A Report on

Problem solving using analytical methods and MATLAB

Submitted for the fulfillment of the CIE (Event-2) for the course

Control Systems - EC540

Submitted by:

Name	Mohamed Farhan Fazal
USN	01JST18EC055
Roll No.	29

Under the guidance of

Dr. Sudarshan Patilkulkarni

Associate Professor
DEPARTMENT OF ELECTRONICS AND COMMUNICATION
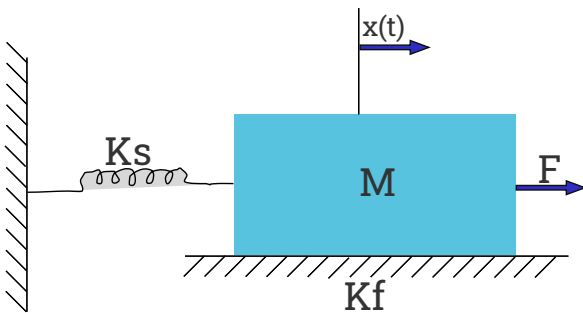SJCE MYSURU- 570006

Problem Statement:

For the mass spring system derive expressions for **ζ** and **ωₙ**. If **M=10 Kg**, determine spring and friction constant for following cases:

1. $\zeta = 0.7$, ωn = 10
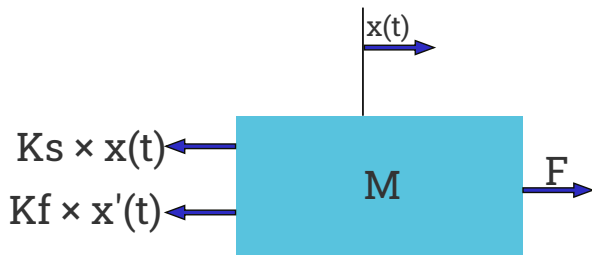2. $\zeta = 0.3$, ωn = 10
3. $\zeta = 0.01$, ωn = 10

Show the plots for step responses and pole-zero locations in MATLAB.

Solution

- Mass Spring System

- Free Body Diagram

According to Newton's Laws of motion:

$$F(t) - \{K_s x(t) + K_f \frac{\mathrm{d}x(t)}{\mathrm{d}t}\} = M \frac{\partial^2 x(t)}{\partial t^2}$$

$$f(t) \Rightarrow u(t) | x(t) \Rightarrow y(t)$$

$$u(t) = M \frac{\partial^2 y(t)}{\partial t^2} + K_s y(t) + K_f \frac{\partial y(t)}{\partial t}$$

In Laplace Domain

$$U(S) = [MS^2 + K_s + K_f S] \times Y(S)$$

$$\frac{Y(S)}{U(S)} = G(S) = \frac{1}{MS^2 + K_f S + K_s}$$

Comparing with the Standard Second Order Equation

$$G(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

$$\omega_n = \sqrt{\frac{K_s}{M}} \text{ and } \zeta = \frac{K_f}{2\sqrt{K_s M}}$$

$$K_s = \omega_n^2 \times M \text{ and } K_f = 2\zeta \times \sqrt{K_s \times M}$$

Case 1:

- **M** = 10 Kg
- ζ = 0.7
- **ω_n** = 10
- $K_s = 1000 N/m$
- $K_f = 140 N/m^2$

Case 2:

- **M** = 10 Kg
- ζ = 0.3
- **ω_n** = 10
- $K_s = 1000 N/m$
- $K_f = 60 N/m^2$

Case 3:

