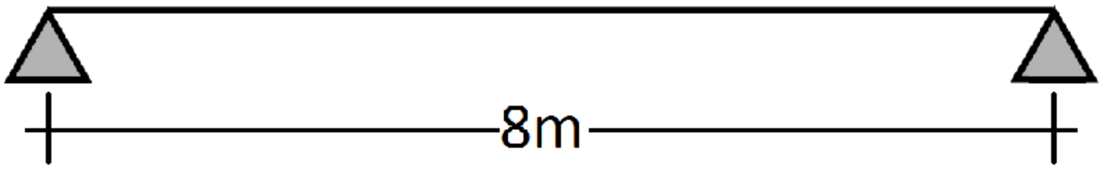
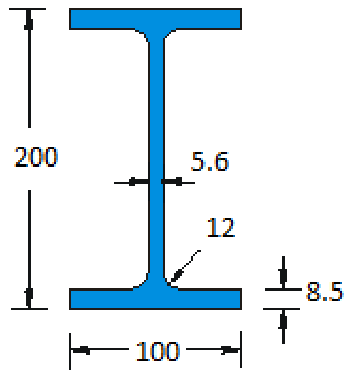
```
clear all
```





We consider a 30 metres long IPE200 beam

We define the physical properties of the beam:

```
L=8;            %m
E=200e9;         %N/m^2
I=1.9430e-05;     %m^4
m=22.4;          %Kg/m

EI=E*I;
```

We investigate the frequency range from 0 to 20 Hz

```
f=linspace(0,300,100000)
```

```
f = 1x100000 double

        0    0.0030    0.0060    0.0090    0.0120    0.0150    0.0180    0.0210 ···
```

```
omega=2*pi*f;
```

$$\gamma = \sqrt[4]{\frac{m}{EI}}\sqrt{\omega}$$

```
gamma=(m/EI)^(1/4)*omega.^(1/2)
```

```
gamma = 1x100000 double

        0    0.0067    0.0095    0.0117    0.0135    0.0150    0.0165    0.0178 ···
```

We consider the usual domain equation:

$$w = \left[A \sin\left(\gamma x\right) + B \cos\left(\gamma x\right) + C \sinh\left(\gamma x\right) + D \cosh\left(\gamma x\right)\right] \cos\left(\omega t + \psi\right)$$

We need to impose the boundary conditions:

Null dispacement in the first pin

$$w|_0 = 0 \quad \Rightarrow \quad B + D = 0$$

Null bending moment in the first pin

$$EI \frac{\partial^2 w}{\partial x^2}|_0 = 0 \quad \Rightarrow \quad EI(-\gamma^2 B + \gamma^2 D) = 0$$

$$\Rightarrow \quad -B + D = 0$$

Null dispacement in the second pin

$$w|_L = 0 \quad \Rightarrow \quad A \sin(\gamma L) + B \cos(\gamma L) + C \sinh(\gamma L) + D \cosh(\gamma L) = 0$$

Null bending moment in the second pin

$$EIw|_L = 0 \quad \Rightarrow \quad -\gamma^2 A \sin(\gamma L) - \gamma^2 B \cos(\gamma L) + \gamma^2 C \sinh(\gamma L) + \gamma^2 D \cosh(\gamma L) = 0$$

$$\Rightarrow \quad -A \sin(\gamma L) - B \cos(\gamma L) + C \sinh(\gamma L) + D \cosh(\gamma L) = 0$$

We can write the system BC matrix:

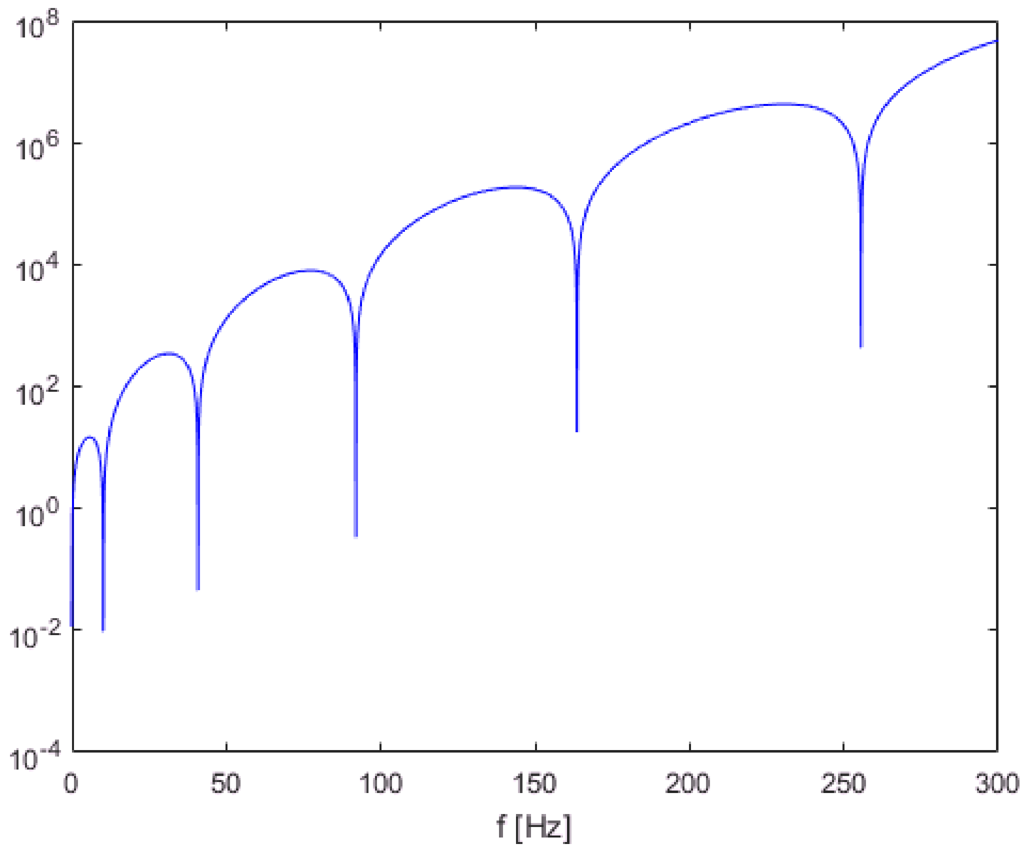
$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ \sin(\gamma L) & \cos(\gamma L) & \sinh(\gamma L) & \cosh(\gamma L) \\ -\sin(\gamma L) & -\cos(\gamma L) & \sinh(\gamma L) & \cosh(\gamma L) \end{pmatrix}$$

```
H=@(gamma) [  0              1            0             1            ;
    0             -1            0             1           ;
    sin(gamma*L)  cos(gamma*L) sinh(gamma*L) cosh(gamma*L);
    -sin(gamma*L) -cos(gamma*L) sinh(gamma*L) cosh(gamma*L)];
```

