Fundamentals of Signals and Systems

Signal — definition, eg.

~~Sxxxx~~ Classification of signals

a) Continuous time and discrete time
b) Analog and digital
c) Deterministic and random
d) Even and odd
e) periodic and non periodic
f) Energy and power

Basic operations on signals

a) Amplitude Scaling
$$y(t) = A\,x(t)$$

b) Addition
$$y(t) = x_1(t) + x_2(t)$$

c) Multiplication
$$y(t) = x_1(t) \cdot x_2(t)$$

Operations on dependent Variable

Operations on independent variable

a) Time scaling

$y_1(n) = x(n)$

$y_2(n) = x(2n)$

$y_3(n) = x(n/2)$

b) Time shifting

$y_1(n) = x(n)$

$y_2(n) = x(n-2)$

$y_3(n) = x(n+2)$

c) folding

$y(n) = x(-n)$

Prob:

1. A discrete time signal is represented as $x(n) = \{ \underset{\uparrow}{1}, 2, 3, 4, 5 \}$

Sketch the following

a) $x(n-2)$ b) $x(n+2)$ c) $x(-n)$

d) $x(-n+2)$ e) $x(-n-2)$

Properties of Systems

a) Linearity b) Causality

c) Time invariance d) Memory

e) Stability

LTI Systems

Convolution Sum

Let $x(n)$ and $h(n)$ be two finite duration sequences.

The convolution sum of $x(n)$ & $h(n)$ is given by

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)$$

Prob:

Let $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$ and

$h(n) = \{\underset{\uparrow}{5}, 6, 7, 8\}$

find $y(n) = x(n) * h(n)$.

Soln:

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(n-k)$$

$x(n)$ & $h(n)$ are purely Causal

$\Rightarrow$ $y(n)$ is also causal.

	5	6	7	8
1	5	6	7	8
2	10	12	14	16
3	15	18	21	24
4	20	24	28	32

$y(n) = \{\underset{\uparrow}{5}, 16, 34, 60, 61, 52, 32\}$

$L_1 = 4$ $\quad L_2 = 4$ $\quad L_1 + L_2 - 1 = 7$

Convolution using Toeplitz Matrix

$y(0) = x(0)\,h(0)$

$y(1) = x(0)\,h(1) + x(1)\,h(0)$

$y(2) = x(0)\,h(2) + x(1)\,h(1) + x(2)\,h(0)$

$\vdots$

$y(6) = x(3)\,h(3)$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \end{bmatrix} = \begin{bmatrix} x(0) & 0 & 0 & 0 \\ x(1) & x(0) & 0 & 0 \\ x(2) & x(1) & x(0) & 0 \\ x(3) & x(2) & x(1) & x(0) \\ 0 & x(3) & x(2) & x(1) \\ 0 & 0 & x(3) & x(2) \\ 0 & 0 & 0 & x(3) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix}$$

$7 \times 1 \qquad\qquad 7 \times 4 \qquad\qquad 4 \times 1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 16 \\ 34 \\ 60 \\ 61 \\ 52 \\ 32 \end{bmatrix}$$

$$\boxed{Ax = b}$$

Ex 2

find Linear Convolution of

$x(n) = \{-2, -1, 0, 1, 2\}$

$h(n) = \{-3, -2, -1, 0\}$

Ans $\{6, 7, 4, -2, -8, -5, -2, 0\}$

$L_1 = 5, \quad L_2 = 4$

$L_1 + L_2 - 1 = 8$

6

De Convolution

$$Ax = b$$

* A & b are known, we have to find x.

* A is not square

* Hence we have to find generalized inverse.

$$A^T A x = A^T b$$

$$x = [A^T A]^{-1} A^T b$$

Z – transform

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

The set of values of z for which summation converges is called ROC.

Eg:

1) $x(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

$$= 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

ROC $|z| > 0$

2) $x(n) = \{1, 1, 2, \underset{\uparrow}{2}\}$

$$X(z) = z^{3} + z^{2} + 2z + 2$$

ROC $|z| < \infty$

3) $x(n) = \{2, 1, \underset{\uparrow}{1}, 2\}$

$$X(z) = 2z^{2} + z + 1 + 2z^{-1}$$

ROC $0 < |z| < \infty$

4) $x(n) = a^n u(n)$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

ROC $|a z^{-1}| < 1$

$\Rightarrow |z| > |a|$

5) $x(n) = -b^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=1}^{\infty} (b^{-1} z)^n$$

$$= \frac{b^{-1} z}{1 - b^{-1} z} = \frac{z}{z - b}$$

ROC $|b^{-1} z| < 1 \Rightarrow |z| < |b|$

6) $x(n) = a^n u(n) - b^n u(-n-1)$

$$X(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n} - \sum_{n=-1}^{-\infty} b^n u(-n-1) z^{-n}$$

$$= \frac{z}{z - a} + \frac{z}{z - b}$$

ROC $|b| > |z| > |a|$

Properties of ROC

1. ROC cannot contain any poles
2. If $x(n)$ is finite and causal, ROC is entire z-plane except $z=0$
3. If $x(n)$ is finite and anticausal ROC is entire z-plane except $z=\infty$
4. If $x(n)$ is double sided & finite, then ROC is entire z-plane except $z=0$ and $z=\infty$
5. If $x(n)$ is causal, infinite then, ROC is $|z| > r_{max}$ where r_{max} is largest magnitude of poles
6. If $x(n)$ is anticausal & infinite then ROC is $|z| < r_{min}$ where r_{min} is smallest magnitude of poles
7. If $x(n)$ is two sided and infinite then ROC is $r_1 < |z| < r_2$ where r_1 and r_2 are magnitude of poles
8. ROC of a stable LTI system contains unit circle in z-plane.

~~Properties of ROC~~

Prob
find Inverse Z- transform of

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

for the following ROC's

a) $|z| > 1$ b) $|z| < 1/3$ c) $1/3 < |z| < 1$

Soln

$$\frac{X(z)}{z} = \frac{1}{3z^2 - 4z + 1} = \frac{1}{3(z - 1/3)(z - 1)}$$

$$= \frac{A}{z - 1/3} + \frac{B}{z - 1}$$

$A = -1/2$ and $B = 1/2$

$$X(z) = \frac{-1/2\, z}{z - 1/3} + \frac{1/2\, z}{z - 1}$$

a) Purely causal

$$x(n) = -\frac{1}{2}\left(1/3\right)^n u(n) + 1/2\, u(n)$$

b) Purely anticausal

$$x(n) = \frac{1}{2}\left(1/3\right)^n u(-n-1) - 1/2\, u(-n-1)$$

c) $-\frac{1}{2}\left(1/3\right)^n u(n) - 1/2\, u(-n-1)$

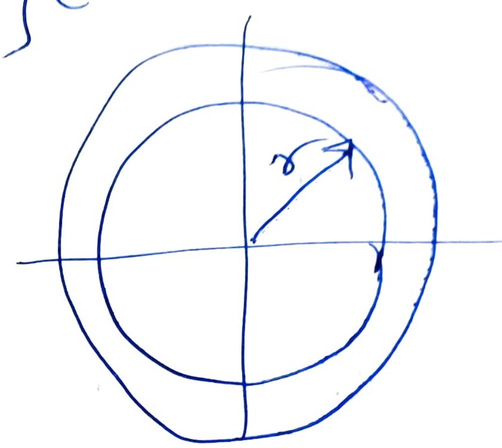
Fourier Transform

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

$x(n)$ is finite duration Causal Signal

$$z = r e^{j\omega}$$

$$X(z) = \sum_{n=0}^{N-1} \left\{ x(n)\, r^{-n} \right\} e^{-j\omega n}$$



if $|r| = 1$

$$X(z)\Big|_{|r|=1} = X(z)\Big|_{z=e^{j\omega}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n} = X(\omega) = X(e^{j\omega})$$

— Fourier transform (DTFT)

— Samples on unit Circle

— exists if $x(n)$ is absolutely summable

— freq. domain representation of the signal

(2)

Problems with FT

— $x(n)$ is discrete function of n

But $X(\omega)$ is continuous function of ω

Inverse FT

$$\rightarrow \quad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega.$$

—— Need for DFT.

