Control Systems

(EC 530)

Brief Notes on

Theory of State-Space Models and Control

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I Introduction

Early classical control design techniques were entirely based on the mathematical representation of the plant to be controlled in the form of input output relationship known as transfer function model. In this technique, output variable of the plant to be controlled is measured using the sensor and the information is fed back to the input through a controller. Controller could then be designed using trial and error method or Evan's root locus technique to meet the design criteria. However, with this approach, one is constrained to place the poles of the closed-loop system only along the root-locus dictated by the configuration of the plant and controller.

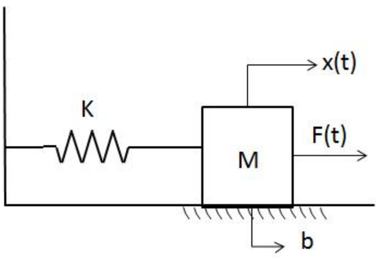
A new modern approach evolved during the 1960s and 1970s, largely due to the efforts of R. E. Kalman, R. E. Bellman, T. Kailath, tried to address the issue mentioned above, by bringing in the concept of state of a system. But more importantly, it gave an entirely fresh perspective on how we view physical systems and in some cases even non-physical systems such as population models, biological systems etc [Refer to works of Ludwig von Bertalanffy]. This led to evolution of an entirely new field of study called systems science.

The new approach is based on a strong notion of '*state of the system*'. State of the dynamical system at time t is described by the collection of values of variables to be controlled(i.e. variables other than the inputs) at a point in time t. Thus, mathematical vector $X(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_n(t) \end{bmatrix}'$ describes the state of the system at time t. State-space model is then a first order differential equation for the state vector and an output equation relating the state and the inputs. It is written in the standard form:

$$\dot{X}(t) = AX(t) + Bu(t) \tag{1}$$
$$y(t) = CX(t) + Du(t). \tag{2}$$

Eq 1 is referred to as state equation and Eq 2 is referred to as output equation.

I.1 State Space Model for a Mechanical System



Force Equation:
$F(t) - b\dot{x}(t) - Kx(t) = M\ddot{x}(t)$

Choose your state space variables such that the system behaviour is completely described. Let $x_1(t)$ be velocity of the mass and $x_2(t)$ be position of the mass.

$$x_1(t) = \dot{x}(t) \qquad x_2(t) = x(t) \qquad Input \;\; u(t) = F(t)$$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} \qquad \dot{X}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \ddot{x}(t) \\ \dot{x}(t) \end{bmatrix}$$

The state equation is

$$\dot{X}(t) = \underbrace{\begin{bmatrix} -\frac{b}{M} & -\frac{k}{M} \\ 1 & 0 \end{bmatrix}}_{\text{A}} X(t) + \underbrace{\begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix}}_{\text{B}} u(t)$$

with $x(t)$, the position of the mass m, as output, the output equation will be

$$y(t) = \underbrace{[0 \quad 1]}_{\text{C}} X(t) + \underbrace{0}_{\text{D}} \quad u(t)$$

I.2 Relation of State-Space Model to Transfer Function Model

We know that transfer function model is a purely input-output model in s- domain under the assumption that initial conditions are zero. Hence we need an expression for $X(s)$:

Consider the state equation,

$$\dot{X}(t) = AX(t) + Bu(t) \tag{3}$$

To obtain $X(s)$, apply Laplace Transform to the state equation,

$$sX(s) - X(0) \quad = \quad AX(s) + BU(s) \tag{4}$$

Assuming intial condition $x(0) = 0$, and taking $X(s)$ terms on one side,

$$\begin{aligned} sX(s) - AX(s) &= BU(s) && (5) \\ [sI - A]\, X(s) &= X(0) + BU(s) && (6) \\ & && (7) \end{aligned}$$

Pre-multiplying the above equation by $[sI - A]^{-1}$:

$$X(s) = [sI - A]^{-1} BU(s). \tag{8}$$

Now applying Laplace transform to the output equation,

$$Y(s) = CX(s) + DU(s) \tag{9}$$

Substituting expression of $X(s)$ obtained from equation 8 in the above equation:

$$\frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D \tag{10}$$

MATLAB function:

[num, den]=ss2tf(A,B,C,D)

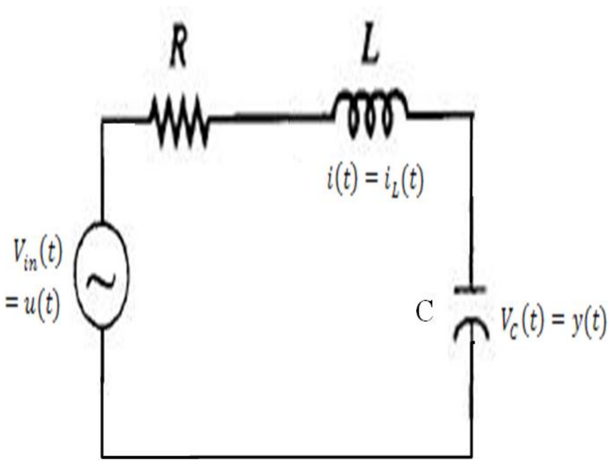
Example: For the above example of mechanical system,

$$\frac{Y(s)}{U(s)} = [0 \quad 1] \begin{bmatrix} s+\frac{b}{M} & \frac{k}{M} \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix} + 0$$

$$\frac{Y(s)}{U(s)} = \frac{[0 \quad 1] \begin{bmatrix} s & \frac{-k}{M} \\ 1 & s+\frac{b}{M} \end{bmatrix} \begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix}}{det \begin{bmatrix} s+\frac{b}{M} & \frac{k}{M} \\ -1 & s \end{bmatrix}}$$

$$= \frac{\frac{1}{M}}{s(s+\frac{b}{M})+\frac{k}{M}} = \frac{\frac{1}{M}}{s^2+\frac{b}{M}s+\frac{k}{M}}$$

I.3 State -Space Modeling of an Electrical System



KVL for circuit is

$$V_{in}(t) = R\, i_L(t) + L\frac{d}{dt}\, i_L(t) + V_C(t)$$

where $\quad i_L(t) = C\frac{d}{dt}\, V_C(t)$

Following the approach used in the above mechanical system example, for the series RLC eclectric circuit, one can write the differential equation in terms of the output variable $V_o(t) = V_c(t)$, i.e.,

$$V_{in}(t) = LC\frac{d^2V_o(t)}{dt^2} + RC\frac{dV_o(t)}{dt} + V_o(t).$$

Then choose the state-variables $x_1(t) = \dot{V}_o(t)$ and $x_2(t) = V_o(t)$. This will result in following state-space model:

$$\dot{X}(t) = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{LC} \\ 1 & 0 \end{bmatrix} X(t) + \begin{bmatrix} \frac{1}{LC} \\ 0 \end{bmatrix} u(t)$$

Now using the relation between state-space model and transfer function, one obtains:

$$\frac{Y(s)}{U(s)} = \frac{V_o(s)}{V_{in}(s)} = \frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}.$$

Alternatively, one can obtain a different state-space model, by defining the state variables in a different manner. Let $x_1(t) = i_L(t)$ and $x_2(t) = V_c(t)$. This definition is more natural in terms of the physical system. With this definition,

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \;\; \& \;\; \dot{X}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{d}{dt} i_L(t) \\ \\ \frac{d}{dt} V_C(t) \end{bmatrix}$$

$$\dot{X} = \underbrace{\begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \\ \frac{1}{C} & 0 \end{bmatrix}}_{\text{A}} X(t) + \underbrace{\begin{bmatrix} \frac{1}{L} \\ \\ 0 \end{bmatrix}}_{\text{B}} u(t)$$

$$y(t) = \underbrace{[0 \quad 1]}_{\text{C}} X(t) + \underbrace{0}_{\text{D}} u(t)$$

State-Space to Transfer Function Model From equation 10

$$\frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D$$

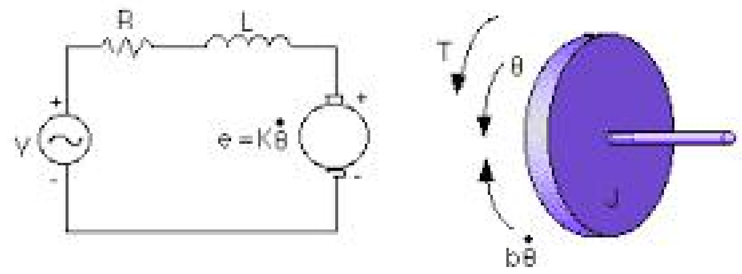
$$\frac{Y(s)}{U(s)} = [0 \quad 1] \begin{bmatrix} s + \frac{R}{L} & \frac{1}{L} \\ \\ \frac{-1}{C} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{L} \\ \\ 0 \end{bmatrix} + 0$$

$$\frac{Y(s)}{U(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

State-space Model of DC Motor

DC Motor is an electro-mechanical device. It operates on both law of motor (where mechanical force/torque is generated in mutually perpendicular directions of current and magnetic field) and law of generator (where voltage gets generated in a mutually perpendicular direction of mechanical force/torque and current). Its various parameters are:

$$\begin{aligned} R &= \text{resistance of armature winding (ohms),} \\ L &= \text{inductance of armature winding (henrys),} \\ i(t) &= \text{armature current (amperes),} \\ e_b &= \text{back emf (volts),} \\ V_{in}(t) &= \text{applied voltage (volts),} \\ k_1 &= \text{motor torque constant (Newton.meters/amps),} \\ k_2 &= \text{back emf constant (volts/rad/sec),} \\ \theta(t) &= \text{motor position (radians)} \\ \omega(t) &= \frac{d\theta(t)}{dt} \text{ motor velocity (rad/sec).} \end{aligned}$$



