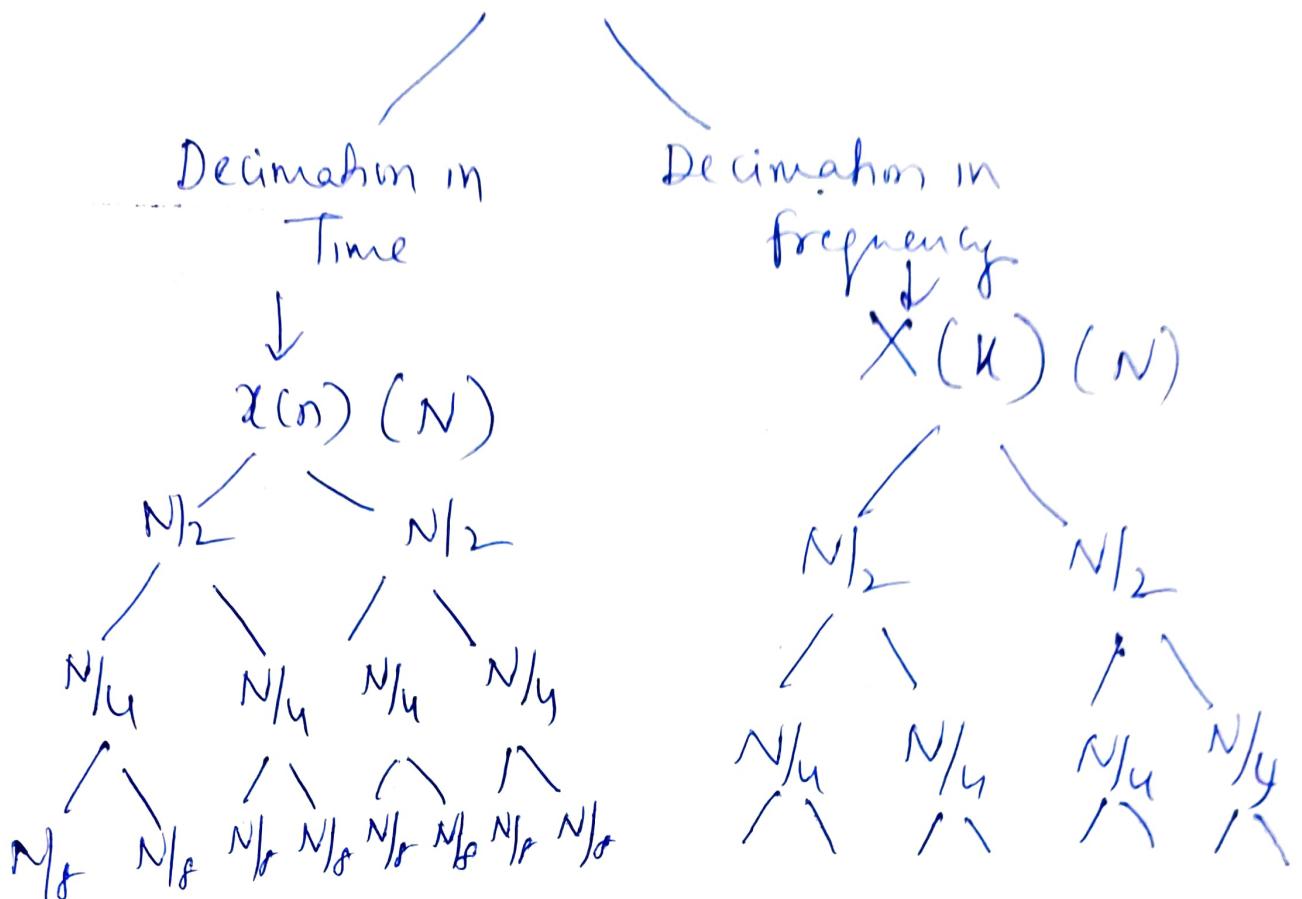


# ①

# Fast Fourier Transform



Radix-2 FFT algorithm .

- Decimate the sequence until we get 2 pt. sequence.

- $\underline{N = 2^x}$

# Radix-2 DFT FFT Algorithm

(2)

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad 0 \leq k \leq N-1$$

$$\Rightarrow X(k) = \underbrace{\sum_{n \text{ even}} x(n) W_N^{kn}} + \underbrace{\sum_{n \text{ odd}} x(n) W_N^{kn}}$$

$n$  even

$$n = 2r$$

$n$  odd

$$n = 2r+1$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_{\frac{N}{2}}^{rk} +$$

$$W_N^k \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_{\frac{N}{2}}^{rk}$$

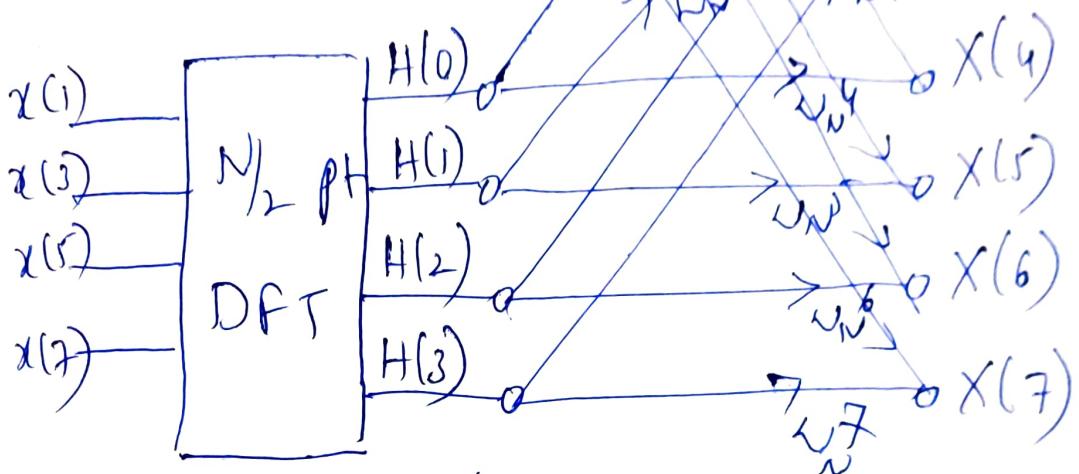
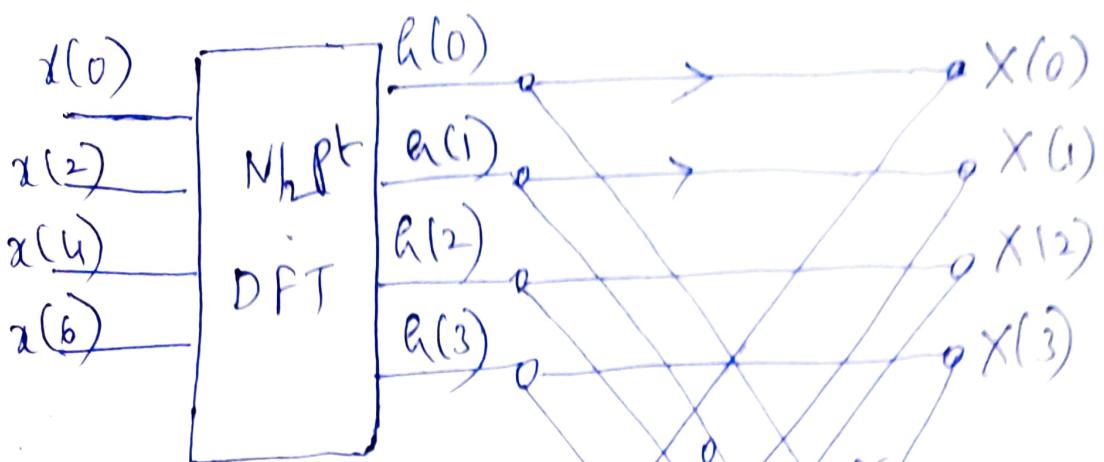
$$\left. \begin{aligned} W_N^{2rk} &= e^{-j \frac{2\pi}{N} rk} \\ &= e^{-j \frac{2\pi}{N/2} rk} \\ &= W_{N/2}^{rk} \end{aligned} \right\}$$

$$= G(k) + W_N^k H(k)$$

$G(k)$  and  $H(k)$  are  $\frac{N}{2}$  point DFT's.

Let's draw SFH. for  $N = 8$

(3)



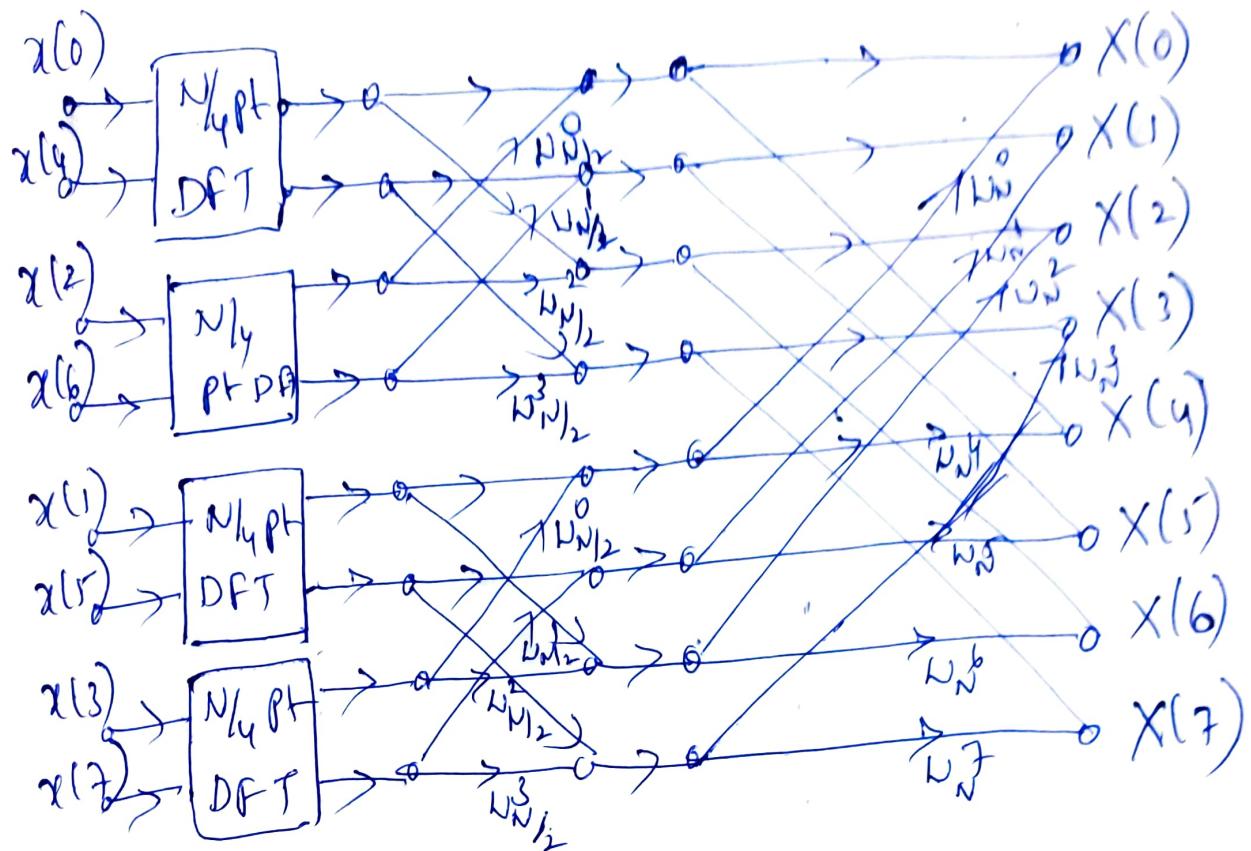
$$h(k) = \sum_{r=0}^{N/2-1} g(r) W_{N/2}^{rk}$$

