Unit-2 → Vector Spaces

Refer { D.C Lay. / Hoffman & Kunze.

A vector space (or linear space) consists of the following

1) a field F of scalars

2) a set V of object called vectors

→ a rule (or operation) called vector addition

The sum of u & v denoted by u+v, is in V

i) $u+v = v+u$

ii) $(u+v)+w = u+(v+w)$

iii) There is a zero vector 0 in V such that $u+0 = u$

iv). For each u in V, there is a vector $-u$ in V such that $u+(-u) = 0$.

→ a operation called scalar multiplication

The scalar multiple of u by c denoted by cu, is in V

v) $c(u+v) = cu + cv$

vi) $(c+d)u = cu + du$

vii) $c(du) = (cd)u$

viii). $1u = u$

A vector space is a composite object consisting of a field a set of vectors and two operations with certain special properties. The same set of vectors may be part of number of distinct vector spaces

Vector spaces as V or when we have to specify with the field we call it has a vector space over the field F

Show that

For $n \geq 0$, the set of polynomials P_n of degree atmost n consists of all polynomomials of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots a_n t^n \quad \text{—} ①$$

$a_0, a_1 \dots a_n$ & t are real numbers is a vector space.

Soln: The degree of P is the highest power of t whose coefficients are not zero.

$p(t) = a_0$ where $a_0 \neq 0 \rightarrow$ degree of polynomial is zero

If all the co-efficients are zero, p is called a zero polynomial

$\therefore$ Zero polynomial exists in the group of polynomials

Let $q(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \dots + b_n t^n$ — ②

addition: $(p+q)(t) = (a_0+b_0) + (a_1+b_1)t + \dots \dots + (a_n+b_n)t^n$

$a_0 \dots a_n$, $b_0 \dots b_n$ are real no. a_0+b_0, $a_1+b_1 \dots$ are also real nos. they exists.

$\therefore (p+q)$ polynomial $\in P_n$.

The scalar multiple of p i.e cp is a polynomial defined by

$$cp(t) = ca_0 + ca_1(t) + \dots + (ca_n)t^n$$

co-efficients are real nos $ca_0 \rightarrow$ exists.

$(Cp)t$ is a polynomial in P_n

to get a negative of polynomial multiply by scalar whose value $= -1 \Rightarrow$

$$(-1)p(t) = -(a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n)$$

$$-p(t) = -a_0 - a_1 t \dots - a_n t^n \in P_n$$

$\therefore P_n$ is a vectorspace.

→ The n-tuple space F^n. Let F be any field, let V be the set of all n-tuples $\alpha = (x_1, x_2 \dots x_n)$ of scalars x_i in F.

If $\beta = (y_1, y_2, \dots y_n)$ with y_i in F, the sum of α and β is defined by

$$\alpha+\beta = (x_1+y_1,\ x_2+y_2, \dots\dots, x_n+y_n)$$

The product of a scalar c and vector α is defined by

$$c\alpha = (cx_1,\ cx_2, \dots\dots, cx_n)$$

It satisfies all the conditions of vector addition & scalar multiplication

→ Check whether the space of $m \times n$ matrices $F^{m \times n}$ is a vector space

$F \rightarrow$ field.

Subspaces

Defn Let V be a vectorspace over the field F. A subspace of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication in V.

The subset W of V is a subspace if

i) for each α & β in W the vector $\alpha+\beta$ is again in W

ii) The 0 vector is in W.

iii) For each α in W the vector $(-\alpha)$ is in W.

iv) For each α in W and each scalar c the vector $c\alpha$ is in W

v) All the commutativity and associativity of vector addition & scalar multiplication are same, since these properties are of the operation on V.

Theorem 1

A non-empty subset W of V is a subspace of V iff for each pair of vectors α, β in W and each scalar c in F the vector $c\alpha+\beta$ is again in W

Proof 1 Suppose that W is a non-empty subset of V such that $c\alpha + \beta \in W$ $\forall$ α, β in W and all scalar c in F

Since W is an non-empty set, ρ is a vector in W.

$\therefore$ $(-1)\rho + \rho = 0$ is in W.

If α is any vector in W and c is any scalar, the vector

$c\alpha = c\alpha + 0$ is in W.

Thus $(-1)\alpha = -\alpha$ is in W

Finally if α & β are in W, then $\alpha + \beta = (1)\alpha + \beta$ is in W

Thus W is a subspace of V

Conversely if W is a subspace of V, α and β are in W & c is a scalar such that $c\alpha+\beta$ is in W.

Alternately **Proof 2**: Suppose W is a subset of a vector space V such that $(c\alpha+\beta)$ is in W for every pairs of vectors α, β in W and every scalars c then we need to show that W is a vector space over F

Since W is a subset of vector space V, it inherits the properties of commutativity, Associativity and all the rules of scalar multiplication from V

It will become a subspace if only we can show that W contains the additive identity '0' and every vector $\alpha \in W$

Suppose $\rho \in W$, then for $c = -1$, $\alpha = \rho$ & $\beta = \rho$ wkt $c\alpha+\beta$ also is in W.

$(-1)\rho + \rho = -\rho + \rho = 0$ is in W

|||^{ly} for ρ in W, we choose $c=-1$, $\alpha=\rho, \beta=0$

We get $c\alpha+\beta = (-1)\rho + 0 = -\rho \in W$

Thus we have shown that $\vec{0}$ vector belongs to W and every vector in W has additive inverse

Here W is a subspace

Conversely, if W is a subspace, then $\alpha, \beta \in W$ & c is in field F means $c\alpha$ is in W & $c\alpha+\beta$ is in W.

