Unit 5: Applications of LA.

Matrices in Engineering.

Linear Algebra in Engineering in 2 ways:

1) Direct way: The physical problem has only a finite no. of pieces. The laws connecting position/velocity are linear.
The laws are expressed by matrix equations.

2) Indirect way: The physical system is continous. The mass density & forces, velocities are function of x, or, x, y or x, y, z. The laws are expressed by differential eq.
To find accurate solution - approximate by finite difference eq or finite element equations.

Equilibrium eq $KU = f$

$K = A^T A$. K - stiffness matrix.

with motion, $M \frac{d^2 u}{dt^2} + KU = f$ - dynamic.

$Kx = \lambda M x$ - λ - eigen values.

Line of Springs:

Three masses m_1, m_2, m_3 connected by a line of springs.

I case - 4 springs with top & bottom fixed.

II - Fixed-free - 3 springs, the lowest mass hangs freely.

I Fixed-fixed - K_0 & $A_0^T C_0 A_0$

II. fixed-free - K_1 & $A_1^T C_1 A_1$

III. Free-free - with no support at either end - K singular

Equations for mass movements (u) & tensions (y):

$u = (u_1, u_2, u_3)$ = movements of masses

$y = (y_1, y_2, y_3, y_4)$ or (y_1, y_2, y_3) = tensions in springs

When mass moves downward, its displacement is positive ($u_i > 0$). For springs tension is positive & compression is negative ($y_i < 0$). In tension, the spring is stretched so it pulls the masses inward.

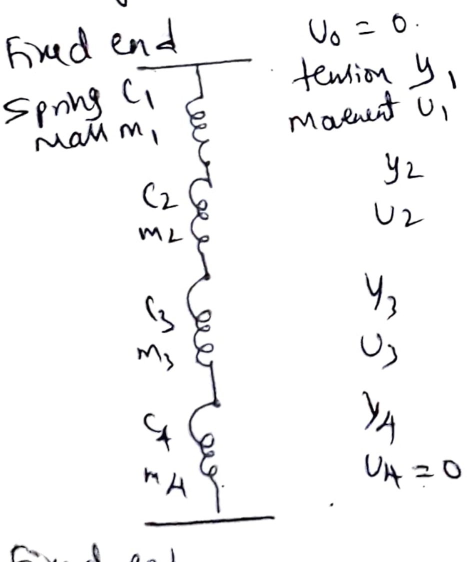
Each spring is controlled by its own Hooke's law

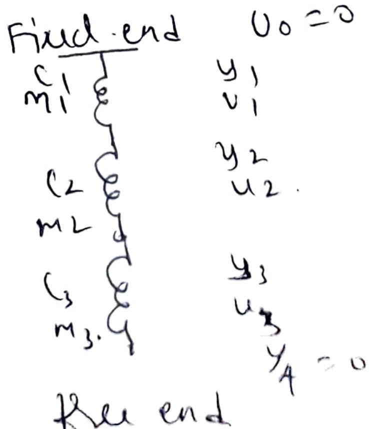
$y = c e_i$

Stretching force = spring constant times stretching distance

$y = ce$ into vector eq $Ku = f$.

The force vector 'f' comes from gravity. The gravitational constant g will multiply each mass to produce forces $f = (m_1 g, m_2 g, m_3 g)$





The matrices K_0, K_1, $K_{singular}$ & $K_{circular}$ have $C =$ [illegible] for simplicity. All spring constants are $c_i = 1$.

Stiffness matrices

$$K_0 = A_0^T A_0 = \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \qquad A_0^T C_0 A_0 = \begin{bmatrix} c_1+c_2 & -c_2 & \\ -c_2 & c_2+c_3 & -c_3 \\ & -c_3 & c_3+c_4 \end{bmatrix}$$

Fixed - fixed. Spring constant included.

$$K_1 = A_1^T A_1 = \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \qquad A_1^T C_1 A_1 = \begin{bmatrix} c_1+c_2 & -c_2 & \\ -c_2 & c_2+c_3 & -c_3 \\ & -c_3 & c_3 \end{bmatrix}$$

Fixed - free.

$$K_{singular} = \begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix}. \qquad K_{circular} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Free - free.

To find stiffness matrices.

Create K in 3 steps.

u = movement of n masses $= (u_1, \ldots, u_n)$

e = Elongations of m springs $= (e_1, \ldots, e_m)$

y = Internal forces in m springs $= (y_1, \ldots, y_m)$

f = External forces on n masses $= (f_1, \ldots, f_n)$.

$e = Au$. $\quad A_{m\times n}$.

$y = Ce \quad C_{m\times m}$

$f = A^T y \quad A^T_{n\times m}$

$$\begin{tikzcd} \boxed{u} \arrow[d, "A"] & \boxed{f} \\ \boxed{e} \arrow[r, "C"] & \boxed{y} \arrow[u, "A^T"] \end{tikzcd}$$

The elongation e is the stretching distance - how far the springs are extended. when it becomes vertical & upright, gravity acts. The masses move down by distances u_1, u_2, u_3.

Each spring is stretched & compressed by the difference in displacements $e_i = u_i - u_{i-1}$ of its ends.

Top fixed ($u_0 = 0$)

Stretching of each spring:

I spring $e_1 = u_1$

Second spring $e_2 = u_2 - u_1$

III $e_3 = u_3 - u_2$

IV $e_4 = -u_3$ ($u_4 = 0$) bottom fixed

If both ends move the same distance, that spring is not stretched: $u_i = u_{i-1}$ & $e_i = 0$.

The matrix of 4 eqs is 4 by 3 difference matrix A and $e = Au$.

Stretching distances $e = Au$ is

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (1)$$

<u>Hook's Law</u>: $y = Ce$. Connects spring elongation e with spring tension y. The constitutive law depends on the material in the spring. A soft spring has small C so moderate force y can produce a large stretching e.

Since each spring has its own law, the matrix in $y = Ce$ is a diagonal matrix C:

$y_1 = C_1 e_1$

$y_2 = C_2 e_2$

$y_3 = C_3 e_3$

$y_4 = C_4 e_4$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \quad (2)$$

Combining $e = AU$ with $y = Ce$ the spring forces are $y = CAU$.

