Julia Sets

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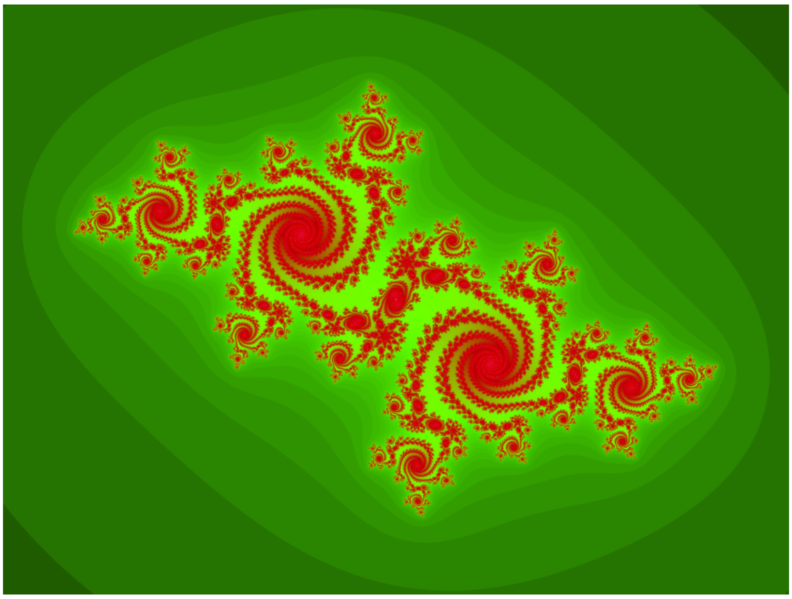

Figure 1: c $= -0.512511498387847167 + 0.521295573094847167i$ [3]

1 Introduction

Julia sets are one of the most beautiful mathematical concepts to emerge out of the 20th century. They are an excellent example of a chaotic dynamical system, and will be the subject of this paper. The Julia set is named after the French mathematician Gaston Julia, who studied these sets extensively in the early 20th century. Julia's work, which was published in 1918 in a paper entitled "Mémoire sur l'itération des fonctions rationnelles" ("Memoir on the Iteration of Rational Functions")[5], delved into the iteration of the quadratic function z^2+c and the intricate boundary properties that emerge. For his work he won the Grand Prix from the French Academy of Sciences in 1918.

Although modestly famous during his time, Gaston Julia's work largely faded into obscurity after his passing. However, the advent of modern computing revitalized interest in his research. Notably, the French-American mathematician Benoit Mandelbrot became the first to visually compute the Julia set, unveiling its intricate and captivating patterns. The visual allure of the Julia set ignited widespread fascination across mathematical circles and popular culture alike. [1]

2 Terminology

2.1 Orbit

To be able to understand the Julia set one must first familiarize themselves with a few terms. The first of which is the orbit of a set. Given a function **F**, then the *orbit* of x_0 under **F** is the set of points:

$$\{x_0, F(x_0), F(F(x_0)), \dots, F^n(x_0)\} \tag{1}$$

The orbit of x_0 is periodic (with period n) if the orbit under **F** eventually repeats after n iterations.

$$\{x_0, F(x_0), \dots, F^{n-1}(x_0), x_0, F(x_0), \dots, F^{n-1}(x_0), x_0 \dots\} \tag{2}$$

The orbit is significant because it determines whether a point is included in the Julia set or not.

2.2 Complex Functions

Understanding complex functions is crucial to understand the Julia set. Complex numbers take the form:

$$z = x + iy \tag{3}$$

Where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. The polar representation of a complex number is:

$$z = r\cos\theta + ir\sin\theta \tag{4}$$

Euler formula is:

$$e^{ix} = \cos x + i\sin x$$

We can combine Euler's formula with Equation 4 to get:

$$z = re^{i\theta} \tag{5}$$

The complex square root of Equation 5 is:

$$\sqrt{z} = \sqrt{r}e^{i\frac{\theta}{2}} = \sqrt{r}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) \tag{6}$$

The square of the Equation 5 is:

$$z^2 = r^2(\cos 2\theta + isin2\theta) = r^2e^{2i\theta} \tag{7}$$

The iteration formula for the Julia set maps the complex square operation (ie: z^2) to another point in the complex plane. Understanding how this operation behaves is important to understanding the Julia set.