Eg Determine the convolution of the sequences $x_1(n) = x_2(n) = \{1, \underset{\uparrow}{1}, 1\}$

Ans

$$x_3(n) = x_1(n) * x_2(n)$$

$$X_3(\omega) = X_1(\omega) \cdot X_2(\omega)$$

$$X_1(\omega) = \sum_{n=-1}^{+1} x_1(n)\, e^{-j\omega n}$$

$$= e^{j\omega} + 1 + e^{-j\omega} = 1 + 2\cos\omega$$

$$X_3(\omega) = (1 + 2\cos\omega)^2 = 1 + 4\cos^2\omega + 4\cos\omega$$

$$= 3 + 4\cos\omega + 2\cos 2\omega$$

$$= 3 + 2(e^{j\omega} + e^{-j\omega}) + (e^{j2\omega} + e^{-j2\omega})$$

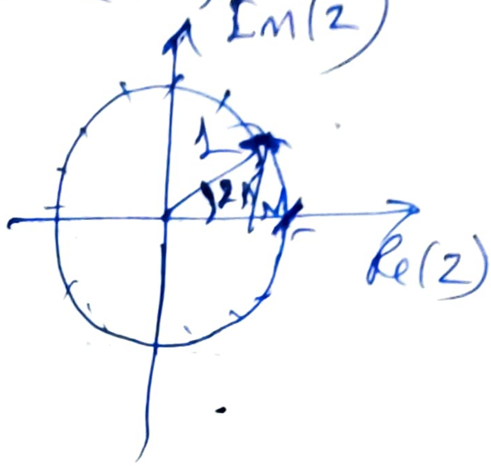
$$X_3(\omega) = e^{-j2\omega} + 2e^{-j\omega} + 3 + 2e^{j\omega} + e^{j2\omega}$$

$$x_3(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$$

DFT

$$X(\omega) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

$x(n)$ is a finite duration sequence in the range $0 \le n \le N-1$



→ $X(\omega)$ is sampled.

→ N equally spaced samples are chosen

→ Sampling interval $\frac{2\pi}{N}$

$$\Rightarrow \quad \omega = \frac{2\pi}{N} k \qquad 0 \le k \le N-1$$

$$X(\omega)\Big|_{\omega = \frac{2\pi}{N}k} = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$\Rightarrow \quad X(k) = \mathrm{DFT}\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

W_N is a complex quantity.

$$W_N = e^{-j\frac{2\pi}{N}}$$

← Twiddle factor.

W_N is periodic with period N ②

$$W_N^{kn+N} = W_N^{kn} \cdot W_N^{N}$$

$$W_N^{N} = e^{-j\frac{2\pi}{N}\cdot N} = e^{-j2\pi} = 1.$$

$$\therefore \quad W_N^{kn+N} = W_N^{kn}$$

<u>Lemma</u>

$$\sum_{n=0}^{N-1} W_N^{kn} = N\,\delta(k) = \begin{cases} N & k=0 \\ 0 & k\neq 0 \end{cases}$$

Proof

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \text{ if } a\neq 1$$

$$\sum_{n=0}^{N-1} W_N^{kn} = \frac{1-W_N^{kN}}{1-W_N^{k}} = \frac{1-1}{1-W_N^{k}} = 0 \text{ if } k\neq 0$$

if $k=0$

$$\sum_{n=0}^{N-1} W_N^{0} = \sum_{n=0}^{N-1} 1 = N$$

Inverse DFT

$$x(n) = \mathrm{IDFT}\{X(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$$0 \le n \le N-1$$

Periodicity of $X(k)$ and $x(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$\Rightarrow X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{(k+N)n}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \cdot \cancel{W_N^{Nn}}^{\,1}$$

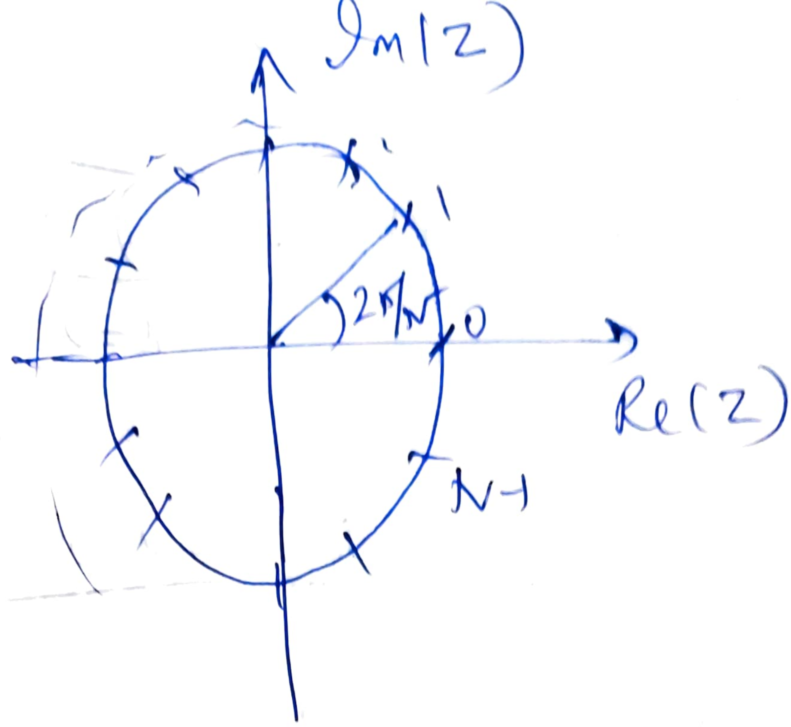
$$W_N^{nN} = e^{-j\frac{2\pi}{N} nN} = 1$$

$$= X(k) \qquad \text{(A)}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$$x(n+N) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-k(N+n)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \cdot \cancel{W_N^{-kN}}^{\,1} = x(n) \qquad \text{(B)}$$



Prob:

1) Compute 4 point DFT of the sequence $x(n) = \{1, 2, 3, 4\}$
$\uparrow$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$X(0) = \sum_{n=0}^{3} x(n) = 1+2+3+4 = 10$$

$$X(1) = \sum_{n=0}^{3} x(n) W_4^{n} = x(0) + x(1) W_4^{1} + x(2) W_4^{2} + x(3) W_4^{3}$$

$$= 1 + 2\times -j + 3\times -1 + 4\times j$$

$$= -2 + 2j$$

$$X(2) = \sum_{n=0}^{3} x(n) W_4^{2n} = x(0) + x(1) W_4^{2} + x(2) W_4^{4} + x(3) W_4^{6}$$

5

$W_4^0 = 1 \quad W_4^1 = e^{-j\frac{2\pi}{4}} = e^{-j\pi/2} = -j$

$W_4^2 = e^{-j\frac{2\pi}{4_2}\times 2} = e^{-j\pi} = -1$

$W_4^3 = e^{-j 2\pi/4 \times 3} = +j$

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1) Periodicity property of W_N^{kn}

$$W_N^{kn+N} = W_N^{kn}$$

$$LHS = W_N^{kn} \cdot W_N^{N} = W_N^{kn} e^{-j\frac{2\pi}{N}\cdot N} = W_N^{kn}$$

2) Symmetry property of W_N^{kn}

$$W_N^{kn+N/2} = -W_N^{kn}$$

$$LHS \; W_N^{kn} W_N^{N/2} = W_N^{kn} e^{-j\frac{2\pi}{\cancel{N}}\times \cancel{N}/2} = -W_N^{kn}$$

If $N = 4$

$W_4^0 = 1 \qquad W_4^1 = -j$

$W_4^2 = -1 \qquad W_4^3 = j$

$W_4^4 = W_4^0 = 1$

$W_4^6 = W_4^2$

$W_4^5 = W_4^9 = W_4^1$

$$X(2) = 1 + 2\times -1 + 3\times 1 + 4\times -1$$
$$= 1 - 2 + 3 - 4 = -2$$

$$X(3) = \sum_{n=0}^{3} x(n) W_4^{3n} = x(0) + x(1) W_4^{3} + x(2) W_4^{6} + x(3) W_4^{9}$$

$$= 1 + 2\times j + 3\times -1 + 4\times -j$$
$$= 1 + 2j - 3 - 4j$$
$$= -2 - 2j.$$

$$\therefore X(k) = \{ \underset{\uparrow}{10}, -2+2j, -2, -2-2j \}$$

2) Compute 4 point DFT of the sequence $x(n) = \{ \underset{\uparrow}{1}, 2, 5, 4 \}$

1) Compute N point DFT of the sequence $x(n) = a^n \quad 0 \le n \le N-1$

Soln

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} (a W_N^{k})^n$$

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \quad \text{for } a \ne 1$$

$$= \frac{1 - a^N \cancel{W_N^{kN}}^{\,1}}{1 - a W_N^{k}}$$

$$= \frac{1 - a^N}{1 - a W_N^{k}}$$

2) Compute N point DFT of the sequence $x(n) = 1 \quad 0 \le n \le N-1$

Soln

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} = \sum_{n=0}^{N-1} 1 \cdot W_N^{kn}$$

$$= \frac{1 - \cancel{W_N^{kN}}^{\,1}}{1 - W_N^{k}} = 0 \text{ for } k \ne 0$$

2

when $k = 0$

$$X(0) = \sum_{n=0}^{N-1} 1 \times 1 = N$$

$$\therefore X(k) = \begin{cases} N & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

$$= N\,\delta(k)$$

3) Compute Inverse DFT of the sequence $X(k) = \{\underset{\uparrow}{2}, 1+j, 0, 1-j\}$