Balance eq: The internal forces from the springs balance the external forces on the masses. Each mass is pulled/pushed by spring force y_j above it. The spring force y_{j+1} plus f_j from gravity.

$\therefore$ $y_j = y_{j+1} + f_j$ & $f_j = y_j - y_{j+1}$.

Force balance: $f = A^T y$

$f_1 = y_1 - y_2$
$f_2 = y_2 - y_3$ is $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$
$f_3 = y_4 - y_3$

Force balance eq $f = A^T y$. (3)

Nature transposes the rows and columns of e-u matrix to produce the f-y matrix. A^T appears along with A.

$KU = f$. stiffness matrix $K = A^T C A$.

$$\left(\begin{array}{l} e = Au \\ y = Ce \\ f = A^T y \end{array} \right) \text{ Combines into } A^T C A U = f \Rightarrow KU = f$$

In elasticity, $e = AU$ is the kinematic eq. The force balance $f = A^T y$ is static equation. The constitutive law is $y = Ce$.

Finite element programs spend major effort on assembling $K = A^T C A$ from thousands of smaller pieces.

If all springs are ~~identic~~ Identical, $C_1 = C_2 = C_3 = C_4$ then $C = I$.

The stiffness matrix reduces to $A^T A$.

with $C = I$

$$K_0 = A_0^T A_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The matrix A from springs is 4 by 3. The triangular matrix L from elimination is square. The stiffness matrix K is assembled from $A^T A$ & then broken up into LL^T. One step is applied mathematics, the other is computational mathematics. Each K is built from rectangular matrices and factored into squared matrices.

Properties of $K = A^T C A$:

* K is tridiagonal, because mass$_3$ is not connected to mass$_1$.

* K is symmetric: because C symmetric & A^T comes with A.

* K is positive definite, because $C_i > 0$ & A has independent columns.

* K^{-1} is a full matrix in eq 15) with all positive entries.

Last property leads to an important fact about $U = K^{-1} f$: If all forces act downwards ($f_j > 0$) then all movements are downward ($u_j > 0$). Positiveness is different from positive definiteness. Here K^{-1} is positive. Both K & K^{-1} are positive definite.

Q1: Suppose all $c_i = c$ & $m_j = m$. Find movements U & tensions y.

All springs are same & all masses are same. But all movements, elongations & tensions will not be same.

K^{-1} includes $1/c$ because A^TCA includes C:

$$U = K^{-1} f = \frac{1}{4c}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3/2 \\ 2 \\ 3/2 \end{bmatrix}$$

The displacement U_2, for the mass in the middle is greater than U_1 & U_2. The force mg divided by force per unit length c gives a length U.

$$e = AU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \frac{mg}{c}\begin{bmatrix} 3/2 \\ 2 \\ 3/2 \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3/2 \\ 1/2 \\ -1/2 \\ -3/2 \end{bmatrix}.$$

Those elongations add to zero because the ends of line are fixed. The sum $U_1 + (U_2 - U_1) + (U_3 - U_2) + (-U_3)$ is certainly zero. For each spring force $y_i \times e_i$ by c. So y_1, y_2, y_3, y_4 are $3/2\,mg$, $\frac{1}{2}mg$, $-1/2\,mg$, $-3/2\,mg$. The upper 2 springs are stretched & lower 2 springs are compressed.

<u>Fixed end and Free end:</u>

Remove 4th spring, All matrices becomes 3 by 3. The new stiffness matrix K_1

$$A_1^T C_1 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Set $C_4 = 0$.

Fixed free.

$$K_1 = A_1^T C_1 A_1 = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

If $c_1 = c_2 = c_3 = 1$ & $C = I$, this is the $-1, 2, -1$ tridiagonal matrix except the last entry 1 instead of 2. The spring at the bottom is free.

Ex 2: All $c_i = c$ & all $m_j = m$ in fixed-free hanging line of springs.

$$K_1 = c\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad K_1^{-1} = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

The force mg from gravity are the same. But the movements from the previous ex because the stiffness matrix has changed.

$$u = K_1^{-1} f = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

The movements are greater. The no. 3 appears in u_1 because all three masses are pulling the first spring down. The next mass moves by plus an additional 2 from the masses below it. The 3 mass drops by $(3+2+1=6)$

The elongations $e = Au$ in the springs display those numbers 3, 2, 1:

$$e = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

The balance eq $A^T y = f$ determines y immediately because $m = n$ & A^T is square.

$$y = (A^T)^{-1} \cdot f = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \begin{bmatrix} 3mg \\ 2mg \\ 1mg \end{bmatrix}$$

Two free ends: K is singular.

The whole line can move, as both ends are free. The matrix A is 2 by 3. short & wide, $e = Au$.

$$\begin{matrix}\text{Free}\\ \text{Free}\end{matrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_2 - u_1 \\ u_3 - u_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Non zero solution to $Au = 0$. The masses can move with no stretching of springs. The whole line can shift by $u = (1,1,1)$ & this leaves $e = (0,0)$. A has dependent columns & vector $(1,1,1)$ is in its nullspace.

$$Au = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ — No stretching}$$

$Au = 0$ certainly leads to $A^TCAu = 0$. So A^TCA is only positive semidefinite, without c_1 & c_4. The pivots will be c_2 & c_3 & no third pivot. The rank is only 2.

$$\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_2 & \\ & c_3 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

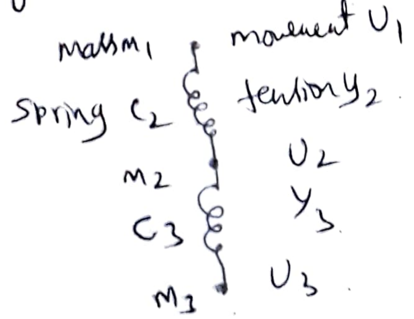
Two eigen values will be positive but $x = (1,1,1)$ is an eigen vector for $\lambda = 0$. The solve $A^TCAu = f$ only for special vector f. The forces have to add to $f_1 + f_2 + f_3 = 0$, the whole line of springs will take off like a rocket.

