Filter design

Filter — selectively changes the waveshape of the signal in a desired manner.

- main objective is
  - → to improve the quality of the signal (eg. to remove noise)
  - → to extract information from signals

Digital filter

→ Implemented on H/W or S/W

— operates on digital I/p to produce digital o/p.

— usually operates on digitized analog signals stored in computer memory

Adv

1) performance does not vary with environmental changes eg: thermal variations
2) frequency response can be more precisely adjusted using a programmable processor.

3) ~~Many~~ Several I/p signals can be filtered without replacing the hardware.

4) Digital filters can be designed to have linear phase.

5) Can be used at very low frequencies.

6) Digital filters are portable.

7) flexible.

Disadvantages

1) Speed limitations

— ADC and DAC are used
Speed depends on Conversion time of ADC, DAC and Speed of processor.

2) finite Word length effects

3) Long design and development time.

Digital filter classification

IIR (Infinite Impulse response)

FIR (finite Impulse response)

Difference Eqn representation of a System with I/P $x(n)$ and O/P $y(n)$

$$\sum_{k=0}^{N} a_k \, y(n-k) = \sum_{k=0}^{M} b_k \, x(n-k)$$

→ IIR System.

FIR System

$$y(n) = \sum_{k=0}^{N-1} b_k \, x(n-k)$$

IIR System

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{\sum_{k=0}^{N-1} a_k z^{-k}}$$

FIR filters
Adv & Disadv

IIR filters
Adv & Disadv.

IIR filter design

Steps

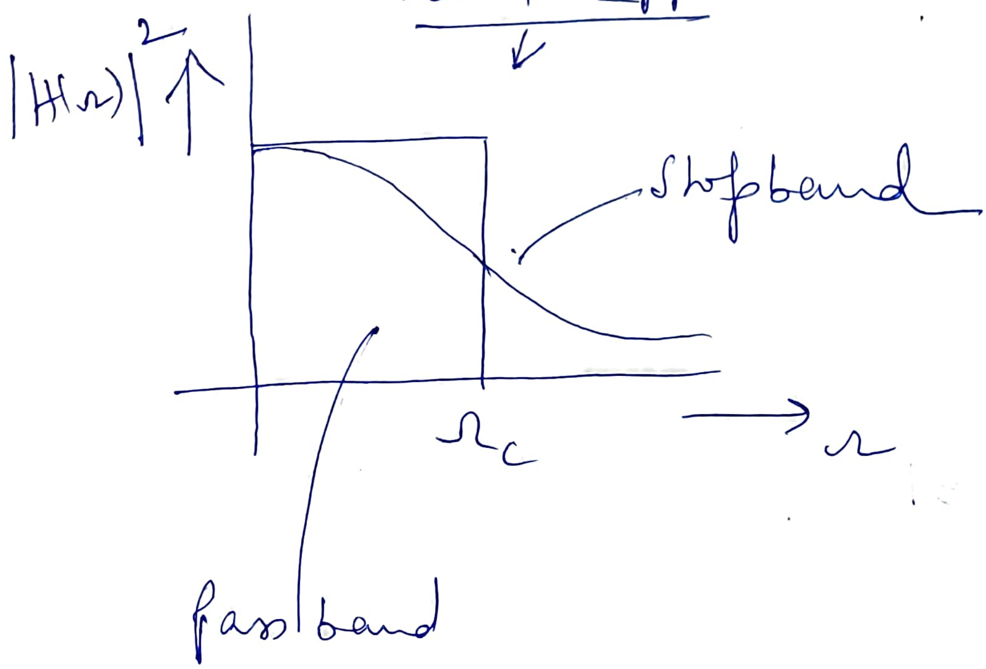
1) Obtain the Specifications of equivalent analog filter.
2) Design the analog filter in accordance with the Specifications
3) Transform the analog filter to an equivalent digital filter.

- analog filters → Butterworth, Chebyshev.

- Low pass filter → prototype filter
- frequency transformations are used to get transfer functions of other filters

Analog filter design using Butterworth Approximation

Ideal LPF

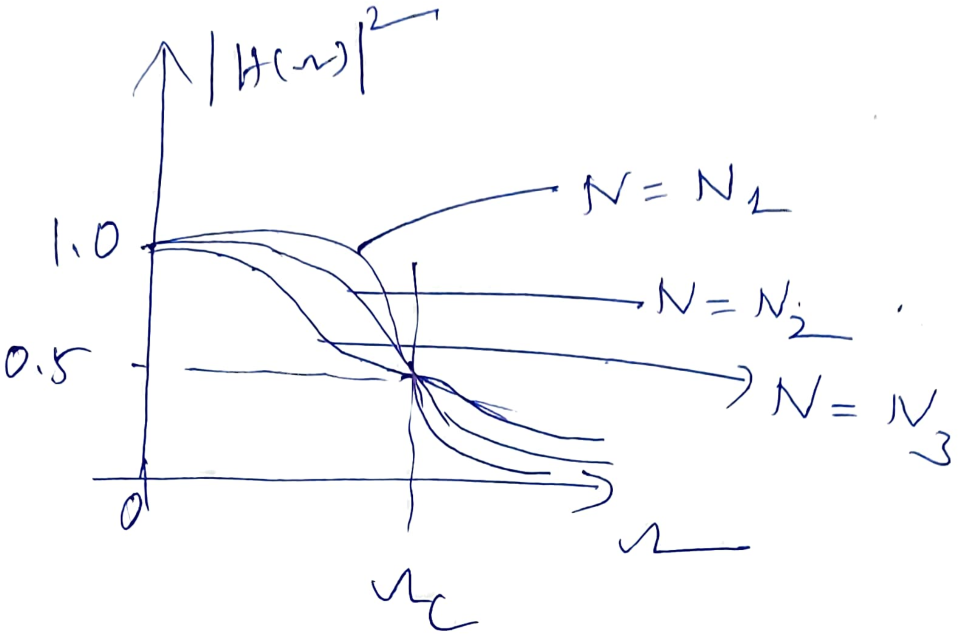


Magnitude squared frequency response of Butterworth filter is given by

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

N is the order

$\Omega_c \rightarrow$ 3 dB Cutoff freq.



Observations

1) $|H(\Omega)|\Big|_{\Omega=0} = 1$ for all N

2) $|H(\Omega)| = \dfrac{1}{\sqrt{2}}$ at $\Omega = \Omega_c$

$\Rightarrow$ $20 \log_{10} |H(\Omega)| = -3.01\ dB$ at $\Omega = \Omega_c$

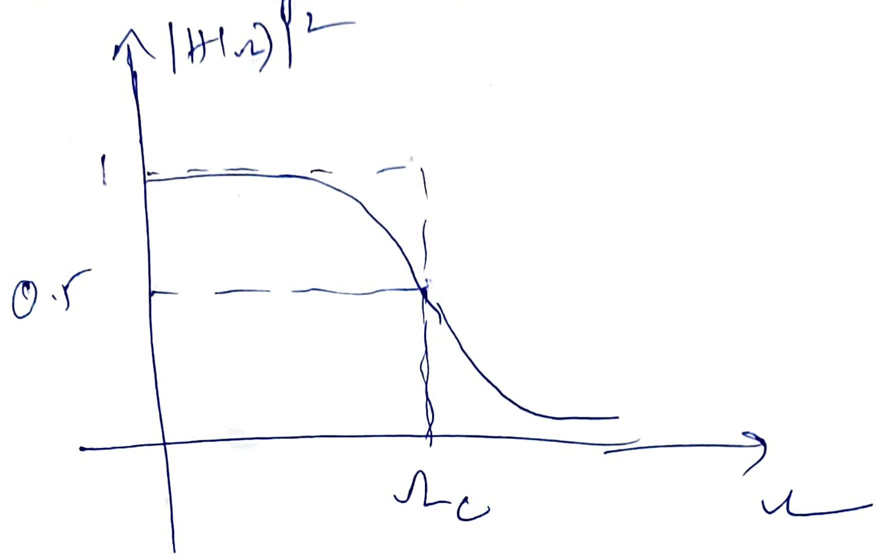
3) $|H(\Omega)| \to 0$ as $\Omega \to \infty$

4) Maximally flat responce in the pass band

$$\left.\frac{d^n |H(\Omega)|}{d\Omega^n}\right|_{\Omega=0} = 0 \quad \text{for} \quad n = 0, 1, \cdots 2N-1$$

(7)