① If V is any vectorspace, W is a subspace of V, the subset consisting of zero vector alone is a subspace of V, called the zero subspace of V.

Prove ?

② In F^n, the set of n-tuples $(x_1, \dots, x_n)$ with $x_1 = 0$ is a subspace however, the set of n-tuples with $x_1 = 1 + x_2$ is not a subspace $(n \geq 2)$

③ The space of polynomial functions over the field F is a subspace of the space of all functions from F into F

4) An $n \times n$ (square matrix) A over the field F is symmetric if $A_{ij} = A_{ji}$ for each i and j. The symmetric matrices form a subspace of the space of all $n \times n$ matrices over F

5) An $n \times n$ (square) matrix A over the field C of complex numbers is Hermitian (or self adjoint) if

$$A_{jk} = \overline{A_{kj}}$$

for each j,k the bar denoting complex conjugation.

A 2×2 matrix is Hermitian iff it has the form

$$\begin{bmatrix} z & x+iy \\ x-iy & w \end{bmatrix}$$ where x, y, z, w are real numbers

The solution space of a system of homogeneous linear equations

Let A be an $m \times n$ matrix over F
then the set of all $n \times 1$ (column) matrices X over F such that
$AX = 0$ is a subspace of the space of all $n \times 1$
matrices over F

To prove this we must S.T $A(cX+Y) = 0$
when $AX = 0$, $AY = 0$ & c is an arbitrary scalar in F

Lemma:- If A is an $m \times n$ matrix over F and B, C are $n \times p$ matrices over F then
$$A(dB+C) = d(AB) + AC \text{ for each scalar } d \text{ in } F$$

Proof:-
$$[A(dB+C)]_{ij} = \sum_{k} A_{ik} (dB+C)_{kj}$$
$$= \sum_{k} [(dA_{ik} B_{kj}) + A_{ik} C_{kj}]$$
$$= d \sum_{k} A_{ik} B_{kj} + \sum_{k} A_{ik} C_{kj}$$
$$= d(AB)_{ij} + (AC)_{ij}$$
$$= [d(AB) + AC]_{ij}$$ → Refer Page - 50

Theorem 2:- Let V be a vectorspace over the field F. The intersection of any collection of subspaces of V is a subspace of V.

Proof:- Let $\{W_a\}$ be a collection of subspaces of V and $\boxed{W = \bigcap_a W_a}$ be their intersection

i.e W contains set of all the elements belonging to every W_a

Each W_a is a subspace it has a zero vector
∴ Intersection of W_a i.e W has a zero vector. W is a non empty set.

Let α and β be vectors in W. & c be a scalar

By defn of W both α & β belongs to each W_a, and because each W_a have $(c\alpha+\beta)$ since it is a subspace it is in every W_a

$\therefore$ $\boxed{c\alpha+\beta \text{ is in } W}$

Hence by Theorem 1 W is a subspace of V

→ Union of collection of subspaces need not be a subspace

Proof: Let us consider the x-axis and y-axis as the subspaces W_1 and W_2 of the vector space R^2

Let $W = W_1 \cup W_2$ union of the x-axis and y-axis

W is not a (subspace) vectorspace $\because$ is not closed under addition.

$\alpha \to (1,0) \in W$, $\beta \to (0,1) \in W$, $c = 1$ scalar

$$c\alpha+\beta = 1(1,0) + (0,1)$$
$$= (1,0) + (0,1)$$
$$= (1,1) \notin W.$$

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Defn of Subspace Spanned by Vectors

Let S be a set of vectors in a vectorspace V. The subspace spanned by S is defined to be the intersection W of all subspaces of V which contains S. When S is a finite set of vectors, $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ we shall simply call W the subspace spanned by the vectors $\alpha_1, \alpha_2 \ldots \alpha_n$.

Theorem 3:- The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

Let $S = \{\alpha_1, \alpha_2 \dots \alpha_m\}$

Proof:- Let W be the subspace spanned by S. If α is a linear combination of vectors in S then

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + \dots c_m\alpha_m$$

We can see that α is in W because W contains the linear combinations vectors in S

Hence if L is the set of all linear combinations of vectors in S., L is in W.

Let α, β belongs to L, then α and β are linear Combination of elements in S.

$$\alpha = \sum x_i\alpha_i = x_1\alpha_1 + x_2\alpha_2 + \dots + x_m\alpha_m$$

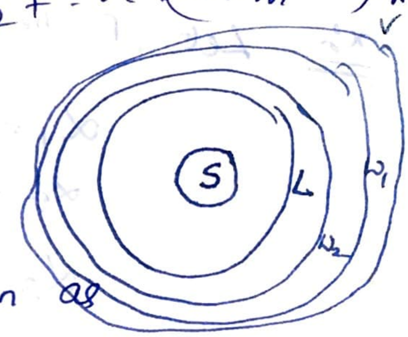
$$\beta = \sum d_i\alpha_i = d_1\alpha_1 + d_2\alpha_2 + \dots + d_m\alpha_m.$$

So $c\alpha+\beta = cx_1\alpha_1 + cx_2\alpha_2 + cx_3\alpha_3 + \dots cx_m\alpha_m + d_1\alpha_1 + d_2\alpha_2 + \dots d_m\alpha_m$

$$= (cx_1 + d_1)\alpha_1 + (cx_2 + d_2)\alpha_2 + \dots (cx_m + d_m)\alpha_m$$

$\therefore c\alpha+\beta \in L$

$\therefore$ L is a subspace of V.



