

Unit 5: Applications of LA

matrices in Engineering.

Linear Algebra in Engineering in 2 ways:

1) Direct way: The physical problem has only a finite no. of pieces. The laws connecting position/velocity are linear. The laws are expressed by matrix equations.

2) Indirect way: The physical system is continuous. The mass density, forces, velocities are functions of x, \dot{x}, y, \dot{y} or $x, y, z, \dot{x}, \dot{y}, \dot{z}$. The laws are expressed by differential eqs. To find accurate solution - approximate by finite difference eqs of finite element equations.

Equilibrium eq $Ku = f$.

$K = AT A$. K - stiffness matrix.

with motion, $M \frac{d^2u}{dt^2} + Ku = f$ - dynamic.

$Kx = \lambda Mx$. λ - eigen values.

Line of Springs:

Three masses m_1, m_2, m_3 connected by a line of spring. With top & bottom fixed.

I call - 4 springs

3 springs, the lowest mass hangs freely.

II - Fixed-free -

K_0 & $A_0^T C_0 A_0$.

III - Fixed-free - K_1 & $A_1^T C_1 A_1$.

IV - Free-free - with no support at either end - K singular.

Equations for mass movements (U) & tension (y):

$U = (U_1, U_2, U_3)$ = movement of masses

$y = (y_1, y_2, y_3, y_4) \& (y_1, y_2, y_3) = \text{tension in springs}$

when mass moves downward, its displacement is positive ($U_i > 0$). For spring tension is positive & compression is negative ($y_i < 0$). In tension, the spring is stretched so it pulls the masses inward.

Each spring is controlled by its own Hooke's law

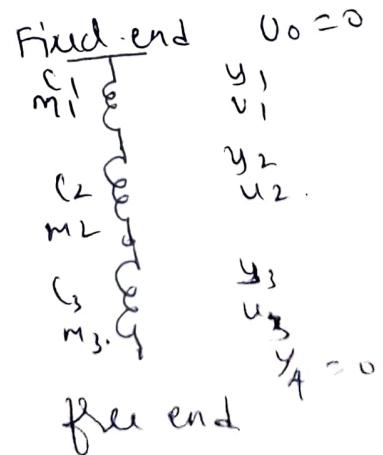
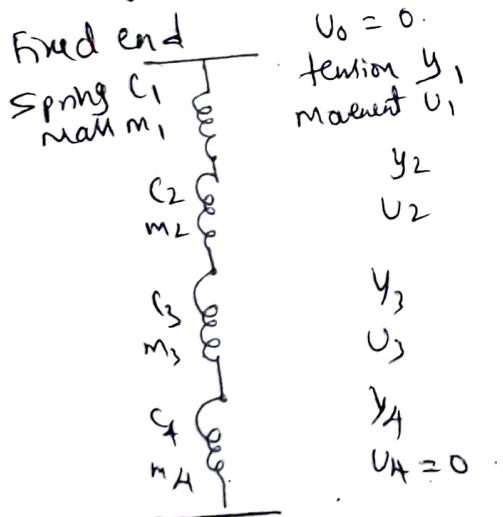
$$y = Ce$$

Stretching force = Spring constant times stretching distance

$$y = Ce \text{ into vector law } Ku = f$$

The force vectors if comes from gravity. The gravitational will multiply each mass to produce.

Constant g
for forces $f = (m_1g, m_2g, m_3g)$



Fixed end

The matrices K_0 , K_1 , K_2 singular & singular have $C_i = 1$.
for simplicity. All spring constants are $C_i = 1$.

Stiffness matrices

$$K_0 = A_0^T A_0 = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & \end{bmatrix}$$

Fixed - fixed.

$$A_0^T C_0 A_0 = \begin{bmatrix} c_1 + c_2 & -c_2 & & \\ -c_2 & c_2 + c_3 & -c_3 & \\ & -c_3 & c_3 + c_4 & \end{bmatrix}$$

Spring constant included.

$$K_1 = A_1^T A_1 = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & \end{bmatrix}$$

Fixed - free.

$$A_1^T C_1 A_1 = \begin{bmatrix} c_1 + c_2 & -c_2 & & \\ -c_2 & c_2 + c_3 & -c_3 & \\ & -c_3 & c_3 & \end{bmatrix}$$

$$K_{\text{singular}} = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 1 & \end{bmatrix}$$

$$K_{\text{circular}} = \begin{bmatrix} 2 & -1 & -1 & \\ -1 & 2 & -1 & \\ -1 & -1 & 2 & \end{bmatrix}$$

Free-free.

To find ~~stiff~~ new matrices.

Create K in 3 steps -

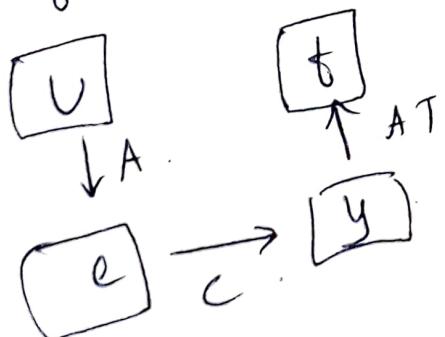
U = movement of n matter = (U_1, \dots, U_n)

e = elongations of n springs = (e_1, \dots, e_m)

y = internal force in m springs = (y_1, \dots, y_m)

f = External force on n matter = (f_1, \dots, f_n)

$$e = AU \quad A_{m \times n}$$



$$y = Ce \quad C_{m \times m}$$

$$f = A^T y \quad A^T_{n \times m}$$

The elongation e is the stretching distance - how far the springs are extended. When it becomes vertical & upright, gravity acts. The matter move down by distances U_1, U_2, U_3 .

Each spring is stretched or compressed by the difference in displacement of its ends.

$$e_i = v_i - v_{i-1} \quad \text{top fixed}$$

Stretching of each spring:

$$\text{I spring } e_1 = v_1 \quad (v_0 = 0)$$

$$\text{Second spring } e_2 = v_2 - v_1$$

$$\text{III } e_3 = v_3 - v_2$$

$$\text{IV } e_4 = -v_3 \quad (v_4 = 0)$$

bottom fixed

If both ends move the same distance, that spring is not stretched: $v_i = v_{i-1}$ & $e_i = 0$.
The matrix of a ~~sys~~ is a 4×3 difference matrix A
and $e = Av$.

Stretching distances

$$e = Av \text{ is } \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (1)$$

Hooke's Law: $y = ce$ connects spring elongation e with spring tension y . The constitutive law depends on the material in the spring. A soft spring has small c , so moderate force y can produce a large stretching e .

