Simple examples of reconstruction of scalar fields from gradient measurements

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Abstract

Consider an surface described by a scalar field $\phi(x, y)$. If we are given a set of gradient measurements at lattice points how could we reconstruct the field?

1 Overview

The reconstruction problem is simple to pose. We have an unknown function $\phi(x, y)$ and a set of gradient measurements at a series of know locations. How would we find the surface $\phi(x, y)$? The process that takes us from the gradient measurements to find the surface is called reconstruction.

Mathematically the reconstruction process uses a set of values of $\nabla\phi$ to find ϕ. Gradient measurements go in and a surface description goes out. The immediate question is how to represent the scalar field ϕ.

We consider two different ways. The first is called a *zonal* reconstruction and it produces a series of values of the field for each zone, or sampling region.

	sample	reconstruction	input	output
1	point	zonal	$\nabla\phi\,(p_i)$	$\phi\,(p_i)$
2	point	modal	$\nabla\phi\,(p_i)$	$\phi(x, y)$
3	area	zonal	$\nabla\phi\,(p_i)$	$\phi\,(p_i)$
4	area	modal	$\nabla\phi\,(p_i)$	$\phi(x, y)$

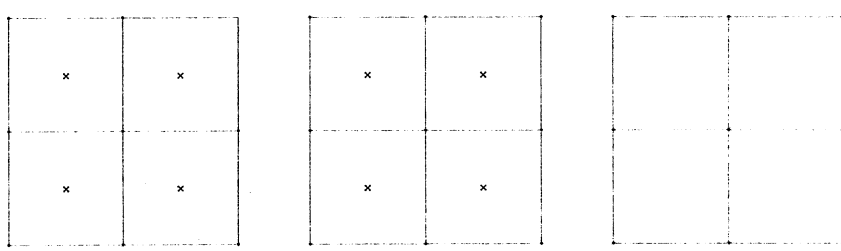

Figure 1: Different geometries for zonal reconstructions. The first two figures on the left correspond to a Hartmann plate measurement: poiint samples of the gradient are taken in the center of the zone and we get predictions for the vertex points. The case on the right is the Shack-Hartmann case. The zone provides an average of the gradient over the zone and predictions for the field are made at the veritices.

2 Reconstruction from samples of the gradient

Consider a basic Hartmann plate as shown below in figure (2). The plate is opaque except for four tiny holes which sample the wavefront. The position of the spot behind the plate is proportional to the gradiant of the surface at the sample location. The gradiant values are the measurements and their location is shown by the blue crosses. The red dots are where the prediction for the surface values will be assigned.

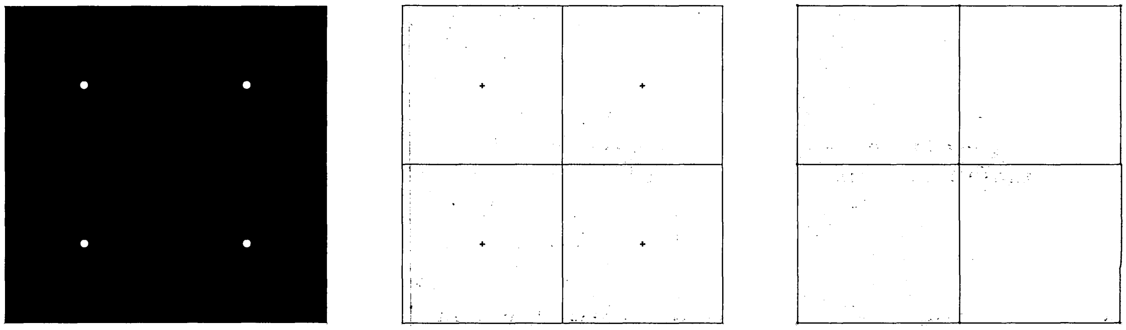

Figure 2: The Hartmann plate. On the left is a Hartmann plate with four holes which sample the wavefront. The figures on the right describe the measurement and prediction coordinates. The middle figure with the crosses marks where the measurements are assigned. The figure on the far right uses red dots to show where the wavefront values are assigned.

In this example the four measurements are

$$\Delta = ((\delta x_1, \delta y_1), (\delta x_2, \delta y_2), (\delta x_3, \delta y_3), (\delta x_4, \delta y_4))$$

where the row-ordering is shown in figure (3) below. The important feature is that we are working in "lenslet space" where the sample region has unit area.

We are now ready to discuss specific models for reconstruction from samples of the gradient of a scalar field.

