Unit - III Linear Transformations

mapping and functions

Let X and Y be arbitrary non-empty sets. Suppose to each element in X there is assigned a unique element of Y, the collection f of such assignments is called a mapping or map from X into Y and is denoted by

$$f : X \to Y.$$

The set X is called domain of the mapping & Y is called the co-domain.

$f(x)$ is the unique element of Y that assigns to $x \in X$

If $f(x) = y$ then we say that 'y' is the image of 'x'.

The pre-image of y is

$$\text{preimage}(y) = \{x \in X : f(x) = y\}$$

The range of f is the set of images of elements in X

The functions from a vector space V to another vector space W, that respect the vector space structures. Such a function will be called a linear transformation

Definition:- Let V and W be two vector spaces. A function

$$T : V \to W$$

is called a linear transformation of V into W, if following two properties are true for all $u, v \in V$ and scalars c

Linear Transformations (T)

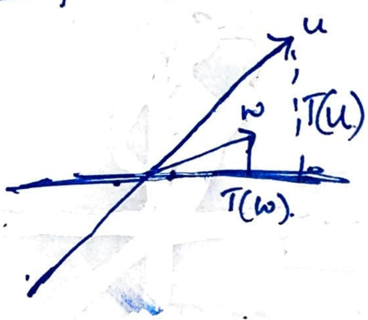
without co-ordinates → no matrix

with co-ordinates → Matrix

$T(v+w) = T(v) + T(w).$

$T(cv) = cT(v)$

Example:- Projection → $T : R^2 \to R^2$



Transformation takes every vector in R^2 i.e in a plane to every vector in R^2

Inputs are mapped to the outputs.

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i) $T(u+v) = T(u) + T(v)$ T preserves additivity

ii) $T(cu) = cT(u)$ → T preserves scalar multiplication

Trivial Example of Linear Transformation

1) Let V & W be two vectorspaces, Define $T : V \to W$ as $T(v) = 0$ for all $v \in V$. $\boxed{0 : V \to W}$

Then T is a linear transformation to be called the zero transformation.

2). Let V be a vector space Define $T : V \to V$ as $T(v) = v$ for all $v \in V \Rightarrow \boxed{I : V \to V}$

Then T is a linear transformation, to be called the identity transformation of V

The difference equation $Ax = b$ & vector equation

$$a_1x_1 + a_2x_2 + \dots a_nx_n = b$$

Matrix equation $Ax = b$ can arise in linear algebra (in applications computer graphics & DSP) in a way that is not directly connected with linear combination of vectors.

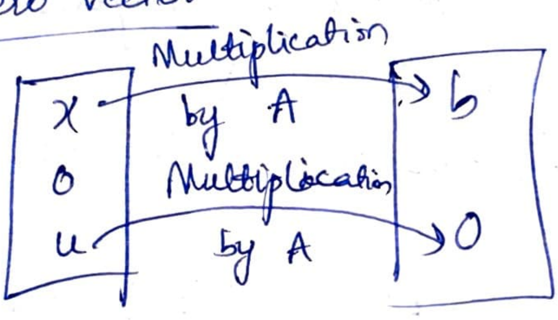
When we think of the matrix A as an object that acts on a vector x by ~~p~~ multiplying to produce a new vector called Ax

Ex:-

$$A = \underbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}_{=b}$$

$$\& \quad A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \text{'0' vector}$$

Multiplication by 'A' transform x into b & transition u into zero vector.



Transformation of vectors by matrix multiplication

Solving $Ax = b$ amounts to find all the vector x in R^4 that transforms into 'b' vector in R^2 (Multiplication by A)

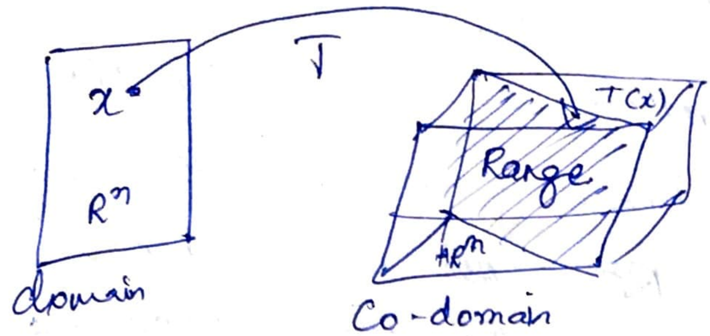
from x to Ax is a transform/function/mapping from one set of vectors to another.

A transformation 'T' from R^n to R^m is a rule that assigned to each vector in R^n a vector $T(x)$ in R^m

$$T: \underset{\text{domain of } T}{R^n} \longrightarrow \underset{\text{Co-domain of } T}{R^m}$$

For x in R^n the vectors $T(x)$ in R^m is called the image of 'x' (under the action of T)

The set of all image $T(x)$ is called the range of 'T'



Range of $T: R^n \to R^m$.

matrix Transformation

Each x in R^n, $T(x)$ is computed as Ax where 'A' is $m \times n$ matrix.

