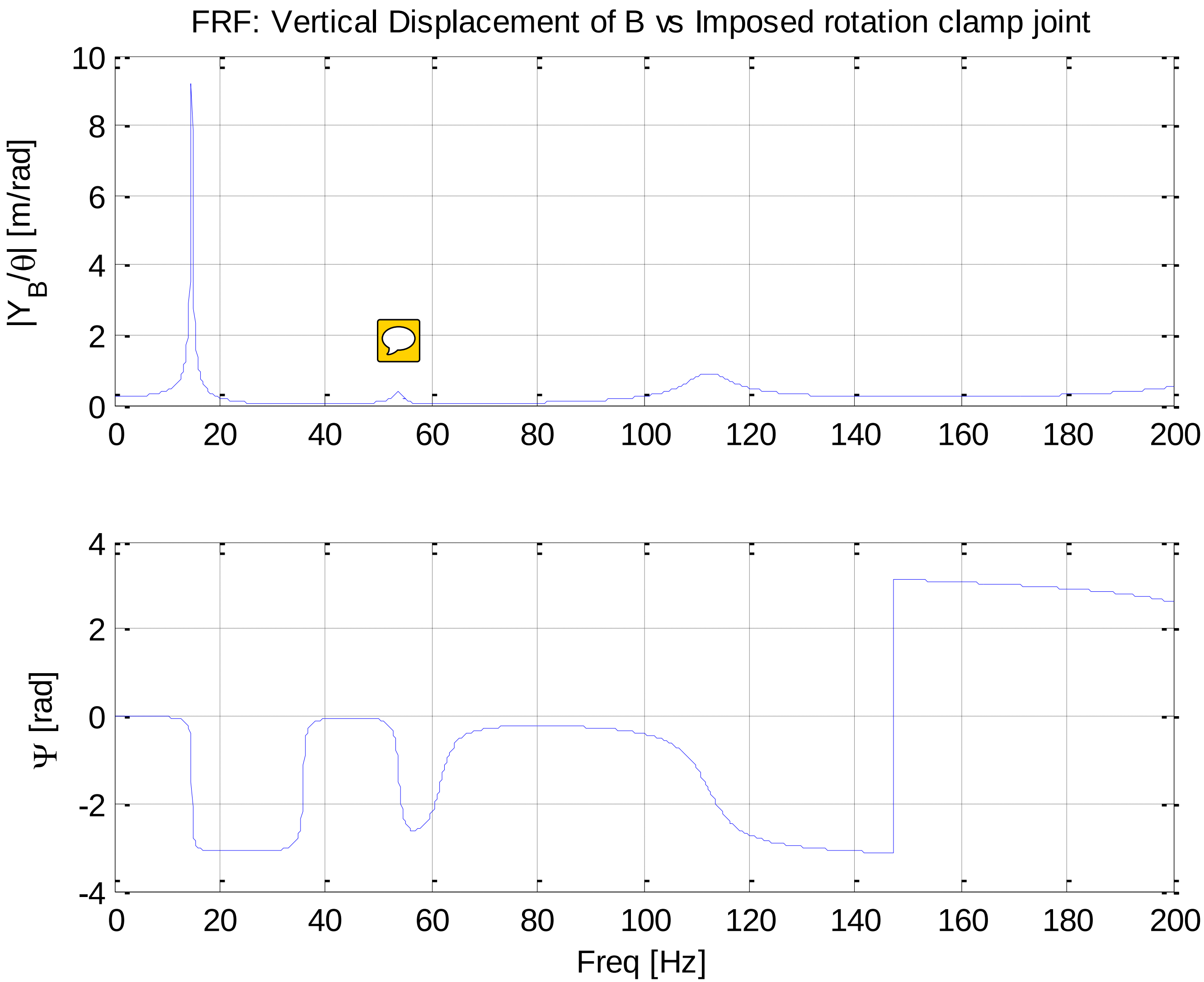
Frequency Response Function

FRF Y_A/θ



Frequency Response Function

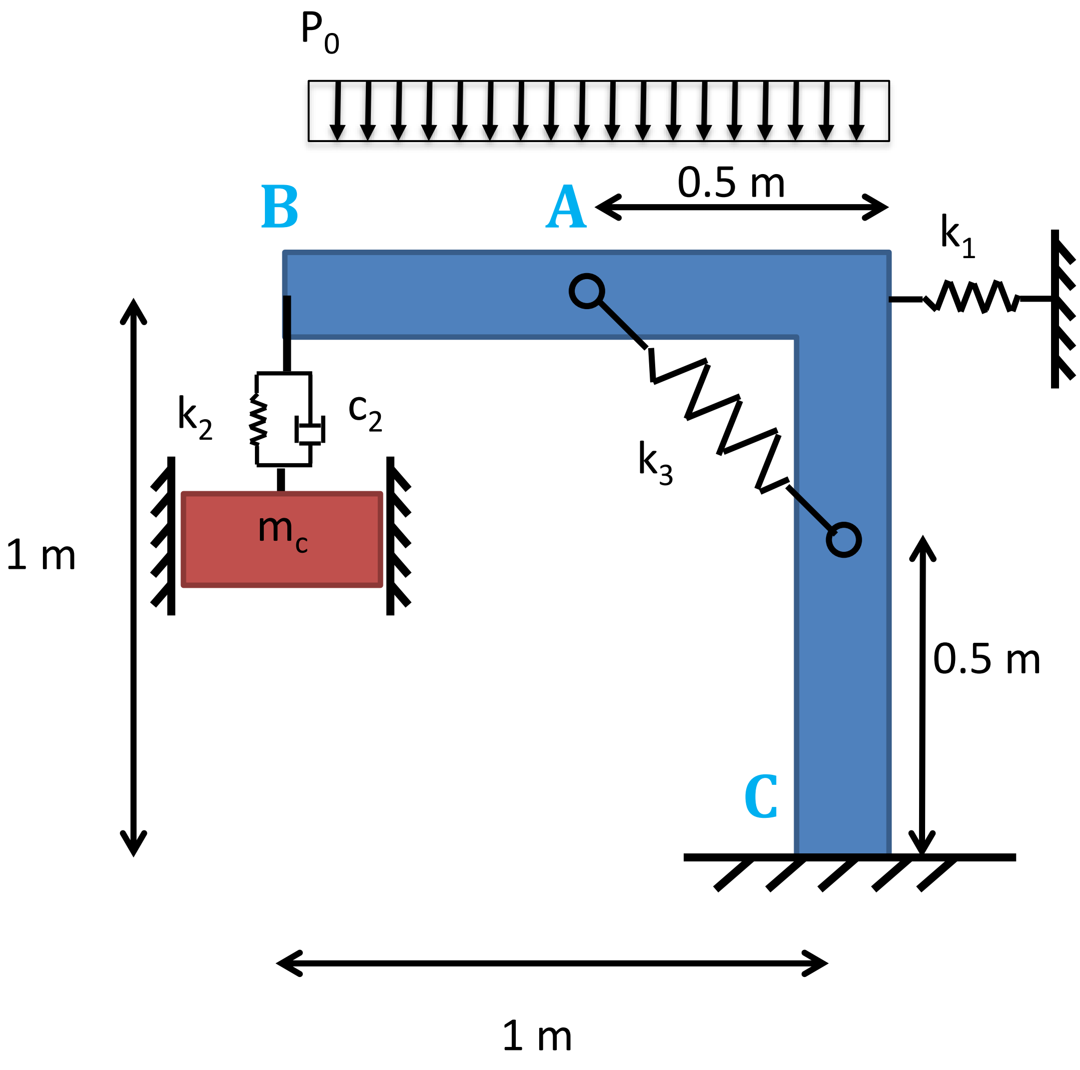
FRF Y_B/θ



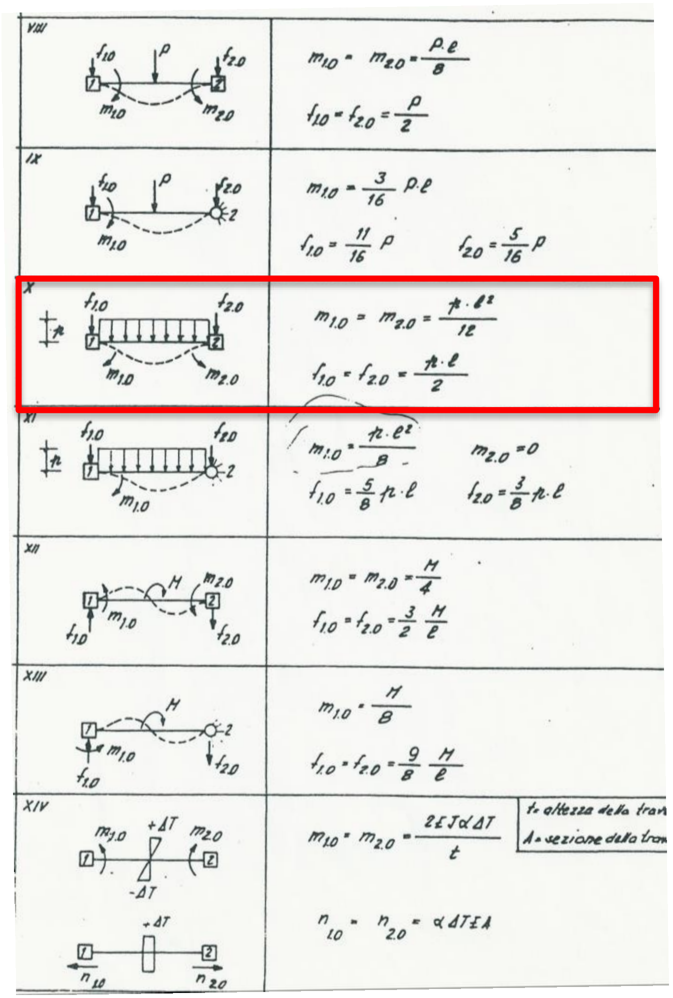
Distributed load

