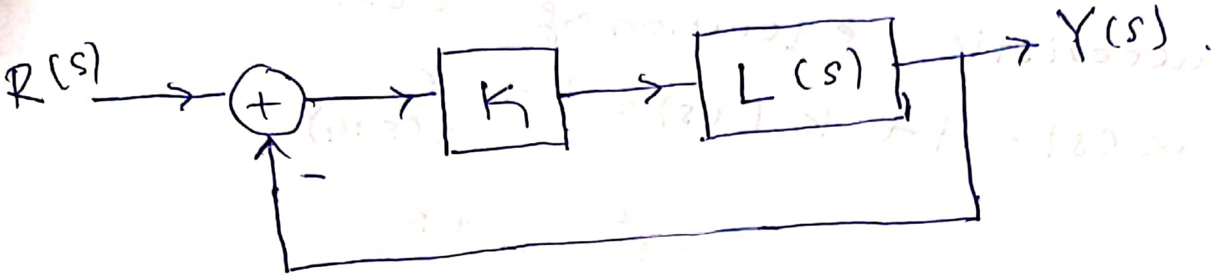
# ROOT-LOCUS CONCEPT.




Let $L(s)$ contain only poles and zeros.

$$L(s) = \frac{b(s)}{a(s)} = \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

Transfer function of Closed-loop system:

$$G_{cl}(s) = \frac{K\,L(s)}{1+KL(s)}$$

$\therefore$ Characteristic equation of closed-loop system is

$$\alpha(s) = 1 + K\,L(s) = 0$$

$$= 1 + K\,\frac{b(s)}{a(s)} = 0$$

$$= a(s) + K\,b(s) = 0$$

**Definition:** In the s-plane, path traced by roots of $1+KL(s) = a(s)+Kb(s)=0$, as $K$ is varied from 0 to infinity, is referred to as ROOT-LOCUS.

Example:

$$L(s) = \frac{1}{s(s+4)}$$

characteristic equation of CLS:

$$\alpha(s) = 1 + K L(s) = 1 + \frac{K}{s(s+4)}$$

$$= s^2 + 4s + K$$

Roots of polynomial are:

$$\alpha(s) = s^2 + 4s + K = 0$$

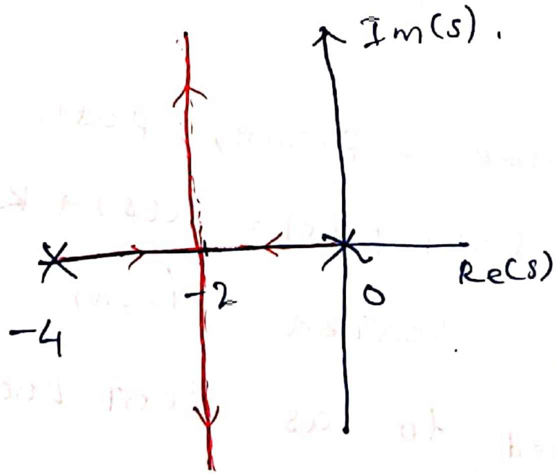
$$s = \frac{-2 \pm \sqrt{16 - 4K}}{2}$$

$$= -2 \pm \sqrt{4 - K}$$

When $0 < K < 4$, roots are on real axis.

When $K = 4$ two roots are at $s = -2$

For $K > 4$, roots are complex-conjugate




For second order, we can thus sketch the root locus by solving for roots.

In order to sketch root-locus for higher system, we should follow the rules given by mathematician Evans.

# Rules for Drawing Root-Locus (RL).

**Rule 1:** RL is always symmetric with respect to real axis of s-plane.

$$L(s) = \frac{b(s)}{a(s)} \quad \begin{array}{l} m = \#\text{ of zeros} \\ n = \#\text{ of poles.} \end{array}$$

Total number of branches $n$.

Total $n$ Branches originate from open-loop pole locations.

Out of $n$, $m$ branches travel towards zeros as $K \to \infty$.

$n-m$ branches travel to infinity as $K \to \infty$.