3 Julia Set

The Julia set works for any polynomial, but this paper will only concern itself with the following function as it relates to the Julia set.

$$Q_c(z) = z^2 + c \tag{8}$$

This function is deceptively simple. On the surface it seems like a simple quadratic formula. But upon examining the orbit of $Q_c(z)$ within the complex plane, the function begins to display a chaotic dynamical system.

The filled Julia set (K_c) is the set of points that remain bounded. It is the set of points that don't diverge to infinity. These points include points that are eventually periodic, points that converge to a fixed point and points that are in the Julia set. The technical definition is that the filled Julia set K_c is the set of all points $z \in \mathbb{C}$ if $\forall z\ \exists \lambda$ s.t. $|Q_c^n(z)| < \lambda\ \forall n$.

The Julia set (J_c) is the set of points whose orbit nether converges to a fixed point, nor reaches a periodic cycle nor diverges to infinity; instead they seem to hover around a certain region indefinitely. This region is capped at 2, as will be proven later on. This set is the boundary of the filled Julia set, such that $J_c \subseteq K_c$

4 Computing The Julia Set

To compute the Julia set one must iterate through the orbit of Q_c, where c is a imaginary number corresponding to the complex plane. For each value of c, the program must determine whether this point diverges to infinity or whether it remains bounded. Determining if a point diverges is computationally difficult and can be exponentially simplified by understanding which values of z diverge every time.

4.1 $|z^2 + c| > |z| > 2$

When $|z| > 2$, it means the orbit of x_0 has escaped $\forall c$. This is due to the following inequality:

$$|z^2 + c| > |z| > 2$$

This inequality means that as long as z is larger than 2, then the quadratic function $|Q_c(z)|$ will output a number larger than z. Each iteration of $Q_c(z)$ is larger than the last so the orbit escapes to infinity. There are two parts to the proof of Equation 4.1: The first part involves when $c \leq 2$.

4.1.1 $|z^2 + c| > |z| > 2 \geq |c|$

We can start of the proof with the triangle inequality:

$$|z^2 - c| \geq ||z^2| - |c||$$

Replacing c with $-c$:

$$|z^2 - (-c)| \geq |z^2| - |-c|$$

$$|z^2 + c| \geq |z^2| - |c| \tag{9}$$

The next part is proving that $|z|^2 - 2 > |z|$. Starting with:

$$|z| > 2 \tag{10}$$

A direct result of Equation 10 is: $\exists\lambda > 0$ s.t. $2|z| = |z| + 2 + \lambda$.[1] This means that:

$$2|z| > |z| + 2 \tag{11}$$

Another result of Equation 10 is:

$$|z|^2 > 2|z| \tag{12}$$

Combining Equations 11 and 12 gets the following:

$$|z|^2 - 2 > |z| \tag{13}$$

Now we attempt to prove that $|z^2| - |c| \geq |z^2| - 2$. Given that:

$$|c| \leq 2$$
$$|c| - |z^2| \leq 2 - |z^2|$$
$$|z^2| - |c| \geq |z^2| - 2 \tag{14}$$

Putting Equations 9, 13, and 14 together we get:

$$|z^2 + c| \geq |z^2| - |c| \geq |z^2| - 2 > |z|$$

Thus:

$$|z^2 + c| > |z| > 2 \tag{15}$$

Q.E.D.

Thus proving that when $c < 2$ and $|z| > 2$, then the orbit will always escape to infinity. This is a crucial proof in the Julia set as it allows one to quickly discern which points are a part of the filled Julia set and which points are not.

4.1.2 $|z| > |c| > 2$

The second part of this theorem constitutes the *Escape Criterion*[2, pg. 235-6]. The Escape Criterion states that the orbit of z under Q goes to infinity if $|z| \geq |c| > 2$. Given

$$|z^2 + c| \geq |z^2| - |c|$$

Since $|z| \geq |c|$

$$|z^2 + c| \geq |z^2| - |z| = |z|(|z| - 1)$$

Given $|z| > 2$, then $\exists\ \lambda > 0$ s.t. $|z| - 1 > \lambda + 1$. Thus we can replace $|z| - 1$ with $\lambda + 1$:

$$|z^2 + c| > |z| - 1 > (1 + \lambda)|z| > |z|$$

Thus proving that $|z^2 + c| > |z| > |c| > 2$ and the *Escape Criterion.*

[1] Can be proved with contradiction.

