Chapter 3

Interest Rate Caps

Interest rate caps are widely traded OTC interest rate derivatives. An interest rate cap is a financial insurance which protects you from having to pay more than a predetermined rate, therefore, a cap is insurance against rising interest rates.

3.1 The Market Practice for Plain Vanilla Caps

In this section we discuss vanilla interest rate caps and the market practice for quoting these instruments. For concreteness suppose the underlying rate is the simply-compounded forward LIBOR rate $L(t;\cdot,\cdot)$ introduced in Sect. 1.1. Let suppose that we are standing at time $t=0$. We consider a fixed set of increasing maturities $x_0,\ x_1,\ldots,x_n$ and we define τ_j, by

$$\tau_j = x_j - x_{j-1}, \quad j = 1, ..., N.$$

The number τ_j is known as the **tenor**.

Definition 14 *We let $P_j(t)$ denote the discount bond price $P(t,x_j)$ and let $L_j(t)$ summarize the notation for a forward LIBOR rate of the type $L(t;x_{j-1},x_j)$, i.e.*

$$L_j(t) = \frac{1}{\tau_j}\left(\frac{P_{j-1}(t)}{P_j(t)} - 1\right) \quad j = 1,\ldots,n.$$

Recall that a *vanilla cap* with *cap rate* K and *resettlement dates* $x_0,\ldots,x_n$ is a contrat which each time x_j gives the holder of the contract the amount

$$h_{\gamma_j}(x_j) = \tau_j(L_j(x_{j-1}) - K)^+, \tag{3.1}$$

where $j = 1, \dots, n$. In fact, the cap is a strip of *caplets.* Note that the forward LIBOR rate $L_j(x_{j-1})$ above is in fact the simply-compounded spot LIBOR interest rate. By definition:

$$L_j(x_{j-1}) := L(x_{j-1}; x_{j-1}, x_j) = L(x_{j-1}, x_j),$$

which is observed already at time x_{j-1}. The payoff h_{γ_j} is determined at the reset date x_{j-1} but not payed out until the settlement date x_j. We also note that the caplet γ_j is a call option on the on the underlying spot rate.

For a very long time, the market practice has been to value caps by using a formal extension of the Black [Ref.] model. This extension is typically obtained by an approximation argument where the short rate at one point in the argument is assumed to be deterministic, while later on in the argument the LIBOR rate is assumed to be stochastic. This is of course logically inconsistent.

Definition 15 (Black's Formula for Caplets.) *The Black-76 formula for the j-caplet with payoff:*

$$h_{\gamma_j}(x_j) = \tau_j (L_j(x_{j-1}) - K)^+,$$

at time $t = 0$ is given by the expression

$$\gamma_j(h_{\gamma_j}, 0) = \tau_j P_j(0) \left\{ L_j(0) N(d_1) - K N(d_2) \right\}, \quad j = 1, \dots, n, \tag{3.2}$$

where

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{L_j(0)}{K}\right) + \frac{1}{2}\sigma_j^2 x_j}{\sigma_j \sqrt{x_j}}, \\ d_2 &= d_1 - \sigma_j \sqrt{x_j}. \end{aligned} \tag{3.3}$$

The constant σ_j is known as the *Black volatility* for the j-caplet, γ_j. In the market, cap prices are not quoted in monetary terms but instead in terms of *implied Black volatilities* and these volatilities can furthermore be quoted as *flat volatilities* or as *forward rate volatilities* but, in this work, we confine ourselves to *flat volatilities.* Suppose we are standing at time $t = 0$ and consider the fixed set of dates $x_0,\ x_1, \dots, x_n$ where $x_0 \geq 0$, and a fixed cap rate K. We assume that, for each $j = 1, \dots, n$, there is a traded cap with resettlement dates $x_0, x_1, \dots, x_j$, and we denote the corresponding observed market price by C_j^o. From this data we can easily compute the market prices for the corresponding caplets by means of the recursion formula

$$\gamma_j^o = C_j^o - C_{j-1}^o, \quad j = 1, \dots, n \tag{3.4}$$

with the convention $C_0^o = 0$. Moreover, note that given market data for caplets we can easily compute the corresponding market data for caps by solving the previous recursion formula (3.4):

$$C_j^o = \sum_{k=1}^{j} \gamma_k^o, \quad j = 1, \dots, n \tag{3.5}$$

Given market price data as above, the implied Black flat volatilities are defined as follows.

Definition 16 *The implied flat volatilities $\bar{\sigma}_1, \dots, \bar{\sigma}_n$ are defined as the solutions of the equations:*

$$C_j^o = \sum_{k=1}^{j} \gamma_k^o(\bar{\sigma}_j), \quad j = 1, \dots, n, \tag{3.6}$$

In other words, the flat volatility $\bar{\sigma}_j$ is the volatility implied by the Black formula if you use the same volatility for each caplet, in the cap with maturity x_j.

