Lesson 8
Gauss Jordan Elimination

Serial and Parallel algorithms

Linear Systems

- A finite **set** of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a system of linear equations or a linear system .

- A sequence of numbers $s_1, s_2, \ldots, s_n$ that satisfies the system of equations is called a solution of the system.

- A system that has **no** solution is said to be inconsistent ; if there is **at least** one solution of the system, it is called consistent.

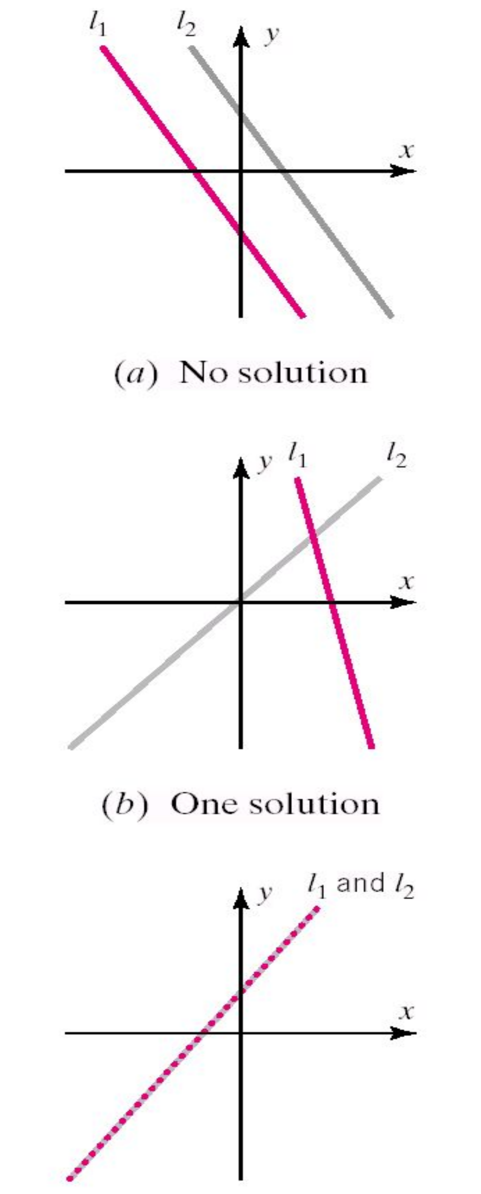
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

↑ An arbitrary system of *m* linear equations in *n* unknowns

Solutions

- *Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.*

- A general system of two linear equations:
  (Figure1.1.1) $$a_1x + b_1y = c_1 \ (a_1, b_1 \text{ not both zero})$$
  $$a_2x + b_2y = c_2 \ (a_2, b_2 \text{ not both zero})$$
  - Two lines may be parallel -> <u>no solution</u>
  - Two lines may intersect at only one point
    -> <u>one solution</u>
  - Two lines may coincide
    -> <u>infinitely many solution</u>



(*a*) No solution

(*b*) One solution

(*c*) Infinitely many solutions

Figure 1.1.1

Systems of Linear Equations

- Systems of linear algebraic equations may represent too much, or too little or just the right amount of information to determine values of the variables constituting solutions.
- Using Gauss-Jordan elimination we can determine whether the system has many solutions, a unique solution or none at all.

Augmented Matrices

- The location of the +'s, the x's, and the ='s can be abbreviated by writing only the rectangular array of numbers.
- This is called the augmented matrix for the system.
- Note: must be written in the same order in each equation as the unknowns and the constants must be on the right.

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + \ldots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + \ldots & + & a_{mn}x_n & = & b_m
\end{array}$$

$$\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\
a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn} & b_m
\end{bmatrix}$$

Elementary Row Operations

- The basic method for solving a system of linear equations is to replace the given system by **a new system that has the same solution set** but which is **easier** to solve.

- Since the **rows** of an augmented matrix correspond to the **equations** in the associated system, the new systems is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns systematically. These are called elementary row operations.
  1. Multiply an equation through by an nonzero constant.
  2. Interchange two equation.
  3. Add a multiple of one equation to another.