3 Minimal coupling

3.1 Functional definition

The first model is the simplest: the minimal coupling model. We assume that the tilt between two points, the measurement, describes the difference between the values of the field. That is

$$\delta x_\mu(p) = \phi(p + \hat{\mu}) - \phi(p) \tag{3.1}$$

where p is the vector containing the coordinates for each vertex (red dot) and $\hat{\mu}$ is a unit vector, either $(1, 0)$ or $(0, 1)$. Think of p as the vector which collapses the problem to a single dimension. Instead of addressing a point using (column, row), we have single index, a counter k. The ordering can be arbitrary and here we use an column major form:

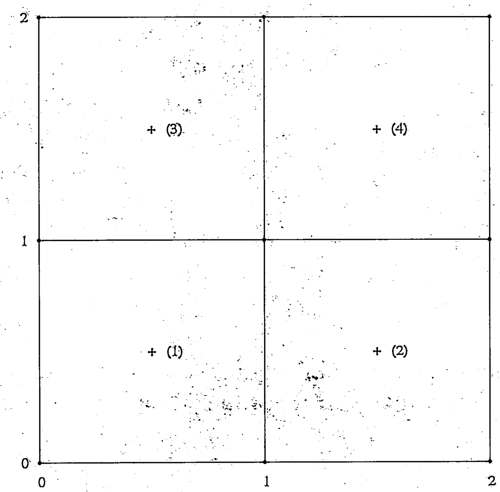

Figure 3: We are working in "lenslet space" where the sample region has unit area.

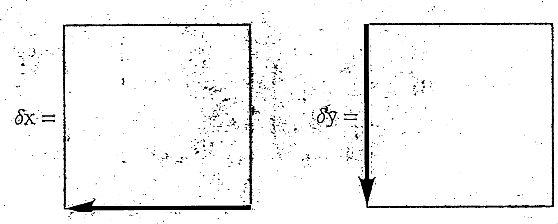

Figure 4: The x measurement is interpreted as the difference between field values which are horizontal neighbors. Similarly, the y measurement is the difference between vertical neighbors.

$$p = ((0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(0,2),(1,2),(2,2))$$

Explicitly the measurements encode the following relationships:

$$\begin{aligned}
\delta x_1 &= \phi(1,0) - \phi(0,0)\\
\delta x_2 &= \phi(2,0) - \phi(1,0)\\
\delta x_3 &= \phi(1,1) - \phi(0,1)\\
\delta x_4 &= \phi(2,1) - \phi(1,1)\\
\delta y_1 &= \phi(0,1) - \phi(0,0)\\
\delta y_2 &= \phi(1,1) - \phi(1,0)\\
\delta y_3 &= \phi(0,2) - \phi(0,1)\\
\delta y_4 &= \phi(1,2) - \phi(1,1).
\end{aligned} \tag{3.2}$$

Notice that we lose a column and a row in this process as show in figure (5). The result is that there is no prediction for the field value ϕ in the upper right-hand corner.

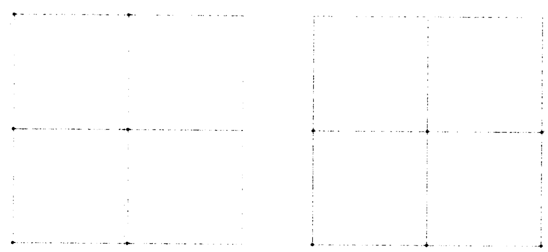

Figure 5: We can't predict the upper right-hand corner

The goal is to find the value of the scalar field ϕ at the red dots. The next step is to assemble the linear systems. The horizontal components are

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \phi(0,0)\\ \phi(1,0)\\ \phi(2,0)\\ \phi(0,1)\\ \phi(1,1)\\ \phi(2,1) \end{pmatrix} = \begin{pmatrix} \delta x_1\\ \delta x_2\\ \delta x_3\\ \delta x_4 \end{pmatrix}; \tag{3.3}$$

the vertical components are

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(0,0)\\ \phi(1,0)\\ \phi(0,1)\\ \phi(1,1)\\ \phi(0,2)\\ \phi(1,2) \end{pmatrix} = \begin{pmatrix} \delta y_1\\ \delta y_2\\ \delta y_3\\ \delta y_4 \end{pmatrix}. \tag{3.4}$$

Of course these two systems are coupled since most points are used in both the horizontal and vertical system. The combined system is

$$A\Phi = \Delta$$

where Δ contains the gradiant and Φ is the solution vector containing the field values at the vertices of the sample zones. In this example the matrices are