3.1.1 IRS and At-The-Money Plain Vanilla Caps

An interest rate swap (henceforth IRS) is a scheme where you exchange a payment stream at a fixed rate of interest for a payment stream at a floating rate (e.g. LIBOR). A payer IRS settled in arrears is specified by:

- a number of future dates $x_0 < x_1 < \dots < x_n$ where $x_j - x_{j-1} \equiv \tau_j$ are the settlement periods and x_n is called the maturity of the swap,
- a fixed rate K; and,
- a nominal value N.

Moreover, plain vanilla IRS satisfy the equidistance condition for the settlement periods; i.e., $\tau \equiv \tau_j$. We recall that cash flows take place just at the settlement dates $x_1, x_2, \dots, x_n$. At this dates, the holder of such an IRS, pays a predetermined amount

$$K \tau N$$

and receives in turn the floating payout

$$L_j(x_{j-1})\tau N.$$

The net cash at x_j is therefore

$$\{L_j(x_{j-1}) - K\}\,\tau N.$$

Without loss of generality we set the notional $N = 1$ and $t = 0$ with $x_0 > 0$. By means of the fundamental arbitrage free pricing formula, starting from the money-market martingale measure $\mathbb{Q}$ we can compute the value of this contract as:

$$\begin{aligned}
\Pi_{sw} &= \sum_{j=1}^{n} \mathbb{E}\left[D(0,x_j)\left\{L_j(x_{j-1}) - K\right\}\tau\right] \\
&= \tau\left(\sum_{j=1}^{n} \mathbb{E}\left[D(0,x_j)L_j(x_{j-1})\right] - K\sum_{j=1}^{n} P_j(0)\right)
\end{aligned} \tag{3.7}$$

where we have used the well-known definition

$$P_j(0) = \mathbb{E}\left[D(0,x_j)\right].$$

Let us consider the first term

$$\mathbb{E}\left[D(0,x_j)L_j(x_{j-1})\right].$$

Fist of all, we may change the risk-neutral measure $\mathbb{Q}$ by means of the *Change of Numeraire Theorem* into the more suitable x_j-forward measure $\mathbb{Q}^{x_j}$. Therefore, we have

$$\Pi_{sw} = \tau\left(\sum_{j=1}^{n} P_j(0)\mathbb{E}^{x_j}\left[L_j(x_{j-1})\right] - K\sum_{j=1}^{n} P_j(0)\right) \tag{3.8}$$

The following result has a crutial role for concluding.

Lemma 1 *For every $j = 1,\ldots,n$, the LIBOR process $L_j(t)$ is a martingale under the corresponding forward measure $\mathbb{Q}^{x_j}$, on the interval $[0, x_{j-1}]$.*

Proof. From Definition 6[1] for the simply-compounded LIBOR forward interest rate $L_j(t)$, we have:

$$\tau L_j(t) := \frac{P_{j-1}(t)}{P_j(t)} - 1.$$

We recall that the process

$$P_{j-1}(t)/P_j(t)$$

is the price of the x_{j-1}-bond in terms of the strictly positive asset $P_j(t)$, which is, by definition, the numeraire for the forward measure $\mathbb{Q}^{x_j}$. The process $P_{j-1}(t)/P_j(t)$ is thus trivially a $\mathbb{Q}^{x_j}$-martingale on the interval $[0, x_{j-1}]$, where the normalized process is well defined. Therefore, $L_j(t)$ is also a $\mathbb{Q}^{x_j}$-martingale on the same interval. □

By using the previous Lemma, we have

$$L_j(t) = \mathbb{E}^{x_j}\left[L_j(s)\middle|\mathcal{F}_t\right] \quad 0 \le t \le s \le x_{j-1};$$

and, in particular,

$$L_j(0) = \mathbb{E}^{x_j}\left[L_j(x_{j-1})\right].$$

By substituting into (3.8):

$$\begin{aligned}
\Pi_{sw} &= \tau\left(\sum_{j=1}^{n} P_j(0)L_j(0) - K\sum_{j=1}^{n} P_j(0)\right) = \\
&= \tau\left[\frac{1}{\tau}\sum_{j=1}^{n} P_j(0)\left(\frac{P_{j-1}(0)}{P_j(0)} - 1\right) - K\sum_{j=1}^{n} P_j(0)\right] = \qquad (3.9)\\
&= \left[\sum_{j=1}^{n}\left(P_{j-1}(0) - P_j(0)\right) - K\tau\sum_{j=1}^{n} P_j(0)\right].
\end{aligned}$$

Finally, the total value Π_{sw} at time $t = 0$ is therefore

$$\Pi_{sw} = P_0(0) - P_n(0) - K\tau\sum_{j=1}^{n} P_j(0). \qquad (3.10)$$

Proposition 4 (General Closed-Formula for Plain Vanilla IRS.) *The total value $\Pi_{sw}(t)$ of a plain vanilla IRS settled in arrears at time time $t \le x_0$ and*

[1] See Chap. 1

notional N is

$$\Pi_{sw}(t) = N\left(P_0(t) - P_n(t) - K\tau\sum_{j=1}^{n} P_j(t)\right). \tag{3.11}$$

In contrast to the *interest rate caps* pricing, which depends on the particular choice of the volatility vector process $\sigma(t,T)$ within the HJM framework, the IRS closed-formula is generic. However, note that the plain vanilla interest rate swaps remain dependent on the term structure of discount bonds $T \to P(t,T)$.

