System of Linear Equations

A linear equation in the variables $x_1 \dots x_n$ is an equation written in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where b and co-efficients $a_1 \dots a_n$ are real or complex No.

A <u>system of linear equations (or linear system)</u> is a collection of one or more linear equations involving the same variables - say $x_1 \dots x_n$

A <u>solution of the system</u> is a list $(s_1, s_2 \dots s_n)$ of number that makes each equation a true statement when the values of $s_1, s_2 \dots s_n$ are substituted for $x_1 \dots x_n$ respectively.

The set of all possible solutions is called the <u>solution set</u> of the linear system. Two linear s/ms are equivalent if they have the same solution set.

<u>Ex:-</u> Find the solution set

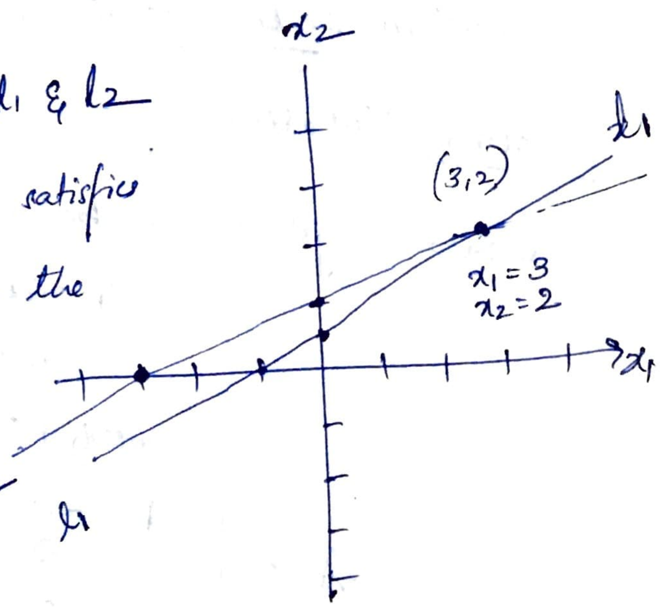
$$+x_1 - 2x_2 = -1$$

$$-x_1 + 3x_2 = 3$$

The graph of these lines are l_1 & l_2

The pair of which satisfies both the equations is the solution for given linear

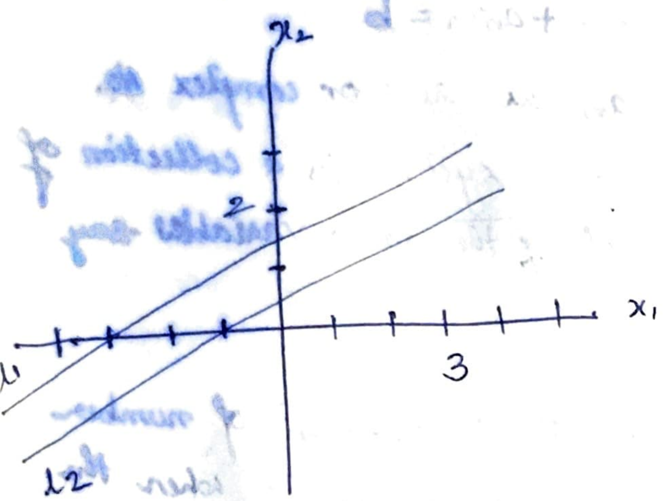
$(3, 2) \rightarrow$ Solution



Two lines need not intersect in a single point. they could be parallel they could co-incide and hence intersect at every point on line.