- **M** = 10 Kg
- ζ = 0.01
- **ω_n** = 10
- $K_s = 1000 N/m$
- $K_f = 2 N/m^2$

MATLAB CODE

Main Code | solution.m

```matlab
clear all;
close all;
clc;

% DECLARING CONSTANTS STATICALLY
% frictionConstant = [140 60 2];
% springConstant = 1000;

% MORE DYNAMIC CONSTANT CALCULATION
massOfBlock = 10;
zetaValues = [0.7, 0.3, 0.01];
naturalFrequency = 10;

springConstant = naturalFrequency^2 * massOfBlock;
frictionConstant = 2*zetaValues*sqrt(springConstant*massOfBlock);

for index = 1:length(frictionConstant)
    figure;
    G = tf([1], [massOfBlock frictionConstant(index) springConstant]);
    response = stepplot(G, "m");
    grid on;
    legend(strcat("ζ = ", num2str(zetaValues(index)), ", ωn = 10"));
    disp(strcat("Time Domain Parameters of transfer function with ", "ζ = ",
num2str(zetaValues(index)), ", ωn = 10"))
    disp(stepinfo(G));

    response.showCharacteristic('PeakResponse');
    response.showCharacteristic('RiseTime');
    response.showCharacteristic('SettlingTime');
    response.showCharacteristic('SteadyState');

    setAxisLimits(axis);
end
```

Helper Snippets

1. setAxisLimits.m

```matlab
% THIS SNIPPET IS TO ADD PADDING TO THE PLOT

function setAxisLimits(axisData)
    % RELATIVE TO THE OVERALL PLOT
    % 0.1 IS 10% AND 0.5 IS 50%
    padding = 0.05;
    axisLength = axisData(2) - axisData(1);
    axisHeight = axisData(4) - axisData(3);
    axis([axisData(1) - padding * axisLength axisData(2) + padding * axisLength
axisData(3) - padding * axisHeight axisData(4) + padding * axisHeight]);
end
```

2. labelOnPlot.m

```matlab
function labelOnPlot(A)
    text(real(A(1)), imag(A(1)), strcat("\rightarrow",num2str(A(1))));
    text(real(A(2)), imag(A(2)), strcat("\rightarrow",num2str(A(2))));
end
```

