A Report on

Problem solving using analytical methods and MATLAB

Submitted for the fulfillment of the CIE (Event-2) for the course

Control Systems - EC540

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Problem Statement:

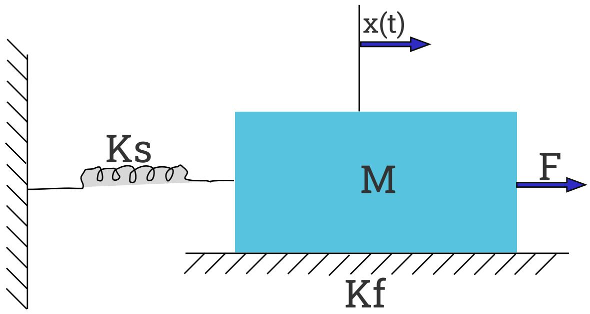
For the mass spring system derive expressions for **ζ** and $\boldsymbol{\omega_n}$. If **M=10 Kg**, determine spring and friction constant for following cases:

1. ζ = 0.7, ωn = 10
2. ζ = 0.3, ωn = 10
3. ζ = 0.01, ωn = 10

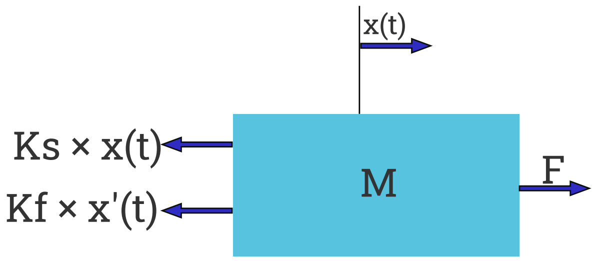
Show the plots for step responses and pole-zero locations in MATLAB.

Solution

- Mass Spring System



- Free Body Diagram



According to Newton's Laws of motion:

$$F(t) - \{K_s x(t) + K_f \frac{\mathrm{d}x(t)}{\mathrm{d}t}\} = M \frac{\partial^2 x(t)}{\partial t^2}$$

$$f(t) \Rightarrow u(t) | x(t) \Rightarrow y(t)$$

$$u(t) = M \frac{\partial^2 y(t)}{\partial t^2} + K_s y(t) + K_f \frac{\partial y(t)}{\partial t}$$

In Laplace Domain

$$U(S) = [MS^2 + K_s + K_f S] \times Y(S)$$

$$\frac{Y(S)}{U(S)} = G(S) = \frac{1}{MS^2 + K_f S + K_s}$$

Comparing with the Standard Second Order Equation

$$G(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

$\omega_n = \sqrt{\frac{K_s}{M}}$ and $\zeta = \frac{K_f}{2\sqrt{K_s M}}$

$K_s = \omega_n^2 \times M$ and $K_f = 2\zeta \times \sqrt{K_s \times M}$

Case 1:

- **M** = 10 Kg
- **ζ** = 0.7
- $\boldsymbol{\omega_n}$ = 10
- $K_s = 1000 N/m$
- $K_f = 140 N/m^2$

Case 2:

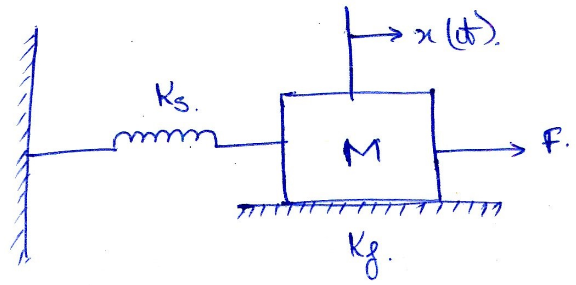
- **M** = 10 Kg
- **ζ** = 0.3
- $\boldsymbol{\omega_n}$ = 10
- $K_s = 1000 N/m$
- $K_f = 60 N/m^2$

Case 3:

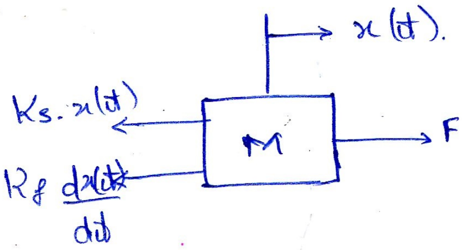
- **M** = 10 Kg
- **ζ** = 0.01
- $\boldsymbol{\omega_n}$ = 10
- $K_s = 1000 N/m$
- $K_f = 2 N/m^2$

Derivation and Calculations

Mass spring system considering the effect of friction.



Free body Diagram.



M → Mass of the block.
Kf → Friction constant.
Ks → Spring constant.
F → Applied Force.
x(t) → Displacement from mean position.