4.1.3 Applying The Escape Criterion

This escape criterion drastically reduces the number of points that need detailed tracking over many iterations, since any point that exceeds this boundary can be immediately excluded from K_c. The proof is significant in the computation of the filled Julia set because it provides a clear, mathematically justified threshold for determining the boundedness of orbits in a computationally efficient manner.

Using the escape criterion, we can derive an algorithm to compute the filled Julia set: Set a maximum number of iterations, N. For each point z on the grid, calculate the first N points in the orbit of z. If at any point up to N the condition $|Q_c^i(z)| > \max(|c|, 2)$ is met, cease further iterations and mark z as white. If $|Q_c^i(z)| <= \max(|c|, 2)\ \forall i \leq N$ mark z as black. Points colored white indicate orbits that escape, while those colored black indicate orbits that do not escape within the first N iterations.

This algorithm does have its limitations. The first is that if N is set too low, then the algorithm may incorrectly label points that escape as being within the filled Julia set. Another issue is with the grid resolution. A coarser grid might miss the finer details of the Julia set, especially for points near the boundary of the Julia set.

4.2 Figure Eights

The Julia set can be computed as the intersection between the curves created by the inverse quadratic function mapped onto the unit circle when $|c| > 2$. In other words the Julia set can be given by

$$J_c = \bigcap_{n \geq 0} Q_c^{-n}(D) \tag{16}$$

where $Q_c^{-n}(D)$ is the preimage of Q_c under (D).

$$Q_c^{-1}(D) = \sqrt{D - c}$$

When $c > 2$ the Julia set will have a disconnected orbit, an thus consist of a scattering of points. We can see from Figure 2 that the Filled Julia set, K_c is minuscule.

5 Cantor Sets

Cantor sets are the set created by removing the middle third within an interval (usually [0, 1] but it can be anything), then removing the middle third from the two intervals you are left with from the first iteration, ad infinitum. Eventually you are left with a set of points that don't touch. This is the Cantor set (Figure 3).

The Cantor set is important because when $|c| > 2$, the Julia set becomes a Cantor set. As proved by the Escape Criterion, the Julia set, J_c is incredibly small since almost all points diverge to infinity. There are a small set of values of c that don't escape to infinity. This "small" set of points has an infinite number of points, and none of these points touch. There isn't a single continuous interval of points within the Julia set, as each point is separate and independent from all other points on the set. Another defining feature when $|c| > 2$ is that the filled Julia is equivalent to the Julia set:

$$\forall c > 2 \quad J_c = K_c$$

Because both the Julia set/filled Julia set and the cantor set possess a disconnected orbit, the Julia set is a Cantor set.

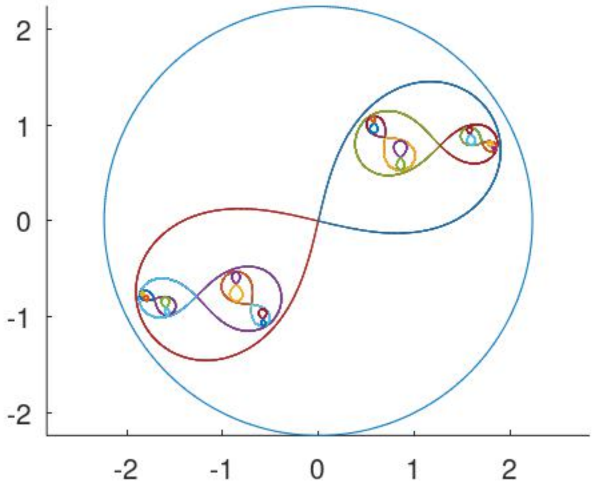

Figure 2: $c = -1 - 2i$

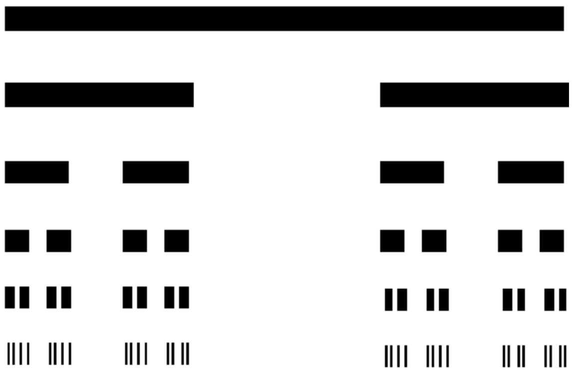

Figure 3: Cantor Set [4]

To illustrate this we can examine Figure 2. As mentioned, the Filled Julia set can be thought of as the result of multiple figure-eight curves and their lobes intersecting with each other. Within these intersections, only single points are present. This arrangement is a Cantor set, where each point represents a unique intersection within the filled Julia set's complex structure [2, pg. 239].

