UNIT - 1: Discrete Fourier Transforms (DFT)[1, 2, 3, 4, 5]

Dr. Manjunatha. P

manjup.jnnce@gmail.com

Professor

Dept. of ECE

J.N.N. College of Engineering, Shimoga

September 11, 2014

Digital Signal Processing: Introduction [1, 2, 3, 4]

- Slides are prepared to use in class room purpose, may be used as a reference material
- All the slides are prepared based on the reference material
- Most of the figures/content used in this material are redrawn, some of the figures/pictures are downloaded from the Internet.
- I will greatly acknowledge for copying the some the images from the Internet.
- This material is not for commercial purpose.
- This material is prepared based on Digital Signal Processing for ECE/TCE course as per Visvesvaraya Technological University (VTU) syllabus (Karnataka State, India).

DSP Syllabus

PART - A

UNIT - 1: Discrete Fourier Transforms (DFT)

- Frequency domain sampling and reconstruction of discrete time signals.
- DFT as a linear transformation, its relationship with other transforms. 7 Hours

The concept of frequency in continuous and discrete time signals

Continuous Time Sinusoidal Signals

- The concept of frequency is closely related to specific type of motion called harmonic oscillation which is directly related to the concept of time.
- A simple harmonic oscillation is mathematically described by:

$$x_a(t) = A cos(\Omega t + \theta), \quad -\infty < t < \infty$$

- The subscript a is used with $x(t)$ to denote an analog signal. A is the amplitude, Ω is the frequency in radians per second(rad/s), and θ is the phase in radians. The Ω is related by frequency F in cycles per second or hertz by

$$\Omega = 2\pi F$$

$$x_a(t) = A cos(2\pi F t + \theta), \quad -\infty < t < \infty$$

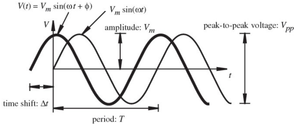


Figure 1: Example of an analog sinusoidal signal

Cycle and Period

- The completion of one full pattern waveform is called a cycle. A period is defined as the amount of time required to complete one full cycle.

Complex exponential signals

$$x_a(t) = Ae^{j(\Omega t+\theta)}$$

where

$$e^{\pm j\phi} \quad = \quad cos\phi \pm jsin\phi$$

$$x_a(t) = Acos(\Omega t+\theta) = \frac{A}{2}e^{j(\Omega t+\theta)} + \frac{A}{2}e^{-j(\Omega t+\theta)}$$

- As time progress the phasors rotate in opposite directions with angular $\pm\Omega$ frequencies radians per second.
- A positive frequency corresponds to counterclockwise uniform angular motion, a negative frequency corresponds to clockwise angular motion.

$Ae^{j\phi}e^{j\omega t_1}$.

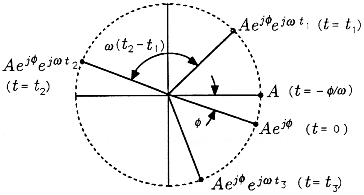


Figure 2: Representation of cosine function by phasor

Discrete Time Sinusoidal Signals

- A discrete time sinusoidal may expressed as

$$x(n) = Acos(\omega n + \theta), \quad -\infty < t < \infty$$

  where n is an integer variable called the sample number.
- A is the amplitude, ω is the frequency in radians per sample(rad/s), and θ is the phase in radians.
- The ω is related by frequency f cycles per sample by

$$\omega = 2\pi f$$

$$x(n) = Acos(2\pi fn + \theta), \quad -\infty < t < \infty$$

- A discrete time signal $x(n)$ is periodic with period $N(N > 0)$ if and only if

$$x(n + N) = x(n) \quad for\ all\ n$$

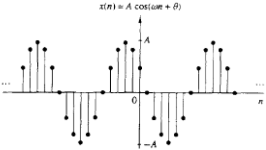


Figure 3: Discrete signal

Periodic and aperiodic (non-periodic) signals.

- A periodic signal consists a continuously repeated pattern. Signal is periodic if it exhibits periodicity i.e.

$$x(t+T) \quad = \quad x(t) \;\; \textit{for all } t$$

- It has a property that it is unchanged by a time shift of T.
- An aperiodic signal changes constantly without exhibiting a pattern or cycle that repeats over the time.

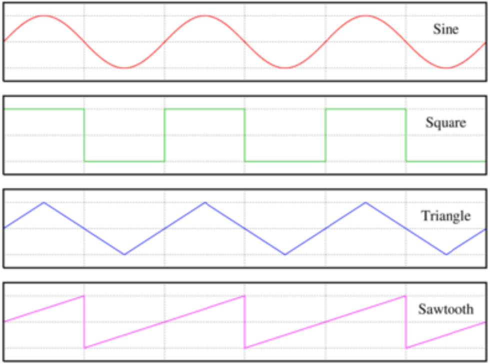


Figure 4: Periodic signals

Periodic and aperiodic (non-periodic) signals.

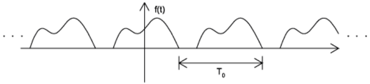


Figure 5: Periodic signal

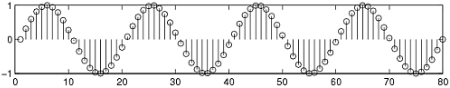


Figure 6: Periodic discrete time signal

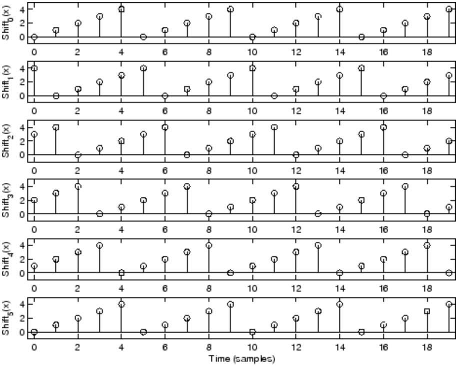


Figure 7: Periodic discrete time signal

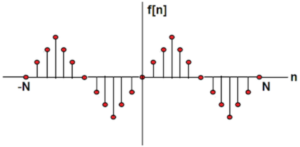


Figure 8: Periodic discrete time

Periodic and aperiodic (non-periodic) signals.

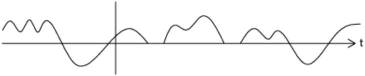


Figure 9: Aperiodic signals

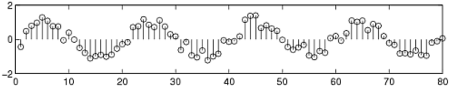


Figure 10: Aperiodic discrete time signals

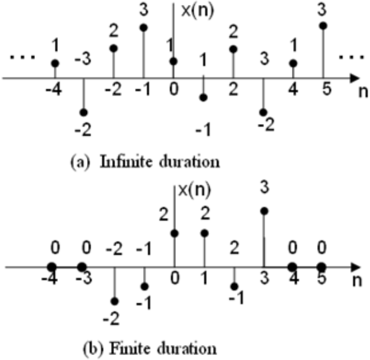


Figure 11: Aperiodic discrete time signals

Periodic and aperiodic (non-periodic) signals.

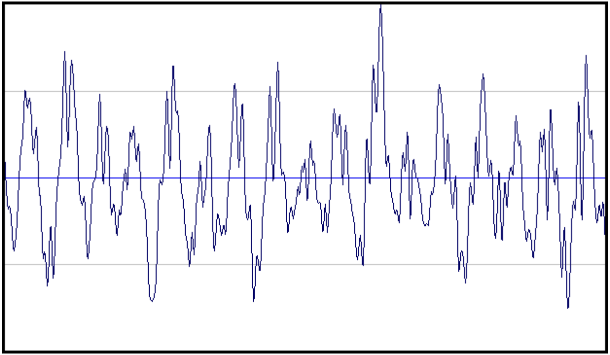


Figure 12: Aperiodic (random) signal

Fourier series

- Sinusoidal functions are wide applications in Engineering and they are easy to generate.
- Fourier has shown that periodic signals can be represented by series of sinusoids with different frequency.
- A signal $f(t)$ is said to be periodic of period T if $f(t) = f(t + T)$ for all t.
- Periodic signals can be represented by the Fourier series and non periodic signals can be represented by the Fourier transform.
- For example square wave pattern can be approximated with a suitable sum of a fundamental sine wave plus a combination of harmonics of this fundamental frequency.
- Several waveforms that are represented by sinusoids are as shown in Figure 14. This sum is called a Fourier series.
- The major difference with respect to the line spectra of periodic signals is that the spectra of aperiodic signals are defined for all real values of the frequency variable ω.

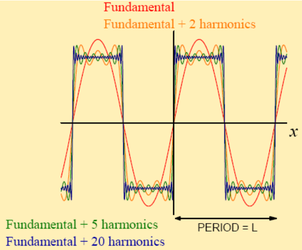


Figure 13: Square Wave from Fourier Series

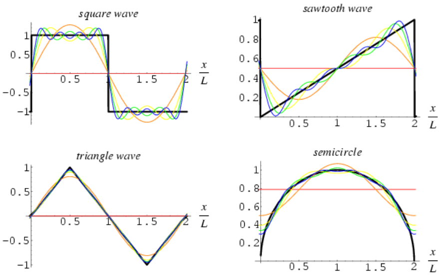


Figure 14: Waveforms from Fourier Series

- Fourier analysis: Every composite periodic signal can be represented with a series of sine and cosine functions with different frequencies, phases, and amplitudes.
- The functions are integral harmonics of the fundamental frequency **f** of the composite signal.
- Using the series we can decompose any periodic signal into its harmonics.