We denote matrix transformation

$$x \to Ax.$$

domain of T is R^n when A has 'n' columns.
Co-domain of T is R^m. when each column of A has m entries

Properties of Linear Transformations

Theorem Let V and W be two vectorspaces. Suppose $T: V \to W$ is a linear transformation. then

1) $T(0) = 0$
2) $T(-v) = -T(v)$ for all $v \in V$
3) $T(u-v) = T(u) - T(v)$ for all $u, v \in V$
4) If $v = c_1v_1 + c_2v_2 + c_3v_3 + \cdots c_nv_n$

Then $T(v) = T(c_1v_1 + c_2v_2 + \cdots + c_nv_n)$

$= c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n)$

Prob:- (1) Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ & $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

define a transformation $T: R^2 \to R^3$ by $T(x) = Ax$

Find (a) $T(u) \longrightarrow$ image of U under the transformation.

(b) Find x in R^2 whose image under T is 'B'

(c) Is there more than one x whose image under T is B

(d) Find if 'c' is in the range of transformation 'T'

Soln $T(x) = Ax = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$

(a) $T(u) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ +1 \\ -9 \end{bmatrix}$

(b) $A : b \Rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \xrightarrow[R_3 + R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix}$

$\downarrow R_2/R_7,\ R_3/2$

$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3 - R_2} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

$2x_2 = -1$ $\boxed{x_2 = -\frac{1}{2}}$

$x_1 - 3x_2 = 3$

$x_1 = 3 + 3\left(-\frac{1}{2}\right) = 3 - \frac{3}{2} \Rightarrow \boxed{x_1 = 1.5}$

(c) x has a unique soln $\therefore$ there is only on x under $T: R^2 \to R^3$ whose image is B.

(d). The vector 'c' is in the range of 'T' if 'c' is the image of some x in R^2 ie if c is in $T(x)$ for some x

Check $Ax = C$ is consistent or not

$$\begin{pmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \xrightarrow[R_2 - 3R_1]{R_3 + R_1} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \xrightarrow[R_3/4]{R_2/7} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$\downarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$\therefore$ It is inconsistent

Thus is not in the range of transformation 'T'

Example: $T : R^2 \to R^3$ defined by $T(x_1, x_2) = (x_1, x_2, x_1 - x_2)$ $x \in R^3$.

Prove that it is a linear transformation

soln: Let $x, y \in R^2$

$x = (x_1, x_2)$ $y = (y_1, y_2)$

To prove that it is a linear tranform we need to s.t

$T(u+v) = T(u) + T(v)$ &

$T(\alpha u) = \alpha T(u)$

$T(x+y) = T((x_1 + y_1), (x_2 + y_2))$

$T(z) = T(z_1, z_2)$ where $\boxed{z_1 = x_1 + y_1 \text{ \& } z_2 = x_2 + y_2}$ (1)

from the defn of $T(z) = (z_1, z_2, z_1 - z_2)$

Sub for z_1 & z_2 from (1)

$T(z) = (x_1 + y_1, x_2 + y_2, x_1 - x_2 + y_1 - y_2)$

$T(z) = (x_1, x_2, x_1 - x_2) + (y_1, y_2, y_1 - y_2)$

$\boxed{T(x+y) = T(x) + T(y)}$ proved

$T(\alpha x) = T(\alpha(x_1, x_2))$

$= T(\alpha x_1, \alpha x_2)$

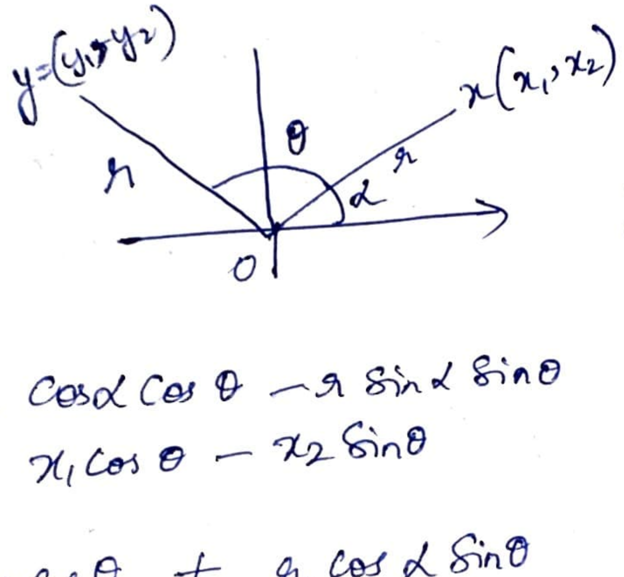
$= (\alpha x_1, \alpha x_2, \alpha(x_1) - \alpha(x_2))$

$= \alpha(x_1, x_2, x_1 - x_2)$

$\boxed{T(\alpha x) = \alpha T(x)}$ proved.