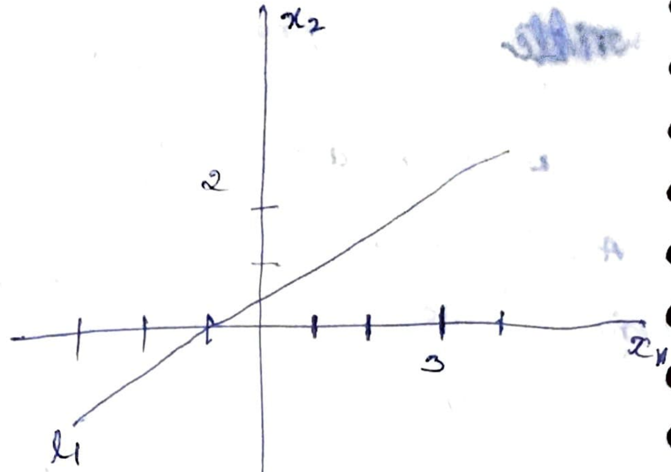
(a) $x_1 - 2x_2 = -1$

$-x_1 + 2x_2 = 3$



(b) $x_1 - 2x_2 = -1$

$-x_1 + 2x_2 = 1$



A system of linear equations has either

no solutions or

exactly one solutions or

infinitely many solutions. or

A system of linear equations is said to be consistent if it has either one solution or infinitely ~~small~~ many solutions, a system is inconsistent if it has no solution.

Matrix Notation

Linear system can be represented in a matrix

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_2 - 8x_3 = 8$$

$$-4x_1 + 5x_2 + 9x_3 = -9$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$ co-efficient matrix.

Any ~~wat~~ set of linear equations can be reduced in the form of $\boxed{Ax = b}$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

[Coefficient matrix ; b] = augmented matrix of the sm

$$\left[\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

An augmented matrix consists of the co-efficient matrix with an added column containing the constants from the right side of the equations.

Solving a Linear System (Gauss Jordan Ellimination)

The basic strategy is to replace one system with an equivalent system (i.e one with the same solution set) that is easier to solve.

Ex:-

$$x_1 - 2x_2 + x_3 = 0 \quad (1)$$
$$2x_2 - 8x_3 = 8 \quad (2)$$
$$-4x_1 + 5x_2 + 9x_3 = -9 \quad (3)$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

$4 \times (1) + (3)$

$$4x_1 - 8x_2 + 4x_3 = 0$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$
$$\overline{-3x_2 + 13x_3 = -9}$$

new eqn (3')

$$\longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$\frac{eqn②}{2}$ $\quad 2x_2 - 8x_3 = 8 \Rightarrow x_2 - 4x_3 = 4$

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$-3x_2 + 13x_3 = -9$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

x_2 in eqn ② is used to elliminate x_2 in eqn ③

eqn ③ + 3[eqn ②] $\Rightarrow$

$$\cancel{3x_2} - 12x_3 = 12$$
$$-\cancel{3x_2} + 13x_3 = -9$$

$$\boxed{x_3 = 3}$$

∴ The new s/m of equation has a triangular form

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$x_3 = 3$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

4 eqn ③ + eqn 2

$$4x_3 = 12$$
$$x_2 - 4x_3 = 4$$
$$x_2 = 16$$

eqn ① − eqn ③

$$x_1 - 2x_2 + \cancel{x_3} = 0$$
$$-\cancel{x_3} = -3$$
$$x_1 - 2x_2 = -3$$

∴

$$x_1 - 2x_2 = -3$$
$$x_2 = 16$$
$$x_3 = 3$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

eqn ① + 2 eqn ②

$$x_1 - 2x_2 = -3$$
$$2x_2 = 32$$
$$x_1 = 29$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

∴ (29, 16, 3) Solution to the given linear eqns

Elementary Row Operations

Row Operations on the augmented matrix. The basic operations in augmented matrix are.

1) Replacement : Replace one row by the sum of itself & a multiple of another row
2) Interchange :- Interchange 2 rows
3) Scaling :- Multiply all the elements in a row by a non-zero constant.

→ Row operations can be applied to any matrix.

→ Two matrices are row equivalent if there is a sequence of elementary row operations that transform one matrix into another.

→ These row operations are reversible.

→ If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

if $\rho(A) = \rho(A:B)$ then s/m is consistent

$\rho(A) \neq \rho(A:B) \rightarrow$ inconsistent

Existence and Uniqueness

. A soln set for a linear s/m consists
- no soln
- one soln
- infinitely small soln.

. If soln exists then s/m is consistent

Determine if the following system is consistent

(1).

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

If the s/m of eqns ie augmented matrix can be reduced to triangular form the system is consistent.

$x_1 - 2x_2 + x_3 = 0$
$x_2 - 4x_3 = 4$
$x_3 = 3$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$x_3 = 3$ has unique soln

from $x_3 \Rightarrow x_2$ can be obtained

|||ly x_1 can be determined uniquely by eqn ①
∴ the soln is unique.