We obtain a 4 x 4 x 50 matrix containing the matrix H for each value of gamma.

We are looking for the value of gama for which the determinant of H is equal to zero

```
for i=1:length(gamma);
    dets(i)=det(H(gamma(i)));
end

semilogy(f,abs(dets),'-b')
hold on, xlabel('f [Hz]')
```



I have five values of f for which the determinant is (close) to zero

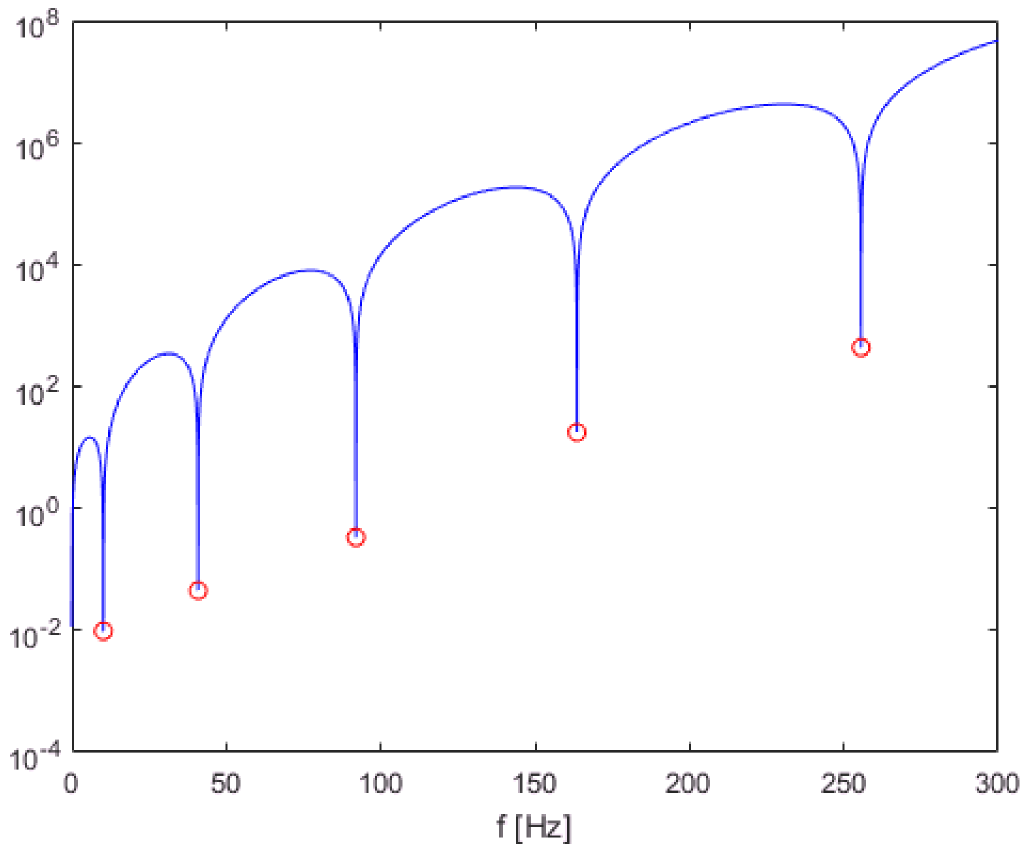
```
i_nat=[];
for i=2:length(dets)-1
    if abs(dets(i)) < abs(dets(i-1)) && abs(dets(i)) < abs(dets(i+1))
        i_nat(end+1)=i;
    end
end

fprintf('Natural frequencies:\n %f\n%f\n%f\n%f\n%f\n',f(i_nat))
```

```
Natural frequencies:
 10.224102
40.890409
92.004920
163.564636
255.569556
```

```
fprintf('Gamma:\n %f\n%f\n%f\n%f\n%f\n',gamma(i_nat))
```

```
Gamma:
 0.392725
0.785393
1.178099
1.570800
1.963500
```

```
plot(f(i_nat),abs(dets(i_nat)),'or')
```