5) Monotonically decreasing in the stopband



6) at $\Omega_c = 1 \Rightarrow$ Normalized LPF

$$|H(\Omega)|^2 = \frac{1}{1+\Omega^{2N}}$$

$$H(j\Omega) \cdot H(-j\Omega) = \frac{1}{1+\Omega^{2N}}$$

Design of Analog Lowpass Butterworth filter

$$H(s) = \frac{\Omega_c^N}{(s-s_0)(s-s_1)\cdots(s-s_{N-1})}$$

1) N
2) Ω_c
3) poles $s_0, s_1, \ldots s_{N-1}$

Order of Lowpass Butterworth filter

The filter specifications are as follows

$$A_p \leq |H(\Omega)| \leq 1 \qquad 0 \leq \Omega \leq \Omega_p$$

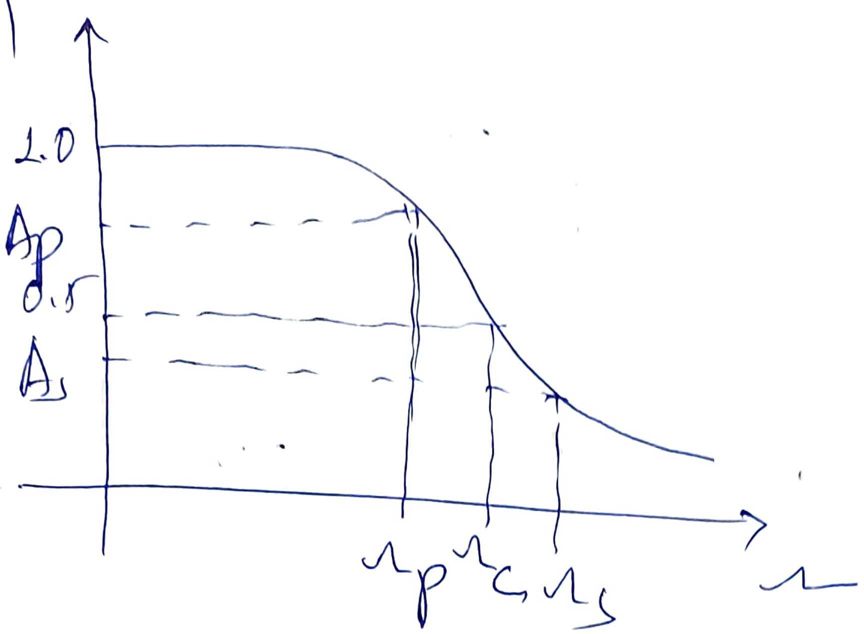
$$|H(\Omega)| \leq A_s \qquad \Omega \geq \Omega_s$$

A_p = passband gain

A_s = stopband gain

Ω_p = passband edge freq

Ω_s = stopband edge freq



pass band gain $\geq A_p$

Stop band gain $\leq A_s$

To find ~~case~~ Minimum value of N

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

at $\Omega = \Omega_p$,

$$A_p^2 = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \longrightarrow (1)$$

and $\Omega = \Omega_s$

$$A_s^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \longrightarrow (2)$$

from (1)

$$\frac{1}{A_p^2} - 1 = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} \rightarrow (3)$$

and from (2)

$$\frac{1}{A_s^2} - 1 = \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \quad — (4)$$

(4) ÷ (3) gives

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{1/A_s^2 - 1}{1/A_p^2 - 1}$$

$$\boxed{N \geq \frac{\log\left[\left(1/A_s^2 - 1\right) / \left(1/A_p^2 - 1\right)\right]}{2 \log\left(\frac{\Omega_s}{\Omega_p}\right)}}$$

<u>To find Ω_c</u>

from (3) $$\frac{\Omega_p}{\Omega_c} = \left[1/A_p^2 - 1\right]^{1/2N} \rightarrow (5)$$

from (4)

$$\frac{\Omega_s}{\Omega_c} = \left[\frac{1}{A_s^2} - 1\right]^{1/2N} \longrightarrow (6)$$

Rearranging terms in (5)

$$\Omega_{cp} = \frac{\Omega_p}{\left[\frac{1}{A_p^2} - 1\right]^{1/2N}}$$

and from (6)

$$\Omega_{cs} = \frac{\Omega_s}{\left[\frac{1}{A_s^2} - 1\right]^{1/2N}}$$

Actual cut off freq is avg of Ω_{cp} and Ω_{cs}

i.e. $\Omega_c = \dfrac{\Omega_{cp} + \Omega_{cs}}{2}$

To determine poles of $H(s)$

at $s = j\Omega$,

$$|H(s)| = |H(-s)|$$

$$s = j\Omega \Rightarrow s^2 = -\Omega^2.$$

$$|H(\Omega)|^2 = H(s).H(-s)$$

$$= \frac{1}{1+\left(\frac{-s^2}{\Omega_c^2}\right)^N}$$

poles of $H(s).H(-s)$ is obtained by equating denominator to zero.

$$1+\left(\frac{-s^2}{\Omega_c^2}\right)^N = 0$$

$$\left(\frac{-s^2}{\Omega_c^2}\right)^N = -1$$

$$\left(\frac{-s^2}{\Omega_c^2}\right) = (-1)^{1/N} \quad \text{--- } (1)$$

Now,

$$e^{j(2k+1)\pi} = -1$$

using this in 1

$$\frac{-s^2}{\Omega_c^2} = e^{j(2k+1)\pi/N}$$

$$s^2 = -\Omega_c^2 \, e^{j(2k+1)\pi/N}$$

$$s = \pm j \, \Omega_c \, e^{j(2k+1)\pi/2N}$$

Now, $e^{j\pi/2} = j$

$$s = \pm \Omega_c \, e^{j(2k+1)\pi/2N + j\pi/2}$$

$$s = \pm \Omega_c \, e^{j(2k+1+N)\pi/2N}$$

$k = 0, 1, \ldots N-1$

$$H(s) = \frac{\Omega_c^N}{(s-s_0)(s-s_1)\cdots(s-s_{N-1})}$$

14

$$N \geq \frac{\log\left[\left(\frac{1}{A_s^2}-1\right) \Big/ \left(\frac{1}{A_p^2}-1\right)\right]}{2 \log \frac{\Omega_s}{\Omega_p}}$$

If A_p and A_s are in dB

$$A_s \text{ in dB} = -20 \log_{10} A_s$$

$$\log_{10} A_s = -\frac{A_s \text{ in dB}}{20}$$

$$A_s = 10^{-\frac{A_s \text{ in dB}}{20}}$$

$$\frac{1}{A_s} = 10^{\frac{A_s \text{ in dB}}{20}}$$

$$\frac{1}{A_s^2} = 10^{\frac{A_s \text{ in dB}}{10}} = 10^{0.1 A_s \text{ in dB}}$$

$$N \geq \frac{\log\left[\frac{10^{0.1 A_s dB}-1}{10^{0.1 A_p dB}-1}\right]}{2 \log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

Frequency Transformations

Analog frequency Transformations

1) Low pass to Low pass

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} s$$

where Ω_{LP} is pass band edge frequency of the desired filter.

2) Low pass to High pass

$$s \rightarrow \frac{\Omega_p \, \Omega_{HP}}{s}$$

Ω_{HP} is the pass band edge frequency of the high pass filter.

3) Low pass to Band pass

$$S \rightarrow \Omega_p \frac{S^2 + \Omega_l \Omega_u}{S(\Omega_u - \Omega_l)}$$

where Ω_l = lower band edge frequency

Ω_u = upper band edge frequency.

4) Low pass to Band stop.

$$S \rightarrow \Omega_p \frac{(\Omega_u - \Omega_l)}{S^2 + \Omega_l \Omega_u}$$

Ω_l = lower band edge frequency

Ω_u = upper band edge frequency

Prob

1) Let $H(s) = \dfrac{1}{s^2+\sqrt{2}s+1}$ represent

transfer function of a low pass filter with cut off freq 1 rad/sec. Use frequency transformations to find the transfer functions of the following analog filters (a) Low pass filter with passband of 10 rad/sec (b) High pass filter with cut off frequency of 10 rad/sec.

Soln

$$H(s) = \frac{1}{s^2+\sqrt{2}s+1} ; \quad \Omega_p = 1 \text{ r/s}$$

a) LP – LP transformation

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} s$$

$$\Omega_p = 1 \text{ r/s} \qquad \Omega_{LP} = 10 \text{ r/s}$$

$$s \rightarrow \frac{s}{10}$$

$$H_1(s) = H(s)\Big/_{s = s/10}$$

$$= \frac{1}{\frac{s^2}{100} + \sqrt{2}\frac{s}{10} + 1} = \frac{100}{s^2 + 10\sqrt{2}s + 100}$$

b) LP → HP transformation

$$s \rightarrow \frac{\Omega_p \Omega_{HP}}{s}$$

$$s \rightarrow \frac{10}{s}$$

$$H_2(s) = H(s)\Big/_{s = \frac{10}{s}} = \frac{1}{\frac{100}{s^2} + \frac{10\sqrt{2}}{s} + 1}$$

$$= \frac{s^2}{s^2 + 10\sqrt{2}s + 100}$$

2) Let $H(s) = \frac{1}{s^2+s+1}$ represent the transfer function of a lowpass filter with pass band of 1 r/s. Use frequency transformation to find the transfer function of the following analog filters

a) High pass filter with cut off frequency 10 rad/sec.

b) Band pass filter with a pass band of 10 r/s and centre freq. 100 r/s.