Thus the $R^2 \to R^3$ given by $(x_1, x_2, x_1 - x_2)$ is a Linear transformation.

⇒ Rotation



$x_1 = r\cos\alpha$

$x_2 = r\sin\alpha$

$y_1 = r\cos(\alpha+\theta)$

$y_2 = r\sin(\alpha+\theta)$

$y_1 = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$

$y_1 = x_1\cos\theta - x_2\sin\theta$

$y_2 = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$

$y_2 = x_2\cos\theta + x_1\sin\theta \Rightarrow x_1\sin\theta + x_2\cos\theta$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$y = Ax$. $\quad T: R^2 \to R^2 \longrightarrow$ Rotation from R^2 to R^2

$T(x) = A(x)$.

hence T is linear transformation

Example Projection along a vector in $\mathbb{R}^n$

Projections in $\mathbb{R}^n$ is a good class of example of L.T.

Suppose $v \in \mathbb{R}^n$ is a vector. Then for $u \in \mathbb{R}^n$

define $\quad \mathrm{Proj}_v(u) = \dfrac{v \cdot u}{\|v\|^2} v$

$\mathrm{Proj}_v : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.

Problems :- Let $T(v_1, v_2, v_3) = (2v_1 + v_2,\ 2v_2 - 3v_1,\ v_1 - v_3)$

i) Compute $T(-4, 5, 1)$ (ii) Compute the preimage of $(4, 1, -1)$

(i) $T(-4,5,1) = (-8+5,\ 10+12,\ -4-1)$

$= (-3,\ 22,\ -5)$

(ii) Let v_1, v_2, v_3 be the pre-image of $(4, 1, -1)$

then $T(v_1, v_2, v_3) = (2v_1 + v_2,\ 2v_2 - 3v_1,\ v_1 - v_3) = (4, 1, -1)$

$$2v_1 + v_2 = 4$$
$$2v_2 - 3v_1 = 1$$
$$v_1 - v_3 = -1$$

Solve the s/m of eqn $A : b$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 4 \\ -3 & 2 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{array}\right] \xrightarrow[R_2 + \frac{3R_1}{2}]{R_3 - \frac{R_1}{2}} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 4 \\ 0 & 7/2 & 0 & 7 \\ 0 & -1/2 & -1 & -3 \end{array}\right] \xrightarrow{R_2/7} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 4 \\ 0 & 1/2 & 0 & 1 \\ 0 & -1/2 & -1 & -3 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 4 \\ 0 & 1/2 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{array}\right] \xrightarrow{R_1/2} \left[\begin{array}{ccc|c} 1 & 1/2 & 0 & 2 \\ 0 & 1/2 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{array}\right] \xrightarrow[R_3(-1)]{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & 1 \\ 0 & 0 & +1 & +2 \end{array}\right]$$

$\boxed{v_1 = 1}$ $\quad \boxed{v_2 = 2}$

$\boxed{v_3 = 2}$.

Preimage of $(4, 1, -1) = (1, 2, 2)$

② Determine whether the $T: R^2 \to R^2$
$T(x,y) = (x^2, y)$ is linear ?

check $T(u+v) = T(u) + T(v)$

$T(\alpha u) = \alpha T(u)$.

③ Let $A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$ Let $T: R^5 \to R^2$ be the linear transformation $T(x) = Ax$

(i) Compute $T(1,0,-1,3,0)$ (ii) Compute preimage, under T of $(-1,8)$

(i). $T(1,0,-1,3,0) = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$

(ii) ~~$(1,8) = T$~~ $T(x_1, x_2, x_3, x_4, x_5) = (-1, 8)$

$$\begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = (-1, 8)$$

$$\left[\begin{array}{ccccc|c} -1 & 2 & 1 & 3 & 4 & -1 \\ 0 & 0 & 2 & -1 & 0 & 8 \end{array}\right] \Rightarrow \begin{bmatrix} 5+2t+3.5s+4u \\ t \\ 4+0.5s \\ s \\ u \end{bmatrix}$$

④ Let $T: R^3 \to R^3$ be a linear transformation such that
$T(1,0,0) = (2,4,-1)$, $T(0,1,0) = (1,3,-2)$ $T(0,0,1) = (0,-2,2)$

Compute $T(-2,4,-1)$

Soln $(-2,4,-1) = -2(1,0,0) + 4(0,1,0) - 1(0,0,1)$

So $T(-2,4,1) = -2T(1,0,0) + 4T(0,1,0) - 1T(0,0,1)$
$= (3,5,-1)$

⇒ A linear transformation $T: V \to V$ can be defined, simply by assigning values of $T(v_i)$ for any basis $(v_1, v_2 \dots v_n)$ of V

In the above problem the values are assigned to std basis $\{e_1, e_2, e_3\}$ of R^3.

(4) Let $T: R^2 \to R^2$ be the linear transformation such that $T(1,1) = (0,2)$ and $T(1,-1) = (2,0)$ (i) Compute $T(1,4)$
ii) Compute $T(-2,1)$.