By Kirchoff's Voltage Law:

$$V_{in}(t) \;=\; L\frac{di(t)}{dt} + Ri(t) + e_b(t). \tag{11}$$

Due to law of generator, back emf is generated in proportion to speed of the motor. i.e.,

$$e_b(t) \;=\; k_2\frac{d\theta(t)}{dt}. \tag{12}$$

Now the torque equation for the motor is:

$$\begin{aligned} T(t) &= b\frac{d\theta(t)}{dt} + J\frac{d^2\theta(t)}{dt^2} &(13)\\ &= b\omega(t) + J\frac{d\omega(t)}{dt}, &(14)\end{aligned}$$

where, due to law of motor:

$$T(t) \;=\; k_1 i(t). \tag{15}$$

By defining the state variables as speed of motor and coil current, i.e.: $x_1(t) = \omega(t) = \frac{d\theta(t)}{dt}$ and $x_2(t) = i(t)$, state-space model will be as follows:

$$\begin{aligned} \dot{X}(t) &= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-b}{J} & \frac{k_1}{J} \\ \frac{-k_2}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t), &(16)\\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. &(17)\end{aligned}$$

By using equation 10,

$$\frac{Y(s)}{U(s)} = \frac{k_1}{(Js+b)(Ls+R)+k_1k_2}. \tag{18}$$

II. Poles and Zeros of the System and State Space Model

From equation 10, numerator polynomial in the transfer function is $b(s) = CAdj([sI - A])B + D|sI - A|$. Hence zeros of the system are roots of $CAdj([sI - A])B + D|sI - A| = 0$. Similarly, denominator polynomial or the characteristic equation of the system is $a(s) = |sI - A|$. Hence poles of the system are roots of $|sI - A| = 0$, which are nothing but eigen values of the system matrix A.

Remark: We also note that there could be some common roots between those of $CAdj([sI - A])B + D|sI - A| = 0$ and $|sI - A| = 0$ in which case zeros and poles of the system will be subset of roots of these polynomial. This situation occurs when the the dimension of the state-space model is higher than the order of the system. Such state-space model is referred to as non-minimal model.

III. Solution of the State Equation.

Since state equation is a first order differential equation for the vector $X(t)$, It is possible to find an expression for $X(t)$ at time t, given the initial condition $X(0)$.

Consider the state equation,

$$\dot{X}(t) = AX(t) + Bu(t) \tag{19}$$

As we know, to solve a differential equation , most convinient tool is Laplace transformorms.

To obtain $X(s)$, apply Laplace Transform to eqn (1),

$$\begin{aligned} sX(s) - X(0) &= AX(s) + BU(s) \\ sX(s) - AX(s) &= X(0) + BU(s) \\ [sI - A]\, X(s) &= X(0) + BU(s) \end{aligned} \tag{20}$$

Pre-multiply the above equation by $[sI - A]^{-1}$:

$$X(s) = [sI - A]^{-1}X(0) + [sI - A]^{-1}]BU(s) \tag{21}$$

(Note: With initial condition $X(0) = 0$, using equation 21 and output equation $Y(s) = CX(s) + DU(s)$, It follows that $\frac{Y(s)}{U(s)} = C[sI - A]^{-1}]B + D)$.)

To obtain the time domain expression for X(t), apply inverse Laplace transform to 21:

$$X(t) = L^{-1}\{[sI - A]^{-1}X(0)\} + L^{-1}\{[sI - A]^{-1}BU(s)\} \tag{22}$$

Now, $L^{-1}\{[sI - A]^{-1}\}$ is defind as $exp(At)$ also written as e^{At}. It is called matrix exponential.

(Note: If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $exp(At) \neq \begin{bmatrix} e^{a_{11}t} & e^{a_{12}t} \\ e^{a_{21}t} & e^{a_{22}t} \end{bmatrix}$).

Eqn. 22 becomes,

$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

If initial time is (t_0) instead of $t = 0$,

$X(t) = e^{A(t-t_0)}X(t_0) + \int_{t_0}^t e^{A(t-t_0-\tau)}Bu(\tau)d\tau$

State equation with zero input is called homogeneous state equation. In that case, state solution is

$$X(t) = e^{A(t-t_0)}X(t_0)$$

and $e^{(A(t-t_0))}$ is termed as state-transition matrix of continuous time LTI system and denoted as $\Phi(t, t_0)$. In general, matrix that causes transition in the state from state at time t_1 to state at time t_2 is $\Phi(t_2, t_1) = e^{A(t_2-t_1)}$.

Example: Given the continuous time state-space model of a system:

$$A = \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$$

with initial condition $X(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, the system is applied with unit step signal. Find the expression for $X(t)$ and hence compute $X(t)$ when $t = 2sec$.

Solution: By using equation 21, we will first determine $X(s)$ and then obtain $X(t)$.

$$\begin{aligned}
[sI - A] &= \begin{bmatrix} s+3 & 0 \\ -1 & s-4 \end{bmatrix} \\
[sI - A]^{-1} &= \begin{bmatrix} s-4 & 0 \\ 1 & s+3 \end{bmatrix} \div (s+3)(s-4).
\end{aligned}$$

$$\begin{aligned}
X(s) &= \begin{bmatrix} \frac{1}{s+3} & 0 \\ \frac{1}{(s+3)(s-4)} & \frac{1}{s-4} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s+3} & 0 \\ \frac{1}{(s+3)(s-4)} & \frac{1}{s-4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\frac{1}{s}\right) \\
&= \begin{bmatrix} \frac{2}{s+3} \\ \frac{2}{(s+3)(s-4)} + \frac{1}{s(s-4)} \end{bmatrix}. \\
&= \begin{bmatrix} \frac{2}{s+3} \\ \frac{-2/7}{(s+3)} + \frac{-1/4}{s} + \frac{2/7+1/4}{s-4} \end{bmatrix}.
\end{aligned}$$

Now, by applying inverse Laplace transform,

$$X(t) = \begin{bmatrix} 2e^{-3t} \\ \frac{-2}{7}e^{-3t} - \frac{1}{4}u_s(t) + (\frac{2}{7} + \frac{1}{4})e^{-4t} \end{bmatrix},$$

and hence, $X(t)$ at time $t = 2sec$,

$$X(2) = \begin{bmatrix} 0.005 \\ 1596.6 \end{bmatrix}.$$

III.2 Properties of Matrix Exponential

1. $exp(At) = L^{-1}[sI - A]^{-1}$
2. $exp(At) = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{2!} + \frac{A^4t^4}{2!}$
3. $L[exp(At)] = [sI - A]^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$

4. Let one eigenvalue of $exp(At) = \mu_i$

$$\mu_i = Eig[exp(At)] = Eig(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{2!} + \frac{A^4t^4}{2!})$$

Let $Eig(A) = \lambda_j$ then,

$$\mu_i = 1 + \lambda_i t + \frac{\lambda_i^2 t^2}{2!} + \cdots$$

$$\boxed{\mu_i = e^{\lambda_i t}}$$

5. Let Eigenvalues of A be distinct i.e. $\lambda_i \neq \lambda_j$ then A is diagonalizable with

$$PAP^{-1} = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

where $P = \begin{bmatrix} p_{l_1} \\ p_{l_2} \\ \vdots \\ p_{l_n} \end{bmatrix}$ & $p_{l_i}A = \lambda_i p_{l_i}$ i.e. p_{l_i} are left eigenvectors (row vectors).

Now,

$$exp(At) = I + P^{-1}\Lambda Pt + P^{-1}\Lambda^2 \frac{Pt^2}{2!} + \cdots$$

$$\begin{aligned} \because \quad A^m &= A, A, \cdots A \\ &= P^{-1}\Lambda P, P^{-1}\Lambda P, \cdots P^{-1}\Lambda P \\ &= P^{-1}\Lambda^m P \\ exp(At) &= P^{-1}[I + \Lambda t + \frac{\Lambda^2 t^2}{2!} + \cdots]P \end{aligned}$$

$$\boxed{exp(At) = P^{-1}exp(\Lambda t)P}$$

For diagonal matrix Λ, it is easy to verify that,

$$exp(\Lambda t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_n t} \end{bmatrix} \quad \text{also a diagonal matrix with } e^{\lambda_i t} \text{ as the elements in diagonal} \tag{23}$$

6. $exp[A(t_1 + t_2)] = exp(At_1).exp(At_2)$

From property 5,

$$\begin{aligned} exp[A(t_1 + t_2)] &= P^{-1}exp[\Lambda(t_1 + t_2)]P \\ &= P^{-1}exp(\Lambda t_1)exp(\Lambda t_2)P \\ &= P^{-1}exp(\Lambda t_1)PP^{-1}exp(\Lambda t_2)P \\ &= exp(At_1).exp(At_2) \end{aligned} \tag{24}$$

7. $\frac{d}{dt}exp(At) = Aexp(At) = exp(At)A$

Proof follows from property 2.