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n cos(n\theta) + \sum_{n=1}^{\infty} b_n sin(n\theta)$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

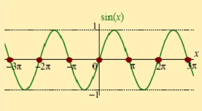
$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) cos(n\theta) d\theta$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) sin(n\theta) d\theta$$

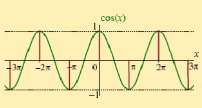
Line spectra, harmonics

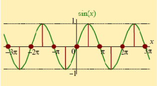
- The fundamental frequency $f_0 = 1/T$. The Fourier series coefficients plotted as a function of n or nf_0 is called a Fourier spectrum.

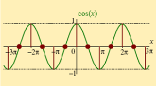
- $\sin n\pi = 0$



- $\cos n\pi = (-1)^n$



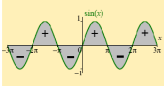




- $\sin n\frac{\pi}{2} = \begin{cases} 0 & , n \text{ even} \\ 1 & , n = 1, 5, 9, ... \\ -1 & , n = 3, 7, 11, ... \end{cases}$
- $\cos n\frac{\pi}{2} = \begin{cases} 0 & , n \text{ odd} \\ 1 & , n = 0, 4, 8, ... \\ -1 & , n = 2, 6, 10, .. \end{cases}$

Areas cancel when
when integrating
over whole periods

- $\int_{2\pi} \sin nx \, dx = 0$
- $\int_{2\pi} \cos nx \, dx = 0$



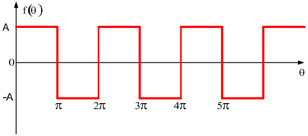


Figure 15: Square Wave

$$f(\theta) = \begin{cases} A & when \quad 0 < \theta < \pi \\ -A & when \quad \pi < \theta < 2\pi \end{cases}$$

$$f(\theta + 2\pi) = f(\theta)$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta \\ &= \frac{1}{2\pi}\left[\int_0^{\pi} f(\theta)d\theta + \int_{\pi}^{2\pi} f(\theta)d\theta\right] \\ &= \frac{1}{2\pi}\left[\int_0^{\pi} Ad\theta + \int_{\pi}^{2\pi} (-A)d\theta\right] = 0 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta \\ &= \frac{1}{\pi}\left[\int_0^{\pi} A\cos n\theta d\theta + \int_{\pi}^{2\pi} (-A)\cos n\theta d\theta\right] \\ &= \frac{1}{\pi}\left[-A\frac{\sin n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[A\frac{\sin n\theta}{n}\right]_{\pi}^{2\pi} = 0 \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta \\
&= \frac{1}{\pi}\left[\int_0^{\pi} A\sin n\theta d\theta + \int_{\pi}^{2\pi}(-A)\sin n\theta d\theta\right] \\
&= \frac{1}{\pi}\left[-A\frac{\cos n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[A\frac{\cos n\theta}{n}\right]_{\pi}^{2\pi} \\
&= \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right] \\
&= \frac{A}{n\pi}\left[1+1+1+1\right] \\
&= \frac{4A}{n\pi} \quad \text{when n is odd}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right] \\
&= \frac{A}{n\pi}\left[-1+1+1-1\right] \\
&= 0 \quad when \quad n \quad is \quad even
\end{aligned}
$$

$$
\frac{4A}{\pi}\left(\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \frac{1}{7}\sin 7\theta + \cdots\right)
$$

$$\frac{4A}{\pi}\left(\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \frac{1}{7}\sin 7\theta + \cdots\right)$$

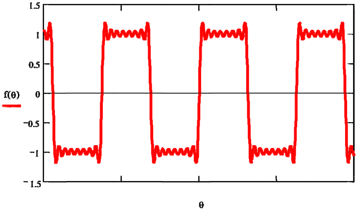


Figure 16: Square Wave from Fourier Series

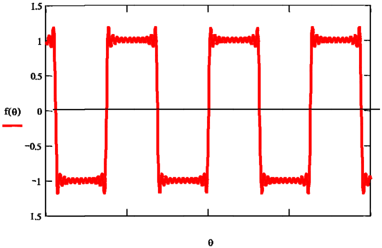


Figure 17: Square Wave from Fourier Series

```
clc;clear all; close all;
f=100;%Fundamental frequency 100 Hz
t=0:.00001:.05;
xsin = sin(2*pi*f*t);
x1 = sin(2*pi*f*t);
x3 = (1/3)*sin(3*2*pi*f*t);
x5 = (1/5)*sin(5*2*pi*f*t);
x7 = (1/7)*sin(7*2*pi*f*t);
x=x1+x3+x5+x7;
subplot(2,1,1)
plot(t,xsin,'linewidth',2);
xlabel('\theta','fontsize',16)
ylabel('sin(\theta)','fontsize',16)
title('Fundamental sinusoidal signal')
subplot(2,1,2)
plot(t,x,'linewidth',2);
xlabel('\theta','fontsize',16)
ylabel('f (\theta)','fontsize',16)
title('Reconstructed square wave by Fourier ')
```

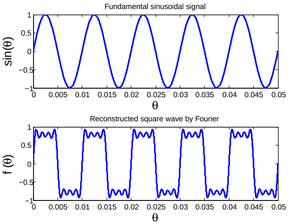


Figure 18: Square Wave

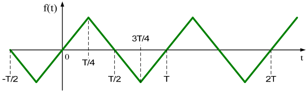


Figure 19: Triangular Wave

$$f(t) = \begin{cases} t & when \quad -\frac{T}{4} \le t \le \frac{T}{4} \\ -t + \frac{T}{2} & when \quad \frac{T}{4} \le t \le \frac{3T}{4} \end{cases}$$

$$\begin{aligned} bn &= \frac{2}{T}\int_0^T f(t)\sin\left(\tfrac{2\pi n}{T}t\right)dt \\ &= \frac{4}{T}\int_0^{T/2} f(t)\sin\left(\tfrac{2\pi n}{T}t\right)dt \\ &= \frac{4}{T}\int_0^{T/4} t\sin\left(\tfrac{2\pi n}{T}t\right)dt + \frac{4}{T}\int_0^{T/4}\left(-t+\frac{T}{2}\right)\sin\left(\tfrac{2\pi n}{T}t\right)dt \\ &= \frac{4}{T}\left[2\left(\frac{T}{2\pi n}\right)^2 \sin\left(\tfrac{\pi n}{2}\right)\right] \\ &= \frac{2T}{2\pi^2 n^2}\sin\left(\tfrac{\pi n}{2}\right) \\ &= 0 \quad when \quad n \quad is \quad even \end{aligned}$$

$$\frac{2T}{\pi^2}\left[\sin\left(\frac{2\pi}{T}t\right) - \frac{1}{3^2}\sin\left(\frac{6\pi}{T}t\right) + \frac{1}{5^2}\sin\left(\frac{10\pi}{T}t\right) - \cdots\right]$$

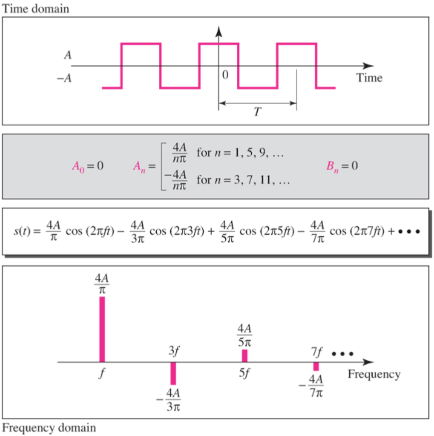


Figure 20: Square Wave[6]

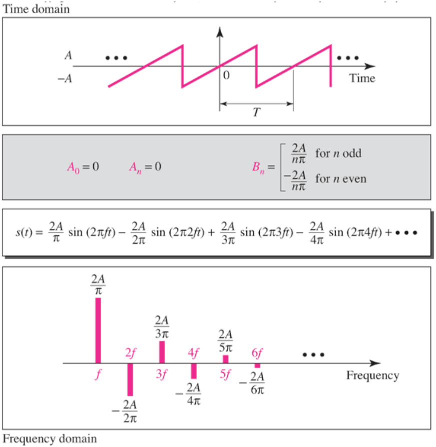


Figure 21: Sawtooth Signal[6]

The Exponential (Complex) Form of Fourier Series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$a_n \cos n\theta + b_n \sin n\theta =$$

$$\begin{aligned}
&= a_n \frac{e^{jn\theta} + e^{-jn\theta}}{2} + b_n \frac{e^{jn\theta} - e^{-jn\theta}}{2j} \\
&= a_n \frac{e^{jn\theta} + e^{-jn\theta}}{2} - jb_n \frac{e^{jn\theta} - e^{-jn\theta}}{2} \\
&= \left(\frac{a_n - jb_n}{2}\right) e^{jn\theta} + \left(\frac{a_n + jb_n}{2}\right) e^{-jn\theta}
\end{aligned}$$

$$let \quad c_n = \left(\frac{a_n - jb_n}{2}\right) \qquad c_{-n} = \left(\frac{a_n + jb_n}{2}\right)$$

$$a_n \cos n\theta + b_n \sin n\theta = c_n e^{jn\theta} + c_{-n} e^{-jn\theta}$$

- In exponential Fourier series only one integral has to be calculated and it is simpler integration.

$$\begin{aligned} f(\theta) &= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{jn\theta} + c_{-n} e^{-jn\theta} \right) \\ &= \sum_{n=-\infty}^{\infty} c_n e^{jn\theta} \end{aligned}$$

where

$$c_n = \left(\frac{a_n - jb_n}{2} \right)$$

The coefficient c_n can be evaluated as.