Soln

a) LP → HP

$$s \rightarrow \frac{\Omega_p \Omega_{HP}}{s} = \frac{10}{s}$$

$$H_1(s) = H(s)\Big|_{s=\frac{10}{s}} = \frac{1}{\left(\frac{10}{s}\right)^2 + \frac{10}{s} + 1}$$

$$= \frac{s^2}{s^2 + 10s + 100}$$

6) Centre freq. $\Omega_0 = \sqrt{\Omega_l \Omega_u}$ (20)

$$\Omega_0 = 100 \text{ r/s}$$

$$\therefore \Omega_l \Omega_u = 100^2 \longrightarrow (1)$$

Pass band $= \Omega_u - \Omega_l = 10. \rightarrow (2)$

$LP \longrightarrow BP$

$$s \longrightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} = \frac{s^2 + 100^2}{s \times 10}.$$

$$= \frac{s^2 + 10000}{10s}$$

$$H_2(s) = H(s) \Big/_{s = \frac{s^2+10000}{10s}}$$

$$= \frac{1}{\left(\frac{s^2+10000}{10s}\right)^2 + \left(\frac{s^2+10000}{10s}\right) + 1}$$

$$= \frac{100s^2}{s^4 + 10s^3 + 20100s^2 + 10^5 s + 10^8}.$$

Analog Butterworth filter design

Prob

1) Given that $|H(\Omega)|^2 = \dfrac{1}{1+16\Omega^4}$,

determine the analog filter transfer function $H(s)$.

Soln

$$|H(\Omega)|^2 = \frac{1}{1+16\Omega^4}$$

$$= \frac{1}{1+\left(\dfrac{\Omega}{1/2}\right)^4} = \frac{1}{1+\left(\dfrac{\Omega}{1/2}\right)^{2\times 2}}$$

$$= \frac{1}{1+\left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}$$

$\therefore$ $\underline{N = 2}$ and $\underline{\Omega_c = 1/2 \text{ r/s}}$

Poles of Butterworth filter

$$P_K = \pm \Omega_c \, e^{j(2K+1+N)\pi/2N}$$

$K = 0, 1, \ldots N-1$

Let us design a normalized filter first

i.e., $\Omega_c = 1$ r/s

$$P_k = \pm e^{j(2k+1+N)\pi/2N}$$

$$k = 0, 1$$

for $k = 0$

$$P_0 = \pm e^{j3\pi/4}$$

$$= \pm \left[-0.707 + j0.707\right]$$

for $k = 1$

$$P_1 = \pm e^{j5\pi/4}$$

$$= \pm \left[-0.707 - j0.707\right]$$

To determine $H_1(s)$

$$H(s) = \frac{1}{(s - P_0)(s - P_1)}$$

$$= \frac{1}{\left[s - (-0.707 + j0.707)\right]\left[s - (-0.707 - j0.707)\right]}$$

$$= \frac{1}{(s+a-jb)(s+a+jb)}$$

$$= \frac{1}{((s+a)^2+b^2)}$$

$$= \frac{1}{[(s+0.707)^2+(0.707)^2]}$$

$$= \frac{1}{(s^2+0.5+1.414s)+0.5}$$

$$H_1(s) = \frac{1}{s^2+1.414s+1}$$

Applying $LP \to LP$ transformation

$$H(s)\Big/_{s \to \frac{s}{0.5}} = \frac{1}{\left(\frac{s}{0.5}\right)^2+1.414 \times \frac{s}{0.5}+1}$$

$$= \frac{(0.5)^2}{s^2+(0.5)\times 1.414s+(0.5)^2}$$

$$H(s) = \frac{0.25}{s^2+0.707s+0.25}$$

Prob Determine the transfer function of Normalized Butterworth filter with order a) 2 b) 3 c) 4.

a) $p_k = \pm \Omega_c \, e^{j(2k+1+N)\pi/2N}$

$k = 0, 1$

$\Omega_c = 1 \; r/s$

$N = 2$

$p_k = \pm e^{j(2k+3)\pi/4}$

Put $k = 0$

$p_0 = \pm e^{j3\pi/4}$

$= \pm(-0.707 + j0.707)$

Put $k = 1$

$p_1 = \pm e^{j5\pi/4}$

$= \pm(-0.707 - j0.707)$

$$H(s) = \frac{1}{(s-p_0)(s-p_1)} = \frac{1}{s^2 + 1.414s + 1}$$

b) $N = 3$

$$P_k = \pm\, \Omega_c\, e^{j(2k+1+N)\pi/2N}$$

$k = 0, 1, 2$

$\Omega_c = 1\ r/s$ and $N = 3$

$$P_k = \pm\, e^{j(2k+4)\pi/6}$$

Put $k = 0$,

$$P_0 = \pm\, e^{j4\pi/6}$$

$$= \pm(-0.5 + j0.866)$$

Put $k = 1$

$$P_1 = \pm\, e^{j6\pi/6}$$

$$= \pm(-1)$$

Put $k = 2$

$$P_2 = \pm\, e^{j8\pi/6}$$

$$= \pm(-0.5 - j0.866)$$

$$H(s) = \frac{1}{(s - P_0)(s - P_1)(s - P_2)}$$

$$H(s) = \frac{1}{(s+1)\left[(s+0.5-j0.866)(s+0.5+j0.866)\right]}$$

$$= \frac{1}{(s+1)\left[(s+0.5)^2+(0.866)^2\right]}$$

$$= \frac{1}{(s+1)(s^2+s+1)}$$

Prob

Given $|H(\Omega)|^2 = \dfrac{1}{1+64\Omega^6}$,

determine the analog filter system function $H(s)$

Examples

1. Design an analog Butterworth filter to meet the following specifications

$$0.8 \le |H(\Omega)| \le 1 \qquad 0 \le f \le 1000\,HZ$$

$$|H(\Omega)| \le 0.2 \qquad f \ge 5000\,HZ$$

Soln

$$A_p = 0.8 \qquad A_s = 0.2$$

$$\Omega_p = 2\pi f_p = 2\pi \times 1000 = 2000\pi \ r/s$$

$$\Omega_s = 2\pi f_s = 2\pi \times 5000 = 10000\pi \ r/s$$

Step 1

To find the order of the filter

$$N \ge \frac{\log\left[\left(\frac{1}{A_s^2}-1\right)\Big/\left(\frac{1}{A_p^2}-1\right)\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\frac{1}{A_p^2} = 1.56$$

$$\frac{1}{A_s^2} = 25$$

$$\ge 1.167$$

$$\boxed{N = 2}$$

Step 2

To find Ω_c

$$\Omega_{cp} = \frac{\Omega_p}{\left[\frac{1}{A_p^2} - 1\right]^{1/2N}}$$

$$= 7263 \text{ r/s}$$

$$\Omega_{cs} = \frac{\Omega_s}{\left[\frac{1}{A_s^2} - 1\right]^{1/2N}}$$

$$= 14196 \text{ r/s}$$

$$\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2} = \underline{\underline{10729 \text{ r/s}}}$$

$$= 3415\pi \text{ r/s}$$

<u>Step 3</u>

To design a normalised 2nd order filter.

$$P_0 = \pm e^{j(2\times 0+1+2)\pi/2\times 2}$$

$$= \pm(-0.707 - j0.707)$$

$$P_1 = \pm e^{j(2\times 1+1+2)\pi/4}$$

$$= \pm(-0.707 + j0.707)$$

$$H_1(s) = \frac{1}{(s+0.707-j0.707)(s+0.707+j0.707)}$$

$$= \frac{1}{(s+0.707)^2+(0.707)^2}$$

$$H_1(s) = \frac{1}{s^2+1.414s+1}$$

<u>Step 4</u>

To design the filter for required Ω_c.

$$LP \rightarrow LP$$

$$s \rightarrow \frac{s}{10729}$$

$$H(s) = \frac{}{\left(\frac{s}{10729}\right)^2 + 1.414\left(\frac{s}{10729}\right)+1}$$

$$= \frac{(10729)^2}{s^2+1.414 \times 10729\, s + (10729)^2}$$

Prob

Design an analog butterworth filter for the following specifications

$$0.8 \leq |H(\Omega)| \leq 1 \qquad 0 \leq \Omega \leq 0.2\pi$$

$$|H(\Omega)| \leq 0.2 \qquad 0.6\pi \leq \Omega \leq \pi$$

Soln

$A_p = 0.8 \qquad A_s = 0.2$

$\Omega_p = 0.2\pi$ r/s $\qquad \Omega_s = 0.6\pi$ r/s

31

Prob: Design a high pass Butterworth filter for the given specifications.
-3 dB passband attenuation at a frequency of $\Omega_p = 1000$ r/s. and at least -15 dB attenuation at 500 r/s.

Solu^n: Specifications of high pass filter to be designed

$A_p = -3$ dB $\quad A_s = -15$ dB

$\Omega_p = 1000$ r/s $\quad \Omega_s = 500$ r/s

Step 1

Design a normalized LPF.

for Equivalent LPF $\Omega_s = 1000$ r/s $\quad \Omega_p = 500$ r/s

order of the filter

$$N \geq \frac{\log\left[\frac{10^{0.1 A_s\,dB} - 1}{10^{0.1 A_p\,dB} - 1}\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\geq \frac{\log[30.9]}{2\log(2)} \geq 2.49$$

$$\boxed{N = 3}$$

Transfer of function of normalized 3rd order Butterworth LPF is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$\therefore$ Transfer function of 3rd order HPF with Ω_c

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_p}{\left[\frac{1}{A_p^2}-1\right]^{1/2N}} + \frac{\Omega_s}{\left[\frac{1}{A_p^2}-1\right]^{1/2N}}\right]$$

$$= \frac{1}{2}\left[\frac{\Omega_p}{\left[10^{0.1 A_p dB}-1\right]^{1/2N}} + \frac{\Omega_s}{\left[10^{0.1 A_s dB}-1\right]^{1/2N}}\right]$$

≈ 1000

frequency transformation is

$$s \rightarrow \frac{1000}{s}$$

$$H(s) = \frac{s^3}{(s+1000)(s^3+1000s+1000^2)}$$

Ans:

Design an analog filter with maximally flat response in the passband and an acceptable attenuation of -2 dB at 20 r/s. The attenuation in the stopband should be more than 10 dB beyond 30 r/s.