8. $det[exp(At)] = exp[trace(At)]$, $trace(At) = \Sigma_{i=1}^{n}\lambda_i$

Then using $\lambda_i = Eig(At)$, $det[exp(At)] = \Pi_{i=1}^{n}\mu_i$ and property 4, proof is complete.

IV. Canonical Models

We emphasize that for the given physical system, there is no one specific state-space model. Depending the choice of state variables, model (A,B,C,D) is different. However the each of these models of the given physical system will result into same transfer function. Some of these models have useful and interesting structures and they are referred to as "canonical" (standard) models.

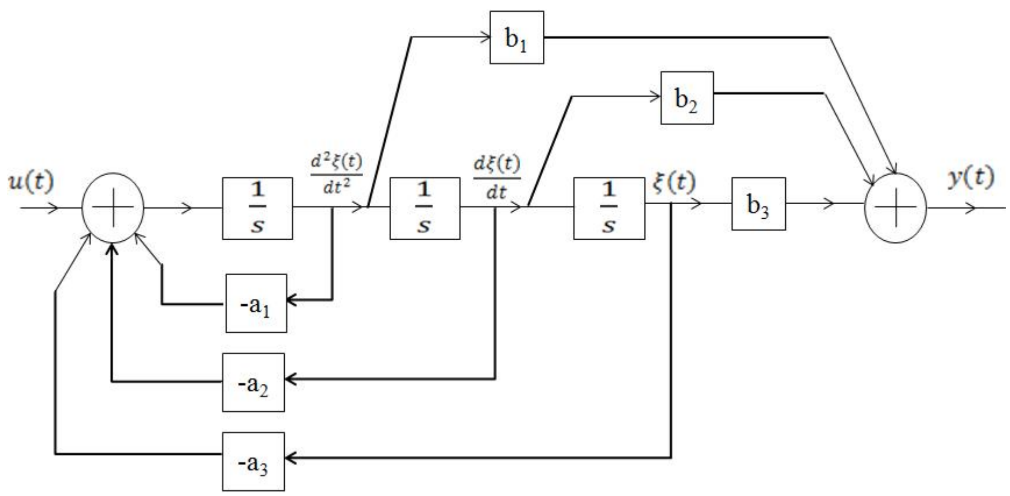
i) Controller Canonical Model

If $\frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C_c = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

is called controller canonical model.
It can be developed by considering intermediate variables $w(t)$, $\dot{\xi}(t)$, and $\ddot{\xi}(t)$ obtained from $\frac{Y(s)}{\xi(s)} . \frac{\xi(s)}{U(s)} = \frac{b(s)}{a(s)}$ with $\frac{\xi(s)}{U(s)} = \frac{1}{a(s)}$ and $\frac{Y(s)}{\xi(s)} = b(s) \Rightarrow \xi(s).a(s) = U(s)$ and $Y(s) = b(s)\xi(s)$.

$$\frac{d^3}{dt^3}\xi(t) = u(t) - a_1\frac{d^2}{dt^2}\xi(t) - a_2\frac{d}{dt}\xi(t) - a_3\xi(t)$$
$$y(t) = b_1\frac{d^2}{dt^2}\xi(t) + b_2\frac{d}{dt}\xi(t) + b_3\xi(t)$$



MATLAB command tf2ss(num, den), always gives A_c, B_c, C_c, D as the default ss model.

Aside;

For given input output relation, there could be many state space models. State space models can be viewed as 'realizations ' of transfer function, i.e. different ways of constructing circuits (using integrators, adders and gain amplifiers) for a given TF. we will soon 'see' that there are infact infinite number of such circuits. But at least one alternate circuit to the one we have now (A, B, C, D) is it's transpose circuit (structure and realization). As we can see,

$$\begin{aligned}\frac{Y(s)}{U(s)} &= (\frac{Y(s)}{U(s)})' = \left(C[sI-A]^{-1}B+D\right)' \\ &= \left(B'([sI-A]^{-1})'C'+D'\right) \\ &= B'\left([sI-A]'\right)^{-1}C'+D \\ &= B'\left([sI-A']^{-1}\right)C'+D\end{aligned} \tag{25}$$

Thus $(A_2 = A^T, B_2 = C^T, C_2 = B^T, D)$ is also another realization / circuit for the given transfer function.

ii) Observer Canonical Model

Thus we expect that the transpose circuit/structure of controler canonical model must also be another realization of given transfer function. This new structure is called observer canonical model.

Thus if $\frac{Y(s)}{U(s)} = \frac{b_1 s^2+b_2 s+b_3}{s^3+a_1 s^2+a_2 s+a_3}$
Then,

$$A_o = A_c^T = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \qquad B_o = C_c^T = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad C_o = B_c^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

iii) Controllability Canonical Model

$$A_{cb} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \qquad B_{cb} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C_{cb} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

where,

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

iv) Observability Canonical Model

$$A_{ob} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \qquad B_{ob} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \qquad C_{ob} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

is transpose of controllability canonical model

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

v) Parallel/Modal Canonical Model

This can be obtained by obtaining the partial fraction expansion of transfer, consisting of first order terms. More on this later!.

IV Similarity Transformation

An operation $Tx(t) = z(t)$ is called co-ordinate transformation or change of basis for the point x(t) in n-dimensional space. For detailed discussion on basis of a vector space, one can refer to any basic linear algebra book, such as [Strang].

Now for the state equation,

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$T^{-1}\dot{z}(t) = AT^{-1}z(t) + Bu(t)$$

will be refering to same process in a different co-ordinate setting. Hence,

$$\dot{z}(t) = TAT^{-1}z(t) + TBu(t) \quad (26)$$
$$y(t) = CT^{-1}z(t) + Du(t) \quad (27)$$

is the new state-space model refering to the same transfrer function.

$$(A, B, C, D) \xrightarrow{T} (TAT^{-1}, TB, CT^{-1}, D)$$

is referred as similarity transformation. Thus if $A_2 = TAT^{-1}, \quad B_2 = TB \quad C_2 = CT^{-1}$ Then,

$$(A, B, C, D) \underset{T^{-1}}{\overset{T}{\rightleftarrows}} (A_2, B_2, C_2, D)$$

<u>Note</u>: Since Invertible T of size $n \times n$ are infinite in number, there are infinite numbr of circuits to realize any given transfer function.

IV.2 <u>Similarity Transformation T to Obtain the Controller Canonical Model</u>

Given the arbitrary model (A, B, C, D)

s.t.

$$\dot{x}(t) = Ax(t) + Bu(t))$$
$$y(t) = Cx(t) + Du(t))$$

To obtain the controller canonical model (A_c, B_c, C_c, D_c)

s.t.

$$\dot{z}(t) = A_c z(t) + B_c u(t)$$
$$y(t) = Cz(t) + Du(t)$$

i.e.

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \vdots \\ \dot{z}_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A_c} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B_c} u(t)$$

We note that

$$Tx(t) = z(t) \qquad TAT^{-1} = A_c$$
$$T\dot{x}(t) = \dot{z}(t) \qquad TB = B_c$$

If Matrix T consists of rows T $= \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}$ then,

$$\left.\begin{array}{rcl} \dot{z}_j(t) & = & T_j\dot{x}(t) \\ z_j(t) & = & T_jx(t) \end{array}\right\} \text{ for } j = 1, 2, \cdots, n.$$

From controller canonical model, for $j = 2, 3, \cdots, n$,

$$\begin{aligned} \dot{z}_j(t) &= z_{j-1}(t) \\ &= T_j\dot{x}(t) \\ &= T_j[Ax(t) + Bu(t)]. \end{aligned} \tag{28}$$

But from controller canonical model, $T_jB = 0 = (B_c)_j$ forj_{th} element of B_c where, $j = 2, 3, \cdots, n$

$\therefore$ Eqn 28 becomes,

$$\dot{z}_j(t) = T_jAx(t) = z_{j-1}(t) = T_{j-1}x(t)$$
$$\Rightarrow\ T_jA = T_{j-1} \qquad for\ \ j = 2, 3, \cdots, n.$$

This means,

$$\begin{array}{rcl} T_nA & = & T_{n-1} \\ T_{n-1}A & = & T_{n-2} \\ T_nAA & = & T_{n-2} \\ T_nA^2 & = & T_{n-2} \end{array}$$

Similarly,

$$\begin{array}{rcl} T_nA^3 & = & T_{n-3} \\ \vdots & & \\ T_nA^{n-2} & = & T_2 \\ T_nA^{n-1} & = & T_1 \end{array}$$

Now using $TB = B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, which is same as,

$$\begin{bmatrix} T_n A^{n-1} \\ T_n A^{n-2} \\ \vdots \\ T_n \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Now rearranging the rows,

$$T_n \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$T_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \mathcal{C}^{-1}$$

where matrix $\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$

Note : On the same lines it can be shown that, observer canonical model can be obtained from the given arbitrary model, i.e.