We also knows that each $\alpha_i \in S$ can be written as

$$\alpha_i = 0\alpha_1 + 0\alpha_2 + 0\cdot\alpha_3 \dots + 1\alpha_i + 0\cdot\alpha_{i+1} + 0\cdot\alpha_{i+2} + 0\alpha_m$$

i.e α_i is also a linear combination of elements in S

Hence each α_i in S is also in L., i.e. $S \leq L$

We have also seen that $L \leq W$

where $W \to$ Subspace spanned by S

i.e $W = \cap W_i$ where each W_i contains S.

Thus L must be equal to W_i i.e L is the subspace spanned by S.

Definition:-

If $S_1, S_2 \dots S_k$ are subsets of a vectorspace V, the set of all sums

$$\alpha_1 + \alpha_2 + \alpha_3 - - - + \alpha_k.$$

of vectors α_i in S_i is called the sum of the subset $S_1, S_2 \dots S_k$ and is denoted by

$$S_1 + S_2 + S_3 + \dots S_k.$$

or

$$\sum_{i=1}^{k} S_i$$

If $W_1, W_2 \dots W_k$ are subspaces of V, then the sum

$$W = W_1 + W_2 + \dots + W_k.$$

is a subspace of V which contains each of the subspaces W_i, it follows from the Thm (3) that W is a subspace spanned by the union of $W_1, W_2 \dots W_k$

Ex:- Let F be a subfield of the field C of complex numbers.

$$\alpha_1 = (1, 2, 0, 3, 0)$$
$$\alpha_2 = (0, 0, 1, 4, 0)$$
$$\alpha_3 = (0, 0, 0, 0, 1)$$

By thm 3 α is in subspace W of F^5 spanned by $\alpha_1, \alpha_2, \alpha_3$

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$$

$$\alpha = c_1(1, 2, 0, 3, 0) + c_2(0, 0, 1, 4, 0) + c_3(0, 0, 0, 0, 1)$$

$$= (c_1, 2c_1, 0, 3c_1, 0) + (0, 0, c_2, 4c_2, 0) + (0, 0, 0, 0, c_3)$$

$$= (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3)$$ c_1, c_2, c_3 are scalars in F

W can be set of all 5-tuples.
$\alpha = (x_1, x_2, x_3, x_4, x_5)$

with x_i in F such that

$$x_2 = 2x_1$$
$$x_4 = 3x_1 + 4x_2$$

$c_1 = -3$, $c_2 = 1$, $c_3 = 2$

$\alpha = (-3, -6, 1, -5, 2)$ is in W, whereas $(2, 4, 6, 7, 8)$ is not in W.

Example

Let F be a subfield of the field C of Complex nos. & Let V be the vectorspace of all 2×2 matrices over F. Let W_1 be the subset V consisting of all matrices of the form

$$\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$$

where x, y, z are arbitrary scalars in F, ~~lets~~

Let W_2 be the subset of V consisting of all matrices of the form $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$

where x and y are arbitrary scalars in F; W_1 & W_2 are subspaces of V.

$\boxed{\begin{array}{l} * \\ W_2 \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \\ \downarrow \\ \text{by } x = 0 \\ y = d. \end{array}}$

Also $V = W_1 + W_2$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

Subspace $W_1 \cap W_2$ consists of all matrices of the form $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$

Linear Combinations

$V_1, V_2 \ldots V_n$ in R^n

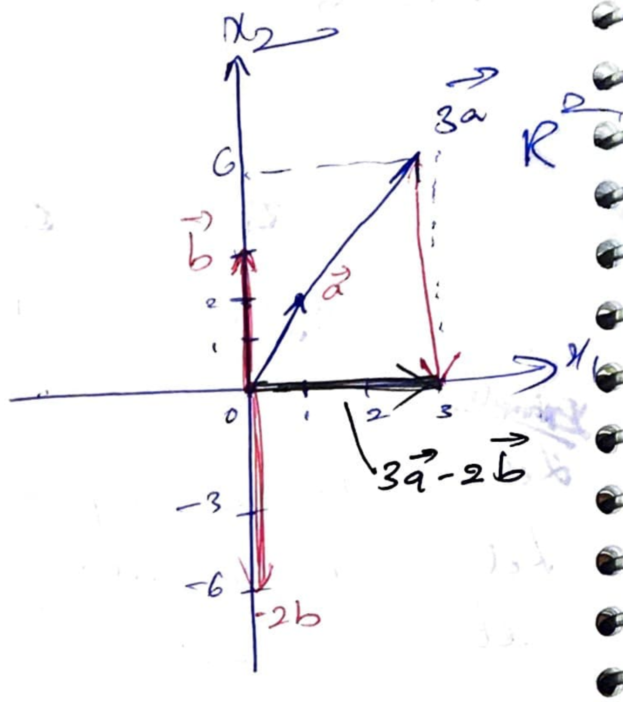
$c_1 V_1 + c_2 V_2 + \ldots + c_n V_n$

$c_1 \longrightarrow c_n \in R$ vectors in R^2

$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

$0 \cdot \vec{a} + 0 \cdot \vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\underbrace{3}_{c_1} \vec{a} + \underbrace{(-2)}_{c_2} \vec{b}$ $\begin{bmatrix} 3 - 0 \\ 6 - 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$



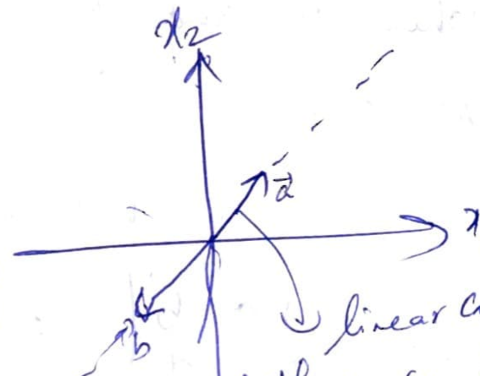
$\vec{a}, \vec{b}$ scaled by an arbitrary multiples. We can generate all vectors in R^2.

$$\boxed{\text{Span}(\vec{a}, \vec{b}) = R^2}$$

i.e any vector in R^2 can be represented by a linear combinations of vector a & b.