2) $x_2 - 4x_3 = 8$
$2x_1 - 3x_2 + 2x_3 = 1$
$5x_1 - 8x_2 + 7x_3 = 1$

Soln Augmented matrix

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Since we have to reduce matrix in triangular form & to obtain x_1 in eqn ①.

Interchange R_1 & R_2

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$R_3 - \frac{5}{2}R_1$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

$R_3 + \frac{R_2}{2}$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

matrix is in triangular form

$2x_1 - 3x_2 + 2x_3 = 1$
$x_2 - 4x_3 = 8$
$0 = 5/2 \Rightarrow x_1(0) + x_2(0) + x_3(0) = 5/2$

which is not true

hence the s/m is inconsistent (no soln)

Describe the solution set of the original s/m

(1) $\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$

(3)

$x_2 + 4x_3 = -5$

$x_1 + 3x_2 + 5x_3 = -2$

$3x_1 + 7x_2 + 7x_3 = 6$

$\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix}$

$R_1 \longleftrightarrow R_2$

$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix}$

$R_3 + 2R_2$

$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$(0)x_1 + (0)x_2 + (0)x_3 = 2$

$0 \neq 2$ ∴ The solution set is empty.

s/m is inconsistent.

Find an equation involving g, h and k that makes this augmented matrix respond to a consistent system.

$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \xrightarrow{R_3 + 2R_1} \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix}$

$R_3 + R_2$

$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$

If $k + 2g + h = 0$ the soln exists.

Row Reduction and Echelon's form $\Longrightarrow$ Gaussian Elimination

Non zero row or column in a matrix means a row or column that contains at least one nonzero entry.

A **leading entry** of a row refers to the leftmost non-zero entry

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties

1) All non zero rows are above any rows of all zeros
2) Each leading entry of a row in a column to the right of the leading entry of the row above it.
3) All the entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the conditions then it is reduced echelon form (or reduced row echelon form)

1) The leading entry in each non-zero row is 1
2) Each leading 1 is the only non-zero entry in its column.

Rank of a Matrix $\rightarrow$ the maximum number of linearly independent column vectors in a matrix or the maximum number of linearly independent row vectors in the matrix.

The rank of a matrix would be zero only if the matrix had no elements. If a matrix had even one element, its maximum rank would be one,

When all the vectors in a matrix are linearly independent the matrix is said to be full rank matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 7 & 8 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\text{Rank} = 2}$$

Ex of Echelon form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \end{bmatrix}$$

Theorem 1 Uniqueness of the Reduced Echelon form.

Each matrix is row equivalent to one and only one reduced echelon matrix.

If a matrix A is row equivalent to an echelon matrix V, then V an echelon form (or row echelon form) of A.

If V is in the reduced echelon form, we ~~have~~ call V is the reduced echelon form of A

Since the reduced echelon form is unique, the leading entries ~~are~~ always in the same positions in any echelon form obtained by a matrix.

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A pivot column is a column of A that contains a pivot position.

Row Reduction Algorithm :-

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Step ~~Pivot column~~

(1) Begin with the leftmost non zero column. This is a pivot column

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

Step (2) Select a non zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

$R_1 \leftrightarrow R_3$

$$\begin{array}{l} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

Step (3) Use row replacement operations to create zeros in all positions below the pivot.

$R_2 - R_1$

$$\begin{array}{l} \text{pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 5 \\ 0 & +2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

Step 4) Ignore the row containing the pivot position & cover all rows if any above it.

Apply the steps 1-3 to the submatrix that remains, Repeat the process until there are no ~~zeros~~. more non zero rows to modify

With Row 1 is covered, column 2 is the next pivot column for step 2, we'll select as pivot the top entry in that column

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

$R_3 - \frac{3}{2}R_2$ $\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ → Row echelon form of a matrix

Pivot

Now R_2 is also covered.

~~Con~~

To convert it into reduced echelon form

Step 5 Beginning with rightmost pivot & working upward & to the left, create zero rows above each pivot. If pivot is not 1, make it one by scaling operation.

