Unit - 4

Eigen Values and Eigen Vectors

Eigen vectors and Eigen values are useful throughout of pure and applied mathematics.

Eigen values are used to study differential equations and continuous dynamical systems, they provide critical information in engg design, and they arise naturally in fields such as physics and chemistry.

EigenVectors and Eigen Values :-

An eigenvector of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ.

A scalar λ is called an eigenvalue of A if there is a non-trivial solution x of $Ax = \lambda x$ such an x is called an eigen vector corresponding to λ

Ex :- Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ Are u & v eigenvectors of A

$$Au = \begin{bmatrix} 1 & 2 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4u$$

u is an eigen vector corresponding to an eigenvalue (-4)

$$Av = \begin{bmatrix} 1 & 2 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$\therefore$ V is not an eigen vector

λ is an Eigenvalue of A iff the equation

$$(A - \lambda I)x = 0 \quad \text{——— (1)}$$

has a non-trivial solution. The set of all solutions of (1) is just the Nullspace of the matrix $(A - \lambda I)$

$\therefore$ This set is a subspace of R^n is called the eigenspace of A corresponding to λ. The eigenspace consists of the zero vector & all eigen vectors corresponding to λ

Prob:- Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ An eigenvalue of A is 2. Find a basis for the corresponding eigenspace

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

Now row Reduce the augmented matrix for $(A-2I)x = 0$

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ 2 is an eigen value of A $\because$ $(A - 2I)x = 0$ has 2 free variables.

$$2x_1 - x_2 + 6x_3 = 0$$

$$x_1 = \frac{x_2}{2} - 3x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The eigenspace ~~shown~~ is a subspace of R^3 (2 Dimensional subspace of R^3)

A basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Theorem:- The eigen values of a triangular matrix are the entries on its main diagonal.

Proof:- $\rightarrow$ Do the proof.

If matrix A has an eigen value of 0 iff $\boxed{Ax = 0x}$ has a non-trivial solution. It has non-trivial soln iff A is not invertible. '0' is an eigenvalue of A iff A is not invertible.

Theorem If $v_1 \dots v_r$ are eigen vectors that correspond to distinct eigenvalues $\lambda_1 \dots \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1 \dots v_r\}$ is linearly independent.

Characteristic equation

The eigen values of a square matrix A is encoded in a special scalar equation called the characteristic equation of A

(i) Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(-6-\lambda) - 9 = 0$$

$$\boxed{\lambda = 3, \ \lambda = -7}$$ Eigen values

Determinants :- Let A be an $n \times n$ matrix, U be any echelon form obtained from A by row replacements and row interchanges. & r be the number of such row interchanges.

Determinant of A, $\det A$ is $(-1)^r$ times the product of the diagonal entries $u_{11} \dots u_{nn}$ in U

If A is invertible then $u_{11} \dots u_{nn}$ are all pivots & u_{ii} have not been scaled to 1.

$$\det A = \begin{cases} (-1)^r \cdot (\text{product of pivots in } U) & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is not invertible} \end{cases}$$

If The number 0 is not an eigenvalue of A & the the determinant of A is not zero then A is invertible

I Characteristic equation of A $\det(A-\lambda I)=0$.

λ is an eigenvalue of an $n\times n$ matrix A iff λ satisfies the characteristic equation $\det(A-\lambda I)=0$

1) Find the characteristic equation of

$$A=\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A-\lambda I)=\begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}=0$$

$$(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)=0$$

$$(5-\lambda)^2(3-\lambda)(1-\lambda)=0$$

or

$$(\lambda-5)^2(\lambda-3)(\lambda-1)=0$$

$$\lambda^4-14\lambda^3+68\lambda^2-130\lambda+75=0$$

$\det(A-\lambda I)$ is a polynomial in λ. If A is an $n\times n$ matrix, then $\det(A-\lambda I)$ is a polynomial degree n called characteristic polynomial of A

In the above example multiplicity 2 because $(\lambda-5)$ occurs two times as a factor of characteristic polynomial

In general the (algebraic) multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic equation

If A and B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $\boxed{P^{-1}AP = B}$

or equivalently, $\boxed{A = PBP^{-1}}$

if $Q = P^{-1}$ then $\boxed{Q^{-1}BQ = A}$

B is also similar to A hence we can say the A and B are similar

Changing A into $P^{-1}AP$ is called a similarity transformation.

Theorem :- If $n \times n$ matrices A and B are similar then they have the same characteristic polynomial and hence the same eigen value (with the same multiplicities.)