Matrices for Linear Transformations

Vectors in R^n are column matrices. The standard basis of R^n is given by

$$B = \{e_1, e_2 \dots e_n\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

Let $T: R^n \to R^m$ be a L.T then

$$T(e_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \quad T(e_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \dots \quad T(e_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

These columns $T(e_1), T(e_2), \dots T(e_n)$ form a $m \times n$ matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

The matrix A has the property that

$$T(v) = Av \quad \text{for all } v \in R^n.$$

This matrix A is called the standard matrix of T

→ Let V be an n-dimensional vectorspace over the field F and let W be an m-dimensional vector space over F. For each pair of ordered basis B and B' for V & W respectively. The function which assigns a linear transformation T in the matrix relative to B, B' is an isomorphism b/w the space L(V, W) & the space of all m×n matrix over field

Kernel And Range of Linear Transformation

Let V, W be two vectorspaces and $T: V \to W$ a linear transformation then Kernel of T or null space of T is denoted by $Ker(T) / N(T)$

$$Ker(T) \text{ or } N(T) = \{ x \in V : T(x) = 0 \} \leq V$$

Ker(T) is the set of all the vectors whose transformation is '0'. i.e $\boxed{T(x) = 0, \forall x \in V}$

Ker(T) is a subspace of V.

Let $x, y \in N(T)$ then $T(x) = 0$ & $T(y) = 0$

$$T(x+y) = T(x) + T(y) = 0 + 0 = 0$$
$$T(\alpha x) = \alpha T(x) = \alpha \cdot 0 = 0$$

Range of T denoted by range(T) is given by

$$R(T) = \{ w \in W \text{ such that } w = T(x) \text{ for some } x \text{ in } V$$

Range of T is a subspace of W

Prove that R(T) is a subspace of W

$0: V \to W$ what is $N(0)$ & Range of 0 is $R(0)$

$$N(0) = V$$
$$R(0) = 0$$

Theorem Let $T: V \to W$ is linear then where V, W are vectorspaces

(1) $T(0) = 0$

(2) $T(u-v) = T(u) - T(v)$ for all $u, v \in V$

(3) $T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k) = \alpha_1 T(u_1) + \dots \alpha_k T(u_k)$.

where $\alpha_1, \alpha_2 \dots \alpha_k \in \mathbb{R}$.

$u_1, u_2, \dots u_k \in V$.

Proof:-

(1) $$T(0) = T(0.0) = 0T(0) = 0.$$

(2) $$T(u-v) = T(u+(-1)v) = T(u) + (-1)T(v)$$
$$= T(u) - T(v).$$

(3) $$T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k) = T(\alpha_1 u_1) + T(\alpha_2 u_2) + \dots T(\alpha_k u_k)$$
$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots \alpha_k T(u_k)$$

Defn Let V and W be vectorspaces over the field F and let T be a linear transformation from V into W. The nullspace of T is the set of all vectors α in V such that $T\alpha = 0$

If V is finite dimensional, the rank of T is the dimension of the range of T and the nullity of T is the dimention of the nullspace of T

Rank Nullity Dimension Theorem (RND theorem)

Theorem: Let V and W be the vectorspaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite-dimensional then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$
$$r + K = n$$

Proof:- Let $\{\alpha_1 \dots \alpha_k\}$ be a basis for N, the null space of T. There are vectors $\alpha_{k+1} \dots \alpha_n$ in V such that $\{\alpha_1, \dots, \alpha_n\}$ is a basis of V.

We shall prove $\{T\alpha_{k+1}, \dots T\alpha_n\}$ is the basis for Range of T

The vectors $T\alpha_{k+1}, \dots T\alpha_n$ certainly span the range of T,

and since $T\alpha_j = 0$, for $j \le k$.

$T\alpha_{k+1} \dots T\alpha_n \rightarrow$ span the range.

Let us check whether these vectors are L.I.

Scalars c_i

$$\sum_{i=k+1}^{n} c_i(T\alpha_i) = 0$$

$$T\left(\sum_{i=k+1}^{n} c_i\alpha_i\right) = 0$$

vector $\boxed{\alpha = \sum_{i=k+1}^{n} c_i\alpha_i}$ is in the null space of T — 1

$\alpha_1 \dots \alpha_k$ basis for N, there must be scalars

$b_1 \dots b_k$ such that $\alpha = \sum_{i=1}^{k} b_i\alpha_i$ —— 2

1 - 2

$$\sum_{i=1}^{k} b_i\alpha_i - \sum_{j=k+1}^{n} c_j\alpha_j = 0$$

$\alpha_1 \dots \alpha_n$ are Linearly independent

$b_1 = b_2 \dots b_k = c_{k+1} \dots c_n = 0$

$r \rightarrow$ rank of T, $k \rightarrow$ nullity of T and n is dimension of T

$T\alpha_{k+1} \dots T\alpha_n \rightarrow$ basis ~~for~~ Range of T

$$r = n - k.$$

$$n = k + r.$$

Theorem

If A is an $m \times n$ matrix with entries in the field F then row rank(A) = column rank(A)