Soln:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad 0 \le n \le N-1$$

$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, W_N^{-kn} \qquad 0 \le n \le 3$$

$$x(0) = \frac{1}{4}\left[2 + (1+j) + (1-j)\right] = 1$$

$x(1) = 0$

$x(2) = 0$

$x(3) = 1$

$x(n) = \{1, 0, 0, 1\}$

$W_4^{-1} = e^{-j\frac{2\pi}{4}\times -1}$

$= j$

$W_4^{-3} = -j$

$W_4^{-5} = j$

Matrix Relation for Computing DFT

Eg 1) Compute 4 point DFT of the sequence

$x(n) = \{\underset{\uparrow}{1}, 2, 1, 0\}$

Solu^n

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \ldots N-1$$

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{kn} \qquad k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

$$X(1) = \sum_{n=0}^{3} x(n) W_4^{n} = x(0) W_4^0 + x(1) W_4^1 + x(2) W_4^2 + x(3) W_4^3$$

$$X(2) = \sum_{n=0}^{3} x(n) W_4^{2n} = x(0) W_4^0 + x(1) W_4^2 + x(2) W_4^4 + x(3) W_4^6$$

$$X(3) = \sum_{n=0}^{3} x(n) W_4^{3n} = x(0) W_4^0 + x(1) W_4^3 + x(2) W_4^6 + x(3) W_4^9$$

2

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$W_4^1 = e^{-j\frac{2\pi}{4}\times 1} = e^{-j\pi/2} = -j$$

$$W_4^0 = 1$$

$$W_4^2 = -W_4^0 = -1$$

$$W_4^3 = -W_4^1 = +j$$

$$W_4^4 = W_4^0 = 1$$

$$W_4^6 = W_4^2 = -1$$

$$W_4^9 = W_4^5 = W_4^1 = -j$$

* Using symmetry and periodicity property of W_N

$$W_N^{kn+N} = W_N^{kn}$$

$$W_N^{kn+N/2} = -W_N^{kn}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

$$\therefore \; X(k) = \{ \underset{\uparrow}{4}, -j2, 0, j2 \}$$

Eg 2 find 4 point DFT of the sequence (3)

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

Soln

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad k = 0, 1, \dots N-1$$

$$X(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn} \qquad k = 0, 1, \dots 3$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

$$\therefore X(k) = \{\underset{\uparrow}{7}, -2-j, 1, -2+j\}$$

IDFT using DFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$n = 0, 1, \ldots N-1$

$$x^*(n) = \frac{1}{N} \left[\sum_{k=0}^{N-1} X(k) W_N^{-kn} \right]^*$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) W_N^{kn}$$

$$= \frac{1}{N} \text{DFT} \left[X^*(k) \right]$$

$$\therefore x(n) = \frac{1}{N} \left\{ \text{DFT} \left[X^*(k) \right] \right\}^*$$

Eg. Find IDFT of 4 point sequence

$X(k) = \{ 4, -j2, 0, j2 \}$

a) using defining equation of IDFT

b) using DFT

Soln a) $$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$n = 0, 1, \cdots N-1$

$$x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, W_4^{-kn}$$

$n = 0, 1, 2, 3$

$$x(0) = \frac{1}{4}\left[X(0) + X(1) + X(2) + X(3)\right]$$

$= 1$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, W_4^{-k}$$

$$= \frac{1}{4}\left[X(0) + X(1) W_4^{-1} + X(2) W_4^{-2} + X(3) W_4^{-3}\right]$$

$= 2$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, W_4^{-2k}$$

$$= \frac{1}{4}\left[X(0) + X(1) W_4^{-2} + X(2) W_4^{-4} + X(3) W_4^{-6}\right]$$

$= 1$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, W_4^{-3k}$$

$$= \frac{1}{4}\left[X(0) + X(1) W_4^{-3} + X(2) W_4^{-6} + X(3) W_4^{-9}\right]$$

$= 0$.

b) $X^*(k) = \{4, j2, 0, -j2\}$

$$DFT[X^*(k)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ j2 \\ 0 \\ -j2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}$$

$$x(n) = IDFT[X(k)] = \frac{1}{N}\left[DFT\{X^*(k)\}\right]^*$$

$$= \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix} = \{\underset{\uparrow}{1}, 2, 1, 0\}$$

Prob:

find IDFT of $\{\underset{\uparrow}{7}, -2-j, 1, -2+j\}$

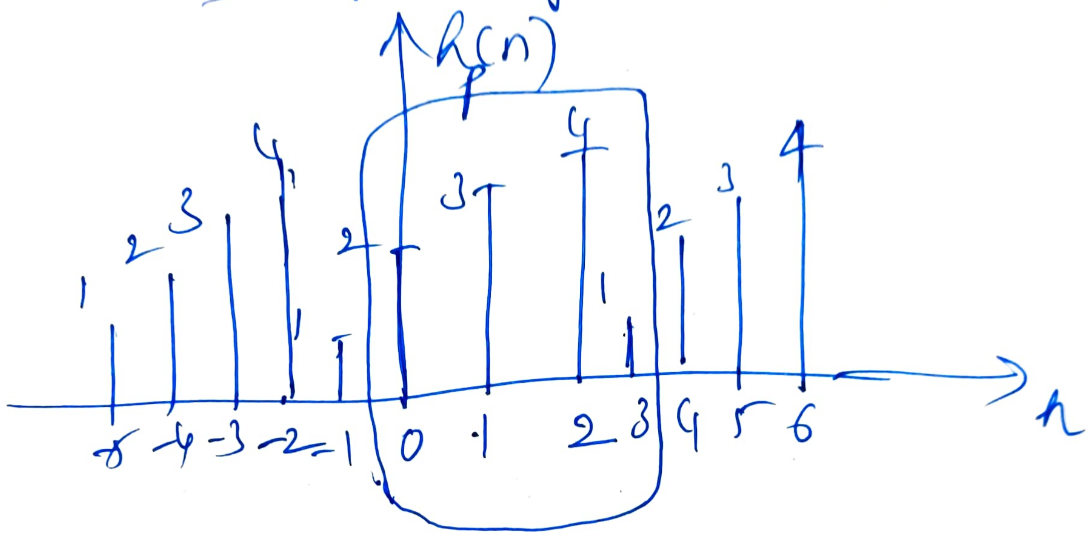
using a) Defining Equation of IDFT

b) using DFT

DFT of a non-Causal sequence

Eg 1) Determine 4 point DFT of the sequence $h(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$

Using Implicit periodicity property



$$h'(n) = \{\underset{\uparrow}{2}, 3, 4, 1\}$$

$$H(k) = \sum_{n=0}^{3} h(n) W_4^{kn} \qquad 0 \le k \le 3$$

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

DFT of a non-Causal sequence

Eg

1. Determine 8 point DFT of the sequence

$$h(n) = \begin{cases} 1/2 & -2 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

plot the magnitude and phase response.

Solu^n^

$$h(n) = \{ 1/2, 1/2, \underset{\uparrow}{1/2}, 1/2, 1/2, 0, 0, 0 \}$$

$$h'(n) = \{ \underset{\uparrow}{1/2}, 1/2, 1/2, 0, 0, 0, 1/2, 1/2 \}$$

$$H(k) = \{ 5/2, 1.207, -1/2, -0.207, 1/2, -0.207, -1/2, 1.207 \}$$

Properties of DFT

1. Conjugate Symmetry

If $x(n)$ is real ~~and~~ then its DFT

$$X(k) = X^*(N-k)$$

Proof:

$$X(k) = \text{DFT}\{x(n)\}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad k = 0, 1 \dots N-1$$

$$X^*(k) = \left\{ \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \right\}^*$$

$$= \sum_{n=0}^{N-1} x^*(n)\, W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x^*(n)\, W_N^{-kn} \cdot W_N^{Nn} \quad \left[\text{Since } W_N^{nN} = 1\right]$$

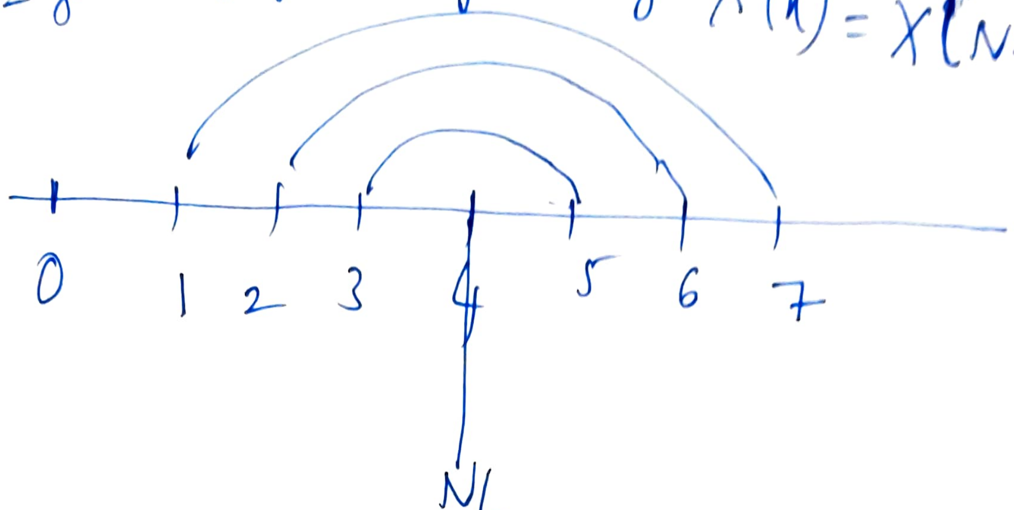
$$= \sum_{n=0}^{N-1} x^*(n)\, W_N^{(N-k)n} \quad \left(\text{Since } x(n) \text{ is real } x^*(n) = x(n)\right)$$

$$X^*(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{(N-k)n} = X(N-k)$$

proved

4

$N = 8$ Conjugate Symmetry $X^*(k) = X(N-k)$



$N/2$.