Circle of Springs:

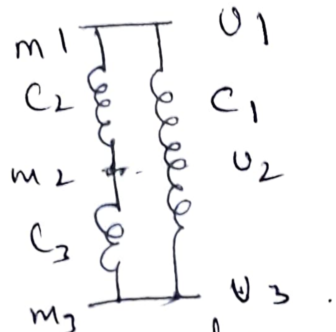
A third spring will complete circle from mass$_3$ back to mass$_1$. This doesn't make K invertible. The new matrix is still singular. The stiffness matrix $K_{circular}$ is not tri diagonal, but it is symmetric & semi definite.

$$A_c^T A_c = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

The only pivots are 2 and 3/2. The eigen values are 3, 3 & 0. The determinant is zero. The nullspace still contains $x = (1, 1, 1)$. when all masses move together.



Free - Free



Circular

The masses can move without stretching springs so $Au = 0$ has non zero solutions.

This movement vector $(1, 1, 1)$ is in nullspace of $A_{circular}$ & $K_{circular}$, even after diagonal matrix C of spring constants is included; the springs are not stretched.

$$(A^T C A)_{circular} = \begin{bmatrix} c_1 + c_2 & -c_2 & -c_1 \\ -c_2 & c_2 + c_3 & -c_3 \\ -c_1 & -c_3 & c_1 + c_3 \end{bmatrix}$$

Continuous ...

Matrix eq are discrete. Differential eq are continuous.

The matrices A & A^T corresponds to derivatives d/dx & $-d/dx$.

$e = AU$ took difference $U_i - U_{i-1}$ & $f = A^T y$ took difference $y_i - y_{i+1}$. The springs are infinitesimally short and those differences become derivatives.

$$\frac{u_i - u_{i-1}}{\Delta x} \text{ is } \frac{du}{dx} \qquad \frac{y_i - y_{i+1}}{\Delta x} \text{ is } -\frac{dy}{dx}.$$

Hooke's Law & force balance at every point x:

$e(x) = AU = \frac{du}{dx}$ | $y(x) = c(x)\,e(x)$ | $A^T y = -\frac{dy}{dx} = f(x)$.

$$A^T C A\, U(x) = f(x).$$

The line of springs becomes an elastic bar:

Solid elastic bar $A^T C A\, u(x) = f(x)$ is

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x).$$

A^TA corresponds to second derivative. A is difference matrix & A^TA is a second difference matrix. The matrix has $-1, 2, -1$ & eq has $-d^2u/dx^2$.

$-U_{i+1} + 2U_i - U_{i-1}$ is a second difference.

First derivative - 3 choice - Forward, Backward & Centered difference

$$\frac{du}{dx} = \frac{u(x+\Delta x) - u(x)}{\Delta x} \text{ or } \frac{u(x) - u(x-\Delta x)}{\Delta x} \text{ or } \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}$$

First derivatives are antisymmetric, the transpose has a minus sign. So second differences are negative definite & changed to $-\frac{d^2u}{dx^2}$.

It starts with differential eqn, formulas for sol upx.
Create discrete matrix K by appr continous problem.

Fixed

Fixed. $AU = \frac{1}{\Delta x}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \simeq \frac{du}{dx}$ with $U_0 = 0$, $U_4 = 0$

Fixed free $A^T y = \frac{1}{\Delta x}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{-dy}{dx}$ with $U_0 = 0$, $y_4 = 0$.

The boundary condition $y_4 = 0$ at free end becomes $du/dx = 0$. Since $y = AU$ corresponds to du/dx. The force balance $A^T y = f$ at that end is 0. The last row of $K_1 U = f$ has entries $-1, 1$ to reflect this condition $du/dx = 0$

The eqn $\frac{-d}{dx}\left(c(x) \frac{du}{dx}\right) = f(x)$. with $U(0) = 0$ and $\left[u(1) \text{ or } \frac{du}{dx}(1)\right] = 0$. Divide the bar into N pieces of length Δx. Replace du/dx by AU & $-\frac{dy}{dx}$ by $A^T y$.
Now A & A^T include $1/\Delta x$. The end conditions are
$U_0 = 0$ & $[U_N = 0 \text{ or } y_N = 0]$.

The 3 steps $-d/dx$ & $c(x)$ & $\frac{d}{dx}$ correspond to A^T, C, A.
$f = A^T y$ $y = Ce$ $e = AU$ give $A^T C A U = f$.

1. Model the problem by differential equation.
2. Discretize the differential eqn to a difference eqn
3. Understand & solve the difference eqn (boundary conditions)
4. Interpret solution, visualize it.

Graphs and Networks.

The model consists of nodes connected by edges. This is called graph. Graphs display function $f(x)$. Graphs of this node-edge lead to matrices. The incidence matrix of a graph tells how n-nodes are connected by m edges. Normally $m > n$.

For any m by n matrix there are 2 fundamental subspaces in R^n & 2 in R^m. They are the row spaces & nullspaces of A & A^T.

Construct directed graph & its incidence matrix. The laws of linear Algebra is Kirchoff's Law.

Entry of an incidence matrix is 0 or 1 or −1. All the pivots & multipliers are ±1. Therefore both factors in $A = LU$ also contain 0, 1, −1. All four subspaces have basis vectors with these exceptionally simple components.

The matrix takes differences in voltage, across 6 edges of a graph. The voltages are. x_1, x_2, x_3, x_4 at the 4 nodes.

Incidence matrix 6 edges 4 nodes.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \text{ reduces to } \xrightarrow[\text{Row echelon}]{} U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nullspace of A & U is the line through $x = (1, 1, 1, 1)$. The column spaces of A & U have dim $r = 3$. The pivot rows are basis for row space.

Every vector in nullspace is perpendicular to every vector in row space. This comes directly from m equations $Ax = 0$. For A & U above $x = (1, 1, 1, 1)$ is perpendicular to all rows and thus to the whole row space. Equal voltages produce no current.