Proof: Let $T : F^{n \times 1} \to F^{m \times 1}$ defined by

$$T(X) = AX$$

~~The~~ the null space of T is a solution space for the s/m

$AX = 0$

(i.e Set of all column matrices X such that $AX = 0$)

The Range of T is the set of all $m \times 1$ column matrices Y such that $AX = Y$ has a solution for X

If $A_1, \ldots A_n$ are columns of matrix A then

$$AX = A_1x_1 + A_2x_2 \cdots \cdots + A_nx_n$$

So that Range of T is the subspace spanned by the columns of A in other words the Range of T is the column space of A

$\therefore$ rank(T) = column rank(A)

If 'S' is the soln space for $AX = 0$ then by RND theorem

$$\text{rank}(T) + \dim(S) = n$$

$$\text{column rank}(A) + \dim(S) = n. \quad \text{—— (1)}$$

If 'r' is the dimension of row space of A, then the solution space S has a basis ~~of~~ of $n-r$ vectors

$$\dim(S) = n - \text{row rank}(A) \quad \text{—— (2)}$$

Comparing (1) & (2)

row rank(A) = Column rank(A)

Theorem:- Let V, W and Z be vectorspaces over the field F. Let T be a linear transformation from V into W and U a linear transformation from W into Z. ⑥

Then the composed function UT defined by

$$(UT)(\alpha) = U(T(\alpha))$$ is a linear transformation from V into Z

Proof:-
$$(UT)(c\alpha+\beta) = U[T(c\alpha+\beta)]$$
$$= U(cT\alpha + T\beta)$$
$$= c[U(T\alpha)] + U(T\beta)$$
$$= c(UT)\alpha + (UT)\beta$$

If V is a vectorspace over the field F, a linear operator on V is a linear transformation from V into V.

Lemma:- Let V be a vector space over the field F, Let U, T_1 & T_2 be the linear operators on V, Let c be an element of F

ⓐ $IU = UI = U$

ⓑ $U(T_1+T_2) = UT_1 + UT_2$

$(T_1+T_2)U = T_1U + T_2U$

ⓒ $c(UT_1) = (cU)T_1 = U(cT_1)$.

Proof:- ⓐ When multiplied with the identity operator the transformation is same

Identity function $I(U) = IU = U$.

(b) $[U(T_1+T_2)]\alpha = U[(T_1+T_2)(\alpha)]$
$$= U(T_1\alpha + T_2\alpha)$$
$$= U(T_1\alpha) + U(T_2\alpha)$$
$$= (UT_1)\alpha + (UT_2)\alpha$$

Thus $\boxed{U(T_1+T_2) = UT_1 + UT_2}$

$[(T_1+T_2)U]\alpha = (T_1+T_2)(U\alpha)$
$$= T_1(U\alpha) + T_2(U\alpha)$$
$$= (T_1U)\alpha + (T_2U)\alpha$$

$$\boxed{\therefore (T_1+T_2)U = T_1U + T_2U.}$$

(c) $c(UT_1) = (cU)T_1 = U(cT_1)$

Consider $[cUT_1](\alpha) = (cU)T_1(\alpha)$
$$= c\,U(T_1(\alpha)) = (cU)T_1(\alpha)$$
$$= c(UT_1(\alpha))$$
$$= c(UT_1)\alpha$$
$$= U(cT_1)\alpha.$$

$$\boxed{\therefore c(UT_1) = (cU)T_1 = U(cT_1)}$$

$\Rightarrow$ If A is a $m \times n$ matrix with entries in F, we have the linear transformation T defined by $T(x) = Ax$ from $F^{n\times 1}$ into $F^{m\times 1}$.

If B is a $p \times m$ matrix ~~matrix~~, we have the linear transformation U from $F^{m\times 1}$ into $F^{p\times 1}$ defined by $U(Y) = BY$

The composition UT is described by

$$
\begin{aligned}
(UT)x &= U(T(x)) \\
&= U(Ax) \\
&= B(Ax) \\
&= (BA)x.
\end{aligned}
$$

The transformation UT is the left multiplication by the product matrix BA

The function T from V into W is called invertible, if there exists a function U from W into V such that UT is the identity function ~~of~~ on V and TU is the identity function on W.

If T is invertible, the function U is unique and and is denoted by T^{-1}.