Prob: Compute 5 point DFT of the sequence $x(n) = \{\underset{\uparrow}{1}, 0, 1, 0, 1\}$ and hence verify symmetry property.

Soln

$W_N = e^{-j\frac{2\pi}{N}}$; $N = 5$; $W_5 = e^{-j\frac{2\pi}{5}}$

$W_5^0 = 1$

$W_5^1 = e^{-j\frac{2\pi}{5}} = 0.309 - j0.951$

$W_5^2 = e^{-j\frac{2\pi}{5}\times 2} = -0.809 - j0.587$

$W_5^3 = e^{-j\frac{2\pi}{5}\times 3} = -0.809 + j0.587$

$W_5^4 = e^{-j\frac{2\pi}{5}\times 4} = 0.309 + j0.951$

$$X(k) = \sum_{n=0}^{4} x(n)\, W_5^{kn}$$

$k = 0, 1, 2, 3, 4$

$= x(0) + x(1)W_5^{k} + x(2)W_5^{2k} + x(3)W_5^{3k} + x(4)W_5^{4k}$

5

$X(k) = 1 + W_5^{2k} + W_5^{4k}$

$X(0) = 1 + 1 + 1 = 3$

$X(1) = 1 + W_5^2 + W_5^4 = 0.5 + j0.364$

$X(2) = 1 + W_5^4 + W_5^8 = 1 + W_5^4 + W_5^3$

$= 0.5 + j1.538$

$X(3) = 1 + W_5^6 + W_5^{12} = 1 + W_5^1 + W_5^2$

$= 0.5 - j1.538$

$X(4) = 1 + W_5^8 + W_5^{16} = 1 + W_5^3 + W_5^1$

$= 0.5 - j0.364$

Verification

Conjugate symmetry property

$X^*(k) = X(N-k)$

$X^*(k) = X(5-k)$

$\therefore X^*(1) = X(4)$ and

$X^*(2) = X(3)$

Hence verified.

Prob 2 HW Compute 5 point DFT of the sequence $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}$ and hence verify conjugate symmetry property.

Properties of DFT

2) Linearity

If $DFT\{x_1(n)\} \rightarrow X_1(k)$

and $DFT\{x_2(n)\} \rightarrow X_2(k)$

then $DFT\{a x_1(n) + b x_2(n)\} \rightarrow a X_1(k) + b X_2(k)$

Example

1) Find 4 point DFT of the sequence

$$x(n) = \cos\left(\frac{\pi}{4} n\right) + \sin\left(\frac{\pi}{4} n\right)$$

using Linearity property.

Solun Let $x(n) = x_1(n) + x_2(n)$

$$x_1(n) = \cos\left(\frac{\pi}{4} n\right) = \left\{1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right\}$$

$$x_2(n) = \sin\left(\frac{\pi}{4} n\right) = \left\{0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}\right\}$$

$$W_4^0 = 1 \qquad W_4^1 = -j$$

$$X_1(k) = DFT\{x_1(n)\}$$

$$X_2(k) = DFT\{x_2(n)\}$$

②

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$X_1(k) = \{ \underset{\uparrow}{1}, 1-j1.414, 1, 1+j1.414 \}$$

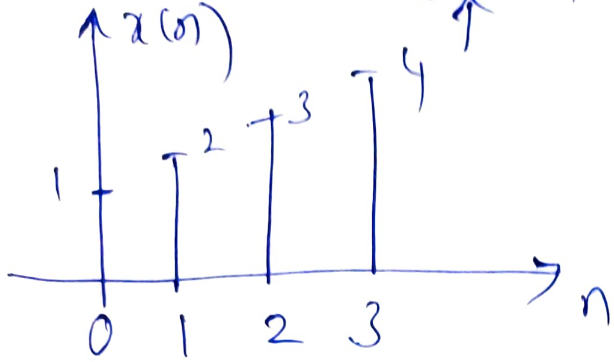
$$\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{bmatrix}$$

$$X_2(k) = \{ \underset{\uparrow}{2.414}, -1, -0.414, -1 \}$$

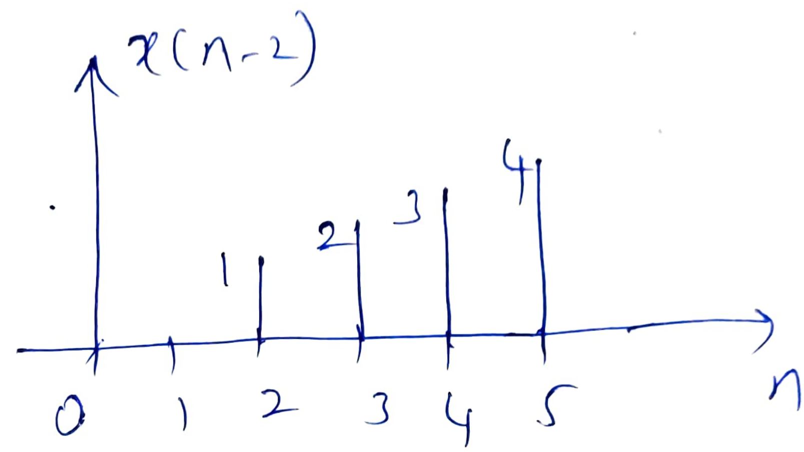
$$\begin{aligned} X(k) &= DFT\{x_1(n)\} + DFT\{x_2(n)\} \\ &= X_1(k) + X_2(k) \\ &= \{ \underset{\uparrow}{3.414}, -j\cancel{0}.414, 0.586, j1.414 \} \end{aligned}$$

Concept of Circular Shift

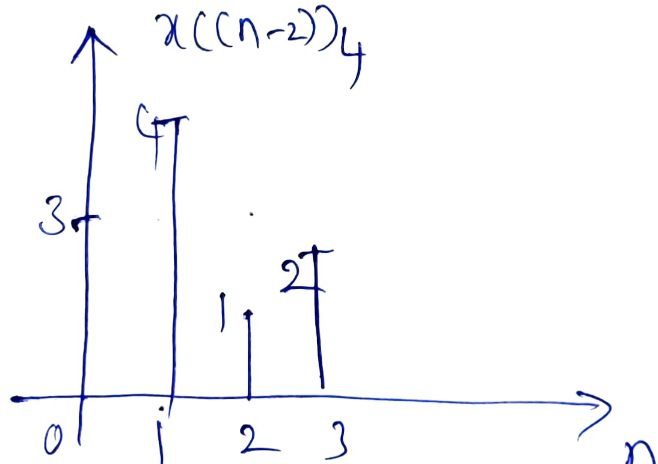
Let $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

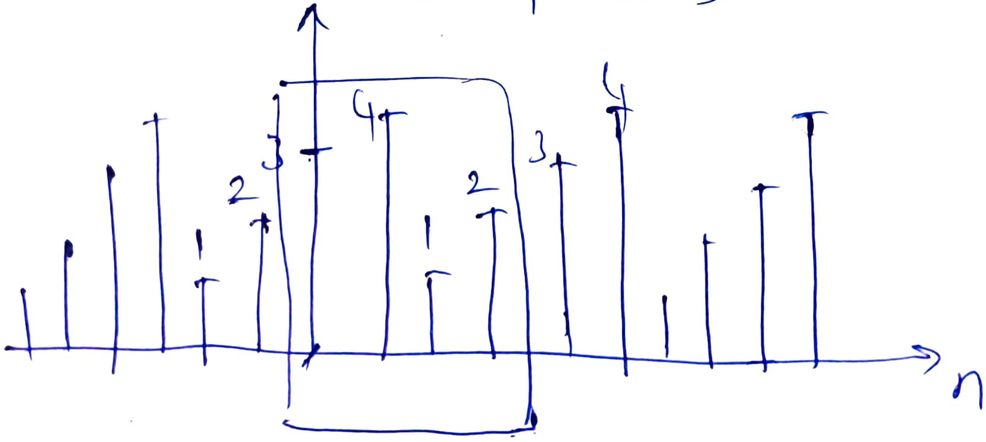


$x(n-2) \Rightarrow$



$x((n-2))_4$





Eg : 2) Let $x(n)$ be an 8 point sequence defined as $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 6, 7, 8\}$

HW

a) Illustrate $x(n)$, $x(n-2)$ and $x((n-2))_8$ graphically.

b) find $X(K)$

c) find DFT $\{x((n-2))_8\}$ using time shift property.