Soln)

$A_p = -2$ dB $\quad A_s = -10$ dB

$\Omega_p = 20$ r/s $\quad \Omega_s = 30$ r/s

$$N \geq \frac{\log\left[\frac{10^{0.1 A_s dB} - 1}{10^{0.1 A_p dB} - 1}\right]}{2 \log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq 3.37$$

$$\boxed{N = 4}$$

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_p}{\left[10^{0.1A_p dB}-1\right]^{1/2N}} + \frac{\Omega_s}{\left[10^{0.1A_s dB}-1\right]^{1/2N}}\right]$$

$$\Omega_c = 22 \text{ r/s}$$

Normalized fourth order Butterworth filter

$$H_1(s) = \frac{1}{\left[(s+0.382)^2+(0.923)^2\right]\left[(s+0.923)^2+(0.382)^2\right]}$$

$$= \frac{1}{(s^2+0.7645+1)(s^2+1.8465+1)}$$

LP → LP transformation

$$s \rightarrow \frac{s}{22}$$

$$H(s) = \frac{(22)^4}{(s^2+3705+484)(s^2+894s+484)}$$

S → Z mapping

1) Approximation of derivatives
2) Impulse Invariance Transformation (IIT)
3) Bilinear transformation
4) pole - zero mapping

Impulse Invariance transformation

$$\underset{H(s)}{[\text{Analog filter}]} \xrightarrow{L^{-1}} h(t) \xrightarrow{\text{Uniform sampling}} h(nT) \xrightarrow{\text{Z-trans}} \underset{(\text{Digital filter})}{H(z)}$$

Let $H(s)$ be the system function of analog filter

Using partial fraction expansion,

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{s - p_k} \rightarrow (1)$$

Taking inverse Laplace transform,

$$h(t) = \sum_{k=1}^{N} C_k e^{p_k t}, \quad \text{for } t \geq 0$$

$$\Rightarrow \sum_{k=1}^{N} C_k e^{p_k t} u(t)$$

Sampling $h(t)$ uniformly gives

$$h(nT) = h(n) = \sum_{k=1}^{N} C_k e^{p_k nT} u(nT)$$

Taking Z-transform

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{p_k T} z^{-1}} \rightarrow (3)$$

from (1) & (3) we can write mapping as

$$\frac{1}{s - p_k} \longrightarrow \frac{1}{1 - e^{p_k T} z^{-1}}$$

$$H(s) = \frac{1}{s - P_k}$$

$\therefore$ $H(s)$ has pole at $s = P_k$.

and $H(z)$ has pole at $z = e^{P_k T}$

analog pole at $s = P_k$ is mapped to digital pole at $z = e^{P_k T}$

$$z = e^{P_k T}$$

$$\Rightarrow z = e^{sT}$$

In polar form $z = re^{j\omega}$ and

$$s = \sigma + j\Omega$$

$$\therefore re^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$re^{j\omega} = e^{\sigma T} \cdot e^{j\Omega T}$$

$\therefore$ Equating real and imaginary parts,

$$\boxed{\begin{aligned} r &= e^{\sigma T} \\ \omega &= \Omega T \end{aligned}} \longrightarrow (4)$$

Mapping Summary

A) from (4) if $\sigma < 0$ then $0 < r < 1$

$\sigma < 0 \Rightarrow$ LHS of s plane

$0 < r < 1 \Rightarrow$ area inside unit circle in Z-plane

$\therefore$ LHS of s plane is mapped inside unit circle in Z-plane

B) from (4) if $\sigma > 0$, then $r > 1$

$\therefore$ RHS of s plane is mapped outside unit circle in Z-plane.

C) from (4) if $\sigma = 0$ then $r = 1$

$\therefore$ $j\Omega$ axis is mapped to unit circle in Z-plane.

Limitations of IIT

Range of ω is $-\pi \leq \omega \leq \pi$

$\omega = \Omega T$

$\therefore -\pi \leq \Omega T \leq \pi$

$-\frac{\pi}{T} \leq \Omega \leq \frac{\pi}{T}$ $\rightarrow$ gets mapped to

$-\pi \leq \omega \leq \pi$

$\therefore$ In general we can write

$\frac{(2k-1)\pi}{T} \leq \Omega \leq \frac{(2k+1)\pi}{T}$

is mapped to $-\pi \leq \omega \leq \pi$

$\Rightarrow$ multiple segments of $j\Omega$ axis are mapped repeatedly on ~~the~~ in the range $-\pi \leq \omega \leq \pi$

$\Rightarrow$ Causes **Aliasing**.

40

Prob

1) The system function of analog filter is given by $H(s) = \frac{1}{(s+1)(s+2)}$

find $H(z)$ using IIT method.

Take ~~[illegible]~~ sampling frequency as 5 samples/sec.

Soln

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = 1, \quad B = -1$$

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

using IIT mapping

$$\frac{1}{s - p_k} \longrightarrow \frac{1}{1 - e^{p_k T} z^{-1}}$$

$$T = \frac{1}{f} = \frac{1}{5} = 0.2 \text{ sec}$$

$$\frac{1}{s+1} \longrightarrow \frac{1}{1 - e^{-1 \times 0.2} z^{-1}}$$

$$\frac{1}{s+2} \longrightarrow \frac{1}{1 - e^{-2 \times 0.2} z^{-1}}$$

$$H(z) = \frac{1}{1-0.818z^{-1}} - \frac{1}{1-0.67z^{-1}}$$

$$= \frac{0.148z^{-1}}{1-1.488z^{-1}+0.548}$$

Prob find $H(z)$ if $H(s) = \dfrac{b}{(s+a)^2+b^2}$

using this result find $H(z)$ when

$$H(s) = \frac{1}{s^2+2s+2}$$

Soln $(s+a)^2+b^2=0$

$$s = -a \pm jb$$

$$s_1 = -a+jb \qquad s_2 = -a-jb$$

$$H(s) = \frac{b}{(s+a-jb)(s+a+jb)}$$

$$= \frac{A}{(s+a-jb)} + \frac{B}{(s+a+jb)} \quad \text{—(1)}$$

$$A = 1/j2 \qquad B = -1/j2$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{p_k T} z^{-1}}$$

$$= \sum_{k=1}^{N} \frac{C_k z}{z - e^{p_k T}}$$

using this in 1

$$H(z) = \frac{1}{j2}\left[\frac{z}{z - e^{(-a+jb)T}} - \frac{z}{z - e^{(-a-jb)T}}\right]$$

$$= \left[\frac{z\, e^{-aT} \sin bT}{z^2 - 2e^{-aT}\cos bT\, z + e^{-2aT}}\right]$$

$$= \frac{e^{-aT} \sin bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT} z^{-2}}$$

Given $H(s) = \dfrac{1}{s^2+2s+2}$

$$H(s) = \frac{1}{(s+1)^2+1^2}$$

$$\Rightarrow \quad H(z) = \frac{e^{-T}\sin T\, z^{-1}}{1-2e^{-T}\cos T\, z^{-1}+e^{-2T}z^{-2}}$$

Prob

find $H(z)$ using IIT. Assume $T=1$

1) $H(s) = \dfrac{2}{(s+1)(s+2)}$

2) $H(s) = \dfrac{10}{s^2+7s+10}$

Bilinear transformation (BLT)

Bilinear transformation is obtained by using trapezoidal formula for numerical integration.