$R_1 - 6R_3$
$R_2 - 2R_3$ $\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

$R_2/2$ $\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

$R_1 + 9R_2$ $\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

$R_1/3$ $\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ Reduced row echelon form

Step 1-4 is called forward phase of row reduction algorithm

Step 5 which produces unique reduced echelon form is called the backward phase of row reduction algorithm

Solns to linear S/m

Row Reduction Algorithm leads directly to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the S/m.

Ex:- $\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$

$$x_1 - 5x_3 = 1$$
$$x_2 + x_3 = 4$$
$$0 = 0$$

x_1 & x_2 corresponding to pivot columns in the matrix are called base variables (leading variables)

$x_3 \rightarrow$ free variable.

If The S/m is consistent, then solution set can be described explicitly by solving the reduced S/m of equations for the basic variables in terms of the free variables.

$$\left.\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \\ x_3 &\text{ is free} \end{aligned}\right\} (5).$$

i.e. for different values of x_3 we get different solution set

Each different choice of x_3 determines a ~~different~~ solution of the S/m, & every soln of the S/m is determined by choice of x_3.

The solution (5) is called general soln of the S/m because it gives an explicit description of all solutions.

The description in (5) are parametric descriptions of solution sets in which the free ~~&~~ variables acts as parameters

Solving a systems amounts to finding a parametric description of the soln set or determining that the soln set is empty

<u>Prob</u> find the general soln of linear s/m with augmented matrix

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Let's define the soln to the s/m if there are free variables & the soln exists. The s/m will have infinitely many solns.

When the s/m is in echelon form & contains no equation of the form $0=b$, with b nonzero, every nonzero equation contain a <u>basic variable with a non zero co-efficient</u>.

Either the basic variables are completely determined (<u>with no free variables</u>) or at least one of the basic variables may be expressed in terms of <u>one or more free variables</u> → infinitely many solns.

→ unique soln

<u>Theorem 2 :- Existence & Uniqueness theorem</u>

A linear system is consistent if and only if the rightmost column of the augmented matrix is <u>not a pivot column</u> i.e iff an echelon form of the augmented matrix has no row of the form

$$[0 \; \ldots \; 0 \;\; b] \quad \text{with } b \text{ nonzero}$$

If a linear s/m is consistent, then the solution set contains either (i) unique with no free variables

ii) infinitely many solns, when there is atleast one free variable

Using Row Reduction to solve a linear S/m

1) Write augmented matrix of the S/m
2) Obtain echelon form of augmented matrix
3) Obtain reduced row echelon form
4) Write S/m of eqns corresponding to matrix in reduced echelon form
5) Rewrite each non-zero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Problems

1) Find the general soln of the S/m

$$x_1 - 2x_2 - x_3 + 3x_4 = 0$$
$$-2x_1 + 4x_2 + 5x_3 - 5x_4 = 3$$
$$3x_1 - 6x_2 - 6x_3 + 8x_4 = 2$$

2) $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$

3) $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row reduction technique to find the soln of S/m of linear equations is also called the Gaussian elimination method

Vector Equations :-

Vectors in R^2

A matrix with only one column is called a column vector or simply a vector.

Ex :- $u = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $v = \begin{bmatrix} .2 \\ .3 \end{bmatrix}$ $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

w_1, w_2 are any real numbers

The set of all vectors with two entries is denoted by R^2

R → real numbers that appears as entries in the vectors

Exponent 2 → that vectors each contain two entries.

→ Two vectors in R^2 are equal iff their corresponding entries are equal.

Vectors in R^2 are ordered pairs of real numbers

Given 2 vectors u and v in R^2 their sum is the vector u + v obtained by adding the corresponding entries of u and v.

Ex $u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $u + v = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Given a vector u and a real number c, the scalar multiple of u by c is a vector cv obtained by multiplying each entry in u by c.

Ex:- $u = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $c = 5$ then $cu = 5\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$

'c' is a scalar.

The operations of scalar multiplication and vector addition can be combined.