1) Let $x(n)$ be a 4 point sequence defined as $x(n) = \{\underset{\uparrow}{4}, 5, -6, 7\}$

Represent the following graphically

(a) $x(n)$ (b) $x(n-3)$ c) $x((n-3))_4$

(d) find $X(K)$.

3) Circular time shift:

If $DFT\{x(n)\} \rightarrow X(k)$ then,

$$DFT\{x((n-m))_N\} \rightarrow W_N^{mk} X(k)$$

Proof: Using the defining equation of IDFT,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$x((n-m))_N = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-k(n-m)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \{X(k) W_N^{km}\} W_N^{-kn}$$

$$x((n-m))_N = IDFT\{X(k) W_N^{km}\}$$

$$\Rightarrow DFT\{x((n-m))_N\} \rightarrow X(k) W_N^{km}$$

Eg: find 4 point DFT of the sequence $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$. Using circular time shift property, find the DFT of the sequence

$$y(n) = x((n-2))_4 :$$

Soln: $X(K) = DFT\{x(n)\}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix}$$

Circular time shift property,

$$DFT\{x((n-m))_N\} = W_N^{mK} X(K)$$

$$\therefore DFT\{x((n-2))_4\} = W_4^{2K} X(K) = Y(K)$$

$$Y(0) = W_4^0 X(0) = 7.$$

$$Y(1) = W_4^{2\times 1} X(1) = W_4^2 X(1) = +2+j$$

$$Y(2) = W_4^{2\times 2} X(2) = W_4^4 X(2) = 1$$

$$Y(3) = W_4^{3\times 2} X(3) = W_4^6 X(3) = 2-j$$

$$\therefore Y(K) = \{7, 2+j, 1, 2-j\}$$

4) <u>Circular frequency shift</u>

If DFT $\{x(n)\} = X(k)$, then

$$IDFT\{X((k-l))_N\} = W_N^{-ln}\, x(n)$$

<u>Proof:</u>

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad 0 \le k \le N-1$$

$$X((k-l))_N = \sum_{n=0}^{N-1} x(n)\, W_N^{(k-l)n}$$

$$= \sum_{n=0}^{N-1} \{x(n)\, W_N^{-ln}\}\, W_N^{kn}$$

$$X((k-l))_N \doteq DFT\{x(n) \cdot W_N^{-ln}\}$$

Eg 1) Find DFT of the sequence

$x(n) = \{\underset{\uparrow}{1}, 2, 1, 0\}$. using circular frequency shift property, find $y(n)$ if

$Y(k) = X((k-2))_4$.

Soln

$$X(k) = \text{DFT}\{x(n)\}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

$$Y(k) = X((k-2))_2$$

$$= \{\underset{\uparrow}{0}, j2, 4, -j2\}$$

$$y(n) = W_N^{-ln} x(n) = W_4^{-2n} x(n)$$

$$y(0) = W_4^{0} x(0) = 1 \times 1 = 1$$

$$y(1) = W_4^{-2} x(1) = -1 \times 2 = -2$$

$$y(2) = W_4^{-4} x(2) = 1 \times 1 = 1$$

$$y(3) = W_4^{-6} x(3) = -1 \times 0 = 0$$

$$y(n) = \{\underset{\uparrow}{1}, -2, 1, 0\}$$

$\hookrightarrow$ IDFT $\{Y(k)\}$

5) DFT of a Complex Conjugate Sequence

Let $x(n)$ be a complex sequence

if $DFT\{x(n)\} = X(k)$ then

$$DFT\{x^*(n)\} = X^*((-k))_N = X^*(N-k)$$

Proof:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

$$X^*(k) = \left\{\sum_{n=0}^{N-1} x(n) W_N^{kn}\right\}^*$$

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \quad \text{— } A$$

Changing k to $-k$, we get

$$X^*(-k) = \sum_{n=0}^{N-1} x^*(n) W_N^{kn}$$

$$\underline{X^*(-k) = DFT\{x^*(n)\}}$$

Changing k to $N-k$ in A, we get

$$X^*(N-k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-(N-k)n}$$

$$= \sum_{n=0}^{N-1} x^*(n)\, \overset{1}{\cancel{W_N^{-Nn}}}\, W_N^{kn}$$

$$X^*(N-k) = \sum_{n=0}^{N-1} x^*(n)\, W_N^{kn}$$

$$\therefore\ X^*(N-k) = DFT\{x^*(n)\} = X^*((-k))_N$$

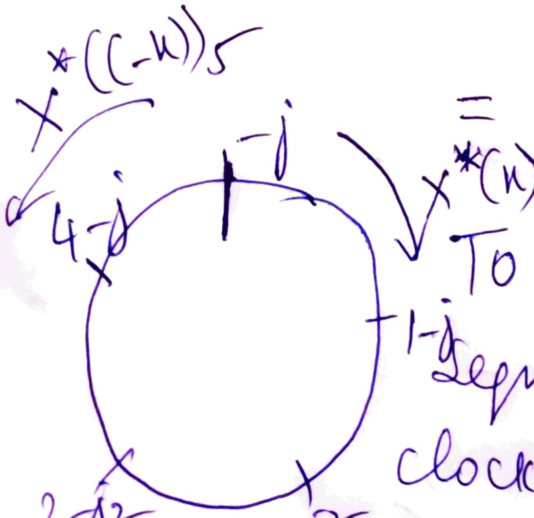
Prob: 5 point DFT of a complex sequence $x(n)$ is given by

$$X(k) = \{\underset{\uparrow}{j},\ 1+j,\ 1+j2,\ 2+j2,\ 4+j\}$$

find $Y(k)$ if $y(n) = x^*(n)$.

Soln

$$Y(k) = DFT\{y(n)\}$$
$$= DFT\{x^*(n)\}$$
$$= X^*((-k))_N = \{\underset{\uparrow}{-j},\ 4-j,\ 2-j2,\ 1-j2,\ 1-j\}$$



To find $X^*((-k))_N$, enter the sequence $X^*(k)$ on a circle in clock wise direction and then read the sequence in anticlock wise direction

6) Circular Convolution in time

$$y(n) = x(n) \circledast h(n)$$
$$= x(n) \; ⓝ \; h(n)$$
$$= x(n) \circledast_N h(n)$$

$$y(n) = \sum_{m=0}^{N-1} x((m))_N \, h((n-m))_N$$

$$\Rightarrow \quad Y(k) = X(k) \, H(k)$$

Proof:

$$y(n) = \text{IDFT}\{Y(k)\}$$
$$= \text{IDFT}\{X(k) \cdot H(k)\}$$
$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \, H(k) \, W_N^{-kn}$$
$$= \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \sum_{l=0}^{N-1} x(l) \, W_N^{lk} \right\} \left\{ \sum_{p=0}^{N-1} h(p) \, W_N^{pk} \right\} W_N^{-kn}$$
$$= \frac{1}{N} \sum_{l=0}^{N-1} x(l) \sum_{p=0}^{N-1} h(p) \sum_{k=0}^{N-1} W_N^{(l+p-n)k}$$

$$= \cancel{\frac{1}{N} \sum_{l=0}^{N-1} x(l) \sum}$$

$$\sum_{k=0}^{N-1} \frac{(l+p-n)k}{N} = N$$

if $l+p-n=0$

or $p = n-l$

$$= \frac{1}{\cancel{N}} \sum_{l=0}^{N-1} x(l)\, h(n-l) \cdot \cancel{N}$$

$$y(n) = \sum_{l=0}^{N-1} x((l))_N\, h((n-l))_N$$

$$= x(n) \circledast h(n)$$

Prob: Find $x_1(n) \circledast x_2(n)$ if

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$ and

$$x_2(n) = \{\underset{\uparrow}{4}, 3, 2, 2\}$$

a) using time domain approach

b) frequency domain approach.