T is invertible if and only if

1) T is 1:1, that is $T\alpha = T\beta$ implies $\alpha = \beta$ $\boxed{\text{injective}}$

(2) T is outo, i.e the range of T is (all of) W
Subjective

for every $w \in W$ $\exists$ $x \in V$ such that $T(x) = W$

Thm Let V and W be vectorspaces over the field F and let T be a linear transformation from V into W. If T is invertible, then the inverse fn T^{-1} is a linear transformation from W into V

Proof: When T is one-one and onto, then there is inverse ~~fun~~ function (T^{-1}) which maps W onto V such that

$T^{-1}T$ is the identity matrix on V

and TT^{-1} is the identity matrix on V

We need to prove if a linear function T is invertible then the inverse T^{-1} is also linear

Let β_1 and β_2 be vectors in W and let c be a scalar

We need to show that

$$T^{-1}(c\beta_1 + \beta_2) = cT^{-1}(\beta_1) + T^{-1}\beta_2 .$$

Let $\alpha_i = T^{-1}\beta_i \cdot \; i = 1, 2$

i.e α_i is a unique vector in V $T\alpha_i = \beta_i$

Since T is linear

$$T(c\alpha_1 + \alpha_2) = cT\alpha_1 + T\alpha_2$$
$$= c\beta_1 + \beta_2 .$$

Thus $c\alpha_1 + \alpha_2$ is the unique vector in V which is sent by T into $c\beta_1 + \beta_2$

thus $$T^{-1}(c\beta_1 + \beta_2) = c\alpha_1 + \alpha_2$$
$$= c(T^{-1}\beta_1) + T^{-1}\beta_2$$

and T^{-1} is linear

$\Rightarrow$ Invertible linear Transformation T from V onto W and an invertible linear transformation U from W onto Z the UT is invertible and $(UT)^{-1} = T^{-1}U^{-1}$

$\Rightarrow$ If T is linear then $T(\alpha - \beta) = T\alpha - T\beta$.

$\therefore T\alpha = T\beta$ iff $T(\alpha - \beta) = 0$.

$\Rightarrow$ Let T be a linear transformation from V into W. Then T is non-singular iff T carries each linearly independent subset of V onto a linearly independent subset of W

$\Longrightarrow$ Let F be a field and let T be the linear operator on F^2 defined by

$$T(x_1, x_2) = (x_1 + x_2, x_1)$$

Then T is non-singular $\because$ if $T(x_1, x_2) = 0$

then $x_2 + x_1 = 0$

$x_1 = 0$

$$\boxed{\therefore x_1 = x_2 = 0}$$

$T \rightarrow$ onto

Let (z_1, z_2) be any vector in F^2

For $(z_1 \,\&\, z_2)$ to be in the Range of T then we must find scalars x_1 & x_2 Such that

$$x_1 + x_2 = z_1$$
$$x_1 = z_2.$$

Soln. $\Rightarrow$ $x_1 = z_2$, $x_2 = z_1 - z_2$.

This is the formula for T^{-1}

i.e. $T^{-1}(z_1, z_2) = (z_2, z_1 - z_2)$

Prob 1 Let T be $R^4 \to R^3$ be a linear Transformation defined by

$$T(x, y, z, t) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t)$$

Find (a) $R(T)$ (b) $N(T)$

Soln

$$T(e_1, e_2, e_3, e_4) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t)$$

$T(1,0,0,0) = (1, 1, 1)$ $\quad$ $T(0,1,0,0) = (-1\ 0\ 1)$

$T(0,0,1,0) = (1, 2, 3)$ $\quad$ $T(0,0,0,1) = (1, -1, -3)$

To find Rank

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} Te_1 & Te_2 & Te_3 & Te_4 \end{bmatrix}$$

$$\dim(M) = \text{rank}(T) = 2$$

(b) $N(T)$

By RND theorem $\quad \dim(V) = \text{rank}(T) + \text{nullity}(T)$

$$4 = 2 + \text{Nullity}(T)$$

$$\boxed{\text{Nullity}(T) = 2}$$

Alt: $N(T)$ can be identified by the no. of free variables

$$\left.\begin{array}{l} x - y + z + t = 0 \\ y + z - 2t = 0 \end{array}\right\} \rightarrow 2 \text{ free variables}$$

$$\boxed{\therefore N(T) = 2}$$

2) $T: R^4 \rightarrow R^3$ where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & 3 \end{bmatrix}$ find $R(T)$ & $N(A)$

$$T(x, y, z, t) = (x + 2y + 3z + t,\ x + 3y + 5z - 2t,\ 3x + 8y + 13z + 3t)$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & 3 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 2 & -3 \\ 0 & 2 & 4 & 0 \end{bmatrix} \rightarrow R(T) = 3$$

By RND theorem

$$N(T) = \dim - R(T)$$
$$= 4 - 3 = 1$$

$$\boxed{N(T) = 1}.$$

Isomorphism → We say the T is isomorphism, if T is one-one and onto. It follows that

T is an isomorphism if $\ker(T) = \{0\}$ and $\text{range}(T) = W$.

Theorem:- Let $T: R^n \rightarrow R^n$ be a linear transformations and let A be the standard matrix of T. Then the following are equivalent,

1. T is invertible
2. T is an isomorphism
3. A is invertible

And if T is invertible, then the standard matrix of T^{-1} is A^{-1}

Proof:- T is isomorphism if T is 1 to 1 and onto

Proof ① - ②.

Suppose T is invertible and T_2 be the inverse.