6 Fractals

The Julia set consists of a large amount of fractals. A *fractal* is a geometric shape that can be split into many different parts. Each part is a mini version of the larger shape; this property is known as *self similarity*. To demonstrate this property in the Julia set, observe Figures 4, 5 and 6. These images are in order of increasing magnification of the Julia set generated using $c = 1$. Notice how each micro image demonstrates self-similarity with the larger macro image.

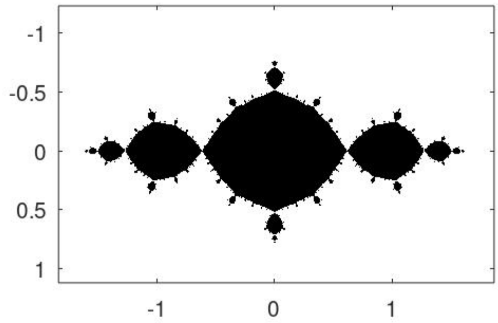

Figure 4: Fractal 1

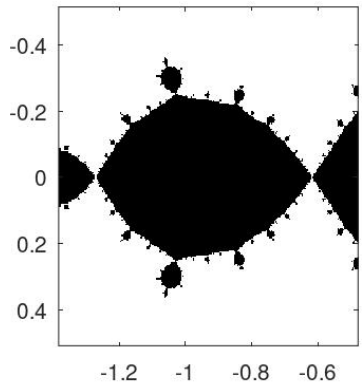

Figure 5: Fractal 2

7 Chaotic Dynamical Systems

A chaotic dynamical system possesses three properties: "unpredictability, indecomposability and an element of regularity" [2, pg. 126]. These conditions of a chaotic system can be restated as follows

1) Sensitive to initial conditions

2) Transitivity

3) Density

The first condition is that the system must be sensitive to initial conditions. The technical definition for this is if $\exists \beta > 0$ s.t. $\forall x$ and $\forall \epsilon > 0$, $\exists y$ within ϵ of x, and $\exists k$ s.t. the distance between $F^k(x)$ and $F^k(y)$ is at least β. This means that For any starting point x and any arbitrarily small distance ϵ around x, there exists another point y within this distance ϵ of x such that after some number of iterations k, the distance between the images of x and y under the system's transformation F is at least β. In simpler terms, it means that even tiny differences in

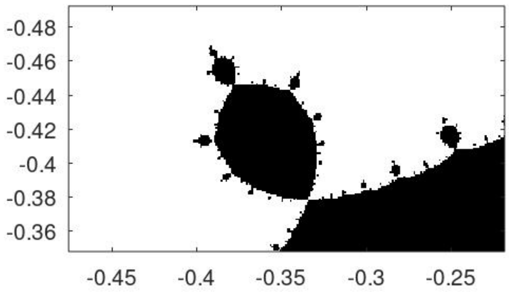

Figure 6: Fractal 3

initial conditions can lead to significantly different outcomes after a certain number of iterations of the system.

For points in the Julia set, small perturbations to the initial values z result in orbits that diverge significantly from each other. This behavior is often demonstrated through the iterative process Q_c, where minor variations in z can determine dramatically different paths, deciding whether an orbit escapes to infinity or remains bounded within certain limits.

The second requirement is the system must be transitive. This means that for any pair of points x and y within the system and for any arbitrarily small distance ϵ, there exists a third point z whose orbit under the system's transformation comes within ϵ of y, starting from x. In simpler terms, it means that the system is able to "mix up" its points sufficiently so that trajectories from any starting point can eventually come arbitrarily close to any other point within the system. This property is crucial for the system to exhibit a complex and unpredictable behavior characteristic of chaos.

The orbits of points under Q_c demonstrate that parts of the Julia set can evolve under iteration to occupy any other part of the set. This mixing behavior shows how the dynamics interconnect different regions of the Julia set, ensuring that the influence of one area can be seen across the entire set.