$$= \sum_{r \text{ even}} + \sum_{r \text{ odd}}$$

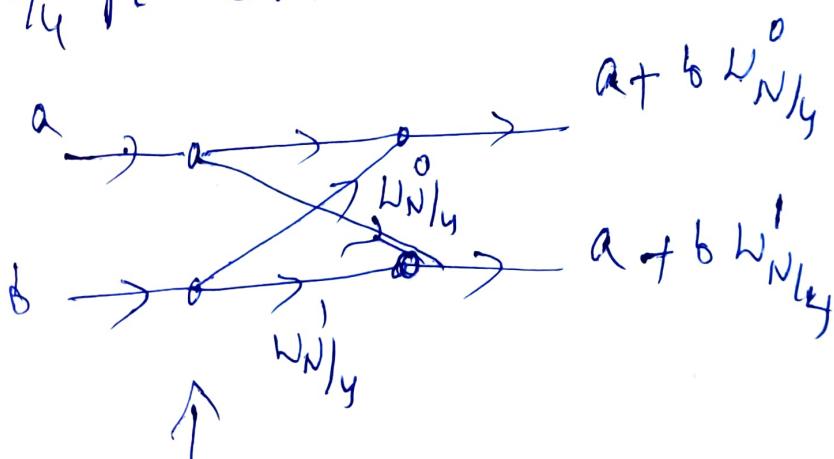
$$= P(k) + W_{N/2}^k Q(k)$$

$$\text{My } H(k) = R(k) + W_{N/2}^k S(k)$$

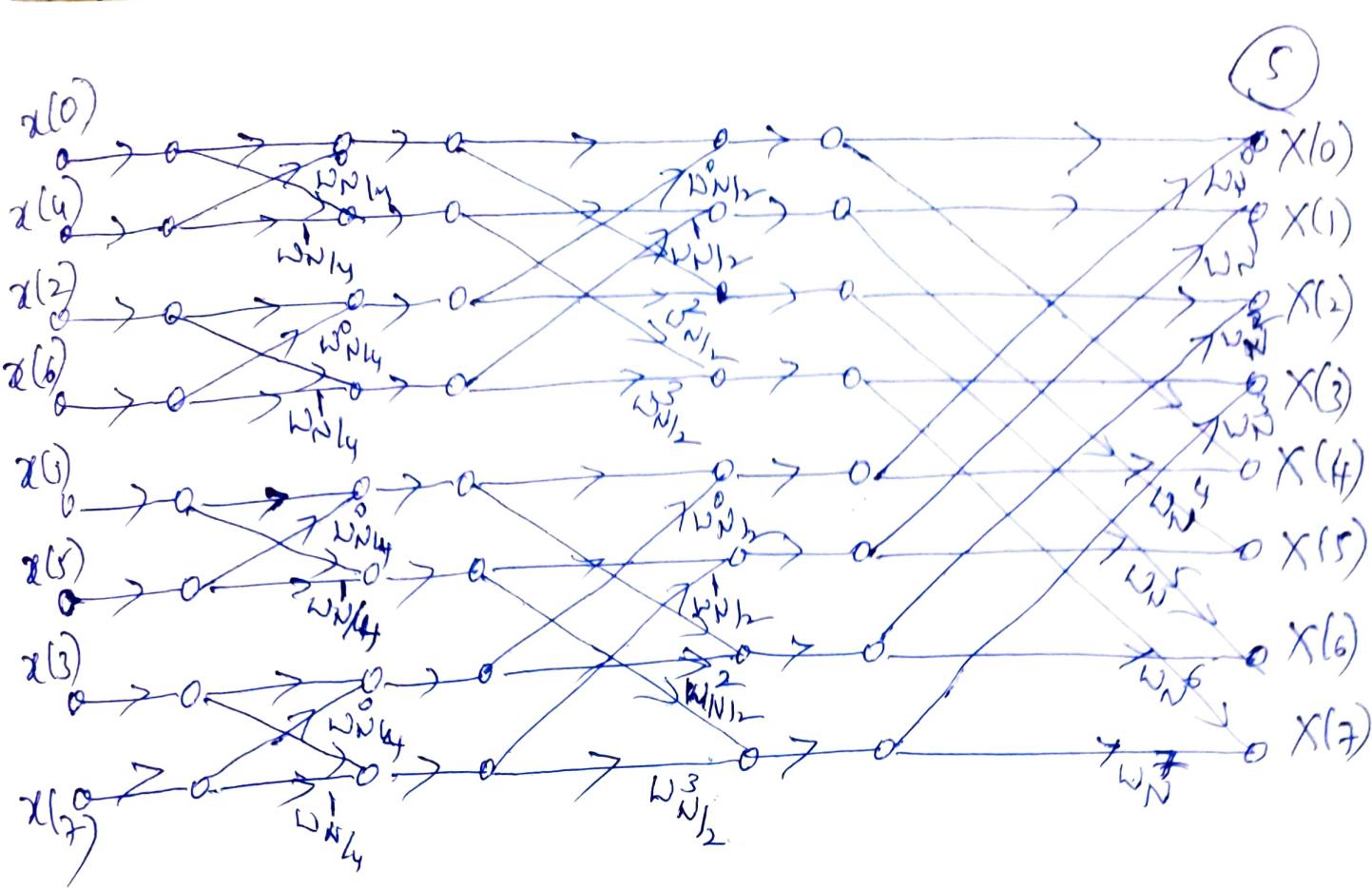
(4)



$N/4$  pt DFT  $\rightarrow$  2 pt. DFT



Butterfly



### Observations

1) O/p date is in bit reversed order

$$x(0) \ 000$$

$$x(4) \ 100$$

$$x(2) \ 010$$

$$x(6) \ 110$$

;

$$X(0) \ 000$$

$$X(1) \ 001$$

$$X(2) \ 010$$

$$X(3) \ 011$$

;

2) o/p date is in proper order (normal order)

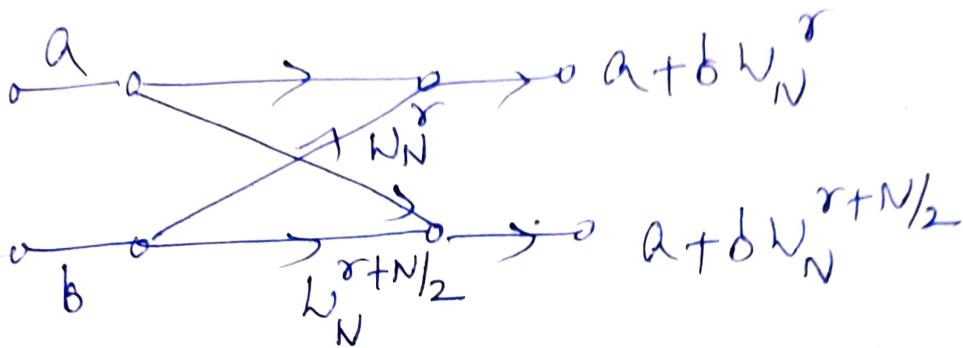
3) Basic Computational block is called butterfly.

Further reduction:

⑥

→ Cooley-Tukey Algorithm

Butterfly Configuration

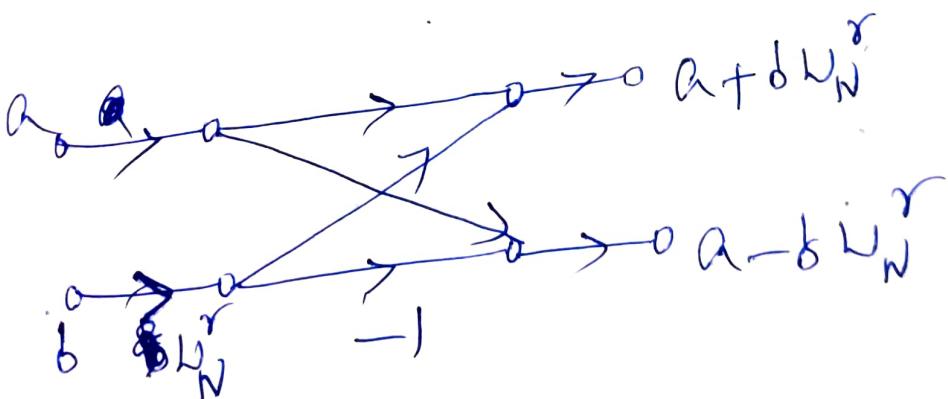


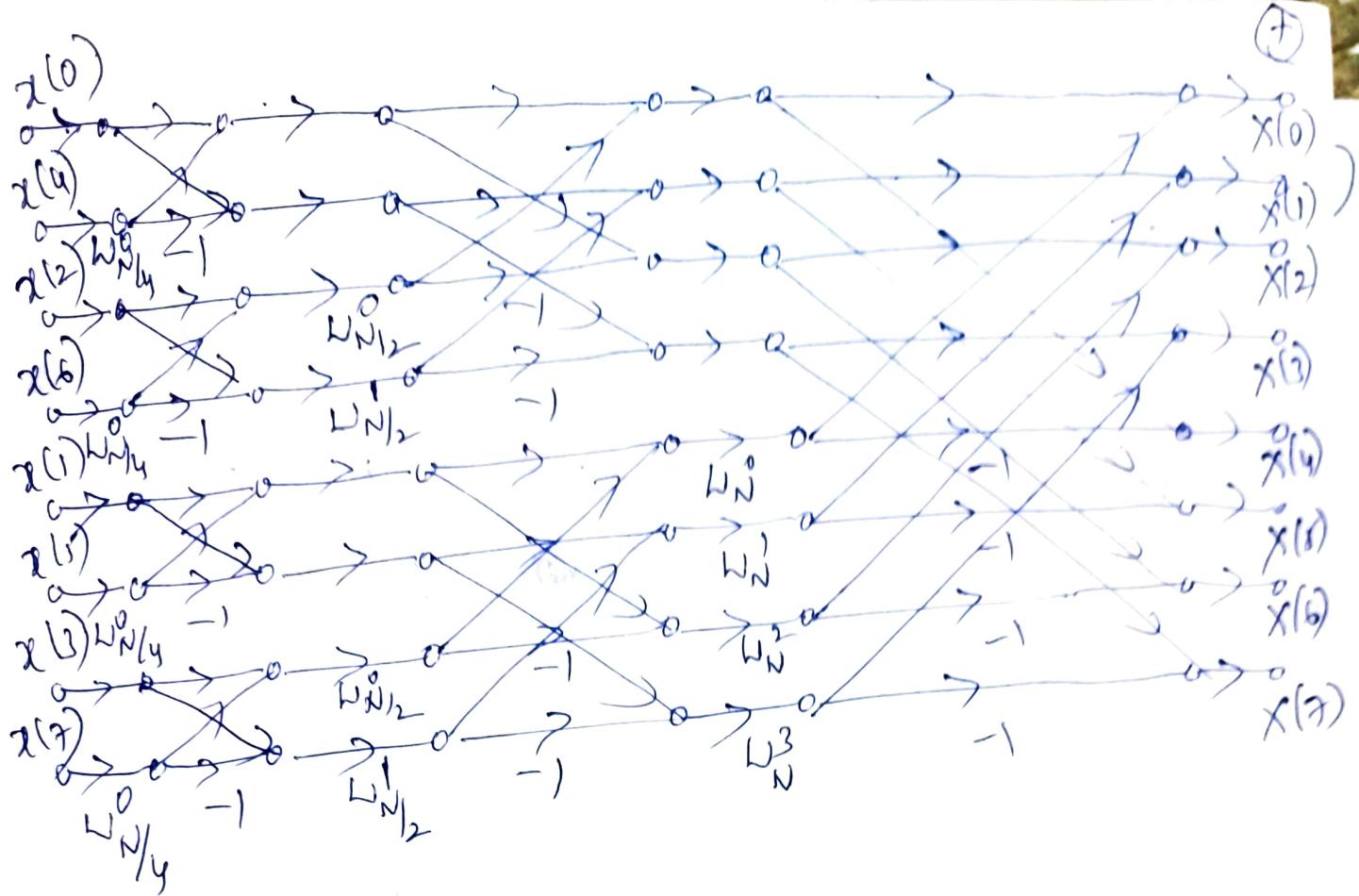
→ Can be further simplified to reduce number of complex multiplications

$$W_N^{r+N/2} = W_N^r W_N^{N/2}$$

$$= -W_N^r$$

$$\left. \begin{aligned} W_N^{N/2} &= e^{-j\frac{2\pi}{N} \frac{N}{2}} \\ &= e^{-j\pi} \\ &= -1 \end{aligned} \right\}$$