Suppose $T(u) = T(v)$ then

$$u = T_2(T(u)) = T_2(T(v)) = v$$

So, T is 1 to 1, Also given $u \in R^n$

We have $u = T(T_2(u))$. So, T onto R^n

So T is an isomorphism.

Proof ② ③ Assume T is an isomorphism then

$$Ax = 0 \Rightarrow T(x) = T(0) \Rightarrow x = 0$$

So $Ax = 0$ has an unique solution.

$\therefore$ A is invertible & ③ follows from ②

Proof ③ to ① Suppose A is invertible

Let $T_2(x) = A^{-1}x$.

Then T_2 is a linear tranformation & T_2 is the inverse of T

Prob Let $T_1 : R^2 \to R^2$ $T_1(x, y) = (x - 2y, 2x + 3y)$

& $T_2 : R^2 \to R^2$ $T_2(x, y) = (y, 0)$

Compute the standard matrices of $T \Rightarrow T_2 \circ T_1$

$$T' = T_1 T_2$$

Soln:- Compute the standard matrix of T_1

With $e_1 = (1, 0)^T$, $e_2 = (0, 1)^T$

$$T_1(e) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad T_1(e_2) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Std matrix of T_1 is

$$A_1 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$$

Compute the standard matrix T_2 with $e_1 = [1,0]^T$ $e_2 = (0,1)^T$

$$T_2(e_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad T_2(e_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The standard matrix of $T = T_2 T_1$ is

$$A_2 A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \qquad T(x,y) = [(2x+3y), 0]$$

Similarly the standard matrix of $T' = T_1 T_2$ is

$$A_1 A_2 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \qquad T'(x,y) = (y, 2y)$$

Theorem:- Let V and W be a finite dimensional vectorspaces over the field F such that $\dim V = \dim W$. If T is a L.T from V into W the following are equivalent

(i) T is invertible

(ii) T is non-singular

iii) T is onto ie Range of T is W

Proof:- Let $n = \dim V = \dim W$.

By RND thm rank(T) + Nullity $(T) = n$

T is non-singular iff nullity $(T) = 0$

$n \to \dim W$ ie Range of T is W iff rank$(T) = n$

$\therefore$ $r + k = n \Rightarrow n + 0 = n$

$\therefore$ T is non-singular iff $T(V) = W$

if it is onto & non-singular then the T is invertible

⇒ Let V, W and Z be a finite dimensional vectorspaces over the field F; Let T be a linear transformation from V into W and U be a linear transformation from W into Z. If B, B' and B'' are ordered basis for the vectorspaces V, W & Z respectively

If $\underline{A}$ is the matrix of T relative to the pair $\underline{B, B'}$ & $\underline{B}$ is the matrix of U relative to the pair $\underline{B', B''}$ then Matrix of the composition UT relative to the pair B, B'' is the product matrix $\boxed{C = BA}$

Let $B = \{\alpha_1 \dots \alpha_n\}$ & $B' = \{\alpha_1' \dots \alpha_n'\}$ then the transformation $[T]_B$ & $[T]_{B'}$ are related?

wkt $\quad [\alpha]_B = P[\alpha]_{B'}$ —— ①

$$[T\alpha]_B = [T]_B [\alpha]_B \quad \text{—— ②}$$

Applying ① for $T\alpha$ vector

$$[T\alpha]_B = P[T\alpha]_{B'} \quad \text{—— ③}$$

Comparing ① ② & 3 we obtain

$$[T]_B P[\alpha]_{B'} = P[T\alpha]_{B'}$$

or

$$[T\alpha]_{B'} = P^{-1}[T]_B P[\alpha]_{B'}$$

$$\therefore \boxed{[T]_{B'} = P^{-1}[T]_B P}$$

⇒ Let V be the space of all polynomial functions from R into R of the form $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3$

ie the space of polynomial function of degree three or less. The differential operator D

Let B be the ordered basis for V Consisting of the four functions f_1, f_2, f_3, f_4 defined by $f_j(x) = x^{j-1}$

$(Df_1)x = 0$, $\quad Df_1 = 0f_1 + 0f_2 + 0f_3 + 0f_4$

$(Df_2)x = 1$, $\quad Df_2 = 1f_1 + 0f_2 + 0f_3 + 0f_4$

$(Df_3)x = 2x$, $\quad Df_3 = 0f_1 + 2f_2 + 0f_3 + 0f_4$

$(Df_4)x = 3x_2$, $\quad Df_4 = 0f_1 + 0f_2 + 3f_3 + 0f_4$.