$$(A, B, C, D) \xrightarrow{T_o} (A_o, B_o, C_o, D)$$

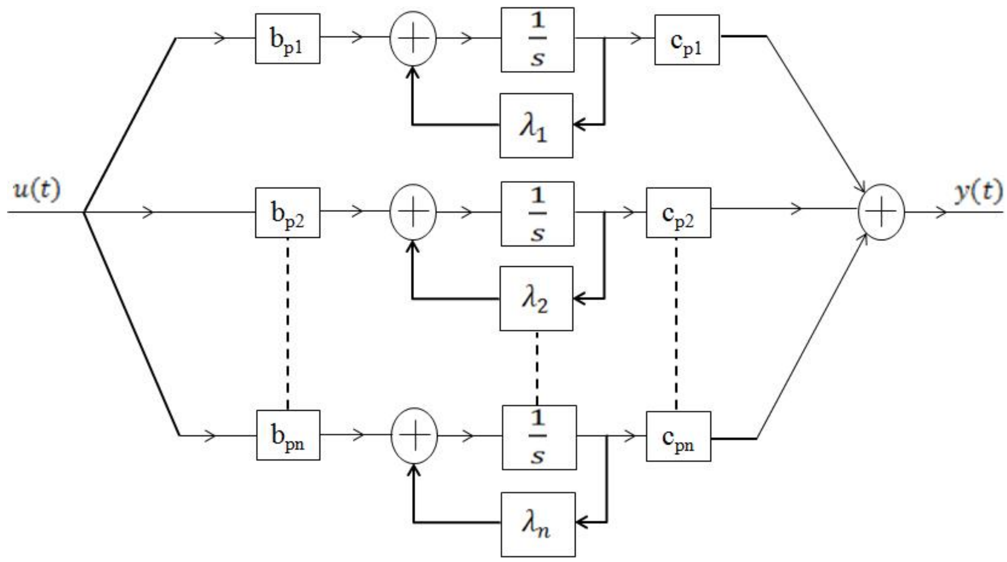
using transformation $T_o^{-1} = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \end{bmatrix}$ with last coloumn $T_n = \mathcal{O}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ and remaining columns $T_l = A^{n-l} T_n, \quad l = 1, 2, \cdots, n-1$, where $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$.

IV.3 Using Similarity Transformation to Obtain Parallel / Modal Canonical structure

(i) Using left eigen Vectors

Use Similarity Transformation matrix $T = P_\ell$ where, $P_\ell = \begin{bmatrix} p_{\ell_1} \\ p_{\ell_2} \\ \vdots \\ p_{\ell_n} \end{bmatrix}$ and $p_{\ell_i} A = \lambda_i p_{\ell_i}$ Thus,

$(A, B, C, D) \xrightarrow{T = P_\ell} (P_\ell A P_\ell^{-1}, P_\ell B, C P_\ell^{-1}, D) = (A_p, B_p, C_p, D)$



$$B_p = \begin{bmatrix} b_{p_1} \\ b_{p_2} \\ \vdots \\ b_{p_n} \end{bmatrix} \quad and \quad C_p = \begin{bmatrix} c_{p_1} & c_{p_2} & \cdots & c_{p_n} \end{bmatrix}$$

(ii) Using right eigen vectors

$$P_r = \begin{bmatrix} p_{r_1} & p_{r_2} & \cdots & p_{r_n} \end{bmatrix}$$

where,

$$A p_{r_i} = \lambda_i p_{r_i}$$

Here similarity transformation matrix $\boxed{T = P_r^{-1}}$

Thus,

$$(A, B, C, D) \xrightarrow{T = P_r^{-1}} (P_r^{-1} A P_r, P_r^{-1} B, C P_r, D) = (A_p, B_p, C_p, D)$$

while obtaining parallel structure, convention is to use normalized eigen vectors to form the matrix P_ℓ (or P_r).

$$\| P_{\ell_i} \|_2 = 1 (or \| P_{r_i} \|_2 = 1)$$

i.e. After obtaining each row eigen vector $p_l A = \lambda p_l$ where, $p_l = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}^T$. Normalized row eigen vector $\hat{p}_l$ is obtained as,

$$vector \ \ \hat{p}_l = \frac{p_l}{\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}}$$

MATLAB Function,
$\begin{bmatrix} P & D \end{bmatrix} = \text{eig}\ (A)$

V. Properties of State-space Model V.1<u>**Property of Controllability**</u> *

State space model (A, B, C, D) is controllable means;
(i) There exists an input to obtain the desired value for state x(t) at time t_f for any given initial time t_0 and value of state at $t_0, x(t)$.

(ii) This also means we can arbitrarily set the initial condition $x(t_0)$.

(iii) Full control over the state , means full control over the system behaviour. (Better than having just control over output, as it happens when transfer function model is used).

Mathematical Test for Controllability A given modal/structure/circuit (A, B, C, D), is said to be fully controllable if the $n \times n$ matrix, $\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$ is invertible.

* MATLAB Function,
$ctrb(A, B)$

<u>Note</u>:

1. Controller Canonical structure is always controllable.
   For eg: when $n = 2$.
$$\mathcal{C} = \begin{bmatrix} B_c & A_c B_c \end{bmatrix} = \begin{bmatrix} 1 & -a_1 \\ 0 & 1 \end{bmatrix}$$

2. Controllability canonical structure is always controllable.

$$n = 2 \qquad A_{co} = \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix} \qquad B_{co} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In fact for any n, controllability matrix $\mathcal{C}$ for controllability canonical structure is an identity matrix

3. If given (A, B, C, D) is not controllable, controller canonical model cannot be obtained using T similarity transformation.

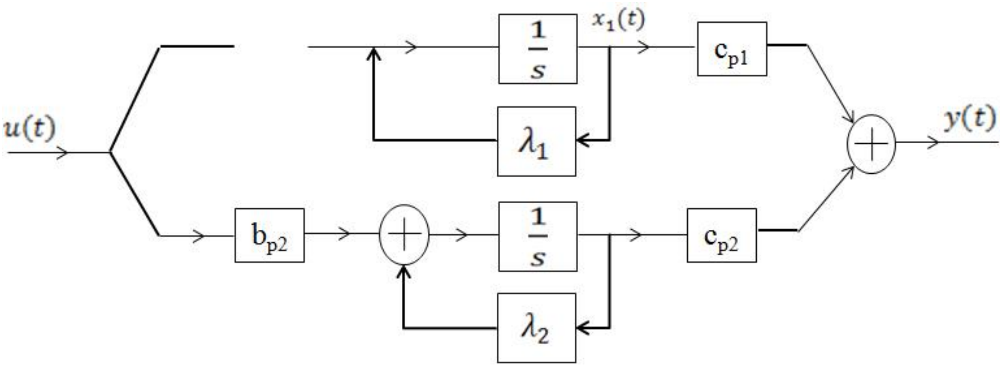
4. If a model (A, B, C, D) is controllable then the new model (A_2, B_2, C_2, D) obtained with any similarity transformation matrix T, is also controllable. This is because, controllability matrix for the new model:

$$\begin{aligned}
\mathcal{C}_2 &= \left[\begin{array}{ccccc} B_2 & A_2B_2 & A_2^2B_2 & \dots & A_2^{n-1}B_2 \end{array}\right] && (29)\\
&= \left[\begin{array}{ccccc} TB & TAT^{-1}TB & TAT^{-1}TAT^{-1}TB & \dots & TAT^{-1}TAT^{-1}TAT^{-1}...TB \end{array}\right] && (30)\\
&= T\left[\begin{array}{ccccc} B & AB_2 & A^2B & \dots & A^{n-1}B \end{array}\right] && (31)\\
&= T\mathcal{C}. && (32)
\end{aligned}$$

Now since both T and $\mathcal{C}$ are invertible $\mathcal{C}_2$ must also invertible.

5. For parallel structure , controllability matrix $\mathcal{C} = \begin{bmatrix} b_{p_1} & \lambda_1 b_{p_1} & \lambda_1^2 b_{p_1} & \cdots & \lambda_1^{n-1} b_{p_1} \\ b_{p_2} & \lambda_2 b_{p_2} & \lambda_2^2 b_{p_2} & \cdots & \lambda_2^{n-1} b_{p_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{p_n} & \lambda_n b_{p_n} & \lambda_n^2 b_{p_n} & \cdots & \lambda_n^{n-1} b_{p_n} \end{bmatrix}$

Hence for system model to be controllable, no element of B_p should be zero. For eg. if $b_{p_1} = 0$



No matter what is your input , state $x_1(t)$ cannot be altered to desired value.
Only those state s can be controlled for which element $b_{p_i} \neq 0$.
In which case, system is said to be (if some $b_{p_i} = 0$) partially controllable. Because in a parallel structure, states are independent of each other, it is possible to 'see' the controllability from circuit property.

V.2 Property of observabiity *

State space Model (A, B, C, D) is observable means,

(i) If we have data about input $u(t)$ and output $y(t)$, it is possible to observe/estimate state value at t_0 and hence $x(t)$ at any time t.

(ii) This means, even if it is not physically possible to measure state value by placing sensors, it is at least possible to design an estimator/observer that provides an estimate of the value of state.

Mathematical Test for Observability A given model/structure/circuit (A, B, C, D) is said to be fully observable, if the $n \times n$ matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is invertible.

