Chapter 2

Forward Rate Models

In the nineties, Heath, Jarrow and Morton (henceforth HJM) [17] proposed a new framework for modeling the entire forward curve directly.

2.1 The Heath-Jarrow-Morton Framework

The stochastic setup is as shown at the end of the Sect. 1.1.3. We consider that there exists a unique equivalent martingale measure $\mathbb{Q}$, associated to the money-market account. Therefore, the T-bond market is complete and there are no arbitrage strategies. Let W be a q-dimensional $\mathbb{Q}$-Wiener process.

We assume that we are given an $\mathbb{R}$-valued and $\mathbb{R}^q$-valued stochastic process $\alpha = \alpha(\omega, t, T)$ and $\sigma = [\ \sigma_1(\omega, t, T)\ \ldots\ \sigma_q(\omega, t, T)\ ]$, respectively, with $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ $\mathcal{F}_T$-adapted processes. We also assume that for $0 \leq t < T < \infty$, the forward rate $F(\cdot, T)$ has a stochastic differential wich under $\mathbb{Q}$ is given by

$$\left\{ \begin{array}{rcl} dF(t,T) & = & \alpha(t,T)dt + \displaystyle\sum_{j=1}^{q} \sigma_j(t,T)dW_j(t) \\ F(0,T) & = & F^o(0,T). \end{array} \right. \tag{2.1}$$

Note that conceptually equation (2.1) is one stochastic differential in the t-variable for every choice of T. Also note that we use the observed forward rate curve $T \mapsto F^o(0,T)$ as the initial condition. This will automatically give us a perfect fit between observed and theoretical T-bond prices at $t = 0$, thus relieving us of the task of inverting the term structure of discount bonds.

Suppose now that we have specified α, σ and $\{F^o(0,T);\ T \geq 0\}$. Then we have specified the entire forward rate structure and thus, by the relation

$$P(t,T) = e^{-Z(t,T)} \tag{2.2}$$

where

$$Z(t,T) = \int_t^T F(t,s)ds \tag{2.3}$$

we have in fact specified the entire term structure of discount bonds

$$\{P(t,T);\ T \geq 0,\ 0 \leq t \leq T\}.$$

We now show how bond price dynamics are induced by a given specification of the forward rate dynamics. By using Itô Lemma in (2.2), we have

$$dP(t,T) = -P(t,T)dZ(t,T) + \frac{1}{2}P(t,T)\left(dZ(t,T)\right)^2, \tag{2.4}$$

and it remains to compute $dZ(t,T)$. We have

$$dZ(t,T) = d\left(\int_t^T F(t,s)ds\right)$$

and this is a situation that is not covered by the standard Itô formula. Let us guess the answer.

Proposition 1 *Consider for $s \in [t_0, T]$ the Itô process defined by*

$$dF(t,s) = \alpha(t,s)dt + \sum_{j=1}^{q} \sigma_j(t,s)dW_j(t) \tag{2.5}$$

with $t \in [t_0, s]$. Then, the dynamics for the stochastic process (2.3) is

$$dZ(t,T) = \left[\left(\int_t^T \alpha(t,s)ds\right) - F(t,t)\right]dt + \sum_{j=1}^{q} \int_t^T \sigma_j(t,s)ds\ dW_j(t). \tag{2.6}$$

Proof. See Appendix A. □

Therefore, by substituting in equation (2.4)

$$\begin{aligned} dP(t,T) &= P(t,T)\left\{\left[r(t) - \int_t^T \alpha(t,s) + \tfrac{1}{2}\sum_{j=1}^{q}\left(\int_t^T \sigma_j(t,s)ds\right)^2\right]dt \right. \\ &- \left. \sum_{j=1}^{q}\left(\int_t^T \sigma_j(t,s)ds\right)dW_j(t)\right\} \end{aligned} \tag{2.7}$$

which may be summarized as the the following:

Corollary 1 *The $\mathbb{Q}$-dynamics for the T-bond price, $P(t,T)$, follows the stochastic differential equation*

$$dP(t,T) = P(t,T)\left[\left(r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right)dt + S(t,T)dW(t)\right], \quad (2.8)$$

where $\|\cdot\|$ denotes the Euclidean norm, and

$$\begin{array}{rcl} A(t,T) & := & -\int_t^T \alpha(t,s)\,ds \\ S_j(t,T) & := & -\int_t^T \sigma_j(t,s)\,ds, \end{array}$$

and we have used the matrix notations

$$\begin{array}{rcl} S(t,T) & = & [\ S_1(t,T) \quad S_2(t,T) \quad \dots \quad S_q(t,T)\ ] \\ W(t) & = & [\ W_1(t) \quad W_2(t) \quad \dots \quad W_q(t)\ ]^T. \end{array}$$

2.1.1 Absence of Arbitrage

Theorem 2 (HJM Drift Condition) *Assume that the family of forward rates is given by (2.1) and that the induced bond market is arbitrage free. Under the martingale measure $\mathbb{Q}$, the process α and σ must satisfy the following relation, for every t and every $T \geq t$.*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)^T\,ds. \quad (2.9)$$

Proof. Since we are modeling the market under the equivalent martingale measure, $\mathbb{Q}$, the discounted T-bond price $\widetilde{P}(t,T)$ have to be a local $\mathbb{Q}$-martingale satisfying the following differential:

$$d\widetilde{P}(t,T) = \widetilde{P}(t,T)S(t,T)dW(t)$$

We now look for the corresponding differential of the ordinary discount bond price, $P(t,T) = B(t)\widetilde{P}(t,T)$. From the Itô Lemma we know that

$$dP(t,T) = P(t,T)\left(r(t)dt + S(t,T)dW(t)\right), \quad (2.10)$$

in other words, as $\mathbb{Q}$ is a martingale measure with the money account B as numeraire, the local rate of return of every asset price under $\mathbb{Q}$ equals the short rate. We thus have

$$A(t,T) + \frac{1}{2}\|S(t,T)\|^2 = 0.$$

Taking the T-derivative of this equation gives us the relation (2.9) $\square$

2.2 From HJM to Short-Rate Models

What is the interplay between the short-rate dynamics and the present HJM framework? Let us consider the simplest one-dimensional HJM model: a constant $\sigma(t,T) \equiv \sigma > 0$. Then, we have under the risk-neutral measure $\mathbb{Q}$

$$dF(t,T) = \sigma^2(T-t)dt + \sigma dW(t),$$

which implies by direct integration

$$F(t,T) = F(0,T) + \frac{\sigma^2}{2}T^2 + \sigma W(T).$$

Hence for the short rates we obtain

$$r(t) = F(t,t) = F(0,t) + \frac{\sigma^2}{2}t^2 + \sigma W(t),$$

and taking the differentials

$$dr(t) = (F(0,t) + \sigma^2 t)dt + \sigma dW(t).$$

The observant reader may identify it with the Ho and Lee model [18]. The main inputs into the HJM framework are the forward rate volatility processes $\sigma_j(t,T)$, and as we have shown the Ho and Lee model is a special case of the general 1-factor HJM framework, corresponding to a particular choice of the volatility process. However, it has remained unclear wheter other short-rate models could be derived within the HJM framework, and whether there exists a systematic approach for generating the short-rate models. In general, we have the following:

Proposition 2 *Suppose that $F(0,T)$, $\alpha(t,T)$ and $\sigma(t,T)$ are differentiable in T with $\int_0^T |\partial_u F(0,u)|\, du < \infty$.*

Then the short-rate process is an Itô process of the form

$$dr(t) = \zeta(t)dt + \sigma(t,t)dW(t),$$

where

$$\zeta(t) = \alpha(t,t) + \partial_t F(0,t) + \int_0^t \partial_t \alpha(s,t)\, ds + \int_0^t \partial_t \sigma(s,t) dW(s)$$

Proof. See Appendix A. □

Remark 1 *For every forward rate model, the arbitrage free price of a derivative security, with T-payoff $h(T)$, will still be given by the general pricing formula*

$$V(h,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\, du} h(T) \middle| \mathcal{F}_t \right],$$

where the short-rate as usual is given by $r(t) = F(t,t)$.