Let $\vec{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ → slope is same

Span$(\vec{a}, \vec{b})$ is the line representing vectors $\vec{a}$ and $\vec{b}$



linear combination of these can be represented only on this line.

Span$(\vec{0}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → its just a point at O.

is a zero vector

Span$(v_1, v_2 \ldots v_n)$ The set of all the vectors linear combinations of vectors

$$\text{Span}(v_1, v_2 \ldots v_n) = \{c_1 v_1, c_2 v_2, \ldots c_n v_n \mid c_i \in \mathbb{R} \text{ for } i \le n\}$$

Consider $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$c_1 \vec{a} + c_2 \vec{b} = \vec{x}$$

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{c_1 = x_1}$$

$$2c_1 + 3c_2 = x_2$$

$$c_2 = \frac{x_2 - 2c_1}{3} \Rightarrow \frac{x_2 - 2x_1}{3}$$

$\therefore$ We can get any value of c_1

Bases and Dimensions (Linearly Independent Sets)

Let V be a vector space over F, An indexed set of vectors $\{v_1, v_2 \ldots v_p\}$ in V said to be linearly independent if the vector equation

$$c_1 v_1 + c_2 v_2 + \ldots\ldots + c_p v_p = 0 \quad \text{—— (1)}$$

have only the trivial solution $c_1 = 0, c_2 = 0 \ldots c_p = 0$

A set $\{v_1, v_2 \ldots v_p\}$ is said to be linearly dependent if eqn (1) has a non-trivial solution, i.e., if there are some scalars $c_1 \ldots c_p$ not all zeros, such that (1) holds. In such a case eqn (1) is called a linear dependence relation among $v_1 \ldots v_p$.

Let V be a vectorspace V over F, A subset S of V is said to be linearly dependent (or simply dependent) if there exists distinct vectors $\alpha_1, \alpha_2 \dots \alpha_n$ in S and scalars $c_1, c_2, \dots, c_n$ in F, not all of which are 0 such that

$$c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = 0$$

A set which is not linearly dependent is called linearly independent. If the set S contains only finitely many vectors $\alpha_1, \alpha_2 \dots \alpha_n$

→ A set of 2 vectors is linearly dependent iff one of the vectors is a multiple of the other.

→ Any set containing the zero vector is linearly dependent

Ex:- The set (sint, cost) is linearly independent in $C[0,1]$ ∴ sint and cost are not multiples of one another as vectors. in $C[0,1]$

Basis :- Let V be a vectorspace, A basis for V is a linearly independent set of vectors in V which spans the space V. The space V is finite dimensional if it has a finite basis

Let H be a subspace of a vectorspace V. An indexed set of vectors $B = \{b_1 \dots b_p\}$ in V is a basis for H if

(i) B is a linearly independent set and

(ii) the subspace spanned by B co-incides with H i.e.

$$H = \text{Span}\{b_1 \dots b_p\}$$

$\Rightarrow$ Let A be an invertible $n \times n$ matrix $A = [a_1 \ldots a_n]$ then the columns of A form a basis for R^n because they are linearly independent and they span R^n, by the Invertible Matrix theorem (27)

$\Rightarrow$ Let $e_1, e_2 \ldots e_n$ be the columns of $n \times n$ identity matrix I_n

i.e $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ .... $e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

The set $\{e_1 \ldots e_n\}$ is called the standard basis for R^n

$\Rightarrow$ Let F be a subfield of the complex numbers In F^3 the vectors: $\alpha_1 = (3, 0 -3)$ $\alpha_2 = (-1, 1, 2)$ $\alpha_3 = (4, 2, -2)$ $\alpha_4 = (2, 1, 1)$ are linearly dependent

$$2\alpha_1 + 2\alpha_2 - \alpha_3 + 0 . \alpha_4 = 0.$$

$\rightarrow$ the set $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$ $v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ Determine $\{v_1, v_2, v_3\}$ is a basis in R_3

If $A = [v_1 \; v_2 \; v_3]$ is invertible, then v_1, v_2, v_3 forms a basis in R^3

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \xrightarrow{R_3 + 2R_1} \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

$\downarrow R_3 + R_2$

3 pivot positions matrix is invertible $\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

$\therefore$ it forms a basis

$\Rightarrow$ Let $S = \{1, t, t^2 \dots t^n\}$ Verify that S forms a basis for P_n. This basis is called the standard basis of P_n

$$C_0 \cdot 1 + C_1 t + C_2 t^2 + \dots C_n t^n = 0(t) \quad (1)$$

A fundamental theorem in algebra says that the only polynomial in P_n with more than n zeros is the zero polynomial.

i.e $c_0, c_1 = \dots = c_n = 0$ (1) holds good only if all $c_0 \dots c_n = 0$

$\therefore$ S is linearly independent.

$\therefore$ this is a basis of P_n

Dimension of a finite dimensional vector space as the number of element in a basis for V it is denoted by dim V.

$\Rightarrow$ Find the dimension of the vectorspace spanned by vectors $(1, 1, -2, 0, -1)$, $(1, 2, 0, -4, 1)$, $(0, 1, 3, -3, 2)$, $(2, 3, 0, -2, 0)$ & find basis for that space.