Computation time

Direct Computation

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad 0 \leq k \leq N-1$$

No. of complex multiplications =  $N^2$

No. of complex additions =  $N(N-1)$

$$N = 8;$$

No. of multiplications =  $8^2 = 64$

No. of additions =  $8 \times 7 = 56$

## Radix-2 DIT FFT algorithm.

(8)

$$\text{No. of stages} = \log_2 N$$

In each stage, No. of butterflies =  $N/2$

In each butterfly, No. of multiplications  
= ~~03~~ 01

In each butterfly, no. of additions  
= 02

$$\therefore \text{Total No. of } X^m = N/2 \log_2 N$$

$$\text{Total No. of } +^m = N \log_2 N$$

$N = 8$ :

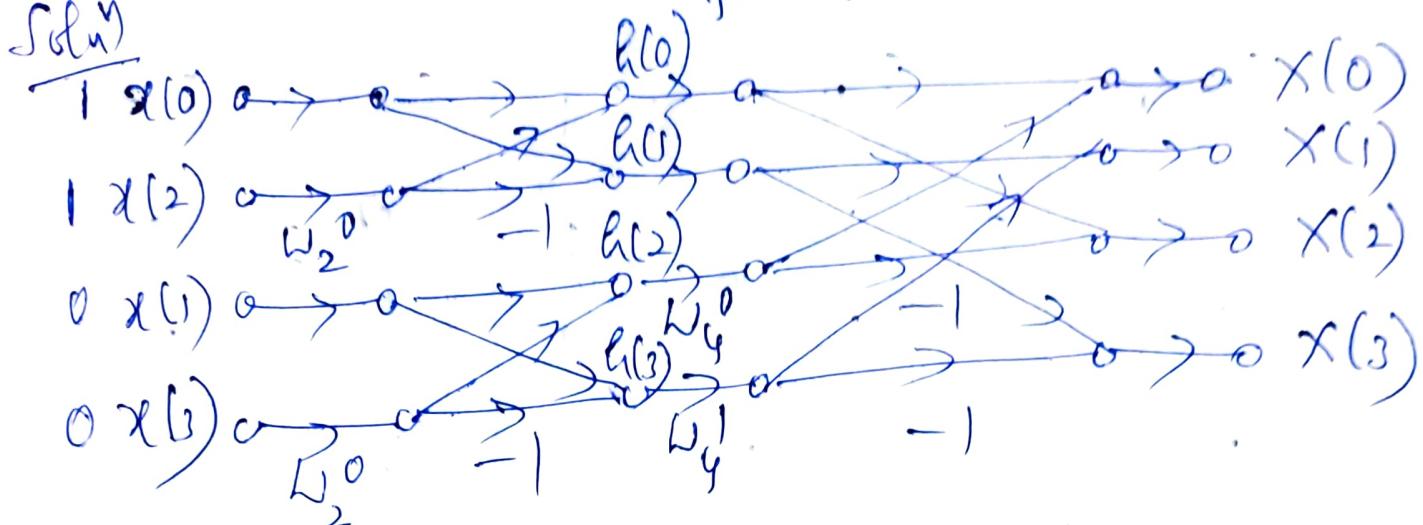
$$\begin{aligned} \text{No. of } X^m &= 4 \log_2 8 = 4 \log_2 2^3 \\ &= 3 \times 4 = \underline{\underline{12}} \end{aligned}$$

$$\text{No. of } +^m = 8 \log_2 8 = 8 \times 3 = \underline{\underline{24}}$$

Ques: 1) Compute 4 point DFT using  
Radix-2 DIT FFT algorithm

Verify the result by direct Computation  
 $x(n) = \{ 1, 0, 1, 0 \}$

Solu<sup>y</sup>



$$h(0) = x(0) + x(2)w_2^0 = 1 + 1 = 2$$

$$h(1) = x(0) - x(2)w_2^0 = 1 - 1 = 0$$

$$h(2) = x(1) + x(3)w_2^0 = 0 + 0 = 0$$

$$h(3) = x(1) - x(3)w_2^0 = 0 - 0 = 0$$

$$X(0) = h(0) + h(2)w_4^0 = 2 + 0 = 2$$

$$X(1) = h(1) + h(3)w_4^1 = 0 + 0 = 0$$

$$X(2) = h(0) - h(2)w_4^0 = 2 - 0 = 2$$

$$X(3) = h(1) - h(3)w_4^1 = 0 - 0 = 0$$

$$X(k) = \{ 2, 0, 2, 0 \}$$

b) Direct Computation

(10)

$$X(k) = \sum_{n=0}^3 x(n) e^{-j\frac{2\pi}{4}kn} \quad 0 \leq k \leq 3$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$X(k) = \{ 2, 0, 2, 0 \}$$

2) Compute 4 point DFT of the sequence  $\{ 1, -3, -2, 5 \}$  using Radix-2 DIT FFT algorithm. Verify the result by direct computation.

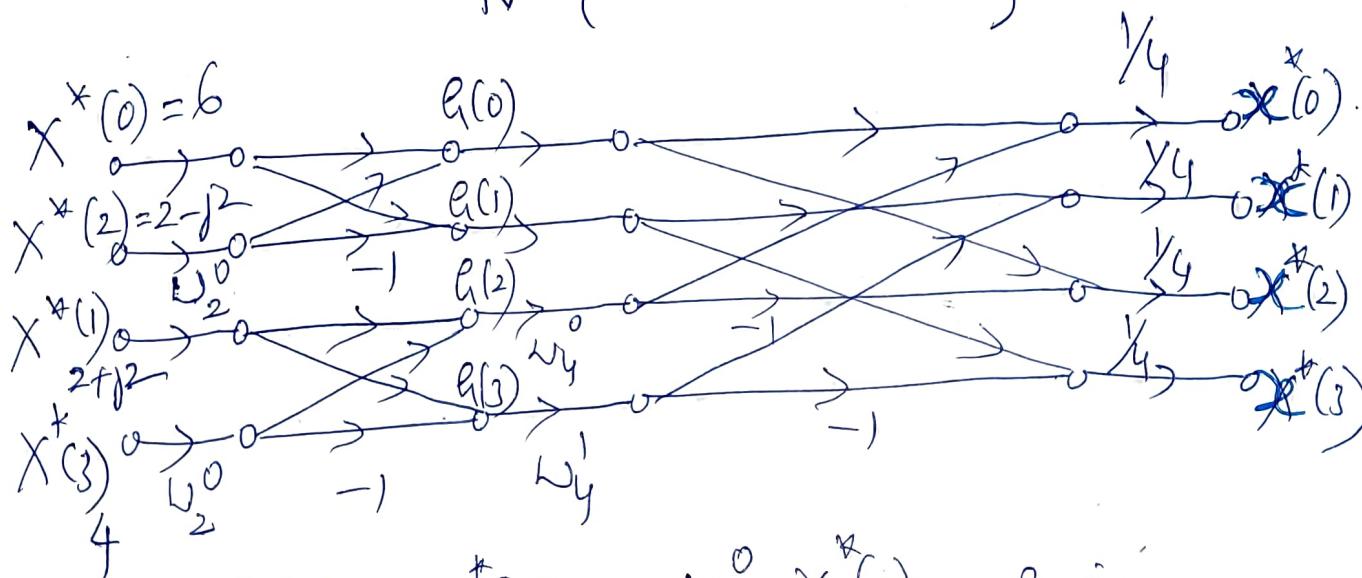
# 11

## Computing IDFT using Radix-2 DIT FFT Algorithm

Prob: find 4 point sequence  $x(n)$  if its DFT samples are  $X(0) = 6$ ,  $X(1) = 2 - j2$ ,  $X(2) = 2 + j2$ ,  $X(3) = 4$ . a) Use Radix-2 DIT FFT algorithm. b) Verify the result by direct computation

Soln

$$x(n) = \frac{1}{N} \left\{ \text{DFT}[X^*(k)] \right\}^*$$



$$x(0) = X^*(0) + \omega_2^0 X^*(2) = 8 - j2$$

$$x(1) = X^*(0) - \omega_2^0 X^*(2) = 4 + j2$$

$$x(2) = X^*(1) + \omega_2^0 X^*(3) = 6 + j2$$

$$x(3) = X^*(1) - \omega_2^0 X^*(3) = -2 + j2$$

(12)

$$x^*(0) = \frac{1}{4} [h(0) + h(2) w_4^0]$$

$$= \frac{1}{4} [8 - j2 + 6 + j2] = \frac{14}{4} = \frac{7}{2}$$

$$x^*(1) = \frac{1}{4} [h(1) + h(3) w_4^1]$$

$$= \frac{1}{4} [4 + j2 + (-j)(-2 + j2)]$$

$$= \frac{1}{4} [4 + j2 + 2j + 2]$$

$$= \frac{1}{4} [6 + j4] = \frac{3 + j2}{2} = \frac{3}{2} + j$$

$$x^*(2) = \frac{1}{4} [h(0) - w_4^0 h(2)]$$

$$= \frac{1}{4} [8 - j2 - 6 - j2] = \frac{1}{4} [2 - j4]$$

$$= \frac{1}{2} - j$$

$$x^*(3) = \frac{1}{4} [h(1) - h(3) w_4^1]$$

$$= \frac{1}{4} [4 + j2 - (-j)(-2 + j2)]$$

$$= \frac{1}{4} [4 + j2 - j2 - 2] =$$

$$= \frac{1}{4} [2] = \frac{1}{2}.$$

$$\therefore x^*(k) = \left\{ \frac{7}{2}, \frac{3}{2} - j, \frac{1}{2} + j, \frac{1}{2} \right\}$$

(13)

b) Direct Computation

$$x(n) = \frac{1}{N} \left\{ DFT [x^*(k)] \right\}^*$$

$$X^*(k) = \{ 6, 2+j2, 2-j2, 4 \}$$

$$\begin{bmatrix} x^*(0) \\ x^*(1) \\ x^*(2) \\ x^*(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 6 \\ 2+j2 \\ 2-j2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 14/4 \\ \cancel{\frac{6+4j}{4}} \\ \cancel{\frac{2-j4}{4}} \\ \cancel{\frac{2}{4}} \end{bmatrix} = \begin{bmatrix} 7/2 \\ 3/2+j \\ 1/2-j \\ 1/2 \end{bmatrix}$$

$$\therefore x(n) = \{ \underset{\uparrow}{7/2}, 3/2-j, 1/2+j, 1/2 \}$$

Prob 2 Compute Circular Convolution  
of the following sequences using Radix-2  
DIT FFT algorithm.

