Econometrics I - TA section

Eric Schulman

The University of Texas at Austin

October 14, 2020

Cheat sheet

Random variables
A random variable is a real-valued outcome; a function from the sample space S i.e. outcomes to the real line $\mathcal{R}$.

- Can be discrete i.e. a dice roll
- Can be cts i.e. a stock price

Cheat sheet

Cumulative distribution function

The distribution function is $F(x) = P[X \leq x]$, the probability of the event $\{X \leq x\}$.

1. $F(x)$ is non-decreasing
2. $F(x) \geq 0$
3. $F(x) \leq 1$

Cheat sheet

Probability density function When $F(x)$ is differentiable, its density is

- $f(x) = \frac{d}{dx}F(x)$
- $F(x) = \int_{-\infty}^{x} f(t)dt$

A function $f(x)$ is a density function if and only if

1. $f(x) \geq 0$ for all x.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Cheat sheet

Expectations

- Discrete random variable $E(X) = \sum_{x \in S} xPr(X = x)$
- $E(x) = \int f(x)dx$

Variance

- $Var(X) = E((X - E(x))^2)$
- $Var(X) = E(X^2) - E(X)^2$

Homework problems

2.2 HIE

X is uniformly distributed. Find the cdf of $Y = \log(X/1 - X)$

In this case, $Y = g(X)$

$$F_y(y) = F_x(g^{-1}(y))$$

2.2 HIE

X is uniformly distributed. Find the cdf of $Y = \log(X/1 - X)$

In this case, $Y = g(X)$

$$F_y(y) = F_x(g^{-1}(y))$$

For the uniform distribution,

$$F_x(x) = x$$

where $0 \leq x \leq 1$

Thus,

$$F_y(y) = g^{-1}(y) = e^y/(e^y - 1)$$

Homework problems

2.6 HIE

Find $E(X)$ and $Var(X)$.

Homework problems

2.6 HIE

Find $E(X)$ and $Var(X)$.

Note that $Var(X) = E(X^2) - E(X)^2$

1. $f(x) = ax^{a-1}$, where $0 < x < 1$, $a > 0$

Homework problems

2.6 HIE

Find $E(X)$ and $Var(X)$.
Note that $Var(X) = E(X^2) - E(X)^2$

1. $f(x) = ax^{a-1}$, where $0 < x < 1$, $a > 0$
   $E(X) = \int_0^1 ax^{a-1}x = \frac{a}{1+a}x^{1+a}|_0^1 = a1 + a$
   $E(X^2) = \int_0^1 ax^{a-1}x^2 = \frac{a}{2+a}x^{2+a}|_0^1 = \frac{a}{2+a}$

Homework problems

2.6 HIE

Find $E(X)$ and $Var(X)$.
Note that $Var(X) = E(X^2) - E(X)^2$

1. $f(x) = ax^{a-1}$, where $0 < x < 1$, $a > 0$
   $E(X) = \int_0^1 ax^{a-1}x = \frac{a}{1+a}x^{1+a}|_0^1 = a1 + a$
   $E(X^2) = \int_0^1 ax^{a-1}x^2 = \frac{a}{2+a}x^{2+a}|_0^1 = \frac{a}{2+a}$
2. $f(x) = 1/n$, $x = 1, 2, ..., n$

2.6 HIE

Find $E(X)$ and $Var(X)$.

Note that $Var(X) = E(X^2) - E(X)^2$

1. $f(x) = ax^{a-1}$, where $0 < x < 1$, $a > 0$
   $E(X) = \int_0^1 ax^{a-1}x = \frac{a}{1+a}x^{1+a}|_0^1 = a1 + a$
   $E(X^2) = \int_0^1 ax^{a-1}x^2 = \frac{a}{2+a}x^{2+a}|_0^1 = \frac{a}{2+a}$
2. $f(x) = 1/n$, $x = 1, 2, ..., n$
   $E(X) = \sum_{x=1}^{n} \frac{x}{n} = \frac{1}{n}\sum_{x=1}^{n} x = \frac{1}{n}\left(\frac{n(n+1)}{2}\right) = (n+1)/2$
   $E(X^2) = \sum_{x=1}^{n} \frac{x}{n} = \frac{1}{n}\sum_{x=1}^{n} x^2 = \frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{(n+1)(2n+1)}{6}$

2.6 HIE

Find $E(X)$ and $Var(X)$.
Note that $Var(X) = E(X^2) - E(X)^2$

1. $f(x) = 3/2(x-1)^2$ where $0 < x < 2$

Homework problems

2.6 HIE

Find $E(X)$ and $Var(X)$.
Note that $Var(X) = E(X^2) - E(X)^2$

1. $f(x) = 3/2(x-1)^2$ where $0 < x < 2$
   $E(X) = \int_0^2 3/2(x-1)^2 x dx = 1$
   $E(X^2) = \int_0^2 3/2(x-1)^2 x^2 dx = \frac{8}{5}$