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta - \frac{j}{2\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) (\cos n\theta - j \sin n\theta) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-jn\theta} d\theta \end{aligned}$$

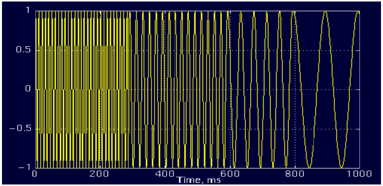


Figure 22: Square Wave[6]

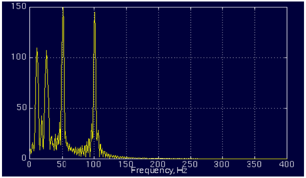
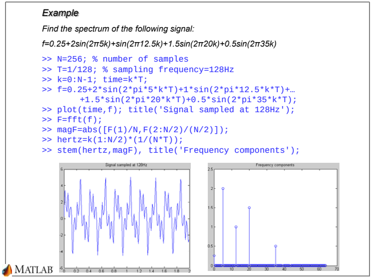


Figure 23: Sawtooth Signal[6]



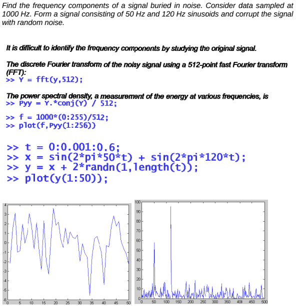
Example

Find the spectrum of the following signal:

f=0.25+2sin(2π5k)+sin(2π12.5k)+1.5sin(2π20k)+0.5sin(2π35k)

```
>> N=256; % number of samples
>> T=1/128; % sampling frequency=128Hz
>> k=0:N-1; time=k*T;
>> f=0.25+2*sin(2*pi*5*k*T)+1*sin(2*pi*12.5*k*T)+…
      +1.5*sin(2*pi*20*k*T)+0.5*sin(2*pi*35*k*T);
>> plot(time,f); title('Signal sampled at 128Hz');
>> F=fft(f);
>> magF=abs([F(1)/N,F(2:N/2)/(N/2)]);
>> hertz=k(1:N/2)*(1/(N*T));
>> stem(hertz,magF), title('Frequency components');
```


Figure 24: Square Wave[6]



Find the frequency components of a signal buried in noise. Consider data sampled at 1000 Hz. Form a signal consisting of 50 Hz and 120 Hz sinusoids and corrupt the signal with random noise.

It is difficult to identify the frequency components by studying the original signal.

The discrete Fourier transform of the noisy signal using a 512-point fast Fourier transform (FFT):

```
>> Y = fft(y,512);
```

The power spectral density, a measurement of the energy at various frequencies, is

```
>> Pyy = Y.*conj(Y) / 512;

>> f = 1000*(0:255)/512;
>> plot(f,Pyy(1:256))

>> t = 0:0.001:0.6;
>> x = sin(2*pi*50*t) + sin(2*pi*120*t);
>> y = x + 2*randn(1,length(t));
>> plot(y(1:50));
```

Figure 25: Sawtooth Signal[6]

 Fourier series

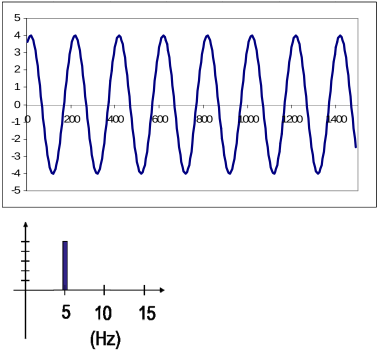


Figure 26: Square Wave[6]

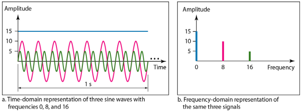


Figure 27: Sawtooth Signal[6]

Fourier Transform

- The Fourier transform is a generalization of the Fourier series representation of functions. The Fourier series is limited to periodic functions, while the Fourier transform can be used for a larger class of functions which are not necessarily periodic. S
- Sinusoidal functions are wide applications in Engineering and they are easy to generate.
- Fourier has shown that periodic signals can be represented by series of sinusoids with different frequency.
- A signal $f(t)$ is said to be periodic of period T if $f(t) = f(t + T)$ for all t.
- Periodic signals can be represented by the Fourier series and non periodic signals can be represented by the Fourier transform.
- For example square wave pattern can be approximated with a suitable sum of a fundamental sine wave plus a combination of harmonics of this fundamental frequency.
- Several waveforms that are represented by sinusoids are as shown in Figure 14. This sum is called a Fourier series.
- The major difference with respect to the line spectra of periodic signals is that the spectra of aperiodic signals are defined for all real values of the frequency variable ω.

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-jn\theta}d\theta$$

$$\theta = \omega t$$

ω is the angular velocity in radians per second.

$$\omega = 2\pi f \quad and \quad \theta = 2\pi ft$$

$$\theta = \frac{2\pi}{T}t \quad and \quad d\theta = \frac{2\pi}{T}dt$$

when $\theta = -\pi$

$$-\pi = \frac{2\pi t}{T} \Rightarrow \quad t = -\frac{T}{2}$$

when $\theta = \pi$

$$\pi = \frac{2\pi t}{T} \Rightarrow \quad t = \frac{T}{2}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega t}dt$$

Relationship from Fourier series to Fourier Transform

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega t}dt$$

As T approaches infinity
ω approaches zero
and n becomes meaningless
$n\omega \Rightarrow \omega \quad \omega \Rightarrow \Delta\omega$
$T \Rightarrow \frac{2\pi}{\Delta\omega}$

$$f(t) = \sum_{n=-\omega}^{\omega} c_\omega e^{j\omega t}$$

$$c_\omega = \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)e^{-j\omega t}dt$$

$$f(t) = \frac{1}{2\pi}\left[\sum_{\omega=-\infty}^{\infty}\int_{-T/2}^{T/2} f(t)e^{-j\omega t}dt\right] e^{j\omega t}\Delta\omega$$

$T \Rightarrow \infty$ $\Delta\omega \Rightarrow d\omega$ and $\sum \Rightarrow \int$

$$f(t) = \frac{1}{2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\mathrm{e}^{(-j\omega t)}dt\right]\mathrm{e}^{(j\omega t)}d\omega$$

$$f(t) = \frac{1}{2}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{(j\omega t)}d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{(-j\omega t)}dt$$

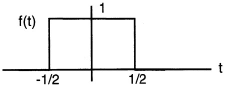


Figure 28: Rectangular Pulse

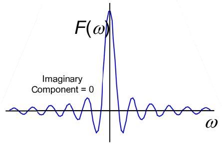


Figure 29: Sinc Function

$$
\begin{aligned}
F(\omega) &= \int_{-1/2}^{1/2} \exp(-j\omega t)dt = \frac{1}{-j\omega}[\exp(-j\omega t)]_{-1/2}^{1/2} \\
&= \frac{1}{-j\omega}[\exp(-j\omega/2) - \exp(j\omega/2)] \\
&= \frac{1}{(\omega/2)} \frac{\exp(j\omega/2) - \exp(-j\omega/2)}{2j} \\
&= \frac{\sin(\omega/2)}{(\omega/2)} \\
&= sinc(\omega/2) \qquad since\ it\ is\ \frac{sinx}{x}\ form
\end{aligned}
$$

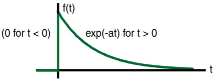


Figure 30: Exponential

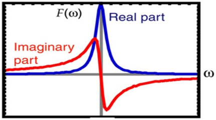
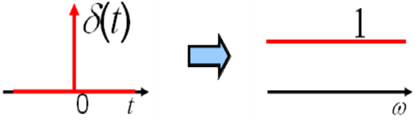
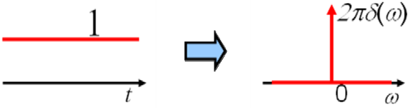


Figure 31: Gaussian

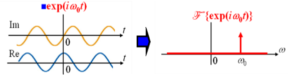
$$\begin{aligned}
F(\omega) &= \int_0^{\infty} \exp(-at)\exp(-j\omega t)dt \\
&= \int_0^{\infty} \exp(-at - j\omega t)dt = \int_0^{\infty} \exp(-[a+j\omega]t)dt \\
&= \frac{-1}{a+j\omega}\exp(-[a+j\omega]t)|_0^{+\infty} = \frac{-1}{a+j\omega}[\exp(-\infty) - \exp(0)] \\
&= \frac{-1}{a+j\omega}[0-1] \\
&= \frac{1}{a+j\omega}
\end{aligned}$$



$$\int_{-\infty}^{\infty} \delta(t)\ \exp(-i\omega\, t)\ dt = \exp(-i\omega\ [0]) = 1$$



$$\int_{-\infty}^{\infty} 1\ \exp(-i\omega\, t)\ dt = 2\pi\ \delta(\omega)$$

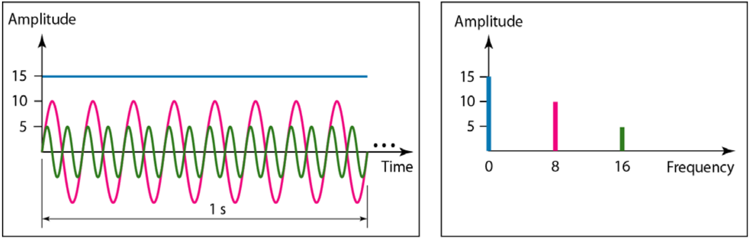


$$F\{\exp(i\omega_0\, t)\} = \int_{-\infty}^{\infty} \exp(i\omega_0\, t)\; \exp(-i\,\omega\, t)\; dt$$

$$= \int_{-\infty}^{\infty} \exp(-i\,[\omega - \omega_0]\; t)\; dt$$

$$= 2\pi\; \delta(\omega - \omega_0)$$

$$
\begin{aligned}
F\left\{\cos(\omega_0 t)\right\} &= \int_{-\infty}^{\infty} \cos(\omega_0 t)\ \exp(-j\,\omega\, t)\ dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left[\exp(j\,\omega_0\ t)\ +\ \exp(-j\,\omega_0\ t)\right]\ \exp(-j\,\omega\ t)\ dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \exp(-j\left[\omega-\omega_0\right] t)\ dt \quad + \quad \frac{1}{2}\int_{-\infty}^{\infty} \exp(-j\left[\omega+\omega_0\right] t)\ dt \\
&= \pi\ \delta(\omega-\omega_0) \quad + \quad \pi\ \delta(\omega+\omega_0)
\end{aligned}
$$



a. Time-domain representation of three sine waves with frequencies 0, 8, and 16

b. Frequency-domain representation of the same three signals

Figure 32: A signal with four different frequencies

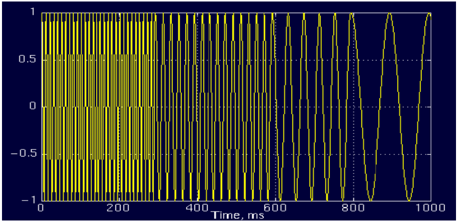


Figure 33: A signal with four different frequency components at four different time intervals