(14)

$$x_1(n) = \left\{ \begin{array}{l} 2, 1, 1, 2 \\ \uparrow \end{array} \right\} \quad x_2(n) = \left\{ \begin{array}{l} 1, -1, -1, 1 \\ \uparrow \end{array} \right\}$$

Soln find  $X_1(k)$  &  $X_2(k)$  using SFL

$$X_1(k) = \{ 6, 1+j, 0, 1-j \}$$

$$X_2(k) = \{ 0, 2+2j, 0, 2-2j \}$$

$$\begin{aligned} X_3(k) &= X_1(k) \cdot X_2(k) \\ &= \{ 0, 4j, 0, -4j \} \end{aligned}$$

find  $x_3(n)$  using SFL.

$$x_3(n) = \left\{ \begin{array}{l} 0, -2, 0, 2 \\ \uparrow \end{array} \right\}$$

## Radix-2 DIF FFT Algorithm

(15)

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad 0 \leq k \leq N-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x(n+\frac{N}{2}) W_N^{k(n+N/2)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{k\frac{N}{2}} \sum_{n=0}^{\frac{N}{2}-1} x(n+\frac{N}{2}) W_N^{kn}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} [x(n) + (-1)^k x(n+\frac{N}{2})] W_N^{kn}$$

Separating odd and even terms of  
 $X(k)$

Even values of  $k \Rightarrow k = 2\tau$ ) (16)

$$X(2\tau) = \sum_{n=0}^{N_2-1} [x(n) + (-1)^{2\tau} x(n+N_2)] W_N^{2\tau n}$$

$$= \sum_{n=0}^{N_2-1} [x(n) + x(n+N_2)] W_{N_2}^{\tau n}$$

→ 1

$$\left| \begin{array}{l} W_N^{2\tau n} \\ = e^{-j \frac{2\pi}{N} 2\tau n} \\ = e^{-j \frac{2\pi}{N_2} \tau n} \\ W_{N_2}^{\tau n} \end{array} \right.$$

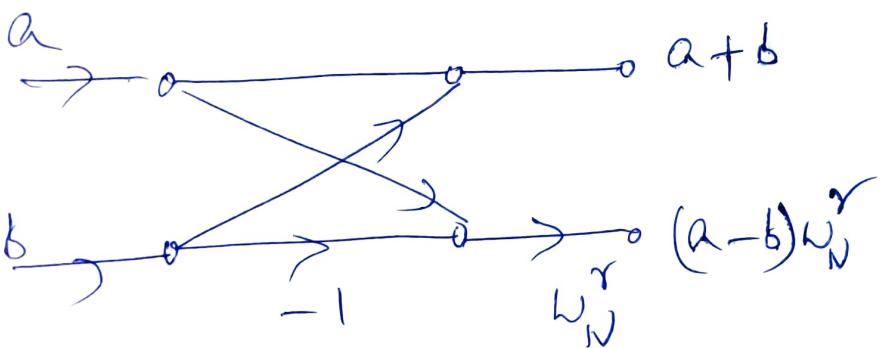
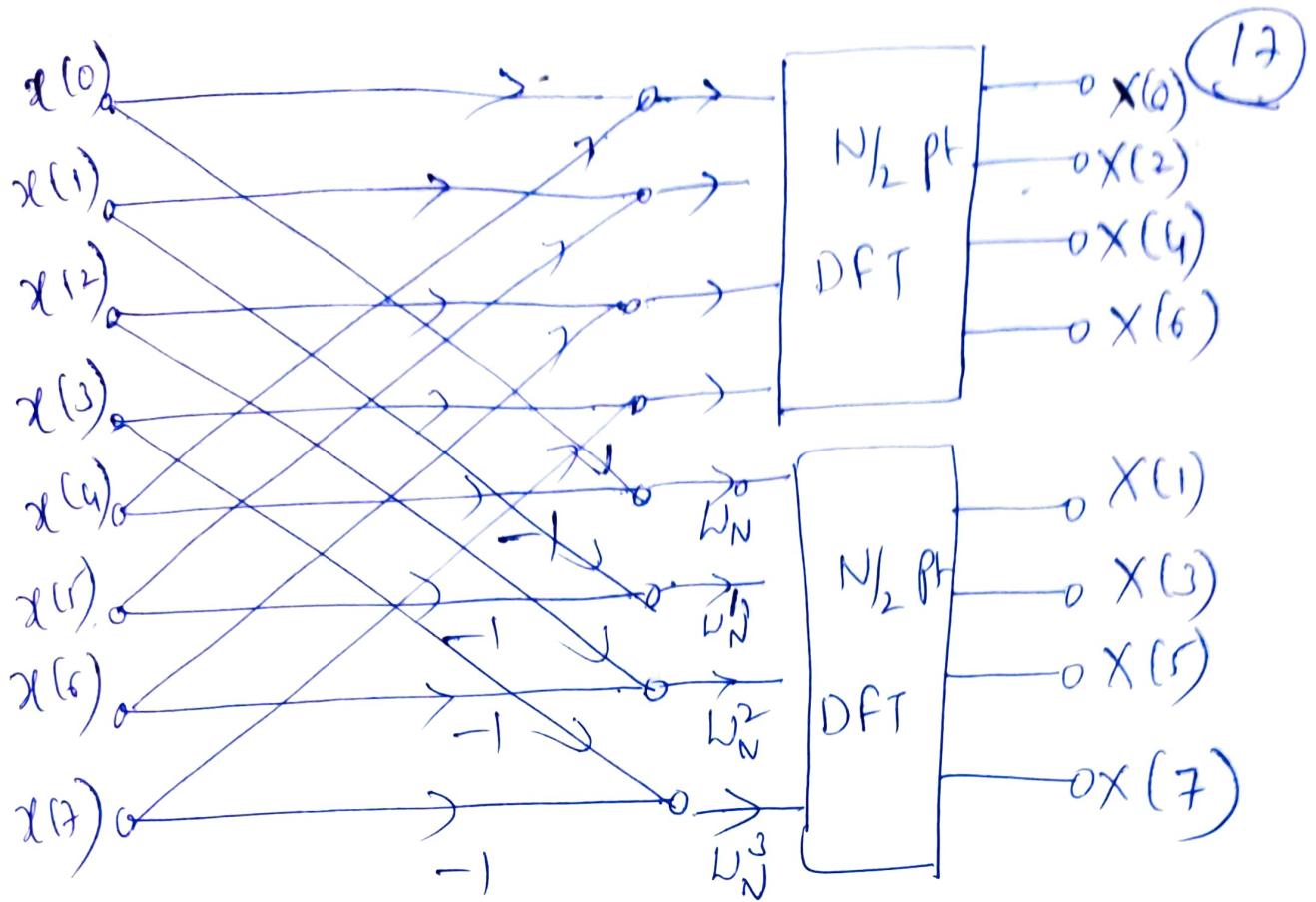
odd values of  $k \Rightarrow k = 2\tau + 1$

$$X(2\tau+1) = \sum_{n=0}^{N_2-1} [x(n) + (-1)^{2\tau+1} x(n+N_2)] W_N^{(2\tau+1)n}$$

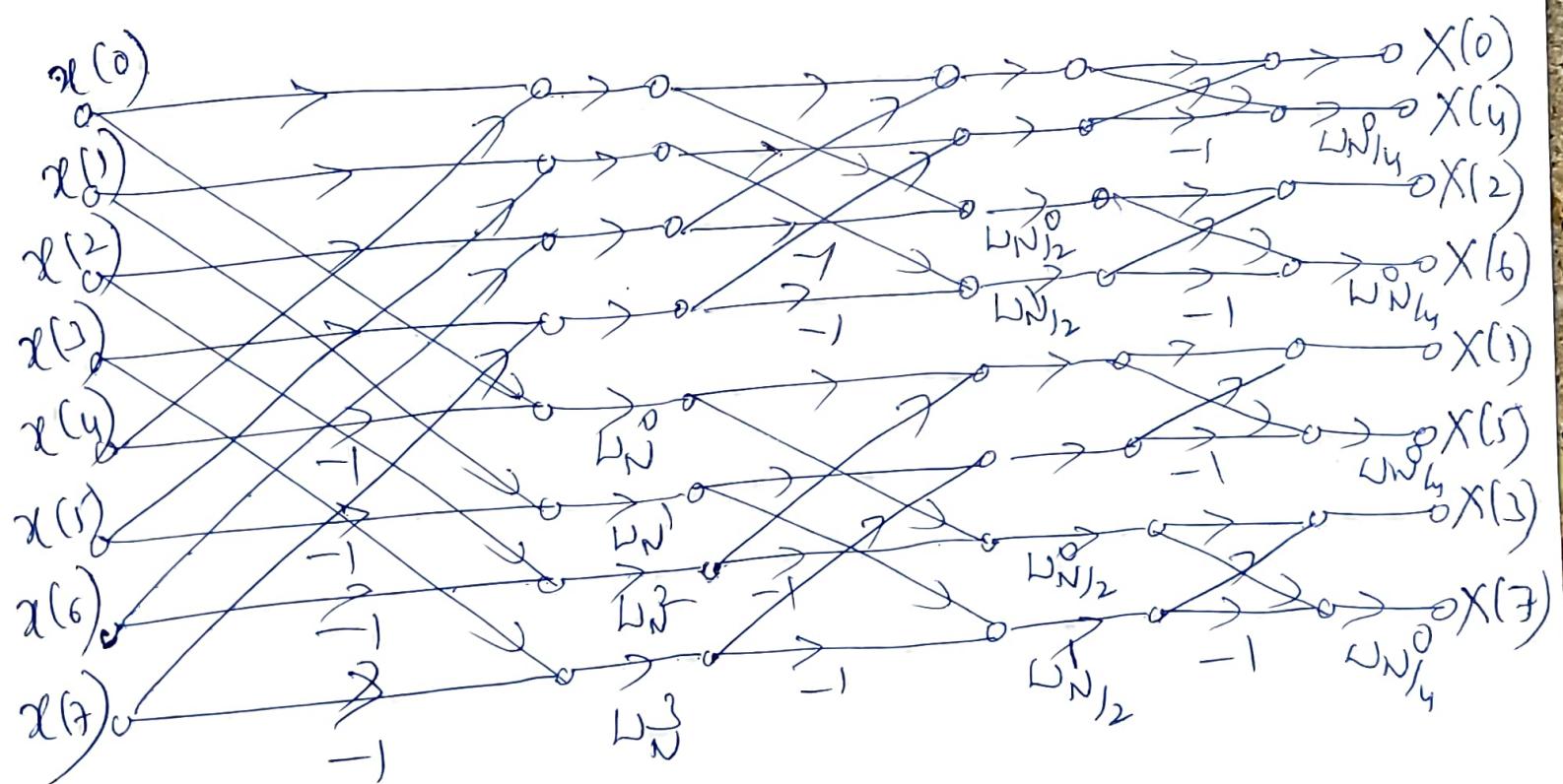
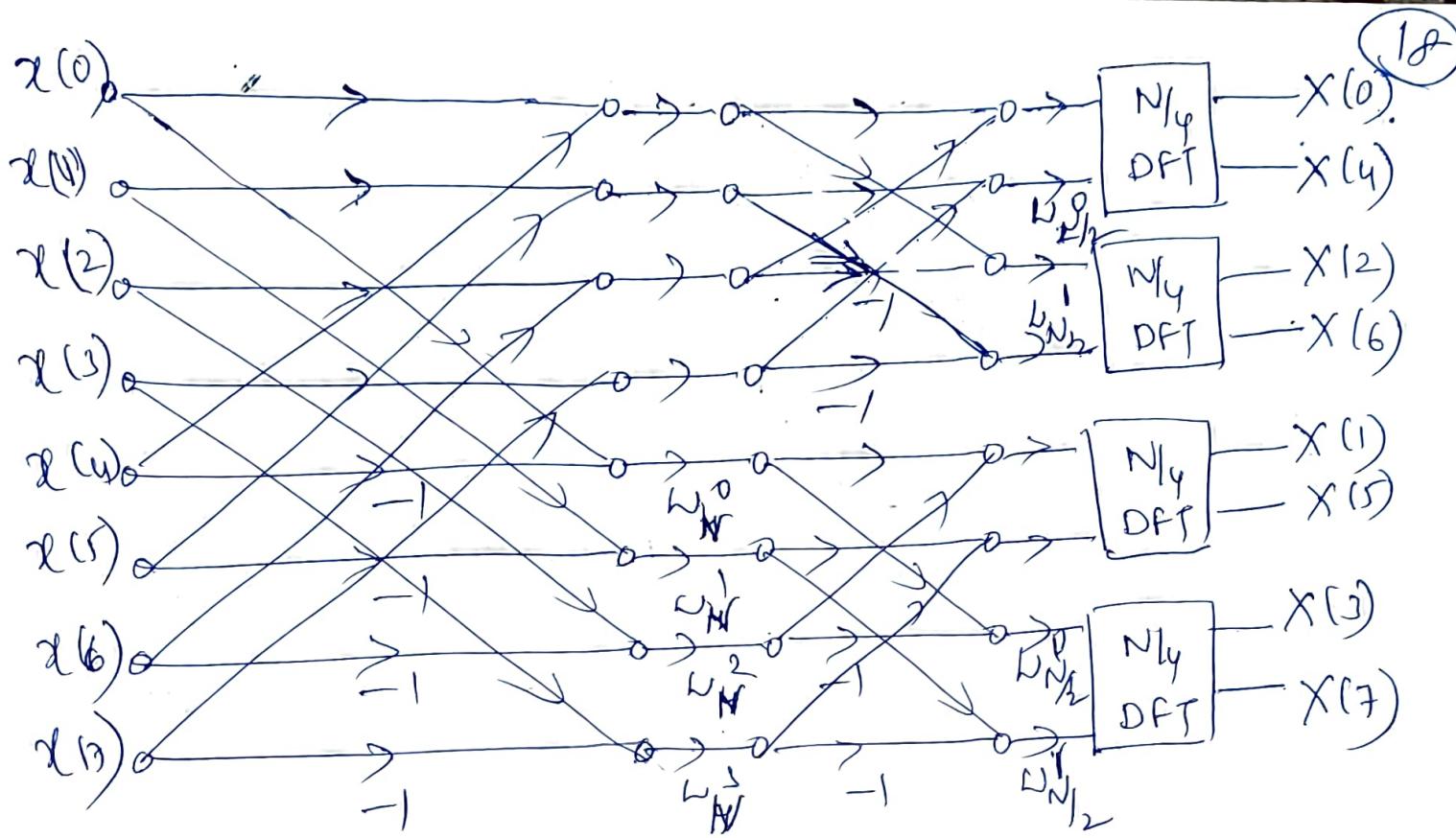
$$= \sum_{n=0}^{N_2-1} \{ [x(n) - x(n+N_2)] W_N^n \} W_N^{2\tau n}$$

$$= \sum_{n=0}^{N_2-1} \{ [x(n) - x(n+N_2)] W_N^n \} W_{N_2}^{\tau n}$$

→ 2



$N/2$  pt DFT is replaced by  
two  $N/4$  pt DFT's.



