Matrix factorization:-

LU Factorization $A = LU$.

The original matrix A becomes the product of two or three special matrices. The first factorization → comes now from elimination.

The factors L and U are triangular matrices. The factorization that comes from elimination is $\boxed{A = LU}$

U → upper triangular matrix $m \times m$ with pivots on its diagonal.

L → lower triangular matrix $m \times m$ with '1' on the diagonal

Ex:- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ we have $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Elementary ~~Elimination~~ matrices.

Ex: $E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$ From A to U

$E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A$ Back from U to A

$\boxed{LU = A}$

3x3 matrix

$(E_{32}E_{31}E_{21})A = U$ (no row exchanges) becomes $(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U$

which is $\boxed{A = LU}$

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}} \Rightarrow \underset{L}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}\underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}}\underset{U}{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}$$

Diagonal matrix

$\boxed{A = LDU}$

Ex:-

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix} = E \text{ (left of A)}$$

E_{32} $\quad$ E_{21}

$EA = U$

(Inverse / Reverse order)

$$\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix} = \begin{bmatrix}1&0&0\\2&1&0\\0&5&1\end{bmatrix} = L \text{ (left of U)}$$

$A = LU$

$\boxed{A = LU}$

If no row exchanges multipliers go directly into L.

How many operations are on $n \times n$ matrix A? or how expensive it is

$n = 100$

$$A = \begin{bmatrix} \cdot & \cdots \\ \vdots & \\ & \end{bmatrix}_{100 \times 100} \rightarrow \begin{bmatrix} \cdot & \cdots \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}$$

$n^2 + \cdots + 1^2 = 2/3\, n^3$ on A.

L & U are sub matrices

$Ax = b$

$L(Ux) = b \Rightarrow Ly = b$

$Ux = y$.

Ex

$$A = \begin{bmatrix}1&2&1\\2&3&3\\-3&-10&2\end{bmatrix}$$

$L \; U = A \Rightarrow A = LU$

$$\begin{bmatrix}1&2&1\\0&-1&1\\0&-4&5\end{bmatrix} \quad \begin{matrix}R_2 - 2R_1\\R_3 + 3R_1\end{matrix} \xrightarrow{R_3 - 4R_2} \begin{bmatrix}1&2&1\\0&-1&1\\0&0&1\end{bmatrix} = U$$

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + 4R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix} = L$$

2) $x_1 + 2x_2 + x_3 = 0$

$2x_1 + 2x_2 + 2x_3 = 3$

$-x_1 - 3x_3 = 2$

$$A \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & -3 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} L$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \qquad \boxed{Ux = y} \quad \boxed{Ly = b}$$

$$\boxed{y_1 = 0}$$

$$-2y_1 + y_2 = 3 \quad \boxed{y_1 = 3}$$

$$-y_1 - y_2 + y_3 = 2 \quad \boxed{y_3 = 5}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$$

$-2x_3 = 5 \Rightarrow \boxed{x_3 = -5/2}$

$-2x_2 = 3 \Rightarrow \boxed{x_2 = -3/2}$

$x_1 + 2x_2 + x_3 = 0$

$x_1 = \frac{5}{2} + \frac{6}{2} = \boxed{11/2 = x_1}$

Alternately, Given the s/m of equations LU factorization Can be found by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$LU = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & -3 \end{bmatrix}$$

equating the corresponding elements and finding out the elements of of L & U matrices.

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2) $A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} \qquad b = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$

Note: (1) Computing LU factorization of A takes about $\frac{2n^3}{3}$ flops (same as the no. of operations reqd for row reduction) where as finding A^{-1} requires about $2n^3$ flops

(2) Solving $Ly = b$ and $Ux = y$ requires about $2n^2$ flops because and $n \times n$ triangular s/m can be solved ~~using~~ in about n^2 flops.

