Econometrics I - TA section

Eric Schulman

The University of Texas at Austin

October 2, 2020

3.6 For the double exponential distribution show

3.6 For the double exponential distribution show

$$f(x;\lambda) = \frac{1}{2\lambda} e^{-|x|/\lambda}$$

- $\int_{-\infty}^{\infty} f(x;\lambda) = 1$

3.6 For the double exponential distribution show
$f(x;\lambda) = \frac{1}{2\lambda}e^{-|x|/\lambda}$

- $\int_{-\infty}^{\infty} f(x;\lambda) = 1$

$$\int_{-\infty}^{\infty} \frac{1}{2\lambda}e^{-|x|/\lambda}dx =$$

$$\int_{-\infty}^{0} \frac{1}{2\lambda}e^{x/\lambda}dx + \int_{0}^{\infty} \frac{1}{2\lambda}e^{-x/\lambda}dx =$$

$$1/2e^{x/\lambda}|_{x=-\infty}^{x=0} + -1/2e^{-x/\lambda}|_{x=0}^{x=\infty} = 1$$

3.6 For the double exponential distribution show

3.6 For the double exponential distribution show

- $E(x) = 0$

3.6 For the double exponential distribution show

- $E(x) = 0$

$$\int_{-\infty}^{\infty} x \frac{1}{2\lambda} e^{-|x|/\lambda} dx =$$

$$\int_{-\infty}^{0} \frac{1}{2\lambda} x e^{x/\lambda} dx + \int_{0}^{\infty} \frac{1}{2\lambda} x e^{-x/\lambda} dx =$$

Need to integrate by parts!

$$1/2(x-a)e^{x/\lambda}|_{x=-\infty}^{x=0} + -1/2(x+a)e^{-x/\lambda}|_{x=0}^{x=\infty} = 0$$

3.6 For the double exponential distribution show

- $E(x) = 0$

$$\int_{-\infty}^{\infty} x \frac{1}{2\lambda} e^{-|x|/\lambda} dx =$$

$$\int_{-\infty}^{0} \frac{1}{2\lambda} x e^{x/\lambda} dx + \int_{0}^{\infty} \frac{1}{2\lambda} x e^{-x/\lambda} dx =$$

  Need to integrate by parts!

$$1/2(x-a)e^{x/\lambda}|_{x=-\infty}^{x=0} + -1/2(x+a)e^{-x/\lambda}|_{x=0}^{x=\infty} = 0$$

- Similar strategy for next part $E(x^2) = 2\lambda^2$

3.11 For the Pareto distribution show

3.11 For the Pareto distribution show
$f(x;\alpha,\beta) = \alpha\beta^{\alpha}x^{-\alpha-1}$ $x \geq \beta$

- $\int_{\beta}^{\infty} f(x;\alpha,\beta) = 1$

3.11 For the Pareto distribution show
$f(x;\alpha,\beta) = \alpha\beta^{\alpha}x^{-\alpha-1}\ x \geq \beta$

- $\int_{\beta}^{\infty} f(x;\alpha,\beta) = 1$

$$\int_{\beta}^{\infty} \alpha\beta^{\alpha}x^{-\alpha-1}dx =$$

$$-\beta^{\alpha}x^{-\alpha}|_{\beta}^{\infty} = 1$$

3.11 For the Pareto distribution show
$f(x;\alpha,\beta) = \alpha\beta^{\alpha}x^{-\alpha-1}\ x \geq \beta$

- $\int_{\beta}^{\infty} f(x;\alpha,\beta) = 1$

$$\int_{\beta}^{\infty} \alpha\beta^{\alpha}x^{-\alpha-1}dx =$$

$$-\beta^{\alpha}x^{-\alpha}|_{\beta}^{\infty} = 1$$

- $E(X) = \dfrac{\alpha\beta}{\alpha-1}$

3.11 For the Pareto distribution show
$f(x;\alpha,\beta) = \alpha\beta^{\alpha}x^{-\alpha-1} \; x \geq \beta$

- $\int_{\beta}^{\infty} f(x;\alpha,\beta) = 1$

$$\int_{\beta}^{\infty} \alpha\beta^{\alpha}x^{-\alpha-1}dx =$$

$$-\beta^{\alpha}x^{-\alpha}|_{\beta}^{\infty} = 1$$

- $E(X) = \dfrac{\alpha\beta}{\alpha-1}$

$$\int_{\beta}^{\infty} \alpha\beta^{\alpha}xx^{-\alpha-1}dx =$$

$$\frac{\alpha\beta^{\alpha}}{1-\alpha}x^{1-\alpha}|_{\beta}^{\infty} = \frac{\alpha\beta}{\alpha-1}$$