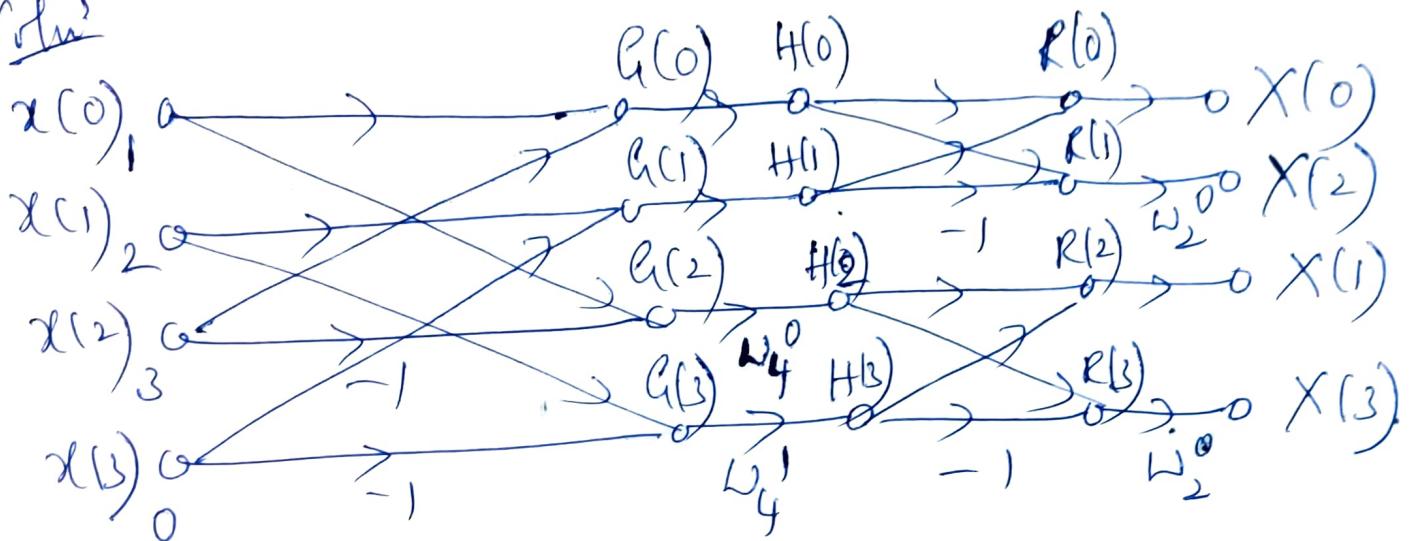
Observations

- 1) If in proper order
- 2) If in bit reversed order

Prob: find 4 point DFT of the sequence  $x(n) = \{1, 2, 3, 0\}$  using radix-2 DIF FFT algorithm. Verify the result by direct computation.

(15)

Solu:



$$h(0) = x(0) + x(2) = 1 + 3 = 4$$

$$h(1) = x(1) + x(3) = 2 + 0 = 2$$

$$h(2) = x(0) - x(2) = 1 - 3 = -2$$

$$h(3) = x(1) - x(3) = 2 - 0 = 2$$

$$H(0) = h(0) = 4 \quad H(1) = h(1) = 2$$

$$H(2) = h(2) \omega_4^0 = -2 \quad H(3) = h(3) \omega_4^1 = -2j$$

$$R(0) = H(0) + H(1) = 4 + 2 = 6$$

$$R(1) = H(0) - H(1) = 4 - 2 = 2$$

$$R(2) = H(2) + H(3) = -2 - 2j$$

$$R(3) = H(2) - H(3) = -2 + 2j$$

(20)

$$x(0) = R(0) = 6$$

$$X(2) = R(1) \omega_2^0 = 2$$

$$X(1) = R(2) = -2 - 2j$$

$$X(3) = R(3) \omega_2^0 = -2 + 2j$$

$$X(k) = \left\{ \begin{matrix} 6, \\ \uparrow \\ -2 - 2j, \\ 2, \\ -2 + 2j \end{matrix} \right\}$$

b)  $X(k) = \sum_{n=0}^{3} x(n) \omega_4^{kn} \quad 0 \leq k \leq 3$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 - 2j \\ 2 \\ -2 + 2j \end{bmatrix}$$

$$X(k) = \left\{ \begin{matrix} 6, \\ \uparrow \\ -2 - 2j, \\ 2, \\ -2 + 2j \end{matrix} \right\}$$

Ques: find IDFT using Radix-2 DIF FFT algorithm. (21)

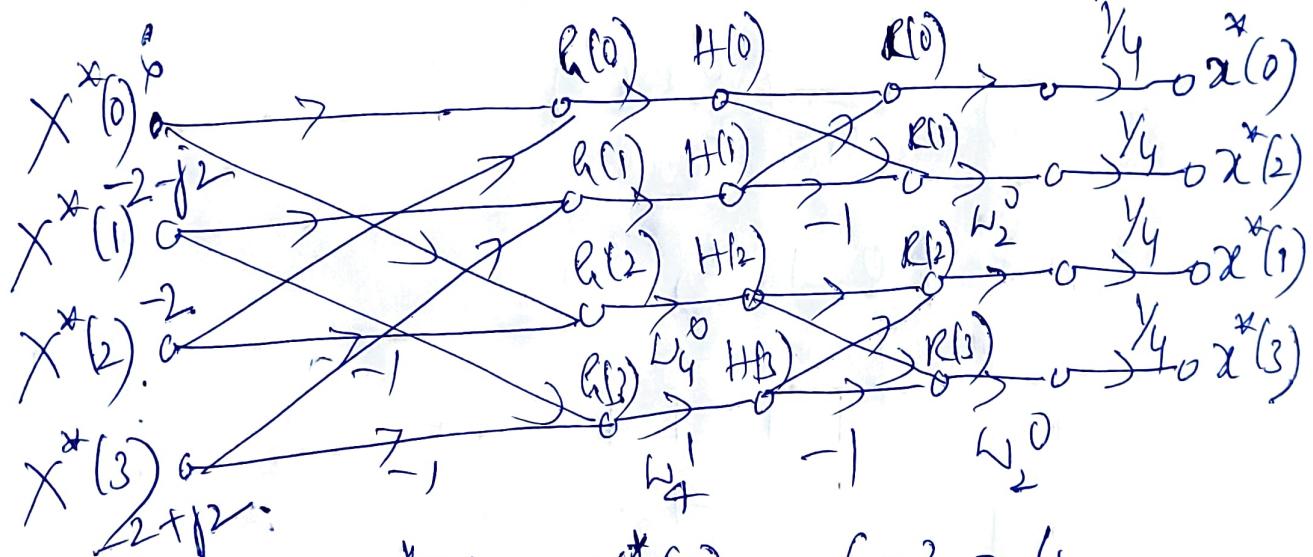
$$X(k) = \{ 6, -2+j2, -2, -2-j2 \}$$

Verify the result by direct computation.

Soln

$$x(n) = \frac{1}{N} \left\{ \text{DFT} [x^*(k)] \right\}^*$$

$$x^*(k) = \{ 6, -2-j2, -2, -2+j2 \}$$



$$R(0) = x^*(0) + x^*(2) = 6 - 2 = 4$$

$$R(1) = x^*(1) + x^*(3) = -2 - j2 - 2 + j2 = -4$$

$$R(2) = x^*(0) - x^*(2) = 6 + 2 = 8$$

$$R(3) = x^*(1) - x^*(3) = -2 - j2 + 2 - j2 = -j4$$

$$H(0) = h(0) \quad H(1) = h(1)$$

$$H(2) = h(2) w_4^0 = h(2) \quad H(3) = w_4^{-1} h(3) = -jx - jy$$

$$= -4$$

(22)

$$R(0) = H(0) + H(1) = 4 - 4 = 0$$

$$R(1) = H(0) - H(1) = 4 + 4 = 8$$

$$R(2) = H(2) + H(3) = 8 - 4 = 4$$

$$R(3) = H(2) - H(3) = 8 + 4 = 12$$

$$x^*(0) = \frac{1}{4} R(0) = 0$$

$$x^*(2) = \frac{1}{4} W_2^0 R(1) = 2$$

$$x^*(1) = \frac{1}{4} R(2) = 1$$

$$x^*(3) = \frac{1}{4} W_2^0 R(3) = 3$$

$$x(n) = \left\{ \begin{array}{c} 0, \\ \uparrow 1, 2, 3 \end{array} \right\}$$

b)

$$\begin{bmatrix} x^*(0) \\ x^*(1) \\ x^*(2) \\ x^*(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & +1 & -j \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 6 \\ -2-j2 \\ -2 \\ -2+j2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x(n) = \left\{ \begin{array}{c} 0, \\ \uparrow 1, 2, 3 \end{array} \right\}$$

## Composite - Radix FFT

(21)

Composite or mixed Radix FFT is used when  $N$  is a composite no. i.e., a no. which has more than one prime factor.

e.g.:  $N = 12$  or  $18$ .

$$\begin{aligned} N &= 18 \\ &= 2 \times 3 \times 3 \end{aligned}$$

Q:  
1) Develop Radix-3 DIT FFT algorithm for Computing DFT for  $N=9$ .

S&W

$$\begin{aligned} N &= 9 = 3 \times 3 = m_1 \times m_2 \\ X(k) &= \sum_{n=0}^{\frac{N}{3}} x(n) W_9^{kn} \\ &= \sum_{n=0}^2 x(3n) W_9^{3nk} + \sum_{n=0}^2 x(3n+1) W_9^{(3n+1)k} \\ &\quad + \sum_{n=0}^2 x(3n+2) W_9^{(3n+2)k} \end{aligned} \quad - (1)$$

$$X(k) = X_1(k) + W_9^k \cdot X_2(k) + W_9^{2k} \cdot X_3(k) \quad - (2)$$

$$X_1(k) = \sum_{n=0}^2 x(3n) w_9^{3nk}$$

(24)

