(1) Is the vector $(3,-1,0,-1)$ in the subspace spanned by $(2,-1,3,2)$, $(-1,1,1,-3)$ & $(1,1,9,-5)$?

Solution:

$\alpha_1 = (2,-1,3,2)$

$\alpha_2 = (-1,1,1,-3)$ $\qquad \alpha = (3,-1,0,-1)$.

$\alpha_3 = (1,1,9,-5)$.

Let $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \alpha$

If c_1, c_2 and c_3 exists and are unique, we can say that the vector α is in the subspace spanned by $\alpha_1, \alpha_2, \alpha_3$.

$\therefore\ c_1(2,-1,3,2) + c_2(-1,1,1,-3) + c_3(1,1,9,-5) = (3,-1,0,-1)$

$$\begin{aligned} 2c_1 - c_2 + c_3 &= 3. \\ -c_1 + c_2 + c_3 &= -1 \\ 3c_1 + c_2 + 9c_3 &= 0. \\ 2c_1 - 3c_2 - 5c_3 &= -1. \end{aligned} \implies \begin{bmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix}.$$

Finding Row reduced matrix

$$R_1 \to \frac{R_1}{2} = \begin{bmatrix} 1 & -1/2 & 1/2 & 3/2 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix} \qquad R_2 \to R_1 + R_2 = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix}$$

$$-3R_1 + R_3 = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{5}{2} & \frac{15}{2} & \frac{-9}{2} \\ 2 & -3 & -5 & -1 \end{bmatrix} \qquad -2R_1 + R_4 = \begin{bmatrix} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1/2 & 3/2 & 1/2 \\ 0 & 5/2 & \frac{15}{2} & \frac{-9}{2} \\ 0 & -2 & -6 & -4 \end{bmatrix}$$

$$R_2 \to 2R_2 = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 3 & 1 \\ 0 & 5/2 & 15/2 & -9/2 \\ 0 & -2 & -6 & -4 \end{bmatrix} \qquad R_3 \to \frac{-5R_2}{\cancel{R}\,2} + R_3 = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -7 \\ 0 & -2 & -6 & -4 \end{bmatrix}$$

$$R_4 \to 2R_2 + R_4 = \begin{bmatrix} 1 & -\frac{1}{2} & 1/2 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad R_3 \to \frac{-R_3}{7} = \begin{bmatrix} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$R_4 \to 2R_3 + R_4 = \begin{bmatrix} 1 & -1/2 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to -R_3 + R_2, \quad R_1 \to \frac{-3R_3}{2} + R_1 = \begin{bmatrix} 1 & -1/2 & 1/2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to \frac{R_2}{2} + R_1 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & \boxed{0 \quad 1} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

C_1 and C_2 are basic variable where as C_3 is Free variable.

$\therefore$ Infinitely many solutions.

$\therefore$ α is not in the subspace spanned by α_1, α_2 and α_3.

(2) Are the vectors linearly independant in R^4? Find the basis for the subspace of R^4 spanned by 4 vectors.

$\alpha_1 = (1, 1, 2, 4)$ $\quad \alpha_2 = (2, -1, -5, 2)$

$\alpha_3 = (1, -1, -4, 0)$ $\quad \alpha_4 = (2, 1, 1, 6)$.

Sol: $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are said to be linearly independant if the reduced row echelon form of the matrix $[\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]$ has only basic variables and no free variables.

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix} \quad R_2 \to -2R_1 + R_2 = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$$

$R_3 \to -R_1 + R_3 = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 2 & 1 & 1 & 6 \end{bmatrix}$ $R_4 \to -2R_1 + R_4 = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$

$R_2 \to -\frac{R_2}{3} = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$ $R_3 \to 2R_2 + R_3 \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & -2 \end{bmatrix}$

$R_4 \to R_2 + R_4 = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_1 \to -R_2 + R_1 = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Two basic variables, two free.

$\therefore \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are not linearly independent.

α_1 and α_2 are the vectors with basic variables.