Ex:- $u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ ~~Eg~~ Find i) $4u + (-3)v$

A Column vector can also be written as $(3, -1) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

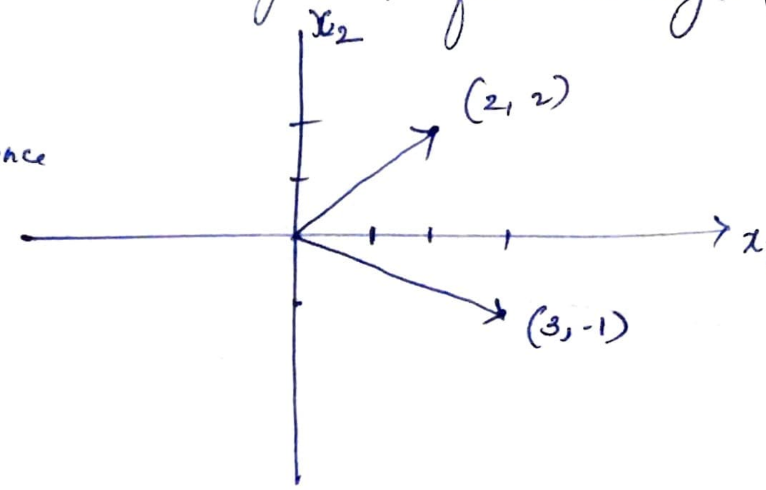
III^{ly} Row vector can be written as $[3 \ -1] \Rightarrow (3 \ -1)$

Geometric Descriptions of R^2

Consider a rectangular co-ordinate s/m in the plane.
We can identify a geometric point (a, b) with column vector $\begin{bmatrix} a \\ b \end{bmatrix}$
R^2 as the set of all points in the plane.

The geometric visualization of a vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is often aided by an arrow (directed line segment) from the origin $(0,0)$ to the pt $(3, -1)$

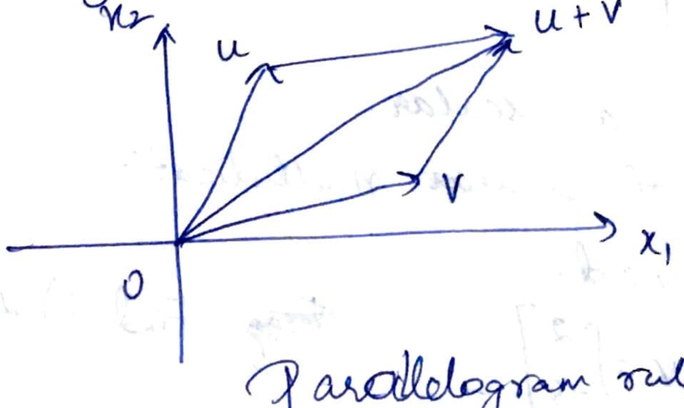
(* There is no special significance for individual points)



The sum of 2 vectors can be found in geometrical representation.

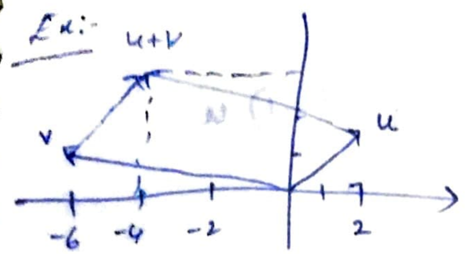
↓

Parallelogram Rule for addition → If u & v ~~are~~ in R^2 are represented as points in the plane, then $u+v$ corresponds to the fourth vertex of the parallelogram whose other vertices are $u, 0$ & v.



Parallelogram rule

Ex:-



$u = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

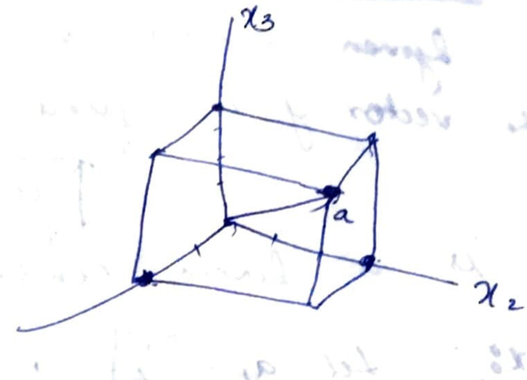
$u + v = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

The length of arrow for 'cu' is $|c|$ times the length of the arrow for u

Vectors in $\mathbb{R}^3$

↳ 3×1 column matrices with three entries. They are represented geometrically by points in three dimensional co-ordinate ~~system~~ Space, with arrows from origin.