Now according to Newton's laws of motion.

Applied Force − Opposing force = Mass x acceleration.

ie $F(t) - \left\{ K_s\, x(t) + K_f \dfrac{d\, x(t)}{dt} \right\} = M \dfrac{d^2}{dt}\, x(t).$

$F(t) \rightarrow$ Input variable $\qquad x(t) \rightarrow$ Output variable.

$\therefore F(t) \rightarrow u(t) \qquad x(t) \rightarrow y(t).$

$$u(t) = M \frac{d^2}{dt^2}\, y(t) + K_s\, y(t) + K_f \frac{d\, y(t)}{dt}.$$

In laplace domain.

$$U(s) = L\left[M \frac{d^2}{dt^2}\, y(t) + K_s\, y(t) + K_f \frac{d\, y(t)}{dt} \right].$$

$$U(s) = M\left[s^2Y(s) - sy(0) + y'(0)\right] + K_s Y(s) + K_f\left[sY(s) - y(0)\right].$$

Assuming the block of mass M is at rest initially.

ie: Initial conditions are '0'.

<u>ie:</u> $y(0) = y'(0) = \underline{\underline{0}}$.

$$U(s) = \left[Ms^2 + K_s + K_f s\right] Y(s).$$

Now $G(s) = \dfrac{Y(s)}{U(s)}$.

$$G(s) = \frac{1}{Ms^2 + K_f s + K_s} = \frac{1/M}{s^2 + \left(\frac{K_f}{M}\right)s + \frac{K_s}{M}}$$

Standard second order equation is given by.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = \frac{K_s}{M} \qquad \omega_n = \sqrt{\frac{K_s}{M}} \qquad \boxed{K_s = \omega_n^2 \times \underline{\underline{M}}}$$

$$2\zeta\omega_n = \frac{K_f}{M} \qquad \zeta = \frac{K_f}{2M}\sqrt{\frac{M}{K_s}} = \frac{K_f}{2\sqrt{M \cdot K_s}}.$$

or $\boxed{K_f = 2\zeta\sqrt{\underline{\underline{K_s \cdot M}}}}$

(a) Case I: $\zeta = 0.7$, $\omega_n = 10$

$K_s = \omega_n^2 \times M = 10 \times 10 \times 10 = \underline{1000}$ N/m

$K_f = 2\zeta\sqrt{K_s M} = 2 \times 0.7\sqrt{1000 \times 10} = 1.4 \times 100 = \underline{140 \text{ N/m}^2}$

Poles are located at: P_1, P_2

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$P_1, P_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$\omega_n = \underline{10}$.

$$\therefore P_1, P_2 = \frac{-20\zeta \pm \sqrt{\zeta^2 - 1} \times 20}{2}$$

$$= -10\zeta \pm 10\sqrt{\zeta^2 - 1}$$

$\zeta = \underline{0.7}$ $\therefore P_1, P_2 = -10(0.7) \pm 10\sqrt{(0.7)^2 - 1}$

$$= -7 \pm \underline{7.141j}.$$

(b) Case II: $\zeta = 0.3$ $\omega_n = 10$.

$\therefore K_s = \omega_n^2 \times M = 10 \times 10 \times 10 = 1000$ N/m.

$K_f = 2 \times 0.3 \times \sqrt{10^4} = 60$ N/m^2

Poles $P_1, P_2 = -10(0.3) \pm 10\sqrt{0.3^2 - 1}$

$P_1, P_2 = -3 \pm \underline{9.539j}$.

(c) Case III: $\zeta = 0.01$ $\omega_n = 10$

$K_s = 1000$ N/m $K_f = 200 \times 0.01 = 2$ N/m^2

$P_1; P_2 = -10(0.01) \pm 10\sqrt{0.01^2 - 1}$

$P_1, P_2 = -0.1 \pm \underline{9.999j}$

MATLAB CODE

Main Code | solution.m

```
clear all;
close all;
clc;

% DECLARING CONSTANTS STATICALLY
% frictionConstant = [140 60 2];
% springConstant = 1000;

% MORE DYNAMIC CONSTANT CALCULATION
massOfBlock = 10;
zetaValues = [0.7, 0.3, 0.01];
naturalFrequency = 10;

springConstant = naturalFrequency^2 * massOfBlock;
frictionConstant = 2*zetaValues*sqrt(springConstant*massOfBlock);

for index = 1:length(frictionConstant)
    figure;
    G = tf([1], [massOfBlock frictionConstant(index) springConstant]);
    response = stepplot(G, "m");
    grid on;
    legend(strcat("ζ = ", num2str(zetaValues(index)), ", ωn = 10"));
    disp(strcat("Time Domain Parameters of transfer function with ", "ζ = ",
num2str(zetaValues(index)), ", ωn = 10"))
    disp(stepinfo(G));

    response.showCharacteristic('PeakResponse');
    response.showCharacteristic('RiseTime');
    response.showCharacteristic('SettlingTime');
    response.showCharacteristic('SteadyState');

    setAxisLimits(axis);
end
```