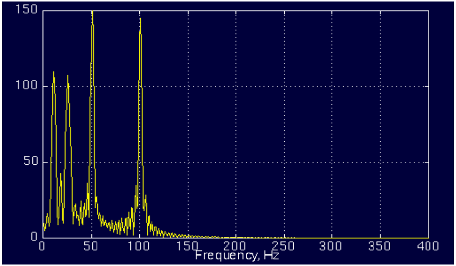


Figure 34: Each peak corresponds to a frequency of a periodic component

Discrete Fourier Transform (DFT)

- Many applications demand the processing of signals in frequency domain.
- The analysis of signal frequency, periodicity, energy and power spectrums can be analyzed in frequency domain.
- Frequency analysis of discrete time signals is usually and most conveniently performed on a digital signal processor.

Applications of DFT:

- Spectral analysis
- Convolution of signals
- Partial differential equations
- Multiplication of large integers
- Data compression

Fourier Series is

$$x(\theta) = a_0 + \sum_{n=1}^{\infty} a_n cos(n\theta) + \sum_{n=1}^{\infty} b_n sin(n\theta)$$

where $a_0 = \frac{1}{2\pi} \int\limits_0^{2\pi} x(\theta) d\theta$

$$a_n = \frac{1}{\pi} \int\limits_0^{2\pi} x(\theta) cos(n\theta) d\theta \quad b_n = \frac{1}{\pi} \int\limits_0^{2\pi} x(\theta) sin(n\theta) d\theta$$

The Exponential (Complex) Form

$$x(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta} \quad where \;\; c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\theta) e^{-jn\theta} d\theta$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \quad where \;\; c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn\omega t} dt$$

Fourier Transform pair is

$$X(\omega) = \int_{-\infty}^{\infty} f(t) e^{(-\omega t)} dt \quad and \quad x(t) = \frac{1}{2} \int_{-\infty}^{\infty} X(\omega) e^{(j\omega t)} d\omega$$

Time Domain	Frequency Domain	Transform
Continuous Periodic	Discrete nonperiodic	Fourier series
Continuous nonperiodic	Continuous nonperiodic	Fourier Transform
Discrete nonperiodic	Continuous nonperiodic	Sequences Fourier Transform
Discrete periodic	Discrete periodic	Discrete Fourier Transform

The Fourier Series for Continuous time Periodic Signals

Synthesis Equation	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi nft}$
Analysis Equation	$c_n = \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-j2\pi nft} dt$

The Fourier Transform for Continuous Time Aperiodic Signals

Synthesis Equation (Inverse transform)	$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$
Analysis Equation (Direct transform)	$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$

The Fourier Series for Discrete time Periodic Signals

Synthesis Equation	$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$
Analysis Equation	$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

The Fourier Transform of Discrete Time Aperiodic Signals

Synthesis Equation (Inverse transform)	$x(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(F) e^{j2\pi F_1 t} dF$
Analysis Equation (Direct transform)	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}$

DFT transforms the time domain signal samples to the frequency domain components.

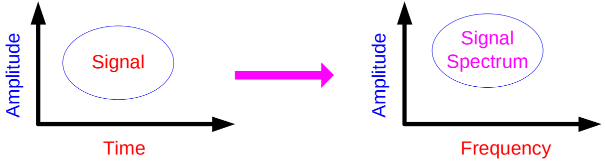


Figure 35: Discrete Fourier Transform

Signal	Types of Transforms	Example Waveform
Continuous and periodic	Fourier Series	sine wave
Continuous and aperiodic	Fourier Series	
Discrete and periodic	Fourier Series	
Discrete and aperiodic	Fourier Series	

Need For Frequency Domain Sampling

- In practical application, signal processed by computer has two main characteristics: It should be Discrete and Finite length
- But nonperiodic sequences Fourier Transform is a continuous function of ω, and it is a periodic function in ω with a period 2π.
- So it is not suitable to solve practical digital signal processing.
- Frequency analysis on a discrete-time signal $x(n)$ is achieved by converting time domain sequence to an equivalent frequency domain representation, which is represented by the Fourier transform $X(\omega)$ of the sequence $x(n)$.
- Consider an aperiodic discrete time signal x(n) and its Fourier transform is

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

- The Fourier transform $X(\omega)$ is a continuous function of frequency and it is not a computationally convenient representation of the sequence.
- To overcome the processing, the spectrum of the signal $X(\omega)$ is sampled periodically in frequency at a spacing of $\delta\omega$ radians between successive samples.
- The signal $X(\omega)$ is periodic with period 2π and take N equidistant samples in the interval $0 \leq \omega \leq 2\pi$ with spacing $\delta = 2\pi/N$.

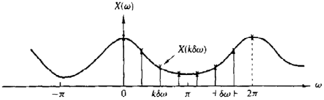


Figure 36: Frequency domain sampling

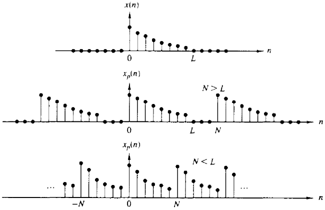


Figure 37: Frequency domain sampling

To Determine The Value Of N

- Now consider $\omega = 2\pi k/N$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \dots N-1$$

$$\begin{aligned}
X\left(\frac{2\pi}{N}k\right) &= \cdots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\
&\quad + \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \cdots \\
&= \sum_{n=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j2\pi kn/N}
\end{aligned}$$

- By changing the index in the inner summation from n to $n - lN$ and interchanging the order of summation

$$\begin{aligned}
X\left(\frac{2\pi}{N}k\right) &= \sum_{n=0}^{N-1}\left[\sum_{n=-\infty}^{\infty} x(n-lN)\right] e^{-j\frac{2\pi}{N}k(n-lN)} \\
&= \sum_{n=0}^{N-1}\left[\sum_{n=-\infty}^{\infty} x(n-lN)\right] e^{-j\frac{2\pi}{N}kn} e^{-j2\pi kl}
\end{aligned}$$

$e^{-j2\pi kl} = 1$ $\because$ both k and l integers

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{n=-\infty}^{\infty} x(n-lN)\right] e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots N-1$$

$$Let \qquad x_p(n) = \sum_{n=-\infty}^{\infty} x(n-lN)$$

- The term $x_p(n)$ is obtained by the periodic repetition of $x(n)$ every N samples hence it is a periodic signal. This can be expanded by Fourier series as

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad n = 0, 1, \cdots N-1$$

- where c_k is the fourier coefficients expressed as

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N} \qquad k = 0, 1, \cdots N-1$$

- Upon comparing

$$c_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right) \qquad k = 0, 1, \cdots N-1$$

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N} \quad n = 0, 1, \cdots N-1$$

- $x_p(n)$ is the reconstruction of the periodic signal from the spectrum $X(\omega)$ (IDFT).
- The equally spaced frequency samples $X\left(\frac{2\pi}{N}\right) \quad k = 0, 1, \cdots N-1$ do not uniquely represent the original sequence when $x(n)$ has infinite duration. When $x(n)$ has a finite duration then $x_p(n)$ is a periodic repetition of $x(n)$ and $x_p(n)$ over a single period is

$$x_p(n) = \begin{cases} x(n) & 0 \le n \le L-1 \\ 0 & L \le n \le N-1 \end{cases}$$

- For the finite duration sequence of length L the Fourier transform is:

$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n} \quad 0 \le \omega \le 2\pi$$

- When $X(\omega)$ is sampled at frequencies $\omega_k = 2\pi k/N \quad k = 0, 1, 2, \ldots N-1$ then

$$X(k) = X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x(n)e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

- The upper index in the sum has been increased from $L-1$ to $N-1$ since x(n)=0 for $n \ge L$

DFT and IDFT expressions are

- DFT expressions is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad k = 0, 1, \ldots N-1$$

- IDFT expressions is

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N} k\right) e^{j2\pi kn/N} \quad n = 0, 1, \cdots N-1$$

- If $x_p(n)$ is evaluated for $n = 0, 1, 2 \ldots N-1$ then $x_p(n) = x(n)$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad n = 0, 1, \cdots N-1$$

DFT as a Linear Transformation

$$X(k) \quad = \quad \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \qquad k = 0, 1, \ldots N-1$$

- Let

$$W_N = e^{-j\frac{2\pi}{N}} \qquad\qquad \textit{is called twiddle factor}$$

$$X(k) \quad = \quad \sum_{x=0}^{N-1} x(n) W_N^{nk} \quad \textit{for} \quad k = 0, 1.., N-1$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & W & W^2 & W^3 & \ldots & W^{N-1} \\ 1 & W^2 & W^2 & W^4 & \ldots & W^{2(N-1)} \\ 1 & W^3 & W^6 & W^9 & \ldots & W^{3(N-1)} \\ \vdots & & & & & \\ 1 & W^{N-1} & W^{N-2} & W^{N-3} & \ldots & W^{(N-1)(N-1)} \end{bmatrix} . \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Periodicity property of W_N

