Chapter 1

Foundations of Interest Rate Theory

1.1 Definitions and Notation

The primary objects of our investigation are pure discount bonds, of various maturities. All payments are assumed to be made in a fixed currency. Moreover, we need some formal definitions.

Definition 1 (Discount Bond.) *A $T-$maturity pure discount bond is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t,T)$. Clearly, $P(T,T) = 1$ for all T.*

Definition 2 (Time to maturity.) *The time to maturity $x = T - t$ is the amount of time expressed in years from the present time t to the maturity time $T > t$.*

Coupon bonds give the owner a payment stream during the interval $[0, T]$. These instruments have the common property, that they provide the owner with a deterministic cash flow, and for this reason they are also known as fixed income instruments.

Pure discount bond prices are the basic quantities in interest-rate theory, and all interest rates can be defined in terms of discount bond prices, as we shall see now. Therefore, they are often used as basic auxiliary quantities from which all rates can be recovered, and in turn discount bond prices can be defined in terms of any given family of interest rates. Notice, however, that interest rates are what is usually quoted in (interbank) financial markets, whereas zero-coupon bonds are theoretical instruments that, as such, are not directly observable in the market. In moving from discount bond prices to interest rates, and vice versa, we need to know two fundamental features of the rates themselves: the compounding type and the day-count convention to be applied in the rate definition. What we mean by "compounding type" will be clear from the definitions below.

Definition 3 (Anually-compounded spot interest rate.) *The annually-compounded spot interest rate prevailing at time t for the maturity T is denoted by $Y(t,T)$ and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $P(t,T)$ units of currency at time t, when reinvesting the obtained amounts once a year. In formulas*

$$Y(t,T) := -\frac{1}{P(t,T)^{T-t}} - 1 \tag{1.1}$$

which implies that bond prices can be expressed in terms of annually-compounded rates as

$$P(t,T) = \frac{1}{(1+Y(t,T))^{T-t}} \tag{1.2}$$

Definition 4 (Continuosly-compounded spot interest rate.) *The continuosly-compounded spot interest rate prevailing at time t for the maturity T is denoted by $R(t,T)$ and is the constant rate at which an investment of $P(t,T)$ units of currency at time t accrues continously to yield a unit amount of currency at maturity T*

$$R(t,T) := -\frac{\log P(t,T)}{T-t} \tag{1.3}$$

The continuously-compounded interest rate is therefore a constant rate that is consistent with the discount bond prices in that

$$e^{R(t,T)(T-t)}P(t,T) = 1 \tag{1.4}$$

from which we can express the bond price in terms of the continuosly compounded rate R:

$$P(t,T) = e^{-R(t,T)(T-t)} \tag{1.5}$$

where $T-t$, the time difference expressed in years. An alternative to continuous compounding is simple compounding, which applies when accruing occurs proportionally to the time of the investment.

Definition 5 (Simply-compounded spot interest rate.) *The simply-compounded spot interest rate prevailing at time t for the maturity T is denoted by $L(t,T)$ and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $P(t,T)$ units of currency at time t, when accruing occurs proportionally to the investment time.*

$$L(t,T) := \frac{1 - P(t,T)}{(T-t)P(t,T)} \tag{1.6}$$

We denote by L such rates because the market LIBOR rates are simply compounded. These are the most important interbank rates and they are considered as a reference for contracts, fixing daily in London (London InterBank Offered Rate). There exist equivalent interbank rates fixing in other markets (e.g. the EURIBOR rate, fixing in Brussels by the European Banking Federation).

Suppose that we are standing at time t, and let us fix two other points in time S and T with $t < S < T$. Let us consider now the project of writing a forward rate agreement at time t which allows us to make an investment of one unit of currency at time S, and have a *deterministic* rate of return, determined at the contract time t, over the interval $[S,T]$. This agreement can be achieved with the following replicating strategy

1. At time t we sell one S-bond. This will give us $P(t,S)$ units of our base currency.

2. With this money we may buy exactly a $\frac{P(t,S)}{P(t,T)}$ amount of T-bonds.

   $$P(t,S) - \frac{P(t,S)}{P(t,T)} P(t,T) = 0 \quad \text{in} \quad t$$

   Note that our net investment at initial time t is zero.