2.3 Forward Measures

Equation (1.18) shows to calculate the arbitrage free price $V(t)$, of a derivative security. The value calculated must, of course, be independent of the choice of numeraire. Consider two numeraires M and N with martingale measures $\mathbb{Q}^M$ and $\mathbb{Q}^N$. Combining the result (1.18) applied to both numeraires yields

$$M(t)\mathbb{E}^M\left[\frac{h(T)}{M(T)} \middle| \mathcal{F}_t \right] = N(t)\mathbb{E}^N\left[\frac{h(T)}{N(T)} \middle| \mathcal{F}_t \right]$$

This expression can be rewritten as

$$\mathbb{E}^M\left[g(T) \middle| \mathcal{F}_t \right] = \mathbb{E}^N\left[g(T)\frac{M(T)/M(t)}{N(T)/N(t)} \middle| \mathcal{F}_t \right] \tag{2.11}$$

where $g(T) = h(T)/M(T)$. Since, h, M and N are general, this result holds for all random payoffs g and all numeraires M and N.

We have now derived a way to express the expectation $g(T)$ under the measure $\mathbb{Q}^M$ in terms of an expectation under the measure $\mathbb{Q}^N$.

Theorem 3 (Change of Numeraire) *Let $\mathbb{Q}^M$ be the equivalent martingale measure with respect to the numeraire $M(t)$. Let $\mathbb{Q}^N$ be the equivalent martingale* *measure with respect to the numeraire $N(t)$. The Radon-Nikodym derivative that changes the equivalent martingale measure $\mathbb{Q}^M$ into $\mathbb{Q}^N$ is given by*

$$\lambda(t) = \frac{d\mathbb{Q}^M}{d\mathbb{Q}^N} = \frac{M(T)/M(t)}{N(T)/N(t)}.$$

The *Change of Numeraire Theorem* is very powerful in the context of pricing interest rate derivatives. Instead of using the value of the money-market account $B(t)$ as a numeraire, the prices of T-bonds can also be used as a numeraire. A very convenient choice is to use the discount bond with maturity T as a numeraire for derivatives which have a payoff $h(T)$ at time T. Assume, on the other hand, that the probability measure $\mathbb{Q}^T$ associated to the numeraire $P(t,T)$ actually exists. Hence, we can apply the *Change of Numeraire Theorem* as follows. Under the measure $\mathbb{Q}^T$ the prices $V(h,t)/P(t,T)$ are martingales for $t < T$. Therefore, applying the definition of a martingale and taking into account that $P(T,T) = 1$, we obtain

$$V(h,t) = P(t,T)\mathbb{E}^T\left[h(T)\middle|\mathcal{F}_t\right] \tag{2.12}$$

The measure $\mathbb{Q}^T$ has another very interesting property, which virtually gave the name T-forward measure. Under the T-forward measure, the instantaneous forward rate, $F(t,T)$ is equal to the expected of the spot interest rate at time T. In formulas

$$F(t,T) = \mathbb{E}^T\left[r(T)\middle|\mathcal{F}_t\right],$$

e.g., see the straightforward arguments followed by Filipovic in [15, Sect. 7.1] or Björk in [4, Sect. 19.4.2]. Note that in this case, the corresponding Radon-Nikodym derivative that changes the T-forward measure $\mathbb{Q}^T$ into the risk-neutral measure (or money-market measure) $\mathbb{Q}$, is

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{P(T,T)/P(t,T)}{B(T)/B(t)} = \frac{D(t,T)}{P(t,T)} = \frac{e^{-\int_t^T r(s)\,ds}}{P(t,T)}. \tag{2.13}$$

2.4 The General HJM Gaussian Model

Consider a general HJM model under the risk-neutral measure $\mathbb{Q}$ specified by (2.1). We also assume that

$$\sigma(t,T) = [\,\sigma_1(t,T)\ \dots\ \sigma_q(t,T)\,]$$

are deterministic functions of (t, T), and hence forward rates $F(t, T)$ are Gaussian distributed. We consider now a European call option, with expiration date T and exercise price K, on an underlying bond with maturity S (where of course $T < S$). The following general pricing formula may be derived:

Proposition 3 (Bond Option Pricing for Gaussian Forward Rates) *The price, at $t = 0$ of the bond option*

$$h(T) = (P(T, S) - K)^+$$

is given by

$$V(h, 0) = P(0, S)N(d_+) - KP(0, T)N(d_-) \tag{2.14}$$

where

$$\begin{aligned} d_\pm &:= \frac{\log\left(\frac{P(0,S)}{KP(0,T)}\right) \pm \frac{1}{2}\vartheta^2(T, S)}{\vartheta(T, S)} , \\ \vartheta^2(T, S) &:= \int_0^T \|\varsigma(u; T, S)\|^2 \, du \, ; \end{aligned} \tag{2.15}$$

and,

$$\varsigma(t; T, S) := S(t, S) - S(t, T) = -\int_T^S \sigma(t, s) \, ds. \tag{2.16}$$

Proof. Let us start with the fundamental arbitrage-free equation

$$V(h, 0) = \mathbb{E}\left[D(0, T)(P(T, S) - K)^+\right],$$

where we are taking the expectations with respect the equivalent martingale measure $\mathbb{Q}$ associated to the money-market numeraire $B(\cdot)$. We decompose it as follows

$$V = \mathbb{E}\left[D(0, T)P(T, S)\mathbb{I}_{\{P(T,S)\geq K\}}\right] - K\mathbb{E}\left[D(0, T)\mathbb{I}_{\{P(T,S)\geq K\}}\right] \tag{2.17}$$

In this case, the Radon-Nikodym derivative that changes S-forward measure $\mathbb{Q}^S$ into the money-market measure $\mathbb{Q}$ will be given by

$$\lambda^S(T) = \frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \frac{P(T, S)/P(0, S)}{B(T)/B(0)} = \frac{D(0, T)P(T, S)}{P(0, S)}.$$

In a similar way note that

$$\lambda^T(T) = \frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{P(T,T)/P(0,T)}{B(T)/B(0)} = \frac{D(0,T)}{P(0,T)},$$

is the conversion factor responsible for changing the T-forward measure $\mathbb{Q}^T$ into the risk-neutral world $\mathbb{Q}$. Substituting into decomposition (2.17), and combining with the measurability at $t = 0$ of $P(0, S)$ and $P(0, T)$ we have

$$V = \mathbb{E}\left[P(0,S)\lambda^S(T)\mathbb{I}_{\{P(T,S)\geq K\}}\right] - K\mathbb{E}\left[P(0,T)\lambda^T(T)\mathbb{I}_{\{P(T,S)\geq K\}}\right]$$

$$V = P(0,S)\mathbb{Q}^S\left(P(T,S) \geq K\right) - KP(0,T)\mathbb{Q}^T\left(P(T,S) \geq K\right)$$

Now we have the value V for the call option in terms of the forward measures $\mathbb{Q}^S$ and $\mathbb{Q}^T$. Let us start with the probability computations referred to the T-forward measure. Note that the probability may be written as

$$\mathbb{Q}^T(P(T,S) \geq K) = \mathbb{Q}^T\left(\frac{P(T,S)}{P(T,T)} \geq K\right) = \mathbb{Q}^T\left(\log\frac{P(T,S)}{P(T,T)} \geq \log K\right)$$

Consider the "discounted" process

$$X_{S,T}(t) := \frac{P(t,S)}{P(t,T)},$$

with terminal value $X_{S,T}(T) = P(T,S)/P(T,T)$. By taking differentials under the risk-neutral measure $\mathbb{Q}$ we have

$$\begin{aligned}
d\left(\frac{P(t,S)}{P(t,T)}\right) &= \frac{1}{P(t,T)}dP(t,S) - \frac{P(t,S)}{(P(t,T))^2}dP(t,T) + dP(t,S)\cdot d\left(\frac{1}{P(t,T)}\right) = \\
&= \{\dots\}dt + X\left\{\left(S(t,S) - S(t,T)\right)dW(t)\right\} \\
dX &= \{\dots\}dt + X\varsigma(t;T,S)dW(t).
\end{aligned}$$

For the second stage we have used equation (2.10) applied to the discount bonds $P(t, S)$ and $P(t, T)$. Recall that $\mathbb{Q}^T$ is a martingale measure and the multidimensional Girsanov's Theorem[1] which locally induces the change into this T-forward measure, does not affect the difusson coefficient of the initially taken differential.

