EC540 Control Systems
Mathematical Modeling of Mechanical Systems

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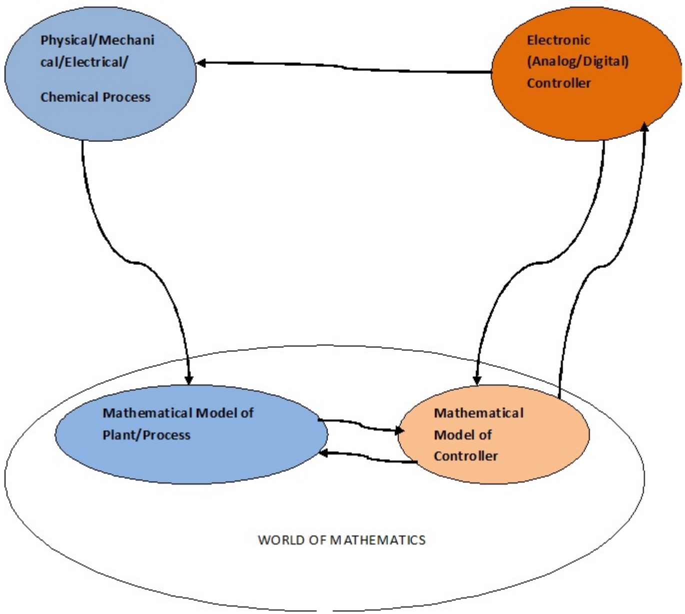
Instructions:

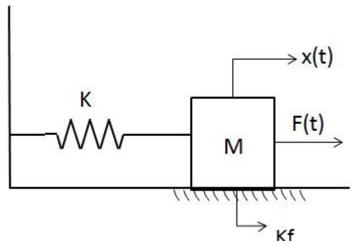
1. Lecture session will be of one hour duration.
2. Fifteen minutes session will be made available for students after the lecture for any question and answers.
3. Regularly review Signals and Systems concepts
4. Regularly visit course webpage.
5. Everyday learn new functions from Octave/Python/MATLAB software
6. Email me on any queries at sudarshan_pk@sjce.ac.in

Why Mathematical Modeling?:

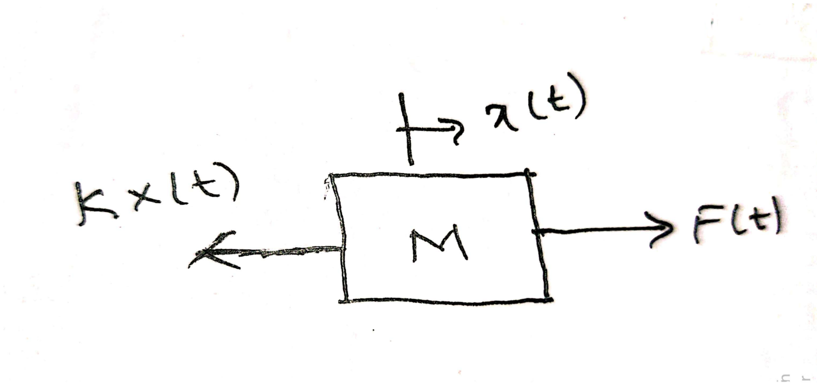


Mathematical Modeling of Mechanical System

Example 1 Consider the mass-spring system on a surface which has no friction; $K_f = 0$; Write the differential equation modeling the behaviour of the system with position of mass as output; $y(t) = x(t)$.



Solution: For mechanical systems involving force and torque First write free body diagram:



Apply Newton's second law of motion :

Applied force − Opposing force = Mass × Acceleration

$$F(t) - Kx(t) = M \times \frac{d^2x(t)}{dt^2}$$

Since ouput is position: $y(t) = x(t)$

$$u(t) - Ky(t) = M \times \frac{d^2y(t)}{dt^2}$$

Applying Laplace transform to above differential equation:

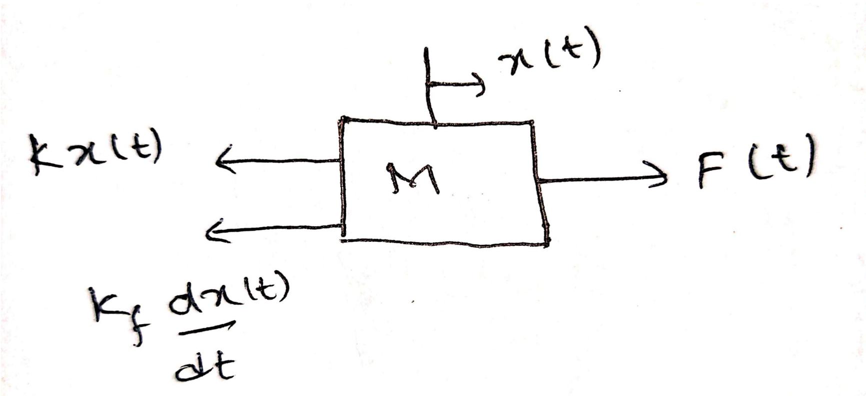
$$\mathcal{L}\{u(t)\} - K\mathcal{L}\{y(t)\} = M \times \mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\}$$

$$U(s) - KY(s) = M[s^2Y(s) - sy(0) - y'(0)]$$

Assuming zero intial conditions:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ms^2 + K}$$