1st we find basis and the dimensions

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 1 & 2 & 0 & -4 & 1 \\ 0 & 1 & 3 & -3 & 2 \\ 2 & 3 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1,\ R_3 - 2R_1} \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 1 & 3 & -3 & 2 \\ 0 & 1 & 4 & -2 & 2 \end{bmatrix}$$

$R_3 - R_2$, $R_4 - R_2$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{bmatrix} \xrightarrow{R_4 - 2R_3} \begin{bmatrix} 1, & 1, & -2, & 0, & -1 \\ 0 & 1 & 2 & -4, & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{Basis}$$

basis $(1, 1, -2, 0, -1)$
$(0, 1, 2, -4, 2)$
$(0, 0, 1, 1, 0)$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ -2 & 0 & 3 & 0 \\ 0 & -4 & -3 & -2 \\ -1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{Operations}} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ \underbrace{0 \quad 0 \quad 0}_{\text{Can't be considered}} & & & 0 \end{bmatrix}$$

$\uparrow \uparrow \uparrow$
Basis.

$(1, 1, -2, 0, -1)$
$(1, 2, 0, -4, 1)$
$(0, 1, 3, -3, 2)$

Independent columns of A :- The only solution $AV = 0$ ie $V = 0$

$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix}$$

Basis for subspace. $\rightarrow$ Independent vector that span the subspace.

Dimension is a number of basis vectors from the subspace

Theorem → Let V be a vectorspace which is spanned by a finite set of vectors $\beta_1, \beta_2 \cdots \beta_m$ then an independent set of vectors V is finite and contains no more than m elements.

Proof :- To prove the theorem ~~...~~ it suffices to show that every subset S of V which contains more than m vectors is linearly dependent

Let S be such a set. i.e $S = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$ where $n > m$.

Since $\beta_1, \beta_2 \cdots \beta_m$ span V, there exists scalars A_{ij} in F such that

$$\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i$$

For any n scalars $x_1, x_2 \cdots x_n$,

We have
$$x_1\alpha_1 + x_2\alpha_2 + \cdots x_n\alpha_n = \sum_{j=1}^{n} x_j\alpha_j$$
$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\beta_i$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{m} x_j(A_{ij}\beta_i)$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{m} (x_j A_{ij})\beta_i$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} (A_{ij}x_j)\beta_i$$

If A is an $m \times n$ matrix & $m < n$, then the homogeneous s/m of linear equations has a non trivial soln

$$\boxed{\sum_{j=1}^{n} (A_{ij}x_j) = 0 \qquad 1 \le i \le m}$$

∴

i.e. $x_1, x_2, \cdots x_n$ not all 0

$x_1\alpha_1 + x_2\alpha_2 + \cdots\cdots + x_n\alpha_n = 0$

This shows that S is a linearly dependent set.

Corollary: If V is a finite dimensional vectorspace, then any two bases of V have the same (finite) number of elements.

Since V is a finite dimensional, it has a finite basis $\{\beta_1, \beta_2, \dots \beta_m\}$

By thm every basis of V is finite & contains no more than m elements.

⇒ Let V be a finite dimensional vectorspace and let $n = \dim V$. Then

(a) any subset of V contains no more than 'n' vectors is linearly dependent

(b) no subset of V which contains fewer than n vectors can span V

Theorem: If W is a subspace of a finite dimensional vector space V, every linearly independent subset of W is finite and is a part of a (finite) basis for W.

Proof: Suppose S_0 is a linearly independent subset of W. If S is linearly independent subset of W containing S_0 then S is also linearly independent subset of V.

~~As~~ V is finite dimensional, S contains no more than $\dim V$ elements

~~So~~ If we extend S_0 to a basis for W.

If S_0 spans W, then S_0 is a basis for W.

If S_0 doesnot span W then find a vector β_1 in W such that

Set $S_1 = S_0 \cup \{\beta_1\}$ is independent.

If S_1 spans W then S_1 is a basis, else we find another vector β_2 .. in W

Such that $S_2 = S_1 \cup \{\beta_2\}$ is independent.

If we continue this way. (not more than $\dim V$ steps)

we reach a set

$$S_m = S_0 \cup \{\beta_1, \beta_2 \ldots \beta_m\}$$

which is a basis for W.

→ If W is a proper subspace of a finite dimensional vector space V, then W is a finite-dimensional and

$$\dim W < \dim V$$

→ In a finite-dimensional vector space V every non-empty linearly independent set of vectors is a part of basis

<u>Theorem</u>:- If W_1 and W_2 are finite dimensional subspaces of a vector space V, ~~$\dim W_1 + \dim W_2$~~ then $W_1 + W_2$ is finite dimensional and

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

<u>Proof</u>:- $W_1 \cap W_2 \longrightarrow$ finite basis $\{\alpha_1, \alpha_2 \ldots \alpha_k\}$

which is a part of a basis

$$\{\alpha_1 \ldots \alpha_k, \beta_1 \ldots \beta_m\} \text{ for } W_1$$

and part of a basis of W_2

$$\{\alpha_1 \ldots \alpha_k, \gamma_1 \ldots \gamma_n\} \text{ for } W_2.$$

The subspace $W_1 + W_2$ is spanned by the vectors

$$\{\alpha_1, \ldots, \alpha_k \; \beta_1, \ldots \beta_m, \gamma_1, \ldots, \gamma_n\}$$

and these vectors form an independent set

i.e $\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0$

then $\quad -\sum z_r \gamma_r = \sum x_i \alpha_i + \sum y_j \beta_j$

this shows that $\boxed{\sum z_r \gamma_r \text{ belongs to } W_1}$

$\sum z_r \gamma_r$ also belongs to W_2 it follows.

