EC540 Control Systems

Recalling Signals and Systems

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Instructions:

1. Lecture session will be of one hour duration.
2. Fifteen minutes session will be made available for students after the lecture for any question and answers.
3. Regularly review Signals and Systems concepts
4. Regularly visit course webpage.
5. Everyday learn new functions from Octave/Python/MATLAB software
6. Email me on any queries at sudarshan_pk@sjce.ac.in

1. Basics of Signals

Signal is any *useful* information about a physical variable such as tempreature, light intensity, sound level, stock market index etc with respect to independent variable time.

Signal is mathematically represented as function $f(t)$ where $-\infty < t < \infty$ indicates time.

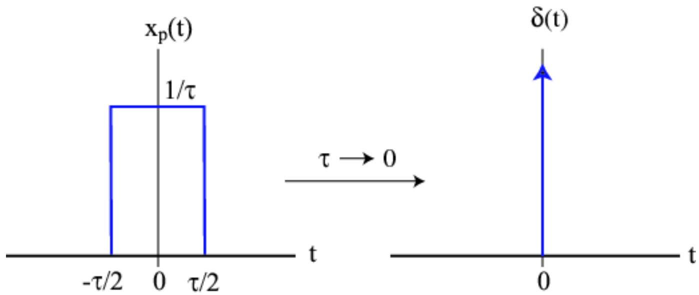
Causal Signal: Signal having $f(t) = 0; t < 0$
Periodic Signal: Signal having $f(t) = f(t + nT)$ where n is any integer. $F_0 = \frac{1}{T}$ is called fundamental frequency of signal.

Standard functions:

Impulse Signal:

$$x_p(t) = \frac{1}{\tau} \qquad \text{for } -\tfrac{\tau}{2} < t < \tfrac{\tau}{2}$$

$$\delta(t) = \lim_{\tau \to 0} x_p(t).$$



Standard functions:

Step Signal:

$$f(t) = u_s(t) = \begin{cases} 1 & t \geq 0 \,, \\ 0 & \text{otherwise} \end{cases}$$

Ramp Signal:

$$f(t) = \begin{cases} t & t \geq 0 \,, \\ 0 & \text{otherwise} \end{cases}$$

Exponential Signal:

$$f(t) = \begin{cases} e^{at} & t \geq 0 \,, \\ 0 & \text{otherwise} \end{cases}$$

At $t = 0$ function value is 1. If $a > 0$ then function value increases exponentially; If $a < 0$ then function value decreases exponentially.

Standard functions:

Sin Signal:

$$f(t) = \begin{cases} sin(\omega_0 t) & t \geq 0 \ , \\ 0 & \text{otherwise} \end{cases}$$

cos Signal:

$$f(t) = \begin{cases} cos(\omega_0 t) & t \geq 0 \ , \\ 0 & \text{otherwise} \end{cases}$$

Complex Exponential Signal:

$$f(t) = \begin{cases} e^{j\omega_0 t} & t \geq 0 \ , \\ 0 & \text{otherwise} \end{cases}$$

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t)$$

Any signal/function $f(t)$ can be represented using the fundamental function of time domain $\delta(t)$

$$f(t) = \int f(\tau)\delta(t-\tau)d\tau.$$

Any signal/function $f(t)$ can be represented using the fundamental function of frequency domain $e^{j\omega t}$

Fourier Series For periodic signals

$$f(t) = \sum_{-\infty}^{\infty} c_k e^{j2\pi k F_0 t} = a_0 + \sum_{k=1}^{\infty} \{a_k \cos(2\pi k F_0 t) - b_k \sin(2\pi k F_0 t$$

$$c_k = \frac{1}{T} \int_0^T f(t) e^{-j2\pi k F_0 t} dt.$$

Alternatively,

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos(2\pi k F_0 t) dt \text{ for } k = 1, 2, \ldots$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin(2\pi k F_0 t) dt \text{ for } k = 1, 2, \ldots$$