Proof: If $B = P^{-1}AP$ then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P$$
$$= P^{-1}[AP - \lambda P] = P^{-1}[A - \lambda I]P$$

$$\det(B - \lambda I) = \det[P^{-1}(A - \lambda I)P]$$
$$= \det(P^{-1}) \det(A - \lambda I) \det(P)$$

$$\det(P^{-1}) \det(P) = \det(P^{-1}P) = \det(I) = 1$$

$$\therefore \det(B - \lambda I) = \det(A - \lambda I)$$

Characteristic Value

Let V be a vectorspace over the field F and let T be a linear operator on V. A characteristic value of T is a scalar c in F such that there is a non-zero vector α in V with $T\alpha = c\alpha$. If c is a characteristic value of T, then

(a) any α such that $T\alpha = c\alpha$ is called a characteristic vector of T associated with the characteristic value c.

(b) the collection of all α such that $T\alpha = c\alpha$ is called the characteristic space associated with c.

Characteristic values are often called characteristic roots, latent roots, eigenvalues, propervalues or spectral values

Prob :- Let A be the real 3×3 matrix $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$
Find the characteristic polynomial & characteristic value

Characteristic Polynomial, Cayley Hamilton Theorem

Cayley-Hamiltion → Let T be a linear operator on a finite dimensional vector space V. If f is the characteristic polynomial for T, then $f(T) = 0$; in other words, the minimal polynomial divides the characteristic polynomial for T

Theorem :- Let T be a linear operator on a finite dimensional space V and let c be a scalar. The following are equivalent.

(i) c is a characteristic value of T

(ii) The operator $(T - cI)$ is singular (non-invertible)

(iii) $\det(T - cI) = 0$

Definition :- If A is an $n\times n$ matrix over the field F, a characteristic value of A in F is a scalar c in F such that the matrix $(A - cI)$ is singular (not invertible)

Since c is a characteristic value of A iff $\det(A-cI)=0$ or equivalently iff $\det(cI-A)=0$

We form the matrix $(xI-A)$ with polynomial entries & consider the polynomial $f=\det(xI-A)$.

Diagonalization → Let A be any n-square matrix it can be represented by a diagonal matrix $D=\mathrm{diag}(k_1, k_2 \ldots k_n)$ such that $u_1 \ldots u_n$ such that

$$Au_1 = k_1u_1$$
$$Au_2 = k_2u_2$$
$$\vdots$$
$$Au_n = k_nu_n$$

In such a case A is said to be diagonalizable.

$$D = P^{-1}AP$$

P is the non-singular matrix whose columns are the basis vectors $u_1, u_2 --- u_n$.

$$A = PDP^{-1}$$

Problems

1) Let $A = \begin{pmatrix} 3 & -4 \\ 2 & -6 \end{pmatrix}$

(a) find all eigen values & vectors

(b) Find the matrix P & D ~~such that P is non-investible~~.

(c) $D = P^{-1}AP$ is diagonal.

$|xI-A|=0$

$$\begin{vmatrix} x-3 & -4 \\ 2 & x+6 \end{vmatrix} = 0$$

$$(x-3)(x+6)+8=0$$
$$x^2+6x-3x-18+8=0$$
$$x^2+3x-10=0$$
$$\boxed{x=2} \text{ or } \boxed{x=-5}$$

Eigen values are 2 & -5

$\lambda = 2$ $\quad m = A - 2I$

$$= \begin{pmatrix} 3 & -4 \\ 2 & -6 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 2 & -8 \end{pmatrix}$$

$$x - 4y = 0$$
$$2x - 8y = 0$$

Let $x = 4, y = 1$ $\quad V_1 = (4, 1)$

$\lambda = 5$ $\quad m = A + 5I = \begin{pmatrix} 3 & -4 \\ 2 & -6 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ 2 & -1 \end{pmatrix}$

$8x - 4y = 0$ $\quad$ Let $x = 1, y = 2$

$2x - y = 0$

$V_2 = (1, 2)$

(b) $P = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad P^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$

(c) $D = P^{-1}AP$

$$= \begin{bmatrix} 2/7 & -1/7 \\ -1/7 & 4/7 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{pmatrix} 4/7 & -2/7 \\ 5/7 & -20/7 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{16-2}{7} & \frac{4-4}{7} \\ \frac{20-20}{7} & \frac{5}{7} - \frac{40}{7} \end{pmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$$

2) Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ (a) find all eigen values & vectors

(b) Find P, $D = P^{-1}AP$

(c) $A^6 = ?$ (d) $f(A) = ?$ where $t^4 - 3t^3 - 6t^2 + 7t + \ldots$

(a) $|xI - A| = \begin{vmatrix} x-2 & -2 \\ -1 & x-3 \end{vmatrix} = 0$ $\quad$ (c) $A^6 = PD^6P^{-1}$

$$(x-2)(x-3) - 2 = 0$$
$$x^2 - 2x - 3x + 6 - 2 = 0$$
$$x^2 - 5x + 4 = 0$$

$x = 4$ or $x = 1$ $\therefore$ Eigen values are 4 & 1

$\boxed{\lambda = 4}$ $m = A - 4I = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ (5)

$$= \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow -2x + 2y = 0 \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x = y$$