When friction coefficient is Nonzero: $K_f \neq 0$
First write free body diagram:



Apply Newton's second law of motion :

Applied force $-$ Opposing force $=$ Mass $\times$ Acceleration

$$F(t) - \left\{ K_f \frac{dx(t)}{dt} + Kx(t) \right\} = M \times \frac{d^2x(t)}{dt^2}$$

Since ouput is position: $y(t) = x(t)$

$$u(t) - K_f \frac{dy(t)}{dt} - Ky(t) = M \times \frac{d^2y(t)}{dt^2}$$

Applying Laplace transform to above differential equation:

$$\mathcal{L}\{u(t)\} - K_f \mathcal{L}\left\{\frac{dy(t)}{dt}\right\} - K\mathcal{L}\{y(t)\} = M \times \mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\}$$

$$U(s) - K_f[sY(s) - y(0)] - KY(s) = M[s^2Y(s) - sy(0) - y'(0)]$$

Assuming zero intial conditions:

$$U(s) = Ms^2Y(s) + K_f sY(s) + KY(s)$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ms^2 + K_f s + K}}$$

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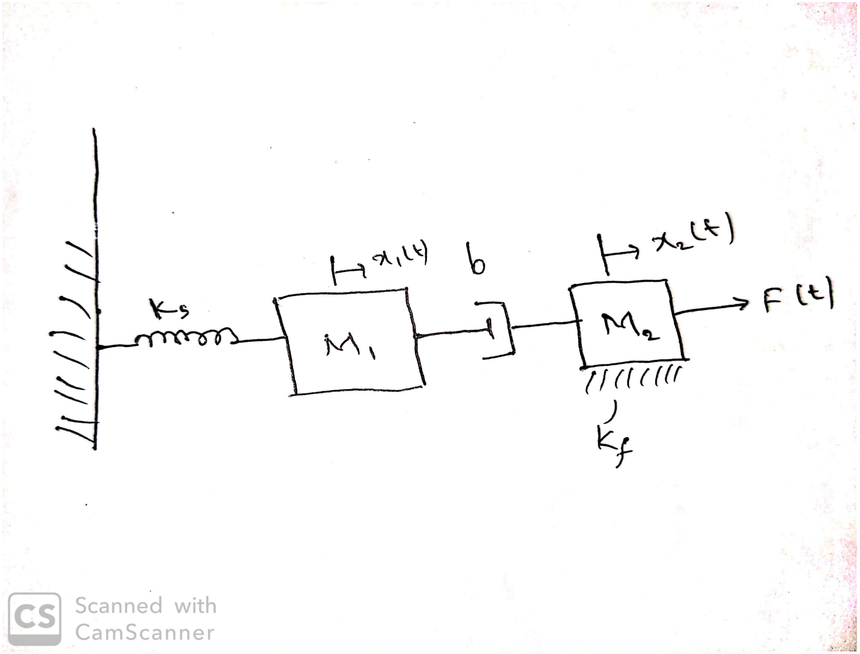
Impulse Response Example $M = 10Kg$, $K_f = 40$, $K_s = 1000$

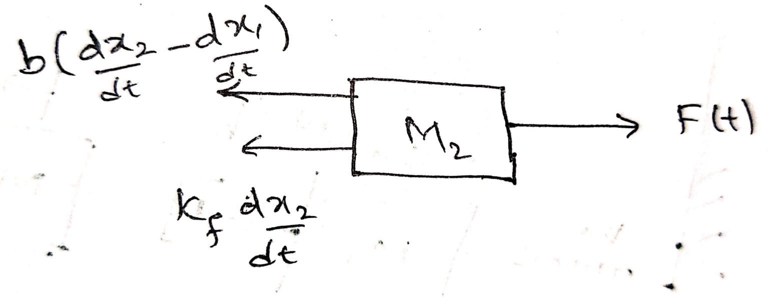
In Octave or MATLAB

G=tf([1],[M Kf K]) impulse(G)

Example of Two Mass System:



Write Free body diagram of mass M_2:

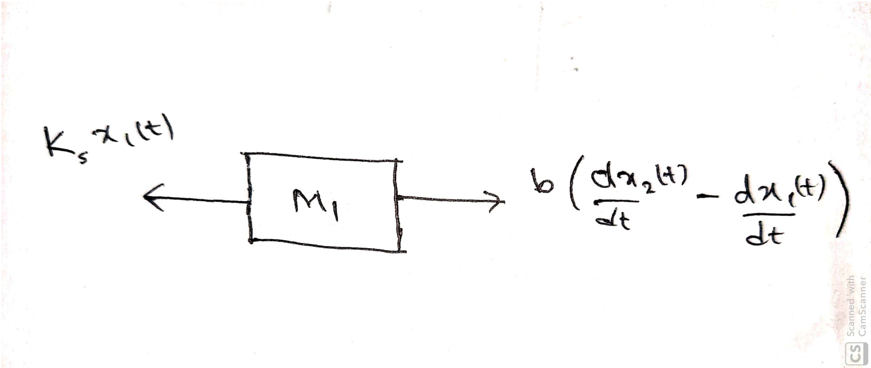


Apply Newton's second law of motion for M_2 :

Applied force $-$ Opposing force $=$ Mass $\times$ Acceleration

$$F(t) - \left\{ b\left(\frac{dx_2(t)}{dt} - \frac{dx_1(t)}{dt}\right) + K_f \frac{dx_2(t)}{dt} \right\} = M_2 \frac{d^2x_2(t)}{dt^2}$$

Write Free body diagram of mass M_1:



$$\left\{b\left(\frac{dx_2(t)}{dt} - \frac{dx_1(t)}{dt}\right)\right\} - K_s x_1(t) = M_1 \frac{d^2 x_1(t)}{dt^2}$$

Apply Laplace transforms to above two differential equations:

$$\mathcal{L}\{F(t)\}$$
$$-\left\{b\left(\mathcal{L}\left\{\frac{dx_2(t)}{dt}\right\}-\mathcal{L}\left\{\frac{dx_1(t)}{dt}\right\}\right)+K_f\mathcal{L}\left\{\frac{dx_2(t)}{dt}\right\}\right\}$$
$$=M_2\mathcal{L}\left\{\frac{d^2x_2(t)}{dt^2}\right\}$$

$$F(s)-b(sX_2(s)-sX_1(s))-K_f sX_2(s)=M_2 s^2 X_2(s)$$

$$\left\{ b \left(\mathcal{L} \left\{ \frac{dx_2(t)}{dt} \right\} - \mathcal{L} \left\{ \frac{dx_1(t)}{dt} \right\} \right) \right\} - K_s \mathcal{L} \left\{ x_1(t) \right\}$$

$$= M_1 \mathcal{L} \left\{ \frac{d^2 x_1(t)}{dt^2} \right\}$$

$$bsX_2(s) - bsX_1(s) - K_s X_1(s) = M_1 s^2 X_1(s)$$

When $y(t) = x_1(t)$ i.e. $Y(s) = X_1(s)$

Then we need to eliminate $X_2(s)$

Get an expression for $X_2(s)$ in terms of $X_1(s)$

$$\boxed{X_2(s) = \frac{M_1 s^2 + bs + K_s}{bs} X_1(s)}$$

Substitute for $X_2(s)$ and get equation only interms of $U(s)$ and $Y(s)$

$$U(s) + bsX_1(s) =$$

$$\{M_2s^2 + K_fs + bs\}\left\{\frac{M_1s^2 + bs + K_s}{bs}\right\}X_1(s)$$

$$U(s) =$$

$$\{M_2s^2 + K_fs + bs\}\left\{\frac{M_1s^2 + bs + K_s}{bs}\right\}X_1(s) - bsX_1(s)$$

$$U(s) = \left\{\frac{(M_2s^2 + K_fs + bs)(M_1s^2 + bs + K_s) - b^2s^2}{bs}\right\}X_1(s)$$

Since $X_1(s) = Y(s)$

$$G(s) = \frac{Y(s)}{U(s)}$$

$$= \frac{bs}{M_1M_2s^4 + s^3(bM_1 + bM_2 + K_fM_1) + s^2(M_2K_s + K_fb) + s(K_fK_s + bK_s)}$$

When $Y(s) = X_2(s)$ we need to eliminate $X_1(s)$

Therefore express $X_1(s)$ in terms of $X_2(s)$

$$X_1(s) = \frac{bs}{M_1 s^2 + bs + K_s} X_2(s)$$

$$U(s) + bs\left\{\frac{bs}{M_1 s^2 + bs + K_s}\right\} Y(s) = \{M_2 s^2 + K_f s + bs\} Y(s)$$

$$U(s) = Y(s)\left\{(M_2 s^2 + K_f s + bs) - \frac{b^2 s^2}{M_1 s^2 + bs + K_s}\right\}$$

Therefore $G(s) = \frac{Y(s)}{U(s)} =$

$$\frac{M_1 s^2 + bs + K_s}{M_1 M_2 s^4 + s^3(M_1 K_f + M_1 b + M_2 b) + s^2(K_f b + K_s M_2) + s(K_f K_s + bK_s)}$$