Now that we know the value for which the system is singular (i.e. admits non-trivial solutions), we can find the modal shapes solving the **reduced system:**

$$\begin{bmatrix} - & - \\ \hat{E}_i & \hat{H}(\gamma_i) \end{bmatrix} \begin{Bmatrix} 1 \\ \hat{C}_i \end{Bmatrix} = 0$$

$$\hat{C}_i = -\hat{H}^{-1}(\gamma_i)\hat{E}_i$$

```
for i_mode=1:length(i_nat)
    fprintf('MODO %i',i_mode)
    gamma_i=gamma(i_nat(i_mode));
    Hi=H(gamma_i)
    Hi_hat=Hi(2:4,2:4)
    Ei_hat=Hi(2:4,1)
    Ci_hat=[1; -Hi_hat\Ei_hat]

    C_hat(:,i_mode)=Ci_hat;
end
```

```
MODO 1
Hi = 4x4 double

         0    1.0000         0    1.0000
         0   -1.0000         0    1.0000
   -0.0002   -1.0000   11.5512   11.5944
    0.0002    1.0000   11.5512   11.5944

Hi_hat = 3x3 double
```

```
   -1.0000         0    1.0000
   -1.0000   11.5512   11.5944
    1.0000   11.5512   11.5944
Ei_hat = 3x1 double

   1.0e-03
         0
   -0.2094
    0.2094
Ci_hat = 4x1 double

    1.0000
   -0.0002
    0.0002
   -0.0002
MODO 2
Hi = 4x4 double

         0    1.0000         0    1.0000
         0   -1.0000         0    1.0000
   -0.0000    1.0000  267.7336  267.7355
    0.0000   -1.0000  267.7336  267.7355
Hi_hat = 3x3 double

   -1.0000         0    1.0000
    1.0000  267.7336  267.7355
   -1.0000  267.7336  267.7355
Ei_hat = 3x1 double

   1.0e-04
         0
   -0.4217
    0.4217
Ci_hat = 4x1 double

    1.0000
    0.0000
   -0.0000
    0.0000
MODO 3
Hi = 4x4 double

   1.0e+03 *
         0    0.0010         0    0.0010
         0   -0.0010         0    0.0010
   -0.0000   -0.0010    6.1959    6.1959
    0.0000    0.0010    6.1959    6.1959
Hi_hat = 3x3 double

   1.0e+03 *
   -0.0010         0    0.0010
   -0.0010    6.1959    6.1959
    0.0010    6.1959    6.1959
Ei_hat = 3x1 double

   1.0e-04
         0
   -0.1357
    0.1357
Ci_hat = 4x1 double

    1.0000
   -0.0000
    0.0000
```

```
   -0.0000
```

MODO 4
Hi = *4x4 double*

```
   1.0e+05 *
         0    0.0000         0    0.0000
         0   -0.0000         0    0.0000
    0.0000    0.0000    1.4338    1.4338
   -0.0000   -0.0000    1.4338    1.4338
```

Hi_hat = *3x3 double*

```
   1.0e+05 *
   -0.0000         0    0.0000
    0.0000    1.4338    1.4338
   -0.0000    1.4338    1.4338
```

Ei_hat = *3x1 double*

```
   1.0e-04
         0
    0.3090
   -0.3090
```

Ci_hat = *4x1 double*

```
    1.0000
   -0.0000
    0.0000
   -0.0000
```

MODO 5
Hi = *4x4 double*

```
   1.0e+06 *
         0    0.0000         0    0.0000
         0   -0.0000         0    0.0000
   -0.0000   -0.0000    3.3179    3.3179
    0.0000    0.0000    3.3179    3.3179
```

Hi_hat = *3x3 double*

```
   1.0e+06 *
   -0.0000         0    0.0000
   -0.0000    3.3179    3.3179
    0.0000    3.3179    3.3179
```

Ei_hat = *3x1 double*

```
   1.0e-04
         0
   -0.3286
    0.3286
```

Ci_hat = *4x1 double*

```
    1.0000
   -0.0000
    0.0000
   -0.0000
```

```
C_hat
```

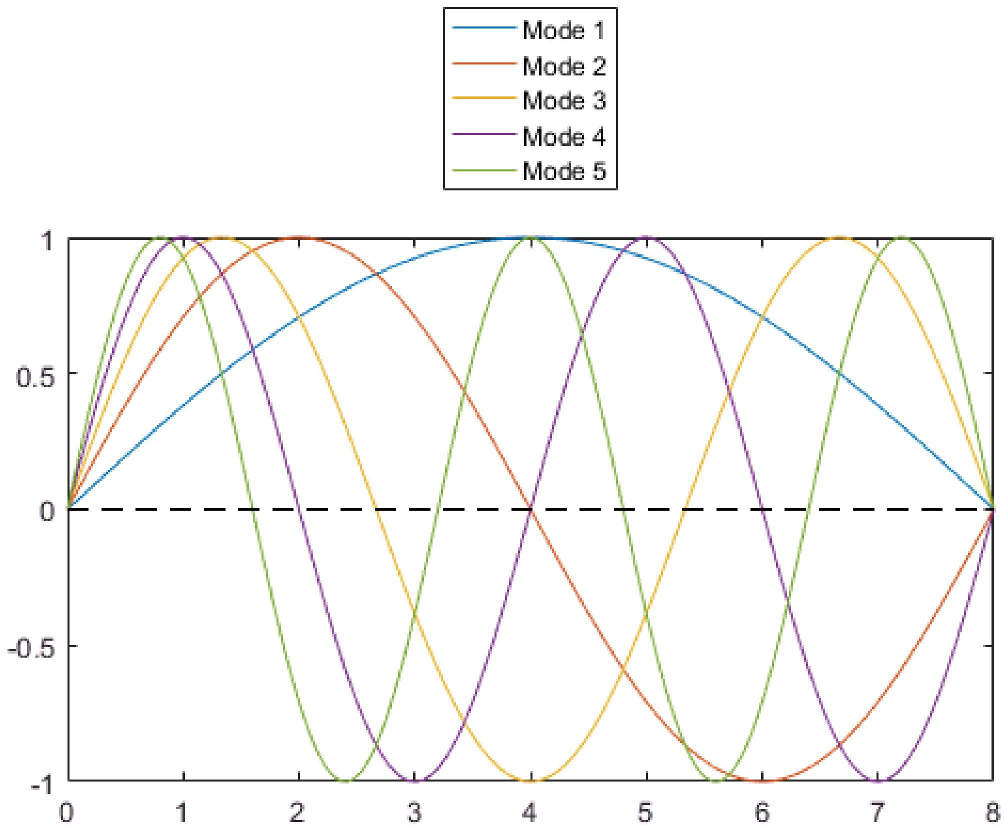
C_hat = *4x5 double*

```
    1.0000    1.0000    1.0000    1.0000    1.0000
   -0.0002    0.0000   -0.0000   -0.0000   -0.0000
    0.0002   -0.0000    0.0000    0.0000    0.0000
   -0.0002    0.0000   -0.0000   -0.0000   -0.0000
```