RESULTS

Command Window Output

```
G =

              1
    ---------------------
    10 s^2 + 140 s + 1000

Continuous-time transfer function.

Time Domain Parameters of transfer function with ζ = 0.7, ωn = 10
         RiseTime: 0.2127
      SettlingTime: 0.5979
       SettlingMin: 9.0010e-04
       SettlingMax: 0.0010
         Overshoot: 4.5986
        Undershoot: 0
             Peak: 0.0010
          PeakTime: 0.4408


G =

              1
    ---------------------
    10 s^2 + 60 s + 1000

Continuous-time transfer function.

Time Domain Parameters of transfer function with ζ = 0.3, ωn = 10
         RiseTime: 0.1324
      SettlingTime: 1.1230
       SettlingMin: 8.6139e-04
       SettlingMax: 0.0014
         Overshoot: 37.1410
        Undershoot: 0
             Peak: 0.0014
          PeakTime: 0.3224


G =

              1
    ---------------------
    10 s^2 + 2 s + 1000

Continuous-time transfer function.

Time Domain Parameters of transfer function with ζ = 0.01, ωn = 10
         RiseTime: 0.1050
```
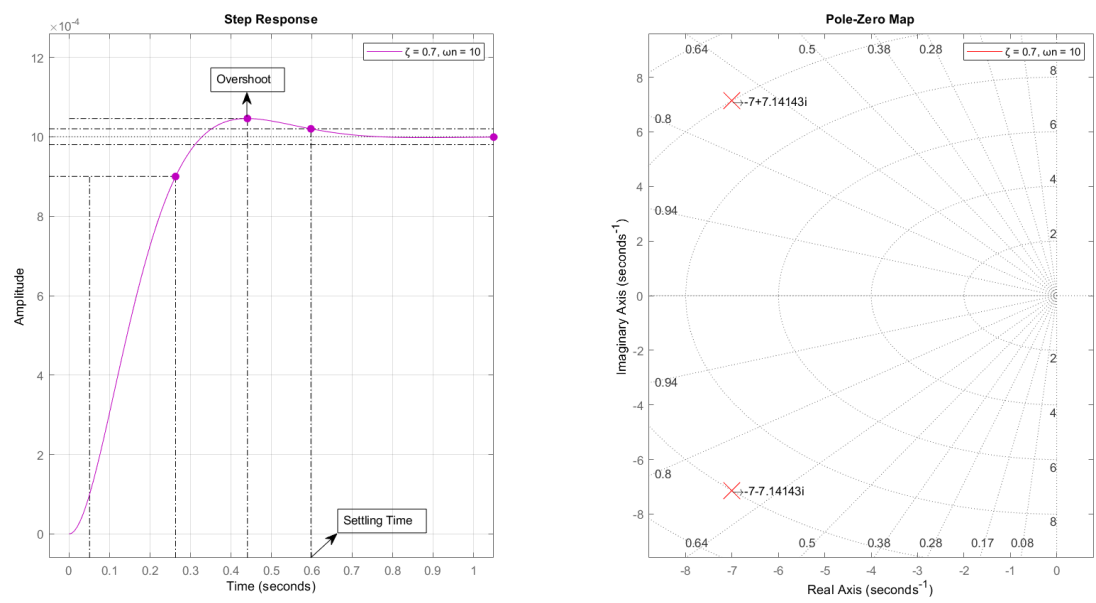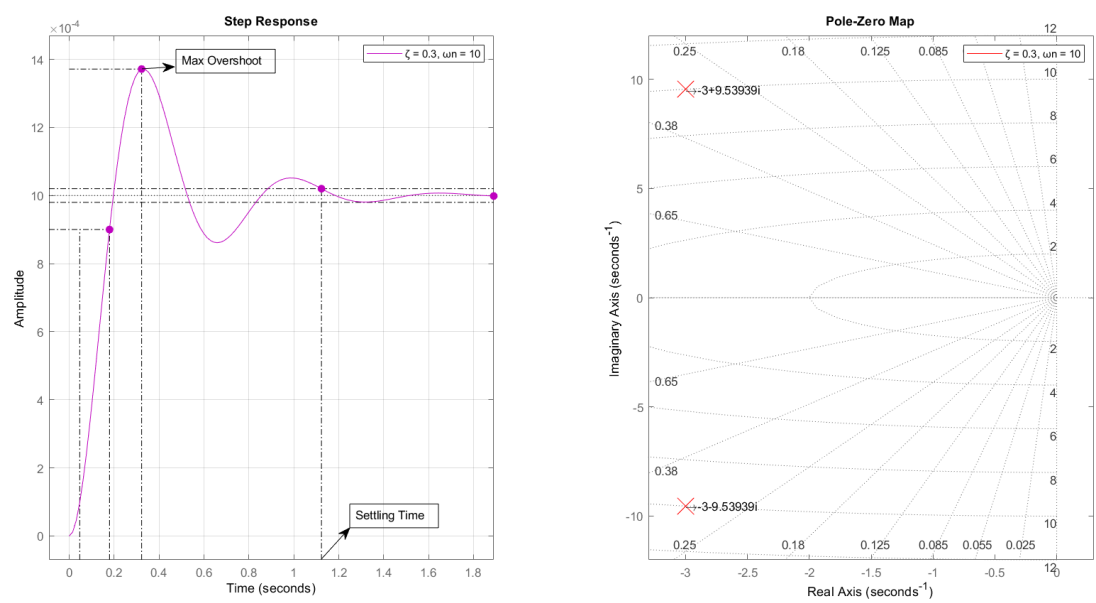
```
SettlingTime: 38.9674
 SettlingMin: 6.0902e-05
 SettlingMax: 0.0020
   Overshoot: 96.9071
  Undershoot: 0
        Peak: 0.0020
    PeakTime: 0.3142
```
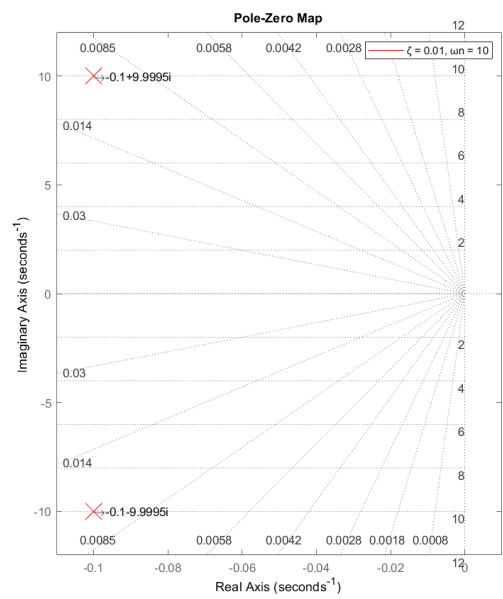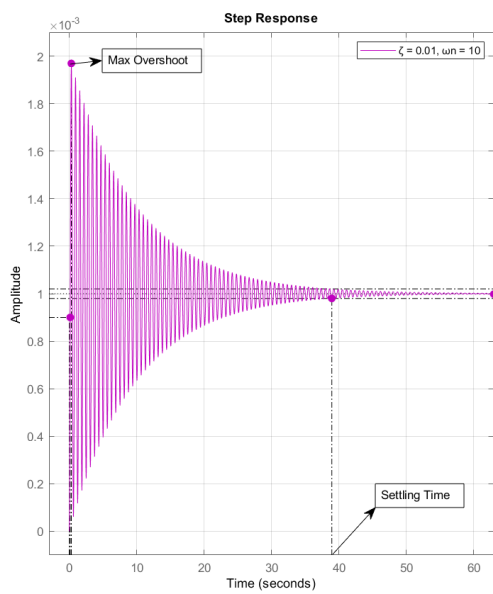
Plots
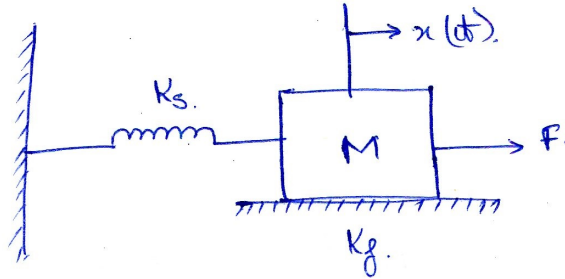
Case 1 Plot