Since each spring has its own law, the matrix is

$y = ce$ is a diagonal matrix C :

$$\begin{aligned} y_1 &= c_1 e_1 & y_1 &= \begin{bmatrix} c_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \\ y_2 &= c_2 e_2 & y_2 &= \begin{bmatrix} 0 & c_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \\ y_3 &= c_3 e_3 & y_3 &= \begin{bmatrix} 0 & 0 & c_3 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \\ y_4 &= c_4 e_4 & y_4 &= \begin{bmatrix} 0 & 0 & 0 & c_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \end{aligned} \quad (2)$$

(combining) $\ell = \mathbf{A}\mathbf{U}$ with $\mathbf{y} = (\mathbf{c} \cdot \text{the Spring force})$
are $\mathbf{y} = \mathbf{C}\mathbf{A}\mathbf{U}$.

Balance eq: The internal forces from the spring balance the external forces on the masses. Each mass is pulled/pushed by spring force y_j above it. The spring force y_{j+1} plus f_j from gravity.

$$\therefore y_j = y_{j+1} + f_j \quad \& \quad f_j = y_j - y_{j+1}$$

force balance: $f_1 = y_1 - y_2$ $f_2 = y_2 - y_3$ $f_3 = y_3 - y_4$

$$f = \mathbf{A}^T \mathbf{y} \quad \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

force balance eq $f = \mathbf{A}^T \mathbf{y}$. (3)

Nature transposes the rows and columns of e-u matrix to produce the $f-y$ matrix. \mathbf{A}^T appears along with \mathbf{A} . stiffness matrix $\mathbf{K} = \mathbf{A}^T \mathbf{C} \mathbf{A}$.

$$\left(\begin{array}{l} \ell = \mathbf{A}\mathbf{U} \\ \mathbf{y} = (\ell) \\ f = \mathbf{A}^T \mathbf{y} \end{array} \right) \text{ combines into } \mathbf{A}^T \mathbf{C} \mathbf{A} \mathbf{U} = f \Rightarrow \mathbf{K}\mathbf{U} = f$$

In elasticity, $\ell = \mathbf{A}\mathbf{U}$ is the kinematic eq. The force balance $f = \mathbf{A}^T \mathbf{y}$ is static equation. The constitutive law is

$\mathbf{y} = (\mathbf{c})$. Finite element programs spend major effort on assembling stiffness matrix from thousand of small pieces. If all springs are ~~identical~~ identical, $c_1 = c_2 = c_3 = c_4 = \text{then } C = I$. It all springs are ~~identical~~ identical, reduces to $\mathbf{A}^T \mathbf{A}$. The stiffness matrix

with $C = I$

$$K_0 = A_0^T A_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The matrix A from Spring is 4 by 3. The triangular matrix L from elimination is square. The stiffness matrix K is assembled from $A^T A$ & then broken up into $L L^T$. One step is applied mathematically, the other is computational mathematics. Each K is built from rectangular matrices and factored into squared matrices.

Properties of $K = A^T C A$:

- * K is tri-diagonal, because m_{33} is not connected to m_{11} .
- * K is symmetric: because C symmetric & A^T comes with A .
- * K is positive definite, because $c_i > 0$ & A has independent columns.
- * K^{-1} is a full matrix in (ex 15) with all positive entries.

Last property leads to an important fact about $U = K^{-1} f$: If all forces act downwards ($f_j > 0$). Then all movements are downward ($u_j > 0$). Positiveness is different from positive definiteness. Here K^{-1} is positive. Both K & K^{-1} are positive definite.

Q1: Suppose all $C_i = C$ & $m_j = m$. Find movements U & tensions y .

All springs are same & all masses are same. But all movements, elongations & tensions will not be same.

K^A includes y_c because $A^T C A$ includes C .

$$U = K^A \cdot f = \frac{1}{4C} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{C} \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

The displacement U_2 , for the mass in the middle is greater than U_1 & U_3 . The force mg divided by force per unit length c gives a length θ .

$$f = A U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \frac{mg}{c} \begin{bmatrix} \frac{3}{2} \\ 2 \\ \frac{3}{2} \end{bmatrix} = \frac{mg}{c} \begin{bmatrix} \frac{3}{2} \\ y_2 \\ -y_2 \end{bmatrix}$$

These elongations add to zero because the end of line $U_1 + (U_2 - U_1) + (U_3 - U_2)$ is certainly zero. The sum $U_1 + (U_2 - U_1) + (-U_3)$ is certainly zero. So y_1, y_2, y_3 are fixed. The sum for each spring from $y_i \propto F_i$ by C .

For each spring from $y_i \propto F_i$ by C . The upper y_A are $\frac{3}{2} mg$, $\frac{1}{2} mg$, $-\frac{1}{2} mg$, $-\frac{3}{2} mg$. The lower 2 springs are compressed.

2 springs are stretched k . Lower 2 springs are compressed.

Fixed end and Free end becomes 3 by 3. The new Fixed end and All matrices becomes 3 by 3.

$$\text{Remove } k^A \text{ spring, All matrices} \\ \text{different matrix } K^A \\ A_1^T C_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$c_4 = 0.$$

$$\text{Find } K_1 = A_1^T C_1 A_1 = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

The $C_1 = C_2 = C_3 = 1$ & $C = 1$, this is the tri-diagonal matrix except for last entry 1 instead of 0. The spring at the bottom is free.

Ex 2: All $C_i = C$ & all $m_j = m$ in fixed-free hanging line.

A spring.

$$K_1 = C \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad K_1^{-1} = \frac{1}{C} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

The force mg from gravity will be same but the movements from the previous eq. because the stiffness matrix has changed.

$$\mathbf{f} = K_1^{-1} \mathbf{f} = \frac{1}{C} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{C} \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}$$

The no. 3 appears in \mathbf{f} , because no. 3 appears in K_1^{-1} .

The movements are greater. The no. 3 is pulling the first spring down because all three masses are pulling the additional 2 from

The next mass moves by plus an additional 2 from (3+2+1=6)

The mass below it. The 3 mass drops by (3+2+1=6)

the masses. $\mathbf{f} = \mathbf{A}\mathbf{U}$. in the springs display true.

The elongations

numbers 3, 2, 1:

$$\mathbf{e} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \frac{mg}{C} \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} = \frac{mg}{C} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

The balance say $A\mathbf{T}\mathbf{y} = \mathbf{f}$. determines \mathbf{y} immediately

The balance say $A\mathbf{T}$ is singular because $m=n$ & $A\mathbf{T}$ is singular in

$$\mathbf{y} = (\mathbf{A}\mathbf{T})^{-1} \cdot \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} mg \\ 2mg \\ 1mg \end{bmatrix} = \begin{bmatrix} 3mg \\ 2mg \\ 1mg \end{bmatrix}$$

Two free ends: \mathbf{R} is singular.