Fourier Transform: For aperiodic signals,

$$\boxed{F(j\omega) = \mathcal{F}\{f(t)\} = \int f(t)e^{-j\omega t}dt}$$

Inverse Fourier Transform:

$$\boxed{f(t) = \frac{1}{2\pi}\int F(j\omega)e^{j\omega t}d\omega}$$

here $-\infty < \omega < \infty$

Definition of Laplace Transform: For any signal,

$$\boxed{F(s) = \int f(t)e^{-st}dt} \qquad \text{where } s = \sigma + j\omega$$

Region of Convergence Values of σ for which integral converges.

$$F(s) = \mathcal{F}\{f(t).e^{-\sigma t}\}$$

$$F(j\omega) = F(s) \text{ when } \sigma = 0$$

Laplace Transform of Standard Functions:

$\mathbf{f(t)}$	$\mathbf{F(s)}$
$\delta(t)$	1
$u_s(t)$	$\frac{1}{s}$
$tu_s(t)$	$\frac{1}{s^2}$
$\frac{t^2}{2}.u_s(t)$	$\frac{1}{s^3}$
$e^{at}.u_s(t)$	$\frac{1}{s-a}$
$t.e^{-at}.u_s(t)$	$\frac{1}{(s+a)^2}$
$\sin\omega_0 t.u_s(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
$\cos\omega_0 t.u_s(t)$	$\frac{s}{s^2+\omega_0^2}$

Properties of Laplace Transform:

1. $\mathcal{L}\{\alpha f_1(t) + \beta f_2(t)\} = \alpha F_1(s) + \beta F_2(s)$. Principle of superposition

2. $\mathcal{L}\{f(t-T)\} = F(s)e^{-sT}$

3. $\mathcal{L}\{f(t)e^{at}\} = F(s-a)$;

4. $\mathcal{L}\{f(at)\} = \frac{1}{|a|}F\left(\frac{s}{a}\right)$

5. $\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$

6. $\mathcal{L}\{\frac{d^n f(t)}{dt^n}\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{n-1}(0)$

7. $\mathcal{L}\{\int f_1(\tau)f_2(t-\tau)d\tau\} = F_1(s)F_2(s)$

8. $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

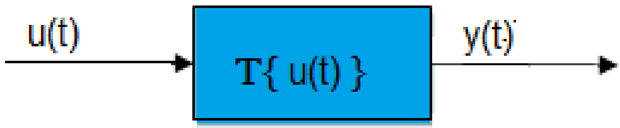
9. $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

2. Basics of Systems

System

$$y(t) = \mathcal{T}\{u(t)\}$$

transforms the input signal $u(t)$ to an output signal based on the operator $\mathcal{T}$



Linear System

$$y(t) = \mathcal{T}\{u(t)\}$$

obeys principle of superposition:

$$\begin{aligned} y(t) &= \mathcal{T}\{u(t)\} \\ &= \mathcal{T}\{a_1u_1(t) + a_2u_2(t)\} \\ &= a_1\mathcal{T}\{u_1(t)\} + a_2\mathcal{T}\{u_2(t)\} \\ &= a_1y_1(t) + a_2y_2(t) \end{aligned}$$

Example: $y(t) = Au(t)$ is linear system.

$$
\begin{aligned}
y_1(t) &= \mathcal{T}\{u_1(t)\} = Au_1(t) \\
y_2(t) &= \mathcal{T}\{u_2(t)\} = Au_2(t) \\
y(t) &= \mathcal{T}\{a_1u_1(t) + a_2u_2(t)\} = A\{a_1u_1(t) + a_2u_2(t)\} \\
&= a_1Au_1(t) + a_2Au_2(t) \\
&= a_1y_1(t) + a_2y_2(t)
\end{aligned}
$$

Example: $y(t) = A\frac{du(t)}{dt}$ is linear system.

$$y_1(t) = \mathcal{T}\{u_1(t)\} = A\frac{du_1(t)}{dt}$$
$$y_2(t) = \mathcal{T}\{u_2(t)\} = A\frac{du_2(t)}{dt}$$
$$\begin{aligned} y(t) = \mathcal{T}\{a_1u_1(t) + a_2u_2(t)\} &= A\frac{d\{a_1u_1(t) + a_2u_2(t)\}}{dt} \\ &= a_1A\frac{du_1(t)}{dt} + a_2A\frac{du_2(t)}{dt} \\ &= a_1y_1(t) + a_2y_2(t) \end{aligned}$$