- $W_N = e^{-j\frac{2\pi}{N}}$
- Let us consider for N=8
- $W_8 = e^{-j\frac{2\pi}{8}1} = e^{-j\frac{\pi}{4}}$

kn	$W_8^{kn} = e^{-\frac{\pi}{4}kn}$	Result
0	$W_8^0 = e^0$	Magnitude 1 Phase 0
1	$W_8^1 = e^{-j\frac{\pi}{4}1} = e^{-j\frac{\pi}{4}}$	Magnitude 1 Phase $-\pi/4$
2	$W_8^2 = e^{-j\frac{\pi}{4}2} = e^{-j\frac{\pi}{2}}$	Magnitude 1 Phase $-\pi/2$
3	$W_8^3 = e^{-j\frac{\pi}{4}3} = e^{-j3\frac{\pi}{4}}$	Magnitude 1 Phase $-3\frac{\pi}{4}$
4	$W_8^4 = e^{-j\frac{\pi}{4}4} = e^{-j\pi}$	Magnitude 1 Phase $-\pi$
5	$W_8^5 = e^{-j\frac{\pi}{4}5} = e^{-j3\frac{\pi}{5}}$	Magnitude 1 Phase $-5\pi/4$
6	$W_8^6 = e^{-j\frac{\pi}{4}6} = e^{-j3\frac{\pi}{2}}$	Magnitude 1 Phase $-3\pi/2$
7	$W_8^7 = e^{-j\frac{\pi}{4}7} = e^{-j7\frac{\pi}{4}}$	Magnitude 1 Phase $-7\pi/4$
8	$W_8^8 = e^{-j\frac{\pi}{4}8} = e^{-j2\pi}$	Magnitude 1 Phase -2π $W_8^8 = W_8^0$
9	$W_8^9 = e^{-j\frac{\pi}{4}9} = e^{-j(2\pi+\frac{\pi}{4})}$	Magnitude 1 Phase $(-2\pi+\pi/4)$ $W_8^9 = W_8^1$
10	$W_8^{10} = e^{-j\frac{\pi}{4}10} = e^{-j(2\pi\frac{\pi}{2})}$	Magnitude 1 Phase $(-2\pi+\pi/2)$ $W_8^{10} = W_8^2$
11	$W_8^{11} = e^{-j\frac{\pi}{4}11} = e^{-j2\pi+\frac{3\pi}{4}}$	$W_8^{11} = W_8^3$

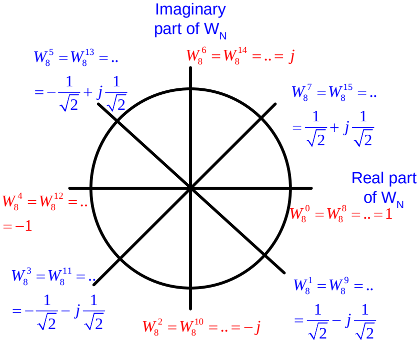


Figure 38: Periodicity of W_N and its values

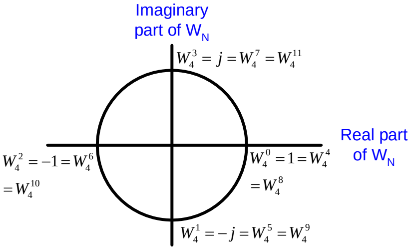


Figure 39: Periodicity of W_N and its values

Find Discrete Fourier Transform (DFT) of x(n) = [2 3 4 4]
Solution:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk} \quad for \quad k = 0, 1.., N-1$$

$$e^{-j\frac{\pi}{2}} = cos\frac{\pi}{2} - jsin\frac{\pi}{2} = -j \quad e^{-j\pi} = cos(\pi) - jsin(\pi) = -1$$

$$e^{-j\frac{3\pi}{2}} = cos\frac{3\pi}{2} - jsin\frac{3\pi}{2} = j \quad e^{-j2\pi} = cos(2\pi) - jsin2(\pi) = 1$$

for k=0,1,2,3

$$X(0) = \sum_{n=0}^{3} x(n)e^{0} = \left[2e^0 + 3e^0 + 4e^0 + 4e^0\right] = [2+3+4+4] = 13$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\frac{2\pi n}{4}} = \left[2e^0 + 3e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2}\right] = [2-3j-4+4j] = [-2+j]$$

$$X(2) = \sum_{n=0}^{3} x(n)e^{\frac{-j4\pi n}{4}} = \left[2e^0 + 3e^{-j\pi} + 4e^{-j2\pi} + 4e^{-j3\pi}\right] = [2-3+4-4] = [-1-0j] = -1$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{\frac{-j6\pi n}{4}} = \left[2e^0 + 3e^{-j3\pi/2} + 4e^{-j3\pi} + 4e^{-j9\pi/2}\right] = [2+3j-4-4j][-2-j]$$

The DFT of the sequence x(n) = [2 3 4 4] is [13, -2+j, -1, -2-j]

- Find Discrete Fourier Transform (DFT) of x(n) = [2 3 4 4]

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk} \quad for \quad k = 0, 1.., N-1$$

- Matlab code for the DFT equation is:

```
clc; clear all; close all
xn=[2 3 4 4]
N=length(xn);
n=0:N-1;
k=0:N-1;
WN=exp(-1j*2*pi/N);
nk=n'*k;
WNnk=WN.^nk;
Xk=xn*WNnk
```

- Matlab code using FFT command

```
clc; clear all; close all
xn=[2 3 4 4]
y=fft(xn)
```

Find DFT for a given a sequence $x(n)$ for $0 \leq n \leq 3$ where $x(0) = 1, x(1) = 2, x(2) = 3, x(3) = 4$

Solution:

x(n) = [1 2 3 4]

for k=0,1,2,3

$$X(0) = \sum_{n=0}^{3} x(n)e^{0} = \left[4e^{0} + 2e^{0} + 3e^{0} + 4e^{0}\right] = [1+2+3+4] = 10$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\frac{2\pi n}{4}} = \left[1e^{0} + 2e^{-j\pi/2} + 2e^{-j\pi} + 4e^{-j3\pi/2}\right] = [1 - j2 - 3 + j4] = [-2 + j2]$$

$$X(2) = \sum_{n=0}^{3} x(n)e^{\frac{-j4\pi n}{4}} = \left[1e^{0} + 2e^{-j\pi} + 3e^{-j2\pi} + 4e^{-j3\pi}\right] = [1-2+3-4] = [-1-0j] = -2$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{\frac{-j6\pi n}{4}} = \left[1e^{0} + 2e^{-j3\pi/2} + 3e^{-j3\pi} + 4e^{-j9\pi/2}\right] = [1 + 2j - 3 - 4j][-2 - j2]$$

The DFT of the sequence x(n) = [1 2 3 4] is $[10, \ -2+j2, \ -2, \ -2-j2]$

Find 8 point DFT for a given a sequence $x(n) = [1, 1, 1, 1]$ assume imaginary part is zero. Also calculate magnitude and phase

Solution:
x(n) = [1 1 1 1]
The 8 point DFT is of length 8. Append zeros at the end of the sequence. x(n) = [1 1 1 1 0 0 0 0]

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ 1 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ 1 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ 1 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ 1 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ 1 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ 1 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix}$$

$$
\begin{aligned}
W_8^0 &= W_8^8 = W_8^{16} = W_8^{24} = W_8^{40} \ldots = 1 \\
W_8^1 &= W_8^9 = W_8^{17} = W_8^{25} = W_8^{33} \ldots = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\
W_8^2 &= W_8^{10} = W_8^{18} = W_8^{26} = W_8^{34} \ldots = -j \\
W_8^3 &= W_8^{11} = W_8^{19} = W_8^{27} = W_8^{35} \ldots = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\
W_8^4 &= W_8^{12} = W_8^{20} = W_8^{28} = W_8^{36} \ldots = -1 \\
W_8^5 &= W_8^{13} = W_8^{21} = W_8^{29} = W_8^{37} \ldots = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
W_8^6 &= W_8^{14} = W_8^{22} = W_8^{30} = W_8^{38} \ldots = j \\
W_8^7 &= W_8^{15} = W_8^{23} = W_8^{31} = W_8^{39} \ldots = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}
\end{aligned}
$$

$$
\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\
1 & j & -1 & -j & 1 & j & -1 & -j \\
1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}
\end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 4 \\ 1 - j(1+\sqrt{2}) \\ 0 \\ 1 + j(1-\sqrt{2}) \\ 0 \\ 1 - j(1-\sqrt{2}) \\ 0 \\ 1 + j(1+\sqrt{2}) \end{bmatrix}
=
\begin{bmatrix} X_R(0) \\ X_R(1) \\ X_R(2) \\ X_R(3) \\ X_R(4) \\ X_R(5) \\ X_R(6) \\ X_R(7) \end{bmatrix}
\begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\ \textit{and}\
\begin{bmatrix} X_I(0) \\ X_I(1) \\ X_I(2) \\ X_I(3) \\ X_I(4) \\ X_I(5) \\ X_I(6) \\ X_I(7) \end{bmatrix}
\begin{bmatrix} 0 \\ -(1+\sqrt{2}) \\ 0 \\ (1-\sqrt{2}) \\ 0 \\ -(1-\sqrt{2}) \\ 0 \\ (1+\sqrt{2}) \end{bmatrix}
$$

Find DFT for a given a sequence $x(n) = [2\ 3\ 4\ 4]$

Solution:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & W^4 & W^6 \\ 1 & W^3 & W^6 & W^9 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$