We can now compute the mode shapes

$$\phi_i(x) = \left[A \sin\left(\gamma_i x\right) + B \cos\left(\gamma_i x\right) + C \sinh\left(\gamma_i x\right) + D \cosh\left(\gamma_i x\right)\right]$$

```
x=linspace(0,L,1000);
dx=x(2);
for i_mode=1:length(i_nat)
    gamma_i=gamma(i_nat(i_mode));
    phi(i_mode,:)= C_hat(1,i_mode)*sin(gamma_i*x)  + C_hat(2,i_mode)*cos(gamma_i*x) +...
        C_hat(3,i_mode)*sinh(gamma_i*x) + C_hat(4,i_mode)*cosh(gamma_i*x);
end

figure
plot(x,phi)
hold on, plot([0 L],[0 0],'--k')
ylim([-1 1])
legend({'Mode 1','Mode 2','Mode 3','Mode 4','Mode 5'},'Location','NorthOutside')
```



Now we can compute the energy functions using the Lagrange equation:

$$E_K = \frac{1}{2}\int_0^L m \frac{\partial w}{\partial t}\frac{\partial w}{\partial t}\, dx =$$

$$= \frac{1}{2}\dot{q}^T \int_0^L m\phi(x)\phi^T(x)\, dx\, \dot{q} =$$

$$= \frac{1}{2}\dot{q}^T[M]\,\dot{q}$$

$$\Rightarrow [M] = \int_0^L m\phi(x)\phi^T(x)\,dx$$

```
M=m*phi*phi'*dx
```

```
M = 5x5 double

   89.5821   -0.0131   -0.0080   -0.0060   -0.0048
   -0.0131   89.6018    0.0022    0.0014    0.0010
   -0.0080    0.0022   89.5996   -0.0000   -0.0002
   -0.0060    0.0014   -0.0000   89.5993   -0.0005
   -0.0048    0.0010   -0.0002   -0.0005   89.5994
```

As you can see **M is diagonal**

$$V = \frac{1}{2}\int_0^L EJ\,\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x^2}\,dx =$$

$$= \frac{1}{2}q^T\int_0^L EJ\,\phi''(x)\phi''^T(x)\,dx\;q =$$

$$= \frac{1}{2}q^T[K]\,q$$

$$\Rightarrow [K] = \int_0^L EJ\,\phi''(x)\phi''^T(x)\,dx$$

```
phi2=diff(phi,2,2)/(dx^2);

K=EI*phi2*phi2'*dx
```

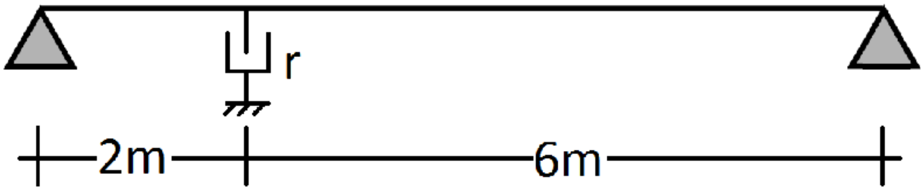
```
K = 5x5 double

   1.0e+08 *
    0.0037   -0.0000   -0.0000   -0.0000   -0.0000
   -0.0000    0.0591    0.0000    0.0000    0.0000
   -0.0000    0.0000    0.2994    0.0000    0.0000
   -0.0000    0.0000    0.0000    0.9463    0.0000
   -0.0000    0.0000    0.0000    0.0000    2.3103
```

As you can see K is diagonal too!

The equation of motion for this system is therefore:

$$[M]\ddot{q} + [K]\dot{q} = 0$$

Now consider the same beam, this time with a damper in the middle:



We need to compute the damping function.

We say that:

$$D = \frac{1}{2} r \left.\frac{\partial w}{\partial t}\right|_{x=2} \left.\frac{\partial w}{\partial t}\right|_{x=2} =$$