Example 1
Using Elementary row Operations(1/4)

$$\begin{aligned} x + \quad y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \xrightarrow{\text{add -2 times the first equation to the second}} \begin{aligned} x + \quad y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned} \xrightarrow{\text{add -3 times the first equation to the third}}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix} \xrightarrow{\text{add -2 times the first row to the second}} \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix} \xrightarrow{\text{add -3 times the first row to the third}}$$

Example 1
Using Elementary row Operations(2/4)

$$\begin{aligned} x + \quad y + \ 2z &= \quad 9 \\ 2y - \ 7z &= -17 \\ 3y - 11z &= -27 \end{aligned} \xrightarrow{\text{multiply the second equation by } \frac{1}{2}} \begin{aligned} x + y + \ 2z &= \quad 9 \\ y - \tfrac{7}{2} z &= -\tfrac{17}{2} \\ 3y - 11z &= \quad 0 \end{aligned} \xrightarrow{\text{add -3 times the second equation to the third}}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix} \xrightarrow{\text{multily the second row by } \frac{1}{2}} \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix} \xrightarrow{\text{add -3 times the second row to the third}}$$

Example 1
Using Elementary row Operations(3/4)

$$\begin{aligned} x+y+2z &= 9 \\ y-\tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned} \xrightarrow{\text{Multiply the third equation by -2}} \begin{aligned} x+y+2z &= 9 \\ y-\tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \xrightarrow{\text{Add -1 times the second equation to the first}}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \xrightarrow{\text{Multily the third row by -2}} \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{\text{Add -1 times the second row to the first}}$$

Example 1
Using Elementary row Operations(4/4)

$$\begin{aligned} x \qquad + \tfrac{11}{2} z &= \tfrac{35}{2} \\ y - \tfrac{7}{2} z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \xrightarrow{\substack{\text{Add } -\frac{11}{2} \text{ times} \\ \text{the third equation} \\ \text{to the first and } \frac{7}{2} \text{ times} \\ \text{the third equation} \\ \text{to the second}}} \begin{aligned} x \qquad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{\substack{\text{Add } -\frac{11}{2} \text{ times} \\ \text{the third row} \\ \text{to the first and } \frac{7}{2} \\ \text{times the third row} \\ \text{to the second}}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- The solution x=1,y=2,z=3 is now evident.

Echelon Forms

- A matrix with the following properties is in reduced row-echelon form, (RREF).

1. If a row does not consist entirely of zeros, then the first nonzero number in the row, called its pivot, equals 1.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the pivot in the lower row occurs farther to the right than the pivot in the higher row.
4. Each column that contains a pivot has zeros everywhere else.

- A matrix that has the first three properties is said to be in row-echelon form.
- A matrix in reduced row-echelon form is of necessity in row-echelon form, but not conversely.

Row-Echelon & Reduced Row-Echelon form

- reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- row-echelon form:

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

More on Row-Echelon and Reduced Row-Echelon form

- All matrices of the following types are in **row-echelon form** (any real numbers substituted for the *'s.) :

$$\begin{bmatrix}1&*&*&*\\0&1&*&*\\0&0&1&*\\0&0&0&1\end{bmatrix},\begin{bmatrix}1&*&*&*\\0&1&*&*\\0&0&1&*\\0&0&0&0\end{bmatrix},\begin{bmatrix}1&*&*&*\\0&1&*&*\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}0&1&*&*&*&*&*&*&*&*\\0&0&0&1&*&*&*&*&*&*\\0&0&0&0&1&*&*&*&*&*\\0&0&0&0&0&1&*&*&*&*\\0&0&0&0&0&0&0&0&1&*\end{bmatrix}$$

- All matrices of the following types are in **reduced row-echelon form** (any real numbers substituted for the *'s.) :

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\begin{bmatrix}1&0&0&*\\0&1&0&*\\0&0&1&*\\0&0&0&0\end{bmatrix},\begin{bmatrix}1&0&*&*\\0&1&*&*\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}0&1&*&0&0&0&*&*&0&*\\0&0&0&1&0&0&*&*&0&*\\0&0&0&0&1&0&*&*&0&*\\0&0&0&0&0&1&*&*&0&*\\0&0&0&0&0&0&0&0&1&*\end{bmatrix}$$

Example 2(a)

Suppose that the augmented matrix for a system of linear equations have been reduced by row operations to the given reduced row-echelon form. Solve the system.