(In which direction? Refer Rule 3)

**Rule 2:** To know which part of real axis is also part of RL.

Real value $s_0$ is on RL if to the right side of $s_0$, # of poles + # of zeros = odd integer.

**Rule 3:** As $K \to \infty$, in which direction poles travel?

They travel along the asymptotes (angles) we draw,

at $\alpha = \dfrac{\Sigma p_i - \Sigma z_i}{n-m}$

with angles $\theta_l = \dfrac{180° + 360°(l-1)}{n-m}$ $\quad l = 1, 2, \dots n-m$

Rule 4: Angle of Departure for Poles
Poles depart from their original location with angle,

For simple pole

$$\phi_l = \Sigma \text{ Angle made by zeros} - \Sigma \text{ Angle made by poles} - 180°$$
$$= \Sigma \psi_i - \Sigma \phi_i - 180°$$

If there are $q$ number of poles at same location

$$q\,\phi_l = \Sigma \text{ Angle made by zeros} - \Sigma \text{ Angle made by poles} - 180° - 360°(l-1). \quad l = 1, 2, \ldots q$$

Angle of arrival towards zero:
( m branches arrive to m zeros)

For simple zero

$$\psi_l = \Sigma \text{ Angle made by poles} - \Sigma \text{ Angle made by zeros} + 180°$$

If there are $q$ number of zeros at same location, then

$$q\,\psi_l = \Sigma \text{ Angle made by poles} - \Sigma \text{ Angle made by zeros} + 180° + 360°(l-1) \quad l = 1, 2, \ldots q.$$

**Rule 5:** Root locus may or may not cross $j\omega$ (imaginary) axis.

To know for what value of K and where it crosses apply Routh-Hurwitz Test to

$$\alpha(s) = 1 + KL(s)$$

**Rule 6:** **Location of Multiple poles**

$$L(s) = \frac{b(s)}{a(s)}$$

Find the roots of

$$b(s)\frac{d\,a(s)}{ds} - a(s)\frac{d\,b(s)}{ds} = 0.$$

If the root $s_\lambda$ is consistent with all the rules (NO violation) then at that location RL breaks into real axis OR breaks out from real axis.

Note: Along RL at any $s_\lambda$ associated K value can be found by magnitude criteria:

$$K = \frac{1}{|L(s_\lambda)|}$$

Example: Sketch the root-locus for

$$L(s) = \frac{s+1}{s^2+4s+8} = \frac{s+1}{(s+2+2j)(s+2-2j)}$$

Rule 1: Here $n=2$, $m=1$.

Total Two Branches of RL.

$m=1$ branch travels towards zero at $s=-1$

$n-m=1$ branch travels towards infinity as $K \to \infty$.

Rule 2: Part of Real axis that is also RL.

For any real $s_0 > -1$ to the right

# poles + # zeros = 0 even number.

$\therefore s_0 > -1$ Not RL.

For any real $-2 < s_0 < -1$

to the right # of poles + # zeros = 1 Odd number.

$\therefore -2 < s_0 < -1$ is RL.

For any real $s_0 < -2$.

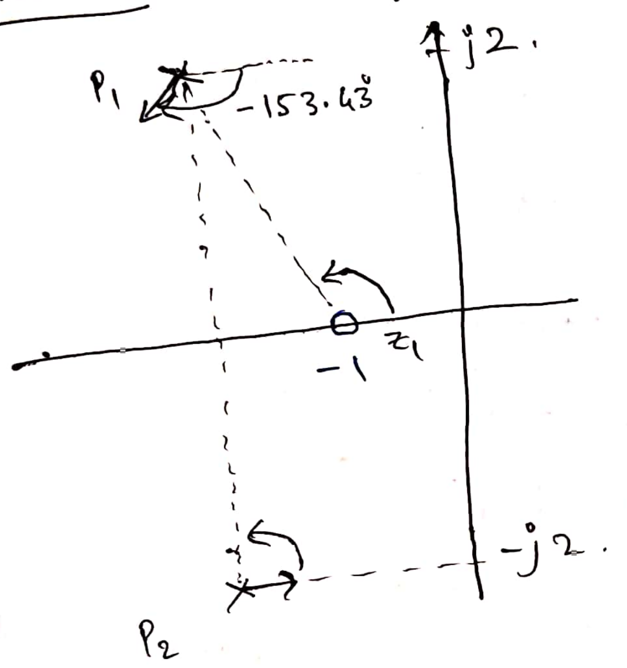
to the right # poles + # zeros = 3 odd number

$\therefore s_0 < -2$ is RL.