$$= x(0) + x(3) w_9^{3k} + x(6) w_9^{6k} \quad \rightarrow (3)$$

$$X_2(k) = \sum_{n=0}^2 x(3n+1) w_9^{3nk}$$

$$= x(1) + x(4) w_9^{3k} + x(7) w_9^{6k} \quad \rightarrow (4)$$

$$X_3(k) = \sum_{n=0}^2 x(3n+2) w_9^{3nk}$$

$$= x(2) + x(5) w_9^{3k} + x(8) w_9^{6k} \quad \rightarrow (5)$$

$$x(0) = X_1(0) + w_9^0 X_2(0) + w_9^0 X_3(0)$$

$$x(1) = X_1(1) + w_9^1 X_2(1) + w_9^2 X_3(1)$$

$$x(2) = X_1(2) + w_9^2 X_2(2) + w_9^4 X_3(2)$$

$$x(3) = X_1(0) + w_9^3 X_2(0) + w_9^5 X_3(0)$$

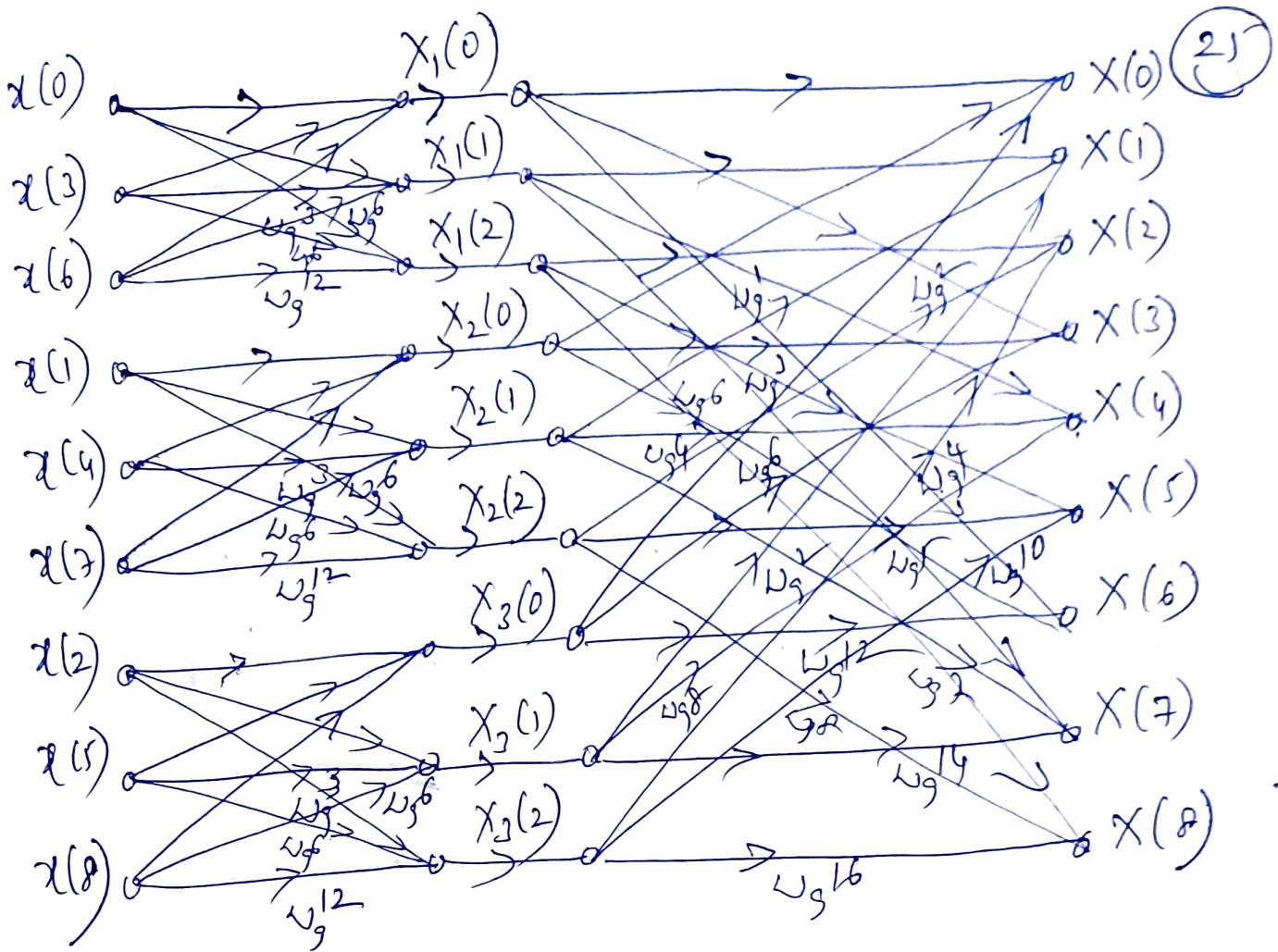
$$x(4) = X_1(1) + w_9^4 X_2(1) + w_9^8 X_3(1)$$

$$x(5) = X_1(2) + w_9^5 X_2(2) + w_9^{10} X_3(2)$$

$$x(6) = X_1(0) + w_9^6 X_2(0) + w_9^{12} X_3(0)$$

$$x(7) = X_1(1) + w_9^7 X_2(1) + w_9^{14} X_3(1)$$

$$x(8) = X_1(2) + w_9^8 X_2(2) + w_9^{16} X_3(2)$$



from  $\mathbb{C}^n$  ②

$$x_1(0) = x(0) + x(3) w_9^0 + x(6) w_9^0$$

$$x_1(1) = x(0) + x(3) w_9^3 + x(6) w_9^6$$

$$x_1(2) = x(0) + x(3) w_9^6 + x(6) w_9^{12}$$

2) Develop DIT FFT algorithm for  $N=6$ . and draw SFG for

(26)

$$(a) N = 6 = 3 \times 2 \quad (b) N = 6 = 2 \times 3.$$

(c) Using the flow graphs Compute  
6 pt. DFT of the sequence

$$x(n) = \{1, -1, 2, -2, 3, -3\} \text{ and}$$

Verify the result by direct computation.

Solution a)  $N = 6 = 3 \times 2 = m_1 \times m_2$   
 $m_1 = 3 \quad m_2 = 2.$

$$X(k) = \sum_{n=0}^5 x(n) W_6^{nk} \quad 0 \leq k \leq 5$$

$$= \sum_{n=0}^1 x(3n) W_6^{2nk} + \sum_{n=0}^1 x(3n+1) W_6^{(3n+1)k}$$

$$+ \sum_{n=0}^1 x(3n+2) W_6^{(3n+2)k}$$

$$= \sum_{n=0}^1 x(3n) W_6^{2nk} + W_6^k \sum_{n=0}^1 x(3n+1) W_6^{3nk} + W_6^{2k} \sum_{n=0}^1 x(3n+2) W_6^{3nk}$$

$$x(k) = X_1(k) + W_6^k X_2(k) + W_6^{2k} X_3(k) \quad (1)$$

$$X_1(k) = \sum_{n=0}^{\infty} x(3n) w_6^{3nk}$$

(27)

$$= x(0) + x(3) w_6^{3k} \rightarrow (2)$$

$$X_2(k) = \sum_{n=0}^{\infty} x(3n+1) w_6^{3nk}$$

$$= x(1) + x(4) w_6^{3k} \rightarrow (3)$$

$$X_3(k) = \sum_{n=0}^{\infty} x(3n+2) w_6^{3nk}$$

$$\text{from } (1) = x(2) + x(5) w_6^{3k} \rightarrow (4)$$

$$X_1(0) = x(0) + x(3)$$

$$X_1(1) = x(0) + x(3) w_6^3$$

from (2)

$$X_2(0) = x(1) + x(4)$$

$$X_2(1) = x(1) + x(4) w_6^3$$

from (4)

$$X_3(0) = x(2) + x(5)$$

$$X_3(1) = x(2) + x(5) w_6^3$$

from ①

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$$X(k) = X_1(k) + w_6^1 X_2(k) + w_6^{24} X_3(k)$$

$$X(0) = X_1(0) + X_2(0) + X_3(0)$$

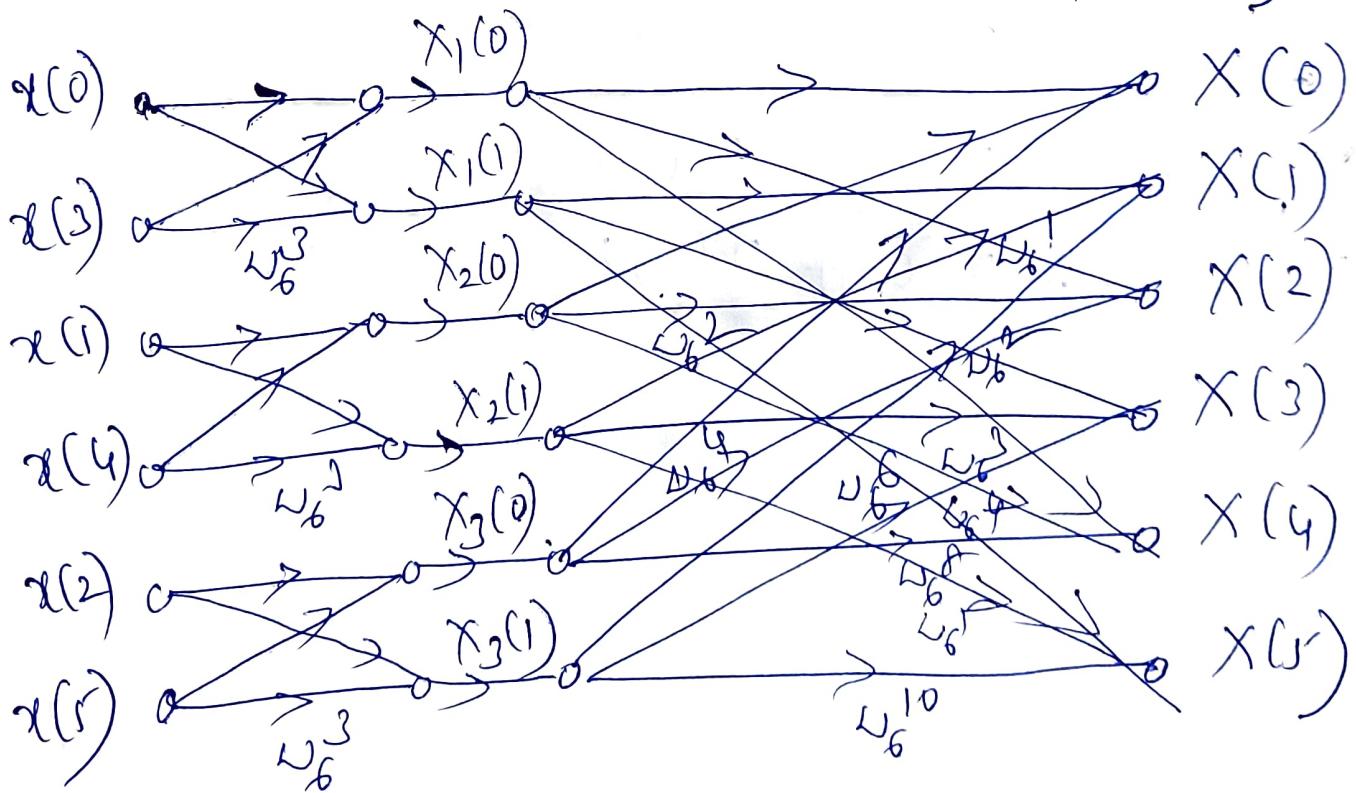
$$X(1) = X_1(1) + X_2(1) + X_3(1)$$

$$X(2) = X_1(0) + w_6^2 X_2(0) + w_6^4 X_3(0)$$

$$X(3) = X_1(1) + w_6^3 X_2(1) + w_6^6 X_3(1)$$

$$X(4) = X_1(0) + w_6^4 X_2(0) + w_6^8 X_3(0)$$

$$X(5) = X_1(1) + w_6^5 X_2(1) + w_6^{10} X_3(1)$$



$$\delta) \quad N = 6 = 2 \times 3 = m_1 \times m_2$$

$$x(k) = \sum_{n=0}^5 x(n) w_6^{nk} \quad 0 \leq k \leq 5$$

$$= \sum_{n=0}^{\frac{5}{2}} x(2n) w_6^{2nk} + \sum_{n=0}^{\frac{5}{2}} x(2n+1) w_6^{(2n+1)k}$$

$$x(k) = \sum_{n=0}^{\frac{5}{2}} x(2n) w_6^{2nk} + w_6^k \sum_{n=0}^{\frac{5}{2}} x(2n+1) w_6^{2nk}$$

$$x(k) = x_1(k) + w_6^k x_2(k) \rightarrow ①$$

$$x_1(k) = \sum_{n=0}^{\frac{5}{2}} x(2n) w_6^{2nk}$$

$$x_1(0) = x(0) + x(2) + x(4)$$

$$x_1(1) = x(0) w_6^0 + x(2) w_6^2 + x(4) w_6^4 \quad A$$

$$x_1(2) = x(0) w_6^0 + x(2) w_6^4 + x(4) w_6^8$$

$$w_6^6 = w_6^0; \quad w_6^8 = w_6^2$$

(30)