$$(a)\ \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Solution

the corresponding system of equations is : $\longrightarrow$

$$\begin{aligned} x \qquad\qquad &= 5 \\ y \qquad &= -2 \\ z &= 4 \end{aligned}$$

Example 2 (b1)

$$\text{(b)}\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

Solution

1. The corresponding system of equations is :

free variables

$$\begin{aligned} x_1 \qquad\qquad + 4x_4 &= -1 \\ x_2 \qquad + 2x_4 &= 6 \\ x_3 + 3x_4 &= 2 \end{aligned}$$

leading variables

Example 2 (b2)

$$x_1 = -1 - 4x_4$$
$$x_2 = 6 - 2x_4$$
$$x_3 = 2 - 3x_4$$

2. We see that the free variable can be assigned an arbitrary value, say t, which then determines values of the leading variables.

3. There are infinitely many solutions, and the general solution is given by the formulas

$$x_1 = -1 - 4t,$$
$$x_2 = 6 - 2t,$$
$$x_3 = 2 - 3t,$$
$$x_4 = t$$

Example 2 (c1)

$$\text{(c)} \begin{bmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

1. The 4th row of zeros leads to the equation places no restrictions on the solutions (why?). Thus, we can omit this equation. $\longrightarrow$

$$\begin{aligned} x_1 + 6x_2 \qquad\qquad + 4x_5 &= -2 \\ x_3 \qquad + 3x_5 &= 1 \\ x_4 + 5x_5 &= 2 \end{aligned}$$

Example 2 (c2)

Solution

2. Solving for the leading variables in terms of the free variables: $\longrightarrow$

$$x_1 = -2 - 6x_2 - 4x_5$$
$$x_3 = 1 - 3x_5$$
$$x_4 = 2 - 5x_5$$

3. The free variable can be assigned an arbitrary value,there are infinitely many solutions, and the general solution is given by the formulas. $\longrightarrow$

$$x_1 = -2 - 6s - 4t \ ,$$
$$x_2 = s$$
$$x_3 = 1 - 3t$$
$$x_4 = 2 - 5t,$$
$$x_4 = t$$

Example 2 (d)

$$\text{(d)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

the last equation in the corresponding system of equation is

$$0x_1 + 0x_2 + 0x_3 = 1$$

Since this equation cannot be satisfied, there is **no solution** to the system.

Elimination Methods (1/7)

- We shall give a step-by-step elimination procedure that can be used to reduce any matrix to reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Elimination Methods (2/7)

- Step1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

- Step2. Interchange the top row with another row, to bring a nonzero entry to top of the column found in Step1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1st and 2nd rows in the preceding matrix were interchanged.

Elimination Methods (3/7)

- Step3. If the entry that is now at the top of the column found in Step1 is a, multiply the first row by 1/a in order to introduce a pivot 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1st row of the preceding matrix was multiplied by 1/2.

- Step4. Add suitable multiples of the top row to the rows below so that all entries below the pivot 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

-2 times the 1st row of the preceding matrix was added to the 3rd row.

Elimination Methods (4/7)

- Step5. Now cover the top row in the matrix and begin again with Step1 applied to the sub-matrix that remains. Continue in this way until the entire matrix is in row-echelon form.

$$\begin{bmatrix} \boxed{1 \quad 2 \quad -5 \quad 3 \quad 6 \quad 14} \\ 0 \quad 0 \quad -2 \quad 0 \quad 7 \quad 12 \\ 0 \quad 0 \quad -5 \quad 0 \quad -17 \quad -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$\begin{bmatrix} \boxed{1 \quad 2 \quad -5 \quad 3 \quad 6 \quad 14} \\ 0 \quad 0 \quad 1 \quad 0 \quad -\frac{7}{2} \quad -6 \\ 0 \quad 0 \quad 5 \quad 0 \quad -17 \quad -29 \end{bmatrix}$$

The 1st row in the sub-matrix was multiplied by -1/2 to introduce a pivot 1.

Elimination Methods (5/7)

- Step5 (cont.)

$$\begin{bmatrix} \boxed{1 \quad 2 \quad -5 \quad 3 \quad 6 \quad 14} \\ 0 \quad 0 \quad 1 \quad 0 \quad -\frac{7}{2} \quad -6 \\ 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{2} \quad 1 \end{bmatrix}$$

-5 times the 1st row of the sub-matrix was added to the 2nd row of the sub-matrix to introduce a zero below the pivot 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

The top row in the sub-matrix was covered, and we returned again Step1.