The whole line can move, at both ends are free. The matrix A is 2×3 . So \mathbf{Q} wide, $\mathbf{Q} = \mathbf{A}\mathbf{U}$

$$\text{Free } \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} v_2 - v_1 \\ v_3 - v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Non zero solution for $\mathbf{A}\mathbf{U} = \mathbf{0}$. The masses can move with no stretching of springs. The whole line can shift by $\mathbf{U} = (1, 1, 1)$ plus leaves $\mathbf{q} = (0, 0)$. A has dependent columns & rows & $(1, 1, 1)$ is in its nullspace.

$$\mathbf{A}\mathbf{U} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \text{no stretching}$$

$\mathbf{A}\mathbf{U} = \mathbf{0}$ (certainly leads to $\mathbf{A}^T \mathbf{C} \mathbf{A}\mathbf{U} = \mathbf{0}$). So $\mathbf{A}^T \mathbf{C} \mathbf{A}$ is only positive semi-definite, without c_1 & c_3 . The rank is only 2.

$\mathbf{G} \propto$ no third pivot.

$$G \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 & -c_3 \\ 0 & -c_2 & c_3 \end{bmatrix}$$

Two eigen values will be positive but $\mathbf{r} = (1, 1, 1)$ is an eigen vector for $\mathbf{A} = \mathbf{0}$. The sides have to add to $\mathbf{f}^T \mathbf{f} = 0$.

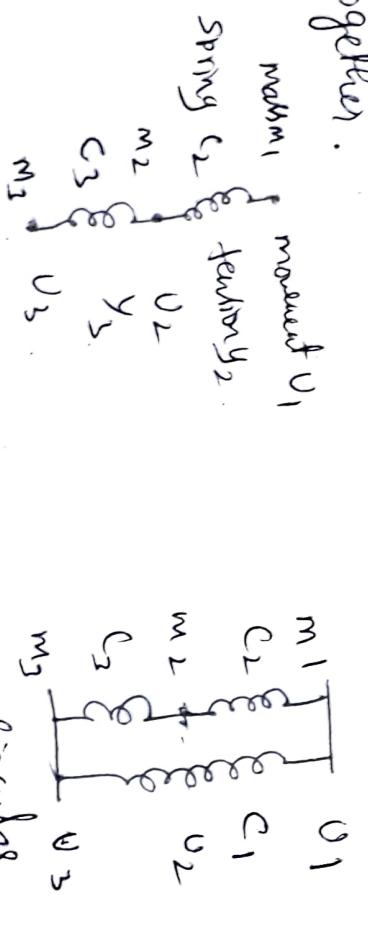
Special solution for $\mathbf{A} = \mathbf{0}$. The forces have to add to $\mathbf{f}^T \mathbf{f} = 0$ for the whole line of spring will take like a crack.

Circle of Springs:

A fixed spring will complete circle from mass back to mass. This doesn't make K invertible. The new matrix is still singular. The stiffness matrix K usually is not tri-diagonal, but it is symmetric & semi-definite.

$$A_c^T A_c = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

The only pivot are 2 and $\frac{1}{2}$. The eigen values are 3, 3 & 0. The determinant is zero. The nullspace still contains $x = (1, 1, 1)$. When all masses move together.



Free-free

can move without stretching spring so.
The masses can move without solutions.

$Av = 0$ has non zero

This makes A not invertible. The nullspace of A is not

empty. Even after diagonal matrix C of spring forces, the system is not stretched.

Contract is included;

$$(A_c^T C A)_\text{wedge} = \begin{bmatrix} c_1 + c_2 & -c_2 & -c_1 \\ -c_2 & c_2 + c_3 & -c_3 \\ -c_1 & -c_3 & c_1 + c_3 \end{bmatrix}$$

continues

Matrix \underline{v} are discrete. Differential \underline{v} are continuous.
 Matrix $A \propto AT$ corresponds to derivatives $\frac{dy}{dx} & -\frac{d^2y}{dx^2}$.
 The matrix $A = Av$ took difference $v_i - v_{i-1}$ & $f = ATy$ took
 difference $y_i - y_{i+1}$. The springs are infinitesimally short
 and these differences become derivatives.

$$\frac{v_i - v_{i-1}}{\Delta x} \text{ is } \frac{du}{dx} \quad \frac{y_i - y_{i+1}}{\Delta x} \text{ is } -\frac{dy}{dx}.$$

Hooke's Law & force balance at every point x :

$$f(x) = Av = \frac{du}{dx} \quad y(x) = c(x) \epsilon(x), \quad ATy = -\frac{dy}{dx} = f(x).$$

$$ATC A v(x) = f(x).$$

The line of spring behaves an elastic bar:
 Solid elastic bar $ATC A v(x) = f(x)$ is

$$\frac{d}{dx} \left(\frac{du}{dx} \right) = f(x). \quad \text{difference matrix } ATA$$

derivative. A is difference matrix & ATA
 corresponds to Second derivative. The matrix has $-1, 2, -1$ & \underline{v} .

ATA corresponds matrix

is a second difference:

$$-\frac{d^2u}{dx^2}.$$

has $-v_{i+1} + 2v_i - v_{i-1}$ is a second difference - Forward, Backward & Centered

$$-v_{i+1} + 2v_i - v_{i-1} \quad \text{choice -}$$

$$\text{First derivative - } 3 \text{ choice - } \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}$$

$$\frac{du}{dx} = \frac{u(x+\Delta x) - u(x-\Delta x)}{\Delta x}.$$

First derivatives are anti-symmetric, the matrix just a mirror sign. So second differences are negative definite & changed to $-\frac{d^2u}{dx^2}$.

It starts with differential eqn, formula for 1st order
Create discrete matrix K by approx continuous problem.

Find $AU = \frac{1}{\Delta x} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \approx \frac{dy}{dx}$ with $U_0 = 0$
 $U_4 = 0$

Fixed free $ATy = \frac{1}{\Delta x} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{-dy}{dx}$ with $y_0 = 0$
 $y_4 = 0$.