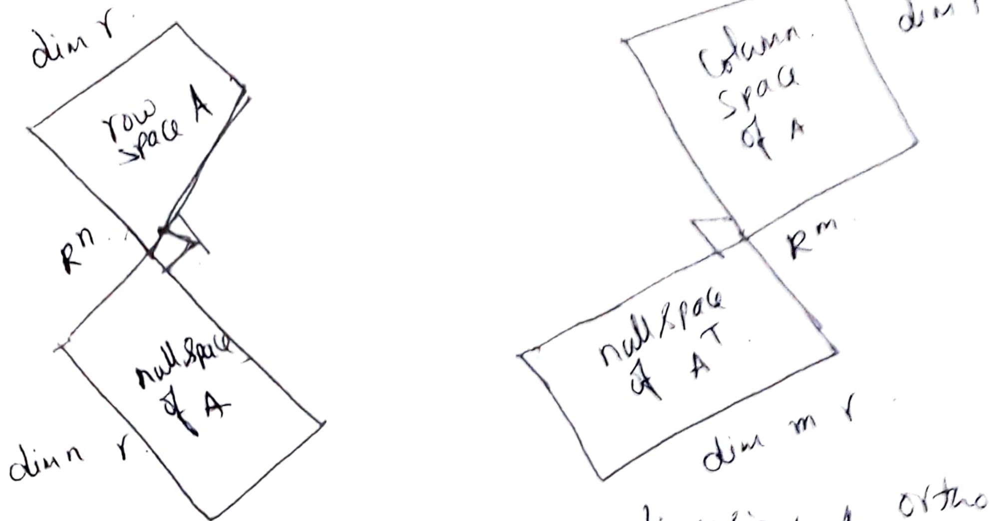


Fig-1 Four subspaces with their dimensions & orthogonality.

A matrix with m by n. Its columns are vectors in R^m. Their linear combinations produce the column space $C(A)$, a subspace of R^m. Those combinations are exactly the matrix vector products Ax.

The Rows of A are vectors in R^n. Their linear combinations produce $C(A^T)$ the column space of A^T row space. The row space becomes $C(A^T)$ the row space.

The nullspace $N(A)$ contains every x that satisfies $Ax = 0$. This is a subspace of R^n. The left-nullspace contains all solutions to $A^T y = 0$. Now y has m components & $N(A^T)$ is a subspace of R^m. $y^T A = 0^T$, combining rows of A to produce zero row. The four subspaces in fig-1, shows R^n on one side & R^m on other. The link between them is A.

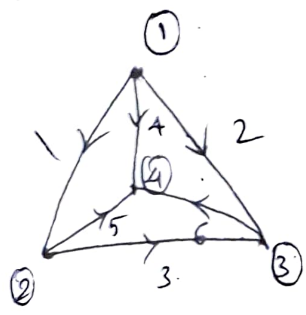
Two central laws of linear Algebra:

$\dim C(A) = \dim C(A^T)$ & $\dim C(A) + \dim N(A) = n$.

when row space has dimension r, the null space has dimension n-r. Elimination leaves these 2 spaces unchanged & echelon form U gives dimension count. There are r rows and columns with pivots. There are n-r free columns without pivots, & those lead to vectors in nullspace.

Directed graphs and Incidence matrices:

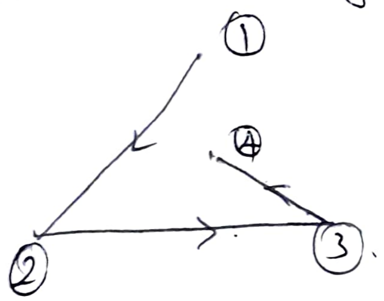
A graph with $m=6$ edges and $n=4$ nodes, so the matrix A is 6 by 4. The entries -1 & 1 the direction of each arrow. The first row $-1, 1, 0, 0$ of A gives a record of first edge from node 1 to node 2.



$$A = \begin{array}{c} \text{nodes} \\ \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \left[\begin{array}{cccc} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \text{ edges}$$

$m=6$ edges $n=4$ nodes.

Row numbers are edge numbers, column no. are node numbers.



$$B = \begin{array}{c} \text{Node} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} +1 & 1 & 0 & 0 \\ 0 & +1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \text{ edges}$$

Tree with 3 edges & 4 nodes & no loops.

The first graph is complete - every pair of nodes is connected by an edge. The second graph is a tree - the graph has no closed loops. Those graphs are the 2 extremes, the maximum number of edges is $\frac{1}{2}n(n-1)$ and minimum (tree) is $m=n-1$.

The rows of B matches non zero rows of U- echelon form. Elimination reduces every graph to a tree. The loops produce zero rows in U.

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

when 2 edges share a node, elimination produces short cut edge without that node. If the graph already has this short cut edge, elimination gives a row of zeros, reduce to the

Rows are dependent when edges form a loop. Independent rows come from trees. The graph is connected & it makes no fundamental difference which way the arrows go. On each edge, flow with arrow is positive. Flow in opposite direction counts as negative. The flow might be a current or signal or force or even oil or gas or water.

For the column space Ax, which is a vector of differences:

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix} \quad -(1)$$

The unknowns x_1, x_2, x_3, x_4 represent potentials or voltages at the nodes. Then Ax gives potential differences or voltage differences across the edges. It is these differences that cause flows.

The null space contains solutions to $Ax = 0$. All 6 potential differences are zero. This means: All potentials are equal. Every x in the nullspace is a constant vector (c, c, c, c). The nullspace of A is a line in R^n - its dimension is $n - r = 1$.

The second incidence matrix B has same nullspace. It contains $(1, 1, 1)$:

$$Bx = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Raise or lower all potentials by same amount C, without changing differences. There is an arbitrary constant in potentials. Raise or lower $f(x)$ function by same amount C, without changing its derivative. There is an arbitrary constant C in integral.