$\therefore \alpha_1$ & α_2 form the basis for the vector space R^4.

$$\text{Basis} = \left(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} \right)$$

(3) Let V be the vector space of polynomials of degree 3 over R. Determine whether u, v & w are linearly independent or not.

$u = t^3 - 3t^2 + 5t + 1$

$v = t^3 - t^2 + 8t + 2.$

$w = 2t^3 - 4t^2 + 9t + 5.$

$$\Longrightarrow \begin{bmatrix} 1 & -3 & 5 & 1 \\ 1 & -1 & 8 & 2 \\ 2 & -4 & 9 & 5 \end{bmatrix}$$

$$R_2 \to -R_1 + R_2 = \begin{bmatrix} 1 & -3 & 5 & 1 \\ 0 & 2 & 3 & 1 \\ 2 & -4 & 9 & 5 \end{bmatrix} \quad R_3 \to -2R_1 + R_3 \begin{bmatrix} 1 & -3 & 5 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & -1 & 3 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2/2 \\ R_3 \to -2R_2 + R_3 \end{matrix} = \begin{bmatrix} 1 & -3 & 5 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & -4 & 2 \end{bmatrix} \quad \begin{matrix} R_3 \to -\frac{R_3}{4} \\ R_2 \to -3\frac{R_3}{2} + R_2 \end{matrix} = \begin{bmatrix} 1 & -3 & 5 & 1 \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

$$-5R_3 + 3R_2 + 2R_1 \to R_1 = \begin{bmatrix} 1 & 0 & 0 & \frac{29}{4} \\ 0 & 1 & 0 & 5/4 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

3 basic variables.

$\therefore$ u, v & w are linearly independant.

$$A = \begin{pmatrix} A1 & B1 & C1 & D1 \\ A2 & B2 & C2 & D2 \\ A3 & B3 & C3 & D3 \\ A4 & B4 & C4 & D4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{pmatrix}$$

m1 = ReducedRowEchelonForm(A)

$$\rightarrow \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

No Solutions. There fore the vector (3,-1,0,-1) doesnt exists in the subspace spanned by α˙1, α˙2, α˙3

	A	B	C	D
1	2	-1	1	3
2	-1	1	1	-1
3	3	1	9	0
4	2	-3	-5	-1
5				
6	Question 1			
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				

$$m1 = \begin{pmatrix} A1 & B1 & C1 & D1 \\ A2 & B2 & C2 & D2 \\ A3 & B3 & C3 & D3 \\ A4 & B4 & C4 & D4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix}$$

$m2 = \mathrm{ReducedRowEchelonForm}(m1)$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$m3 = \begin{pmatrix} A7 & B7 & C7 & D7 \\ A8 & B8 & C8 & D8 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \end{pmatrix}$$

$l1 = \mathrm{Dimension}(m3)$

$\rightarrow \{2, 4\}$

$a = \mathrm{MatrixRank}(m3)$

$\rightarrow 2$

	A	B	C	D
1	1	1	2	4
2	2	-1	-5	2
3	1	-1	-4	0
4	2	1	1	6
5				
6				
7	1	1	2	4
8	2	-1	-5	2
9				
10	Question 2			
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
21				
22				
23				
24				
25				

$w(t) = 2t^3 - 4t^2 + 9t + 5$

$v(t) = t^3 - t^2 + 8t + 2$

$u(t) = t^3 - 3t^2 + 5t + 1$

$$A = \begin{pmatrix} A1 & B1 & C1 & D1 \\ A2 & B2 & C2 & D2 \\ A3 & B3 & C3 & D3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -3 & 5 & 1 \\ 1 & -1 & 8 & 2 \\ 2 & -4 & 9 & 5 \end{pmatrix}$$

$m1 = \text{ReducedRowEchelonForm}(A)$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 7.25 \\ 0 & 1 & 0 & 1.25 \\ 0 & 0 & 1 & -0.5 \end{pmatrix}$$