Consider an analog system.

$$y(t) = \frac{dx(t)}{dt} \longrightarrow (1)$$

$$x(t) \longrightarrow \boxed{s} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

$$H(s) = \frac{Y(s)}{X(s)} = s \longrightarrow (2)$$

(1) is now integrated b/w the limits $(nT-T)$ and nT.

$$\int_{nT-T}^{nT} y(t)\,dt = \int_{nT-T}^{nT} \frac{dx(t)}{dt}\,dt.$$

$$\int_{nT-T}^{nT} y(t)\,dt = x(nT) - x(nT-T) \longrightarrow (3)$$

using trapezoidal rule on LHS of (3), we get

$$\frac{T\left[y(nT) + y(nT-T)\right]}{2} = x(nT) - x(nT-T)$$
$\rightarrow$ (4)

using $x(nT) = x(n)$ in (4) we get

$$\frac{T}{2}\left[y(n) + y(n-1)\right] = x(n) - x(n-1)$$
$\rightarrow$ (5)

Applying Z- transform to (5), we get

$$\frac{T}{2}\left[Y(z) + z^{-1}Y(z)\right] = X(z) - z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{T}\left[\frac{(1-z^{-1})}{(1+z^{-1})}\right] \rightarrow (6)$$

Comparing (2) and (6), we get

s to z mapping as

$$\boxed{s \rightarrow \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

To obtain the relationship b/w analog and digital frequencies

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \text{ --- } \star$$

Evaluating $\star$ at $s = j\Omega$ and $z = e^{j\omega}$, we get,

$$j\Omega = \frac{2}{T}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right]$$

$$= \frac{2}{T}\left[\frac{e^{+j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}}\right]$$

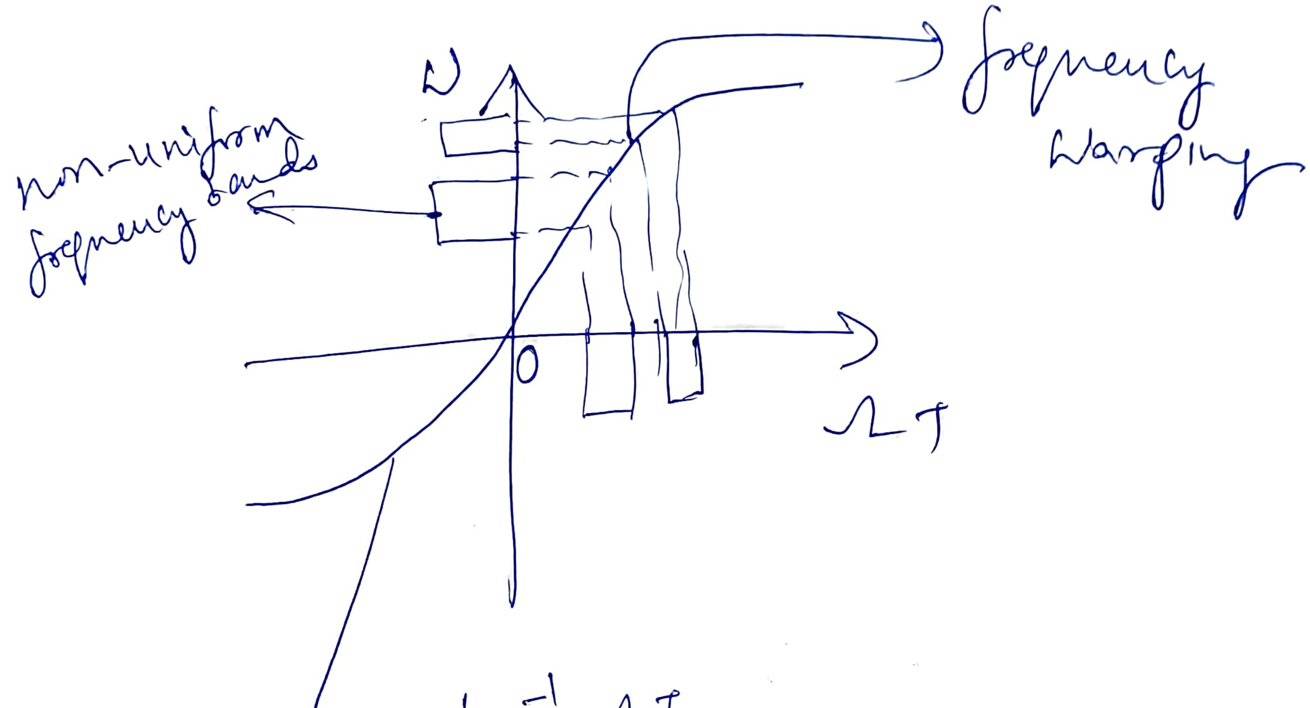
$$= \frac{2}{T}\,\frac{2j\sin\omega/2}{2\cos\omega/2}$$

$$j\Omega = j\frac{2}{T}\tan\omega/2$$

$$\Rightarrow \boxed{\Omega = \frac{2}{T}\tan\omega/2}$$

8) ~~Ae~~ $\omega = 2\tan^{-1}\frac{\Omega T}{2}$

$\Rightarrow$ relationship b/w analog and digital frequencies is highly non linear.



$\omega = \frac{2}{\cancel{T}}\tan^{-1}\frac{\Omega T}{2}$.

* The non linear relationship b/w ω and Ω is called frequency warping
* because of frequency warping, the evenly spaced bands of analog filter are mapped to unevenly spaced bands of digital filter
* Lower frequency bands are expanded and higher frequency bands are Compressed.

S plane to Z-plane mapping

4A

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$\cancel{\sigma + j\Omega = \frac{2}{T}\left[\frac{1-(re^{jw})^{-1}}{1+}\right]}$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

$$= \frac{2}{T}\left[\frac{re^{jw}-1}{re^{jw}+1}\right]$$

$$= \frac{2}{T}\left[\frac{r(\cos w + j\sin w)-1}{r(\cos w + j\sin w)+1}\right]$$

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos w} + j\,\frac{2r\sin w}{1+r^2+2r\cos w}\right]$$

$$\Rightarrow \quad \left.\begin{aligned} \sigma &= \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos w}\right] \\ \Omega &= \frac{2}{T}\left[\frac{2r\sin w}{1+r^2+2r\cos w}\right] \end{aligned}\right\} \text{— } A$$

Mapping Summary

* from Eqn ① of Ⓐ, if $r = 1$ then
$\sigma = 0 \Rightarrow j\Omega$ axis in s plane
is mapped to unit circle in z-plane

* from Eqn ① of Ⓐ, if $r > 1$, then
$\sigma > 0 \Rightarrow$ RHS of s plane is
mapped to outside the unit circle

* from Eqn ① of Ⓐ if $r < 1$, then
$\sigma < 0 \Rightarrow$ LHS of s plane is
mapped to inside the unit circle.

$\Rightarrow$ A stable analog filter is
converted to a stable digital filter

Adv. of BLT

* Mapping of frequency is one-one
  NO Aliasing
* Stable analog filter is converted to a stable digital filter

disadv of BLT

* Warping effect.

Prob:

1) Convert the analog filter

$$H(s) = \frac{2}{(s+1)(s+3)}$$ into a digital

filter using BLT. Take $T = 0.1$ sec.

Sol^n
Mapping from s to z in BLT is

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\therefore H(z) = \frac{2}{\left\{\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]+1\right\}\left\{\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]+3\right\}}$$

51

$T = 0.1$ sec

$\frac{2}{T} = 20.$

$$\therefore H(z) = \frac{2(1+z^{-1})^2}{483 - 794z^{-1} + 323z^{-2}}$$

2) Convert 2nd order normalized filter to digital filter using BLT. Take $T = 1$ sec

Soln

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$H(z) = \frac{1}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right\}^2 + \sqrt{2}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right\} + 1}$$

$$= \frac{0.127(1+2z^{-1}+z^{-2})}{1 - 0.766z^{-1} + 0.277z^{-2}}$$

Ω

3) Transform analog filter

$$H(s) = \frac{s+0.1}{(s+0.1)^2+9} \text{ into digital}$$

filter using BLT. The digital filter should have resonant frequency

$$\omega_r = \pi/4$$

soln'

$$(s+0.1)^2+9 = (s+0.1-j3)(s+0.1+j3)$$

$$\Rightarrow s = -0.1 \pm j3$$

$$= \sigma \pm j\Omega$$

$$\Rightarrow \underline{\Omega = 3} \text{ r/s}$$

$$\omega_r = \pi/4$$

$$\Omega = \frac{2}{T} \tan\frac{\omega}{2}$$

$$\Rightarrow T = \frac{2}{\Omega} \tan\frac{\omega}{2}$$

$$T = 0.2761 \text{ sec}$$

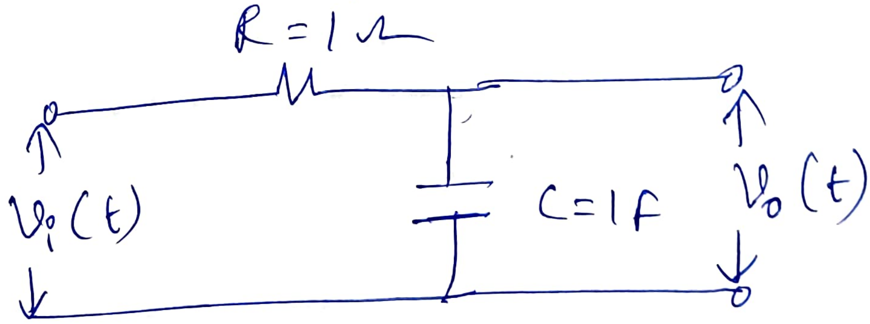
$$\& \quad \frac{2}{T} = 3.621$$

$$H(z) = \frac{3.621\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left[3.621\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1\right]^2+9}$$

$$= \frac{0.1629\left(1-0.053z^{-1}-0.95z^{-2}\right)}{1-0.36z^{-1}+0.94z^{-2}}$$

Prob

4) Obtain digital filter equivalence of the analog filter shown in fig using
a) IIT b) BLT. Assume sampling frequency $F_s = 8f_c$ where f_c is cut off frequency of the filter