$\therefore \boxed{\sum z_r \gamma_r = \sum c_i \alpha_i}$ for certain scalars $c_1 \dots c_t$.

$\because$ the set

$\{\alpha_1 \dots \alpha_k, \gamma_1 \dots \gamma_n\}$ is independent

each of the scalar $z_r = 0$ thus.

$$\sum x_i \alpha_i + \sum y_j \beta_j = 0.$$

Since $\{\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_m\}$ is also an independent set.

i.e each $x_i = 0$ & $y_i = 0$

Thus

$\{\alpha_1, \dots \alpha_k, \beta_1, \dots \beta_m, \gamma_1 \dots \gamma_n\}$ is a basis for $W_1 + W_2$.

$$\therefore \dim W_1 + \dim W_2 = k + m + k + n$$

$$= k + (m + k + n)$$

$$\boxed{\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)}$$

Co-ordinates

If V is a finite dimensional vector space, an ordered basis for V is a finite sequence of vectors which is linearly independent and spans V.

If the sequence $(\alpha_1 \dots \alpha_n)$ is an ordered basis for V, then the set $\{\alpha_1, \dots \alpha_n\}$ is a basis for V.

$B = \{\alpha_1, \dots \alpha_n\}$

Suppose V is a finite dimensional vectorspace over the field F that $B = \{\alpha_1, \dots, \alpha_n\}$

is an ordered basis for V.

Given α in V, there is a unique n-tuple $(x_1 \dots x_n)$ of scalars such that

$$\alpha = \sum_{i=1}^{n} x_i \alpha_i \longleftarrow (1),$$

The n-tuple is unique because if we also have

$$\alpha = \sum_{i=1}^{n} z_i \alpha_i$$

$$\sum_{i=1}^{n} (x_i - z_i)\alpha_i = 0$$

& the linear independence of the α_i tells us that $x_i - z_i = 0$ for each i.

$x_i \to i^{th}$ co-ordinate of α relative to the ordered basis.

$$B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$$

$$\beta = \sum_{i=1}^{n} y_i \alpha_i$$

then $\alpha + \beta = \sum_{i=1}^{n} (x_i + y_i)\alpha_i$

i^{th} corordinate of $(\alpha + \beta)$ in the ordered bases. is $(x_i + y_i)$

Co-ordinate matrix of α relative to the ordered basis B:

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow [\alpha]_B \to \text{indicate the co-ordinate matrix}$$

If V is finite(n) dimensional. and that

$B = \{\alpha_1, \dots \alpha_n\}$ and $B' = \{\alpha_1', \dots \alpha_n'\}$

are two ordered bases for V.

There are unique scalars P_{ij} such that

$$\alpha_j' = \sum_{i=1}^{n} P_{ij}\alpha_i \quad 1 \le j \le n.$$

Let $x_1' \cdots x_n'$ be the co-ordinates of a given vector α in the ordered basis B' then

$$\alpha = x_1'\alpha_1' + \cdots + x_n'\alpha_n'$$

$$= \sum_{j=1}^{n} x_j'\alpha_j' = \sum_{j=1}^{n} x_j'\left(\sum_{i=1}^{n} P_{ij}\alpha_i\right)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} (P_{ij}x_j')\alpha_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij}x_j'\right)\alpha_i \quad \text{———} \; 2$$

Compare 1 & 2.

$$x_i = \sum_{j=1}^{n} P_{ij}x_j' \quad 1 \le i \le n$$

Let P be the $n \times n$ matrix whose i,j entry is the scalar P_{ij} and let X and X' be the co-ordinates matrices of the vector α in the ordered basis B and B'

Then we can write $\boxed{X = PX'}$ ——— 3

B and B' are linearly independent sets.

$\boxed{X = 0}$ iff $\boxed{X' = 0}$

$\therefore$ P is invertible.

$\times$ by with P^{-1} 3 : $P^{-1}X = P^{-1}PX'$

$$\Rightarrow \boxed{X' = P^{-1}X}$$

$$[\alpha]_B = P[\alpha]_{B'}$$

$$[\alpha]_{B'} = P^{-1}[\alpha]_B$$

Theorem 7 Let V be an n-dimensional vector space over the field F, and let B and B' be two ordered bases of V. Then there is a unique, necessarily invertible, $n\times n$ matrix P with entries in F such that

(i) $[\alpha]_B = P[\alpha]_{B'}$

(ii) $[\alpha]_{B'} = P^{-1}[\alpha]_B$

for every vector α in V. The columns of P are given by

$$P_j = [\alpha_j']_B \quad j=1,\dots,n$$

Theorem 8 Suppose P is an $n\times n$ invertible matrix over F. Let V be an n-dimensional vector space over F, and let B be an ordered basis of V. Then there is a unique ordered basis B' of V such that

i) $[\alpha]_B = P[\alpha]_{B'}$

ii) $[\alpha]_{B'} = P^{-1}[\alpha]_B$ for every vector α in V.

Proof:- Let B consist of the vectors $\alpha_1, \dots \alpha_n$. If $B' = \{\alpha_1', \dots, \alpha_n'\}$ is an ordered basis of V for which (i) is valid, it is clear that

$$\alpha_j' = \sum_{i=1}^{n} P_{ij}\alpha_i$$

Thus we need only S.T the vectors α_j', defined by these eqns, form a basis

Assume $Q = P^{-1}$ then

$$\sum_j Q_{jk}\alpha_j' = \sum_j Q_{jk} \sum_{i=1}^{n} P_{ij}\alpha_i$$

$$= \sum_j \sum_i P_{ij} Q_{jk}\alpha_i$$

$$= \sum_i \left(\sum_j P_{ij} Q_{jk} \right) \alpha_i$$

$$= \alpha_k$$

Thus the subspace spanned by the set

$$B' = \{\alpha'_1, \ldots, \alpha'_n\}$$

Contains B and hence equals V. Thus B' is a basis, from its defn. it is clear that (i) is valid & hence also (ii).