$$= \frac{1}{2} \dot{q}^T r \phi(x = 2) \phi^T(x = 2) \dot{q} =$$

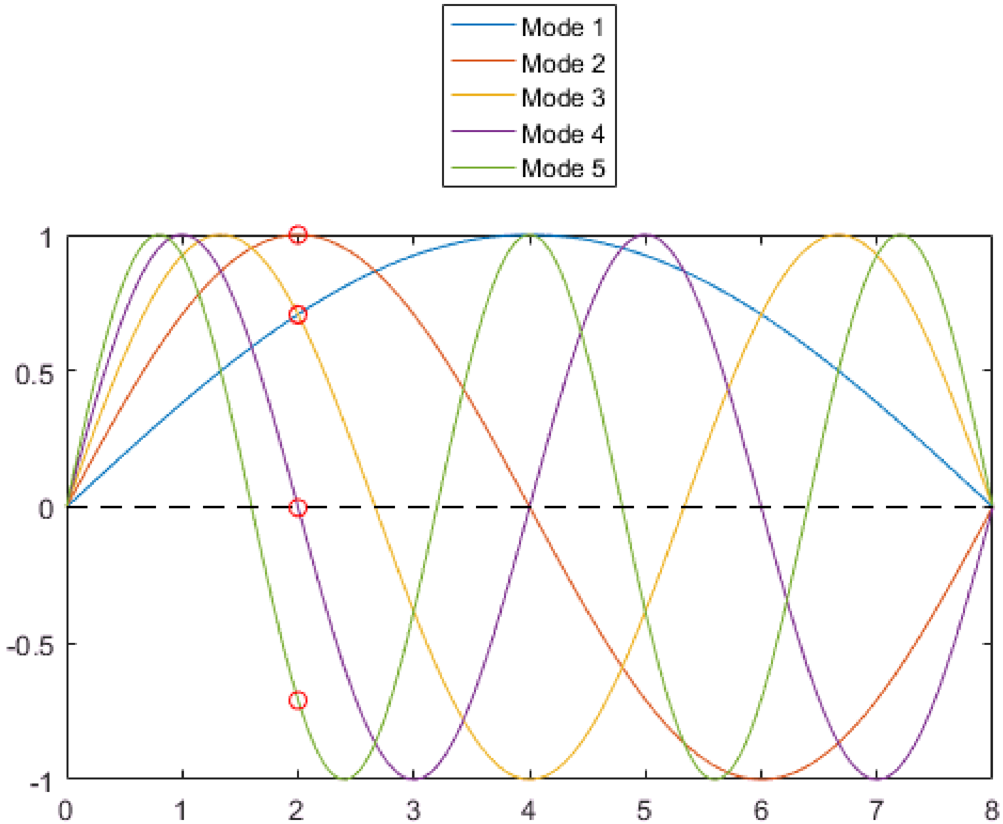
$$= \frac{1}{2} \dot{q}^T [C] \dot{q}$$

$$\Rightarrow [C] = r\, \phi(x = 2)\, \phi^T(x = 2)$$

```
xDamper=2;
r=.10 * 2 * M(1,1) * omega(i_nat(1));

[~,i_damper]=min(abs(x-xDamper));
phi_damper=phi(:,i_damper)
```

```
phi_damper = 5x1 double

    0.7075
    1.0000
    0.7054
   -0.0031
   -0.7099
```

```
plot(xDamper,phi_damper,'or')
```




```
R=r*phi_damper*phi_damper'*dx
```

```
R = 5x5 double

   1.0e+03 *
    4.6130    6.5205    4.5998   -0.0204   -4.6287
    6.5205    9.2170    6.5020   -0.0288   -6.5427
    4.5998    6.5020    4.5867   -0.0203   -4.6155
   -0.0204   -0.0288   -0.0203    0.0001    0.0204
   -4.6287   -6.5427   -4.6155    0.0204    4.6444
```

R IS NOT DIAGONAL!!

We want to compute the eigenvalues and eigenvectors of the damped system.

We use the the system state variable

$$z = \begin{Bmatrix} \dot{q} \\ q \end{Bmatrix}$$

to write the system as

$$\begin{aligned} [M]\ddot{q} \quad + [R]\dot{q} \quad + [K]q \quad &= 0 \\ [M]\dot{q} \qquad\qquad\quad - [M]\dot{q} \quad &= 0 \end{aligned}$$

that allow the system to be written as

$$\begin{bmatrix} [M] & [0] \\ [0] & [M] \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} [R] & [K] \\ -[M] & [0] \end{bmatrix} \begin{Bmatrix} \dot{q} \\ q \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[B]\dot{z} + [C]z = 0$$

$$[A] = -[B]^{-1}[C]$$

$$\dot{z} - [A]z = 0$$

```
B=[M,              zeros(size(M));
    zeros(size(M)), M             ];
C=[ R, K             ;
    -M, zeros(size(M))];
A=-inv(B)*C;
```

we can use this equation to write the eigenvalue problem

$$[[A] - \lambda[I]]Z = 0$$

```

[phi_damped,lamba_damped]=eig(A,'vector')
```

```
phi_damped =
   0.0038 - 0.0318i   0.0038 + 0.0318i  -0.0222 + 0.0844i  -0.0222 - 0.0844i   0.9965 + 0.0000i   0.9965
   0.0054 - 0.0460i   0.0054 + 0.0460i  -0.0360 + 0.1483i  -0.0360 - 0.1483i   0.0299 - 0.0763i   0.0299
   0.0042 - 0.0364i   0.0042 + 0.0364i   0.9843 + 0.0000i   0.9843 + 0.0000i   0.0036 - 0.0104i   0.0036
  -0.0000 + 0.0002i  -0.0000 - 0.0002i  -0.0001 + 0.0002i  -0.0001 - 0.0002i  -0.0000 + 0.0000i  -0.0000
   0.9977 + 0.0000i   0.9977 + 0.0000i  -0.0043 + 0.0117i  -0.0043 - 0.0117i  -0.0004 + 0.0014i  -0.0004
  -0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0001 + 0.0000i   0.0001 - 0.0000i  -0.0062 - 0.0134i  -0.0062
  -0.0000 - 0.0000i  -0.0000 + 0.0000i   0.0003 + 0.0001i   0.0003 - 0.0001i  -0.0012 + 0.0001i  -0.0012
  -0.0000 - 0.0000i  -0.0000 + 0.0000i  -0.0001 - 0.0017i  -0.0001 + 0.0017i  -0.0002 + 0.0000i  -0.0002
   0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 - 0.0000i   0.0000
  -0.0000 - 0.0006i  -0.0000 + 0.0006i   0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 - 0.0000i   0.0000

lamba_damped =
   1.0e+03 *

  -0.0255 + 1.6021i
  -0.0255 - 1.6021i
  -0.0235 + 0.5703i
  -0.0235 - 0.5703i
  -0.0284 + 0.0615i
  -0.0284 - 0.0615i
  -0.0514 + 0.2419i
  -0.0514 - 0.2419i
  -0.0000 + 1.0277i
  -0.0000 - 1.0277i
```

We have 10 eigen-vectors and 10 eigen-values complex coniugated two by two. We keep only one of each

```
phi_damped=phi_damped(:,1:2:end);
lamba_damped=lamba_damped(1:2:end);
```