- $W_N = e^{-\frac{2\pi}{N}} = e^{-\frac{2\pi}{4}} = e^{-\frac{\pi}{2}} = -j$
- $W^2 = -j^2 = -1,\ W^3 = -j^3 = j$
- Using the property of periodicity of W $W^p = W^{P+r.N} = j$ with basic period $N = 4$
- $W^4 = W^{4-4} = W^0 = 1,\ W^6 = W^{6-4} = W^2 = -1,\ W^9 = W^{9-2.4} = W^1 = -j$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ -2+j \\ -1 \\ -2-j \end{bmatrix}$$

- The DFT of the sequence x(n) = [2 3 4 4] is $[13,\ -2+j,\ -1,\ -2-j]$

Find DFT for a given a sequence $x(n) = [1\ 2\ 3\ 4]$

Solution:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & W^4 & W^6 \\ 1 & W^3 & W^6 & W^9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- $W_N = e^{-\frac{2\pi}{N}} = e^{-\frac{2\pi}{4}} = e^{-\frac{\pi}{2}} = -j$
- $W^2 = -j^2 = -1,\ W^3 = -j^3 = j$
- Using the property of periodicity of W $W^p = W^{P+r.N} = j$ with basic period $N = 4$
- $W^4 = W^{4-4} = W^0 = 1,\ W^6 = W^{6-4} = W^2 = -1,\ W^9 = W^{9-2.4} = W^1 = -j$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

- The DFT of the sequence x(n) = [1 2 3 4] is $[10,\ -2+j2,\ -2,\ -2-j2]$

Inverse DFT: Find the IDFT for $X(k) = [10, \ -2+j2, \ -2, \ -2-j2]$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{\frac{2\pi}{N}kn} \quad for \quad n = 0, 1.., N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W^{*kn} \quad for \quad n = 0, 1.., N-1 \quad where \quad W^{*} = e^{\frac{2\pi}{N}}$$

$$\begin{aligned} x(0) &= \frac{1}{4}\sum_{k=0}^{N-1} X(k)e^{j0} = X(0)e^{j0} + X(1)e^{j0} + X(2)e^{j0} + X(3)e^{j0} \\ &= \frac{1}{4}(10 + (-2+j2) - 2 + (-2-j2)) = 1 \end{aligned}$$

$$\begin{aligned} x(1) &= \frac{1}{4}\sum_{k=0}^{N-1} X(k)e^{j\frac{k\pi}{2}} = X(0)e^{j0} + X(1)e^{j\frac{\pi}{2}} + X(2)e^{j\pi} + X(3)e^{j\frac{3\pi}{2}} \\ &= \frac{1}{4}(X(0) + jX(1) - X(2) - jX(3) \\ &= \frac{1}{4}(10 + j(-2+j2) - (-2) - j(-2-j2)) = 2 \end{aligned}$$

$$\begin{aligned}
x(2) &= \frac{1}{4}\sum_{k=0}^{N-1} X(k)e^{j\frac{k\pi}{2}} = X(0)e^{j0} + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi} \\
&= \frac{1}{4}(X(0) - X(1) + X(2) - X(3) \\
&= \frac{1}{4}(10 - (-2 + j2) + (-2) - (-2 - j2)) = 3
\end{aligned}$$

$$\begin{aligned}
x(1) &= \frac{1}{4}\sum_{k=0}^{N-1} X(k)e^{j\frac{k\pi 3}{2}} = X(0)e^{j0} + X(1)e^{j\frac{3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{j\frac{9\pi}{2}} \\
&= \frac{1}{4}(X(0) - jX(1) - X(2) + jX(3) \\
&= \frac{1}{4}(10 - j(-2 + j2) - (-2) + j(-2 - j2)) = 4
\end{aligned}$$

Matlab command used to calculate the Inverse DFT is *ifft*

```
x=[10 -2+j2 -2 -2-j2]
y=ifft(x)
```

Find the Discrete Fourier Transform of the following signal: x(n), n = 0,1,2,3 = [1, 1, -1, -1].

Solution:

N=4 The matrix notation is

$$X = T.f$$

where T is matrix of the transform with elements $T_{kn} = W_N^{kn}\ k, n = 0, 1.., N-1$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2-2j \\ 0 \\ 2+2j \end{bmatrix}$$

The DFT of the sequence x(n) = [1 1 -1 -1] is $[0,\ 2-j2,\ 0,\ 2+j2]$

Find the Inverse Discrete Fourier Transform of the following signal: x(n), n = 0,1,2,3 = [0, 2-2j, 0, 2+2j].

Solution:

The IDFT of the discrete signal X(k)is x(n):

$N = 4$ and $W_4 = e^{-\pi/2}$

$$x(n) \quad = \quad \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad for \quad n = 0, 1.., N-1 \quad where \quad W = e^{-\frac{2\pi}{N}}$$

N=4 The matrix notation is

$$[W_N] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad [W_N^*] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 0 \\ 2-2j \\ 0 \\ 2+2j \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

The IDFT of the sequence $X(k) = [0,\ 2-j2,\ 0,\ 2+j2]$ is [1 1 -1 -1]

Find DFT for a given a sequence x[0]=1, x[1]=2, x[2]=2, x[3]=1, x[n]=0 otherwise: x = [1,2,2,1]

Solution:
x(n) = [1 2 2 1]

for k=0,1,2,3

$$X(0) = \sum_{n=0}^{3} x(n)e^{0} = \left[1e^{0} + 2e^{0} + 2e^{0} + 1e^{0}\right] = [1+2+2+1] = 6$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\frac{2\pi n}{4}} = \left[1e^{0} + 2e^{-j\pi/2} + 2e^{-j\pi} + 1e^{-j3\pi/2}\right] = [1 - j2 - 2 + j1] = [-1 - j1]$$

$$X(2) = \sum_{n=0}^{3} x(n)e^{\frac{-j4\pi n}{4}} = \left[1e^{0} + 2e^{-j\pi} + 2e^{-j2\pi} + 1e^{-j3\pi}\right] = [1 - 2 + 2 - 1] = [0] = 0$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{\frac{-j6\pi n}{4}} = \left[1e^{0} + 2e^{-j3\pi/2} + 2e^{-j3\pi} + 1e^{-j9\pi/2}\right] = [1 + 2j - 2 - 1j][-1 + j1]$$

The DFT of the sequence x(n) = [1 2 2 1] is $[6, \ -1-j1, \ 0, \ -1+j1]$

Find IDFT for a given a sequence X[0]=6, X[1]=-1-j1, X[2]=0, X[3]=-1+j1, X[n]=0 otherwise:
$x = [6, \ -1-j1, \ 0, \ -1+j1]$

Solution:
$x(n) = [6, \ -1-j1, \ 0, \ -1+j1]$

for k=0,1,2,3

$$X(0) = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{0} = \left[6e^{0} + (-1-j1)e^{0} + 0e^{0} + (-1+j1)e^{0}\right] = \frac{1}{4}[6-1-j1+0-1+j1] = 1$$

$$X(1) = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{j\frac{2\pi n}{4}} = \left[6e^{0} + (-1-j1)e^{j\pi/2} + 0e^{j\pi} + (-1+j1)e^{j3\pi/2}\right] =$$
$$\frac{1}{4}[6-j+1+j+1] = [2]$$

$$X(2) = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{\frac{j4\pi n}{4}} = \left[6e^{0} + (-1-j1)e^{j\pi} + 0e^{j2\pi} + (-1+j1)e^{j3\pi}\right] =$$
$$\frac{1}{4}[6+(-1-j1)(j)+0+(-1+j)(-j)] = \frac{1}{4}[6-j1+1+0+1+j] = [2]$$

$$X(3) = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{\frac{-j6\pi n}{4}} = \left[6e^{0} + (-1-j1)e^{j3\pi/2} + 0e^{j3\pi} + (-1+j1)e^{j9\pi/2}\right] =$$
$$\frac{1}{4}[6+(-1-j1)(-j)+0+(-1+j)(j)] = \frac{1}{4}[6+j1-1+0-1-j] = [1]$$

The IDFT of the sequence $[6, \ -1-j1, \ 0, \ -1+j1]$ is $x(n) = [1 \ 2 \ 2 \ 1]$

Continuous Time Fourier Transform (CTFT)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Discrete Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Unit sample $\delta(n)$

$$x(n) = \begin{cases} 1 & for \quad n = 0 \\ 0 & for \quad n \neq 0 \end{cases}$$

$$\begin{aligned} X(k) &= \sum_{x=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk} \\ &= \sum_{x=0}^{N-1} x(0) e^{0} = 1 \times 1 = 1 \end{aligned}$$

Find the N Point DFT of $x(n) = a^n \quad for \quad 0 \leq n \leq N-1$

$$
\begin{aligned}
X(k) &= \sum_{x=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk} \\
&= \sum_{x=0}^{N-1} a^n e^{-j\frac{2\pi}{N}nk} = \sum_{x=0}^{N-1} (ae^{-j\frac{2\pi k}{N}})^n
\end{aligned}
$$

$$
X(k) = \frac{1 - a^N e^{-j2\pi k}}{1 - ae^{-j2\pi k}/N} \quad \left(Using\ series\ expansion\ \sum_{k=0}^{N-1} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a} \right)
$$

$e^{-j2\pi k} = 1$

$$
X(k) = \frac{1 - a^N}{1 - ae^{-j2\pi k}/N}
$$

$$
x[n] = (0.5)^n\, u[n] \quad 0 \leq n \leq 3
$$

$$
X(k) = \frac{1 - (0.5)^4}{1 - 0.5e^{-j2\pi k}/4} = \frac{0.9375}{1 - 0.5e^{-j\pi/2k}}
$$

Find the 4 Point DFT of $x(n) = cos(\frac{n\pi}{4})$

Solution:

$x(0) = cos(0) = 1$
$x(1) = cos(\frac{1\pi}{4}) = 0.707$
$x(2) = cos(\frac{2\pi}{4}) = 0$
$x(3) = cos(\frac{3\pi}{4}) = -0.707$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0.707 \\ 0 \\ -0.707 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - j1.414 \\ 1 \\ 1 + j1.414 \end{bmatrix}$$

If the length of $x[n]$ is N=4, and if its 8-point DFT is: $X_8[k]$
$k = 0..7 = [5, 3-\sqrt{2}j, 3, 1-\sqrt{2}j, 1, 1+\sqrt{2}j, 3, 3+\sqrt{2}j]$, find the 4 point DFT of the signal $x[n]$.