Case 2 Plot

Case 3 Plot

Derivation and Calculations

Mass spring system considering the effect of friction.

K_s.

$x(t)$

M

F.

K_f.

Free body Diagram.

$x(t)$.

$K_s. x(t)$

M

F
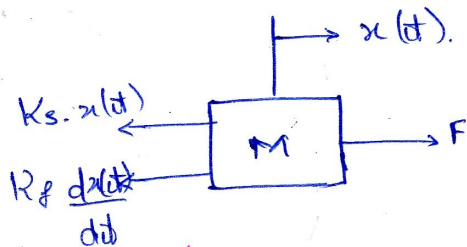
$K_f \frac{d x(t)}{dt}$

$M \rightarrow$ Mass of the block.

$K_f \rightarrow$ friction constant.

$K_s \rightarrow$ spring constant.

$F \rightarrow$ Applied Force.

$x(t) \rightarrow$ Displacement from mean position.

~~M~~ Now according to Newton's laws of motion.

Applied Force − Opposing force = Mass × acceleration.

ie $F(t) - \left\{ K_s\, x(t) + K_f \frac{d\, x(t)}{dt} \right\} = M \frac{d^2}{dt} x(t).$

$F(t) \rightarrow$ Input variable $\qquad x(t) \rightarrow$ Output variable.

$\therefore F(t) \rightarrow u(t) \qquad\qquad x(t) \rightarrow y(t).$

$u(t) = M \frac{d^2}{dt^2} y(t) + K_s\, y(t) + K_f \frac{d\, y(t)}{dt}.$

In laplace domain.