Since we are interested only in the first 5 lines of the previous system, we keep only the first 5 values of each eigen-vector

```
phi_damped=phi_damped(1:5,:);
```

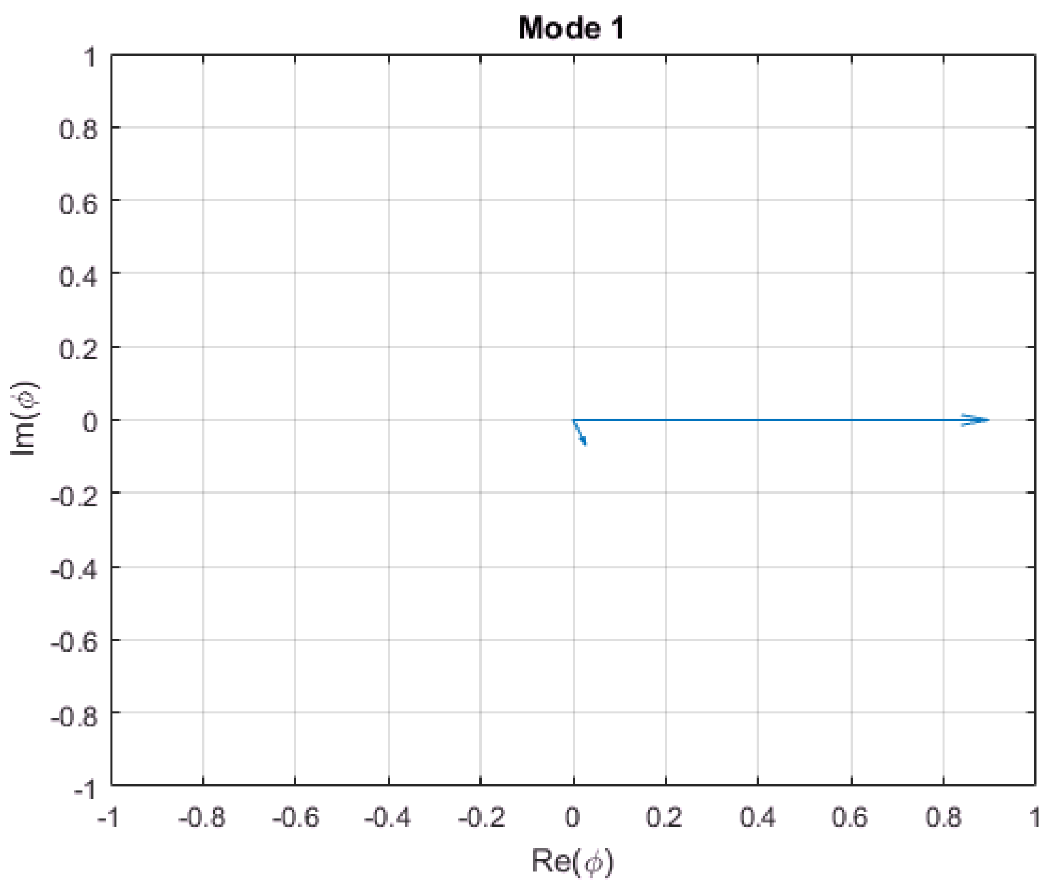
Now we sort the eigen-values smallest-to-biggest and the eigen-vectors accordingly.

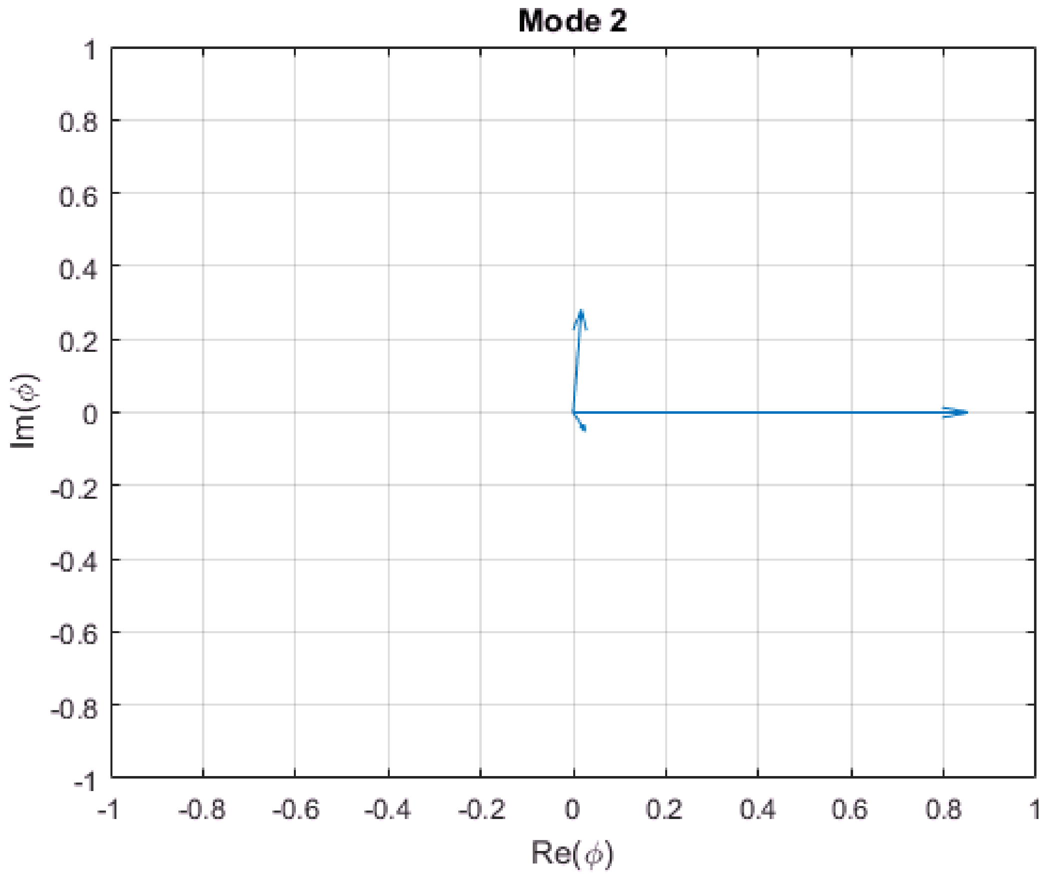
```
[lamba_damped,sortInd]=sort(lamba_damped);
phi_damped=phi_damped(:,sortInd)
```