We want to compute the FRF of the displacement in A due to a distributed load on the horizontal element.

Ho do we implement the distributed load in the FE code?

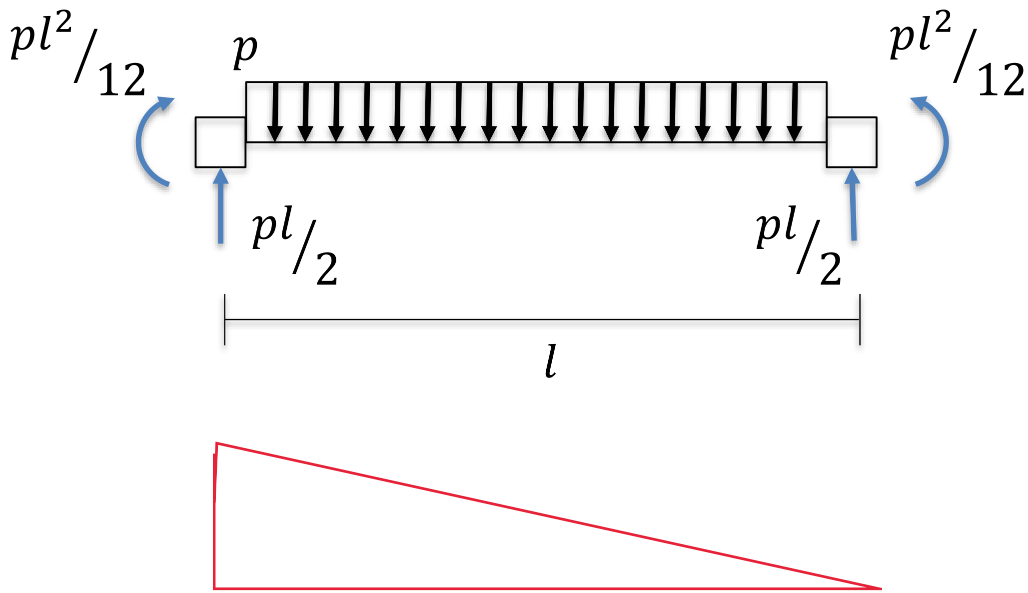


Distributed load



Flexibility coefficients – engineering manual

The distributed load on a beam can be repleaced by the equivalent forces at the nodes. These can be computed solving the constrains reaction of a fixed-fixed beam subject to the same load.



Frequency Response Function

FRF Y_A/p