$U(s) = L\left[M \frac{d^2}{dt^2} y(t) + K_s\, y(t) + K_f \frac{d\, y(t)}{dt} \right]$

$$U(s) = M[s^2 Y(s) - sy(0) + y'(0)] + K_s Y(s) + K_f[sY(s) - y(0)]$$

(Assuming the block of mass M is at rest initially.

i.e: Initial conditions are '0'

i.e: $y(0) = y'(0) = \underline{\underline{0}}$.

$$U(s) = [Ms^2 + K_s + K_f s] Y(s).$$

Now $\quad G(s) = \dfrac{Y(s)}{U(s)}$.

$$G(s) = \frac{1}{Ms^2 + K_f s + K_s} = \frac{1/M}{s^2 + \left(\dfrac{K_f}{M}\right)s + \dfrac{K_s}{M}}$$

Standard second order equation is given by.

$$G(s) = \frac{\omega_n^2}{s^2 + 2Z\omega_n s + \omega_n^2}$$

$$\omega_n^2 = \frac{K_s}{M} \qquad\qquad \omega_n = \sqrt{\frac{K_s}{M}} \qquad \boxed{K_s = \omega_n^2 \times \underline{\underline{M}}}$$

$$2Z\omega_n = \frac{K_f}{M} \qquad Z = \frac{K_f}{2M}\sqrt{\frac{M}{K_s}} = \frac{K_f}{2\sqrt{M \cdot K_s}}.$$

$$\text{or} \quad \boxed{K_f = 2Z\sqrt{K_s \cdot M}}.$$

(a) Case 1: $Z = 0.7$, $\omega_n = 10$

$$K_s = \omega_n^2 \times M = 10 \times 10 \times 10 = \underline{\underline{1000}} \ N/m$$

$$K_f = 2Z\sqrt{K_s M} = 2 \times 0.7 \sqrt{1000 \times 10} = 1.4 \times 100 = \underline{\underline{140 \ N/m^2}}$$

Poles are located at: P_1, P_2

$$s^2 + 2Z\omega_n s + \omega_n^2$$

$$P_1, P_2 = \frac{-2Z\omega_n \pm \sqrt{4Z^2\omega_n^2 - 4\omega_n^2}}{2}$$

$\omega_n = \underline{\underline{10}}$.

$$\therefore P_1, P_2 = \frac{-20Z \pm \sqrt{Z^2 - 1} \times 20}{2}$$

$$= -10Z \pm 10\sqrt{Z^2 - 1}$$

$Z = \underline{\underline{0.7}}$ $\therefore P_1, P_2 = -10(0.7) \pm 10\sqrt{(0.7)^2 - 1}$

$$= \underline{\underline{-7 \pm 7.141j}}$$

(b) Case 11: $Z = 0.3$ $\omega_n = 10$.

$$\therefore K_s = \omega_n^2 \times M = 10 \times 10 \times 10 = 1000 \ N/m.$$

$$K_f = 2 \times 0.3 \times \sqrt{10^4} = 60 \ N/m^2$$

Poles $P_1, P_2 = -10(0.3) \pm 10\sqrt{0.3^2 - 1}$

$$P_1, P_2 = \underline{\underline{-3 \pm 9.539j}}$$

(c) Case 111: $Z = 0.01$ $\omega_n = 10$

$$K_s = 1000 \ N/m \qquad K_f = 200 \times 0.01 = 2 \ N/m^2$$

$$P_1, P_2 = -10(0.01) \pm 10\sqrt{0.01^2 - 1}$$

$$P_1, P_2 = \underline{\underline{-0.1 \pm 9.999j}}$$

Conclusion

From the above results and observations, we can conclude the following

- The system in all the 3 cases is **stable** as the poles of the system in all the 3 cases are to the *left* of the Imaginary Axis on the *S-plane*
- Friction co-efficient K_f is directly proportial to ζ , hence as ζ decreses, K_f also decreases.
- As the value of friction co-efficient decreses, that is when the effect of friction decreases ,the number of oscillations increses, which means, there is an increase in the **settling time**, and also the **maximum overshoot** is relatively *higher* as seen in case 3 with *ζ=0.01*.