**Rule 3:** When $n - m = 1$.

As $K \to \infty$, only one branch goes along $180^\circ$ direction.

**Rule 4:** Angle of departure from $P_1 = -2 + j2$.




$\phi_1$ = Angle made by $z_1$

− Angle made by $P_2$

$- 180^\circ$.

$= 116.56 - 90^\circ - 180^\circ$

$= -153.43$.

Angle of departure from $P_2 = -2 - j2$.

$\phi_2$ = Angle made by $z_1$ to $P_2$.

− Angle made by $P_1$ to $P_2$

$- 180^\circ$

$= -116.56^\circ - (-90^\circ) - 180^\circ$

$= -206.56^\circ$

$= +153.43^\circ$

Angle of arrival to zero at $s=-1$

$\psi_1$ = Angle made by $P_1$ + Angle made by $P_2$ + $180°$

$= -\tan^{-1}2 + \tan^{-1}2 + 180°$

$= 180°$ arrives from horizontal left.

Rule 5: RL crosses imaginary axis?

Apply RH test to

$\alpha(s) = 1 + KL(s) = 0$

$= s^2 + 4s + 8 + Ks + K$

$= s^2 + (4+K)s + 8 + K.$

when $K > 0$ both necessary & sufficient condns satisfied.

$\therefore$ RL always remains on left half of s-plane.

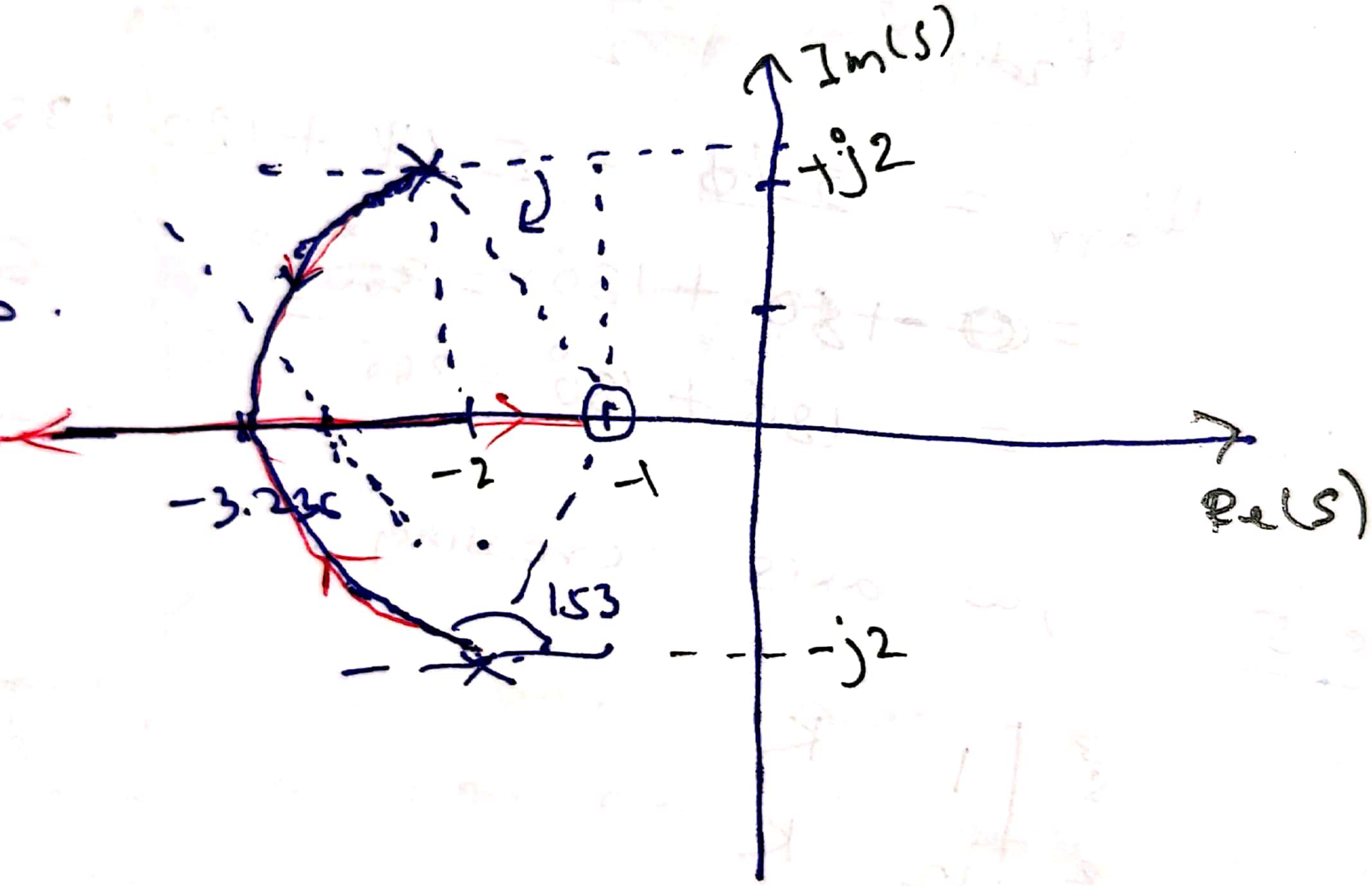
Rule 6: $b(s) = s+1$ $\quad a(s) = s^2 + 4s + 8$