Helper Snippets

1. setAxisLimits.m

```
% THIS SNIPPET IS TO ADD PADDING TO THE PLOT

function setAxisLimits(axisData, padding)
    % RELATIVE TO THE OVERALL PLOT
    % 0.1 IS 10% AND 0.5 IS 50%

    arguments
        axisData;
        padding = 0.05; % PADDING DEFALUTS TO 5%
    end

    axisLength = axisData(2) - axisData(1);
    axisHeight = axisData(4) - axisData(3);
    axis([axisData(1) - padding * axisLength axisData(2) + padding * axisLength
axisData(3) - padding * axisHeight axisData(4) + padding * axisHeight]);
end
```

2. labelOnPlot.m

```
function labelOnPlot(A)
    text(real(A(1)), imag(A(1)), strcat("\rightarrow",num2str(A(1))));
    text(real(A(2)), imag(A(2)), strcat("\rightarrow",num2str(A(2))));
end
```

RESULTS

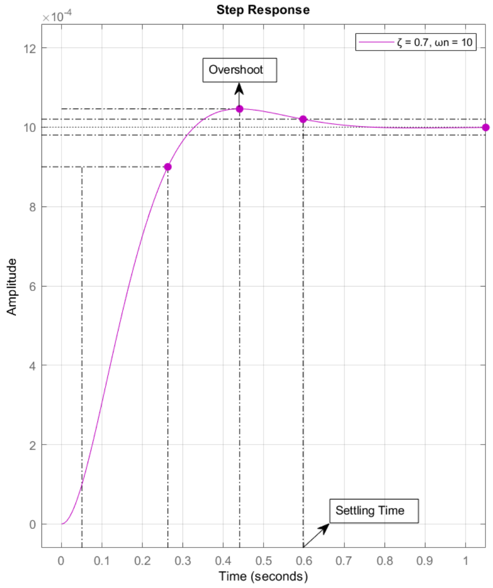
Command Window Output

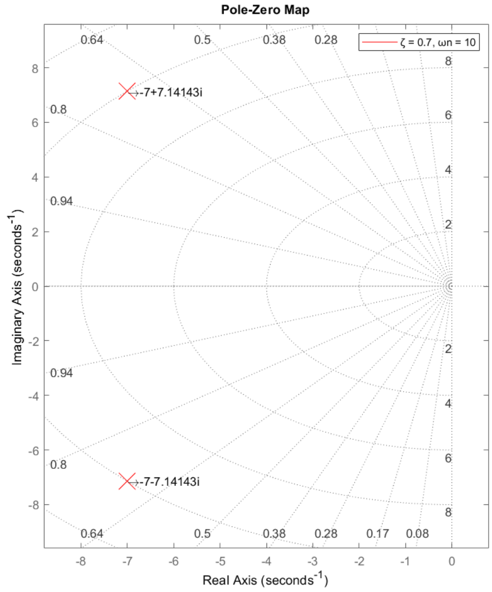
```
G =
 
            1
  ---------------------
  10 s^2 + 140 s + 1000
 
Continuous-time transfer function.

Time Domain Parameters of transfer function with ζ = 0.7, ωn = 10
        RiseTime: 0.2127
    SettlingTime: 0.5979
     SettlingMin: 9.0010e-04
     SettlingMax: 0.0010
       Overshoot: 4.5986
      Undershoot: 0
            Peak: 0.0010
        PeakTime: 0.4408


G =
 
           1
  --------------------
  10 s^2 + 60 s + 1000
 
Continuous-time transfer function.

Time Domain Parameters of transfer function with ζ = 0.3, ωn = 10
        RiseTime: 0.1324
    SettlingTime: 1.1230
     SettlingMin: 8.6139e-04
     SettlingMax: 0.0014
       Overshoot: 37.1410
      Undershoot: 0
            Peak: 0.0014
        PeakTime: 0.3224


G =
 
           1
  -------------------
  10 s^2 + 2 s + 1000
 
Continuous-time transfer function.

Time Domain Parameters of transfer function with ζ = 0.01, ωn = 10
        RiseTime: 0.1050
```

```
    SettlingTime: 38.9674
     SettlingMin: 6.0902e-05
     SettlingMax: 0.0020
       Overshoot: 96.9071
      Undershoot: 0
            Peak: 0.0020
        PeakTime: 0.3142
```