⇒ Let F be a field and let $\alpha = (x_1, x_2, \ldots x_n)$ be vectors in F^n. If B is a standard ~~basis~~ ordered basis of F^n,

$B = \{\epsilon_1, \epsilon_2 \ldots \epsilon_n\}$ The co-ordinate matrix of the vector α in the basis B is given by

$$[\alpha]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let R be the field of real numbers and let θ be a fixed real no. The matrix

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$[\alpha]_B = P[\alpha]_{B'}$

$[\alpha]_{B'} = P^{-1}[\alpha]_B$

is invertible with inverse,

$$P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

∴ for each θ the set $B' \rightarrow \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$ is a basis for R^2

This basis is obtained by rotating the standard basis through the angle θ

α (x_1, x_2)

$$[\alpha]_{B'} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{bmatrix}$$

$x'_1 = x_1\cos\theta + x_2\sin\theta$

$x'_2 = x_2\cos\theta - x_1\sin\theta$

$\Rightarrow$ Let F be a subfield of the complex numbers The matrix.

$$P = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 8 \end{bmatrix}$$

Find the co-ordinate of vector $\alpha(x_1', x_2', x_3')$ in given Basis B'

$$[\alpha]_B = P[\alpha]_{B'}$$

If P is invertible (P^{-1}) then $[\alpha]_{B'} = P^{-1}[\alpha]_B$

$$P^{-1} = \left(\begin{array}{ccc|ccc} -1 & 4 & 5 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|ccc} -1 & 0 & 11 & 1 & -2 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1/-1,\ R_2/2,\ R_3/8} \left(\begin{array}{ccc|ccc} 1 & 0 & -11 & -1 & 2 & 0 \\ 0 & 1 & -3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/8 \end{array}\right) \xrightarrow{R_1 + 11R_3,\ R_2 + \frac{3}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 11/8 \\ 0 & 1 & 0 & 0 & 1/2 & 3/16 \\ 0 & 0 & 1 & 0 & 0 & 1/8 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -1 & 2 & 11/8 \\ 0 & 1/2 & 3/16 \\ 0 & 0 & 1/8 \end{pmatrix}$$

$\therefore \alpha_1' = (1, 0, 0)$ $\quad \alpha_2' = (4, 2, 0)$ $\quad \alpha_3' = (5, -3, 8)$

forms a Basis B' of F^3

Co-ordinates of x_1', x_2', x_3' of the vectors

$\alpha = \{x_1, x_2, x_3\}$. ⟵ given basis.

$$[\alpha]_{B'} = P^{-1}[\alpha]_B$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -1 & 2 & 11/8 \\ 0 & 1/2 & 3/16 \\ 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_2 + 11/8\, x_3 \\ 1/2\, x_2 + 3/16\, x_3 \\ 1/8\, x_3 \end{bmatrix}$$

1) Show that the vectors $\alpha_1 = (1,1,0,0)$ $\alpha_2 = (0,0,1,1)$ (33)
$\alpha_3 = (1,0,0,4)$ $\alpha_4 = (0,0,0,2)$

form a basis for R^4. Find the co-ordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

Soln $P = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ or $A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$ & check for L.I.

$[\alpha]_{B'} = P^{-1}[\alpha]_B$

$P = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$ vectors are represented in columns

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix} \xrightarrow{R_4 - R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$\xrightarrow{R_4 + 4R_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow[R_4/2]{R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Linearly independent}$$

$\therefore \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ forms a basis.

Co-ordinates for standard Basis

$\epsilon_1 = (1,0,0,0) = \sum x_i \alpha_i$ $\boxed{[x]_{B'} = P^{-1}[\epsilon_1]_B}$

Std Basis. $B = \{\epsilon_1\ \epsilon_2\ \epsilon_3\ \epsilon_4\}$

$[\alpha]_{B'} = P^{-1}[\alpha_B]$

$[\epsilon_1]_{B'} = P[\alpha]_B$.

$(1,0,0,0) = x_1(1,1,0,0) + x_2(0,0,1,1) + x_3(1,0,0,4) + x_4(0,0,0,2)$

$(1,0,0,0) = x_1 + x_3,\ x_1,\ x_2,\ x_2 + 4x_3 + 2x_4)$

$x_1 + x_3 = 1 \Rightarrow \boxed{x_3 = 1}$

$0 + 4 + 2x_4 = 0 \Rightarrow x_4 = -2$

$x_1 = 0.$

$x_2 = 0$ $[\epsilon_1]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$

Coordinates for ϵ_2

$(0,1,0,0) = x_1(1,1,0,0) + x_2(0,0,1,1) + x_3(1,0,0,4) + x_4(0,0,0,2)$

$(0,1,0,0) = (x_1 + x_3,\ x_1,\ x_2,\ x_2 + 4x_3 + 2x_4)$

$\therefore$ $\boxed{x_1 = 1}$ $\boxed{x_2 = 0}$ $\boxed{x_3 = -1}$

$x_2 + 4x_3 + 2x_4 = 0$

$-4 + 2x_4 = 0$ $\boxed{x_4 = 2}$

$[\epsilon_2]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ Coordinates $\Rightarrow P^{-1}\epsilon_2$