$$X_2(k) = \sum_{n=0}^2 x(2n+1) \omega_6^{2nk}$$

$$X_2(k) = x(1) + x(3) \omega_6^{2k} + x(5) \omega_6^{4k}$$

$$X_2(0) = x(1) + x(3) + x(5)$$

$$X_2(1) = x(1) + x(3) \omega_6^2 + x(5) \omega_6^4 \quad \left. \right\} \text{B}$$

$$X_2(2) = x(1) + x(3) \omega_6^4 + x(5) \omega_6^8 \quad \left. \right\}$$

$$X(k) = X_1(k) + \omega_6^k X_2(k)$$

$$X(0) = X_1(0) + X_2(0)$$

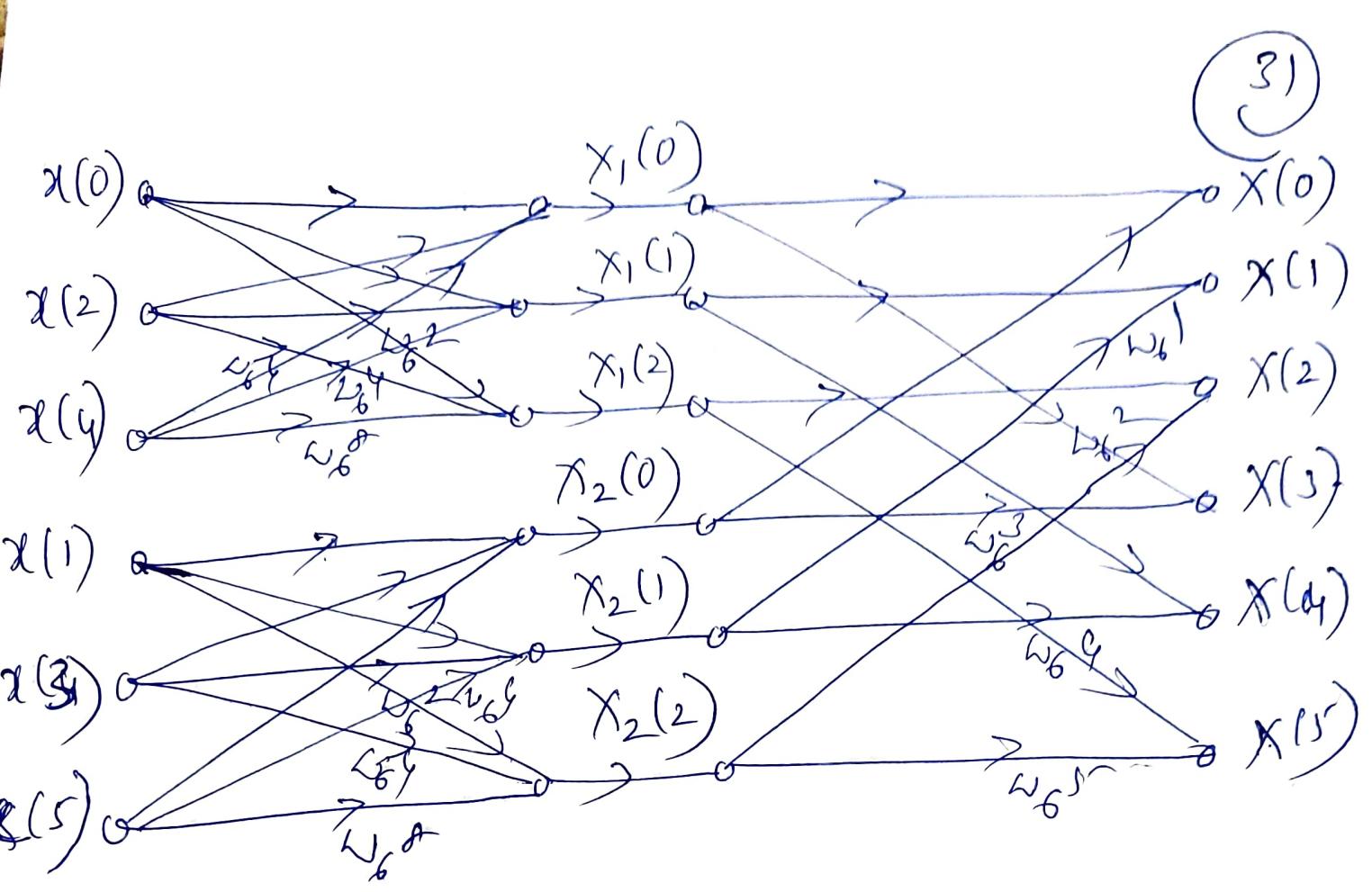
$$X(1) = X_1(1) + \omega_6^1 X_2(1)$$

$$X(2) = X_1(2) + \omega_6^2 X_2(2)$$

$$X(3) = X_1(0) + \omega_6^3 X_2(0)$$

$$X(4) = X_1(1) + \omega_6^4 X_2(1)$$

$$X(5) = X_1(2) \neq \omega_6^5 X_2(2)$$



# Fast Convolution techniques/

(32)

## Lemniscate Techniques.

- a) overlap-add method
- b) overlap-save method

### a) overlap-add method

Prob i) Compute  $y(n) = x(n) * h(n)$  using overlap-add method and verify the result by direct convolution

$$h(n) = \{1, 1, 1\}$$

$$x(n) = \{1, 2, 0, -3, 4, 2, -1, 1, -2, 3, 4, 2, 1, -3\}$$

Soln

$$M = 3$$

$$L = 2^M = 2^3 = 8$$

$$L = M + N - 1 \Rightarrow N = 6$$

Iteration 1

$$h_1(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$$

$$x_1(n) = \{1, 2, 0, -3, 4, 2, 0, 0\}$$

$N-1$  zeros

$M-1$

zeros

$$X_1(k) = 1 + 2w_8^k + 0 - 3w_8^{3k} + 4w_8^{4k} + \frac{33}{2}w_8^{5k}$$

$$H_1(k) = 1 + w_8^k + w_8^{2k}$$

$$Y_1(k) = X_1(k) \cdot H_1(k)$$

$$Y_1(k) = 1 + 3w_8^k + 3w_8^{2k} - w_8^{3k} + w_8^{4k} \\ + 3w_8^{5k} + 6w_8^{6k} + 2w_8^{7k}$$

$$y_1(n) = \{1, 3, 3, -1, 1, 3, 6, 2\}$$

Irrahm 2

$$h_2(n) = \{1, 1, 1, \underbrace{0, 0, 0, 0, 0}\} \quad \text{N-1 zeros}$$

$$x_2(n) = \{-1, 1, -2, 3, 4, 2, \underbrace{0, 0}\} \quad \text{M-1 zeros}$$

$$X_2(k) = -1 + w_8^k - 2w_8^{2k} + 3w_8^{3k} + 4w_8^{4k} + 2w_8^{5k}$$

$$H_2(k) = 1 + w_8^k + w_8^{2k}$$

$$Y_2(k) = X_2(k) \cdot H_2(k)$$

$$= -1 + 0 - 2w_8^{2k} + 2w_8^{3k} + 5w_8^{4k} + 9w_8^{5k} \\ + 6w_8^{6k} + 2w_8^{7k}$$

$$y_2(n) = \{-1, 0, -2, 2, 5, 9, 6, 2\}$$

## Ibrahim

N-1 zero

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$$h_3(n) = \{1, 1, 1, \underline{0}, 0, 0, 0, 0\}$$

$$\chi_3(n) = \{ 1, -3, 0, 0, 0, 0, 0, 0, 0 \}$$

$$H_3(\mu) = 1 + \omega_8^{\mu} + \omega_8^{2\mu}$$

$$X_3(\mu) = 1 - 3 \mu^{\frac{1}{\mu}}$$

$$Y_3(u) = X_3(u) \cdot H_3(u)$$

$$= 1 - 2w_f^k - 2w_f^{2k} - 3w_f^{3k}$$

$$y_3(n) = \{1, -2, -2, -3\}$$

To find  $y(n)$

$$y(n) = y_1(n) + y_2(n-n) + y_3(n-2n)$$

$$= y_1(n) + y_2(n-6) + y_3(n-12)$$

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$$\begin{array}{r}
 2 \ 1 \ 3 \ 3 -1 \ 1 \ 3 \quad (6) \quad (2) \\
 \underline{-} \quad (1) \quad (0) \quad -2 \ 2 \ 5 \ 9 \quad (6) \quad (2) \\
 \hline
 \{1 \ 3 \ 3 -1 \ 1 \ 3 \ 5 \ 2 \ -2 \ 2 \ 5 \ 9 \ 7 \ 0 \ -2 \ -3\}
 \end{array}$$

$$L_1 = 3, \quad L_2 = 14$$

(35)

$$\text{Length of } y(n) = L_1 + L_2 - 1$$

$$= 3 + 14 - 1 = 16$$

### Direct Computation

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \\ -3 & 0 & 2 \\ 4 & -3 & 0 \\ 2 & 4 & -3 \\ -1 & 2 & 4 \\ 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & -2 & 1 \\ 4 & 3 & -2 \\ 2 & 4 & 3 \\ 1 & 2 & 4 \\ -3 & 1 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \\ 1 \\ 3 \\ 5 \\ 2 \\ -2 \\ 2 \\ 5 \\ 9 \\ 7 \\ 0 \\ -2 \\ -3 \end{bmatrix}$$

Prob 2

(36)

Compute  $x(n) * h(n) = y(n)$  using overlap-add method and verify the result by direct computation.

$$h(n) = \{1, 2, 3\}$$

$$x(n) = \{1, 4; 3, 2, 5, 4, 3, 2, 1, -5, -4, 4, 5, 2, 3\}$$

# Mixed - Radix DFT

(37)

Prob: Develop a DIF FFT algorithm for ~~decomposition~~ Computing DFT for  $N=6$  and draw the signal flow graph for (a)  $N = 2 \times 3$  (b)  $N = 3 \times 2$

Solutions a)  $N = 6 = 2 \times 3$

$$X(k) = \sum_{n=0}^5 x(n) W_6^{nk} \quad 0 \leq k \leq 5$$

$$= \sum_{n=0}^2 x(n) W_6^{nk} + \sum_{n=3}^5 x(n) W_6^{nk}$$

$$= \sum_{n=0}^2 x(n) W_6^{nk} + \sum_{n=0}^2 x(n+3) W_6^{(n+3)k}$$

$$X(k) = \sum_{n=0}^2 [x(n) + x(n+3) W_6^{3k}] W_6^{nk} \quad \rightarrow ①$$

Put  $n = 2k$

$$X(2k) = \sum_{n=0}^2 [x(n) + x(n+3) W_6^{6k}] W_6^{2nk}$$

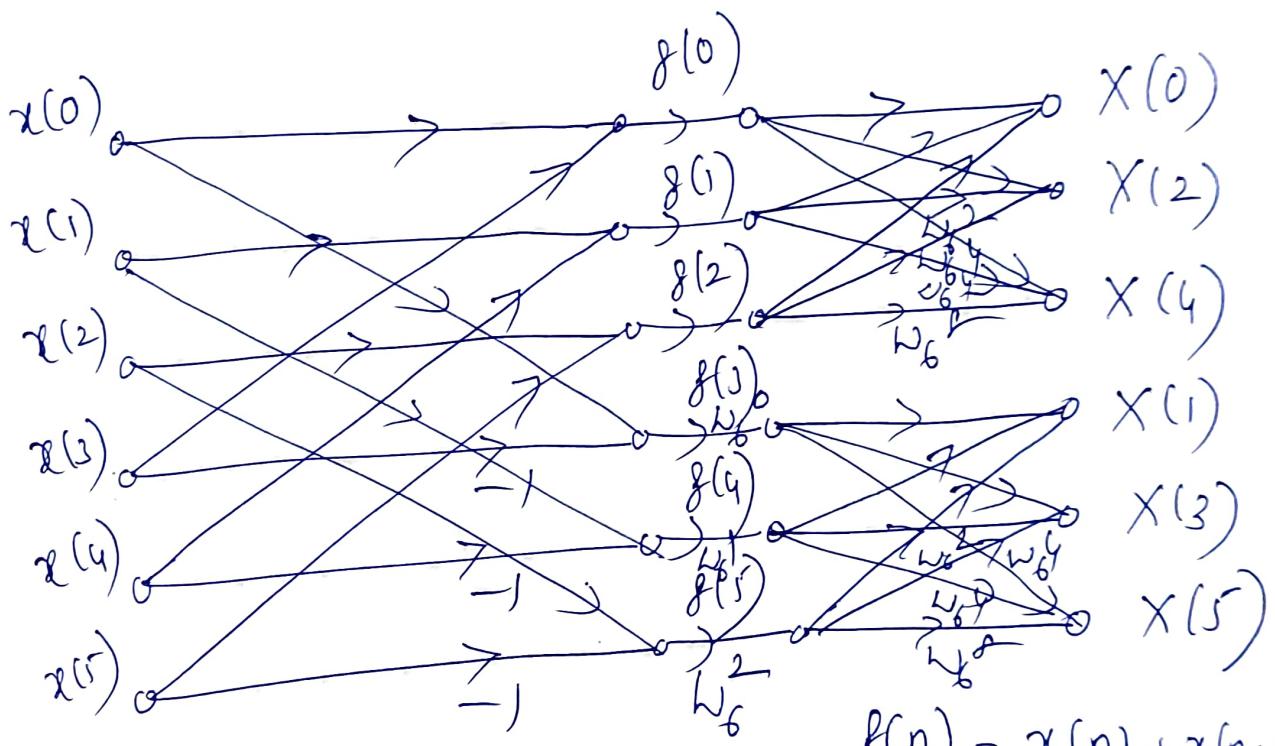
$$= \sum_{n=0}^2 [x(n) + x(n+3)] W_3^{nk} \rightarrow ②$$