$\lambda = 1$ $\quad m = A - I = \begin{bmatrix} 2-1 & 2 \\ 1 & 3-1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$$x + 2y = 0 \qquad x = -2y$$

$$v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(b) $P = [v_1 \; v_2] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ $\qquad P^{-1} = \frac{1}{-3} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$

$D = P^{-1} A P$ $\qquad P^{-1} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}$

$$D = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} + \frac{2}{3} & \frac{2}{3} + \frac{6}{3} \\ \frac{2}{3} - \frac{1}{3} & \frac{2}{3} - \frac{3}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 & 8/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) $A^2 - 5A + 4I = 0$

$$A^2 = 5A - 4I = 5\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 10 \\ 5 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 6 & 10 \\ 5 & 11 \end{bmatrix}$$

$$A^6 = \begin{pmatrix} 6 & 10 \\ 5 & 11 \end{pmatrix}\begin{pmatrix} 6 & 10 \\ 5 & 11 \end{pmatrix}\begin{pmatrix} 6 & 10 \\ 5 & 11 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 10 \\ 5 & 11 \end{pmatrix}\begin{pmatrix} 86 & 170 \\ 85 & 171 \end{pmatrix} = \begin{pmatrix} 1366 & 2730 \\ 1365 & 2731 \end{pmatrix}$$

(c) $A^6 = PD^6P^{-1}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4096 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}$$

$$= \begin{pmatrix} 4096 & 2 \\ 4096 & -1 \end{pmatrix}\begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1366 & 2730 \\ 1365 & 2731 \end{pmatrix}.$$

(d) $f(t) = t^4 - 3t^3 - 6t^2 + 7t + 3$

function value of eigen values,

$f(4) = -1$, $f(1) = 2$.

$$f(A) = P f(D) P^{-1}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}$$

$$= \begin{pmatrix} -1 & 4 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$$

Diagonalization :-

Eigen value - Eigen vector infn contained within a matrix A can be displayed in a useful factorization of the form

$$A = PDP^{-1}$$

The factorization enables us to compute A^k quickly for large values of k

$$A^k = PD^kP^{-1}$$

$\Rightarrow$ If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ then $D^2 = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$

$$D^3 = D D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

In general $D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$ for $k \geq 1$

<u>Ex:-</u> Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ Find a formula for A^k given that $A = PDP^{-1}$

where $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

<u>Soln</u>:- $P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDDP^{-1}$$

$$= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A^3 = PDP^{-1}A^2 = PDP^{-1}PD^2P^{-1} = PD^3P^{-1}$$

$\therefore$ $k \geq 1$ $A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot 5^k - 3^k & 5^k - 3^k \\ 2\cdot 3^k - 2\cdot 5^k & 2\cdot 3^k - 5^k \end{bmatrix}$$

A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix. i.e $\exists$ $\boxed{A = PDP^{-1}}$ for some invertible matrix P and diagonal matrix D

The diagonalization theorem

An $n\times n$ matrix A is diagonalizable iff A has n linear independent eigen vectors.

$A = PDP^{-1}$ with D a the Diagonal matrix iff the columns of P are n linearly independent eigenvectors of A.

In this case the diagonal entries of D are eigenvalues of A, corresponds respectively, to the eigenvectors in P.

A is diagonalizable iff there are enough eigenvectors to form a basis of $\mathbb{R}^n$, then we call such a basis an eigenvector basis.

Proof:- P is any $n\times n$ matrix with columns $v_1 \dots v_n$

& if D is any diagonal matrix with diagonal entries $\lambda_1 \dots \lambda_n$ then

$$AP = A[v_1 \; v_2 \dots v_n] = [Av_1 \; Av_2 \dots Av_n] \quad (1)$$

while

$$PD = P\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = [\lambda_1 v_1 \; \lambda_2 v_2 \dots \lambda_n v_n] \quad (2)$$

Suppose if A is diagonalizable and $A = PDP^{-1}$ then right multiplying P we get $\boxed{AP = PD}$

Equating (1) & (2)

$$[Av_1 \; Av_2 \dots Av_n] = [\lambda_1 v_1 \; \lambda_2 v_2 \dots \lambda_n v_n] \quad (3)$$

equating each columns $\quad A_1 v_1 = \lambda_1 v_1 \quad A_2 v_2 = \lambda_2 v_2 \dots A_n v_n = \lambda_n v_n \quad (4)$

Since P is invertible its columns $v_1 \dots v_n$ must be linearly independent, these columns are nonzero

(4) shows that $\lambda_1 \dots \lambda_n \rightarrow$ eigen values

$v_1 \dots v_n \rightarrow$ eigen vectors corresponding to eigen values $\lambda_1 \dots \lambda_n$

Finally given any n eigenvectors $V_1 \ldots V_n$ use them to Construct the columns of P and use corresponding eigenvalues $\lambda_1 \ldots \lambda_n$ to construct D.

