EC540 Control Systems

Mathematical Modeling of Electro-Mechanical Systems-II

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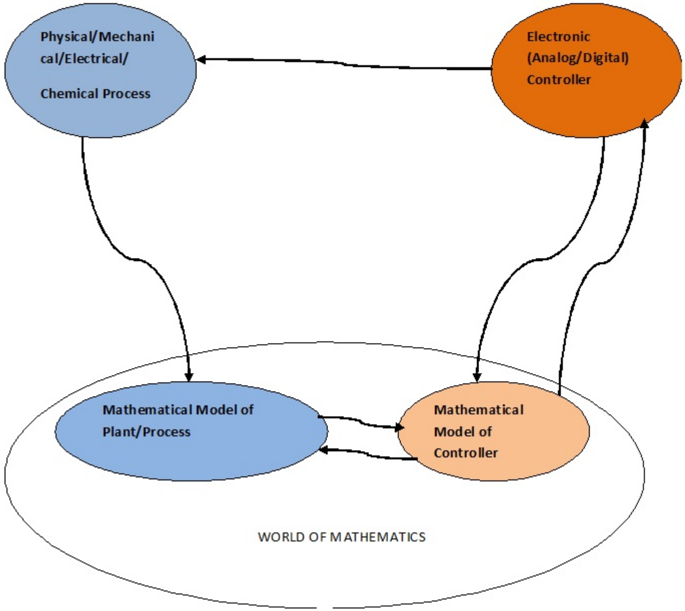
Instructions:

1. Lecture session will be of one hour duration.
2. Fifteen minutes session will be made available for students after the lecture for any question and answers.
3. Regularly review Signals and Systems concepts
4. Regularly visit course webpage.
5. Everyday learn new functions from Octave/Python/MATLAB software
6. Email me on any queries at sudarshan_pk@sjce.ac.in

Why Mathematical Modeling?:



1. Mathematical Model of Electromechanical System

Most electromechanical systems are modeled using following two Faraday's laws:

Law of Motor If a current carrying conductor is arranged such that it is perpendicular to the direction of magnetic field then it results in mechanical force is developed and hence dispacement of conductor occurs in the mutually perpendicular direction as per right hand thumb rule.

$$F(t) = Bli(t)$$

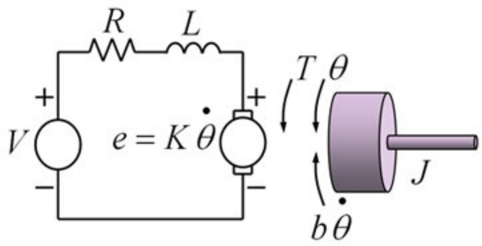
F is force in Newton, B is field strength, l is length of conductor in meters and i(t) is current in Amps

Law of Generator If a moving conductor is placed in the direction perpendicular to magnetic field then an emf is generated in the mutually perpendicular direction.

$$e(t) = Bl\frac{dx(t)}{dt}$$

Example:Mathematical Modelling of a D C Motor

Consider a DC Motor:



DC Motor is an electro-mechanical device. Its operation is based on both law of motor (where mechanical force/torque is generated in mutually perpendicular directions of current and magnetic field) and law of generator (where voltage gets generated in a mutually perpendicular direction of mechanical force/torque and current).

Its various parameters are:

$$
\begin{aligned}
R &= \text{resistance of armature winding (ohms)}, \\
L &= \text{inductance of armature winding (henrys)}, \\
i(t) &= \text{armature current (amperes)}, \\
e_b &= \text{back emf (volts)}, \\
V_{in}(t) &= \text{applied voltage (volts)}, \\
k_1 &= \text{motor torque constant (Newton.meters/amps)}, \\
k_2 &= \text{back emf constant (volts/rad/sec)}, \\
\theta(t) &= \text{motor position (radians)} \\
\omega(t) &= \frac{d\theta(t)}{dt} \text{ motor velocity (rad/sec)}.
\end{aligned}
$$

Equations of DC Motor Its Equations are as follows:

$$\begin{aligned} T(t) &= k_1 i(t) \\ V_{in}(t) &= L\frac{di(t)}{dt} + Ri(t) + e_b(t) \\ e_b(t) &= k_2\frac{d\theta(t)}{dt} \\ T(t) &= b\frac{d\theta(t)}{dt} + J\frac{d^2\theta(t)}{dt^2} \end{aligned} \tag{1}$$

DC Motor

Input to the system, Applied voltage: $u(t) = V_{in}(t)$.

Output of the system, Angular velocity: $y(t) = \frac{d\theta(t)}{dt}$.

$$\begin{array}{rcl} T(s) & = & k_1 I(s) \\ U(s) & = & (Ls+R)I(s) + E_b(s) \\ E_b(s) & = & k_2 Y(s) \\ T(s) & = & bY(s) + JsY(s) \end{array} \qquad (2)$$

Rewriting Equation (2)

$$U(s) = (Ls+R)\frac{(Js+b)}{k_1}Y(s) + k_2 Y(s)$$

Therefore Transfer Function: $\frac{Y(s)}{U(s)} = \frac{k_1}{(Js+b)(Ls+R)+k_1k_2}$

State Space Model

Input to the system, Applied voltage: $u(t) = V_{in}(t)$.
Output of the system, Angular velocity: $y(t) = \frac{d\theta(t)}{dt}$.
By defining the state variables as speed of motor and coil current, i.e.: $x_1(t) = \omega(t) = \frac{d\theta(t)}{dt}$ and $x_2(t) = i(t)$, state-space model will be as follows:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-b}{J} & \frac{k_1}{J} \\ \frac{-k_2}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t), \tag{3}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{4}$$