$a = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$



Vectors in $\mathbb{R}^n$

If n is a positive integer, R^n denotes the collection ~~of~~ of all lists (or ordered n-tuples) of n real number usually written in $n \times 1$ column matrices

$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$ The vectors whose entries are all zero is called zerovector & is denoted by '0'

Equality of vector in R^n & operations of scalar multiplication & vectors addition in R^n are similar to vectors in R^2

Algebraic Properties of ~~Rn~~ R^n.

For all u, v, w in R^n and all scalars c & d.

(i) $u + v = v + u$

(ii) $(u + v) + w = u + (v + w)$

iii) $u+0 = 0+u = u$

iv) $u+(-u) = -u+u = 0$ where $(-u) = (-1)u$

v) $c(u+v) = cu+cv$

vi) $(c+d)u = cu+du$

vii) $c(du) = (cd)u$

viii) $1u = u$

Linear Combination of Vectors

Given vectors $v_1, v_2, v_3 \dots v_p$ and given scalars $c_1, c_2 \dots c_p$ the vector y is defined by

$$\boxed{y = c_1v_1 + c_2v_2 + c_3v_3 \dots c_pv_p.}$$

is a linear combination of $v_1 \dots v_p$ with weights $c_1 \dots c_p$.

Ex:- Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ Determine whether b can be generated (or written) as a linear combination of a_1 & a_2. i.e $\boxed{x_1a_1 + x_2a_2 = b}$ if the vector eqn has a soln find it.

Soln

$$x_1\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1+2x_2 \\ -2x_1+5x_2 \\ -5x_1+6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

The vectors are equal iff only if the corresponding elements are equal

$\therefore$ $x_1 + 2x_2 = 7$

$-2x_1 + 5x_2 = 4$

$-5x_1 + 6x_2 = -3$

Row Reduction of augmented matrix

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \xrightarrow[R_3+5R_1]{R_2+2R_1} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \xrightarrow{R_2/9} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix}$$

$$\xrightarrow{R_3-16R_2} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution exists. $x_1 = 3$

$x_2 = 2$

i.e $3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

The augmented matrix: $\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$

$\uparrow \quad \uparrow \quad \uparrow$

$a_1 \quad a_2 \quad b$

we can also write $[a_1 \ a_2 \ b]$

A vector equation $x_1 a_1 + x_2 a_2 + \dots x_n a_n = b$ has the same soln set as the linear sys whose augmented matrix is $[a_1 \ a_2 \ \cdots \ a_n \ b]$

b can be generated by a linear combination of $a_1, \dots a_n$ iff there exists a solution to the linear system.

Theorem 3 If A is an $m \times n$ matrix, with columns $a_1 \dots a_n$ and if b is in $\mathbb{R}^m$ then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation.

$$x_1 a_1 + x_2 a_2 + x_3 a_3 + \dots + x_n a_n = b.$$

which in turn has the same solution set as the S/m of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n & b \end{bmatrix}.$$

Solution Sets of Linear Systems

Homogeneous Linear Systems

A system of linear equations is said to be homogeneous if it can be written in the form $Ax = 0$, where A is an $m \times n$ matrix and '0' is the zero vector in $\mathbb{R}^m$.

Such a system $Ax = 0$ always has at least one solution, i.e $x = 0$ (The zero vector in $\mathbb{R}^n$).

This zero soln is usually called the trivial soln.

For a given equation $Ax = 0$, the imp question is whether there exists a non-trivial soln, i.e an non-~~zero~~ zero vector x that satisfies $Ax = 0$

The homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the equation has atleast one free variable.

Ex: (1) Determine if the following homogenous system has a non-trivial soln.

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$-3x_1 - 2x_2 + 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0.$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2+R_1} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \xrightarrow{R_3+3R_2} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2/3} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-5R_2} \begin{bmatrix} 3 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable

$$x_1 - 4/3\, x_3 = 0$$
$$x_2 = 0$$
$$0 = 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = x_3 V \qquad V = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

~~every~~ soln of $Ax=0$ → is a scalar multiple of V. The trivial soln is obtained by choosing $x_3=0$. Geometrically, the solution set is a line through 0 in R^3.

② Solve $10x_1 - 3x_2 - 2x_3 = 0$

Solutions of Non-homogeneous System:-

When a nonhomogeneous linear system has many solutions,

1) $Ax = b \qquad b \neq 0$

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \qquad b = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$A \to$ matrix of co-efficients. Row operations on $[A\ b]$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

Applications of Linear Systems

500 equations in (s/m with 500 variables)

→ A homogeneous system Economics.