The third requirement is the system has to be dense. For A to be dense in B, given that A $\subseteq$ B, then for any point $b \in B$, there is a point $a \in A$ arbitrarily close to b. This statement asserts that a subset A is said to be dense within a larger set B if, for every point b belonging to B, there exists at least one point a in A that is arbitrarily close to b. In simpler terms, it means that A fills up B in such a way that every point in B is either directly in A or can be approached as closely as desired by selecting a point from A.

Repelling points in particular, are dense in the Julia Set. [2, pg. 246] This means that if you take a slice of the Julia set, no matter how small the slice is, you would be able to find points that are repelling.

Because the Julia set satisfies these properties, it is a chaotic dynamical system.

7.1 Chaos When $c = 0$

It can be directly proven that the Julia set is a chaotic dynamical system when $c = 0$. When $c = 0$, the equation mapped by the Julia is $Q_c(z) = z^2 = r^2e^{2i\theta}$ (Equation 7). This is the angle double function. Furthermore the Julia set is contained on the unit circle. In other words the Julia set for $c = 0$ is the set of all z such that $|z| = 1$. When $|z| < 1$ the orbit decreases until it reaches zero and when $|z| > 1$ the orbit diverges to infinity. But when $|z| = 1|$ the orbit remains on the unit circle but is chaotic.

To demonstrate that periodic points are densely distributed on the unit circle, we consider a segment of the circle defined by angles θ_1 and θ_2, where $\theta_1 < \theta < \theta_2$. The goal is to find a periodic point within this segment under the mapping $Q_0(z) = z^2$ on the unit circle.

Given a point $z_0 = e^{i\theta}$ on the unit circle, the n-th iteration of Q_0 is $Q_0^n(e^{i\theta}) = e^{i2^n\theta}$. For z_0 to be a periodic point with period n, it must satisfy:

$$e^{i2^n\theta} = e^{i\theta}.$$

This leads to the equation:

$$2^n\theta = \theta + 2k\pi,$$

for some integers k, n. Solving for θ, we get:

$$\theta = \frac{2k\pi}{2^n - 1}.$$

Here, k and n are integers. Setting n and selecting k from 0 to $2^n - 2$ results in points whose angles are $\frac{2k\pi}{2^n-1}$. These angles represent $2^n - 1$ equidistant points around the unit circle, as these are the $2^n - 1$st roots of unity.

To ensure at least one of these points falls within the interval $\theta_1 < \theta < \theta_2$, it is necessary that:

$$\frac{2\pi}{2^n - 1} < \theta_2 - \theta_1.$$

Choosing n such that this inequality holds guarantees that the interval contains at least one point of the form $\frac{2k\pi}{2^n-1}$, making it a periodic point with period n. This dense distribution of periodic points means any segment of the unit circle can potentially contain a periodic point of this map.

Transitivity in the context of the squaring map Q_0, which squares each point on the unit circle, operates as previously described. Specifically, any open arc defined by $\theta_1 < \theta < \theta_2$ on the circle is doubled in arc length with each application of Q_0. Therefore, the operation Q_0^n multiplies the arc length by 2^n.

Given this exponential increase, $\exists n$ large enough that the image of the arc from θ_1 to θ_2 spans the entire unit circle, enveloping any other arc on the circle. This not only establishes transitivity, showing that any segment can lead to any other through repeated application of Q_0, but it also illustrates the map's sensitive dependence on initial conditions. The latter is evident since points that are initially close on the circle can rapidly evolve to occupy positions that are on opposite sides of the circle, underscoring the chaotic nature of this dynamic.

Thus we have proved that $Q_0^n(z)$ is a chaotic dynamical system.

8 Conclusion

In conclusion, Julia sets intricately weave chaos and order into a complex dynamical system. The incredible abstractions generated by this dynamical system have long enraptured the minds of many mathematicians. This essay attempted to give an overview of the Julia set, starting from the orbits of a simple quadratic function and evolving into the chaotic dynamical systems. Along the way, fractals, the Cantor set, and complex functions were explained as each topic relates to the Julia set. Meticulous examination of the Julia set offers insights into the nature of chaos, of which this essay has only scratched the surface of.

Furthermore, the relationship between Julia sets and the Mandelbrot set highlights the interconnectedness of mathematical concepts and the power of visual representation in understanding abstract phenomena. By mapping the Julia set across the complex plane, the Mandelbrot set provides a visual manifestation of the underlying dynamics, inviting exploration and discovery.

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