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Interpolation

Interpolation is like a mathematical game! To understand its concept, consider the following two examples.

Example 1:

x	1	2	3	4
$f(x)$	1	4	?	16

Can you guess what is actually $f(3)$?

We noticed that the above function is
$f(x) = x^2$ and so $f(3) = 9$.

Example 2:

x	0	1	2	3
$f(x)$	1	3	?	55

Can you guess what is actually $f(2)$? It's hard – Isn't it? That's where this new 'art' of interpolation comes in. It will derive the unknown function $f(x)$ for us!

Interpolation means to insert or add something and in numerical computing, it means to find new data points, using the given or known data points.
It is actually a powerful method of construction or estimating a function $f(x)$ from the known distinct data points such that the function passes through them.

Through two distinct points, we can construct a unique polynomial of degree one, that is, a linear polynomial (or a straight line). In general, through n distinct points, we can construct a unique polynomial of degree $n-1$.

Lagrange Interpolation:

Suppose that we are given the following two distinct data points.

x	x_0	x_1
$f(x)$	$f(x_0)$	$f(x_1)$

So Lagrange linear polynomial (of degree one) is defined as

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1),$$

where $L_0(x)$ and $L_1(x)$ are called Lagrange's Fundamental Polynomials and are defined as

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

Similarly, for three distinct data points,

x	x_0	x_1	x_2
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$

Lagrange quadratic polynomial (of degree two) is defined as

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2),$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}, \quad L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

and $$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

Task: For the following four data points, write a cubic Lagrange polynomial (of degree three).

x	x_0	x_1	x_2	x_3
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Example: Find the Lagrange polynomial for the following data.

x	$x_0 = 0$	$x_1 = 1$	$x_2 = 3$
$f(x)$	$f(x_0) = 1$	$f(x_1) = 3$	$f(x_2) = 55$

In addition, interpolate or find $f(2)$.

Solution: Since we are given three data points, so Lagrange quadratic polynomial of degree two is given by

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2),$$

where

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(-1)(-3)} = \frac{1}{3}(x^2-4x+3),$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1)(-2)} = \frac{1}{2}(3x-x^2),$$

and $$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x)(x-1)}{(3)(2)} = \frac{1}{6}(x^2-x).$$

Therefore,

$$P_2(x) = \frac{1}{3}(x^2-4x+3)\,(1) + \frac{1}{2}(3x-x^2)\,(3) + \frac{1}{6}(x^2-x)(55).$$

We next take L.C.M. to write

$$P_2(x) = \frac{1}{6}(2x^2 - 8x + 6 + 27x - 9x^2 + 55x^2 - 55x)$$

$$= \frac{1}{6}(48x^2 - 36x + 6)$$

$$= 8x^2 - 6x + 1 = f(x).$$

So our required function is

$$f(x) = 8x^2 - 6x + 1.$$

Now,

$$f(2) = 8(2)^2 - 6(2) + 1 = 21.$$

Note: We can always check that our derived function is correct. For this purpose, we need to put the given values of x (i.e., $x = 0, 1$ and 3) in our above derived function and we must get the same given values of $f(x)$ (i.e., $f(x) = 1$, 3 and 55 respectively).

Task: Find the Lagrange polynomial for the following four data points.

x	-1	2	4	6
$f(x)$	-9	0	56	208

In addition, interpolate or find $f(3)$.

Note: In this task, we do not need to find $L_1(x)$. But why? Think!

Look carefully at $f(x_1)$. Did you notice something?

Numerical Differentiation

The general method for deriving the numerical differentiation formula is to differentiate the interpolation polynomial. The Lagrange's quadratic interpolation polynomial is given by

$$P(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2,$$

$$P(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2.$$

Differentiating both sides with respect to x, we obtain

$$P'(x) = L_0{}'(x)y_0 + L_1{}'(x)y_1 + L_2{}'(x)y_2,$$

$$P''(x) = L_0{}''(x)y_0 + L_1{}''(x)y_1 + L_2{}''(x)y_2$$

and so on.

This concept will be clearer by considering the following example and task.

Example: Perform the numerical differentiation to find the first derivative, at $x = 2$, of the following data, using the Lagrange's interpolation formula.

$$y(0) = 1, y(1) = 3, y(3) = 55.$$

Solution: The Lagrange's quadratic polynomial, for the given data, is given by

$P_2(x) = P(x) = 8x^2 - 6x + 1.$ (That has been derived before.)

Differentiating both sides with respect to x, we obtain
$P'(x) = 16x - 6.$

Therefore, we obtain

$P'(2) = 16(2) - 6 = 26.$

Task: Perform the numerical differentiation to find the first derivative, at $x = 3$, of the following data, using the Lagrange's interpolation formula.

$$y(-1) = -9, y(1) = -7, y(2) = 0, y(5) = 117.$$

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