Summarize

1) Row reduction algorithm is also called as Gaussian Ellimination method. Solving the s/m of equation by obtaining row reduction algorithm of the augmented matrix $[A:b]$

2) Find the solution to s/m of equations, by augmenting the co-efficient matrix with b. & obtaining the soln to the s/m is called Gauss Jordan method

$[A:b]$ → Co-efficient matrix is in reduced row echelon form (i.e identity matrix)

3) Inverse of a matrix using elementary row operations are also called as Gauss Jordan method

LDU Factorization

steps
1. Row reduce matrix through identity matrix
2. Keep track of row operations
3. Factor out the main diagonal

$A = LDU$

L — lower triangular matrix

D — diagonal matrix

U — upper triangular matrix

Ex:- $A = \begin{bmatrix} 3 & 9 \\ 15 & 49 \end{bmatrix} \xrightarrow{R_2 - 5R_1} \begin{bmatrix} 3 & 9 \\ 0 & 4 \end{bmatrix}$

$\underset{D}{\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$L \xrightarrow{R_2 + 5R_1} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

$A = \underset{L}{\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}$

(2)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix} \xrightarrow[R_2 - 2R_1]{R_3 + 3R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -4 & 5 \end{bmatrix} \xrightarrow{R_3 - 4R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow U$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + 4R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 + 2R_1}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(3)

$$A = \begin{bmatrix} 4 & -20 & -12 \\ -8 & 45 & 44 \\ 20 & -105 & -79 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -20 & -12 \\ -8 & 45 & 44 \\ 20 & -105 & -79 \end{bmatrix} \xrightarrow[R_3 - 5R_1]{R_2 + 2R_1} \begin{bmatrix} 4 & -20 & -12 \\ 0 & 5 & 20 \\ 0 & -5 & -19 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 4 & -20 & -12 \\ 0 & 5 & 20 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\quad D \qquad\qquad U$$

$$L \Rightarrow I \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\downarrow R_2 - 2R_1$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -1 & 1 \end{bmatrix} \xleftarrow{R_3 + 5R_1} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

(7) $\begin{bmatrix} 2 & 4 & -6 \\ -2 & -8 & -10 \\ -6 & -32 & -66 \end{bmatrix}$

Fields

~~A set is said to be field w.r.t the given binary operations $*$ and $*'$ if $(F, *)$ and $(F, *')$ is an abelian group and satisfies distributive property~~

Let F denotes either a set of real numbers or the set of complex numbers.

1) Addition is commutative

$$x+y = y+x$$

$\forall\ x, y$ in F

2) Addition is associative

$$x+(y+z) = (x+y)+z. \quad \forall\ x,y,z \text{ in } F$$

3) There is a unique element 0 (zero) in F such that

$x+0 = x$ for every x in F

4) To each x in F there corresponds a unique element $(-x)$ in F such that $x+(-x) = 0$

5) Multiplication is commutative,

$$xy = yx \quad \forall\ x,y \text{ in } F$$

6) Multiplication is associative

$$(xy)z = x(yz) \quad \forall\ x,y,z \text{ in } F$$

7) There exists a unique non-zero element 1 (one) in F such that $x1 = x$ for every x in F

8) To each non-zero x in F there corresponds a unique element x^{-1} or $(1/x)$ in F such that $xx^{-1} = 1$

9) Multiplication distributes over addition; i.e $x(y+z) = xy+xz$ $\forall\ x,y,z$ in F

(10)

Set F of objects x, y & z & 2 operations on the elements of F

1st operation addition, associates with each pair of elements x, y in F an element $(x+y)$ in F.

2nd operation :- multiplication, associates with each pair x, y an element xy in F

Set F together with these two operations is then called Field.

$$(F, +, \cdot)$$

With usual operations of addition and multiplication, the set C of complex numbers is a field, as in the set R of real number.

A subfield of the field C is a set F of complex numbers which is itself a field under the usual operations of addition and multiplication of complex numbers.

This means 0 & 1 are in the set F, & that if x & y are elements of F, so are $(x+y)$, $-x$, xy and x^{-1} (if $x \neq 0$).

~~An exam~~ Subfield is the field R of real numbers (Real no's is a complex no $(a+ib)$ with $b=0$)

→ Positive integers } is a field ?
→ Set of Integers. }

→ Check whether the set of rational numbers is a subfield of C