3. At time S the S-bond expires, so we must to pay out one monetary unit of our currency.

4. At time T each T-bond expires paying one unit of currency, so we will receive the payoff $P(t,S)/P(t,T) \cdot 1$.

5. The real effect of this strategy is that, based on the contract agreed at t, for an investment of one unit of currency we have received in turn $P(t,S)/P(t,T)$ at time T.

Now the following crutial definition is well motivated by the implementation of the financial strategy above.

Definition 6 (Simply-compounded forward interest rate.) *The simple forward rate for the period* $[S, T]$ *contracted at* $t < S < T$*, is defined as*

$$L(t; S, T) := \frac{1}{T - S}\left(\frac{P(t, S)}{P(t, T)} - 1\right)$$

Or, in other words, the simple forward rate L, is the solution to the equation

$$1 + (T - S)L = \frac{P(t, S)}{P(t, T)}$$

Moreover, it is straightforward to recover the spot definition making the assignment $t = S$, i.e. the spot rates are forward when the time of the agreement coincides with the start of the interval over which the interest rate is effective.

The simple forward rate $L(t; T, S)$ may be viewed as an estimate of the future spot rate $L(T, S)$.

When the maturity of the forward rate collapses towards its expiry, we have the notion of *instantaneous forward rate*. Let us consider the limit

$$\begin{aligned}
\lim_{\Delta T \to 0^+} L(t; T, T + \Delta T) &= -\lim_{\Delta T \to 0^+} \frac{P(t, T + \Delta T) - P(t, T)}{P(t, T + \Delta T)\Delta T} \\
&= -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T} \\
&= -\frac{\partial \log P(t, T)}{\partial T}
\end{aligned} \tag{1.7}$$

This leads to the following.

Definition 7 (Instantaneous forward interest rate.) *The instaneous forward interest rate prevailing at time* t *for the maturity* $T > t$ *is denoted by* $F(t, T)$ *and is defined as*

$$F(t, T) := \lim_{\Delta T \to 0^+} L(t; T, T + \Delta T) = -\frac{\partial \log P(t, T)}{\partial T}, \tag{1.8}$$

so that we also have

$$P(t, T) = \exp\left(-\int_t^T F(t, u)\, du\right) \tag{1.9}$$

Clearly for this notion to make sense, we need to asume smoothness of the discount bond price function $T \mapsto P(t,T)$ for all T's.

Intuitively, the instantaneous forward rate $F(t,T)$ is a forward interest rate at time t whose maturity is very close to its expiry T, say $F(t,T) \approx L(t;T,T+\Delta T)$ with ΔT small.

1.2 Interest-Rate Curves

A fundamental curve that can be obtained from the market data of interest rates is the zero-copupon curve at a given date t. This curve is the graph of the function mapping maturities into rates at times t. More precisely:

Definition 8 (Zero-rate curve.) *The zero-rate curve at time t is the graph of the function*

$$T \mapsto \begin{cases} L(t,T) & t < T \leq t+1 \\ Y(t,T) & T > t+1 \end{cases} \tag{1.10}$$

Such a zero-coupon curve is also called the *term structure of interest rates* (TSIR) at time t. By definition (1.10), it is a plot at time t of simply-compounded interest rates for all maturities T up to one year and of annually-compounded rates for maturities T larger than one year. Recall that at times it may be considered the