* MATLAB Function,
$obsv(A, C)$

Remarks:

1. Observer cononical model is always observable.
   For eg: when $n = 2, \mathcal{O} = \begin{bmatrix} 1 & 0 \\ -a_1 & 1 \end{bmatrix}$ is invertible.

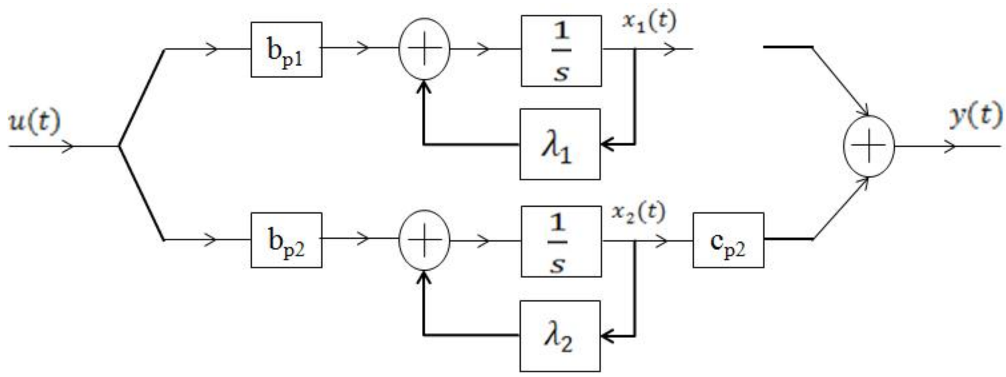
2. Observability canonical Structure is always observable (In fact in this case $\mathcal{O} = \mathcal{I}$).

3. If a model (A, B, C, D) is not observable, observer canonical model cannot be obtained using similarity transformation, in fact T_o does NOT exist.

4. If a model (A, B, C, D) is observable then the new model (A_2, B_2, C_2, D) obtained with *any* similarity transformation matrix T, is also observable. It can be shown by working on similar lines as Remark 4 of controllability property.

5. For parallel structure, observability matrix $\mathcal{O} = \begin{bmatrix} c_{p_1} & c_{p_2} & \cdots & c_{p_n} \\ \lambda_1 c_{p_1} & \lambda_2 c_{p_2} & \cdots & \lambda_n c_{p_n} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{n-1} c_{p_1} & \lambda_2^{n-1} c_{p_2} & \cdots & \lambda_n^{n-1} c_{p_n} \end{bmatrix}$

   Hence for the system model to be observable, no element of C_p should be zero. For eg. if first element of C_p is is zero,



It is not possible to 'observe' state $x_1(t)$ from the output.

Some Additional Remarks

1. If given (A, B, C, D) is not observable (or not conrollable) then $C[sI - A]^{-1}B + D$ will have at least one factor in numerator that can be cancelled.
   den

2. If A, B, C is both controllable and obsrervable, A, B, C is said to be minimal. All such minimal A, B, C, Ds give irreducible $C\left[sI - A\right]^{-1} B + D$. They are all related by similarity transformation.

Example: Consider the DC motor example with angular velocity and armature current as the state variables;i.e. $x_1(t) = \omega(t) = \frac{d\theta(t)}{dt}$, $x_2(t) = i(t)$, input $u(t) = V_{in}(t)$ and output variable $y(t) = \omega(t)$, then the state-space model parameters are:

$$A = \begin{bmatrix} \frac{-b}{J} & \frac{k_1}{J} \\ \frac{-k_2}{L} & \frac{-R}{L} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Therefore, controllability matrix $\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_1}{L} \\ \frac{1}{L} & \frac{-R}{L^2} \end{bmatrix}$ and $|\mathcal{C}| \neq 0$, hence model is controllable. Now the observability matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-b}{J} & \frac{k_1}{J} \end{bmatrix}$ and $|\mathcal{O}| \neq 0$, hence model is observable. It is possible to estimate the values of other state variable armature current with the knowledge of output variable angular velocity.

Since the model is both *controllable* and *observable* it is also *minimal* which means there are no redundant state variables and $\frac{Y(s)}{U(s)} = C\left[sI - A\right]^{-1} B + D = \frac{k_1}{(Js+b)(Ls+R)+k_1k_2}$ will have no pole zero cancellation and hence transfer function will be irreducible.

For the same DC motor, let us consider angular position, angular velocity and armature current as state variables,i.e, $x_1(t) = \theta(t)$, $x_2(t) = \frac{d\theta(t)}{dt}$, $x_3(t) = i(t)$, input $u(t) = V_{in}(t)$ and output $y(t) = \omega(t) = \frac{d\theta(t)}{dt}$. Then state-space model parameters are:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{-b}{J} & \frac{k_1}{J} \\ 0 & \frac{-k_2}{L} & \frac{-R}{L} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Then the controllability matrix will be

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{k_1}{JL} \\ 0 & \frac{k_1}{JL} & \frac{-bk_1}{J^2L} - \frac{Rk_1}{JL^2} \\ \frac{1}{L} & \frac{-R}{L^2} & \frac{-k_1k_2}{JL^2} + \frac{R^2}{L^3} \end{bmatrix},$$

and $|\mathcal{C}| \neq 0$ hence model is controllable.

However, the observability matrix will be

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{-b}{J} & \frac{k_1}{J} \\ 0 & \frac{b^2}{J^2} - \frac{k_1k_2}{JL} & \frac{-bk_1}{J^2} - \frac{Rk_1}{JL} \end{bmatrix}$$

and $|\mathcal{O}| = 0$ hence model is NOT observable. It is NOT possible to estimate the value of angular position with mere knowledge/measurements of angular velocity.

Therefore for the output variable $y(t) = \omega(t)$, this model is NOT minimal which means there is at least one redundant variable and transfer function $\frac{Y(s)}{U(s)} = C\left[sI - A\right]^{-1} B + D = \frac{sk_1}{s((Js+b)(Ls+R)+k_1k_2)}$ will have pole zero cancellation and hence it is said to be reducible.

But if angular position is the variable of interest in your application and you have a position sensor to measure the angular position of the given DC motor, then $y(t) = \theta(t)$, and hence with same state variables, now $C = \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right]$ and observability matrix will be

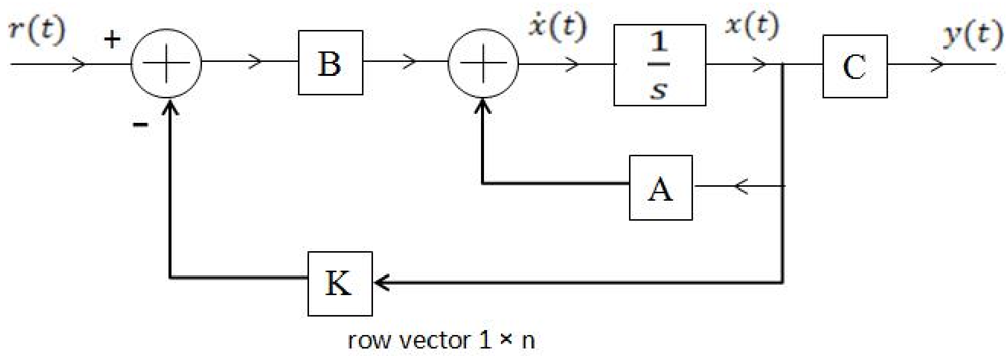
$$\mathcal{O} = \left[\begin{array}{c} C \\ CA \\ CA^2 \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-b}{J} & \frac{k_1}{J} \end{array}\right],$$

and $|\mathcal{O}| \neq 0$ hence model is fully observable. It is possible to estimate the value of angular velocity and armature current with mere knowledge/measurements of angular position. Transfer function obtained from state-space model $\frac{Y(s)}{U(s)} = C\left[sI - A\right]^{-1} B + D = \frac{k_1}{s((Js+b)(Ls+R)+k_1k_2)}$ will have NO pole zero cancellation and hence will be irreducible.

VI. <u>State</u> <u>Variable</u> <u>Feedback</u>

In case of the classical transfer function model, output is measured using a senor and it is fedback to the reference signal. The error signal (between the actual output and the reference) is passed through a controller to provide the control law for the plant. The closed-loop pole locations are in this case constrained along the root-locus determined by the structure of plant and controller.

However, in case of the state-space model of the plant, one can think about all the state variables (which is the internal information of the system) to be fedback to the reference. We will also see later that, in this case, there is more freedom to place the closed-loop poles. If (A, B) is controllable then, theoretically, the new pole locations of closed loop system can be in any arbitrary location, not just along any specific root locus.



From Figure , new control law will be:

$$u(t) = r(t) - BKx(t).$$

The state equation for the closed-loop system becomes,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(r(t) - BKx(t)) \\ &= (A - BK)x(t) + Br(t.) \end{aligned} \tag{33}$$

Hence the state space model of the closed loop system is $(A_{c_\ell}, B_{c_\ell}, C_{c_\ell}, D) = (A-BK, B, C, D)$.
eig (A) are open-loop pole locations, eig $(A - BK)$ are closed - loop pole locations.
$a(s) = \det(sI - A)$, is the open-loop characteristic equation.
$\alpha(s) = \ det[sI - (A - BK)]$, is the closed-loop characteristic equation.