Co-ordinates for ϵ_3

$(0,0,1,0) = \sum_{i=1}^{4} x_i \alpha_i$

$= (x_1 + x_3,\ x_1,\ x_2,\ x_2 + 4x_3 + 2x_4)$

$x_1 = 0$, $x_3 = 0$

$x_2 = 1$ $\quad 1 + 2x_4 = 0$

$x_4 = -1/2$

$\therefore$ $[\epsilon_3]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{bmatrix}$

Finally Co-ordinates of ϵ_4

$(0,0,0,1) = \sum_{i=1}^{4} x_i \alpha_i$

$(0,0,0,1) = (x_1 + x_3,\ x_1,\ x_2,\ x_2 + 4x_3 + 2x_4)$

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$

$x_2 + 4x_3 + 2x_4 = 1$

$x_4 = 1/2$

$[\epsilon_4]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$

2) Find the co-ordinate matrix of the vector $(1,0,1)$ in the basis of C^3 consisting of the vectors $(2i,1,0)$, $(2,-1,1)$, $(0,1+i,1-i)$ in that order.

$\alpha_1 = (2i,1,0)$ $\quad \alpha_2 = (2,-1,1)$ $\quad \alpha_3 = (0,1+i,1-i)$

~~So~~ ordered Basis $B'(\alpha_1,\alpha_2,\alpha_3)$.

$$\cancel{(1,0,1) = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3}$$

$$\cancel{= x_1(2i,1,0) + x_2(2,-1,1) + x_3(0,1+i,1-i)}$$

$$(1,0,1) = (2ix_1 + 2x_2,\ x_1 - x_2 + x_3 + ix_3,\ x_2 + 3x_3 - 3ix_3)$$

$$\cancel{P = \begin{pmatrix} 2i & 1 & 0 \\ 2 & -1 & 1 \\ 0 & 1+i & 1-i \end{pmatrix}}$$

$$P^{-1} = \begin{bmatrix} \frac{1-i}{2} & -i & -1 \\ \frac{-i}{2} & -1 & i \\ \frac{i-1}{4} & \frac{i+1}{2} & 1 \end{bmatrix}$$

$$P = (\alpha_1 \quad \alpha_2 \quad \alpha_3) = \begin{bmatrix} 2i & 2 & 0 \\ 1 & -1 & 1+i \\ 0 & 1 & 1-i \end{bmatrix}$$

Find $P^{-1} = \begin{bmatrix} \frac{1-i}{4} & \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{1-i}{4} & \frac{-1-i}{2} & \frac{-1+i}{2} \\ 0 & 0 & 1+i/2 \end{bmatrix}$

$$\begin{pmatrix} \frac{-1-i}{2} \\ i/2 \\ \frac{i+3}{4} \end{pmatrix}$$

$$[\alpha']_{B'} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = P^{-1}[\alpha]_B = \begin{bmatrix} \frac{1-i}{4} & \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{1-i}{4} & \frac{-1-i}{2} & \frac{-1+i}{2} \\ 0 & 0 & \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1-3i}{4} \\ \frac{-1+i}{4} \\ \frac{1+i}{2} \end{bmatrix} \qquad [\alpha]_{B'} =$$

Summary of Row - Equivalence

If A is an $m \times n$ matrix over the field F the row vectors of A are the vectors $\alpha_1, \dots \alpha_m$ in F^n defined by

$$\alpha_i = (A_{i1}, A_{i2} \dots A_{in}).$$

and that the row space of A is the subspace of F^n spanned by these vectors. The row rank of A is the dimension of the row space of A.

If P is a $k \times m$ matrix over F, then the product $B = PA$ is $k \times n$ matrix whose row vectors $\beta_1 \dots \beta_k$ are linear combinations

$$\beta_i = P_{i1}\alpha_1 + P_{i2}\alpha_2 + \dots P_{im}\alpha_m$$

of the row vectors of A. Then the row space of B is a subspace of the row space of A.

If P is an $m \times m$ invertible matrix, then B is row equivalent to A so that symmetry of row equivalence or the equation $\boxed{A = P^{-1}B}$ implies that the row space of A is a subspace of the row space of B

Theorem 9:- Row equivalent matrices have the same row space

Theorem 10 :- Let R be a non-zero row reduced echelon matrix. Then the non-zero row vectors of R form a basis for the row space of R.

Proof:- Let $\rho_1 \dots \rho_r$ be the non-zero row vectors of R

$$\rho_i = (R_{i1}, R_{i2} \dots R_{in})$$

Certainly these vectors span the row space of R. we need to prove that they are linearly independent. (35)

Since R is a row reduced echelon matrix, there are positive integers $k_1, \ldots k_r$ such that for $i \leq r$

(a) $R(i,j) = 0$ if $j < k_i$

(b) $R(i, k_j) = \delta_{ij}$

(c) $k_1 < k_2 \cdots < k_r$

β is $(b_1 \ldots b_n)$ is a vector in the row space of R

$$\beta = c_1\rho_1 + \cdots + c_r\rho_r$$

Then we claim that $c_j = b_{k_j}$

$$b_{k_j} = \sum_{i=1}^{r} c_i R(i, k_j)$$

$$= \sum_{i=1}^{r} c_i \delta_{ij}$$

$$= c_j$$

if $\beta = 0$ i.e if $c_1\rho_1 + \cdots + c_r\rho_r = 0$ then c_j must be the k_jth co-ordinate of the zero vector so that $c_j = 0$ $j = 1, \ldots r$ thus $\rho_1 \ldots \rho_r$ are linearly independent

$\Longrightarrow$ Let A and B be $m \times n$ ~~square~~ matrices over the field F. Then A and B are row-equivalent iff they have the same row space.