Solution:

- The samples of $X_s[k]$ are eight equally spaced samples from the frequency spectrum of the signal $x[n]$:
- More precisely, they are samples from the spectrum for the following frequencies: $\omega T = \left[0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\right]$
- With 4-point DFT of x[n] we get 4 samples from the spectrum of x[n]: $\omega T = \left[0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right]$
- By comparing the two sets of frequencies:
  $X_4[0] = X(e^{j0}) = X_s[0] = 5$
  $X_4[1] = X(e^{j\pi/2}) = X_s[2] = 3$
  $X_4[2] = X(e^{j\pi}) = X_s[4] = 1$
  $X_4[3] = X(e^{j3\pi/2}) = X_s[6] = 3$

Find the Fourier Transform of the sequence

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 4 \\ 0 & else \end{cases}$$

$$X\left(e^{jw}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

$$X\left(e^{j\omega}\right) = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

Consider a causal sequence x[n] where;

$$x[n] = (0.5)^n\, u[n]$$

Its DTFT $X\left(e^{jw}\right)$ can be obtained as

$$X\left(e^{jw}\right) = \sum_{n=-\infty}^{\infty} (0.5)^n\, u[n] e^{-jwn} = \sum_{n=0}^{\infty} (0.5)^n\, (1) e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \left(0.5e^{jw}\right)^n = \frac{1}{1 - 0.5e^{-jw}}$$

Find the N Point DFT of $x(n) = 4 + cos^2(\frac{2\pi n}{N})$

Solution:
$x(0) = 4 + cos^2(0) = 5$
$x(1) = 4 + cos^2(\frac{2\pi 1}{10}) = 4.6545$
$x(2) = 4 + cos^2(\frac{2\pi 2}{10}) = 4.09549$
$x(3) = 4 + cos^2(\frac{2\pi 3}{10}) = 4.09549$
$x(4) = 4 + cos^2(\frac{2\pi 4}{10}) = 4.09549$
$x(5) = 4 + cos^2(\frac{2\pi 5}{10}) = 5$
$x(6) = 4 + cos^2(\frac{2\pi 6}{10}) = 4.6545$
$x(7) = 4 + cos^2(\frac{2\pi 7}{10}) = 4.09549$
$x(8) = 4 + cos^2(\frac{2\pi 8}{10}) = 4.09549$
$x(9) = 4 + cos^2(\frac{2\pi 9}{10}) = 4.6545$
$x(n) = x(N - n)$
Cosine function is even function
$x(n) = x(-n)$

$$X(k) = \sum_{n=0}^{N-1} x(n) cos\left(\frac{2\pi kn}{N}\right) \qquad 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{N-1} \left[4 + cos^2(\frac{2\pi n}{N})\right] cos\left(\frac{2\pi kn}{N}\right) \qquad 0 \le k \le N-1$$

Relationship of the DFT to other Transforms

Relationship to the Fourier series coefficients of periodic sequence

- DFT expression is

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad k = 0, 1, \ldots N-1 \tag{1}$$

- IDFT expression is

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \quad n = 0, 1, \cdots N-1 \tag{2}$$

- Fourier series is

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}nk} \quad -\infty \leq n \leq \infty \tag{3}$$

- Fourier series coefficients are expressed as:

$$c_k = \frac{1}{N}\sum_{k=0}^{N-1} x_p(n)e^{-j\frac{2\pi}{N}nk} \quad k = 0, 1.., N-1 \tag{4}$$

- By comparing $X(k)$ and c_k fourier series coefficients has the form of a DFT. $x(n) = x_p(n) \quad 0 \leq n \leq N-1$
- $X(k) = Nc_k \quad 0 \leq n \leq N-1$
- Fourier series has the form of an IDFT

Relationship to the Fourier transform of an aperiodic sequence (DFT and DTFT)

- Fourier transform $X(\omega)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)^{-j\omega n} \quad -\infty \leq n \leq \infty \tag{5}$$

$$X(k) = X(\omega|_{\omega=2\pi k/N}) = \sum_{n=-\infty}^{\infty} x(n)^{-j\frac{2\pi}{N}nk} \quad -\infty \leq n \leq \infty \tag{6}$$

- DFT coefficients are expressed as:

$$x_p(n) = \sum_{n=-\infty}^{\infty} x(n - lN) \tag{7}$$

- $x_p(n)$ is determined by aliasing x(n) over the interval $0 \leq n \leq N-1$. The finite duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n) & 0 \leq n \leq N-1 \\ 0 & Otherwise \end{cases}$$

- The relation between $\hat{x}(n)$ *and* $x(n)$ exist when x(n) is of finite duration
- $x(n) = \hat{x}(n) \quad 0 \leq n \leq N-1$

Relationship to the Z Transform

- Z transform of the sequence x(n) is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{8}$$

Sample X(z) at N equally spaced points on the unit circle. These points will be

$$Z_k = e^{j2\pi k/N} \quad k = 0, 1, \cdots N-1 \tag{9}$$

$$X(z)|z_k = e^{j2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N} \tag{10}$$

- If x(n) is causal and has N number of samples then

$$X(z)|z_k = e^{j2\pi k/N} = \sum_{n=0}^{\infty} x(n) e^{-j2\pi kn/N} \tag{11}$$

- This is equivalent to DFT X(k)

$$X(k) = X(z)|z_k = e^{j2\pi k/N} \tag{12}$$

Parseval's Theorem

- Consider a sequence x(n) and y(n)

$$x(n) \overset{DFT}{\leftrightarrow} X(k)$$

$$y(n) \overset{DFT}{\leftrightarrow} Y(k)$$

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k) \tag{13}$$

- When x(n)=y(n)

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2 \tag{14}$$

- This equation give the energy of finite duration sequence it terms of its frequency components

Determine the DFT of the sequence for N=8, $h(n) = \begin{cases} \frac{1}{2} & -2 \leq n \leq 2 \\ 0 & otherwise \end{cases}$

Plot the magnitude and phase response for N=8

Solution:

$$h(n) = \left\{ \frac{1}{2}, \frac{1}{2}, \underset{\uparrow}{\frac{1}{2}}, \frac{1}{2}, \frac{1}{2} \right\}$$

Consider a sequence x(n) and its DFT is

$$x(n) \overset{DFT}{\leftrightarrow} X(k)$$

$$x_p(n) \overset{DFT}{\leftrightarrow} X(k)$$

where $x_p(n)$ is the periodic sequence of x(n) in this example x(n) is of h(n) and is of

$$h(n) = \left\{ \frac{1}{2}, \frac{1}{2}, \underset{\uparrow}{\frac{1}{2}}, \frac{1}{2}, \frac{1}{2} \right\}$$

There are 5 samples in h(n) append 3 zeros to the right side of the sequence h(n)

$$h(n) = \left\{ \frac{1}{2}, \frac{1}{2}, \underset{\uparrow}{\frac{1}{2}}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0 \right\}$$

The DFT of $h(n)$ and $h_p(n)$ is $h(n) \overset{DFT}{\leftrightarrow} H(k)$ $\quad h_p(n) \overset{DFT}{\leftrightarrow} H(k)$

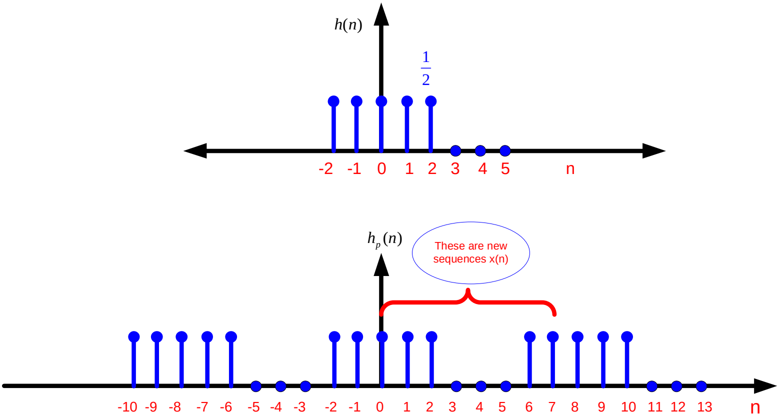


Figure 40: Plot of $h(n)$ and $h_p(n)$

- The value of h(n) from the Figure is represented as

$$h(n) = \begin{cases} h_p(n) & 0 \leq n \leq N-1 \\ 0 & Otherwise \end{cases}$$

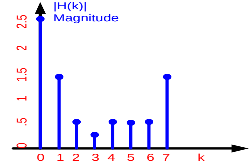
The new sequence $h(n)$ from $h_p(n)$ is

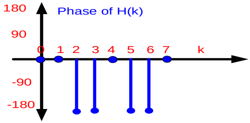
$$h(n) = \left\{ \underset{\uparrow}{\frac{1}{2}}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, \frac{1}{2} \right\}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ 1 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ 1 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ 1 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ 1 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ 1 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ 1 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix}$$