Calculus adds $+C$ to indefinite integrals. Graph theory adds (C, C, C, C) to the vector x of potentials. Linear algebra adds any vector x_n in the nullspace to one particular solution of $Ax = b$. The $+C$ disappears in calculus when integral starts at a known point $x = a$. Similarly nullspace disappears when set $x_4 = 0$. The unknown x_4 is removed & so are the fourth columns of A & B. Node 4 is grounded.

2) The row space contains all combination of 6 rows. Its dimension is certainly not 6. The eqn $r + (n-r) = n$ must be $3+1 = 4$. The rank $r = 3$.

v is in the row space iff it is perpendicular to $(1, 1, 1, 1)$ in the null space.

The vector $v = (0, 1, 2, 3)$ fails this test - its components add to 6. The vectors $(-6, 1, 2, 3)$ passes test. It lies in the row space because its component add to zero. It equals $6(\text{row}_1) + 5(\text{row}_3) + 3(\text{row}_6)$.

Each row of A adds to zero. This must be true for every vector in the rowspace.

3) The column space contains all combinations of 4 columns. Three independent columns, since there were 3 independent rows. The first 3 columns are independent. But 4 columns add to zero vector, which is $(1, 1, 1, 1)$ is in nullspace.

The components of Ax add to zero around every loop. when b is in column space of A,

Kirchoff's law : $b_1 + b_3 - b_2 = 0$.

$Ax = b$ can be solved exactly when components of b satisfy all the same dependencies as the rows of A. Then elimination leads to $0 = 0$ & $Ax = b$ is consistent.

4) The left null space contains solutions to $A^T y = 0$. Its dimension is $m - r = 6 - 3$:

Current Law (KCL)

$$A^T y = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

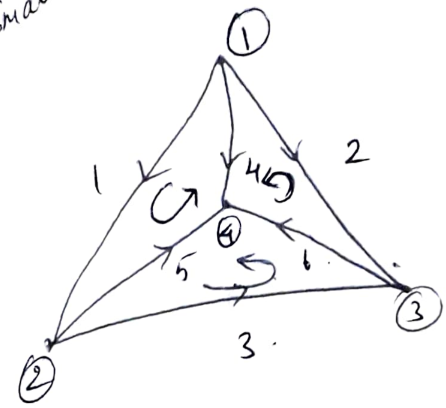
The true no. of equations $r = 3$ and not $n = 4$. The net flow into node 1 is zero. The First eq $-y_1 - y_2 - y_4 = 0$. Flow into the node fourth equation says that $y_4 + y_5 + y_6 = 0$. $A^T y = 0$ – fundamental: minus flow out is zero.

Kirchoff's Current law : Flow in equals flow out at each node.

When current of forces are in equilibrium, equation to solve is $A^T y = 0$. The matrix in this balance eq is transpose of incidence matrix A.

Sol: to $A^T y = 0$. The easiest way is to flow around a loop. If a unit of current goes around big triangle, results in $y = (1, -1, 1, 0, 0, 0)$: $\rightarrow$ forward edge 1, 3, backward 2. Every loop current is a solution to the current law. Around the loop flow in equals flow out at every node. A smaller loop goes forward on edge 1, forward on 5, back on 4. Then $y = (1, 0, 0, -1, 1, 0)$ is in left nullspace.

The 3 smaller loops in graph are independent. The big triangle seems to give fourth y, but it is sum of flows around small loops. The small loops give a basis for left nullspace



$$\begin{bmatrix}1\\0\\0\\1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\\1\\1\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

Small loops — big loop

Summary: The incidence matrix A comes from connected graph with n nodes & m edges. The row space & column space have dimensions $n-1$. The nullspaces of A and A^T have dimension 1 and $m-n+1$:

* The constant vectors $(c, c, \dots c)$ make up the nullspace of A.
* There are $r = n-1$ independent rows, using edges from any tree.
* voltage law: The components of Ax add to zero around every loop.
* current law: $A^Ty = 0$ is solved by loop currents. $N(A^T)$ has dimension $m-r$. There are $m-r = m-n+1$. independent loops in the graph.

For every graph in a plane, LA yields Euler's formula:

$$(\text{No. of nodes}) - (\text{no. of edges}) + (\text{No. of small loops}) = 1.$$

i.e. $n - m + (m-n+1) = 1$.

A single triangle has (3 nodes) − (3 edges) + (1 loop).

Networks and $A^T C A$

In real n/w, the current y along an edge is product of 2 numbers. One no. is difference b/w potentials x at the ends of the edge. This difference is Ax & it drives the flow. The other number is the 'Conductance' C - which measures how easily flow gets through.

In physics and engineering, c is decided by material. For electrical currents, C is high for metal & low for plastics. For Superconductors C is nearly infinite. For elastic stretching, C might be low for metal & higher for plastics. In economics, C measures the capacity of an edge or its cost.

The graph is known from its connectivity matrix A, connection b/w nodes & edges. A n/w goes further & assigns a conductance C to each edge. These numbers $C_1 \cdots C_m$ go into the conductance matrix C which is diagonal.

For each n/w of resistors, $C = 1/\text{(resistance)}$. Ohm's law connects the current y_1 on edge 1 to potential difference $x_2 - x_1$ between the nodes:

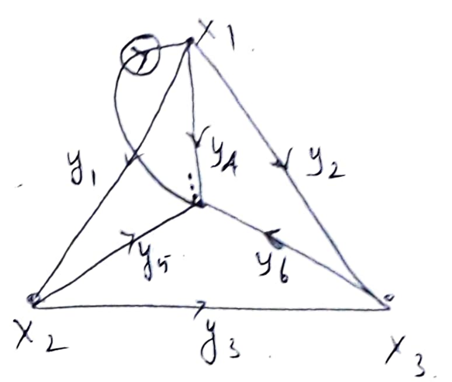
Ohm's Law: current along edge = conductance times potential difference.

Ohm's Law for all m currents is $y = -CAx$. The vector Ax gives the potential differences, & C multiplies by the conductances. Combining ohm's Law with Kirchoff's current Law. $A^T y = 0$. i.e. $A^T C A x = 0$. This is almost the central equation for n/w flows. The N/w needs power from outside a voltage source or current source - to make something happen.