[1] See [Ref.] for a detailed discussion of it.

Therefore we have

$$dX_{T,S}(t) = X_{T,S}(t)\varsigma(t;T,S)dW^T(t).$$

Let us introduce the auxiliary process:

$$Y_{T,S}(t) = \log X_{T,S}(t)$$

By means of the multidimensional Itô Lemma, it is not difficult to prove that the random variable $Y_{T,S}(T)$ distributes like

$$Y_{T,S}(T) \sim \mathcal{N}\left(\log\frac{P(0,S)}{P(0,T)} - \frac{1}{2}\vartheta^2(T,S), \vartheta^2(T,S)\right),$$

where $\vartheta^2(T,S) = \int_0^T \|\varsigma(u;T,S)\|^2\, du$. Now the computation of the probability under the T-forward measure is straightforward:

$$\mathbb{Q}^T(P(T,S) \geq K) = \mathbb{Q}^T(Y_{T,S}(T) \geq \log K) = N(d_-)$$

For the pending probability $\mathbb{Q}^S$, first, note the following:

$$\mathbb{Q}^S(P(T,S) \geq K) = \mathbb{Q}^S\left(\frac{P(T,T)}{P(T,S)} \leq \frac{1}{K}\right) = \mathbb{Q}^S\left(\log\frac{P(T,T)}{P(T,S)} \leq -\log K\right).$$

It is enough to introduce the auxiliary processes,

$$W_{T,S}(t) := \frac{P(t,T)}{P(t,S)},$$

and,

$$Z_{T,S}(t) := \log W_{T,S}(t),$$

for concluding that $Z_{T,S}(T)$ distributes like

$$Z_{T,S}(T) \sim \mathcal{N}\left(\log\frac{P(0,T)}{P(0,S)} - \frac{1}{2}\vartheta^2(T,S), \vartheta^2(T,S)\right),$$

and then

$$\mathbb{Q}^S(Z_{T,S}(T) \leq -\log K) = N(d_+).$$

□

Corollary 2 *The price at $t = 0$ of the put option*

$$h(T) = (K - P(T, S))^+$$

is given by

$$\Pi(h, 0) = KP(0, T)N(-d_-) - P(0, S)N(-d_+) \tag{2.18}$$

where the quantities $d_\pm$ are completely determined by the identities (2.15) to (2.16).

Proof. First consider the difference between the call and the put option at time $t = 0$. Under the risk-neutral martingale measure we know

$$\begin{aligned} V - \Pi &= \mathbb{E}\left[D(0, T)\left\{(P(T, S) - K)^+ - (K - P(T, S))^+\right\}\right] \\ &= \mathbb{E}\left[D(0, T)(P(T, S) - K)\right] \end{aligned}$$

By definition we have $\mathbb{E}[D(0, T)K] = KP(0, T)$. However, we have the problem of the correlation between the discounting factor and the payoff factor for the first term $\mathbb{E}[D(0, T)P(T, S)]$. We can circumvent this problem by using *Change of Numeraire Theorem* in an identical way to that shown in Proposition 3. Recall first that the likelihood

$$\lambda^S(T) = \frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \frac{P(T, S)/P(0, S)}{B(T)/B(0)} = \frac{D(0, T)P(T, S)}{P(0, S)},$$

induces the change of the S-forward measure into the risk-neutral measure. Thus we have:

$$\mathbb{E}\left[D(0, T)P(T, S)\right] = \mathbb{E}\left[\lambda^S(T)P(0, S)\right] = \mathbb{E}^S\left[P(0, S)\right],$$

and then the *Put-Call Parity Relation*:

$$V - \Pi = P(0, S) - KP(0, T), \tag{2.19}$$

is finally inferred. □