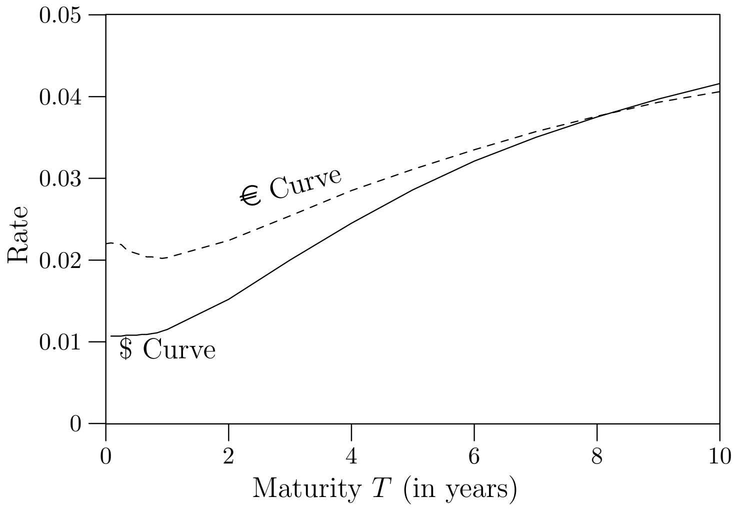

Figure 1.1: Zero-rate curves on July 1, 2003. The normal line corresponds to US dollar rates and the dashed one to the euro rates.

sample for rates with different compounding conventions, such as for example

$$T \mapsto R(t,T), \quad T > t$$

Definition 9 (Discount bond curve.) *The Discount bond curve at time t is the graph of the function*

$$T \mapsto P(t,T), \quad T < t \tag{1.11}$$

which, because of the positivity of interest rates, is a T-decreasing function starting from $P(t,t) = 1$. Two examples of such a curve ara shown in 1.2

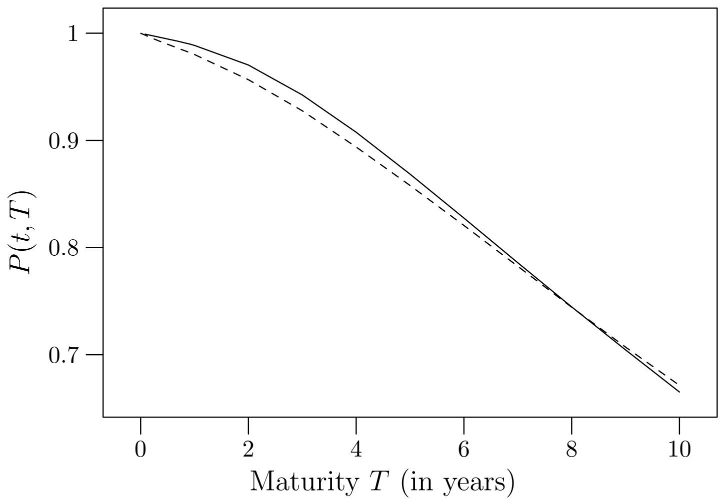

Figure 1.2: Term structure of discount bonds on July 1, 2003. The normal line belongs to the US dollar discount bond curve and the dashed one to the euro.

1.2.1 The Short Rate and the Money-Market Account

Definition 10 (Short rate.) *The instantaneous spot interest rate, also referred as the short rate, is the continuously-compounded insterest rate when time to maturity collapses to zero:*

$$r(t) = \lim_{\Delta t \to 0} R(t, t + \Delta t) \tag{1.12}$$

Let us work out this limit

$$
\begin{aligned}
\lim_{\Delta t \to 0} R(t, t+\Delta t) &= -\lim_{\Delta t \to 0} \frac{\log P(t, t+\Delta t)}{(t+\Delta t)-t} \\
&= -\lim_{\Delta t \to 0} \frac{\log P(t, t+\Delta t) - \log P(t,t)}{(t+\Delta t)-t} \\
&= -\left.\frac{\partial \log P(t,\theta)}{\partial \theta}\right|_{\theta=t} \\
&= F(t,t)
\end{aligned} \tag{1.13}
$$

The next definition we consider is the definition of a money-market account. A money-market account represents a locally riskless investment, where profit is accrued continuously at the short rate prevailing in the market at every instant.