So matrix D of the ordered basis B is

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ Let V be a finite dimensional vectorspace over the field F and let $B\{\alpha_1, \dots \alpha_n\}$ and $B' = \{\alpha_1', \alpha_2' \dots \alpha_n'\}$ be ordered bases for V. Suppose T is a linear operator on V. If $P = [P_1, \dots P_n]$ is the $n\times n$ matrix with columns

$$P_j = [\alpha_j']_B \text{ then}$$

$$\boxed{[T]_{B'} = P^{-1}[T]_B P}$$

Alternatively if U is an invertible operator on V defined by

$U\alpha_j = \alpha_j'$, $j = 1 \dots n$

$$[T]_{B'} = [U]_B^{-1}[T]_B[U]_B$$

Problems :- Let T be the linear operator on R^2 defined by

(1) $T(x_1, x_2) = (x_1, 0)$ Find the $[T]_{B'} \Rightarrow B' = \{\epsilon_1', \epsilon_2'\}$

$\epsilon_1' (1,1)$

$\epsilon_2' (2,1)$

$T(e_1) = T(1,0) = (1,0)$, $\quad T(e_2) = T(0,1) = (0,0)$

$$[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\epsilon_1' = e_1 + e_2$

$\epsilon_2' = 2e_1 + e_2$

$$\Rightarrow P \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Short computation $P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

$$[T]_{B'} = P^{-1}[T]_B P$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$[T]_{B'} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

2) Let V be the space of polynomial functions from R into R which have degree less than of equal to 3, Let D Be the differentiation operator on V and let

$B = \{f_1, f_2, f_3, f_4\}$ be the ordered basis for V

$f_i(x) = x^{i-1}$,

Let t be a real number and define $g_i(x) = (x+t)^{i-1}$

find the transformation wrt the new basis $B' = \{g_1, g_2, g_3, g_4\}$

$$g_1(x) = (x+t)^0 = 1$$

$$g_2(x) = (x+t)$$

$$g_3(x) = (x+t)^2 = x^2 + 2xt + t^2$$

$$g_4(x) = (x+t)^3 = x^3 + 3xt(x+t) + t^3$$

$$= x^3 + 3x^2 t + 3xt^2 + t^3$$

$$\boxed{\begin{matrix} f_1 \to x^0 & f_2 \Rightarrow x^1 \\ f_3 \to x^2 & f_4 = x^3 \end{matrix}}$$

$$g_1 = f_1$$

$$g_2 = tf_1 + f_2$$

$$g_3 = t^2 f_1 + 2tf_2 + f_3$$

$$g_4 = t^3 f_1 + 3t^2 f_2 + 3tf_3 + f_4$$

$$D(g_1) = 1f_1 + 0f_2 + 0f_3 + 0f_4$$

$$D(g_2) = tf_1 + 1f_2 + 0\cdot f_3 + 0\cdot f_4$$

$$D(g_3) = t^2f_1 + 2tf_2 + f_3 + 0f_4$$

$$D(g_4) = t^3f_1 + 3t^2f_2 + 3tf_3 + f_4$$

$$P = \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 1 & 3t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Take the inverse of the matrix P

$$P^{-1} = \begin{bmatrix} 1 & -t & t^2 & -t^3 \\ 0 & 1 & -2t & 3t^2 \\ 0 & 0 & 1 & -3t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[D]_{B'} = P^{-1}[D]_B P$$

$$= \begin{bmatrix} 1 & -t & t^2 & -t^3 \\ 0 & 1 & -2t & 3t^2 \\ 0 & 0 & 1 & -3t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 1 & +3t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -t & t^2 & -t^3 \\ 0 & 1 & -2t & 3t^2 \\ 0 & 0 & 1 & -3t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 2 & 6t \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

⇒ Let A and B be $n \times n$ matrices over the field F. We say that B is similar to A over F if there is an invertible $n \times n$ matrix P over the F such that $B = P^{-1}AP$

(3) Let T be the linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + 2x_2, -x_1 + 2x_2 + 4x_3)$$

(a) What is the matrix of T in the standard ordered basis of R^3?

(b) What is the matrix of T in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$ and $\alpha_3 = (2, 1, 1)$

(c) Prove that T is invertible and give a rule for T^{-1} like the one which defines T

The Null space of A : Solving $Ax = 0$

$A_{m \times n} \rightarrow$ square or rectangular matrices

Subspace containing all soln $Ax = 0$

for non invertible matrices there are non-zero solution to $Ax = 0$, Each soln x belongs to the nullspace of A

The nullspace of A consists of all solutions to $Ax = 0$. These vectors x are in R^n. The null space containing all solutions of $Ax = 0$ is denoted by $N(A)$.

(1) Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. This matrix is singular

$Ax = 0$

$\xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$$\text{Soln} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

The nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ contains all multiples of $S = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ Special soln

The nullspace consists of all combinations of the special solns

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix}$$

$$\boxed{[\alpha]_B = P[\alpha]_{B'}}$$

$$[\alpha]_{B'} = P^{-1}[\alpha]_B$$

$$P^{-1} = [P \;\; I)$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_4 - R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_4 + 4R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{array}\right] \xrightarrow{R_4/2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1 + R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{(-1)R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$\{E_1\}_{B'}$

$$[\alpha]_{B'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \checkmark$$

$$(\alpha)_{B'} = P^{-1}(\alpha)_B$$