(1) Is the vector $(3,-1,0,-1)$ in the subspace spanned by $(2,-1,3,2)$, $(-1,1,1,-3)$ & $(1,1,9,-5)$?

Solution:

$\alpha_1 = (2,-1,3,2)$

$\alpha_2 = (-1,1,1,-3)$ $\quad\quad \alpha = (3,-1,0,-1)$.

$\alpha_3 = (1,1,9,-5)$.

Let $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \alpha$

If c_1, c_2 and c_3 exists and are unique, we can say that the vector α is in the subspace spanned by $\alpha_1, \alpha_2, \alpha_3$.

$\therefore$ $c_1(2,-1,3,2) + c_2(-1,1,1,-3) + c_3(1,1,9,-5) = (3,-1,0,-1)$

$$2c_1 - c_2 + c_3 = 3.$$
$$-c_1 + c_2 + c_3 = -1$$
$$3c_1 + c_2 + 9c_3 = 0.$$
$$2c_1 - 3c_2 - 5c_3 = -1.$$

$$\Longrightarrow \begin{bmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix}.$$

Finding Row reduced matrix

$$R_1 \to \frac{R_1}{2} = \begin{bmatrix} 1 & -1/2 & 1/2 & 3/2 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix} \quad R_2 \to R_1 + R_2 = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix}$$

$$-3R_1 + R_3 = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{5}{2} & \frac{15}{2} & \frac{-9}{2} \\ 2 & -3 & -5 & -1 \end{bmatrix} \quad -2R_1 + R_4 = \begin{bmatrix} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1/2 & 3/2 & 1/2 \\ 0 & 5/2 & \frac{15}{2} & \frac{-9}{2} \\ 0 & -2 & -6 & -4 \end{bmatrix}$$

$$R_2 \to 2R_2 = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 3 & 1 \\ 0 & 5/2 & 15/2 & -9/2 \\ 0 & -2 & -6 & -4 \end{bmatrix} \quad R_3 \to \frac{-5R_2}{2} + R_3 = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -7 \\ 0 & -2 & -6 & -4 \end{bmatrix}$$

$$R_4 \to 2R_2 + R_4 = \begin{bmatrix} 1 & -\frac{1}{2} & 1/2 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad R_3 \to \frac{-R_3}{7} = \begin{bmatrix} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$R_4 \to 2R_3 + R_4 = \begin{bmatrix} 1 & -1/2 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to -R_3 + R_2 \\ R_1 \to \frac{-3R_3}{2} + R_1 \end{matrix} = \begin{bmatrix} 1 & -1/2 & 1/2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to \frac{R_2}{2} + R_1 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & \boxed{0} & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

C_1 and C_2 are basic variable where as C_3 is Free variable.

$\therefore$ Infinitely many solutions.

$\therefore$ α is not in the subspace spanned by α_1, α_2 and α_3.

(2) Are the vectors linearly independant in R^4? Find the basis for the subspace of R^4 spanned by 4 vectors.

$\alpha_1 = (1, 1, 2, 4)$ $\alpha_2 = (2, -1, -5, 2)$

$\alpha_3 = (1, -1, -4, 0)$ $\alpha_4 = (2, 1, 1, 6)$.

Sol: $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are said to be linearly independant if the reduced row echelon form of the matrix $[\alpha_1 \alpha_2 \alpha_3 \alpha_4]$ has only basic variables and no free variables.

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix} \quad R_2 \to -2R_1 + R_2 = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$$

$$R_3 \to -R_1 + R_3 = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 2 & 1 & 1 & 6 \end{bmatrix} \quad R_4 \to -2R_1 + R_4 = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

$$R_2 \to -\frac{R_2}{3} = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix} \quad R_3 \to 2R_2 + R_3 \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

$$R_4 \to R_2 + R_4 = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to -R_2 + R_1 = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Two basic variables, two free.

$\therefore \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are not linearly independent.

α_1 and α_2 are the vectors with basic variables.

$\therefore \alpha_1$ & α_2 form the basis for the vector space R^4.

$$\text{Basis} = \left(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} \right)$$

(3) Let V be the vector space of polynomials of degree 3 over R.
Determine whether u, v & w are linearly independent or not.

$u = t^3 - 3t^2 + 5t + 1$

$v = t^3 - t^2 + 8t + 2$

$w = 2t^3 - 4t^2 + 9t + 5$

$$\Longrightarrow \begin{bmatrix} 1 & -3 & 5 & 1 \\ 1 & -1 & 8 & 2 \\ 2 & -4 & 9 & 5 \end{bmatrix}$$

$$R_2 \to -R_1 + R_2 = \begin{bmatrix} 1 & -3 & 5 & 1 \\ 0 & 2 & 3 & 1 \\ 2 & -4 & 9 & 5 \end{bmatrix} \quad R_3 \to -2R_1 + R_3 \begin{bmatrix} 1 & -3 & 5 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & -1 & 3 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2/2 \\ R_3 \to -2R_2 + R_3 \end{matrix} = \begin{bmatrix} 1 & -3 & 5 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & -4 & 2 \end{bmatrix} \quad \begin{matrix} R_3 \to -\frac{R_3}{4} \\ R_2 \to -3\frac{R_3}{2} + R_2 \end{matrix} = \begin{bmatrix} 1 & -3 & 5 & 1 \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

$$-5R_3 + 3R_2 + 2R_1 \to R_1 = \begin{bmatrix} 1 & 0 & 0 & \frac{29}{4} \\ 0 & 1 & 0 & 5/4 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

3 basic variables.

$\therefore$ u, v & w are linearly Independent.