3.11 For the Pareto distribution show

3.11 For the Pareto distribution show

- $Var(X^2) = \dfrac{\alpha\beta^2}{(\alpha-2)(\alpha-1)^2}$

3.11 For the Pareto distribution show

- $Var(X^2) = \dfrac{\alpha\beta^2}{(\alpha-2)(\alpha-1)^2}$

$$E(X^2) = \int_\beta^\infty \alpha\beta^\alpha x^2 x^{-\alpha-1} dx =$$

$$\frac{\alpha\beta^\alpha}{2-\alpha} x^{2-\alpha}|_\beta^\infty = \frac{\alpha\beta^2}{\alpha-2}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{\alpha\beta^2}{(\alpha-2)(\alpha-1)^2}$$

Cheat Sheet

- Marginal densities:

$$f(x) = \int_Y f(x, y) dy$$

Cheat Sheet

- Marginal densities:

$$f(x) = \int_Y f(x, y) dy$$

- Conditional densities:

$$f(y|x) = \frac{f(x, y)}{f(x)}$$

Cheat Sheet

- Marginal densities:

$$f(x) = \int_Y f(x,y)dy$$

- Conditional densities:

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

- Independence

$$f(x)f(y) = f(x,y)$$

Cheat Sheet

When integrating multivariate functions. Always write down the set your trying to integrate over ... Double check this equation

- Fubini's theorem

$$\int_{(x,y)\in X\times Y} f(x,y)d(x,y) = \int_{y\in Y}\int_{x\in X} f(x,y)dxdy$$
$$= \int_{x\in X}\int_{y\in Y} f(x,y)dydx$$

- Will need to use this when x and y are not independent

Cheat Sheet

- Law of iterated expectations

$$E(E(X|Y)) = E(X)$$

Cheat Sheet

- Law of iterated expectations

$$E(E(X|Y)) = E(X)$$

Why? Look at discrete version for intuition
Note, that $E(X|Y) = \sum_{x \in supp(X)} xPr(x|y)$

Cheat Sheet

- Law of iterated expectations

$$E(E(X|Y)) = E(X)$$

Why? Look at discrete version for intuition
Note, that $E(X|Y) = \sum_{x \in supp(X)} xPr(x|y)$

$$E(E(X|Y)) = \sum_{y \in supp(Y)} \left(\sum_{x \in supp(X)} xPr(x|y) \right) Pr(y)$$

Cheat Sheet

- Law of iterated expectations

$$E(E(X|Y)) = E(X)$$

Why? Look at discrete version for intuition
Note, that $E(X|Y) = \sum_{x \in supp(X)} xPr(x|y)$

$$E(E(X|Y)) = \sum_{y \in supp(Y)} \left(\sum_{x \in supp(X)} xPr(x|y) \right) Pr(y)$$

$$= \sum_{y \in supp(Y)} \sum_{x \in supp(X)} xPr(x|y)Pr(y)$$

$$E(x) = \sum_{y \in supp(Y)} \sum_{x \in supp(X)} xPr(x,y)$$

4.1 Let $f(x, y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

1. Verify $f(x, y)$ is a valid density

4.1 Let $f(x, y) = 1/4$ for $-1 \le x \le 1$ and $-1 \le y \le 1$

1. Verify $f(x, y)$ is a valid density

$$\int_{-1}^{1}\int_{-1}^{1} f(x, y)dxdy = \frac{1}{4}xy|_{x=-1}^{x=1}|_{y=-1}^{y=1} = 1$$

   Also, $f(x, y) \ge 0$

2. Find the marginal density for X

4.1 Let $f(x,y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

1. Verify $f(x,y)$ is a valid density

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy = \frac{1}{4}xy|_{x=-1}^{x=1}|_{y=-1}^{y=1} = 1$$

   Also, $f(x,y) \geq 0$

2. Find the marginal density for X

$$\int_{-1}^{1} f(x,y)dx = \frac{1}{4}x|_{x=-1}^{x=1} = \frac{1}{2}$$

3. Find the conditional density of Y given $X = x$

4.1 Let $f(x,y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

1. Verify $f(x,y)$ is a valid density

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy = \frac{1}{4}xy|_{x=-1}^{x=1}|_{y=-1}^{y=1} = 1$$