VI.1 Controller Design for Controller Canonical Model

For given (A, B, C, D) pole-placement problem is the problem of designing controller K, such that open-loop pole locations $p_1, p_2, \cdots, p_n$ are moved to new desired locations $\hat{p_1}, \hat{p_2}, \cdots \hat{p_n}$ after state variable feed back. For controller canonical model, open-loop char eqn.

$$det(sI - A_c) \Rightarrow \text{where } A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$$

and closed-loop model parameter,

$$A_{c_\ell} = A_c - B_c K_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} k_{c_1} & k_{c_2} & \cdots & k_{c_n} \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 - k_{c_1} & -a_2 - k_{c_2} & \cdots & -a_{n-1} - k_{c_{n-1}} & -a_n - k_{c_n} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We note that, A_{c_ℓ} also has controller canonical structure. Hence,

$$det(sI - A_{c_\ell}) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

$$= s^n + (a_1 + k_{c_1}) s^{n-1} + (a_2 + k_{c_2}) s^{n-2} + \cdots + a_n + k_{c_n}$$

Thus if,

$$k_c = \begin{bmatrix} k_{c_1} & k_{c_2} & \cdots & k_{c_n} \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}, \quad a = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

then,

$$\boxed{k_c = \alpha - a}$$

VI.2 Controller Design for Arbitrary Model (A, B, C, D)

We know that desired closed-loop characteristic equation $\alpha(s)$ can be achieved with controller canonical model i.e. $\alpha(s) = \ det[sI - (A_c - B_c K_c)]$.

Now, if the given arbitrary model (A, B, C, D) is controllable then it is possible to find the similarity transformation matrix T such that,

$$(A, B, C, D) \xrightarrow{T} (TAT^{-1}, TB, CT^{1-,D}) = (A_c, B_c, C_cD)$$

Then it is possible to achieve,

$$\alpha(s) = \det[sI - (A_c - B_cK_c)] = det[sI - (A - BK)]$$

by using appropriate K. What is $K =?$. To know, substitute for A_c, B_c,

$$\alpha(s) = \det[sI - (TAT^{-1} - TBK_c)]$$

$$= \ det[sTT^{-1} - (TAT^{-1} - TBK_cTT^{-1})]$$

$$= \cancel{\det}(T)det[sI - (A - BK_cT)]\cancel{det}(T^{-1})$$

$\therefore$ With $\boxed{K = K_cT}$ we can achieve same $\alpha(s)$

<u>Remark</u>: procedure to obtain T is in page 12.

MATLAB Function,
$K = place(A, B, P)$

Ackermann's formula,

$$K = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \mathcal{C}^{-1}\alpha(A)$$

MATLAB Function,
$K = acker(A, B, P)$

<u>VII. OBSERVER</u> is a circuit that gives an estimate of state value of the plant $x(t)$. This circuit is used when it is not possible to measure the actual state value $x(t)$ directly by placing sensors.

<u>VII.1 **Open- Loop Observer**</u> is a circuit built in lab that imitates the plant using it's mathematical model.

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t)$$

But since the state value cannot be measured initial condition $\hat{x}(t_0)$ cannot be set same as$x(t_0)$. If error $\tilde{x} = x(t) - \hat{x}(t)$

$$\dot{\tilde{x}}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$$

$$= Ax(t) - A\hat{x}(t)$$

$$= A\tilde{x}(t)$$

$$\tilde{x}(t) = \exp[A(t - t_0)]\tilde{x}(t_0)$$

If $Re[eig(A)] > 0$, $\lim_{t\to\infty} \tilde{x}(t) \to \infty$.

If $Re[eig(A)] > 0$ but close to imaginary axis, it will take long time for $\hat{x}(t)$ to converge to $x(t)$. Hence open-loop observer is not useful in many cases.

VII.2 Closed-Loop Observer

In the closed loop observer, error in actual output and estimated output is fed bask to the state stimator dynamics through a vector L.

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L[y(t) - \hat{y}(t)] \\ &= A\hat{x}(t) + Bu(t) + LCx(t) - LC\hat{x}(t) \\ &= (A - LC)\hat{x}(t) + Bu(t) + LCx(t)\end{aligned}$$

By choosing L, error in output hence error in estimation of state can be controlled. Ideally, we would like error to approach zero as soon as possible.
Problem of designing L is dual to the problem of designing K. We achive it by first designing L_o for observer canonical model.

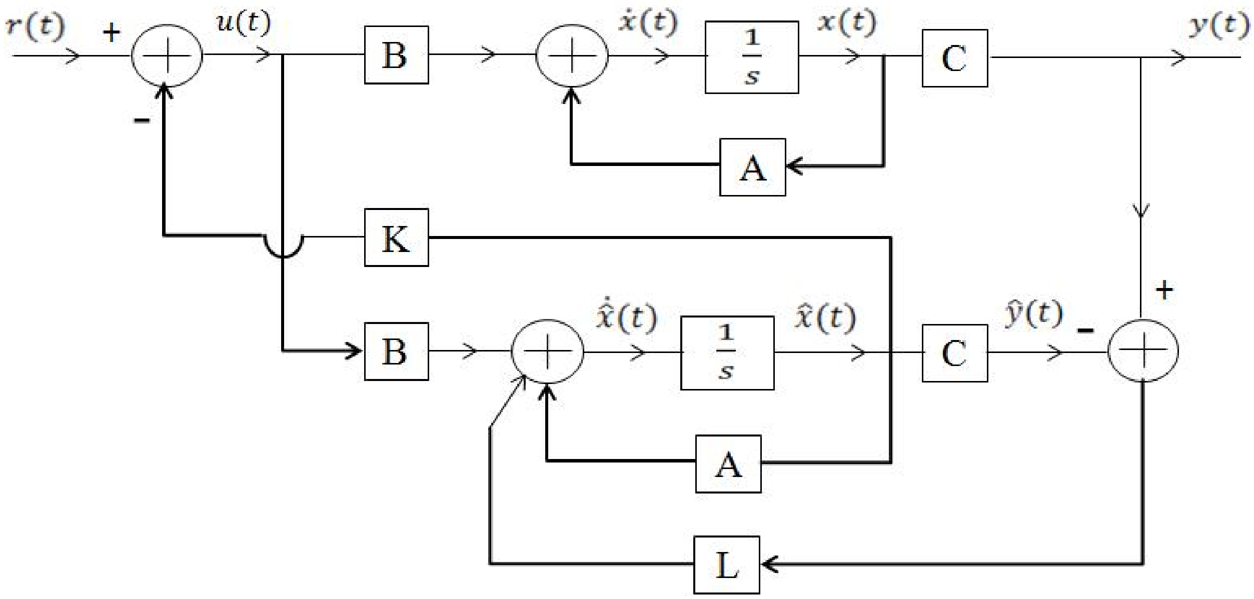
$$\begin{tikzcd}
(A,B,C,D) \arrow[r, "T_o"] \arrow[d, "L"', "\text{Feedback}"] & (A_o,B_o,C_o,D) \arrow[d, "L_o"', "\text{Feedback}"] \\
(A-LC,B,C,D) & (A_o - L_oC_o, B_o, C_o, D) \arrow[l, "T_o^{-1}"']
\end{tikzcd}$$

It is easy to show that,

$$\boxed{L = T_o^{-1}L_o}$$

Procedure to obtain T_o^{-1} is on page 14.

VIII. Combined System With Controller & Observer



Let us define the state vector for the entire combined system with both observer and controller,

$$X_{co}(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}.$$

Noting,

$$u(t) = r(t) - k\hat{x}(t).$$

state space model for combined system,

$$\dot{x}_{co}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r(t) \tag{34}$$

with initial condition,

$$X_{co}(t_0) = \begin{bmatrix} x(t_0) \\ \hat{x}(t_0) \end{bmatrix}$$

$$y(t) \;\; = \;\; \begin{bmatrix} C & 0 \end{bmatrix} x_{co}(t) \tag{35}$$

In order to determine the poles of the entire combined system, we consider its characteristic equation:

$$\alpha_{co}(s) = \det(sI - A_{co}) = \det \begin{bmatrix} sI - A & BK \\ -LC & sI - A + BK + LC \end{bmatrix}$$

By doing the operation $Row1 = Row1 - Row2$

$$= \det \begin{bmatrix} sI - A + LC & -(sI - A + LC) \\ -LC & sI - A + BK + LC \end{bmatrix}$$

Now do Col 2 = Col 1 + Col 2

$$= \det \begin{bmatrix} sI - A + LC & 0 \\ -LC & sI - A + BK \end{bmatrix}$$

For triangular matrices

$$\alpha_{co}(s) = \det[sI - (A - LC)] \det[sI - (A - BK)].$$

Remark: (1) If controller K is designed for $A - BK$ to be stable and observer L is designed for $A - LC$ to be stable, then combined system $(A_{co}, B_{co}, C_{co}, D)$ is stable.

(2) K and L can be designed separately. Usually we want eigen values of $(A - LC)$ (OR the poles of the closed-loop observer) to be faster (farther left in s-plane) than the eigen values of $(A - BK)$ (OR poles of the closed-loop control system with state variable feedback, without any observer).

Example: The linearized equations of the simple pendulum are $\ddot{\theta}(t) + \omega_o^2\theta(t) = u(t)$.