The boundary condition $y_4 = 0$ at free end becomes $\frac{dy}{dx} = 0$. Since $y = AU$ (correspond to $\frac{dy}{dx}$). The last row $\frac{dy}{dx} = 0$ balance $ATy = f$ at that end is 0. The last row $f(x) = b$ has entries 1, 1 to reflect this condition $\frac{dy}{dx} = 0$

The eqn $\frac{-d}{dx} \left((\Delta x) \frac{du}{dx} \right) = f(x)$. with $U(0) = 0$ and $\int u(1) - \frac{du}{dx}(1) \Big] = 0$. Divide the box into n pieces of length Δx . Replace du/dx by $AU + \frac{-dy}{dx}$ by ATy .

length Δx . Replace du/dx by $AU + \frac{-dy}{dx}$ by ATy . The end conditions are

Now $A \times AT$ include $\frac{dy}{dx}$. The end conditions are

$$U_0 = 0 \quad \& \quad [U_N = 0 \quad \& \quad y_N = 0]$$

The 3 steps $\frac{-d}{dx} \approx \frac{dy}{dx}$ or $\frac{dy}{dx}$ correspond to AT, COA .

$$f = ATy \quad y = Ce \quad e = AU \text{ give } ATeAU = f.$$

1. Model the problem by differential equation.

2. Discretize the differential eqn to a difference eqn.

3. Understand & solve the difference eqn (boundary conditions)

4. Interpret solution, visualize it.

Graph and networks:

The model consists of nodes connected by edges. This is called Graph display function $f(x)$. Graphs of tree node edge graph to matrices. The incidence matrix of a graph tells how n -nodes are connected by m edges. Normally $m > n$.

For any m by n matrix there are 2 fundamental subspaces in \mathbb{R}^n & 2 in \mathbb{R}^m . They are the row spaces & nullspaces

$$A \& A^T.$$

of directed graph & its incidence matrix. The law of Kirchoff's Law.

Linear Algebra is incident matrix is $0 \& 1 \& -1$. All the pivots & entry of an incidence matrix is $0 \& 1 \& -1$. All the factors in $A = LV$ also multiplication are ± 1 . Therefore both factors have both rows contain $0, 1, -1$. All four subspace have both vectors.

contain exceptionally simple components with zero differences in voltage, across 6 edges of a graph. The voltages are x_1, x_2, x_3, x_4 at the 4 nodes.

graph. The voltage matrix 6 edges

Incidence.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

row reduction.

reduces to $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The nullspace of $A \& U$ have dim $\gamma = 3$. The pivot vector

The column space of $A \& U$ have dim $\gamma = 4$.

The basis for row space. The column space is perpendicular to every vector in nullspace in equations $AX = 0$. Every vector in nullspace is directly from m equations. This vector $x = (1, 1, 1, 1)$ is perpendicular to all rows of $A \& U$ above & in row space. It is final voltage vector & thus to the whole row space.

and thus to the whole row space.

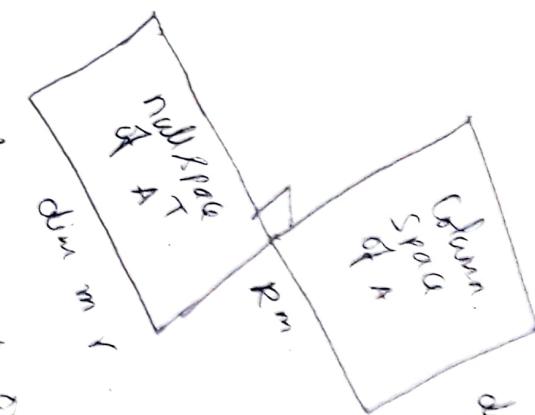
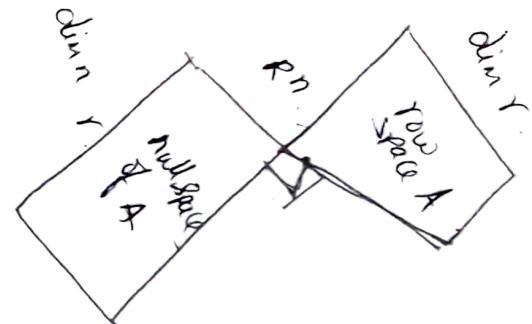


Fig-1 Four Subspaces with their dimensions & orthogonality.

A matrix with m by n . Its columns are vectors in \mathbb{R}^m which produce the column space ($C(A)$, or $\text{Col}(A)$). These linear combinations are exactly the matrix with r rows of \mathbb{R}^m . More combinations are made by multiplying A with vectors in \mathbb{R}^n .

The rows of A are vectors in \mathbb{R}^m . Their linear combinations produce the row space (Row space of A^T) $C(A^T)$. The nullspace of A contains every vector x that satisfies $Ax = 0$ true if x is zero vector. The nullspace of A^T contains all solutions to $A^T y = 0$. Now y has n components & $N(A^T)$ is a subspace of \mathbb{R}^n .

The four subspaces shown in Fig-1, shown \mathbb{R}^n on one side & \mathbb{R}^m on other. The link between them is A . The four subspaces are orthogonal to each other. This link is called linear algebra.

$$\text{Two central laws of linear Algebra: } \dim(C(A)) = \dim(C(A^T)) + \dim(N(A)) = n.$$

$$\dim(C(A)) = \dim(C(A^T)) + \dim(N(A)) = n.$$

When row space has dimension r , the nullspace has dimension $n-r$. Elimination leaves two linear equations. There are $n-r$ pivot columns from A which dimension count. These are $n-r$ free columns and columns with pivots. These lead to zero in nullspace. without pivots.

Initial graphs and Incident matrices

with $m=6$ edges and $n=4$ nodes, so the matrix A is
 a graph with entries -1 by 1 the direction of each rainbow. The
 6 by 4 \rightarrow , $1, 0, 0$ of A gives a record of first edge from
 node 1 to node 2 .
 \rightarrow $\textcircled{1}$ $\textcircled{2}$ $\textcircled{1}$ $\textcircled{4} \rightarrow 1$

$A =$
 $\begin{bmatrix} 0 & 1 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{bmatrix}$
 edges:
 $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$
 nodes:
 ①
 ②
 ③
 ④

n = 6 edges
 n = 4 nodes.
 numbers are edge numbers, column no. all n
 Node
 1
 2
 3
 4
 5
 6
 7
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$m=6$ edges $n=4$ nodes. 0 0
 Row numbers are edge numbers, column no. are node numbers.
 ① 1 2 3 4 ↗

Tree with 3 edges & 4 nodes & no loops.
A vertex is connected.

The first graph is complete - every pair of nodes is connected by an edge. The second graph is a tree - the graph has no cycles. These graphs are the 2 extremes, the maximum number of edges is $\frac{1}{2}n(n-1)$ and minimum (tree) is $m = n-1$.

The rows of B matches non-zero rows of U -echelon form
 Elimination reduces every graph to a tree. The loops produce
 zero rows in U .

$$\begin{array}{c}
 \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right] \\
 \downarrow \\
 \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]
 \end{array}$$

when an edge share a node, elimination produces 8 short cut edges without that node. If the graph already has this short cut edge, elimination gives a sum of zeros, reduce to the

Row are dependent when a edge from a loop. Independent row come from trees. The graph is connected & it makes no fundamental difference which way the arrows go. On each edge, flow with arrow is positive. Flow in opposite direction counts as negative. The flow might be a current or signal of flow of even oil or gas or water.