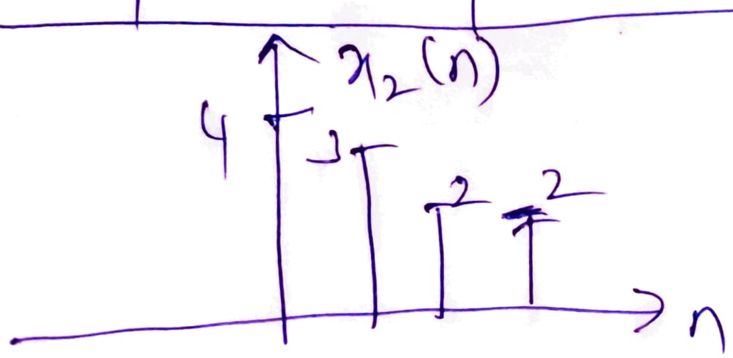
Soln a)

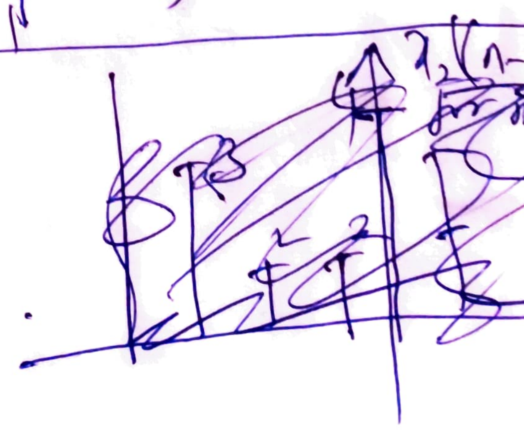
$$y(n) = \sum_{m=0}^{N-1} x_1((m))_N \, x_2((n-m))_N$$

$$0 \le n \le N-1$$

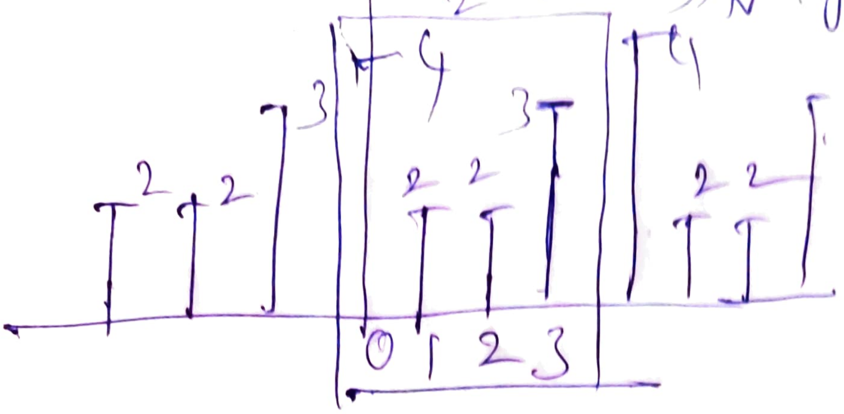
(i) Tabular approach

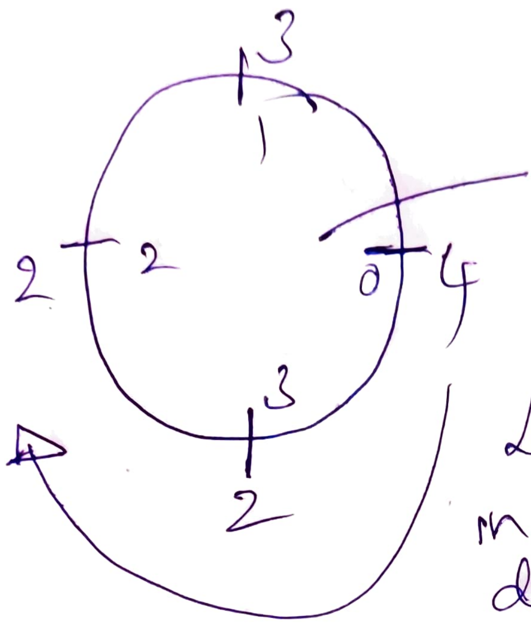
n	$x_1((m))_N$	$x_2((n-m))_N$	$y(n)$
0	(1 2 3 1)	(4 2 2 3)	17
1	(1 2 3 1)	(3 4 2 2)	19
2	(1 2 3 1)	(2 3 4 2)	22
3	(1 2 3 1)	(2 2 3 4)	19





$x_2((n-m))_N$ for $n=0$





$x_2((n-m))_N$

for $n=0$

Locate the samples in anticlock wise direction and read in clock wise direction.

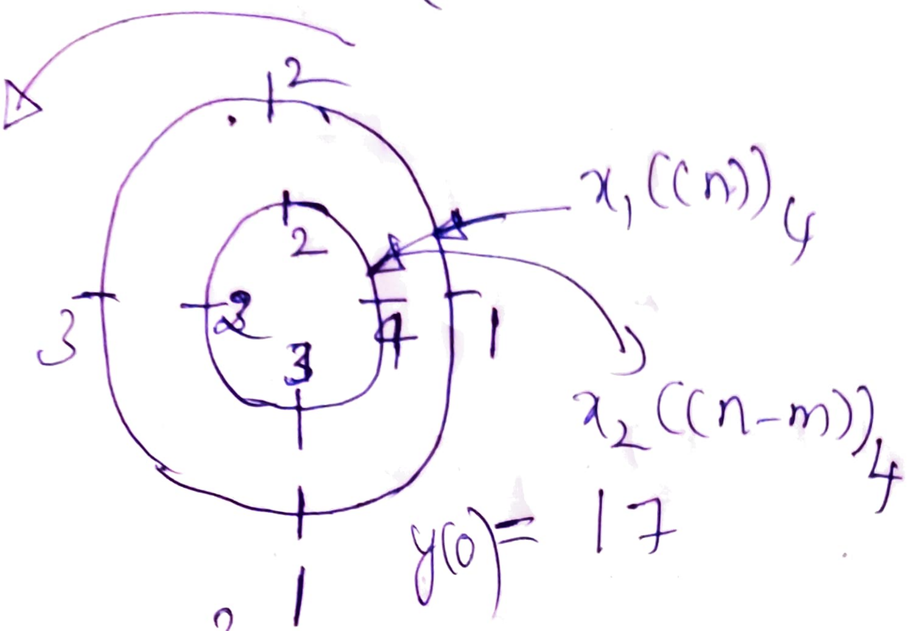
(ii) <u>Matrix approach</u>

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 & 3 \\ 3 & 4 & 2 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

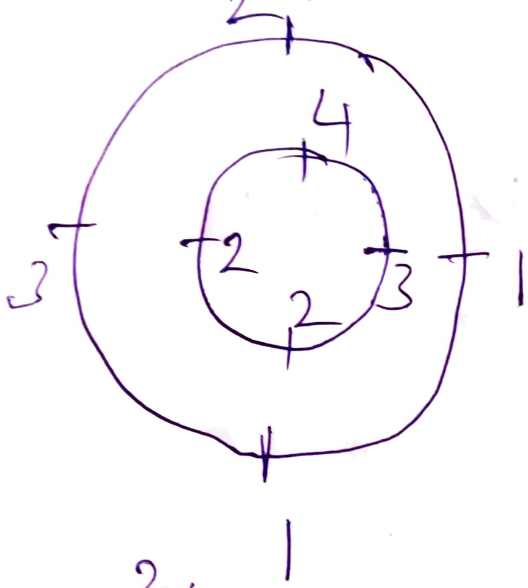
(iii) Circles method

$x_1(n) = \{1, 2, 3, 1\}$

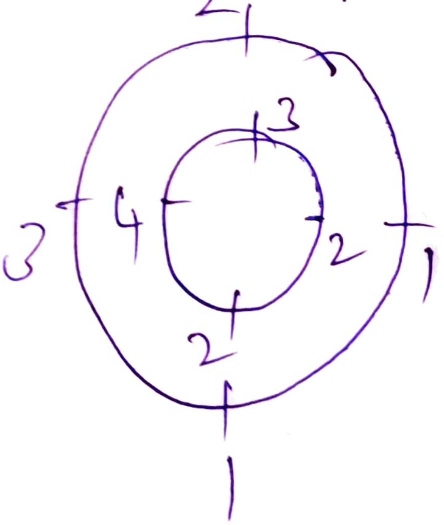
$x_2(n) = \{4, 3, 2, 2\}$



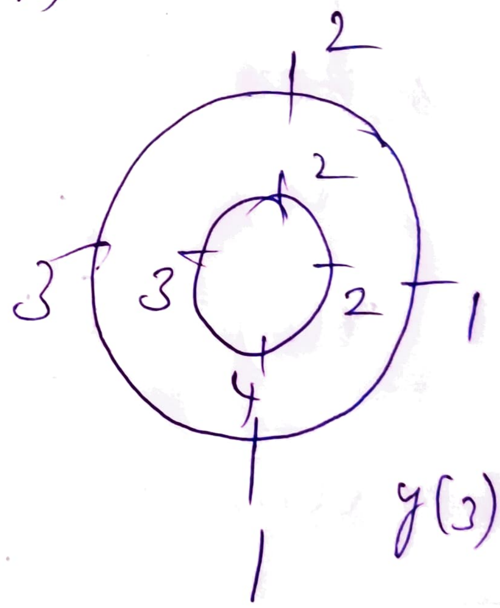
$y(0) = 17$



$y(1) = 19$



$y(2) = 22$



$y(3) = 19$

$y(n) = \{\underset{\uparrow}{17}, 19, 22, 19\}$

8

b) frequency domain approach

$$X_1(k) = \sum_{n=0}^{3} x_1(n) W_4^{kn} = 1 + 2W_4^{k} + 3W_4^{2k} + W_4^{3k}$$