(a) Write the equations of motion in state-space form.

(b) Design an estimator (observer) that reconstructs the state of the pendulum given measurements of $\dot{\theta}(t)$. Assume $\omega_o = 5rad/sec$ and pick the estimator roots at $s = -10 \pm j10$.

(c) Design a controller gain vector for state-variable feedback such that closed-loop poles are at $-4 \pm j4$.

(d) Write the state-space model of the combined system.

Solution: (a) State-space model for the given simple-pendulum can be obtained by defining the state variables: $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$. Since angular velocity $\dot{\theta}(t)$ is measurable (using some sensor) it becomes the output of the system. Thus:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix}. \end{aligned} \tag{36}$$

(b)To design an observer, first we need to test for observability property of the given model with $A = \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Thus:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix} \tag{37}$$

is invertible. Therefore given model is observable.

$$\mathcal{O}^{-1} = \begin{bmatrix} 0 & -1 \\ 25 & 0 \end{bmatrix} \div 25 = \begin{bmatrix} 0 & \frac{-1}{25} \\ 1 & 0 \end{bmatrix}.$$

Now we find T_o^{-1}, inverse of the similarity matrix operator T_o that converts the given model into observer canonical structure:

$T_o^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$. Now $T_2 = \mathcal{O}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{25} \\ 0 \end{bmatrix}$ and $T_1 = AT_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Hence $T_o^{-1} = \begin{bmatrix} 0 & \frac{-1}{25} \\ 1 & 0 \end{bmatrix}$.

Now the open-loop characteristic equation is $a(s) = s^2 + a_1 s + a_2 = det(sI - A) = s^2 + 25$. Hence $a_1 = 0, a_2 = 25$.

And the required characteristic equation for the closed-loop observer is: $\alpha(s) = s^2+\alpha_1 s+\alpha_2 = (s + 10 + j10)(s + 10 - j10) = s^2 + 20s + 200$. Hence $\alpha_1 = 20, \alpha_2 = 200$.

Thus $L_o = \begin{bmatrix} \alpha_1 - a_1 \\ \alpha_2 - a_2 \end{bmatrix} = \begin{bmatrix} 20 - 0 \\ 200 - 25 \end{bmatrix}$, and hence, $L = T_o^{-1} L_0 = \begin{bmatrix} -7 \\ 20 \end{bmatrix}$. It can be verified that $det(sI - (A - LC)) = s^2 + 20s + 200$.

(c)To design the controller gain vector, we first test for the controllability property of the given model:

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is invertible, therefore given model is controllable. Hence,

$$\mathcal{C}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now we find the similarity matrix operator T that converts the given model into controller canonical structure:

$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ where $T_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $T_1 = T_2 A = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

Hence $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

And the required characteristic equation for the closed-loop system with controller is: $\alpha(s) = s^2 + \alpha_1 s + \alpha_2 = (s + 4 + j4)(s + 4 - j4) = s^2 + 8s + 32$.

Therefore, controller gain vector for controller canonical model is: $K_c = \begin{bmatrix} \alpha_1 - a_1 & \alpha_2 - a_2 \end{bmatrix} = \begin{bmatrix} 8 - 0 & 32 - 25 \end{bmatrix}$.

Therefore, controller gain vector for given model is:

$$K = K_c T = \begin{bmatrix} 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 8 \end{bmatrix}.$$

It can be verified that $det(sI - (A - BK)) = s^2 + 8s + 32$.

(d)To obtain state-space model for the combined system, with $x_1(t) = \theta(t), x_2(t) = \dot{\theta}(t), x_3(t) = \hat{\theta}(t)$ and $x_4(t) = \dot{\hat{\theta}}(t)$, we find BK, LC and $A - BK - LC$.

$$BK = \begin{bmatrix} 0 & 0 \\ 7 & 8 \end{bmatrix}.$$

$$LC = \begin{bmatrix} 0 & -7 \\ 0 & 20 \end{bmatrix}.$$

$$A - BK - LC = \begin{bmatrix} 0 & 8 \\ -32 & -28 \end{bmatrix}.$$

Thus combined system model according to equations 34 and 35 is:

$$A_{co} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -25 & 0 & -7 & -8 \\ 0 & -7 & 0 & 8 \\ 0 & 20 & -32 & -28 \end{bmatrix}, B_{co} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, C_{co} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

It can be easily verified using MATLAB that eigen values of A_{co} are in fact: $-4 \pm j4$ and $-10 \pm j10$.

Exercise Examples with Solutions

1. Given the following (A, B), test for the observability of the transpose structure. obtain the similarity transformation matrix T_o to obtain the observer canonical structure from that transpose structure.

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \qquad \underline{\underline{sol}}: \begin{array}{c} \text{Transpose Structue} \\ (A_2, C_2) = (A^T, B^T). \end{array}$$

For observability of transpose structure

$$\mathcal{O} = \begin{bmatrix} C_2 \\ C_2A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 2.5 & 0 \end{bmatrix} \text{ is invertible.}$$

$\therefore$ Transpose structure is observable $\quad \because \quad C_2A_2 = B^TA^T = (AB)^T$

For finding T_o such that,

$$(A_2, B_2, C_2) \xrightarrow{T_o} (A_o, B_o, C_o)$$

$T_2 =$ Last col $(\mathcal{O}^{-1})$ and $T_1 = A^T T_2$ and $T_o^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$

$$\mathcal{O}^{-1} = \begin{bmatrix} 0 & -0.5 \\ -2.5 & 1 \end{bmatrix} \div (-1.25) = \begin{bmatrix} 0 & 0.4 \\ 2.0 & -0.8 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0.4 \\ -0.8 \end{bmatrix} \qquad T_1 = \begin{bmatrix} 1.2 \\ -0.4 \end{bmatrix} = A_2T_1$$

2. With T as similarity transformation: $(A, B, C, D) \xrightarrow{T} (A_2, B_2, C_2, D)$ show that transfer function is preserved.
Transfer Function for the model (A_2, B_2, C_2, D) will be

$$\frac{Y(s)}{U(s)} = C_2[sI - A_2]^{-1}B_2 + D$$

$$= \frac{C_2Adj(sI - A_2)B_2 + D}{\det(sI - A_2)}$$

$$= \frac{\tilde{b}(s)}{\tilde{a}(s)}$$

We need to show that $\frac{\tilde{b}(s)}{\tilde{a}(s)} = \frac{b(s)}{a(s)}$.
Now consider denominator polynomial,

$$\begin{aligned} \tilde{a}(s) &= \det(sI - A_2) = \det(sTT^{-1} - TAT^{-1}) \\ &= \det(T)\det(sT^{-1} - AT^{-1}) \\ &= \det(T)\det(sI - A)\det(T^{-1}) \\ &= \det(sI - A) \\ &= a(s) \end{aligned}$$

And numerator polynomial,

$$\begin{aligned}
\tilde{b}(s) &= CT^{-1}[Adj(sI - TAT^{-1})]TB + D \\
&= CT^{-1}Adj(sTT^{-1} - TAT^{-1})TB + D \\
&= CT^{-1}Adj[T(sT^{-1} - AT^{-1})]TB + D \\
&= CT^{-1}Adj(T^{-1})Adj(sI - A)Adj(T)TB + D \\
&= CT^{-1}T[\det(T^{-1})]Adj(sI - A)(T^{-1})TB\det(T) + D \\
&= CAdj(sI - A)B + D \\
&= b(s)
\end{aligned}$$

3. For matrix exponential $\exp(At)$, show that

$$\frac{d}{dt}[\exp(At)] = A\exp(At) = \exp(At)A$$

$$\frac{d}{dt}[\exp(At)] = \frac{d}{dt}\{I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\}$$

$$= 0 + A + A^2 frac(2t)2! + A^3 frac(3t^2)3! + \cdots$$

$$= A\{I + At + \frac{A^2t^2}{2!} + \cdots\}$$

$$= \{I + At + \frac{A^2t^2}{2!} + \cdots\}A$$

$$= A\exp(At) = \exp(At)A$$

4. Given the following (A, C), test for the controllability of the transpose structure. Obtain the similarity transformation matrix T to obtain the controller canonical structure from that transpose structure.