U = 7.25

-5 10

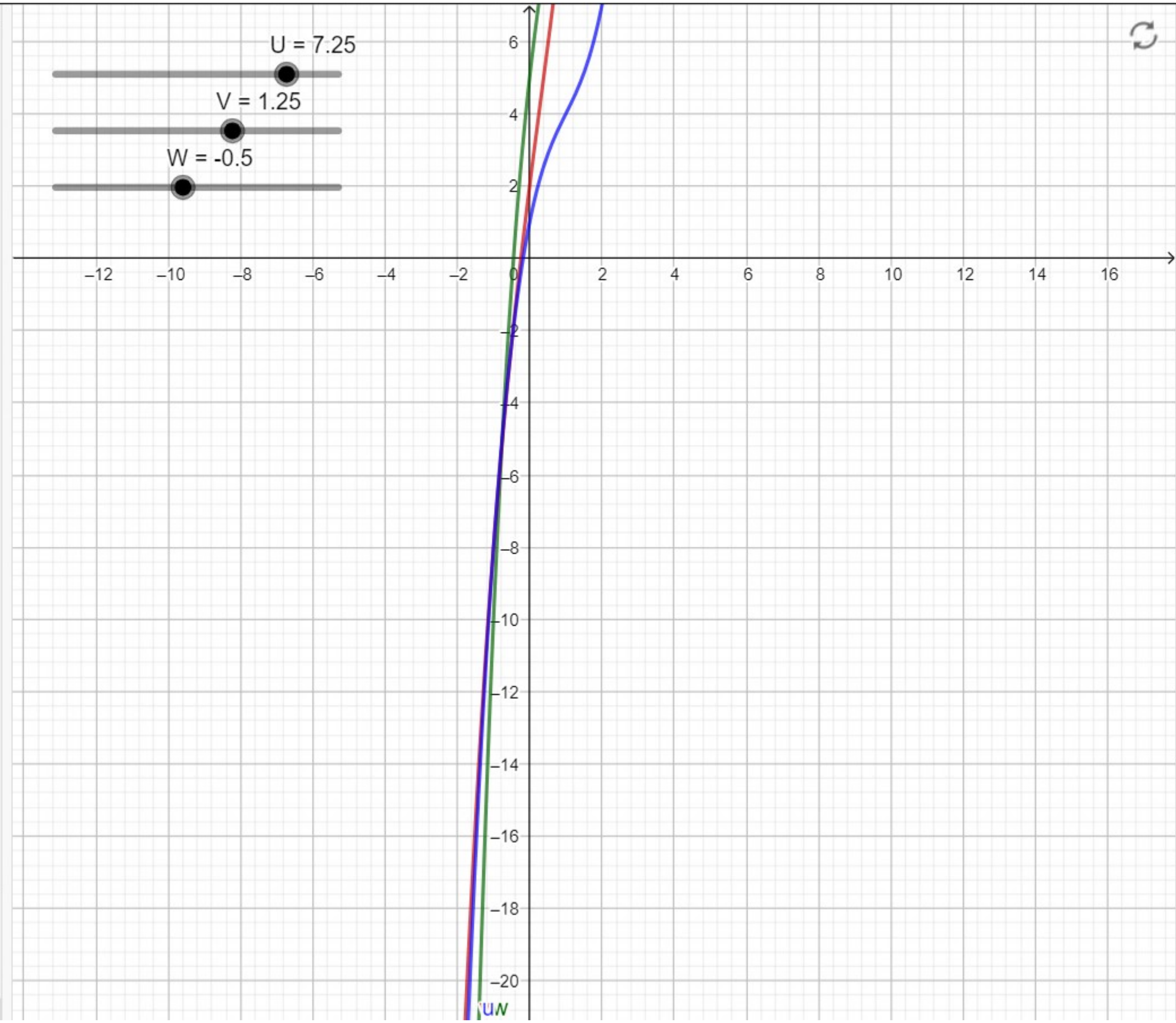
V = 1.25

-5 5

W = -0.5

-5 5

Number of free variables = 0



	A	B	C
1	1	-3	5
2	1	-1	8
3	2	-4	9

Question 3