For the column space Ax , which is a vector of differences:

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

The unknowns x_1, x_2, x_3, x_4 represent potentials of voltage at the nodes. Then Ax gives potential differences that cause flow differences across the edges.

The null space contains solutions to $Ax = 0$. All 6 potential differences are zero. This means: All potentials are equal. Every x in the nullspace is a constant vector (c, c, c, c, c, c) . Every x of A is a line in \mathbb{R}^n - its dimension is $n-r=1$. The nullspace of A is a one-dimensional line in \mathbb{R}^6 .

The second incident matrix B has same nullspace. It contains $(1, 1, 1)$.

$$Bx = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now all potentials by same amount (, without changing
calculus). There is an arbitrary constant in potentials. Rate of
change function by same amount C, without changing its
form. There is an arbitrary constant C in integral.

Calculus adds $+C$ to indefinite integrals. Graph theory adds
 (c_1, c_2, c_3) to the vector \mathbf{x} of potentials. In fact
 C adds any vector \mathbf{x}_n in the nullspace to one
of graph. Solution of $A\mathbf{x} = \mathbf{b}$. The $+C$ disappears in
particular when integral starts at a known point $\mathbf{x} = \mathbf{a}$. Seen last
calculus when set $\mathbf{x}_4 = 0$. The unknown \mathbf{x}_4 is
nullspace disappears when removed \mathbf{x}_4 . So are the fourth columns of A & B. Note A.
is grounded.

The row space contains all combinations of 6 rows. Its
dimension is certainly not 6. The for $\mathbf{y} + (\mathbf{n}-\mathbf{y}) = \mathbf{n}$ must be
 $3+1 = 4$.
The rank $r = 3$.
 \mathbf{y} is in the row space iff. it is perpendicular to $(1, 1, 1, 1)$
in the null space.

The vector $\mathbf{v} = (0, 1, 2, 3)$ fails this test - its components add
 $+ 6$. The vector $(-6, 1, 2, 1)$ passes test. It lies in the
row space because its component add to zero.
 $6(\text{rows}) + 5(\text{rows}) + 3(\text{rows})$. This must be true for every
row of A add to zero.

Each row of A adds to zero in the row space.
This is the row space contains all combinations of 4 columns.
3) The column space contains all independent
columns. Since there were 3 independent
columns. But 4 columns.
The first 3 columns are independent. The fourth is in nullspace.
Add to zero results, which is $(1, 1, 1, 1)$.

The component of Ax add to zero around every loop.

when b is in column space of A ,
 Kirchoff's law: $b_1 + b_3 - b_2 = 0$
 $Ax = b$ can be solved exactly when components of b satisfy all the same dependencies as the rows of A . Then elimination leads to $0 = 0$ & $A^T y = 0$. It's solution contains solutions to $A^T y = 0$.

4) The left null space dimension is $m-y = 6-3$:
 $A^T y = \begin{bmatrix} -1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

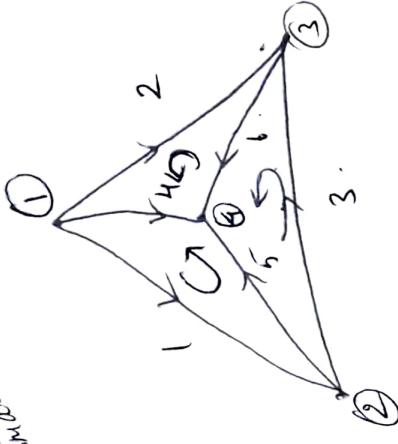
Current Law (KCL)

The true no. of equations $y=3$ and not $n=4$.
 First eq. $-y_1 - y_2 - y_4 = 0$. Flow into node 1
 satisfies equation says that $y_4 + y_5 + y_6 = 0$.
 Minus flow out is zero. $A^T y = 0$ - inconsistent.
 Kirchoff's Current Law: Flow in equals flow out at each node.

When current of nodes are in equilibrium, equation to solve is $A^T y = 0$. The matrix is transpose of when current of nodes are in equilibrium, equation to solve is $A^T y = 0$. The matrix A .

in circle around a loop. $A^T y = 0$. The earliest way is to flow around big triangle, which is a unit of current goes around big triangle, backword 2 loop. If a unit of current goes forward edge 1, 3, backward 2, $y = (1, 1, 1, 0, 0, 0)$. \rightarrow forward edge 1, 3, backward 2 loop. Current is a solution to the current law. A smaller loop has enough flow out at every node. A smaller loop goes forward on edge 1, forward on 5, backward on 4. The loop goes forward on edge 1, forward on 5, backward on 4. Then $y = (1, 0, 0, -1, 1, 0)$ is in left nullspace.

The 3 small loops in graph are independent. The big triangle
is sum of these 3 small loops. but it is sum of those around.
Hence + sign. similarly, the small loops give a basis for the nullspace
of small loops.



$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Small loops big loop

Summary: The incidence matrix A comes from column space with n nodes & m edges. The row space & column space with dimension $n-1$. The nullspace of A and A^T have dimension $n-m+1$.

* and $n-m+1$ linearly independent vectors $(c_1, c_2, \dots, c_{n-m})$ make up the nullspace of A .

* The last one is c_{n-m+1} is called the fundamental circuit, which edges form a tree.

* True iff $c_{n-m+1} = 0$. The components of Ax add to zero around every loop.

* Kirchhoff law: The components of A^Ty is produced by loop currents. $N(A^T)$ has $m-n+1$ independent loops.

* Current law: $A^Ty = 0$ is produced by loop currents. There are $m-n+1$ independent loops in the graph.

In graph theory, this formula is called Euler's formula.

If graph in a plane, $\frac{1}{2}A$ yields Euler's formula $(\text{no. of small loops}) = 1$.

If graph is not in a plane, $\frac{1}{2}A$ yields Euler's formula $(\text{no. of edges}) - (\text{no. of nodes}) + (\text{no. of edges}) = 1$.

i.e. $n-m+(m-n+1) = 1$.

A single triangle has $(3 \text{ nodes}) - (3 \text{ edges}) + (1 \text{ loop})$.

Networks and ATCA

In real nw, the current y along an edge is product of a number one no. is different b/w potentials x at the ends of the edge. This difference is Ax & it drives the flow. The other number is the 'conductance' c - which measures how easily flow gets through.

In physics and engineering, c is decided by material. For electrical currents, c is high for metal & low for plastics. For Superconductors c is nearly infinite. For elastic stretching, c might be low for metal & higher for plastics. In economics, c measures the capacity of an edge of its cost.