By eqn (1) – (3) $AP = PD$

Eigen vectors are linearly independent, then P is invertible

& $AP = PD \Rightarrow \boxed{A = PDP^{-1}}$

Problems

(1) Diagonalize the following matrix if possible

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Find invertible matrix P

Diagonal matrix D

Such that $A = PDP^{-1}$

$\det(A - \lambda I) = 0 = (\lambda I - A) = 0$

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0$$

$(\lambda - 1)(\lambda + 2)^2 = 0$

$\boxed{\lambda = 1, \lambda = -2}$

$$A - I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 3 & 3 & 0 & 0 \\ -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x + y = 0$

$-y + (-z) = 0$

$x = -y$

$\boxed{y = -z} \Rightarrow z = -y$

let $y = 1$ $\quad V_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$|A+2I| = \begin{bmatrix} -1 & 3 & 3 \\ -3 & -7 & -3 \\ 3 & 3 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 3 & 3 & 0 \\ -3 & -7 & -3 & 0 \\ 3 & 3 & -1 & 0 \end{array}\right] \xrightarrow[R_2 - 3R_1]{R_3 + 3R_1} \left[\begin{array}{ccc|c} -1 & 3 & 3 & 0 \\ 0 & -16 & -12 & 0 \\ 0 & 12 & 8 & 0 \end{array}\right]$$

$$\xrightarrow[R_3/4]{R_2/4} \left[\begin{array}{ccc|c} -1 & 3 & 3 & 0 \\ 0 & -4 & -3 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] \xrightarrow{R_3 + \frac{3}{4}R_2} \left[\begin{array}{ccc|c} -1 & 3 & 3 & 0 \\ 0 & -4 & -3 & 0 \\ 0 & 0 & -1/4 & 0 \end{array}\right]$$

$\lambda = -2$

$$[A+2I] = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix}$$

$$[A+2I \,\vdots\, 0] = \left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$3x + 3y + 3z = 0$$

$$x + y + z = 0$$

$$x = -y - z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \; v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [v_1 \;\; v_2 \;\; v_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Find P^{-1} and check $A = PDP^{-1}$

or. $\boxed{AP = PD}$ check

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

2) Diagonalize the following matrix if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix} \qquad \lambda = 1, -2, -2$$

An $n \times n$ matrix with n distinct eigenvalues is ~~diagonalizable~~ diagonalizable

As in prob 2 we have seen that an $n \times n$ matrix need not have 'n' distinct values to be diagonalizable.

Lemma :- Let $T \in L(V)$ over F be a polynomial over F if $Tx = \lambda x$ for $\lambda \in F$ then $f(T) \cdot x = f(\lambda) x$.

Proof :- $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_5 t^5$

$f(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_5 T^5$

$f(T) x = a_0 I x + a_1 T x + a_2 T^2 x \cdots + a_5 T^5 x$

$T^2 x = T(Tx) = T(\lambda x) = \lambda(Tx) = \lambda^2 x$

$\therefore f(T) x = a_0 I x + a_1 \lambda x + a_2 \lambda^2 x + a_3 \lambda^3 x + \cdots a_5 \lambda^5 x$

$= x(a_0 I + a_1 \lambda + a_2 \lambda^2 + \cdots + a_5 \lambda^5)$

$\boxed{f(T) x = f(\lambda) x}$

Defn: Let T be a linear operator on the finite dimensional space V. We say T is diagonalizable if there is a basis for V each vector of which is a characteristic vector of T.

If there is an ordered basis $B = \{\alpha_1 \ldots \alpha_n\}$ for V in which each α is a characteristic vector of T, then the matrix of T in the ordered basis B is diagonal if

$$T\alpha_i = c\alpha_i$$

$$[T]_B = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ & & & & c_n \\ 0 & 0 & \cdots & - & \end{bmatrix}$$

Lemma: Let T be a linear operator on the finite dimensional space V. Let $c_1 \ldots c_k$ be the distinct characteristic values of T and let W_i be the space of characteristic vectors associated with the characteristic value c_i. If $W = W_1 + W_2 \ldots + W_k$ then

$$\dim W = \dim W_1 + \dim W_2 + \ldots + \dim W_k.$$

In fact if B_i is an ordered basis for W_i then $B = \{B_1 \ldots B_k\}$ is an ordered basis for W.