Soln

$$V_o(t) = \frac{X_C}{R+X_C} V_i(t)$$

$$\frac{V_o(t)}{V_i(t)} = \frac{X_C}{R+X_C}$$

Taking Laplace Transforms, (54)

$$H(s) = \frac{1/s}{1 + 1/s} = \frac{1}{s+1}$$

Cut off frequency $f_c = \frac{1}{2\pi RC}$

$$f_s = 8 f_c = \frac{8}{2\pi RC} = \frac{8}{2\pi \times 1 \times 1}$$

$$f_s = 1.273 \text{ Hz}$$

$$T = 1/f_s = 0.7855 \text{ sec}$$

a) $\frac{1}{s - p_k} \longrightarrow \frac{1}{1 - e^{p_k T} z^{-1}}$

$\frac{1}{s+1} = H(s)$
$\Rightarrow s = -1$
$\Rightarrow p_k = -1$

$$H(z) = \frac{1}{1 - e^{-1 \times 0.7855} z^{-1}} = \frac{1}{1 - 0.456 z^{-1}}$$

b) $s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$

$$= \frac{2}{0.7855}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$\therefore H(z) = \frac{1}{2.55\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + 1} = \frac{0.282(1 + z^{-1})}{1 - 0.442 z^{-1}}$$

Digital Butterworth filter design

Prob 1) Design a Butterworth filter using BLT for the following specifications:

$$0.8 \le |H(\omega)| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le 0.2 \qquad 0.6\pi \le \omega \le \pi$$

Solu^n: $A_p = 0.8$ $\qquad A_s = 0.2$

$\omega_p = 0.2\pi$ $\qquad \omega_s = 0.6\pi$

Step 1. To obtain specifications of Corresponding analog filter:

$A_p = 0.8$ $\qquad A_s = 0.2$

$\boxed{\text{Take } \frac{2}{T} = 1}$

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2}$$

$$= 0.325 \text{ r/s}$$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2}$$

$$= 1.376 \text{ r/s}$$

Step 2

To determine order N of the filter.

$$N = \frac{\log\left[\dfrac{\left(\frac{1}{A_s^2}-1\right)}{\left(\frac{1}{A_p^2}-1\right)}\right]}{2\log\dfrac{\Omega_s}{\Omega_p}}$$

$$= 1.3$$

Take $\boxed{N=2}$

Step 3 To determine Ω_c.

$$\Omega_{cp} = \frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{1/2N}}$$

$$\Omega_{cs} = \frac{\Omega_s}{\left(\frac{1}{A_s^2}-1\right)^{1/2N}}$$

$$\Omega_c = \frac{\Omega_{cp}+\Omega_{cs}}{2} = 0.5 \text{ r/s}$$

Step 4

finding System function of normalised LPF

$$P_k = \pm \Omega_c \, e^{j(2k+1+N)\pi/2N}$$

$N = 2 \qquad \Omega_c = 1$ r/s

$$P_0 = \pm(-0.707 + j0.707)$$

$$P_1 = \pm(-0.707 - j0.707)$$

$$H_1(s) = \frac{1}{(s-P_0)(s-P_1)} = \frac{1}{s^2 + 1.414s + 1}$$

Step 5

To find the required H(s) using frequency transformation

$$LP \rightarrow LP \quad ; \quad s \rightarrow \frac{\Omega_p}{\Omega_{Lp}} s = \frac{s}{0.5}$$

$$H(s) = H_1(s)\Big|_{s \rightarrow \frac{s}{0.5}} = \frac{0.25}{s^2 + 0.707s + 0.25}$$

Step 6

finding $H(z)$ by applying BLT.

$$s \rightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\frac{2}{T} = 1$$

$$s \rightarrow \left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = H(s)\Big/_{s=\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{0.25}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.707\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.25}$$

$$H(z) = \frac{0.127\left(1+2z^{-1}+z^{-2}\right)}{1-0.766z^{-1}+0.2772z^{-2}}$$

Prob 2:

A digital LPF is required to meet the following specifications:

a) An acceptable passband attenuation of -1.9328 dB

b) Passband edge frequency of $\omega_p = 0.2\pi$ rad

c) Stopband attenuation of -13.9794 dB or higher beyond 0.6π rad.

The filter must have maximally flat frequency response in the passband.

Find $H(z)$ using IIT:

Soln

$A_p = -1.9328$ dB $\omega_p = 0.2\pi$

$A_s = -13.9794$ dB $\omega_s = 0.6\pi$

Step 1: To obtain specifications of corresponding analog filter.

Take $T = 1$ sec.

$A_p = -1.9328$ dB

$A_s = -13.9794$ dB

$$\Omega = \frac{\omega}{T} \quad ; \quad T = 1 \text{ sec}$$

$$\Omega_p = \frac{\omega_p}{T} = 0.2\pi$$

$$\Omega_s = \frac{\omega_s}{T} = 0.6\pi$$

Step 2

To determine order N of the filter

$$N = \frac{\log\left[\dfrac{10^{0.1 A_s dB} - 1}{10^{0.1 A_p dB} - 1}\right]}{2 \log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$N = 1.7$$

Take $\boxed{N = 2}$

Step 3

To determine Ω_c

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_p}{\left[10^{0.1 A_p dB} - 1\right]^{1/2N}} + \frac{\Omega_s}{\left[10^{0.1 A_s dB} - 1\right]^{1/2N}}\right]$$

$$\Omega_c = 0.726 \text{ r/s}$$

Step 4

To determine the system function of normalized LPF

$$P_k = \pm \Omega_c \, e^{j(2k+1+N)\pi/2N}$$

$N = 2$; $\Omega_c = 1$ rad/s

$$P_0 = \pm(-0.707 + j0.707)$$

$$P_1 = \pm(-0.707 - j0.707)$$

$$H_1(s) = \frac{1}{(s-P_0)(s-P_1)} = \frac{1}{s^2 + 1.414s + 1}$$

Step 5

To find the required $H(s)$ using frequency transformation

LP → LP

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} s = \frac{s}{0.726}$$

$$H(s) = H_1(s)\Big/_{s=\frac{s}{0.726}} = \frac{0.526}{s^2+1.03s+0.526}$$

$$= \frac{1.026 \times 0.513}{(s+0.513)^2+(0.513)^2}$$

Step 6

finding $H(z)$ by applying IIT

$$\frac{b}{(s+a)^2+b^2} = \frac{e^{-aT}\sin bT z^{-1}}{1-2e^{-aT}\cos bTz^{-1}+e^{-2aT}z^{-2}}$$

$T = 1$ sec $\quad a = 0.513 \quad b = 0.513$

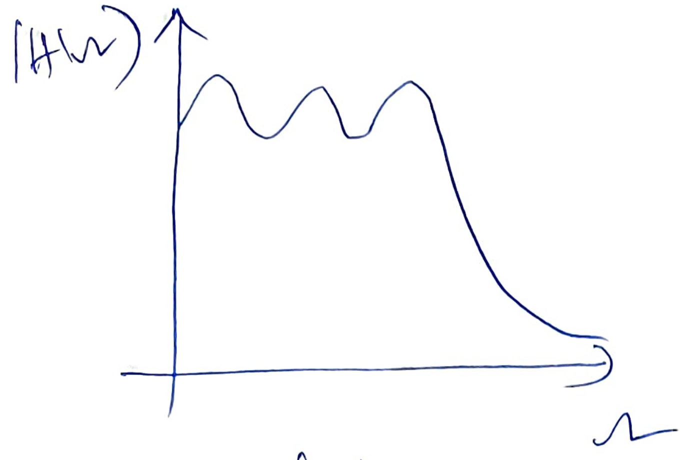
$$H(z) = \frac{0.302z^{-1}}{1-1.043z^{-1}+0.36z^{-2}}$$

Chebyshev filter design

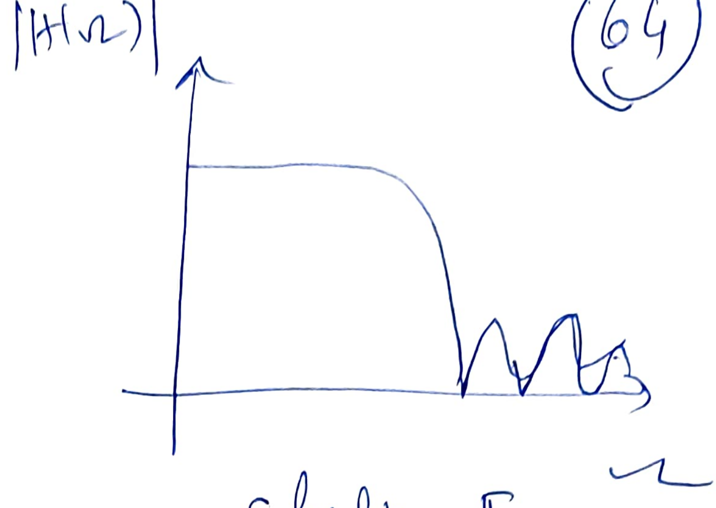
Two types
a) Cheby – I
b) Cheby – II

a) Cheby – I
- All pole filters
- usually called as chebyshev filters
- ripples in the pass band
- steeper roll off than Butterworth filters

b) Cheby – II
- pole – zero filter
- does not roll off so fast as Cheby – I
- usually called Inverse Chebyshev filter
- ripples in the stop band



