MATHEMATICS

Linear Algebra Cheat Sheet

VECTOR OPERATIONS

Vector Basics

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{\|\mathbf{r}\|} \quad \text{(Unit Vector)}$$

$$\|\mathbf{r}\| = \sqrt{\mathbf{r}_1^2 + \mathbf{r}_2^2 + \cdots + \mathbf{r}_n^2} \quad \text{(Norm)}$$

Dot Product

$$\mathbf{r} \cdot \mathbf{s} = \mathbf{r}_i \mathbf{s}_i + \cdots + \mathbf{r}_n \mathbf{s}_n$$

$$\mathbf{r} \cdot \mathbf{s} = \|\mathbf{r}\|\|\mathbf{s}\| \cos\theta$$

- If $\theta = 90°$ then $\mathbf{r} \cdot \mathbf{s} = 0$
- If $\theta = 0°$ then $\mathbf{r} \cdot \mathbf{s} = \|\mathbf{r}\|\|\mathbf{s}\|$
- If $\theta = 180°$ then $\mathbf{r} \cdot \mathbf{s} = -\|\mathbf{r}\|\|\mathbf{s}\|$

Projections

$$\frac{\mathbf{r} \cdot \mathbf{s}}{\|\mathbf{r}\|} \quad \text{(Scalar Projection of } \mathbf{s} \text{ onto } \mathbf{r}\text{)}$$

$$\frac{\mathbf{r} \cdot \mathbf{s}}{\mathbf{r} \cdot \mathbf{r}}\mathbf{r} = \frac{\mathbf{r} \cdot \mathbf{s}}{\|\mathbf{r}\|\|\mathbf{r}\|}\mathbf{r} \quad \text{(Vector Projection of } \mathbf{s} \text{ onto } \mathbf{r}\text{)}$$

Basis

A basis is a set of n vectors that:

- are not linear combinations of each other;
- span the space.

The space is then n-dimensional.

A vector is independent of one or more vectors if it cannot be represented as a linear combination of them. This means:

$$\mathbf{v}_1 \neq c_1 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n$$

MATRICES

Identity

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{(Identity Matrix)}$$

Determinant (2x2 Matrix)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- The determinant indicates how much the transformation can dilate or compress the space.
- $\det(I)$ is always equal to 1.
- If the vectors that make up a matrix are linearly dependent, the determinant will be equal to zero.

Inverse (2x2 Matrix)

$$A^{-1}A = I$$

The product between the matrix and its inverse is always the identity matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(Inverse Calculation)}$$

A square matrix is invertible only if its determinant is non-zero. If the determinant is zero, the matrix is called **singular** and does not have an inverse.

There is a method known as "Gauss elimination" that can be used to find the inverse of a matrix. This method involves reducing the original matrix to a "stepped reduced" form by elementary operations on the rows.

Matrix Multiplication

Summation convention for multiplying matrices a and b:

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

$$AB = \begin{bmatrix} \sum_{k=1}^n A_{1k}B_{k1} & \cdots & \sum_{k=1}^n A_{2k}B_{kn} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^n A_{mk}B_{k2} & \cdots & \sum_{k=1}^n A_{mk}B_{kn} \end{bmatrix}$$

Change of Basis

If we have a transformation matrix B where the columns are the new basis vectors in the original coordinate system:

$$B\mathbf{r} = \mathbf{s}$$

and consequently:

$$B^{-1}\mathbf{s} = \mathbf{r}$$

If a matrix A is **orthonormal** (all the columns are of unit size and orthogonal to eachother) then:

$$A^T = A^{-1}$$

Gram-Schmidt Process for Constructing an Orthonormal Basis

Start with n linearly independent basis vectors $\mathbf{v} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Then:

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{e}_1)\mathbf{e}_1 \quad \text{so} \quad \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}$$

and so on for $\mathbf{u}_3$ being the remnant part of $\mathbf{v}_3$ not composed of the preceding $\mathbf{e}$-vectors, etc...

TRANSFORMATIONS & REFLECTIONS

Here is the approach for transforming vectors into a higher space using reflections and linear transformations:

1. **Transformation into the Reflection Plane Base (E^{-1})**: the vector of interest is transformed into the basis of the plane of reflection.
2. **Application of the Reflection or Desired Transformation (T_E)**: the desired reflection or other transformation in the plane of the object is performed.
3. **Return to Original Base (E)**: the transformed vector is returned to the original base.

So the transformed vector ($\mathbf{r}'$) is obtained through the following formula: $\mathbf{r}' = E \cdot T_E \cdot E^{-1}\mathbf{r}$.

This approach allows manipulating vectors in a higher space by combining reflections and linear transformations, offering a powerful and systematic method for examining and manipulating geometric objects in this context.

EIGENSTUFF

Eigenvalues and Eigenvectors

The **eigenvectors** (or characteristic vectors) of a matrix are nonzero vectors that, when multiplied by that matrix, result only scaled by a value known as the **eigenvalue** (or characteristic value).

There is a fundamental relationship that is of great importance, namely, $Ax = \lambda x$, where A is a square matrix of dimension $n \times n$, x is a nonzero vector, and λ is a scalar (the eigenvalue) associated with x. This equation can be rewritten as $(A - \lambda I)x = 0$, where I is the identity matrix of dimension $n \times n$.

To find the eigenvalues λ of the matrix A we can solve $\det(A - \lambda I) = 0$. This characteristic polynomial is zero, and the solutions of this polynomial are the eigenvalues of the matrix A.

Once the eigenvalues have been found, we can calculate the eigenvectors associated with each eigenvalue by solving the equation $(A - \lambda I)x = 0$. The eigenvectors are the nonzero vectors x that satisfy this equation.

To summarize, eigenvalues and eigenvectors are important because they provide crucial information about the behavior of the linear transformation represented by the matrix A.

Eigenvalue Decomposition

Given a matrix T, we would like to perform repeated transformations of a vector v. Unfortunately, if we perform $T^n v$, the computation can be computationally quite onerous. What we do is to perform eigenvalue decomposition to simplify the repeated transformations.

1. **Eigenvector Matrix (C)**: Find the eigenvectors associated with T and arrange them as columns in matrix C:

$$C = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

2. **Diagonal Eigenvalue Matrix (D)**: Organize the eigenvalues of T along the main diagonal of matrix D, and zeros elsewhere:

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

3. **Inverse of the Eigenvector Matrix (C^{-1})**: Compute the inverse of matrix C.

Now, the eigenvalue decomposition of matrix T is expressed as:

$$T = C \cdot D \cdot C^{-1}$$

So, decomposition to eigenvalues allows us to represent a **matrix as the product of three matrices** and this allows us to simplify operations like raising T to a power n, where we can use the decomposition to obtain T^n as:

$$T^n = C \cdot D^n \cdot C^{-1}$$

Here, D^n is the matrix obtained by raising each eigenvalue of T to the power n while maintaining the diagonal structure.

Using this decomposition, we can significantly simplify the calculations associated with T and its powers, making the analysis and application of repeated transformations easier. See the graph below for a better understanding:

$$\begin{tikzcd} \mathbf{v} \arrow[r, "T^n"] \arrow[d, "C^{-1}"'] & T^n\mathbf{v} \\ {[\mathbf{v}]_E} \arrow[r, "D^n"'] & {[T^n\mathbf{v}]_E} \arrow[u, "C"'] \end{tikzcd}$$