   Also, $f(x,y) \geq 0$

2. Find the marginal density for X

$$\int_{-1}^{1} f(x,y)dx = \frac{1}{4}x|_{x=-1}^{x=1} = \frac{1}{2}$$

3. Find the conditional density of Y given $X = x$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{1/4}{1/2} = 1/2$$

4.1 Let $f(x, y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

- Find $E(Y|X = x)$

4.1 Let $f(x,y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

- Find $E(Y|X = x)$

$$E(Y|X = x) = \int_{-1}^{1} yf(y|x)dy = \int_{-1}^{1} 1/2ydy = 1/4y^2|_{y=-1}^{y=1} = 0$$

4.1 Let $f(x, y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

- Determine $P(X^2 + Y^2 < 1)$

4.1 Let $f(x, y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

- Determine $P(X^2 + Y^2 < 1)$
  Start with Fubini's theorem

$$\int_{(x,y) \in X \times Y} f(x, y) d(x, y)$$

4.1 Let $f(x, y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

- Determine $P(X^2 + Y^2 < 1)$
  Start with Fubini's theorem

$$\int_{(x,y) \in X \times Y} f(x, y) d(x, y)$$

  But, what do we integrate over? Well fixing y,
  $x \in \left[-\sqrt{1-y^2}, \sqrt{1-y^2}\right]$. And, $y \in [0, 1]$

4.1 Let $f(x,y) = 1/4$ for $-1 \le x \le 1$ and $-1 \le y \le 1$

- Determine $P(X^2 + Y^2 < 1)$
  Start with Fubini's theorem

$$\int_{(x,y)\in X\times Y} f(x,y)d(x,y)$$

But, what do we integrate over? Well fixing y, $x \in \left[-\sqrt{1-y^2}, \sqrt{1-y^2}\right]$. And, $y \in [0,1]$

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{4}dxdy = \int_{-1}^{1}\frac{1}{2}\sqrt{1-y^2}dy$$

Using a change of variable let $y = sin(\theta)$, $dy = cos(\theta)d\theta$ i.e. $\sqrt{1-y^2} = cos(\theta)$. Doing the algebra, we get $\frac{\pi}{4}$

4.1 Let $f(x, y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

- Determine $P(X^2 + Y^2 < 1)$
  Start with Fubini's theorem

$$\int_{(x,y) \in X \times Y} f(x, y) d(x, y)$$

But, what do we integrate over? Well fixing y,
$x \in \left[-\sqrt{1 - y^2}, \sqrt{1 - y^2}\right]$. And, $y \in [0, 1]$

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{4} dxdy = \int_{-1}^{1} \frac{1}{2} \sqrt{1 - y^2} dy$$

Using a change of variable let $y = sin(\theta)$, $dy = cos(\theta)d\theta$ i.e.
$\sqrt{1 - y^2} = cos(\theta)$. Doing the algebra, we get $\dfrac{\pi}{4}$

4.1 Let $f(x, y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

- Determine $P(|X + Y| < 2)$

4.1 Let $f(x, y) = 1/4$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

- Determine $P(|X + Y| < 2)$
  What do we integrate over? Fixing y, $x \in [-2 - y, 2 - y]$.
  And, $y \in [-1, 1]$

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{4} dxdy = \int_{-1}^{1} \frac{1}{2} dy = 1$$

4.6 $f(x,y) = \begin{cases} cxy & x, y \in [0,1], x + y \leq 1 \\ 0 & \text{o.w.} \end{cases}$

1. Find the value of c such that $f(x,y)$ is a joint PDF

$$\int_0^1 \int_0^{1-y} f(x,y)dxdy = 1$$

$$\int_0^1 \frac{c(1-y)^2 y}{2} dxdy = 1$$

$$c/24 = 1$$

$$c = 24$$

4.6 $f(x, y) = \begin{cases} cxy & x, y \in [0, 1], x + y \leq 1 \\ 0 & \text{o.w.} \end{cases}$

- Find the marginal distribution of X and Y

$$f(x) = \int_0^{1-x} f(x, y) dydx = 12(1 - x)^2 x$$

$$f(y) = \int_0^{1-y} f(x, y) dydx = 12(1 - y)^2 y$$

- Are X and Y independent? No. $f(x)f(y) \neq f(x, y)$

4.7 Let X and Y have density $f(x, y) = exp(-x - y)$ for $x > 0$ and $y > 0$. Find the marginal density of X and Y. Are X and Y independent or dependent?