Definition 17 (Forward Swap Rate.) *The forward swap rate (also called par swap rate) $K_{sw}(t)$ at time $t \leq x_0$ which gives the "fair value" $\Pi_{sw}(t) = 0$:*

$$K_{sw}(t) = \frac{P_0(t) - P_n(t)}{\tau\sum_{j=1}^{n} P_j(t)}.$$

Let $t = 0$ again for simplicity and suppose, as above, that $x_0 > 0$, then:

Remark 2 *A plain vanilla cap is said to be at-the-money (ATM henceforth) if*

$$K = K_{sw}(0) = \frac{P_0(0) - P_n(0)}{\tau\displaystyle\sum_{j=1}^{n} P_j(0)}. \tag{3.12}$$

3.2 Caps under The General HJM Gaussian Model

Let us now turn to the problem of rigorously pricing the caplet. Remember that the payoffs on settlement dates $x_1, \ldots, x_n$ are:

$$h_{\gamma_j}(x_j) = \tau_j(L_j(x_{j-1}) - K)^+.$$

Note that such an stream of payoffs for the corresponding strip of caplets γ_j, is equivalent in terms of pricing to those

$$h_{\gamma_j}(x_{j-1}) = \tau_j\mathbb{E}\left[D(x_{j-1}, x_j)(L_j(x_{j-1}) - K)^+ \middle| \mathcal{F}_{x_{j-1}}\right],$$

which are received at fixing dates $x_0, \ldots, x_{n-1}$. Because of $\mathcal{F}_{x_{j-1}}$-mesurability of $L_j(x_{j-1})$, they can also be expressed as

$$\begin{aligned} h_{\gamma_j}(x_{j-1}) &= \tau_j (L_j(x_{j-1}) - K)^+ \mathbb{E}\left[D(x_{j-1}, x_j) \middle| \mathcal{F}_{x_{j-1}}\right] \\ &= \tau_j P_j(x_{j-1})(L_j(x_{j-1}) - K)^+. \end{aligned} \tag{3.13}$$

From Definition 5 for the simply-compounded LIBOR spot rate $L_j(x_{j-1})$, we know:

$$L_j(x_{j-1}) := \frac{1}{\tau_j}\left(\frac{1}{P_j(x_{j-1})} - 1\right),$$

and after some trivial algebra

$$\begin{aligned} h_{\gamma_j}(x_{j-1}) &= \tau_j P_j(x_{j-1}) \left(\frac{1}{\tau_j}\left(\frac{1}{P_j(x_{j-1})} - 1\right) - K\right)^+ \\ &= \left(P_j(x_{j-1})\left(\frac{1}{P_j(x_{j-1})} - 1\right) - K\tau_j P_j(x_{j-1})\right)^+ \\ &= (1 - (1 + \tau_j K) P_j(x_{j-1}))^+ \end{aligned} \tag{3.14}$$

we may finally write the following representation for the stream of payoffs:

$$h_{\gamma_j}(x_{j-1}) = (1 + \tau_j K)\left(\kappa - P_j(x_{j-1})\right)^+ \tag{3.15}$$

where $\kappa = (1 + \tau_j K)^{-1}$.

Consequently we see that a j-caplet is equivalent to $(1 + \tau_j K)$ put options on an underlying x_j-bond, where the exercise date of the option is at x_{j-1} and the exercise price is κ. An entire cap contract can thus be viewed as a portfolio of put options, and we may use the results on Corollary 2 of Sect. 2.4 to compute the theoretical price and, in particular, to price it under the General Gaussian HJM model.

Proposition 5 (Caplet Pricing for Gaussian Forward Rates.) *The price at $t = 0$ of the j-caplet, γ_j, with payoff:*

$$h_{\gamma_j}(x_j) = (L_j(x_{j-1}) - K)^+$$

is given by

$$\gamma_j(h_{\gamma_j}, 0) = (1 + \tau_j K)\left\{\kappa P_{j-1}(0) N(-d_-) - P_j(0) N(-d_+)\right\} \tag{3.16}$$

where

$$
\begin{aligned}
d_{\pm} \;&:=\; \frac{\log\left(\frac{P_j(0)}{\kappa P_{j-1}(0)}\right) \pm \frac{1}{2}\vartheta^2(0, x_{j-1})}{\vartheta(0, x_{j-1})}\,, \\
\vartheta^2(0, x_{j-1}) \;&:=\; \int_0^{x_{j-1}} \|\varsigma(u; x_{j-1}, x_j)\|^2 \, du\;; \\
\varsigma(t; x_{j-1}, x_j) \;&:=\; -\int_{x_{j-1}}^{x_j} \sigma(t, s)\, ds.
\end{aligned}
\tag{3.17}
$$

and $\kappa = (1 + \tau_j K)^{-1}$.