The graph is known from its connectivity matrix A , connection b/w nodes & edges. A nw goes further & assigns a conductance c to each edge. These numbers c_1, \dots, c_m go into the conductance matrix C which is diagonal.

For each nw of resistor, $(= Y_{resistor})$. Ohm's law connects the current y , on edge i , to potential difference $x_2 - x_1$ between the nodes:

Ohm's Law: current along edge = conductance times potential difference.

Ohm's Law for all m currents is $y = -CAx$. The vector Ax gives the potential differences, & C multiplies by the conductances. Combining Ohm's Law with Kirchoff's Current Law, $A^T y = 0$, i.e. $A^T C A x = 0$. This is almost the central equation for nw flows. The nw needs power from outside a voltage source of current source - to make something happen.

limit theory, charge from x_1 to $-x_1$, the flow is from potential to lower potential. This is current from node to higher when $x_1 - x_2$ is positive - where as Ax was normalized to node $x_2 - x_1$. The minus sign in physical and electrical field $y_2 - y_1$ is a plus sign in nodal expression of electric field.

In electric structures $y = CAx$ is Hooke's law. The stress y is elasticity C times (stretching Ax) - if heat conduction is a temperature gradient. If oil flows it is a pressure. It is a second law for heat transfer equation.

gradient. There is a similar law for heat transfer

Kirchhoff's Law: Charge from $A^T y = 0$ to

to balance the source & from $A^T y = 0$

Flow into each node still outside. Flow with its

source flow out. The node with its

conductances $c_1 \dots c_6$ and shows

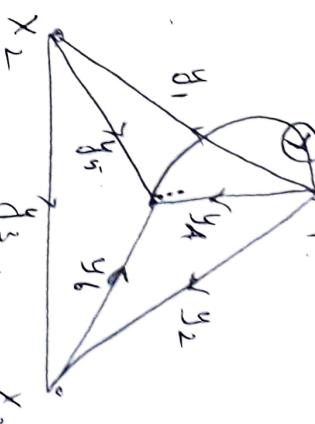
the lowest source going into node 1.

The source comes out at node A & keep

the balance.

All conductances are $c=1$, so that $C = I$. A crossed y_A goes directly from node 1 to node A. other wires goes long way from node 1 $\rightarrow 2 \rightarrow 4$ ($y_1 - y_5$) node 1 $\rightarrow 3 \rightarrow 4$ ($y_2 - y_6$). Find 6 values by using Special value for symmetry $A^T C A$. Since $C = I$ the matrix is $A^T A$, graph Laplacian matrix:

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$



Last matrix is not invertible. We cannot solve for all three potentials because $(1, 1, 1, 1)$ is in nullspace. One node has to be grounded. Setting $x_4 = 0$ removes from 4th row and column, thus

leaves 3×3 invertible matrix.

Solve $\bar{A}^T \bar{A} \bar{x} = \bar{y}$ for unknown potentials x_1, x_2, x_3 with

source s into node 1:

$$\text{voltages} \begin{bmatrix} -3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ 0 \end{bmatrix} \text{ gives } \begin{bmatrix} -x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s/4 \\ s/4 \\ s/4 \end{bmatrix}$$

Ohm's law $y = -C \bar{A} \bar{x}$ yields 6 currents. $C = I$ b $x_4 = 0$

$$\text{currents} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = - \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} s/4 \\ s/4 \\ s/4 \\ s/4 \\ s/4 \\ s/4 \end{bmatrix} = \begin{bmatrix} -s/4 \\ s/4 \\ 0 \\ s/2 \\ s/4 \\ s/4 \end{bmatrix}$$

Half current goes directly on edge 1. i.e. $y_4 = y_2$. No current comes from node 2 to node 3. Symmetry indicated $y_3 = 0$ & now sd proves it.

The same matrix $\bar{A}^T \bar{A}$ appears in last square. Notice ditributes free currents to minimize the node volt. Justifies choice \hat{x} to minimize the last square error.

Markov matrices, population & Economics:

largest eigen value is real & positive & so is its eigenvector.

economics, ecology & population dynamics & random walks

In Markov $A_{max} = 1$ population $A_{max} > 1$ consumption $A_{max} < 1$.

Analyse long-run powers of A .

Markov matrices multiply a positive vector $v_0 = (a, 1, -a)$ again & again by this A :

$$\text{Markov matrix } A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}, \quad v_1 = Av_0, \\ v_0 = A^2v_0.$$

After K steps, $A^K v_0$. The vector v_1, v_2, v_3, \dots will approach a steady state $v_\infty = (.6, .4)$. This final outcome does not depend on starting vector. For every v_0 converge to v_∞ .

The Steady State ~~for~~

$$\text{with eigenvalue } 1: \quad \text{Steady State} \quad \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$$

Steady State,

Not Markov $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has unattainable steady state.

$$B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The steady state $v_0 = (0, 1)$ will give $v_1 = (0, 2)$ $v_2 = (0, 4)$. The second component are doubled.

Markov matrix: * Every column of A adds to 1.

Ex: The fraction of rental cars in Denver starts at $\frac{1}{50} = 0.02$. The fraction outside Denver is .98. Every month 80% of Denver cars stay in Denver. Also 5% of outside cars come in. This means that the fractions $v_0 = (0.02, 0.98)$ are multiplied by A .

$$\text{First month } A = \begin{bmatrix} .8 & .05 \\ .2 & .95 \end{bmatrix} \text{ lead to } v_1 = Av_0 = A \begin{bmatrix} .02 \\ .98 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} .065 \\ .935 \end{bmatrix}$$

Next month. $v_2 = Av_1 = (.09875, .90125)$ This is A^2v_0 .

The diagonal matrix N is simple. The eigenvectors matrix is constant then: A^K equals $S N^K S^{-1}$. The new application to markov matrices uses the eigenvalues and eigenvectors. v_0 is an eigen vector corresponding to $\lambda = 1$.

$A^K v_0$ gives fraction in and out of Denver after K steps.

The eigen values are $\lambda = 1 \vee 0.75$.

$$Ax = \lambda x \quad A \begin{bmatrix} .2 \\ .8 \end{bmatrix} = 1 \begin{bmatrix} .2 \\ .8 \end{bmatrix} \quad A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.75 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The starting vector v_0 combines x_1, v_0x_2 , in this case with coefficient 1 & 0.75:

$$\text{Combination of eigenvalues } v_0 = \begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + .75 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now multiply A to find v_1 .

$$v_1 = 1 \begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)(.18) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Every month, another .75 multiplies the vector x_2 .