$$b(s)\frac{d}{ds}a(s) - a(s)\frac{d}{ds}b(s) = 0$$

$$(s+1)(2s+4) - (s^2+4s+8)(1) = 0.$$

$$s^2 + 2s - 4 = 0.$$

$$s_{1,2} = \underbrace{-1+\sqrt{5}}_{\text{Violates Rule 2.}} \quad \text{and} \quad \underset{\text{valid.}}{-1-\sqrt{5}}$$


Im(s)
+j2
-j2
Re(s)
-3.236
-2
-1
153

<u>Example:</u> $L(s) = \dfrac{(s+1)^2}{s^3\,(s+4)}$

<u>Rule: 1:</u> RL is symmetric w.r.t. real axis.

Here $n = \#$ of poles $= 4$

$m = \#$ of zeros $= 2$

Therefore, Total $n = 4$ branches.

As $K \to \infty$, $m = 2$ branches go to zeros located at $s = -1$

$n - m = 2$ branches go to infinity in the direction decided according to Rule 3.

<u>Rule 2:</u> For Real $(s) > 0$, to the right

$\#P + \#Z = 0$ Even $\therefore$ NOT RL.

For $-1 <$ Real $(s) < 0$, to the right,

$\#P + \#Z = 3 + 0 = 3$ odd

$\therefore$ RL.

$-4 <$ Real $(s) < -1$ to the right

$\#P + \#Z = 3 + 2 = 5$ Odd

$\therefore$ RL.

Real$(s) < -4$ to the right

$\#P + \#Z = 4 + 2 = 6$ Even

$\therefore$ NOT RL.

**Rule 3:** Draw asymptotes at point

$$\alpha = \frac{\Sigma p_i - \Sigma z_i}{n-m} = \frac{-4-(-1-1)}{4-2} = -1$$

with angles

$$\theta_1 = \frac{180}{2} = 90^\circ$$

$$\theta_2 = \frac{180+360}{2} = 270^\circ$$

**Rule 4:** 3 poles at $s=0$, $P_1, P_2, P_3$.
$P_4$ at $s=-4$. (single)

$3\phi_l$ = Angle made by zero − Angle made by pole $P_4$ $- 180^\circ - 360^\circ(l-1)$