Proof: The space $W = W_1 + W_2 + \ldots + W_k$ is the subspace spanned by all the characteristic vectors of T.

Sum W of all subspaces W_i then

$$\dim W < \dim W_1 + \ldots + \dim W_k$$

because of linear relations which may exist between vectors in the various spaces.

Lemma states that Characteristic spaces associated with different characteristic (9)
values are independent of one another.

$\forall \beta_i \in W_i$ ~~are linearly in~~ assume $\beta_1 + \cdots + \beta_k = 0$

Let f be any polynomial. Since $T\beta_i = c_i\beta_i$

wkt $f(T)x = f(c)x$

$$0 = f(T)0 = f(T)\beta_1 + \cdots f(T)\beta_k$$
$$= f(c_1)\beta_1 + \cdots f(c_k)\beta_k.$$

Choose polynomials $f_1 \cdots f_k$ such that

$$f_i(c_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$0 = f_i(T)0 = \sum_j \delta_{ij}\beta_j$$

$\boxed{0 = \beta_i}$ $\rightarrow \therefore \beta_i = 0$ for each i

Let B_i be an ordered basis for W_i

Let B be the sequence $B = (B_1 \cdots B_k)$.

Then B spans the subspace $W = W_1 + \cdots + W_k$.

B is a linear independent sequence of vectors

$\because$ Linear relation b/w the vectors in B will have the form $\beta_1 + \cdots + \beta_k = 0$. $\rightarrow$ β_i is the linear combination of the vectors in B_i

Since each B_i is linearly independent.

Theorem:- Let T be a linear operator on a finite dimensional space V. Let $c_1 \cdots c_k$ be the distinct characteristic values of T and let W_i be the null space of $(T - c_iI)$. The following are equivalent (i) T is diagonalizable

(ii) The characteristic polynomial for T is

$$f = (x-c_1)^{d_1} \cdots\cdots (x-c_k)^{d_k}.$$

and $\dim W_i = d_i \quad i = 1, 2 \dots k.$

(iii) $\dim W_1 + \dots + \dim W_k = \dim V.$

Annihilating Polynomials

collection of polynomials p which annihilate T in the sense that $p(T) = 0$ is an ideal polynomial

Suppose T is a linear operator on an n-dimensional space. then $1, T, T^2 \dots T^{n^2}$ are (n^2+1) vectors in $L(V,V)$ whose dimension is n^2. Thus there is a polynomial $f \in F(x)$ such that $f(T) = 0$. Hence the set $Ann(T) = \{ g \in F[x] \mid g(T) = 0 \}$ is non-zero

$L(V, V)$ the space of linear operators on V.

Let T be a linear operator on a finite dimensional vectorspace V over the field F. The minimal polynomial for T is the (unique) monic generator of the ideal polynomials over F which annihilate T

[Monic Polynomial is a single variable polynomial in in which the leading co-efficient is equal to 1

$$x^n + c_{n-1}x^{n-1} + \dots \dots + c_2 x^2 + c_1 x]$$

The minimal polynomial for the linear operator T is uniquelly determined by these three properties.

(i) p is a monic polynomial over the scalar field F

(ii) $p(T) = 0$

(iii) No polynomial over F which annihilates T has smaller degree than p has.

If A is an $n \times n$ matrix over F, we define the minimal polynomial for A in an analogous way, as the unique monic generator of the ideal of all polynomials over F which annihilate A. If the operator T is represented in some ordered basis by matrix A the T & A have the same minimal polynomial. $f(T) = 0$ iff $f(A) = 0$

Similar matrices have the same minimal polynomial

$$f(P^{-1}AP) = P^{-1} f(A) P$$

If f is a monic polynomial over F.

$$f = x^k + \sum_{j=0}^{k-1} a_j x^j$$

then $f(A) = 0$ merely says that we have a linear relation between the power of A

$$A^k + a_{k-1} A^{k-1} + \dots a_1 A + a_0 I = 0.$$

The degree of the minimal polynomial is the least positive integer k such that there is a linear relation, $A \dots A^k$

Let T be a linear operator on an n-dimensional vectorspace V (A be an $n \times n$ matrix). The characteristic & minimal polynomials for T [for A] have the same roots except for multiplicities.