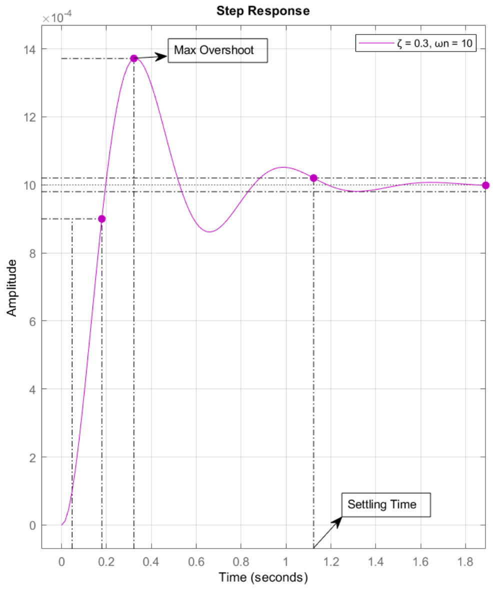
Plots

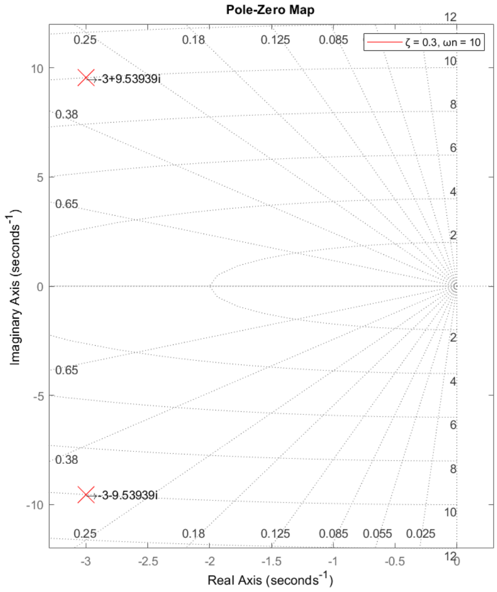
Case 1 Plot



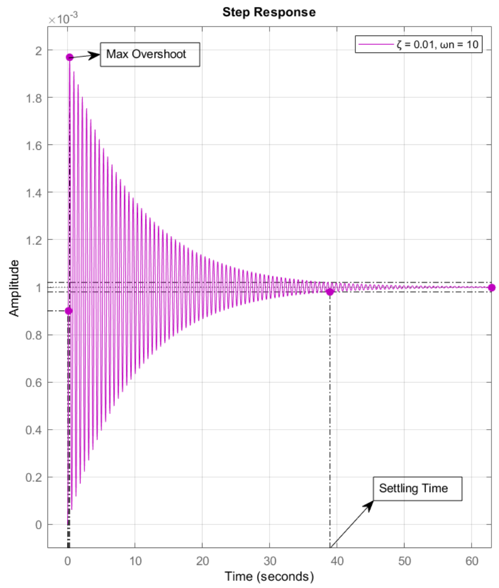


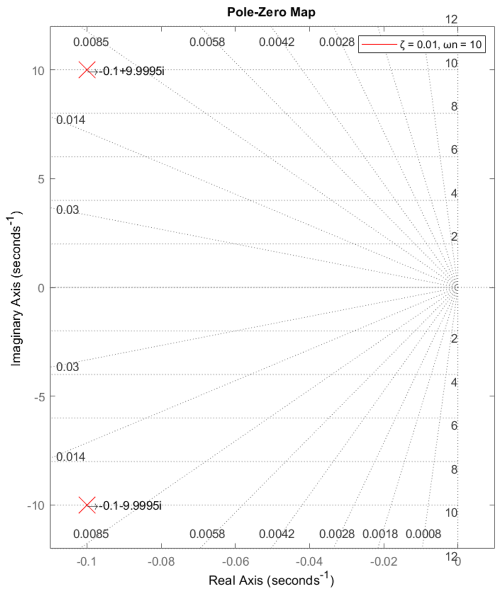
Case 2 Plot





Case 3 Plot





Conclusion/Inference

From the above results and observations, we can conclude the following

- The system in all the 3 cases is **stable** as the poles of the system in all the 3 cases are to the ***left*** of the Imaginary Axis on the ***S-plane***
- Friction co-efficient K_f is directly proportial to ζ , hence as ζ decreses, K_f also decreases.
- As the value of friction co-efficient decreses, that is when the effect of friction decreases ,the number of oscillations increses, which means, there is an increase in the **settling time**, and also the **maximum overshoot** is relatively ***higher*** as seen in case 3 with ***ζ=0.01***.