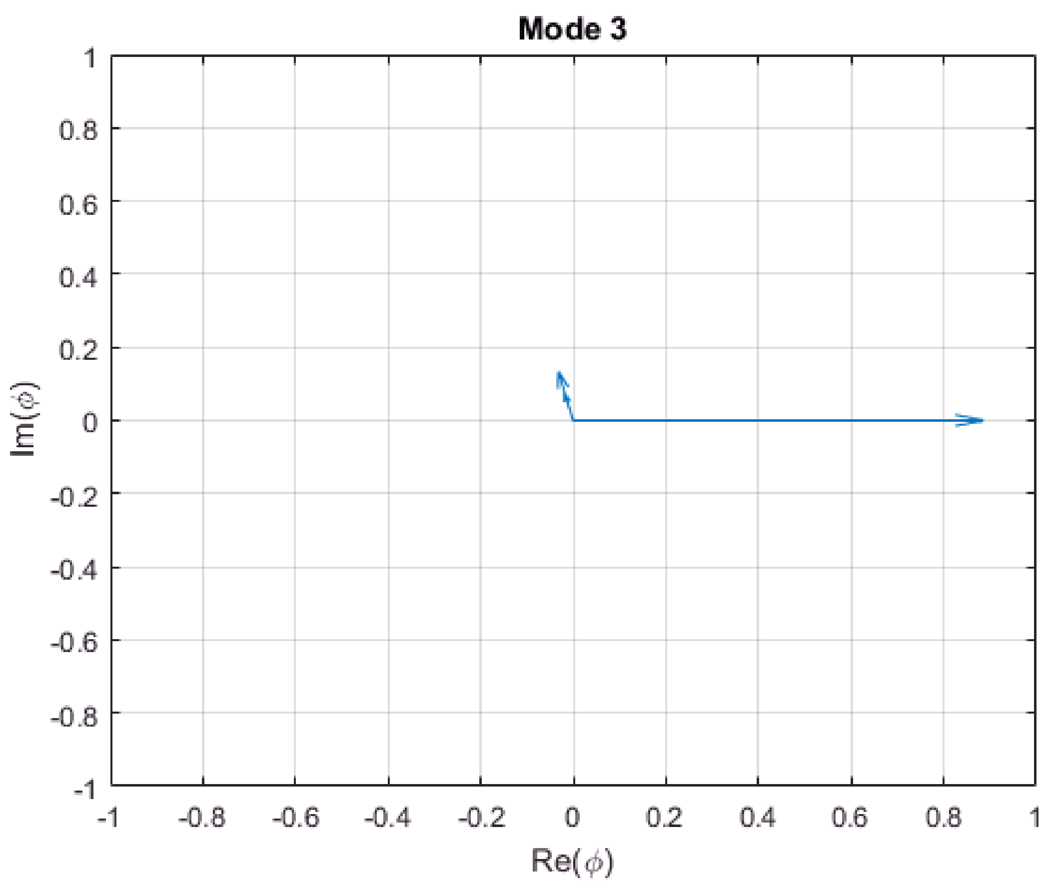
```
phi_damped =
   0.9965 + 0.0000i   0.0193 + 0.3128i  -0.0222 + 0.0844i   0.0001 - 0.0002i   0.0038 - 0.0318i
   0.0299 - 0.0763i   0.9474 + 0.0000i  -0.0360 + 0.1483i   0.0000 - 0.0003i   0.0054 - 0.0460i
   0.0036 - 0.0104i   0.0294 - 0.0571i   0.9843 + 0.0000i   0.0001 - 0.0003i   0.0042 - 0.0364i
  -0.0000 + 0.0000i  -0.0000 + 0.0001i  -0.0001 + 0.0002i   1.0000 + 0.0000i  -0.0000 + 0.0002i
  -0.0004 + 0.0014i  -0.0028 + 0.0066i  -0.0043 + 0.0117i   0.0000 - 0.0001i   0.9977 + 0.0000i
```