 Problems and Solutions on DFT

$$
\begin{aligned}
W_8^0 &= W_8^8 = W_8^{16} = W_8^{24} = W_8^{40} \ldots = 1\\
W_8^1 &= W_8^9 = W_8^{17} = W_8^{25} = W_8^{33} \ldots = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\\
W_8^2 &= W_8^{10} = W_8^{18} = W_8^{26} = W_8^{34} \ldots = -j\\
W_8^3 &= W_8^{11} = W_8^{19} = W_8^{27} = W_8^{35} \ldots = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\\
W_8^4 &= W_8^{12} = W_8^{20} = W_8^{28} = W_8^{36} \ldots = -1\\
W_8^5 &= W_8^{13} = W_8^{21} = W_8^{29} = W_8^{37} \ldots = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\\
W_8^6 &= W_8^{14} = W_8^{22} = W_8^{30} = W_8^{38} \ldots = j\\
W_8^7 &= W_8^{15} = W_8^{23} = W_8^{31} = W_8^{39} \ldots = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}
\end{aligned}
$$





$$
\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\\ X(4)\\ X(5)\\ X(6)\\ X(7) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\\
1 & -j & -1 & j & 1 & -j & -1 & j\\
1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\\
1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\\
1 & j & -1 & -j & 1 & j & -1 & -j\\
1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}
\end{bmatrix}
=
\begin{bmatrix} \frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ 0\\ 0\\ 0\\ \frac{1}{2}\\ \frac{1}{2} \end{bmatrix}
$$

$$
\begin{bmatrix} 2.5\\ 1.207\\ -0.5\\ -0.207\\ 0.5\\ -0.207\\ -0.5\\ 1.207 \end{bmatrix}
=
\begin{bmatrix} 2.5\angle 0\\ 1.207\angle 0\\ 0.5\angle -180\\ 0.207\angle -180\\ 0.5\angle 0\\ 0.207\angle -180\\ 0.5\angle -180\\ 1.207\angle 0 \end{bmatrix}
$$

The unit sample response of the first order recursive filter is given as $h(n) = a^n u(n)$

i) Determine the Fourier transform $H(\omega)$

ii) DFT H(k) of h(n)

iii) Relationship between $H(\omega)$ and $H(k)$

$$\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} a^n u(n)e^{-j\omega n} \\ &= \sum_{n=0}^{\infty}(ae^{-j\omega})^n \qquad \because u(n) = 0 \;\; for \;\; n < 0 \end{aligned}$$

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & for \;\; a = 1 \\ \frac{1-a^N}{1-a} & for \;\; a \neq 1 \end{cases}$$

$$H(\omega) = \frac{(ae^{-j\omega})^0 - (ae^{-j\omega})^{\infty+1}}{1 - ae^{-j\omega}} = \frac{1}{1 - ae^{-j\omega}}$$

- The DFT of $h(n)$ and $h_p(n)$ is

$$h(n) \overset{DFT}{\leftrightarrow} H(k) \qquad h_p(n) \overset{DFT}{\leftrightarrow} H(k)$$

- where $h_p(n)$ is related as

$$h_p(n) = \sum_{l=-\infty}^{\infty} h(n - lN)$$

- Consider l=-p

$$h_p(n) = \sum_{l=\infty}^{-\infty} h(n + pN) \qquad h_p(n) = \sum_{l=-\infty}^{\infty} h(n + pN)$$

- N point DFT H(k) in terms of $h_p(n)$ is

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-j\frac{2\pi}{N}kn}$$

$$H(k) = \sum_{n=0}^{N-1} \left[\sum_{n=-\infty}^{\infty} h(n + pN) \right] e^{-j\frac{2\pi}{N}kn}$$

$$
\begin{aligned}
H(k) &= \sum_{n=0}^{N-1}\left[\sum_{n=-\infty}^{\infty} a^{(n+pN)}u(n+pN)\right]e^{-j\frac{2\pi}{N}kn}\\
&= \sum_{n=0}^{N-1}\left[\sum_{n=0}^{\infty} a^{(n+pN)}\right]e^{-j\frac{2\pi}{N}kn} \quad \because u(n)=0 \ \ for \ \ n<0\\
&= \sum_{n=0}^{N-1}\left[\sum_{n=0}^{\infty} a^{n}a^{pN}\right]e^{-j\frac{2\pi}{N}kn}
\end{aligned}
$$

Interchanging the summations

$$H(k)=\sum_{n=0}^{\infty}a^{pN}\sum_{n=0}^{N-1}a^{n}e^{-j\frac{2\pi}{N}kn}$$

$$\sum_{k=0}^{N}a^{k}=\frac{1-a^{N}}{1-a}$$

$$\sum_{p=0}^{\infty}a^{pN}=\sum_{p=0}^{\infty}a^{N^{p}}=\frac{1-(a^{N})^{\infty+1}}{1-a^{N}}=\frac{1}{1-a^{N}}$$

$$\sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} (ae^{-j\frac{2\pi}{N}k})^n = \frac{(ae^{-j2\pi k/N})^0 - (ae^{-j2\pi k/N})^N}{1-(ae^{-j2\pi k/N})}$$

$$\sum_{n=0}^{N-1} (ae^{-j\frac{2\pi}{N}k})^n = \frac{1-a^N e^{-j2\pi k}}{1-(ae^{-j2\pi k/N})} = \frac{1-a^N}{1-(ae^{-j2\pi k/N})} \quad \because e^{-j2\pi k} = 1$$

$$\begin{aligned} H(k) &= \frac{1}{1-a^N}\frac{1-a^N}{1-(ae^{-j2\pi k/N})} \\ &= \frac{1}{1-(ae^{-j2\pi k/N})} \end{aligned}$$

$$H(\omega) = = \frac{1}{1-ae^{-j\omega}} \quad and \quad H(k) = \frac{1}{1-(ae^{-j2\pi k/N})}$$

$$H(k) = H(\omega)|_{\omega=2\pi k/N}$$

Compute the DFT of the following finite length sequence of length N $x(n) = u(n) - u(n-N)$

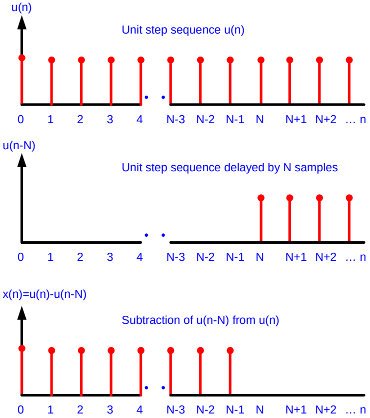


Figure 41: Generation of $x(n) = u(n) - u(n-N)$

- The value of x(n) as shown in Figure is represented as

$$x(n) = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & Otherwise \end{cases}$$

- DFT expression is

$$
\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad k = 0, 1, \ldots N-1 \\
&= \sum_{n=0}^{N-1} 1e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} (e^{-j2\pi k/N})^n \quad (1) \quad \left[\sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a}\right]
\end{aligned}
$$

$$X(k) = \frac{1-e^{-j2\pi k}}{e^{-j2\pi k/N}} = \frac{1-1}{e^{-j2\pi k/N}} = 0$$

When k=0 From the expression (1)

$$X(k) = \sum_{n=0}^{N-1} (1)^n = N$$

$$X(k) = \begin{cases} 0 & when \quad k \neq 0 \\ N & when \quad k = 0 \end{cases}$$

$$X(k) = N\delta(k)$$

 Problems and Solutions on DFT

If x(n)=[1,2,0,3,-2, 4,7,5] evaluate the following i) X(0) ii) X(4) iii) $\sum_{k=0}^{7} X(k)$ iv) $\sum_{k=0}^{7} |X(k)|^2$

- X(0) is

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

- with k=0 and N=8

$$X(0) = \sum_{n=0}^{N-1} x(n) = 1 + 2 + 0 + 3 - 2 + 4 + 7 + 5 = 20$$

- X(4) is

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

- with k=4 and N=8

$$X(4) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi 4n/8} = \sum_{n=0}^{N-1} x(n)e^{-j\pi n} = \sum_{n=0}^{N-1} x(n)(-1)^n$$

$$X(4) == 1 - 2 + 0 - 3 - 2 - 4 + 7 - 5 = -8$$

- iii) $\sum_{k=0}^{7} X(k)$
- We Know the IDFT expression as

$$x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k) e^{j2\pi kn/N}$$

- With n=0 and N=8 it becomes

$$x(0) = \frac{1}{8} \sum_{n=0}^{N-1} X(k)$$

$$\therefore \quad \sum_{n=0}^{N-1} X(k) = 8x(0) = 8 \times 1 = 8$$

 Problems and Solutions on DFT

- The value of $\sum_{k=0}^{7} |X(k)|^2$ is
- The expression for Parseval's theorem is

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

- $\sum_{k=0}^{7} |X(k)|^2$is related as

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

- N=8 Then

$$\sum_{n=0}^{7} |x(n)|^2 = \frac{1}{8} \sum_{k=0}^{7} |X(k)|^2$$

$$\sum_{k=0}^{7} |X(k)|^2 = 8 \sum_{n=0}^{7} |x(n)|^2 = 8[1 + 4 + 0 + 9 - 4 + 4 + 49 + 25] = 864$$

Thank You

References

J. G. Proakis and D. G. Monalakis, *Digital signal processing Principles Algorithms & Applications*, 4th ed. Pearson education, 2007.

Oppenheim and Schaffer, *Discrete Time Signal Processing*. Pearson education, Prentice Hall, 2003.

S. K. Mitra, *Digital Signal Processing*. Tata Mc-Graw Hill, 2004.

L. Tan, *Digital Signal Processing*. Elsivier publications, 2007.

J. S. Chitode, *Digital signal processing*. Technical Pulications.

B. Forouzan, *Data Communication and Networking*, 4th ed. McGraw-Hill, 2006.