Definition 11 (Money-market account.) *We define $B(t)$ to be the value of a money-market account at time $t \geq 0$. Assume that $B(0) = 1$, and that the money-market account evolves according to the following differential equation:*

$$dB(t) = r(t)B(t)dt, \qquad B(0) = 1, \tag{1.14}$$

where $r(t)$ is a positive stochastic process, i.e.,

$$B(t) = \exp\left(\int_0^t r(s)\, ds\right). \tag{1.15}$$

1.3 A Brief Note on Martingale Modeling

Throughout this work we consider a continuous trading economy, with a finite trading interval given by $[0, \Theta]$. The uncertainty is modelled by the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ where Ω denotes a sample space, with elements $\omega \in \Omega$; $\mathcal{F}$ denotes a σ-algebra on Ω; and $\mathbb{P}$ denotes a probability measure in $(\Omega, \mathcal{F})$. The uncertainty is resolved over $[0, \Theta]$ according to the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$.

We consider a financial market $S = [\ S_0\ S_1\ \ldots\ S_n\ ]^T$ with a riskless investment, S_0, or money market account given by (1.14), and n risky assets which all follow Itô processes driven by a q-dimensional Wiener-Einstein process, W,

$$dS_i = S_i(\mu_i dt + \sigma_i \cdot dW), \quad S_i(0) > 0, \quad i = 1, \ldots, n.$$

the appreciation rates μ_i and the volatility row vectors $\sigma_i = [\ \sigma_{i1}\ \ldots\ \sigma_{iq}\ ]$ are assumed to be $\mathcal{F}_t$-adapted, intuitively, this means that they all depend on past values but not on future. They also satisfy the integrability conditions

$$\int_0^\Theta |\mu_i|dt < \infty, \int_0^\Theta ||\sigma_i||^2 dt < \infty \quad i = 1, \ldots, n \tag{1.16}$$

almost surely.

A continuous time *trading strategy* is any $\mathbb{R}^{n+1}$-valued $\mathcal{F}_t$-adapted stochastic process

$$\phi(t) = [\ \phi_0(t) \ \ldots \ \phi_n(t)\ ]$$

where $\phi_i(t)$ denotes the holdings in the asset i at time t. The asset holdings $\phi_i(t)$ are furthermore assumed to satisfy similar regularity conditions as the presented in (1.16).

Its correponding *value process* is

$$V(\phi, t) = \phi(t) \cdot S(t) = \sum_{i=0}^{n} \phi_i(t) S_i(t)$$

The *portfolio* or trading strategy ϕ, is called *self-financing* when

$$V(\phi, t) = V(\phi, 0) + \sum_{i=0}^{n} \int_0^t \phi_i(s) dS_i(s), \quad t \in [0, \Theta], \tag{1.17}$$

where $\int \phi_i(s)dS_i(s)$ denote Itô integrals. Hence, a self-financing trading strategy is a trading strategy that requires nor generates funds between time 0 and time Θ.

1.3.1 Martingale Measures, Derivative Securities and Arbitrage

All prices above are interpreted as being given in terms of some a priori given *numeraire*, or monetary basis. Tipically this *numeraire* is the domestic currency like €, but we may, of course, equally express all prices denominated in some other *numeraire*. In fact, any asset which has strictly positive prices for all $t \in [0, \Theta]$ is a *numeraire*.