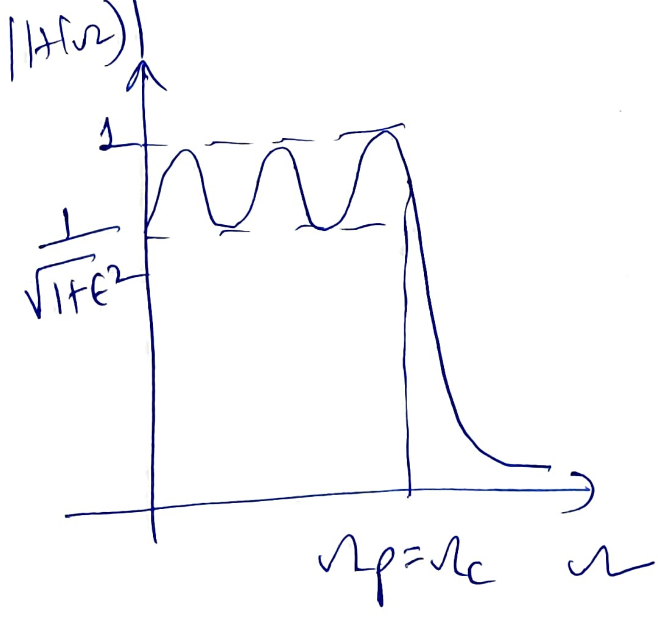
Cheby – I



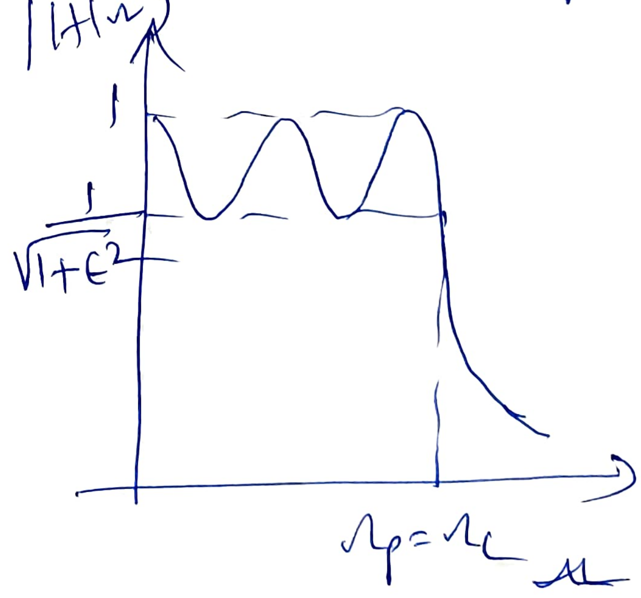
Cheby – II

Chebyshev – I filters

- These filters are all pole filters
- In the pass band, these filters show equiripple behaviour and have monotonic characteristic in the stopband



N odd



N Even

Comparison b/w Butterworth and Chebyshev filters.

a) frequency response
b) order
c) Transition band
d) phase response
e) poles of H(s) - location.

Chebyshev polynomials

$$C_N(\Omega) = \cos(N \cos^{-1}(\Omega)) \quad \text{for } |\Omega| \le 1$$
$$= \cosh(N \cosh^{-1}(\Omega)) \quad \text{for } |\Omega| > 1$$

for $N = 0 \Rightarrow C_0(\Omega) = \cos 0 = 1$

for $N = 1 \Rightarrow C_1(\Omega) = \cos[\cos^{-1}\Omega] = \Omega$

Higher order Chebyshev polynomials are obtained using the recursive formula

$$\boxed{C_N(\Omega) = 2\Omega\, C_{N-1}(\Omega) - C_{N-2}(\Omega)}$$

Prob:

Find Chebyshev polynomials for $N = 2, 3, 4, 5$.

Solution

$$C_N(x) = 2x\, C_{N-1}(x) - C_{N-2}(x)$$

$N = 2$ $\quad C_2(x) = 2x\, C_1(x) - C_0(x)$

$$C_2(x) = 2x^2 - 1$$

$N = 3$ $\quad C_3(x) = 2x\, C_2(x) - C_1(x)$

$$= 4x^3 - 3x$$

$N = 4$ $\quad C_4(x) = 8x^4 - 8x^2 + 1$

$N = 5$ $\quad C_5(x) = 16x^5 - 20x^3 + 5x$

Observation

for $x >> 1$, first term dominates

and $\boxed{C_N(x) = 2^{N-1} x^N}$

Magnitude function of Chebyshev filter is given by

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2(\Omega)} \quad \text{— (A)}$$

where ϵ = ripple factor

$C_N(\Omega)$ = Chebyshev polynomial of order N.

for normalized filter $\Omega_p = \Omega_c = 1$ r/s

Observations

a) for $|\Omega| \leq 1$,

ripple in the passband $= 1 - \dfrac{1}{\sqrt{1+\epsilon^2}}$

b) at $\Omega = 1$,

$C_N^2(\Omega) = 1$. $[C_N(1) = 1$ always$]$

at $\Omega = 1$, (A) can be written as

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2} \quad \text{or} \quad |H(\Omega)| = \frac{1}{\sqrt{1+\epsilon^2}}$$

$$\therefore \quad A_p = \frac{1}{\sqrt{1+\epsilon^2}}$$

or

$$\epsilon = \sqrt{\frac{1}{A_p^2} - 1} \quad \text{——} \; (B)$$

for $\Omega >> 1$, $\epsilon^2 C_N^2(\Omega) >> 1$

$$\therefore \quad |H(\Omega)|^2 = \frac{1}{\epsilon^2 C_N^2(\Omega)}$$

$$|H(\Omega)| = \frac{1}{\epsilon\, C_N(\Omega)}$$

$$|H(\Omega)| \text{ in dB} = 20 \log_{10} 1 - 20 \log_{10}(\epsilon\, C_N(\Omega))$$

$$= 0 - 20 \log_{10}\left[\epsilon\, 2^{N-1} \Omega^N\right]$$

$$= -20 \log \epsilon - 20\,(N-1) \log_{10} 2$$

$$- 20 N \log \Omega$$

$\Omega >> 1 \Rightarrow$ stop band. $\therefore \Omega = \Omega_s$

$\therefore |H(\Omega)|$ in dB $= -20\log_{10}\epsilon - 20(N-1)\log_{10}2$

$- 20N \log \Omega_s'$

$= -20\log_{10}\epsilon - 6(N-1)$

$- 20N \log_{10}\Omega_s'$

Ω_s' = Normalized stopband Edge freq.

<u>poles of H(s)</u>

$$S_k = \sigma_k + j\Omega_k$$

$$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\cos\left(\frac{2k-1}{2N}\right)\pi$$

$k = 1, 2, 3 \ldots N$

System function H(s) of Chebyshev filter

$$H(s) = \frac{k}{(s-s_1)(s-s_2)\cdots(s-s_N)}$$

$$= \frac{k}{s^N + b_{N-1} s^{N-1} + b_{N-2} s^{N-2} + \cdots + b_0}$$

Constant k is given by

$$k = \begin{cases} b_0 & \text{for } N \text{ odd} \\ \dfrac{b_0}{\sqrt{1+\epsilon^2}} & \text{for } N \text{ even} \end{cases}$$

Prob:

1) Design analog Chebyshev filter to meet the following specifications.

passband ripple 1 dB $0 \le \Omega \le 10$ r/s

Stopband attenuation -60 dB $\Omega \geq 50$ r/s

Solution

$\Omega_p = 10$ r/s $A_p = -1$ dB   71

$\Omega_s = 50$ r/s $A_s = -60$ dB

Step 1:

Normalised specifications.

$\Omega_p' = \frac{10}{10} = 1$ r/s $A_p = -1$ dB

$\Omega_s' = \frac{\Omega_s}{\Omega_p} = \frac{50}{10} = 5$ r/s $A_s = -60$ dB

Step 2

To determine ϵ.

$$\epsilon = \sqrt{\frac{1}{A_p^2} - 1} = \sqrt{10^{0.1 A_p dB} - 1}$$

$$= 0.508$$

Step 3

To determine order N

$$|H(\Omega)| \text{ in dB} = -20 \log_{10} \epsilon - 6(N-1) - 20N \log \Omega_s'$$

$$-60 = -20 \log(0.5) - 6(N-1) - 20N \log 5$$

$$\boxed{N = 3.9 = 4}$$

$$H_1(s) = \frac{K}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)}$$

$$s_1 = -0.139 + j0.983$$

$$s_2 = -0.337 + j0.407$$

$$s_3 = -0.337 - j0.407$$

$$s_4 = -0.139 - j0.983$$

$$H_1(s) = \frac{K}{(s+0.139-j0.983)(s+0.139+j0.983)(s+0.337-j0.407)(s+0.337+j0.407)}$$

$$= \frac{K}{\left[(s+0.139)^2+(0.983)^2\right]\left[(s+0.337)^2+(0.407)^2\right]}$$

$$H_1(s) = \frac{K}{(s^2+0.278s+0.985)(s^2+0.674s+0.279)}$$

$$b_0 = 0.985 \times 0.279$$

$$K = \frac{b_0}{\sqrt{1+\epsilon^2}} = \frac{0.274}{\sqrt{1+(0.508)^2}}$$

$$K = 0.244$$

$$H_1(s) = \frac{0.244}{(s^2+0.278s+0.985)(s^2+0.674s+0.279)}$$

Step 4

Applying frequency transformation

LP — LP

$$s \rightarrow \frac{\Omega_p}{\Omega_{LP}} s = \frac{s}{10}$$

$$H(s) = H_1(s)\Big/_{s=\frac{s}{10}} = \frac{0.244}{\left(\left(\frac{s}{10}\right)^2+0.278\times\frac{s}{10}+0.985\right)\left(\left(\frac{s}{10}\right)^2+0.674\times\frac{s}{10}\times 0.279\right)}$$

$$H(s) = \frac{0.244\times10^4}{(s^2+2.78s+98.5)(s^2+6.74s+27.9)}$$

Prob 2 Design an analog chebyshev filter to meet the following specification (73)