Homework problems

2.11 HIE Suppose X has density $f(x) = e^{-x}$ on $x > 0$. Set $Y = -\log(X)$. Find the density of Y

Homework problems

2.11 HIE Suppose X has density $f(x) = e^{-x}$ on $x > 0$. Set $Y = -\log(X)$. Find the density of Y

Just do a change of variables

$$f_y(y) = \left| \frac{dg^{-1}(y)}{dy} \right| f_x(g^{-1}(y))$$

For this problem, $g^{-1}(y) = e^{-y}$ and $\dfrac{dg^{-1}(y)}{dy} = -e^{-y}$

Thus

$$f_y(y) = e^{-e^{-y}} e^{-y}$$

Homework problems

2.14 HIE Show that if X is a continuous random variable, then $\min E|X - a| = E|X - m|$ where m is the median

Homework problems

2.14 HIE Show that if X is a continuous random variable, then $\min E|X-a| = E|X-m|$ where m is the median
What is the median? $F^{-1}(.5) = m$. As a result we should show $F(a) = .5$

Homework problems

2.14 HIE Show that if X is a continuous random variable, then $\min E|X-a| = E|X-m|$ where m is the median

2.14 HIE Show that if X is a continuous random variable, then $\min E|X-a| = E|X-m|$ where m is the median

Split the integral up into 2 sets $\{X < a\}$ and $\{X \geq a\}$

$$\int_{-\infty}^{a}(a-x)f(x)dx + \int_{a}^{\infty}(x-a)f(x)dx$$

Take derivative of this expression, to find the maximum. Use Leibniz integral rule

$$(a-x)f(x)|_{x=a} + \int_{-\infty}^{a} f(x)dx + (x-a)f(x)|_{x=a} - \int_{a}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{a} f(x)dx - \int_{a}^{\infty} f(x)dx = 0$$

Thus $F(a) = 1 - F(a)$ and $F(a) = .5$ as required.

Homework problems

3.2 For the Binomial distribution show

$$\sum_{x=0}^{n} \pi(x; n, p) = 1$$

3.2 For the Binomial distribution show

$$\sum_{x=0}^{n} \pi(x; n, p) = 1$$

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} =$$

$$([1-p] + [p])^n = 1$$

From the binomial theorem

Homework problems

3.2 For the Binomial distribution show

$$E(X) = np$$

Homework problems

3.2 For the Binomial distribution show

$$E(X) = np$$

$$\sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

Factoring out an *np*, this gives:

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!(x-1)!} p^{x-1} (1-p)^{n-x}$$

Can apply the binomial theorem to

$$np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x} = np(p + [1-p])^{n-1} = np$$

Homework problems

3.2 For the Binomial distribution show

$$Var(X) = np(1-p) = np - np^2$$

Homework problems

3.2 For the Binomial distribution show

$$Var(X) = np(1-p) = np - np^2$$

Need to calculate $E(X^2)$

$$\sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$

Can use a similar factoring trick

$$= n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2}(1-p)^{n-x} + np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1}(1-p)^{n-x}$$

Can apply the binomial theorem to get

$$E(X^2) = n(n-1)p^2 + np$$

Hence

$$Var(X) = E(X^2) - E(X)^2 = n(n-1)p^2 + np - [(np)^2] = np(1-p)$$

Homework problems

3.3 For the Poisson distribution show

$$\sum_{x=0}^{\infty} \pi(x;\lambda) = 1$$

Homework problems

3.3 For the Poisson distribution show

$$\sum_{x=0}^{\infty} \pi(x;\lambda) = 1$$

$$\sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

Turns out Taylor expansion of e^{λ} is

$$e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

So,
$e^{\lambda} e^{-\lambda} = 1$
Taylor expansions are your friend... super under-rated.

Homework problems

3.3 For the Poisson distribution show

$$E(X) = \lambda$$

Homework problems

3.3 For the Poisson distribution show

$$E(X) = \lambda$$

$$E(X) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!}$$

Need to get rid of that x... Note the index of the sum changed

$$= e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

Homework problems

3.3 For the Poisson distribution show

$$Var(X) = \lambda$$

Homework problems

3.3 For the Poisson distribution show

$$Var(X) = \lambda$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!}$$

Need to get rid of x^2 apply the factoring trick 2x now

$$= e^{-\lambda} \lambda \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!} =$$

Repeat that trick to get

$$= e^{-\lambda} (\lambda + 1) \lambda \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$Var(X) = E(X^2) - E(X)^2 = [\lambda + \lambda^2] - [\lambda]^2 = \lambda$$