$$X_2(k) = 4 + 3W_4^{k} + 2W_4^{2k} + 2W_4^{3k}$$

$$Y(k) = X_1(k) X_2(k)$$

$$= 17 + 19 W_4^{k} + 22 W_4^{2k} + 19 W_4^{3k}$$

$$y(n) = \{\underset{\uparrow}{17}, 19, 22, 19\}$$

HW Find $y(n) = x(n) \circledast h(n)$ using

a) time domain approach

b) frequency domain approach

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

$$h(n) = \{\underset{\uparrow}{5}, 6, 7, 8\}$$

dsp 9

1) find circular convolution of the following sequences using

a) time domain approach

b) frequency domain approach

$x_1(n) = \{\underset{\uparrow}{2}, 3, 1, 1\}$ and $x_2(n) = \{\underset{\uparrow}{1}, 3, 5, 3\}$

2) find circular convolution of the following sequences

$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

$x_2(n) = \{\underset{\uparrow}{5}, 6, 7\}$

Solu) Make the lengths equal

$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

$x_2(n) = \{\underset{\uparrow}{5}, 6, 7, 0\}$

(2)

3) For the sequences $x_1(n) = \{\underset{\uparrow}{2}, 1, 1, 2\}$

and $x_2(n) = \{\underset{\uparrow}{1}, -1, 1, -1\}$

a) Compute Circular Convolution $[x_3(n)]$

b) Compute linear Convolution $[x_4(n)]$

c) Compute linear Convolution using Circular Convolution $[x_5(n)]$

d) Compute Circular Convolution using linear Convolution $[x_6(n)]$

Sol^n a)

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

b)

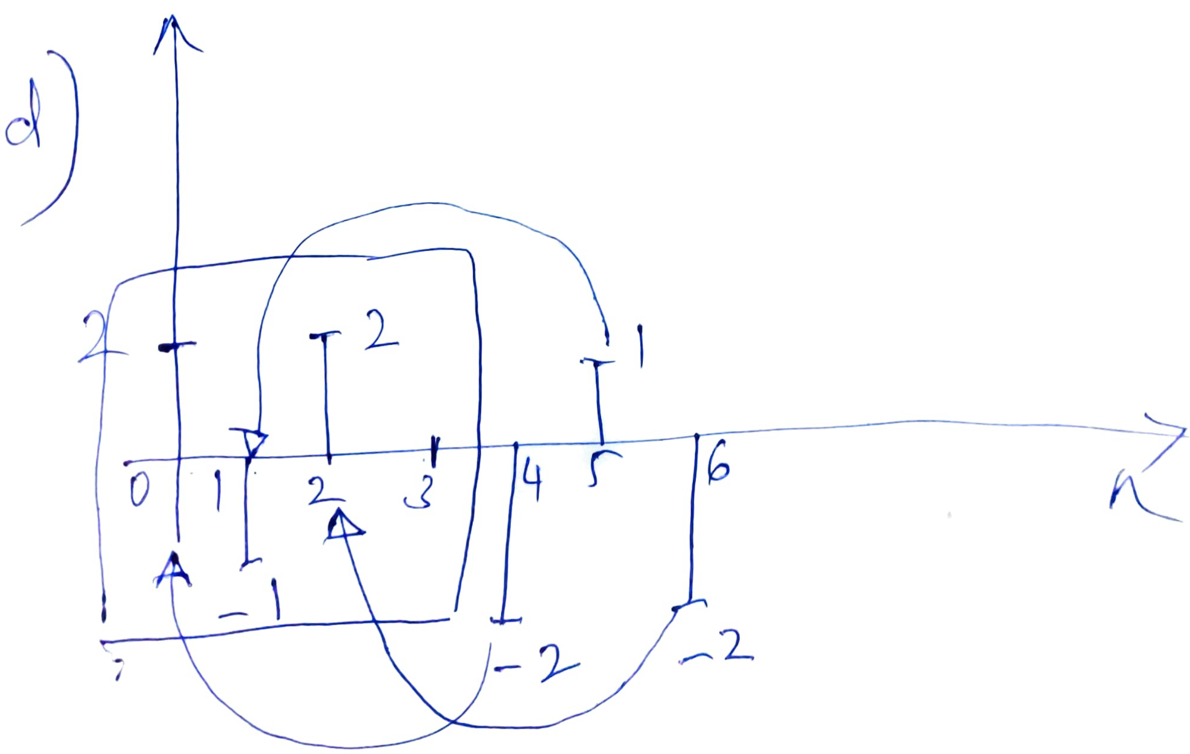
$$\begin{bmatrix} x_4(0) \\ x_4(1) \\ x_4(2) \\ x_4(3) \\ x_4(4) \\ x_4(5) \\ x_4(6) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \\ -2 \\ 1 \\ -2 \end{bmatrix}$$

3

c) $$\begin{bmatrix} x_5(0) \\ x_5(1) \\ x_5(2) \\ x_5(3) \\ x_5(4) \\ x_5(6) \\ x_5(7) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 0 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 0 \\ -2 \\ 1 \\ -2 \end{bmatrix}$$

$$x_1(n) = \{2, 1, 1, 2, 0, 0, 0\}$$

$$x_2(n) = \{1, -1, 1, -1, 0, 0, 0\}$$

d) 

$$x_6(n) = \{2+-2, -1+1, 2-2, 0\}$$
$$= \{0, 0, 0, 0\}$$

HW Repeat (3) for the following sequences:

$$x_1(n) = \{ \underset{\uparrow}{1}, 1, 2, 5 \}$$

$$x_2(n) = \{ \underset{\uparrow}{3}, 2, 4 \}$$

1) For the sequences $x_1(n) = \cos\left(\frac{2\pi n}{N}\right)$ and $x_2(n) = \sin\left(\frac{2\pi n}{N}\right)$ $0 \le n \le N-1$.

find the N point circular convolution

$$x_3(n) = x_1(n) \circledast x_2(n).$$

Sol^n

$$x_3(n) = \sum_{m=0}^{N-1} x_1((m))_N \, x_2((n-m))_N$$

$$\Rightarrow \quad X_3(k) = X_1(k)\, X_2(k)$$

$$x_3(n) = \text{IDFT}\{X_3(k)\}$$

$$x_1(n) = \cos\frac{2\pi n}{N}$$

$$= \frac{1}{2}\left[e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}\right]$$

$$= \frac{1}{2}\left[W_N^{-n} + W_N^{n}\right]$$

②

$$x_2(n) = \sin \frac{2\pi n}{N}$$

$$= \frac{1}{2j}\left[e^{j\frac{2\pi n}{N}} - e^{-j\frac{2\pi n}{N}}\right]$$

$$= \frac{1}{2j}\left[W_N^{-n} - W_N^{n}\right]$$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, W_N^{kn}$$

$$= \frac{1}{2}\left\{\sum_{n=0}^{N-1} W_N^{-n} W_N^{kn} + \sum_{n=0}^{N-1} W_N^{n} W_N^{kn}\right\}$$

$$= \frac{1}{2}\left\{\sum_{n=0}^{N-1} W_N^{(k-1)n} + \sum_{n=0}^{N-1} W_N^{(k+1)n}\right\}$$

$$= \frac{1}{2}\left\{N\delta(k-1) + N\delta(k+1)\right\} \quad \text{— ①}$$

$$\sum_{n=0}^{N-1} W_N^{kn} = N\delta(k)$$

$$X_2(k) = \frac{1}{2j}\left\{N\delta(k-1) - N\delta(k+1)\right\} \quad \text{— ②}$$

$$X_3(k) = X_1(k) \cdot X_2(k)$$

$$= \frac{N^2}{4j}\left\{\delta(k-1) - \delta(k+1)\right\} \quad \text{— ③}$$

(3)

$$X_3(k) = \frac{N}{2} \left\{ \frac{N}{2j} \left[\delta(k-1) - \delta(k+1) \right] \right\} \quad -(4)$$

$$\therefore x_3(n) = \frac{N}{2} \sin\left(\frac{2\pi n}{N}\right) \quad 0 \le n \le N-1$$

Multiplication in time

$$DFT\{x_1(n)\, x_2(n)\} = \frac{1}{N}\left[X_1(k) \circledast_N X_2(k)\right]$$

Proof :

$$DFT\{x_1(n) \cdot x_2(n)\} = \sum_{n=0}^{N-1} \{x_1(n)\, x_2(n)\} W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x_1(n) \left[\frac{1}{N} \sum_{l=0}^{N-1} X_2(l)\, W_N^{-ln}\right] W_N^{kn}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_2(l) \left\{ \sum_{n=0}^{N-1} x_1(n)\, W_N^{(k-l)n} \right\}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_2((l))_N\, X_1((k-l))_N = \frac{1}{N}\{X_1(k) \circledast X_2(k)\}$$

4

Eg: find $X_3(k)$, if $x_3(n) = x_1(n) \cdot x_2(n)$.

where $x_1(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

and $x_2(n) = \cos\left(\frac{\pi}{4} n\right) \quad 0 \le n \le 7$.