Example:

$y(t) = \mathcal{T}\{u^2(t)\}$ is a Nonlinear system

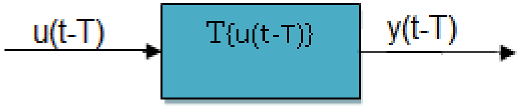
$$\begin{aligned}
y_1(t) &= \mathcal{T}\{u_1(t)\} = u_1^2(t) \\
y_2(t) &= \mathcal{T}\{u_2(t)\} = u_2^2(t) \\
y(t) &= \mathcal{T}\{a_1u_1(t) + a_2u_2(t)\} = (a_1u_1(t) + a_2u_(t))^2 \\
&= a_1^2u_1^2(t) + a_2^2u_2^2(t) + 2a_1a_2u_1(t)u_2(t) \\
&\neq a_1y_1(t) + a_2y_2(t).
\end{aligned}$$

Example: Is $y(t) = mu(t) + c$ a linear system?

$$
\begin{aligned}
y_1(t) &= \mathcal{T}\{u_1(t)\} = mu_1(t) + c \\
y_2(t) &= \mathcal{T}\{u_2(t)\} = mu_2(t) + c \\
y(t) &= \mathcal{T}\{a_1u_1(t) + a_2u_2(t)\} \\
&= m(a_1u_1(t) + a_2u_2(t)) + c \\
&\neq a_1y_1(t) + a_2y_2(t).
\end{aligned}
$$

Time Varying/Invariant System

$$
\begin{aligned}
y(t) &= \mathcal{T}\{u(t)\} \\
y_2(t) &= \mathcal{T}\{u(t-T)\} \\
&\neq y(t-T) \text{ Time varying system} \\
&= y(t-T) \text{ Time invariant system}
\end{aligned}
$$



Example: $y(t) = \mathcal{T}\{u(t)\} = tu(t)$ is time varying system.

$$y_2(t) = \mathcal{T}\{u(t-T)\} = tu(t-T)$$
$$y_2(t) \neq y(t-T) = (t-T)u(t-T).$$

$y(t) = \mathcal{T}\{u(t)\} = A\frac{du(t)}{dt}$ and $y(t) = \mathcal{T}\{u(t)\} = mu(t) + c$ are examples of Time invariant systems.

Linear Time-Invariant System (LTI) satisfies both linearity and timeinvariance property.
$y(t) = Au(t)$, $y(t) = A\frac{d^n u(t)}{dt^n}$, $y(t) = \int Au(\tau)d\tau$ are examples.
Any linear combinations of above examples are also LTI systems.

Impulse Response and Convolution Let us denote $g(t) = \mathcal{T}\{\delta(t)\}$; is called impulse response of the system.

$$
\begin{aligned}
y(t) &= \mathcal{T}\{u(t)\} \\
&= \mathcal{T}\left\{\int u(\tau)\delta(t-\tau)d\tau\right\} \\
&= \int u(\tau)\mathcal{T}\{\delta(t-\tau)\}d\tau \quad \text{due to linearity} \\
&= \int u(\tau)g(t-\tau)d\tau \quad \text{due to time invariance.}
\end{aligned}
$$

is called convolution integral.

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$$\boxed{a_0 \frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_n y(t) = b_0 \frac{d^m u(t)}{dt^m} + \ldots + b_m}$$

$$\boxed{\left(a_0 s^n + a_1 s^{n-1} + \ldots + a_n\right) Y(s) = \left(b_0 s^m + b_1 s^{m-1} + \ldots + b_m\right)}$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_n}.}$$

$$G(s) = \mathcal{L}\{g(t)\}; \qquad Y(s) = \mathcal{L}\{y(t)\} \qquad U(s) = \mathcal{L}\{u(t)\}$$

Since $y(t) = \int u(\tau)g(t-\tau)d\tau$

By property of laplace transform of convolution integral:

$$\boxed{Y(s) = G(s)US(s)}$$