In circuit theory, change from Ax to $-Ax$. The flow is from higher potential to lower potential. There is current from node 1 to node 2 when $x_1 - x_2$ is positive, whereas Ax was constructed to yield $x_2 - x_1$. The minus sign in physics and electrical engineering is a plus sign in mechanical engineering & economics.

For elastic structures $y = CAx$ is Hooke's Law. The stress y is (elasticity c) times (stretching Ax). For heat conduction Ax is a temperature gradient. For oil flows it is a pressure gradient. There is a similar law for least squares regression in statistics.

Kirchoff's Law changes from $A^T y = 0$ to $A^T y = f$ to balance the source f from outside. Flow into each node still equals flow out. The n/w with its conductances $C_1 \dots C_6$ and shows the current source going into node 1. The source comes out at node 4 to keep the balance.



Source S into node 1.

Q1: All conductances are $C=1$, so that $C = I$. A current y_4 travels directly from node 1 to node 4. Other current goes long way from node $1 \to 2 \to 4$ $(y_1 = y_5)$, node $1 \to 3 \to 4$ $(y_2 = y_6)$. Find 6 currents by using special rules for symmetry $A^T C A$. Since $C = I$ the matrix is $A^T A$, graph Laplacian matrix:

$$\begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Last matrix is not invertible. We cannot solve for all four potentials because $(1,1,1,1)$ is in nullspace. One node has to be grounded. Setting $x_4 = 0$ removes from 4th row and column, this leaves 3×3 invertible matrix.

Solve $A^TCAx = f$ for unknown potentials x_1, x_2, x_3 with Source S into node 1:

$$\text{voltages} \quad \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S \\ 0 \\ 0 \end{bmatrix} \text{ gives } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S/2 \\ S/4 \\ S/4 \end{bmatrix}$$

Ohm's Law $y = -CAx$ yields 6 currents. $C = I$ & $x_4 = 0$

$$\text{Currents} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = - \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} S/2 \\ S/4 \\ S/4 \\ 0 \end{bmatrix} = \begin{bmatrix} S/4 \\ S/4 \\ 0 \\ S/2 \\ S/4 \\ S/4 \end{bmatrix}$$

Half current goes directly on edge 4. i.e. $y_4 = S/2$. No current crosses from node$_2$ to node$_3$. Symmetry indicated $y_3 = 0$ & now sol. proves it.

The same matrix A^TA appears in least squares. Nature distributes the currents to minimize the heat loss. Statistics chooses $\hat{x}$ to minimize the least squares error.

Markov matrices, Population & Economics:

The largest eigen value is real & positive & so is its eigenvector.

In economics, ecology & population dynamics & random walks

Markov $\lambda_{max} = 1$ population $\lambda_{max} > 1$ consumption $\lambda_{max} < 1$.

λ_{max} controls powers of A.

Markov matrices

Multiply a positive vector $v_0 = (a, 1, -a)$ again & again by this A:

Markov matrix $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$. $u_1 = A v_0$.

$v_0 = A v_1 = A^2 u_0$.

After k steps, $A^k v_0$. The vectors $v_1, v_2, v_3 \ldots$ will approach a steady state $v_\infty = (.6, .4)$. This final outcome doesnot depend on starting vectors. For every v_0 converge to v_∞.

The steady state eq. $A v_\infty = v_\infty$ makes v_∞ an eigenvector with eigenvalue 1:

Steady state $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$

Steady state,

Not Markov $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has unattractive steady state

$B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The starting vector $v_0 = (0, 1)$ will give $v_1 = (0, 2)$ & $v_2 = (0, 4)$. The second component are doubled.

Markov matrix:
* Every entry of A is non-negative
* Every column of A adds to 1.

Ex1: The fraction of rental cars in Denver starts at $1/50 = .02$. The fraction outside Denver is $.98$. Every month 80% of Denver cars stay in Denver. Also 5% of outside cars come in. This means that the fractions $u_0 = (0.02, 0.98)$ are multiplied by A:

First month $A = \begin{bmatrix} .8 & .05 \\ .2 & .95 \end{bmatrix}$ lead to $u_1 = A u_0 = A \begin{bmatrix} .02 \\ .98 \end{bmatrix}$

$$u_1 = \begin{bmatrix} .065 \\ .935 \end{bmatrix}$$

Next month. $u_2 = A u_1 = (.09875, .90125)$ This is $A^2 u_0$.

The diagonal matrix Λ^k is simple. The eigenvectors matrix S connects them: A^k equals $S \Lambda^k S^{-1}$. The new application to Markov matrices uses the eigenvalues and eigenvectors. u_∞ is an eigenvector corresponding to $\lambda = 1$.

$A^k u_0$ gives fractions in and out of Denver after k steps.
The eigenvalues are. $\lambda = 1$ & 0.75.

$$Ax = \lambda x \qquad A\begin{bmatrix} .2 \\ .8 \end{bmatrix} = 1\begin{bmatrix} .2 \\ .8 \end{bmatrix} \quad \& \quad A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.75\begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

The starting vector u_0 combines x_1 & x_2, in this case with coefficients 1 & 0.18:

Combination of eigenvectors $u_0 = \begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + .18\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now multiply A to find u_1.

$$u_1 = 1\begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)(.18)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Every month, another $.75$ multiplies the vector x_2

The eigenvector x_1 is unchanged:

After k steps $\quad u_k = A^k u_0 = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + (0.75)^k (0.18) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The eigenvector x_1 with $\lambda = 1$ is the steady state. The other eigenvector x_2 disappears because $|\lambda| < 1$. The more steps leads $u_\infty = (0.2, 0.8)$. In the limit, $2/10$ of cars are in Denver & $8/10$ are outside.

If A is a positive markov matrix ($a_{ij} > 0$, each column add to 1) then $\lambda_1 = 1$ is larger than any other eigen value. The eigenvector x_1 is the steady state:

$$u_k = x_1 + c_2(\lambda_2)^k x_2 + \dots + c_n(\lambda_n)^k x_n \text{ always approaches}$$

$$u_\infty = x_1.$$

Ex. 2: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has no steady state because $\lambda_2 = -1$.