Problems :-

① Find the characteristic polynomial & minimal polynomial given $A = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$$|A - \lambda I| = \begin{bmatrix} 4-\lambda & -2 & 2 \\ 6 & -3-\lambda & 4 \\ 3 & -2 & 3-\lambda \end{bmatrix} \Rightarrow (t-2)(t-1)^2$$

Characteristic Polynomial of A. $f(t) = (t-2)(t-1)^2$

Minimal polynomial of A $= m(t) = (t-1)(t-2)$

(2) Find the minimal polynomial for $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{vmatrix}$$

$$= (7-\lambda)\left[(7-\lambda)(4-\lambda) - 4\right] - 4\left[4(4-\lambda) - 4\right] - 1(-16 + 4(7-$$

$$(7-\lambda)\left[28 - 4\lambda - 7\lambda + \lambda^2 - 4(7-\lambda) - 4(16 - 4\lambda - 4) - (-16 + 28 - 4\lambda)\right]$$

$$(3-\lambda)(\lambda - 3)(\lambda - 12) = 0$$

$$\lambda = 3, 3, 12$$

$m(x)$ is minimal polynomial

$$M(x) = (x-3)(x-12)$$

$$= x^2 - 15x + 36$$

$$m(A) = A^2 - 15A + 36 = 0$$

(3) Find the minimal polynomial of a matrix represented by a linear operator $T : R^4 \to R^4$ defined by $T(x, y, z, t) = (x+y, 2y, z+t, -2z+4t)$ wrt to standard basis

$$T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \implies A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 2 & 4-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)\{(1-\lambda)(4-\lambda)-2\} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(4-5\lambda+\lambda^2-2) = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(\lambda^2-5\lambda+2) = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(\lambda-2)(\lambda-3) = 0$$

roots are 1, 2, 2, 3 Eigen values.

The minimal polynomial can be.

$$p(x) = (x-1)(x-2)(x-3)$$

$$p(A) = (A-I)(A-2I)(A-3I)$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ~~≠ 0~~ $p(A)$ is the minimal polynomial.

$\therefore$ $p(x) = (x-1)(x-2)(x-3)$ is ~~not~~ the minimal polynomial

hence the ~~minimal polynomial has degree at least 4~~

~~then the characteristic polynomial is the minimal polynomial~~

4) Find all the eigen values and a basis of each of eigensp of the operator $T: R^3 \to R^3$ defined by

$T(x, y, z) = (x-y,\ 2x+3y+2z,\ x+y+2z)$.

Is T diagonalizable? If so diagonalise the matrix

5) Find the minimal polynomial of a matrix represented by a linear operator.

$T: R^3 \to R^3$ defined by

$T(x, y, z) = (2x+y,\ y-z,\ 2y+4z)$

Invariant Subspaces

Let T be a vectorspace and T a linear operator on. If W is a subspace of V, we say that W is invariant under if for each vector α in W the vector $T\alpha$ is in W, i.e. if $T(W)$ is contained in W.

If T is a linear operator on V then V is invariant under T, as is the zero subspace. The range of T & the nullspace of T are also invariant under T

Unit-4 Inner Product Spaces

Definition:- Let V be a real vectorspace. Suppose to each pair of vectors $u, v \in V$ there is assigned a real number denoted by $\langle u, v\rangle$. This function is called a (real) inner product on V if it satisfies the following axioms:

1) Linear Property: $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$
2) Symmetric Property :- $\langle u, v\rangle = \langle v, u\rangle$
3) Positive Definite Property :- $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$

The vectorspace V with an inner product is called a (real) inner product space.

Using ① & ② [symmetry axiom] we obtain

$$\langle u, cv_1 + dv_2\rangle = \langle cv_1 + dv_2, u\rangle = c\langle v_1, u\rangle + d\langle v_2, u\rangle.$$
$$= c\langle u, v_1\rangle + d\langle u, v_2\rangle$$

That is the inner product function is also linear in its second position (variable)

$$\langle a_1u_1 + \cdots a_ru_r, v\rangle = a_1\langle u_1, v\rangle + a_2\langle u_2, v\rangle + \cdots + a_r\langle u_r, v\rangle$$

and $\langle u, b_1v_1 + b_2v_2 + \cdots + b_sv_s\rangle = b_1\langle u, v_1\rangle + b_2\langle u, v_2\rangle + \cdots + b_s\langle u, v_s\rangle$

Combining these two properties yields the following general formula

$$\left\langle \sum_{i=1}^{r} a_iu_i, \sum_{j=1}^{s} b_jv_j \right\rangle = \sum_{i=1}^{r} \sum_{j=1}^{s} a_ib_j\langle u_i, v_j\rangle$$

Example: Let V be an inner product space. Then by the linearity property of the innerproduct.