Suppose that, for some $p \leq n$, the p-asset is a *numeraire* . The prices of other assets $i \neq p$ denominated in S_p are called the *relative prices* or *discounted prices* and we denote them by

$$\widetilde{S}_i := S_i/S_p.$$

We denote the *relative value process* as well by

$$\widetilde{V} := \frac{V}{S_p} = \sum_{i \neq p}^{n} \phi_i \widetilde{S}_i$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space from the beginning of this section. Consider now the set that contains all probability measures $\mathbb{Q}^*$ such that:

1. $\mathbb{Q}^* \sim \mathbb{P}$, i.e. both measures have the same null-sets;
2. the relative processes $\widetilde{S}_i$ are martingales under $\mathbb{Q}^*$ for all i, i.e. for $t \leq s$

$$\widetilde{S}_i(t) = \mathbb{E}^{\mathbb{Q}^*}[\widetilde{S}_i(s)|\mathcal{F}_t].$$

The measures $\mathbb{Q}^*$ are called *equivalent martingale measures*. Suppose we pick one particular equivalent martingale measure $\mathbb{Q}^*$.

Definition 12 (Derivative security.) *Is any $\mathcal{F}_t$-measurable random variable $h(T)$ such that*

$$\mathbb{E}^*(|h(T)|) < \infty,$$

where $\mathbb{E}^*$ denotes expectation under the equivalent martingale measure $\mathbb{Q}^*$.

Hence, derivative securities are those assets for which the expectation of the payoff is well defined. If we can find a self-financing trading strategy ϕ such that $\widetilde{V}(\phi, T) = h(T)$ with probability one, the derivative is said to be *attainable*. The self-financing trading strategy is then called a *replicating strategy*. If in an economy *all* derivative securities are attainable, the economy is called *complete*.

An *arbitrage portfolio* is a self-financing trading strategy ϕ, with

$$\mathbb{P}[\widetilde{V}(\phi, T) \geq 0] = 1, \quad \text{with} \quad \widetilde{V}(\phi, 0) < 0,$$

thus, an arbitrage trading strategy is capable to produce a "free lunch", because with initial negative costs we obtain at terminal time a non-negative value of the portfolio denominated in the chosen numeraire.

Theorem 1 (Unique Equivalent Martingale Measure.) *A continuos trading economy is free of arbitrage trading strategies and every derivative security* *is attainable, i.e. the market is complete, if for every choice of numeraire there exists a unique martingale measure.*

Proof. See [15]. □

Thus for a given numeraire M with unique martingale measure $\mathbb{Q}^M$, the value of a self-financing trading strategy

$$\widetilde{V}(\phi, t) = \frac{V(\phi, t)}{M(t)}$$

is a $\mathbb{Q}^M$-martingale. Hence, for a replicating strategy ϕ_h that replicates the derivative security $h(T)$ we obtain

$$\mathbb{E}^M\left[\frac{h(T)}{M(T)}\Big|\mathcal{F}_t\right] = \mathbb{E}^M\left[\frac{V(\phi_h, T)}{M(T)}\Big|\mathcal{F}_t\right] = \frac{V(\phi_h, t)}{M(t)}$$

where the last equality follows from the definition of a martingale. Combining the first and the last expression yields

$$V(\phi_h, t) = M(t)\mathbb{E}^M\left[\frac{H(T)}{M(T)}\Big|\mathcal{F}_t\right] \tag{1.18}$$

This formula can be used to determine the value at time $t < T$ for any derivative security $h(T)$. In particular, absence of arbitrage and market completeness implies the existence of the unique probability measure $\mathbb{Q}^B$, equivalent to the physical $\mathbb{P}$, under which the price of any discount bond, or T-bond, appropiately discounted by the market account, $S_0(t) = B(t)$, is a $\mathbb{Q}^B$-martingale.

$$\widetilde{P}(t, T) := \frac{P(t, T)}{B(t)} = \mathbb{E}^B\left[\frac{P(T, T)}{B(T)}\Big|\mathcal{F}_t\right] = \mathbb{E}^B\left[e^{-\int_0^T r(u)du}P(T, T)\Big|\mathcal{F}_t\right]$$

Combining this fact with the fact that a T-bond is a derivative security which has price 1 at its maturity we can write the well-known *arbitrage-free pricing* formula

$$P(t, T) = \mathbb{E}^B\left[e^{-\int_t^T r(s)ds}\Big|\mathcal{F}_t\right]. \tag{1.19}$$