Soln

$$X_1(k) = \sum_{n=0}^{7} x_1(n) W_8^{kn}$$

$$= \sum_{n=0}^{7} 1 \times W_8^{kn} = \sum_{n=0}^{7} W_8^{kn}$$

$$= \begin{cases} 0 & k \neq 0 \\ 8 & k = 0 \end{cases}$$

$$\sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$X_1(k) = \{\underset{\uparrow}{8}, 0, 0, 0, 0, 0, 0, 0\}$$

$$X_2(k) = \sum_{n=0}^{7} x_2(n) W_8^{kn} = \sum_{n=0}^{7} \cos\left(\frac{\pi}{4} n\right) W_8^{kn}$$

$$= \frac{1}{2} \sum_{n=0}^{7} \left[e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n} \right] W_8^{kn}$$

(5)

$$= \frac{1}{2}\sum_{n=0}^{7}\left[W_8^{-n} + W_8^{n}\right]W_8^{kn}$$

$e^{j\pi/4\, n}$

$= e^{j\frac{2\pi n}{8}}$

$= W_8^{-n}$

$$= \frac{1}{2}\sum_{n=0}^{7}\left\{W_8^{(k-1)n} + W_8^{(k+1)n}\right\}$$

$$= \frac{1}{2}\left\{\sum_{n=0}^{7} W_8^{(k-1)n} + \sum_{n=0}^{7} W_8^{(k+1)n}\right\}$$

$$= \frac{1}{2}\left[8\delta(k-1) + 8\delta(k+1)\right]$$

$\sum_{n=0}^{N-1} W_N^{(k-k_0)n}$

$= N\delta(k-k_0)$

$$= 4\delta(k-1) + 4\delta(k+1)$$

$$= \begin{cases} 4 & k=1 \\ 4 & k=-1 \text{ or } -1+8=7 \\ 0 & \text{otherwise} \end{cases}$$

$$X_2(k) = \{\underset{\uparrow}{0}, 4, 0, 0, 0, 0, 0, 4\}$$

Now, DFT $\{x_1(n)\, x_2(n)\}$ (6)

$$X_3(k) = \frac{1}{N}\left[X_1(k) \circledast X_2(k)\right]$$

$$\begin{bmatrix} X_3(0) \\ X_3(1) \\ X_3(2) \\ X_3(3) \\ X_3(4) \\ X_3(5) \\ X_3(6) \\ X_3(7) \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 \\ 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

$\therefore X_3(k) = \{\underset{\uparrow}{0}, 4, 0, 0, 0, 0, 0, 4\}$

dsp 11

Parseval's theorem

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

Proof:

LHS

$$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} x^*(n)\, x(n)$$

$$= \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \right\}^* x(n)$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \sum_{k=0}^{N-1} X^*(k)\, W_N^{nk} \right\} x(n)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) \left\{ \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \right\}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k)\, X(k) = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$ RHS

Prob

1. find the energy of the 4 point sequence a) using time domain approach b) using frequency domain approach

$$x(n) = \sin\left(\frac{2\pi}{N} n\right) \quad 0 \le n \le 3$$

Soln

a) $x(n) = \sin\left(\frac{2\pi}{4} n\right) = \sin\left(\frac{\pi}{2} n\right)$ $\quad 0 \le n \le 3$

$= \{0, 1, 0, -1\}$

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = 1^2 + 1^2 = 2 \text{ Joules}$$

b) $X(k) = \text{DFT}\{x(n)\}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2j \\ 0 \\ +2j \end{bmatrix}$$

$$E = \frac{1}{4} \sum_{k=0}^{3} |X(k)|^2 = \frac{1}{4}[+4+4]$$

$= 2$ Joules

(3)

2) Verify parseval's theorem for the sequence $x(n) = \{\underset{\uparrow}{1}, 3, 5, 3\}$

Soln $x(n) = \{\underset{\uparrow}{1}, 3, 5, 3\}$

$$E = \sum_{n=0}^{3} |x(n)|^2 = 1^2 + 3^2 + 5^2 + 3^2$$

$$= 1 + 9 + 25 + 9 = 44 \text{ Joules} \quad -(1)$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \\ 0 \\ -4 \end{bmatrix}$$

$$E = \frac{1}{4} \sum_{k=0}^{3} |X(k)|^2 = \frac{1}{4}\left[12^2 + (-4)^2 + 0 + (-4)^2\right]$$

$$= \frac{1}{4}\left[144 + 16 + 0 + 16\right]$$

$$= \frac{176}{4} = 44 \text{ Joules} \quad -(2)$$

(1) = (2)

Hence verified

(4)

(3) ~~Sol~~ If $x(n) = \{1, 2, 0, 3, -2, 4, 7, 5\}$

find the following

a) $X(0)$ b) $X(4)$ c) $\sum_{k=0}^{7} X(k)$

d) $\sum_{k=0}^{7} |X(k)|^2$

Solⁿ a) $$X(k) = \sum_{n=0}^{7} x(n) W_N^{kn}$$

put $k=0$

$$X(0) = \sum_{n=0}^{7} x(n) = 1+2+0+3-2+4+7+5$$
$$= 20$$

b) put $k = N/2$

$$X(N/2) = \sum_{n=0}^{7} x(n) W_N^{n \cdot N/2}$$

$$= \sum_{n=0}^{7} (-1)^n x(n)$$

$$W_N^{nN/2} = e^{-j2\pi/N \times nN/2}$$
$$= e^{-j\pi n}$$
$$= (-1)^n$$

$$\therefore X(4) = \sum_{n=0}^{7} (-1)^n x(n)$$

(5)

$$X(4) = x(0) - x(1) + x(2) - x(3) + x(4) - x(5) + x(6) - x(7)$$
$$= 1 - 2 + 0 - 3 - 2 - 4 + 7 - 5$$
$$= \underline{\underline{-8}}.$$

c) Using definition of IDFT,

$$x(n) = \frac{1}{N} \sum_{k=0}^{7} X(k) W_N^{-kn} \qquad 0 \leq n \leq 7$$

put $n = 0$

$$x(0) = \frac{1}{8} \sum_{k=0}^{7} X(k)$$

$$\therefore \sum_{k=0}^{7} X(k) = 8 \times x(0) = 8 \times 1 = \underline{\underline{8}}$$

d) $$\sum_{n=0}^{7} |x(n)|^2 = \frac{1}{8} \sum_{k=0}^{7} |X(k)|^2$$

$$\sum_{k=0}^{7} |X(k)|^2 = 8 \sum_{n=0}^{7} |x(n)|^2$$

$$= 8(1 + 4 + 0 + 9 + 4 + 16 + 49 + 25) = 864$$

Appln of DFT in Linear filtering

(6)

Ex 1) An FIR filter has an impulse response of $h(n) = \{\underset{\uparrow}{1}, 2\}$. Determine the response of the filter to the i/p sequence $x(n) = \{\underset{\uparrow}{1}, 2, 3\}$ using DFT and IDFT.

Verify the result by direct computation of linear convolution.

Soln

$L_1 = 2 \quad L_2 = 3$

Length of o/p sequence $2+3-1 = 4$

$h_1(n) = \{1, 2, 0, 0\}$

$x_1(n) = \{1, 2, 3, 0\}$

$$X_1(k) = \sum_{n=0}^{3} x_1(n)\, W_4^{kn}$$

$$= 1 + 2W_4^{k} + 3W_4^{2k}$$

$$H_1(k) = \sum_{n=0}^{3} h_1(n)\, W_4^{kn}$$

$$= 1 + 2W_4^{k}$$

$$Y(K) = X_1(K) \cdot H_1(K)$$

$$= 1 + 4W_4^{K} + 7W_4^{2K} + 6W_4^{3K}$$

$$\therefore y(n) = \{\underset{\uparrow}{1}, 4, 7, 6\}$$

Direct Computation

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 6 \end{bmatrix}$$

Prob: Determine the o/p $y(n)$ for a system with $h(n) = \{1, 2\}$ to an input $x(n) = \{1, 2, 0, 4\}$ using a) Circular Convolution b) DFT and IDFT c) Verify the result by direct Computation