→ Balancing Chemical Equations. $2H_2 + O_2 \rightarrow 2H_2O$

→ Network flow.

↳ S/ms of linear equations arise naturally when Scientists, engineers, or economists study the flow of some quantity through a network.

Inverse of Matrix

An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that $CA = I$ and $AC = I$

where $I = I_n$, the $n \times n$ identity matrix

$A^{-1}A = I$ and $AA^{-1} = I$

Theorem 4 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ If $ad - bc \neq 0$ then A is invertible &

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$ then A is not invertible

if determinant $= 0$ then matrix is singular matrix

If determinant of a matrix is not equal to zero then it is called as non-singular matrix

Theorem 5 If A is an invertible $n \times n$ matrix. Then for each b in R^n, the equation $Ax = b$ has unique soln. $x = A^{-1}b$ (14)

Proof: Let b in R^n soln exists $\because$ if $A^{-1}b$ is substituted for x, then

$$Ax = A(A^{-1}b) = (AA^{-1})b$$

$$= Ib = b$$

$\therefore$ $A^{-1}b$ is a solution.

To prove that soln is unique, show that if u is an solution then; 'u' must be equal $A^{-1}b$

if $Au = b$

x^{ly} by A^{-1}

$$A^{-1}Au = A^{-1}b$$

$$Iu = A^{-1}b$$

$$\boxed{u = A^{-1}b}$$

Ex:- Using inverse of a matrix Solve the given s/m of equation.

$$3x_1 + 4x_2 = 3$$
$$5x_1 + 6x_2 = 7.$$

Soln

$$Ax = b$$

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$x = A^{-1}b = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

m^b

(a) If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

(b) If A and B are $n \times n$ invertible matrices, then so is (AB) and the inverse of (AB) is the product of the inverses of A and B in the reverse order. i.e

$$(AB)^{-1} = B^{-1}A^{-1}$$

(c) If A is an invertible matrix, then so is A^T, & the inverse of A^T is the transpose of A^{-1}

i.e $(A^T)^{-1} = (A^{-1})^T$

$$AA^{-1} = I \Rightarrow (AA^{-1})^T = I^T = (A^{-1})^T A^T = I$$

Elementary matrices

An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If an elementary row operations is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where $m \times m$ matrix E is created by performing the same row operation on I_m.

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transform E back into I

If $\Rightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \Rightarrow E_1^{-1}$ to I_3 $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{bmatrix}$

$R_3 = R_3 + 4R_1$

Theorem 7 An $n \times n$ matrix A is invertible iff A is row equivalent to I_n and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}

Proof:- If A is invertible $Ax = b$ has a solution for each b

A has a pivot position in each row.

Since A is square, the n-pivot positions must be on the diagonal, which implies that reduced echelon form of A is I_n.

$$A \sim E_1 A \sim E_2(E_1 A) \sim \cdots E_p(E_{p-1} \cdots E_1 A) = I_n.$$

i.e $\quad E_p \cdots E_1 A = I_n$

An Algorithm for finding A^{-1} :- → ~~Gauss-Jordan Method~~

If we place A and I side by side to form an augmented matrix $[A \;\; I]$ then Row operations on this ~~product~~ matrix produce identical operations on A and on I

$$[A \;\; I] \xrightarrow{\text{Row reduced}} [I \;\; A^{-1}]$$

If A is row equivalent ? else A does not have inverse.

Problem (1) Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ if it exists.

$$[A \;\; I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$\underset{\sim}{R_3-4R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \underset{\sim}{R_3+3R_2}\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$

$\xrightarrow{R_3/2}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \underset{\sim}{R_2-2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$

R_1-3R_3

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

2) ~~Find~~ $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$ Find A^{-1} if it exists

3) Find the soln. for the s/m of linear equations using matrix inverse. (a)

$$3x - 2y + 8z = 9$$
$$-2x + 2y + z = 3$$
$$x + 2y - 3z = 8$$

(b)

$$2y + 3z = 7$$
$$3x + 6y - 12z = -3$$
$$5x - 2y + 2z = -7$$