$A_p = 2.5\ dB$ $\Omega_p = 20\ r/s$

$A_s = 30\ dB$ $\Omega_s = 50\ r/s.$

Soln

$$\epsilon = \sqrt{10^{0.1 A_p} - 1} = 0.882$$

$$\Omega_p' = 1 \qquad \Omega_s' = \frac{50}{20} = 2.5\ r/s$$

$$N = 3.$$

Design of Digital Chebyshev filters

1) Design a digital low pass Chebyshev filter using Bilinear Transformation to meet the following specifications.

$$0.75 \le |H(\omega)| \le 1 \qquad 0 \le \omega \le 0.25\pi$$

$$|H(\omega)| \le 0.23 \qquad 0.63\pi \le \omega \le \pi$$

$\boxed{\text{BLT} \rightarrow \text{Assume } T = 2\text{ sec}}$

Soln

$A_p = 0.75 \qquad \omega_p = 0.25\pi$ rad

$A_s = 0.23 \qquad \omega_s = 0.63\pi$ rad

Step 1

To obtain specifications of corresponding analog filter

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 0.413 \text{ r/s}$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 1.52 \text{ r/s}$$

$$A_p \text{ in dB} = -20\log_{10}A_p = 2.5 \text{ dB}$$

$$A_s \text{ in dB} = -20\log_{10}A_s = 12.76 \text{ dB}$$

Step 2 To find Normalized frequency values (11)

$$\Omega_p' = \frac{0.413}{0.413} = 1 \text{ r/s}$$

$$\Omega_s' = \frac{1.52}{0.413} = 3.68 \text{ r/s}$$

Step 3

To find ϵ

$$\epsilon = \left[10^{0.1 A_p dB} - 1\right]^{1/2}$$

$$= 0.88$$

Step 4 To find order N

$$|H(\Omega)| \text{ in dB} = -20 \log_{10} \epsilon - 6(N-1) - 20 N \log \Omega_s'$$

$$\Rightarrow N = 1.146$$

$$\boxed{N = 2}$$

Step 3 To find $H_1(s)$

$$s_k = \sigma_k + j\Omega_k$$

$$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = +\cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\cos\left(\frac{2k-1}{2N}\right)\pi$$

$$s_1 = -0.357 + j0.792$$
$$s_2 = -0.357 - j0.792$$

$$H_1(s) = \frac{k}{(s-s_1)(s-s_2)}$$

$$= \frac{k}{(s+0.357+j0.792)(s+0.357-j0.792)}$$

$$= \frac{k}{\left[(s+0.357)^2+(0.792)^2\right]} = \frac{k}{s^2+0.714s+0.754}$$

$$k = \frac{b_0}{\sqrt{1+\epsilon^2}} = \frac{0.754}{\sqrt{1+(0.88)^2}} = 0.566$$

$$H_1(s) = \frac{0.566}{s^2 + 0.7145 + 0.754}$$

Step 4 Apply frequency transformation

$$s \rightarrow \frac{\Omega_p}{\Omega_{Lp}} s = \frac{s}{0.413}$$

$$H(s) = \frac{0.566}{\left(\frac{s}{0.413}\right)^2 + 0.714 \times \left(\frac{s}{0.413}\right) + 0.754}$$

$$= \frac{0.096}{s^2 + 0.293 s + 0.128}$$

Step 5 To find $H(z)$

Mapping from s to z is

$$s \rightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\frac{2}{T} = 1$$

$$\therefore s = \left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\therefore H(z) = H(s) \Big/_{s = \frac{1-z^{-1}}{1+z^{-1}}} = \frac{0.096(1+z^{-1})^2}{(1-z^{-1})^2 + 0.293(1-z^{-2}) + 0.128(1+z^{-1})^2}$$

$$\therefore H(z) = \frac{0.096\,(1+z^{-1})^2}{(1+z^{-2}-2z^{-1}) + 0.293 - 0.293z^{-2} + 0.128\,(1+z^{-2}+2z^{-1})}$$

$$= \frac{0.096\,(1+z^{-1})^2}{1.421 - 1.744z^{-1} + 0.835z^{-2}}$$

$$H(z) = \frac{0.063\,(1+z^{-1})^2}{1 - 1.22z^{-1} + 0.587z^{-2}}$$

Prob

2) Design a low pass Chebyshev filter using Impulse Invariance transformation for satisfying the following constraints.

$\omega_p = 0.162$ rad $\quad \omega_s = 1.63$ rad

pass band ripple = 3 dB

stop band attenuation = 30 dB.

$\boxed{\text{IIT} \rightarrow \text{Assume } T = 1\,\text{sec}}$

Soln $\omega_p = 0.162$ rad $\quad A_p = -3\,dB$ (79)

$\omega_s = 1.63$ rad $\quad A_s = -30\,dB$

Step 1 To obtain specifications of equivalent analog filter

$$\Omega_p = \frac{\omega_p}{T} = 0.162 \text{ r/s}$$

$$\Omega_s = \frac{\omega_s}{T} = 1.63 \text{ r/s}$$

$$A_p = -3\,dB, \quad A_s = -30\,dB$$

Step 2 Normalizing frequency values

$$\Omega_p' = \frac{0.162}{0.162} = 1 \text{ r/s}$$

$$\Omega_s' = \frac{1.63}{0.162} = 10.06 \text{ r/s}$$

Step 3 To find ϵ

$$\epsilon = \left[10^{0.1 A_{p\,dB}} - 1\right]^{1/2} = 0.997$$

Step 4 To find N

$$|H(\Omega)| \text{ in dB} = -20 \log_{10} \epsilon - 6(N-1) - 20N \log_{10} \Omega_s'$$

$$N \geq 1.38$$

$$\boxed{N = 2}$$

Step 5 To find $H_1(s)$

$$H_1(s) = \frac{K}{(s-s_1)(s-s_2)}$$

$$s_k = \sigma_k + j\Omega_k$$

$$\sigma_k = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\sin\left(\frac{2k-1}{2N}\right)\pi$$

$$\Omega_k = \cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\cos\left(\frac{2k-1}{2N}\right)\pi$$

$$s_1 = -0.322 + j\,0.777$$

$$s_2 = -0.322 - j\,0.777$$

$$H_1(s) = \frac{k}{(s+0.322-j0.777)(s+0.322+j0.777)}$$

$$= \frac{k}{\left[(s+0.322)^2+(0.777)^2\right]}$$

$$= \frac{k}{s^2+0.644s+0.707}$$

$$k = \frac{b_0}{\sqrt{1+\epsilon^2}} = \frac{0.707}{\sqrt{1+(0.997)^2}}$$

$$k = 0.5$$

$$\therefore H_1(s) = \frac{0.5}{s^2+0.644s+0.707}$$

<u>Step 6</u> Applying frequency transformation

$LP \to LP$

$$s \to \frac{\Omega_p}{\Omega_{LP}} s = \frac{s}{0.162}$$

$$H(s) = H_1(s) \Big/_{s = \frac{s}{0.162}}$$

82

$$H(s) = \frac{0.5}{\left(\frac{s}{0.162}\right)^2 + 0.644\left(\frac{s}{0.162}\right) + 0.707}$$

$$= \frac{0.013}{s^2 + 0.104\,s + 0.018} = \frac{0.052 \times 0.25}{\left[(s+0.52)^2 + (0.25)^2\right]}$$

<u>Step 7</u> To find $H(z)$ using IIT

Mapping from $s \to z$

$$\frac{b}{(s+a)^2 + b^2} = \frac{e^{-aT} \sin bT \, z^{-1}}{1 - 2e^{-aT} \cos bT \, z^{-1} + e^{-2aT} z^{-2}}$$

$$s^2 + 0.104s + 0.018 = (s + 0.52 - j0.25)(s + 0.52 + j0.25)$$

$$= \left[(s+0.52)^2 + (0.25)^2\right]$$

$a = 0.52, \quad b = 0.25, \quad T = 1$

$$\therefore H(z) = \frac{0.594 \times 0.25 \, z^{-1}}{1 - 2 \times 0.594 \times 0.97 \, z^{-1} + 0.35 \, z^{-2}}$$

$$H(z) = \frac{0.15 z^{-1}}{1 - 1.15 z^{-1} + 0.35 z^{-2}}$$