The eigenvector x_1 is unchanged:

$$\text{After } K \text{ steps} \quad v_K = A^K v_0 = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + (0.75)^K \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The eigenvalue λ_1 with $\lambda=1$ is the steady state. The other eigenvalues disappear because $|\lambda| < 1$. The mole steps have $v_0 = (0.2, 0.8)$. In the limit, $2/10$ cars are in Denver & $8/10$ are outside.

A is a positive markov matrix ($a_{ij} > 0$, each column adds to 1). If $\lambda_1 = 1$ is larger than any other eigenvalue. The eigenvalue x_1 is the steady state:

$$v_K = x_1 + c_2(\lambda_2)^K x_2 + \dots + c_n(\lambda_n)^K x_n \text{ always approaches}$$

$$v_0 = x_1.$$

Ex. 2: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has no steady state because $\lambda_2 = -1$.

This matrix sends all cars from inside Denver to outside & vice versa. The power A^K alternates between A & I . The reason is that $\lambda_2 = -1$ at every step & ...

correct $x_2 = (-1, 1)$ will be multiplied.

but smaller: No steady state. No steady state: No steady state.

Approach the rank one matrix that has the steady state.

The power A^K approach 0 in every column.

At each time step, half of group goes to group₂ and

Ex: At each time step, half of group also split in half.

then half goes to group₂. The other groups also split in half.

Take one step from the starting population P_1, P_2, P_3 :

more. Take one step from the starting population P_1, P_2, P_3 :

$$\text{new population} \quad v_1 = Av_0 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} P_2 + \frac{1}{2} P_3 \\ \frac{1}{2} P_1 + \frac{1}{2} P_3 \\ \frac{1}{2} P_1 + \frac{1}{2} P_2 \end{bmatrix}$$

A is markov matrix. A contains zeros, which just double in

Ex 2. But after 2 steps in this new A , the zeros disappear from A^2 .

$$2 \text{ Step matrix} \quad V_2 = A^2 V_0 = \begin{pmatrix} -2 & Y_4 & Y_4 \\ Y_4 & Y_2 & Y_4 \\ Y_4 & Y_4 & Y_2 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

The eigen values of A are $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -1$. For $\lambda = 1$, eigen vector $X_1 = (Y_3, Y_3, Y_3)$ will be the steady state. When 3 equal population split in half & more, the populations are again equal. Starting from $V_0 = (8, 16, 32)$, the matrix chain approaches its steady state:

$$V_0 = \begin{bmatrix} 8 \\ 16 \\ 32 \end{bmatrix} \quad V_1 = \begin{bmatrix} 24 \\ 20 \\ 12 \end{bmatrix} \quad V_2 = \begin{bmatrix} 16 \\ 19 \\ 22 \end{bmatrix} \quad V_3 = \begin{bmatrix} 20 \\ 19 \\ 17 \end{bmatrix}$$

The total population: $8 + 16 + 32 = 56$ at every step. The steady state is 56 times (Y_3, Y_3, Y_3) . The three populations approaching, but never reaching, their final limits $56/3$.

Perron - Frobenius Theorem:

One matrix theorem: dominance. The Perron Frobenius theorem applies when all $a_{ij} \geq 0$. There is no requirement that last row add to 1.

Perron - Frobenius for $A \geq 0$. All numbers in $Ax = t_{\max}$

are strictly positive.

Proof: Number t , such that $Ax \geq t x$, $x \neq 0$. Consistency holds $Ax = t_{\max} x$.

If $Ax \geq t_{\max} x$ is not an equality, multiply by A . Because A is positive, that produces a strict inequality $A^2 x > t_{\max} Ax$. Therefore $Ay > t_{\max} y$ and t_{\max} could be increased. This contradiction forces equality $Ax = t_{\max} x$.

The eigenvalue λ is positive because on the left side of that equality, Ax is sum to be positive.

No eigenvalue can be larger than $|z|$, suppose $Az = \lambda z$. Since λ and z may involve negative or complex numbers, $|z|$ is non-negative vector, so $|\lambda|$ is one of the possible candidates, $|\lambda|$ cannot exceed $|z|$ which must be. Also $\text{tr}(A)$ cannot exceed $|z|$.

population growth:

Divide population into 3 age groups: age < 20, 20-39 & 40-59. At year T , the sizes of those groups are n_1, n_2, n_3 . Twenty years later, the sizes have changed for 2 reasons:

1. Reproduction $n_1^{\text{new}} = F_1 n_1 + F_2 n_2 + F_3 n_3$ gives the old generation.
2. Survival $n_2^{\text{new}} = P_1 n_1$ and $n_3^{\text{new}} = P_2 n_2$ gives the old generation.

The fertility rates are F_1, F_2, F_3 (F_2 largest). The Leslie matrix A might look like this

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}_{\text{new}} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} .04 & 1.1 & .01 \\ .98 & 0 & 0 \\ 0 & .92 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

In a realistic model, A will change with time, include A^k group age ≥ 60 .

The matrix has $A \geq 0$ but not $A > 0$. The Perron-Frobenius theorem applies because $A^2 > 0$. The largest eigenvalue is $\lambda_{\max} = 1.06$.

Starting from $n_2 = 1$ in the middle generation:

$$\begin{aligned} \text{eig}(A) &= 1.06 \\ A^2 &= \begin{bmatrix} 1.08 & 0.05 & 0.0 \\ 0.04 & 1.09 & 0.01 \\ .90 & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0.1 & 1.19 & .01 \\ 0.06 & 0.05 & 0.0 \\ 0.04 & 0.99 & 0.01 \end{bmatrix} \end{aligned}$$

A fast start $v_0 = (0, 1, 0)$. The middle group will reproduce
 1.1 & survive 0.92. The newest & oldest generation are in
 $v_1 = (1.1, 0, 0.92) = \text{column 2 of } A$. Then $v_2 = Av_1 = A^2 v_0$
 but asymptotic growth rate λ_{max} is same from every start.
 Its eigenvalues $\lambda = (0.63, 0.58, 0.51)$ show all 3 groups growing
 steadily together.

Linear Algebra in Economics: The consumption matrix
 The consumption matrix tells how much of each input goes into
 unit of output. This describes manufacturing side of economy.

Consumption matrix: There are n industries like Chemical, food & oil.
 To produce unit of Chemical may require 0.2 units of Chemical,
 0.3 units of food & 0.4 units of oil. These numbers go into row 1
 of Consumption matrix A:

$$\begin{bmatrix} \text{Chemical o/p} \\ \text{food o/p} \\ \text{oil o/p} \end{bmatrix} = \begin{bmatrix} .2 & .3 & .4 \\ .4 & .4 & .1 \\ .5 & .1 & .3 \end{bmatrix} \begin{bmatrix} \text{Chemical i/p} \\ \text{food i/p} \\ \text{oil i/p} \end{bmatrix}$$

Row 2 shows the inputs to produce food - heavy use of Chemical
 & food, not so much oil. Row 3 of A shows input consumed by
 refine a unit of oil.