$l = 1, 2, 3.$

$\therefore$ $P_1$ at $s=0$ departs with angle

$3\phi_1 = (0+0) - 0 - 180^\circ$

$\phi_1 = -60^\circ.$

$P_2$ at $s=0$ departs with angle

$3\phi_2 = (0+0) - 0 - 180^\circ - 360$

$\phi_2 = -180^\circ$

$P_3$ at $s=0$ departs with angle

$3\phi_3 = 0 - 0 - 180^\circ - 720^\circ$

$\phi_3 = -300^\circ = +60^\circ.$

$P_4$ at $s=-4$ departs with angle

$\phi_4 = 2\times180^\circ - 3\times180^\circ - 180^\circ$

$= -540^\circ + 180^\circ = -360^\circ = 0^\circ$

## Angle of arrival to zeros

Root-locus branches arrive to two zeros at $s=-1$;

$$2\psi_l = (3\times 180^\circ + 0) + 180^\circ + 360^\circ(l-1)$$

Angles made by poles $\qquad l = 1, 2.$

$$2\psi_1 = 720^\circ$$

$\psi_1 = 360^\circ = 0^\circ$ arrives from right

$$2\psi_2 = 1080^\circ$$

$\psi_2 = 540^\circ = 180^\circ$ arrives from left.

Rule 5: Imaginary axis crossing.

$$\alpha(s) = 1 + KL(s) = s^3(s+4) + K(s+1)^2 = 0$$

$$= s^4 + 4s^3 + Ks^2 + 2Ks + K$$

For $K>0$, necessary conditions satisfied.

| | | | |
|---|---|---|---|
| $s^4$ | $1$ | $K$ | $K$ |
| $s^3$ | $4$ | $2K$ | $0$ |
| $s^2$ | $\frac{4K-2K}{4}$ | $K$ | $0$ |
| $s$ | $\frac{K(K-4)}{K/2}$ | $0$ | $0$ |
| $1$ | $K$ | | |

For $0 < K < 4$, two roots are in right half of s-plane

($\because$ two sign changes in first coloumn)

For $K=4$ two roots will be on imaginary axis.

$s = \pm j\omega_0$

$$(j\omega_0)^4 + 4(j\omega_0)^3 + 4(j\omega_0)^2 + 8j\omega_0 + 4 = 0$$

$$\omega_0^4 - j4\omega_0^3 - 4\omega_0^2 + 8j\omega_0 + 4 = 0$$

$$\omega_0^4 - 4\omega_0^2 + 4 = 0$$

$$8\omega_0 - 4\omega_0^3 = 0$$

$$\Rightarrow \omega_0 = \pm\sqrt{2}.$$

For $K > 4$, all first coloumn entries in Routh Table are positive.

$\therefore$ All roots will be in left half of s-plane.

Rule 6: $L(s) = \frac{b(s)}{a(s)} = \frac{(s+1)^2}{s^3(s+4)}$

$$b(s)\frac{d}{ds}a(s) - a(s)\frac{d}{ds}b(s) = 0$$

$$(s+1)^2\left[4s^3 + 12s^2\right] - (s^4+4s^3)(2s+2) = 0$$

$$s^2(s+1)\left\{(s+1)\left[4s+12\right] - (s^2+4s)(2)\right\} = 0$$

$s=0$, $s=-1$ are location of multiple roots consistent with all other rules.

$s=0$ is Location when $K=0$

$s=-1$ is Location as $K \to \infty$

$$4s^2+16s+12 - 2s^2 - 8s = 0$$

$$2s^2 + 8s + 12 = 0$$

$$s^2 + 4s + 6 = 0$$

$$s = \frac{-4}{2} \pm \frac{\sqrt{16-12(2)}}{2}$$

$$= -2 \pm j\sqrt{2}$$

Invalid NOT consistent with other rules.

We need at least 4 poles to be in complex plane.

But we know 2 poles will remain on real axis.

Only two poles will attain complex values.

Root Locus for

$$L(s) = \frac{(s+1)^2}{s^3(s+4)}$$