This matrix sends all cars from inside Denver to outside & viceversa. The powers A^k alternate between A & I. The second eigenvector $x_2 = (-1, 1)$ will be multiplied by $\lambda_2 = -1$ at every step & does not smaller: No steady state.

The powers A^k approach the rank one matrix that has the steady state in every column.

Ex 3: At each time step, half of $group_1$ goes to $group_2$ and other half goes to $group_3$. The other groups also split in half & move. Take one step from the starting populations P_1, P_2, P_3:

new populations $\quad u_1 = Au_0 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}P_2 + \frac{1}{2}P_3 \\ \frac{1}{2}P_1 + \frac{1}{2}P_3 \\ \frac{1}{2}P_1 + \frac{1}{2}P_2 \end{bmatrix}$

A is markov matrix. A contains zeros, which gave trouble in Ex 2. But after 2 steps in this new ex, the zeros disappear from A^2.

2 Step matrix. $V_2 = A^2 v_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$

The eigen values of A are $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -1$. For $\lambda = 1$, eigen vector $x_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ will be the steady state. when 3 equal population split in half & move, the population are again equal. Starting from $v_0 = (8, 16, 32)$, the Markov Chain approaches its steady state:

$$v_0 = \begin{bmatrix} 8 \\ 16 \\ 32 \end{bmatrix} \quad v_1 = \begin{bmatrix} 24 \\ 20 \\ 12 \end{bmatrix} \quad v_2 = \begin{bmatrix} 16 \\ 18 \\ 22 \end{bmatrix} \quad v_3 = \begin{bmatrix} 20 \\ 19 \\ 17 \end{bmatrix}$$

The total population $8 + 16 + 32 = 56$ at every step. The steady state is 56 times $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The three populations approaching, but never reaching, their final limits $56/3$.

Person - Frobenius Theorem:

One matrix theorem dominates. The person Frobenius theorem applies when all $a_{ij} \geq 0$. There is no requirement that columns add to 1.

Person - Frobenius for $A > 0$. All numbers in $Ax = \lambda_{max}$ are strictly positive.

Proof: Number - t, such that $Ax \geq tx$, $x \neq 0$. For largest value t_{max}, equality holds $Ax = t_{max} x$.

If $Ax \geq t_{max} x$ is not an equality, multiply by A. Because A is positive that produces a strict inequality $A^2 x > t_{max} Ax$. Therefore the positive vector $y = Ax$ satisfies $Ay > t_{max} y$ and t_{max} could be increased. This contradiction forces equality $Ax = t_{max} x$.

The eigenvector x is positive because on the leftside of that equality, Ax is sure to be positive.

No eigenvalue can be larger than t_{max}, suppose $Az = \lambda z$. Since λ and z may involve negative or complex numbers,

$|\lambda||z| = |Az| \leq A|z|$ by the triangle inequality. This $|z|$ is non-negative vector, so $|\lambda|$ is one of the possible candidates t, Therefore $|\lambda|$ cannot exceed t_{max} which must be λ_{max}.

Population growth:

Divide population into 3 age groups: age < 20, $20-39$ & $40-59$. At year T the sizes of those groups are n_1, n_2, n_3. Twenty years later, the sizes have changed for 2 reasons:

1. Reproduction $n_1^{new} = F_1 n_1 + F_2 n_2 + F_3 n_3$ gives a new generation.
2. Survival $n_2^{new} = P_1 n_1$ and $n_3^{new} = P_2 n_2$ gives the older generation.

The fertility rates are F_1, F_2, F_3 (F_2 largest). The Leslie matrix A might look like this

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}^{new} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} .04 & 1.1 & .01 \\ .98 & 0 & 0 \\ 0 & .92 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

In a realistic model, A will change with time, include 4th group age ≥ 60.

The matrix has $A \geq 0$ but not $A > 0$. The Perron-Frobenius theorem applies because $A^3 > 0$. The largest eigenvalue is $\lambda_{max} = 1.06$ in the middle generation;

Starting from $n_2 = 1$

$$eig(A) = \begin{matrix} 1.06 \\ -1.01 \\ -0.01 \end{matrix} \quad A^2 = \begin{bmatrix} 1.08 & 0.05 & 0.0 \\ 0.04 & 1.08 & .01 \\ .90 & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0.1 & 1.19 & .01 \\ 0.06 & 0.05 & 0.0 \\ 0.04 & 0.99 & 0.01 \end{bmatrix}$$

A fast start $v_0 = (0, 1, 0)$. That middle group will reproduce 1.1 & survive 0.92. The newest & oldest generations are in $v_1 = (1.1, 0, 0.92) =$ column 2 of A. Then $v_2 = Av_1 = A^2 v_0$ is second column of A^2. The early numbers depend a lot on v_0, but asymptotic growth rate λ_{max} is same from every start. Its eigenvector $x = (0.63, 0.58, 0.51)$ shows all 3 groups growing steadily together.

<u>Linear Algebra in Economics</u>: The consumption matrix

The consumption matrix tells how much of each input goes into a unit of output. This describes manufacturing side of economy.

<u>Consumption matrix</u>: There are n industries like chemical, food & oil. To produce unit of chemicals may require 0.2 units of chemical, 0.3 units of food & 0.4 units of oil. These numbers go into row 1 of consumption matrix A:

$$\begin{bmatrix} \text{chemical o/p} \\ \text{food o/p} \\ \text{oil o/p} \end{bmatrix} = \begin{bmatrix} .2 & .3 & .4 \\ .4 & .4 & .1 \\ .5 & .1 & .3 \end{bmatrix} \begin{bmatrix} \text{chemical i/p} \\ \text{food i/p} \\ \text{oil i/p} \end{bmatrix}$$

Row 2 shows the inputs to produce food - heavy use of chemicals & food, not so much oil. Row 3 of A shows inputs consumed to refine a unit of oil.