Leftmost nonzero column in the new sub-matrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The first (and only) row in the new sub-matrix was multiplied by 2 to introduce a pivot 1.

- The **entire** matrix is now in **row-echelon form**.

Elimination Methods (6/7)

- Step6. Beginning with last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the pivot 1's.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

7/2 times the 3rd row of the preceding matrix was added to the 2nd row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

-6 times the 3rd row was added to the 1st row.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5 times the 2nd row was added to the 1st row.

- The **last** matrix is in **reduced row-echelon form**.

Elimination Methods (7/7)

- Step1~Step5: the above procedure produces a row-echelon form and is called Gaussian elimination.
- Step1~Step6: the above procedure produces a reduced row-echelon form and is called Gaussian-Jordan elimination.
- Every matrix has **a unique reduced row-echelon** form but a row-echelon form of a given matrix is not unique.

Example 4
Gauss-Jordan Elimination(1/4)

- Solve by Gauss-Jordan Elimination

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2\mathrm{x}_5 \qquad\quad &= \ \ 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= \ \ 5 \\
2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 - 18x_6 &= \ \ 6
\end{aligned}
$$

- Solution:

The augmented matrix for the system is

$$
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{bmatrix}
$$

Example 4
Gauss-Jordan Elimination(2/4)

- Adding -2 times the 1st row to the 2nd and 4th rows gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

- Multiplying the 2nd row by -1 and then adding -5 times the new 2nd row to the 3rd row and -4 times the new 2nd row to the 4th row gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Example 4
Gauss-Jordan Elimination(3/4)

- Interchanging the 3rd and 4th rows and then multiplying the 3rd row of the resulting matrix by 1/6 gives the row-echelon form.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Adding -3 times the 3rd row to the 2nd row and then adding 2 times the 2nd row of the resulting matrix to the 1st row yields the reduced row-echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 4
Gauss-Jordan Elimination(4/4)

- The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

- Solving for the leading variables in terms of the free variables

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

- We assign the free variables, and the general solution is given by the formulas:

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ \ x_6 = \tfrac{1}{3}$$

Back-Substitution

- It is sometimes preferable to solve a system of linear equations by using Gaussian elimination to bring the augmented matrix into row-echelon form **without continuing all the way to the reduced row-echelon form**.
- When this is done, the corresponding system of equations can be solved by by a technique called back-substitution.
- Example 5

Example 5
Ex4 solved by Back-substitution(1/2)

- From the computations in Example 4, a row-echelon form from the augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- To solve the corresponding system of equations

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2\mathrm{x}_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

- Step1. Solve the equations for the leading variables.

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2\mathrm{x}_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Example5
Ex4 solved by Back-substitution(2/2)

- Step2. Beginning with the bottom equation and working upward, successively substitute each equation into all the equations above it.
  - Substituting x6=1/3 into the 2nd equation

$$x_1 = -3x_2 + 2x_3 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \tfrac{1}{3}$$

  - Substituting x3=-2 x4 into the 1st equation

$$x_1 = -3x_2 + 2x_3 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \tfrac{1}{3}$$

- Step3. Assign free variables, the general solution is given by the formulas.

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ \ x_6 = \tfrac{1}{3}$$

Example 6
Gaussian elimination(1/2)

- Solve $\begin{aligned} x+\ \ y+2z&=9 \\ 2x+4y-3z&=1 \\ 3x+6y-5z&=0 \end{aligned}$ by Gaussian elimination and back-substitution.
- Solution
  – We convert the augmented matrix $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$
  – to the row-echelon form $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$
  – The system corresponding to this matrix is

$$x+y+2z=9,\ \ y-\tfrac{7}{2}z=-\tfrac{17}{2},\ \ z=3$$

Example 6
Gaussian elimination(2/2)

- Solution
  - Solving for the leading variables

$$x = 9 - y - 2z,$$
$$y = -\tfrac{17}{2} + \tfrac{7}{2}z,$$
$$z = 3$$

  - Substituting the bottom equation into those above

$$x = 3 - y,$$
$$y = 2,$$
$$z = 3$$

  - Substituting the 2nd equation into the top

$$x = 1,\ \ y = 2,\ \ z = 3$$