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

For transpose structure,

$$(A_2, B_2) = (A^T, C^T).$$

$$A_2 = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathcal{C}(A_2, B_2) = \begin{bmatrix} B_2 & A_2B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix} \quad \textit{invertible}$$

$$(A_2, B_2) \;\; \textit{Controllable}$$

For finding T such that,

$$(A_2, B_2, C_2) \xrightarrow{T} (A_c, B_c, C_c)$$

$$T_2 = LastRow(\mathcal{C}^{-1}) = LastRow\{\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix} \div 5\} = \begin{bmatrix} \frac{-1}{5} & \frac{1}{5} \end{bmatrix}$$

$$T_1 = T_2 A_2 = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{-1}{5} & \frac{1}{5} \end{bmatrix}$$

5. Show that transfer function is preserved for transpose structure of (A, B, C, D).

$$\frac{Y(s)}{U(s)} = (\frac{Y(s)}{U(s)})^T = \{C[sI - A]^{-1}B + D\}^T$$

$$= \{C[sI - A]^{-1}B\}^T + D^T$$

$$\begin{aligned} &= B^T\{[sI - A]^{-1}\}^T C^T + D && \because \quad (xyz)^T = z^T y^T x^T \\ &= B^T\{[sI - A]^T\}^{-1} C^T + D && \because \quad (x^{-1})^T = (x^T)^{-1} \\ &= B^T[(sI)^T - A^T]^{-1} C^T + D && \because (X - Y)^T = X^T - Y^T \\ &= B^T[sI - A^T]^{-1} C^T + D && \because \quad (I)^T = I \end{aligned}$$

$$\therefore \quad TF(A, B, C, D) = TF(A^T, C^T, B^T, D).$$

6. For matrix exponential $\exp(At)$ shows that

$$\exp A(t_1 + t_2) = \exp(At_1).\exp(At_2)$$

L H S $\Longrightarrow I + A(t_1 + t_2) + \frac{A^2(t_1+t_2)^2}{2!} + \frac{A^3(t_1+t_2)^3}{3!} + \cdots$

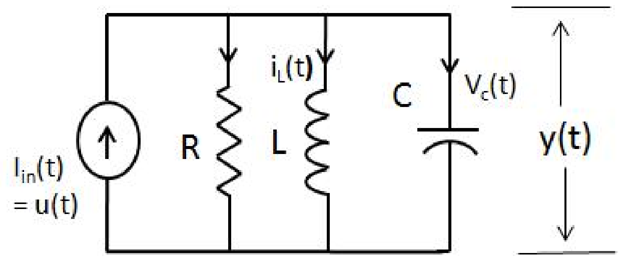
$= I + At_1 + At_2 + \frac{A^2t_1^2}{2!} + \frac{A^2t_2^2}{2!} + A^2t_1t_2 + \frac{A^3t_1^3}{3!} + \frac{A^3t_2^3}{3!} + \frac{A^3t_1^2t_2}{2!} + \frac{A^3t_1t_2^2}{2!} + \cdots$

R H S $\Longrightarrow (I + At_1 + \frac{A^2t_1^2}{2!} + \frac{A^3t_1^3}{3!} + \cdots)(I + At_2 + \frac{A^2t_2^2}{2!} + \frac{A^3t_2^3}{3!} + \cdots)$

$= (I + At_2 + \frac{A^2t_2^2}{2!} + \frac{A^3t_2^3}{3!} + \cdots) + At_1(I + At_2 + \frac{A^2t_2^2}{2!} + \frac{A^3t_2^3}{3!} + \cdots) + \frac{A^2t_2^2}{2!}(I + At_2 + \frac{A^2t_2^2}{2!} + \frac{A^3t_2^3}{3!} + \cdots)$

$$= LHS.$$

7. For the circuit shown obtain one non-canonical and one canonical SS model (Mere re-arrangement of variables $x_1, x_2, \cdots, x_n$ as $x_n, x_{n_1}, \cdots, x_1$ cannot be considered as another model).



Let $\quad i_L(t) = x_1(t) \qquad v_L(t) = x_2(t) \qquad x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$

$$\dot{x}(t) = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x(t) + \begin{bmatrix} 0 & \frac{1}{L} \end{bmatrix} u(t)$$

using, $\quad L\frac{d}{dt}i_L(t) = v_c(t)$

$$C\frac{d}{dt}v_c(t) = I_{in}(t) - i_L(t) - \frac{v_c(t)}{R}$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)$$

To get controller canonical model, find T

$$T_2 = \text{Last Row } (\mathcal{C}^{-1}) = \text{Last Row } \{\begin{bmatrix} 0 & \frac{1}{LC} \\ \frac{1}{C} & -\frac{1}{RC^2} \end{bmatrix}^{-1}\}$$

$$= LastRow\,\{\begin{bmatrix} \frac{1}{RC^2} & \frac{1}{LC} \\ \frac{1}{C} & 0 \end{bmatrix} LC^2\}$$

$$= \begin{bmatrix} LC & 0 \end{bmatrix}$$

$$T_1 = T_2 A = \begin{bmatrix} 0 & C \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & C \\ LC & 0 \end{bmatrix}$$

$$A_c = TAT^{-1} = \begin{bmatrix} 0 & C \\ LC & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{LC} \\ \frac{1}{c} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -\frac{1}{R} \\ 0 & C \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{LC} \\ \frac{1}{C} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C_c = CT^{-1} = \begin{bmatrix} LC & 0 \end{bmatrix}$$

It can also be obtained from $G(s) = \frac{Y(s)}{U(s)} = C[sI - A]^{-1}B$.

8. Under what conditions parallel structure is observable ?

$$A_p = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad C_p = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

$$\text{Observabillity Matrix} \quad \mathcal{O} = \begin{bmatrix} C_p \\ C_p A_p \\ \cdots \\ C_P A_p^{n-1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ c_1\lambda_1 & c_2\lambda_2 & \cdots & c_n\lambda_n \\ c_1\lambda_1^2 & c_2\lambda_2^2 & \cdots & c_n\lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ c_1\lambda_1^{n-1} & c_2\lambda_2^{n-1} & \cdots & c_n\lambda_n^{n-1} \end{bmatrix}$$

If all $c_j \neq 0$ then $\mathcal{O}$ is invertible and model (A_p, B_p, C_p) will be observable. However, even if any one of the $c_j = 0$, $\mathcal{O}$ will not be invertible and model (A_p, B_p, C_p) will be NOT be observable.

9. Is $\exp(At) * \exp(-At)$ operation commutative? prove your statement.

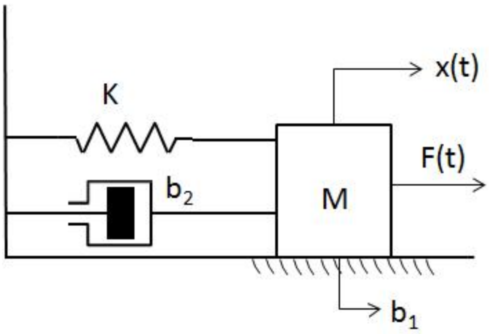
$$\begin{aligned}
\mathcal{L}\{\exp(At) * \exp(-At)\} &= [sI - A]^{-1}[sI + A]^{-1} \\
&= \{(sI + A)(sI - A)\}^{-1} \\
&= \{s^2 I + As - As - A^2)\}^{-1} \\
&= [s^2 I - A^2)\}^{-1}. \qquad (38)
\end{aligned}$$

Now consider:

$$\begin{aligned}
\mathcal{L}\{\exp(-At) * \exp(At)\} &= [sI + A]^{-1}[sI - A]^{-1} \\
&= \{(sI - A)(sI + A)\}^{-1} \\
&= [s^2 I - A^2)]^{-1}, \qquad (39)
\end{aligned}$$

Hence $\exp(At) * \exp(-At)$ operation is commutatitive.

10. For the mechanical system shown, obtain one non-canonical and one canonical model. (mere re-arrangment of variables, i.e. calling $x_1, x_2, \cdots, x_n$ as $x_{n-1_1}, \cdots x_{n_1}, x_1$ cannot be considered another model).



Force Equation:
$F(t) - (b_1 + b_2)\dot{x} - Kx(t) = M\ddot{x}(t)$

Let $\quad x_1(t) = x(t) \;\; x_2(t) = \dot{x} \;\; u(t) = F(t) \;\; y(t) = x(t) \;\; x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{(b_1+b_2)}{M} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \qquad I$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

$$x_1(t) = M\dot{x}(t) \qquad x_2(t) = Mx(t) \qquad x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$$

$$\dot{x}(t) = \begin{bmatrix} -\frac{(b_1+b_2)}{M} & -\frac{K}{M} \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \qquad II$$

$$y(t) = \begin{bmatrix} 0 & \frac{1}{M} \end{bmatrix} x(t)$$

II, is controller canonical model.
I, is non-canonical model but just a scaled model of controller canonical model.

To get a more arbitrary non-canonical model, use any similarity transformation Eqn. $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ apply on (A_c, B_c, C_c).

11. Under what conditions, parallel structure is controllable?

$$(A_p, B_p, C_p) \equiv \begin{bmatrix} \lambda_1 & 0 & \cdots & o \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} B_p & A_pB_p & \cdots & A_p^{n-1}B_p \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & \cdots & \lambda_1^{n-1} b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 & \cdots & \lambda_2^{n-1} b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & \cdots & \lambda_n^{n-1} b_n \end{bmatrix}$$

If any $b_j = 0$, $\mathcal{C}$ will not be invertible.

$\therefore b_j \neq 0$ for every j.

12. what is $Eig[\exp(At)]$? show with steps.
<u>Hint</u>: If f is polynomial function , for any spuare matrix $M, Eig[f(M)] = f[Eig(M)]$.

$$\exp(At) = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$Eig[\exp(At)] = Eig[I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots]$$

$$\mu_i = Eig(I) + Eig(At) + Eig(\frac{A^2t^2}{2!}) + \cdots$$

$$= 1 + \lambda_i(A)t + \frac{\lambda_i^2(A)(t^2)}{2!} + \frac{\lambda_i^3(A)(t^3)}{3!} + \cdots$$

$$\boxed{\mu_i = \underline{\underline{e}}^{\lambda_i t}}$$

$Eig(At) = Eig(A)t$, with t being scalar.