$$\text{p.w } k = 2k+1$$

(38)

$$x(2k+1) = \sum_{n=0}^2 [x(n) + x(n+3)] w_6^{3(2k+1)} w_6^{(2k+1)n}$$

$$= \sum_{n=0}^2 \{ [x(n) - x(n+3)] w_6^n \} w_3^{nk} \rightarrow (3)$$



from (2)

$$x(0) = f(0) + f(1) + f(2)$$

$$x(2) = f(0) + f(1) w_6^2 + f(2) w_6^4$$

$$x(4) = f(0) + f(1) w_6^4 + f(2) w_6^8$$

$f(n) = x(n) + x(n+3)$
$f(0) = x(0) + x(3)$
$f(1) = x(1) + x(4)$
$f(2) = x(2) + x(5)$

(39)

$$g(3) = x(0) - x(3)$$

$$g(4) = x(1) - x(4)$$

$$g(5) = x(2) - x(5)$$

$$X(1) = g(3) + g(4) + g(5)$$

$$X(3) = g(3) + g(4) w_6^1 w_6^2 + g(5) w_6^2 w_6^4$$

$$X(5) = g(3) + g(4) w_6^1 w_6^4 + g(5) w_6^2 w_6^2.$$

b)  $N = 6 = 3 \times 2$

$$X(k) = \sum_{n=0}^5 x(n) w_6^{kn} \quad 0 \leq k \leq 5$$

$$= \sum_{n=0}^1 x(n) w_6^{kn} + \sum_{n=2}^3 x(n) w_6^{kn} + \sum_{n=4}^5 x(n) w_6^{kn}$$

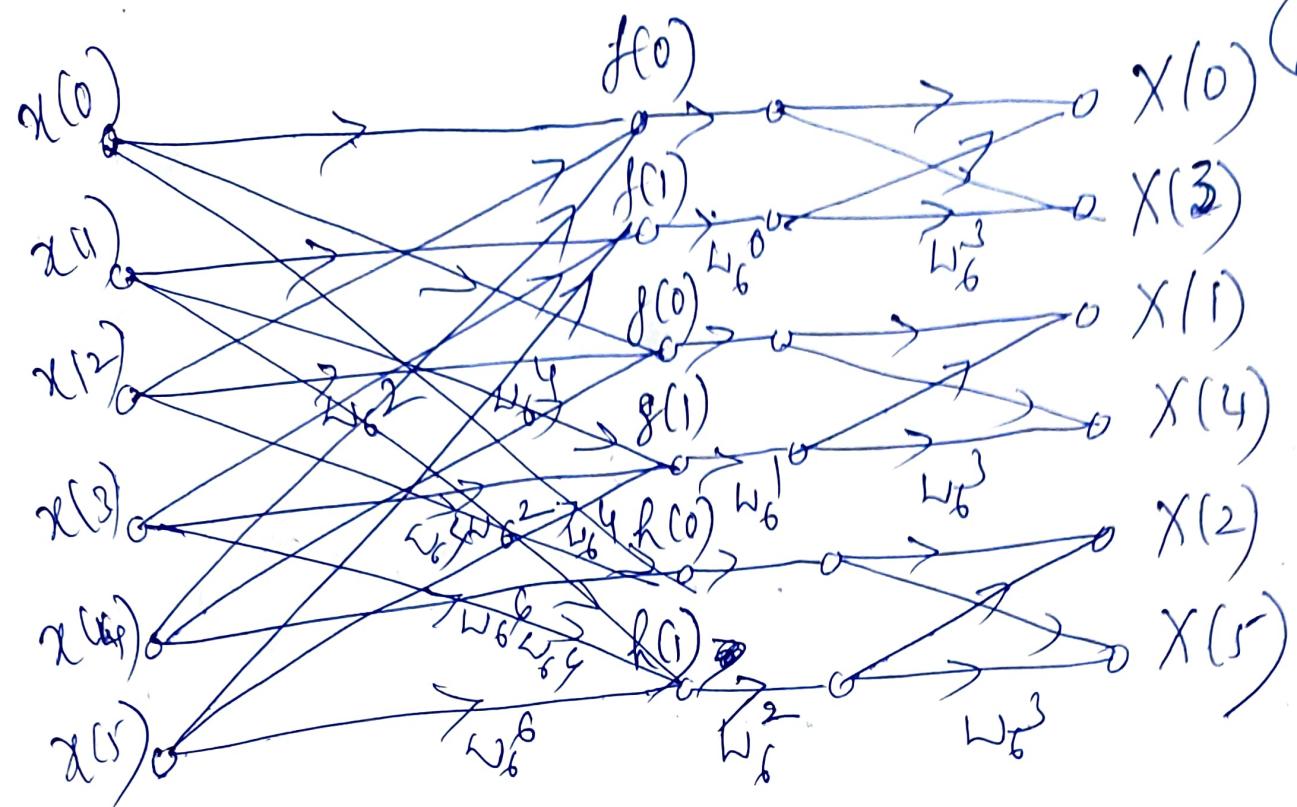
$$X(k) = \sum_{n=0}^1 \{x(n) + x(n+2) w_6^{2k} + x(n+4) w_6^{4k}\} w_6^{kn}$$

$$X(3k) = \sum_{n=0}^1 [x(n) + x(n+2) + x(n+4)] w_6^{3nk}$$

$$X(3k+1) = \sum_{n=0}^1 [x(n) + x(n+2) w_6^2 + x(n+4) w_6^4] w_6^1 w_6^{3nk}$$

$$X(3k+2) = \sum_{n=0}^1 [x(n) + x(n+2) w_6^4 + x(n+4) w_6^6] w_6^{2n} w_6^{3nk}$$

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$$f(n) = x(n) + x(n+2) + x(n+4)$$

$$g(n) = x(n) + x(n+2) w_6^2 + x(n+4) w_6^4$$

$$h(n) = x(n) + x(n+2) w_6^4 + x(n+4) w_6^6$$

$$x(0) = f(0) + f(1) w_6^0 \quad \left| \begin{array}{l} x(3) = f(0) + f(1) w_6^0 w_6^3 \\ x(4) = g(0) + g(1) w_6^1 w_6^1 \end{array} \right.$$

$$x(1) = g(0) + g(1) w_6^1 \quad \left| \begin{array}{l} x(4) = g(0) + g(1) w_6^1 w_6^1 \\ x(5) = h(0) + h(1) w_6^2 w_6^2 \end{array} \right.$$

$$x(2) = h(0) + h(1) w_6^2 \quad \left| \begin{array}{l} x(5) = h(0) + h(1) w_6^2 w_6^2 \end{array} \right.$$

(41)

## Overlap-Save method

prob: Compute  $y(n) = x(n) * h(n)$  using overlap-save method. Verify the result by direct computation.

$$h(n) = \{1, 1, 1\}$$

$$x(n) = \{1, 2, 0, -3, 4, 2, -1, 1, -2, 3, 4, 2, 1, -3\}$$

Solu<sup>n</sup>

$$M = 3$$

$$L = 2^M = 2^3 = 8$$

$$L = M + N - 1 \Rightarrow \underline{\underline{N=6}}$$

Iteration 1 N+1 Zeros

$$h_1(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$$

$$x_1(n) = \{0, 0, 1, 2, 0, -3, 4, 2\}$$

M-1 zeros.

$$H_1(k) = 1 + w_8^k + w_8^{2k}$$

$$X_1(k) = w_8^{2k} + 2w_8^{3k} + 0 \cancel{- 3w_8^{5k}} + 4w_8^{6k} + 2w_8^{7k}$$

$$Y_1(k) = 6 + 2w_8^k + w_8^{2k} + 3w_8^{3k} + 3w_8^{4k} \cancel{- w_8^{5k}} + w_8^{6k} + 3w_8^{7k}$$

(42)

$$y_1(n) = \{ \textcircled{6}, 2, 1, 3, 3, -1, 1, 3 \}$$

discarded

~~$y_1(n)$~~   $y_1'(n) = \{ 1, 3, 3, -1, 1, 3 \}$

Iteration 2

$$h_2(n) = \{ 1, 1, 1, \underbrace{0, 0, 0, 0, 0}_{N-1 \text{ zeros}} \}$$

$$x_2(n) = \{ \textcircled{4}, 2, -1, 1, -2, 3, \textcircled{4}, 2 \}$$

Last two values of  $x_1(n)$ 

$$H_2(k) = 1 + w_8^k + w_8^{2k}$$

$$X_2(k) = 4 + 2w_8^k - w_8^{2k} + w_8^{3k} - 2w_8^{4k} \\ + 3w_8^{5k} + 4w_8^{6k} + 2w_8^{7k}$$

$$Y_2(k) = X_2(k) \cdot H_2(k)$$

$$= 10 + 8w_8^k + 5w_8^{2k} + 2w_8^{3k} - 2w_8^{4k} \\ + 2w_8^{5k} + 5w_8^{6k} + 9w_8^{7k}$$

$$y_2(n) = \{ \textcircled{10}, 8, 5, 2, -2, 2, 5, 9 \}$$

discarded

$$y_2^1(n) = \{ 5, 2, -2, 2, 5, 9 \}$$

(43)

IIR

$$h_3(n) = \{ 1, 1, 1, \underbrace{0, 0, 0, 0, 0}_{N-1 \text{ zeros}} \}$$

$$x_3(n) = \{ \underbrace{4, 2}, 1, -3, 0, 0, 0, 0 \}$$

Last two samples of  $x_2(n)$ .

$$H_3(k) = 1 + w_8^k + w_8^{2k}$$

$$X_3(k) = 4 + 2w_8^k + w_8^{2k} - 3w_8^{3k}$$

$$Y_3(k) = \begin{cases} 4 + 6w_8^k + 7w_8^{2k} + 0 \\ \quad - 2w_8^{4k} - 3w_8^{5k} \end{cases}$$

$$y_3(n) = \{ \underbrace{4, 6}, 7, 0, -2, -3 \}$$

discarded

$$y_3^1(n) = \{ 7, 0, -2, -3 \}$$

To find  $y(n)$

$$y(n) = y_1^1(n) + y_2^1(n-6) + y_3^1(n-12)$$

$$= \{ 1, 3, 3, -1, 1, 3, 5, 2, -2, 2, 5, 9, \\ 7, 0, -2, -3 \}$$

Prb: Compute  $y(n) = x(n) * h(n)$  (44)  
using overlap - save method and  
Verify your answer by direct  
Computation

$$h(n) = \{2, -3\}$$

$$x(n) = \{4, 5, 6, 7, -8, -9, -10, -7, -6, 5, 4, 3, 2, 1\}$$