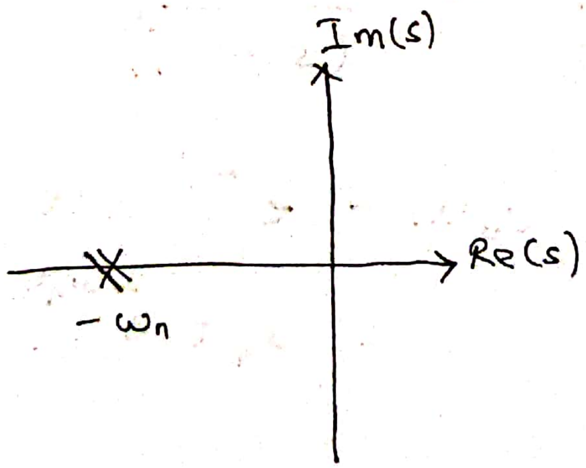
Case III $\xi = 1$ Critically Damped

$$G(s) = \frac{\omega_n^2}{s^2 + 2\omega_n \xi s + \omega_n^2}$$

$$= \frac{\omega_n^2}{(s+\omega_n)^2}$$

Poles of system $\Rightarrow$ Roots of $(s+\omega_n)^2 = 0$

Two Poles are at $P_{1,2} = -\omega_n$ on Real Axis.



Impulse Response of the system

$$g(t) = \omega_n^2 \, t \cdot e^{-\omega_n t} \quad \text{for } t \geq 0.$$

$g(t) \to 0$ as $t \to \infty$.

Step Response of SSOS when $\xi = 1$

$$Y(s) = G(s) \cdot U(s)$$

$$= \frac{1}{s} \cdot \frac{\omega_n^2}{(s^2 + 2\omega_n s + \omega_n^2)}$$

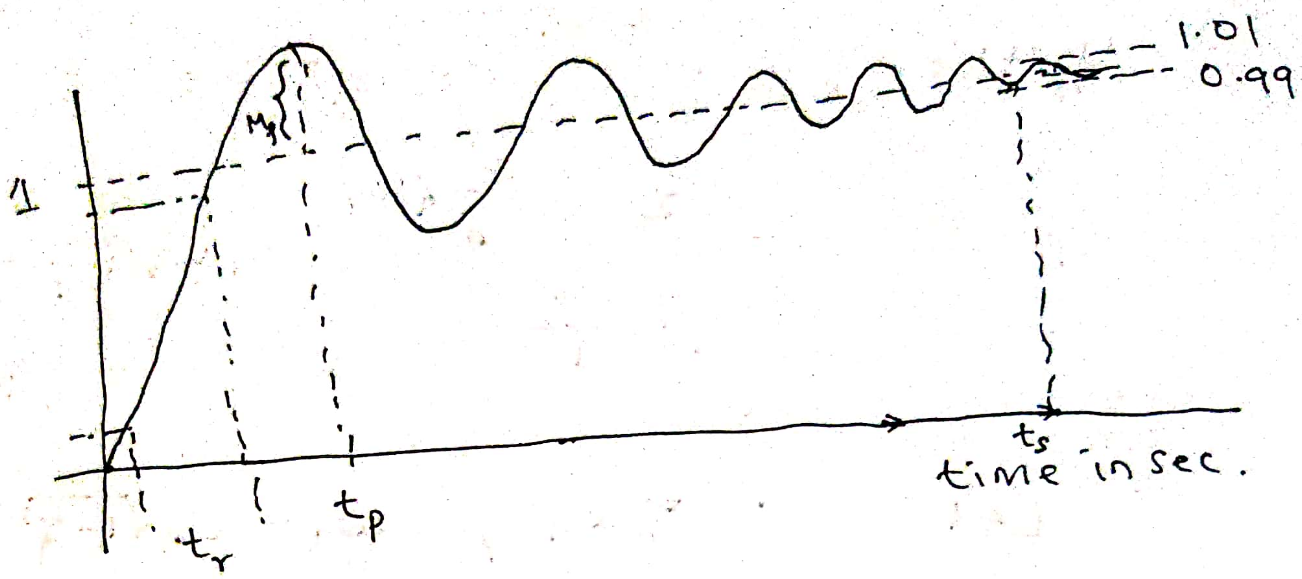
$$= \frac{A}{s} + \frac{Bs + C}{(s^2 + 2\omega_n s + \omega_n^2)}$$

$A = 1 \quad B = -1 \quad C = -2\omega_n$

$$Y(s) = \frac{1}{s} - \frac{(s+\omega_n)}{(s+\omega_n)^2} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$\therefore \quad y_s(t) = u_s(t) - e^{-\omega_n t} u_s(t) - \omega_n t \, e^{-\omega_n t} \cdot u_s(t)$$

Time Domain Specifications:



Rise Time: t_r Time taken by the step response to rise from 5% of steady state value to 95% of steady state value.

$$t_r \approx \frac{1.8}{\omega_n}$$

Peak Time: t_p. Time at which step response of the system is maximum value.

$$y_s(t_p) = y_{max}$$

$$y_s(t) = 1 - e^{-\xi\omega_n t} \cos \omega_n\sqrt{1-\xi^2}\, t$$

$$- \frac{\xi}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin \omega_n\sqrt{1-\xi^2}\, t$$

for $t \geq 0$.

Let $\sigma = \xi\omega_n$, $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$y_s(t) = 1 - e^{-\sigma t} \cos \omega_d t - \frac{\sigma}{\omega_d} e^{-\sigma t} \sin \omega_d t$$

$$\frac{dy_s(t)}{dt} = 0 \quad \text{when} \quad t = t_p.$$

$$\therefore \quad 0 = 0 - \left\{ e^{-\sigma t} (-\sin \omega_d t)\, \omega_d - \sigma e^{-\sigma t} \cos \omega_d t \right\}$$

$$- \frac{\sigma}{\omega_d} \left\{ e^{-\sigma t} \cos \omega_d t\, \omega_d - \sigma e^{-\sigma t} \sin \omega_d t \right\}$$

$$0 = \sin \omega_d t \left\{ e^{-\sigma t} \right\} \left\{ 1 + \frac{\sigma^2}{\omega_d} \right\}$$

$$\boxed{\omega_d t_p = \pi} \qquad \boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

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Overshoot in % $\quad M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$.

Definition $\quad M_p = \dfrac{y_{max} - y_{ss}}{y_{ss}} \times 100$

y_{max} = Maximum of $y_s(t)$

y_{ss} = steady state value of $y_s(t)$

$$y_{max} = y_s(t_p)$$

$$= 1 + e^{-\xi\omega_n t_p}$$

$$= 1 + e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$\therefore$ When $y_{ss} = 1$

$$M_p = \frac{1 + e^{-\pi\xi/\sqrt{1-\xi^2}} - 1}{1} \times 100$$

$$\boxed{M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}}$$

Settling Time: t_s Time at which step response enters 99% of y_{ss}.

When $y_{ss} = 1$

$y_s(t) = 0.99$ or 1.01.

$$e^{-\xi \omega_n t_s} \approx 0.01$$

$$\boxed{t_s \approx \frac{4.6}{\xi \omega_n}}$$