DC Motor

Input to the system, Applied voltage: $u(t) = V_{in}(t)$.
Output of the system, Angular Position: $y(t) = \theta(t)$.

$$\begin{aligned} T(s) &= k_1 I(s) \\ U(s) &= (Ls+R)I(s) + E_b(s) \\ E_b(s) &= k_2 s Y(s) \\ T(s) &= bsY(s) + Js^2 Y(s) \end{aligned} \tag{5}$$

Rewriting Equation (5)

$$U(s) = (Ls+R)\frac{(Js^2+bs)}{k_1}Y(s) + k_2 s Y(s)$$

Therefore Transfer Function:

$$\frac{Y(s)}{U(s)} = \frac{k_1}{(Js^2+bs)(Ls+R) + k_1 k_2 s}$$

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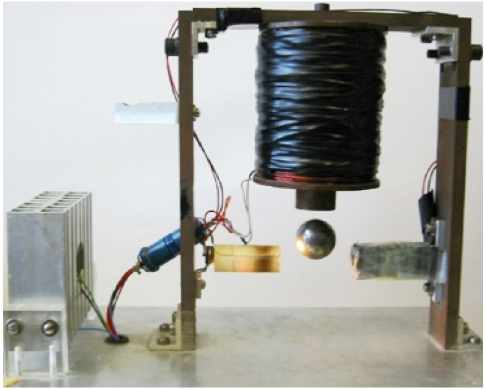
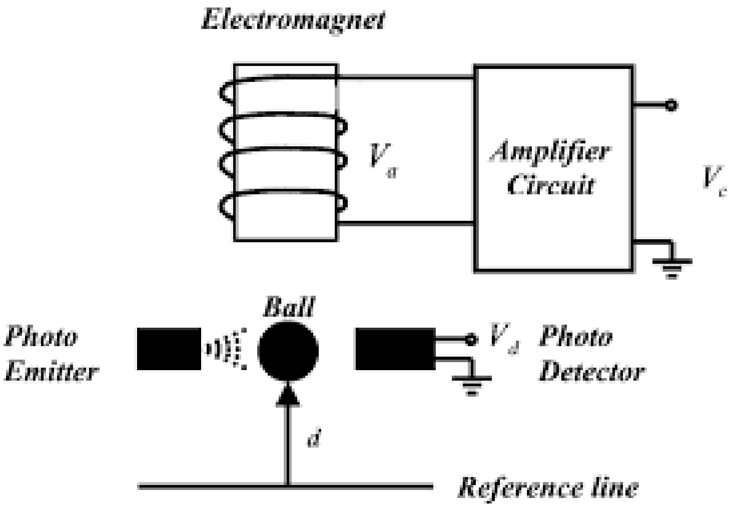
By defining the state variables as speed of motor and coil current, i.e.: $x_1(t) = \omega(t) = \frac{d\theta(t)}{dt}$, $x_2(t) = \theta(t)$ and $x_3(t) = i(t)$, state-space model will be as follows:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \frac{-b}{J} & 0 & \frac{k_1}{J} \\ 1 & 0 & 0 \\ \frac{-k_2}{L} & 0 & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t), \tag{6}$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}. \tag{7}$$

Mathematical Modeling Magnetic Levitation




Electromagnet
V_a
Amplifier
Circuit
V_c
Ball
Photo
Emitter
V_d
Photo
Detector
d
Reference line

Mathematical Modeling
Magnetic Levitation
Equations:

$x_1(t) = h(t)$ gap between coil and metal ball, $x_2(t) = \dot{h}(t)$, $x_3(t) = i(t)$ coil current $u(t) = V_{in}(t)$, $y(t) = h(t)$. β=magnetic force constant depends on number of coil windings

Non linear equations describing the system are:

$$m\frac{d^2h(t)}{dt^2} = mg - \beta(\frac{i^2}{h^2}) \tag{8}$$

$$V_{in}(t) = Ri(t) + L\frac{di(t)}{dt} - 2\beta\frac{dh(t)}{dt}\frac{i}{h^2} \tag{9}$$

Equilibrium point is the current i_0 that holds the ball at h_0 where derivative are zero:

$$i_0 = \sqrt{\frac{gmh_0^2}{\beta}}.$$

Linearized equations using Taylor series approximation are:

$$m\frac{d^2h(t)}{dt^2} = \frac{2\beta i_0^2}{h_0^3}i(t) - \frac{2\beta i_0}{h_0^2}h(t) \tag{10}$$

$$V_{in}(t) = Ri(t) + L\frac{di(t)}{dt} - \frac{2\beta i_0}{h_0^2}\frac{dh(t)}{dt} \tag{11}$$

Linearzation of state-space model

$$\dot{x}_1(t) = f_1(x_1, x_2, x_3, t, u)$$
$$\dot{x}_2(t) = f_2(x_1, x_2, x_3, t, u)$$
$$\dot{x}_3(t) = f_3(x_1, x_2, x_3, t, u)$$

$$\dot{X}(t) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{bmatrix} u(t) \quad (12)$$

Above matrices are evaluated at equilibrium point $(h_0, 0, i_0)$ to get constant matrices A, B.

Mathematical Modeling
Magnetic Levitation: State-space Model

$$\begin{bmatrix} \frac{dh(t)}{dt} \\ \frac{d^2h(t)}{dt^2} \\ \frac{di(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\beta i_0^2}{m h_0^3} & 0 & -2\frac{\beta i_0}{m h_0^2} \\ 0 & \frac{2\beta i_0}{L h_0^2} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} h(t) \\ \frac{dh(t)}{dt} \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t) \tag{13}$$