This economy meet demands y_1, y_2, y_3 for chemicals, food, oil. The inputs p_1, p_2, p_3 will have to be higher - because part of p is consumed in producing y. The input is p and consumption is Ap, which leaves the o/p $p - Ap$. This net production is what meets demand y:

Geometric series: $S = (I - A)^{-1} = I + A + A^2 + A^3 + \cdots$

multiply S by A

$S - AS = I$ which is $(I-A)S = I$. The series adds to $S = (I-A)^{-1}$ if it converges, and it converges if all eigenvalues of A have $|\lambda| < 1$.
$A \geq 0$, All terms of series are nonnegative. Its sum is $(I-A)^{-1} \geq 0$.

Ex 1) $A = \begin{bmatrix} .2 & .3 & .4 \\ .4 & .4 & .1 \\ .5 & .1 & .3 \end{bmatrix}$ has $\lambda_{max} = 0.9$ & $(I-A)^{-1} = \frac{1}{93}\begin{bmatrix} 41 & 25 & 27 \\ 33 & 36 & 24 \\ 34 & 23 & 36 \end{bmatrix}$

This economy is productive. A is small compared to I, because λ_{max} is 0.9.
Start $p = (I-A)^{-1} y$. Then Ap is consumed in production, leaving $p - Ap$. This is $(I-A)p = y$ & demand is met.

Ex 2) $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ has $\lambda_{max} = 2$ & $(I-A)^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

This consumption matrix is too large. Demands can't be met, because production consumes more than it yields. Thus series $I + A + A^2 \ldots$ doesn't converge to $(I-A)^{-1}$ because $\lambda_{max} > 1$. The series is growing while $(I-A)^{-1}$ is actually negative.

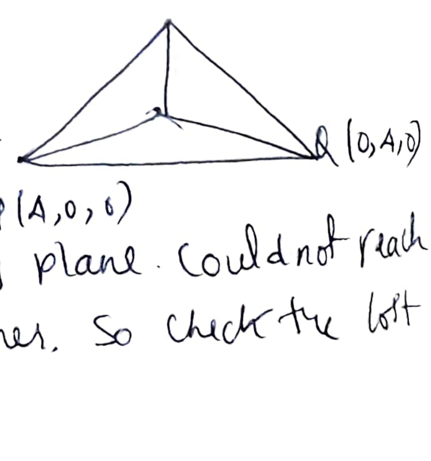
Linear Programming: For $Ax = b$, If there are any solution $x \geq 0$ to $Ax = b$, Linear programming picks the solution $x^* \geq 0$ that minimizes the cost.
The cost is $c_1x_1 + \ldots c_nx_n$. The winning vector x^* is non-negative solution of $Ax = b$ that has smallest cost.
The linear programming starts with a matrix A & 2 vectors b & c:
i) A has $n > m$; Ex: $A = [1 \quad 1 \quad 2]$
ii) b has m components for m equations $Ax = b$; $b = [4]$
iii) The cost vector c has n components: $c = [5 \quad 3 \quad 8]$
Then the problem is to minimize $c \cdot x$ subject to requirements $Ax = b$ & $x \geq 0$.
Minimize $5x_1 + 3x_2 + 8x_3$ subject to $x_1 + x_2 + 2x_3 = 4$ & $x_1, x_2, x_3 \geq 0$.

Look at constraint: $Ax = b$ & $x \geq 0$, The equation $x_1 + x_2 + 2x_3 = 4$ gives a plane in 3 dimensions. The non negativity $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ chops the plane down to triangle.

Inside that triangle, all components of x are positive. On the edges of PQR, one component is zero. At the corners P, Q & R, 2 components are zero. The optimal solution x^* will be one of those corners.

The triangle contains all vectors x that satisfy $Ax = b$ and $x \geq 0$. Those x's are called feasible points & triangle is feasible set. These points are the allowed candidates in the minimization of $c \cdot x$ which is the final step.

Find x^* in the triangle PQR to minimize the cost $5x_1 + 3x_2 + 8x_3$. That vectors that have zero cost lie on plane $5x_1 + 3x_2 + 8x_3 = 0$. That plane doesnot meet the triangle. Increase cost C untill the plane $5x_1 + 3x_2 + 8x_3 = C$ does meet triangle. As C increases, parallel plane moving toward the triangle.



The first plane $5x_1 + 3x_2 + 8x_3 = c$ to touch the triangle has minimum cost c. The point where it touches is the solution x^*. This touching must be one of corners P, Q, R. A moving plane could not reach inside of triangle before it touches a corner. So check the cost $5x_1 + 3x_2 + 8x_3$ at each corner.

$P = (4, 0, 0)$ costs 20 $Q = (0, 4, 0)$ costs 12 $R = (0, 0, 2)$ costs 16.

The winner is Q, Then $x^* = (0, 4, 0)$ solves problem.

If the cost vector c is changed, the parallel planes are tilted. For small changes, Q is still winner. For cost $c \cdot x = 5x_1 + 4x_2 + 7x_3$ the optimum x^* moves to $R = (0, 0, 2)$. The minimum cost is now $7 \cdot 2 = 14$.

Note 1: Linear programming maximize profit instead of minimizing cost. The parallel plane start with a large value of c, instead of a small value. They move toward the origin. as c gets smaller. The first touching point is still a corner.

Note 2: The requirements $Ax = b$ & $x \geq 0$ could be impossible to satisfy. The equation $x_1 + x_2 + x_3 = -1$ cannot be solved with $x \geq 0$. That feasible set is empty.

Note 3: The feasible set is unbounded. If the requirement is $x_1 + x_2 - 2x_3 = 4$ the large positive vector $(100, 100, 98)$. The plane $Ax = b$ is no longer chopped off to a triangle. The two corners P & Q are still candidates for x^* but R moved to infinity.

Note 4: with an unbounded feasible set, the minimum cost could be $-\infty$. Suppose cost is $-x_1 - x_2 + x_3$. Then the vector $(100, 100, 98)$ cost $c = -102$. The vector $(1000, 1000, 998)$ costs $c = -1002$. In realistic not possible.