Singular Value Decomposition

(A^TA) is symmetric

$(A^TA)^T = A^{TT}A^T = A^TA$

So $A_{n \times p} = U \, \Sigma_{n \times p} \, V^T_{p \times p}$

$U \rightarrow$ ~~orthogonal~~ matrix $\rightarrow$ orthonormal Matrix

$\Sigma \rightarrow$ diagonal matrix

$V \rightarrow$ Eigen vector matrix. with normalised Column.

(1) Find the 'SVD of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$A^TA \rightarrow$ characteristic value or eigen value

$$|A^TA - \lambda I| = |A^TA - xI| \quad \begin{vmatrix} 9-x & -9 \\ -9 & 9-x \end{vmatrix} = 0$$

$$(9-x)^2 - 81 = 0$$

$$x(x-18) = 0$$

$$x = 0, 18$$

Arrange eigen values in descending order

$\lambda_1 = 18 \quad \lambda_2 = 0$

$\therefore$ Singular values are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{18} = 3\sqrt{2}$ (only +ve value)

$\sigma_2 = \sqrt{\lambda_2} = 0$

Order of $A \Rightarrow 3 \times 2$

Order of Diagonal matrix Σ = Order of A.

$= 3 \times 2$

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \quad D \rightarrow \text{diagonal matrix.}$$

$$\therefore \Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}_{3 \times 2}$$

$$\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$V^T \rightarrow \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$

$\boxed{\lambda_1 = 18}$

$$(A^T A - 18I)X = 0$$

$$\left(\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} - \begin{bmatrix} 18 & 0 \\ 0 & 18 \end{bmatrix} \right) X = 0$$

$$\begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 = -x_2 \qquad \therefore v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Normalize $\quad v = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\boxed{\lambda_2 = 0}$ $\quad (A^TA - 0I)X = 0$

$$A^TAX = 0$$

$$\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 = x_2$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Normalize $v_2 \Rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\therefore V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\boxed{V^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

$$u_1 = \frac{Av_1}{\|Av_1\|} = \frac{1}{\sigma_1} A v_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} +1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$= \frac{1}{3\sqrt{2}} \begin{bmatrix} -2/\sqrt{2} \\ +4/\sqrt{2} \\ -4/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/3 \\ +2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

if $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

$u_2 = \frac{Av_2}{\|Av_2\|} = \frac{1}{\sigma_2} Av_2$ 'Not possible $\sigma_2 = 0$ (17)

Consider $\langle u_1, x\rangle = 0 \rightarrow$ to be orthogonal

$x = (x_1, x_2, x_3)$

$\langle u_1, x\rangle$

$-\frac{1}{3}x_1 + \frac{2x_2}{3} - \frac{2}{3}x_3 = 0$

$x_1 - 2x_2 + 2x_3 = 0$

$x_1 = -2x_3 + 2x_2$

$x_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $x_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

x_2 & x_3 are not orthogonal

$\therefore$ Consider x_2 as one vector. u_2

$u_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ & then orthogonalize this vector using Gram Schmitt's process.

$u_3 = x_3 - \frac{\langle x_3, u_2\rangle}{\|u_2\|^2} u_2 = (-2,0,1) + \frac{4}{5}(2,1,0)$

$\langle x_3, u_2\rangle = -4$

$= \begin{pmatrix} -2/5 \\ 4/5 \\ 1 \end{pmatrix}$

Normalize u_2 & u_3

$u_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}$

$u_3 = \begin{pmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{pmatrix}$

$$u = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & \frac{2}{\sqrt{5}} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{pmatrix}$$

$$A = \underset{3\times3}{U} \; \underset{3\times2}{\Sigma} \; \underset{2\times2}{V^T}$$

$$= \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ +2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

② $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

$\lambda_1 = 25 \quad \lambda_2 = 9 \quad \lambda_3 = 0$

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & -1/3 \end{bmatrix}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & 1/\sqrt{18} & -1/3 \end{bmatrix}$$