4.7 Let X and Y have density $f(x, y) = exp(-x - y)$ for $x > 0$ and $y > 0$. Find the marginal density of X and Y. Are X and Y independent or dependent?

$$f_x(x) = \int_0^\infty exp(-x - y)dy$$

$$f_x(x) = \int_0^\infty exp(-x - y)dy = -exp(-x - y)|_{y=0}^{y=\infty}$$

$$f_x(x) = exp(-x)$$

Similarly, $f_y(y) = exp(-y)$

$$f(x, y) = f_x(x)f_y(y)$$

so yes. They are independent

4.8 Let X and Y have density $f(x, y) = 1$ on $0 < x < 1$ and $0 < y < 1$. Find the density function of $Z = XY$.
A lot of you noticed, that there are "two areas" to integrate over. Comes from LIE.

$$Pr(XY \leq z) = Pr(XY \leq z \cap Y \leq z) + Pr(XY \leq z \cap Y > z)$$

$$Pr(XY \leq z \cap Y \leq z) = \int_0^z \int_0^1 1dxdy = z$$

$$Pr(XY \leq z \cap Y > z) = \int_z^1 \int_0^{z/y} 1dxdy = -zln(z)$$

Thus, we get $F(z) = z - zln(z)$ and $f(z) = -ln(z)$ where $z \in (0, 1]$

4.9 Let X and Y have density $f(x,y) = 12xy(1-y)$ for $0 < x < 1$ and $0 < y < 1$. Are X and Y independent or dependent?

$$f(x) = \int_0^1 12xy(1-y)dy = 2x$$

$$f(y) = \int_0^1 12xy(1-y)dx = 6(1-y)y$$

So, yes they are independent

$$f(y)f(x) = 6(1-y)y2x = 12(1-y)y2x = f(x,y)$$

4.14 Let $X_1 \sim gamma(r, 1)$ and $X_2 \sim gamma(s, 1)$ be independent. Find the distribution of $Y = X_1 + X_2$.

4.14 Let $X_1 \sim gamma(r, 1)$ and $X_2 \sim gamma(s, 1)$ be independent. Find the distribution of $Y = X_1 + X_2$.

First, derive the moment generating function of the gamma distribution

$$M(t; s, 1) = Ee^{tX} = \int_0^\infty e^{tx} f(x; s, 1) dx$$

$$= \int_0^\infty e^{tx} \frac{1}{\Gamma(s)} x^{s-1} e^{-x} dx$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-(1-t)x} dx =$$

$$\frac{1}{\Gamma(s)} \frac{\Gamma(s)}{(1-t)^s} = \frac{1}{(1-t)^s}$$

4.14 By using the property of independent random variables, we know

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

So if $X_1 \sim gamma(s, 1), X_2 \sim gamma(s, 1)$,

$$M_{X_1+X_1}(t) = \frac{1}{(1-t)^s}\frac{1}{(1-t)^r} = \frac{1}{(1-t)^{r+s}}$$

You can see the MGF of the product is still in the format of Gamma distribution. Finally we can get
$X_1 + X_2 \sim gamma(r + s, 1)$

Suppose that the distribution of Y conditional on $X = x$ is $N(x, x^2)$ and the marginal distribution of X is $U[0, 1]$

1. Find $E(Y)$

Suppose that the distribution of Y conditional on $X = x$ is $N(x, x^2)$ and the marginal distribution of X is $U[0, 1]$

1. Find $E(Y)$

$$E(Y) = E[E(Y|X)] = E(x) = \int_0^1 x dx = 1/2$$

2. Find $Var(Y)$

Suppose that the distribution of Y conditional on $X = x$ is $N(x, x^2)$ and the marginal distribution of X is $U[0, 1]$

1. Find $E(Y)$

$$E(Y) = E[E(Y|X)] = E(x) = \int_0^1 x dx = 1/2$$

2. Find $Var(Y)$

$$E(Y^2) = E[E(Y^2|X)] = E\left[E(Y^2|X) - E(Y|X)^2 + E(Y|X)^2\right] =$$

$$E\left[Var(Y|X) + E(Y|X)^2\right] = E(x^2 + x^2) = \int_0^1 2x^2 dx = 2/3$$

$$E(Y^2) - E(Y)^2 = 2/3 - 1/4 = 5/12$$