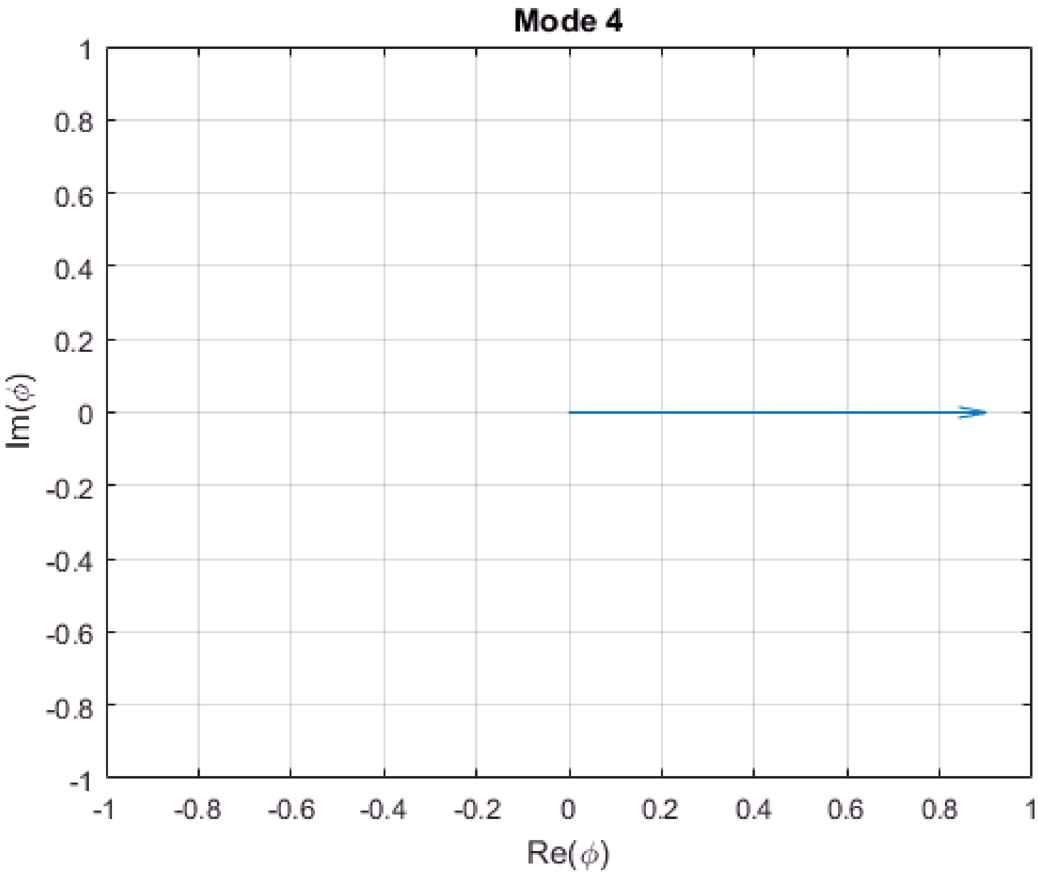
We can plot these eigen-vetors in the complex plane:

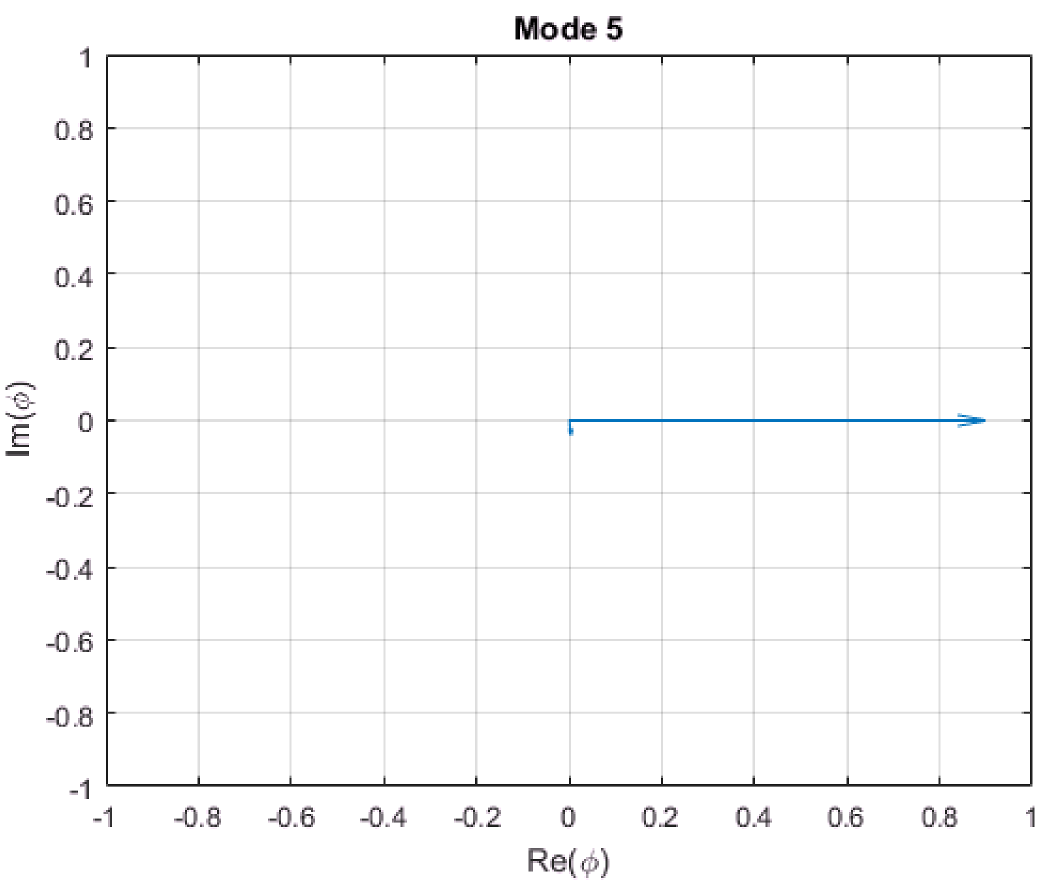
```
for i=1:5
    figure
    quiver(zeros(5,1),zeros(5,1),real(phi_damped(:,i)),imag(phi_damped(:,i)))
    title(sprintf('Mode %i',i))
    xlim([-1 1]), ylim([-1 1])
    xlabel('Re(\phi)'), ylabel('Im(\phi)')
    grid on
end
```




Mode 2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Im(ϕ)
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Re(ϕ)


Mode 3
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Im(ϕ)
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Re(ϕ)





Consider Mode 2:

The damped mode will be:

$$w_2(x, t) = \phi(x)^T \phi_{D,2}\, q(t) = \phi(x)^T \phi_{D,2}\, e^{i\omega_1 t}$$

since $\phi_{D,2}$ is complex, also the vibrating mode will be complex, with some components out of phase with respect to the "pure" mode. To look at it we need to see it in the time domain:

(run modeAnimation.m)