This economy has demands y_1, y_2, y_3 for Chemical, food, oil.
 The inputs p_1, p_2, p_3 will have to be higher - because part of p
 is consumed in producing y . The input is p and Consumption
 is A^P , which leaves the O/P $P - A^P$. This net production
 is A^P , which leaves the O/P $P - A^P$. This net production

which needs demand y :

$$\text{Geometric Series: } S = (\Gamma - A)^{-1} = \Gamma + A + A^2 + A^3 + \dots$$

$S = A S = I$. which is $(I - A)S = I$. The sum of all eigenvalues of $A \geq 0$, all terms of sum are nonnegative. The sum is $(I - A)^{-1} \geq 0$.

If A have $\{\lambda\} < 1$,
 $A \geq 0$, all terms of sum are nonnegative. The sum is $(I - A)^{-1} \geq 0$.

$$(2) \quad A = \begin{bmatrix} 1.2 & 1.5 & 1.4 \\ 1.4 & 1.4 & 1.1 \\ 1.5 & 1.1 & 1.3 \end{bmatrix} \text{ has } \text{Avane} = 0.9 \vee (I - A)^{-1} = \frac{1}{0.9} \begin{bmatrix} 41 & 25 & 24 \\ 33 & 36 & 24 \\ 34 & 23 & 36 \end{bmatrix}$$

This economy is productive. A is small compared to I , because there is 0.9.

Start $P = (I - A)^{-1}y$ then AP is consumed in production, having $P - AP$. This is $(I - A)P = y$ if demand is met.

$$(3) \quad A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \text{ has } \text{Avane} = 2 \vee (I - A)^{-1} = -\frac{1}{3} \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

This consumption matrix is too large. Demands can't be met, because production exceeds more than it yields. Then $\text{Sum } I + A + A^2 - \dots$ doesn't converge to $(I - A)^{-1}$ because $\text{Avane} > 1$. The series is growing while.

(2-A) \dagger is actually negative.

(2-A) \dagger is actually negative.

Linear programming: For $Ax = b$, If there are any solution $x \geq 0$

linear programming picks the solution $x^* \geq 0$ that

minimizes the cost:

The cost is $c_1 x_1 + \dots + c_n x_n$. The winning reckons x^* is non-negative

solution of $Ax = b$ that has smallest cost.

The linear programming

$\triangleright A$ has $n > m$: Ex: $A = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$

i) b has m components for m equations $Ax = b$, $b = [4]$

ii) C has n components for n components: $C = \begin{bmatrix} 5 & 3 & 8 \end{bmatrix}$

iii) The cost vector c has n components C is subject to requirements $Ax = b$, $x \geq 0$.

Then the problem is to minimize $C^T x$ subject to $x_1 + 2x_2 + 3x_3 = 4$ & $x_1, x_2, x_3 \geq 0$.

Minimize $5x_1 + 3x_2 + 8x_3$.

Look at constraint: $Ax = b$ & $x \geq 0$, the equation $x_1 + x_2 + 2x_3 = 4$ gives a plane in 3 dimensions. The nonnegativity $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ chops the plane down to triangle.

Inside that triangle, all component of x are positive on the edges of PQR , one component is zero. At the corners P , Q & VR , 2 components are zero. The Optimal Solution x^* will be one of those corners.

The triangle contains all vector x that satisfy $Ax = b$ and $x \geq 0$. These x 's are called feasible point & triangle is feasible set. These points are not allowed. Candidates in the minimization of $C \cdot x$ which is the final step

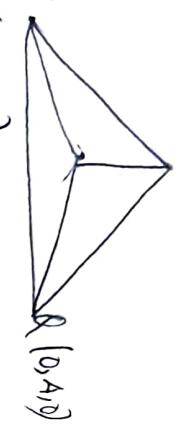
Find x^* in the triangle PQR to minimize the cost $5x_1 + 3x_2 + 8x_3$.

The vector that have zero cost lie on plane $5x_1 + 3x_2 + 8x_3 = 0$. That plane doesn't meet the triangle. Increase cost until the plane $5x_1 + 3x_2 + 8x_3 = c$ does meet triangle. As c increases, plane $5x_1 + 3x_2 + 8x_3 = c$ moves toward the triangle.

$$R = (0, 0, 2)$$

parallel plane: moving parallel plane: moving $5x_1 + 3x_2 + 8x_3 = c$ to touch.

The first plane: $5x_1 + 3x_2 + 8x_3 = c$. The point where



triangle has minimum cost c . This touching plane $5x_1 + 3x_2 + 8x_3 = c$ it touches in the solution x^* . This touching plane could reach must be one of corners P , Q , R . A moving plane could reach inside of triangle before it touches a corner. So check the cost

inside of triangle at each corner.

$$5x_1 + 3x_2 + 8x_3 \text{ at each corner}$$

$$P = (1, 0, 0) \text{ cost } 20 \quad Q = (0, 4, 0) \text{ cost } 12 \quad R = (0, 0, 2) \text{ cost } 16.$$

The winner is Q . Then $x^* = (0, 4, 0)$ solves problem.

If the cost vector C is changed, the parallel planes are tilted. For small changes, Q is still winner. For cost $(C \cdot x = 5x_1 + 3x_2 + 8x_3)$ for optimum x^* moves to $R = (0, 0, 2)$. The minimum cost is now $F \cdot x = 14$.

linear programming maximize profit instead of minimizing cost.
initial parallel plane start with a large value of c , instead of a small
to move toward the origin. as c gets smaller. The first
value. Touching point is still a corner.

Note 2: The requirements $Ax = b$ & $x \geq 0$ could be impossible to
satisfy. The equation $x_1 + x_2 + x_3 = 1$ cannot be solved with $x \geq 0$.

The feasible set is empty.

Note 3: The feasible set is unbounded. If the revised set is
 $x_1 + x_2 - 2x_3 = 4$ the large positive vector $(100, 100, 98)$. The plane
chopped off to a triangle. The new corner point
 $Ax = b$ is no longer candidate for x^* but R moved to infinity.

We still candidate for x^* but the minimum cost could be
unbounded. feasible set, the vector $(100, 100, 99)$
Note A: with an unbounded feasible set, the vector $(100, 100, 99)$
Suppose cost is $-x_1 - x_2 + x_3$. Then the vector $(1000, 1000, 998)$ costs $C = -1002$. In
cost $C = -102$. The vector $(1000, 1000, 998)$ is
nonnegative and feasible.