$$\langle 3u_1 + 5u_2 - 6u_3, v\rangle = 3\langle u_1, v\rangle + 5\langle u_2, v\rangle - 6\langle u_3, v\rangle$$

$$\langle u, 4v_1 - 7v_2 - 2v_3\rangle = 4\langle u, v_1\rangle - 7\langle u, v_2\rangle - 2\langle u, v_3\rangle$$

$$\langle 2u - 5v, 4u + 6v\rangle = 8\langle u, u\rangle + 12\langle u, v\rangle - 20\langle v, u\rangle - 30\langle v, v\rangle$$

$$= 8\langle u, u\rangle - 8\langle u, v\rangle - 30\langle v, v\rangle$$

Norm of a vector :-

By the third axiom (3) of an innerproduct $\langle u, u\rangle$ is non-negative for any vector u. Then its positive square root exists:

$$\|u\| = \sqrt{\langle u, u\rangle} \longrightarrow \text{norm or length of } u.$$

If $\|u\| = 1$ or equivalently, if $\langle u, u\rangle = 1$ then 'u' is called a unit vector & is said to be normalized

Every non-zero vector v in V can be multiplied by the reciprocal of its length to obtain the unit vector:

$$\hat{v} = \frac{1}{\|v\|}(v)$$

which is a positive multiple of v. This process is called normalizing v

Examples of Inner Product Spaces

Euclidean n-space R^n

Consider the vectorspace R^n. The dot product or scalar pr[oduct] in R^n is defined by

$$u \cdot v = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

where $u=(a_i)$ and $v=(b_i)$

This function defines an innerproduct in R^n is defined by

The norm $\|u\|$ of vector $u=(a_i)$ in the space follows:

$$\|u\| = \sqrt{u \cdot u} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

By the Pythagorean theorem, the distance from the origin O in R^3 to the point $P(a,b,c)$ is given by $\sqrt{a^2+b^2+c^2}$ which is same as above defined norm of the vector $v=(a,b,c)$ in R^3.

Pythagorean theorem is a consequence of the axioms of Euclidean geometry, the vectorspace R^n with the above inner product & norm is called <u>Euclidean n-space</u>

Problems

1) Let $u=(1,3,-4,2)$, $v=(4,-2,2,1)$, $w=(5,-1,-2,6)$ in R^4

Find (i) $\langle u+v, w\rangle$, $\langle u,w\rangle$, $\langle v,w\rangle$ hence

Prove that $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$

(ii) Find the norm of the vectors u, v, w

(iii) Normalize the vectors u, v & w.

Soln

$$\langle u,w\rangle = 1(5) + 3(-1) + (-4)(-2) + 2(6)$$
$$= 22$$

$$\langle v,w\rangle = 4(5) - 2(-1) + 2(-2) + 1(6)$$
$$= 24$$

$\langle u+v, w\rangle \Rightarrow \quad u+v = (5, 1, -2, 3)$

$$\langle u+v, w\rangle = 5(5) + 1(-1) - 2(-2) + 3(6)$$
$$= 46$$

$$\therefore \langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$$

ii] $\|u\| = \sqrt{1^2+3^2+(-4)^2+2^2} = \sqrt{30}$

$\|v\| = \sqrt{4^2+(-2)^2+2^2+1} = \sqrt{25} = 5$

$\|w\| = \sqrt{5^2+(-1)^2+(-2)^2+6^2} = \sqrt{66}$

iii) $\hat{u} = \frac{u}{\|u\|} = \left(\frac{1}{\sqrt{30}}, \frac{3}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{2}{\sqrt{30}}\right)$

$\hat{v} = \frac{v}{\|v\|} \left(\frac{4}{5}, \frac{-2}{5}, \frac{2}{5}, \frac{1}{5}\right)$

$\hat{w} = \frac{w}{\|w\|} \left(\frac{5}{\sqrt{66}}, \frac{-1}{\sqrt{66}}, \frac{-2}{\sqrt{66}}, \frac{6}{\sqrt{66}}\right)$.

Note:- If the vectors in R^n will be represented by column vectors, i.e by $n \times 1$ column matrices. In such a case $\langle u, v\rangle = u^T v$. defines the usual inner product on R^n.

Function Space $C[a,b]$ and Polynomial Space $P(t)$

The notation $C[a,b]$ is used to denote the vectorspace all continuous functions on the closed interval $[a,b]$ i.e. where $a \le t \le b$.

Then the following is an innerproduct on $C[a,b]$:

$$\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$$

where $f(t)$ and $g(t)$ are now continuous functions on $[a,b]$. It is called the usual innerproduct on $C[a,b]$

The vectorspace $P(t)$ of all polynomials is a subspace of $C[a,b]$ for any interval $[a,b]$ and hence the above is also an innerproduct on $P(t)$

Prob: Consider the vectorspace $P(t)$ with innerproduct

$$\langle f, g\rangle = \int_0^1 f(t)\,g(t)\,dt$$

Let $f(t) = t+2$ and $g(t) = (6t-5)$ then find $\langle f, g\rangle$, $\langle f, f\rangle$, $\langle g, g\rangle$

$$f(t)\,g(t) = (t+2)(6t-5) = 6t^2+7t-10$$

$$\langle f, g\rangle = \int_0^1 (6t^2+7t-10)\,dt = 2t^3 + \frac{7}{2}t^2 - 10t \Big|_0^1$$

$$\boxed{\langle f, g\rangle = -4.5}$$

$$[f(t)]^2 = f(t)f(t) = t^2+4t+4$$

$$\|f\|^2 = \langle f, f\rangle = \int_0^1 (t^2+4t+4)\,dt$$

$$= \frac{t^3}{3} + \frac{4t^2}{2} + 4t \Big|_0^1$$

$$= \frac{1}{3} + 2 + 4 = 19/3$$

$$\|f\| = \sqrt{19/3}$$

$$\langle g, g\rangle \Rightarrow [g(t)]^2 = g(t)\,g(t) = (6t-5)^2$$

$$= 36t^2 - 60t + 25$$

$$\therefore \|g^2\|\ \langle g, g\rangle = \int_0^1 (36t^2 - 60t + 25)\,dt$$

$$= 12t^3 - 30t^2 + 25t\Big|_0^1 = 12 - 30 + 25$$

$$= 7$$

$$\|g\| = \sqrt{7}.$$

Matrix Space $M = M_{m,n}$

Consider the vectorspace $M = M_{m,n}$ of all real $m \times n$ matrices.
An innerproduct is defined on M by

$$\langle A, B \rangle = tr(B^T A)$$

where tr stands for trace, the sum of the diagonal elements.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ then

$$\langle A, B \rangle = tr(B^T A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$$ → Sum of products of corresponding entries A & B.

$$\|A\|^2 = \langle A, A \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$$ → Sum of all the squares of the entries of A

1) Consider the vectorspace $M = M_{2,3}$ with innerproduct defined by $\langle A, B \rangle = tr(B^T A)$

Given $A = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -6 & 5 \\ 1 & 4 & -3 \end{bmatrix}$

Find $\langle A, B \rangle$, $\|A\|$.

$$\Rightarrow tr\langle B^T A \rangle = \begin{bmatrix} 2 & 1 \\ -6 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 8 - 30 + 30 + 7 + 32 - 27$$

$$= 20$$

$\langle A, B \rangle$ = Sum of products of corresponding items

$$= 4(2) + 5(-6) + 6(5) + 7(1) + 8(4) + 9(-3)$$

$$= 20$$

$\|A\|^2 = \langle A, A \rangle$ = Sum of Square of all elements of A

$$= 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 = 271$$

$$\|A\| = \sqrt{271}$$

Hilbert Space

Let V be the vectorspace of all infinite sequences of real numbers $(a_1, a_2, a_3 \ldots)$ satisfying

$$\sum_{i=1}^{\infty} a_i^2 = a_1^2 + a_2^2 \ldots < \infty$$ that is the sum converges

Addition & scalar multiplication are defined in V Component wise i.e if

$$u = (a_1, a_2 \ldots) \quad \text{and} \quad v = (b_1, b_2 \ldots)$$

Then $u + v = (a_1 + b_1, a_2 + b_2 \ldots)$ and $ku = (ka_1, ka_2, \ldots)$

An inner product is defined in V by

$$\langle u, v \rangle = a_1 b_1 + a_2 b_2 + \ldots$$

The above sum converges absolutely for any pair of points in V. This inner product space is called <u>I_2- space</u> or <u>Hilbert space</u>

<u>Cauchy-Schwarz Inequality, Applications:-</u>

Cauchy Schwarz inequality theorem states that, for any vectors u and v in an innerproduct space V,

$$\langle u, v \rangle^2 \leq \langle u, u \rangle \langle v, v \rangle$$

or equivalently

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$

→ Consider any real $a_1 \ldots a_n, b_1 \ldots b_n$. Then by the Cauchy-Schwarz inequality,

$$(a_1 b_1 + a_2 b_2 \ldots + a_n b_n)^2 \leq (a_1^2 + a_2^2 - \cdots + a_n^2)(b_1^2 + b_2^2 \ldots b_n^2)$$

i.e $(u \cdot v)^2 \leq \|u\| \|v\|$ where $u = (a_i)$ & $v = (b_j)$

→ Let f and g be any real continuous functions defined on the unit interval $0 \leq t \leq 1$. Then Cauchy Schwarz inequality

$$(\langle f, g \rangle)^2 = \left[\int_0^1 f(t) g(t)\, dt\right]^2 \leq \left[\int_0^1 f^2(t)\, dt \int_0^1 g^2(t)\, dt = \|f\|^2 \|g\|^2\right.$$

$$= \|f\|^2 \|g\|^2$$

V is the innerproduct space $C[0,1]$ of all continuous functions on the unit interval $0 \leq t \leq 1$