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(0,0) \\ \phi(1,0) \\ \phi(2,0) \\ \phi(0,1) \\ \phi(1,1) \\ \phi(2,1) \\ \phi(0,2) \\ \phi(1,2) \end{pmatrix} = \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \\ \delta y_1 \\ \delta y_2 \\ \delta y_3 \\ \delta y_4 \end{pmatrix} \tag{3.5}$$

3.2 Normal equations

This linear system may appear to soluble on the basis of the dimension two vectors, 4×1. This is a lucky coincidence. For a plate with c columns and r rows there are $n = cr$ measurements and $m = (c+1)(r+1)$ plaquette vertices. This means that the measurement vector Δ has dimension $n \times 1$, the solution vector ϕ has dimension $m - 1 \times 1$. The matrix A is fat (vs. tall) and must have dimension $m - 1 \times n$.

For a realistic system, consider $c = 30$ and $r = 40$. The data vector has dimension 1200×1 and the solution vector has dimension 1270×1, and the A matrix is 1200×1270.

So we need to form the normal equations

$$A^T A \Phi = A^T \Delta$$

which in this case are

$$\begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(0,0) \\ \phi(1,0) \\ \phi(2,0) \\ \phi(0,1) \\ \phi(1,1) \\ \phi(2,1) \\ \phi(0,2) \\ \phi(1,2) \end{pmatrix} = \begin{pmatrix} \delta x_2 - \delta x_1 \\ \delta x_3 - \delta x_2 \\ \delta y_1 - \delta x_4 \\ \delta y_2 - \delta y_1 \\ \delta x_4 - \delta x_1 \\ \delta y_1 - \delta x_2 \\ \delta y_3 - \delta x_4 \\ \delta y_4 - \delta y_1 \end{pmatrix} \tag{3.6}$$

3.3 Solution

which we solve as

$$\Phi = \left(A^T A\right)^+ A^T \Delta \tag{3.7}$$

by finding the pseudoinverse which is

$$
(A^T A)^+ = \frac{1}{64}\begin{pmatrix}
33 & 5 & -3 & 5 & -7 & -15 & -3 & -15 \\
5 & 25 & 17 & -7 & -3 & -11 & -15 & -11 \\
-3 & 17 & 73 & -15 & -11 & -19 & -23 & -19 \\
5 & -7 & -15 & 25 & -3 & -11 & 17 & -11 \\
-7 & -3 & -11 & -3 & 17 & 9 & -11 & 9 \\
-15 & -11 & -19 & -11 & 9 & 65 & -19 & 1 \\
-3 & -15 & -23 & 17 & -11 & -19 & 73 & -19 \\
-15 & -11 & -19 & -11 & 9 & 1 & -19 & 65
\end{pmatrix}. \tag{3.8}
$$

The results are

$$
\begin{aligned}
\Phi &= (A^T A)^+ A^T \Delta \\
&= \frac{1}{16}\begin{pmatrix}
-7 & -2 & -3 & -2 & -7 & -3 & -2 & -2 \\
5 & -2 & 1 & -2 & -3 & -7 & -2 & -2 \\
5 & 14 & 1 & -2 & -3 & -7 & -2 & -2 \\
-3 & -2 & -7 & -2 & 5 & 1 & -2 & -2 \\
1 & -2 & 5 & -2 & 1 & 5 & -2 & -2 \\
1 & -2 & 5 & 14 & 1 & 5 & -2 & -2 \\
-3 & -2 & -7 & -2 & 5 & 1 & 14 & -2 \\
1 & -2 & 5 & -2 & 1 & 5 & -2 & 14
\end{pmatrix}
\begin{pmatrix}
\delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \\ \delta y_1 \\ \delta y_2 \\ \delta y_3 \\ \delta y_4
\end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix}
-7\delta x_1 - 2\delta x_2 - 3\delta x_3 - 2\delta x_4 - 7\delta y_1 - 3\delta y_2 - 2\delta y_3 - 2\delta y_4 \\
5\delta x_1 - 2\delta x_2 + \delta x_3 - 2\delta x_4 - 3\delta y_1 - 7\delta y_2 - 2\delta y_3 - 2\delta y_4 \\
5\delta x_1 + 14\delta x_2 + \delta x_3 - 2\delta x_4 - 3\delta y_1 - 7\delta y_2 - 2\delta y_3 - 2\delta y_4 \\
-3\delta x_1 - 2\delta x_2 - 7\delta x_3 - 2\delta x_4 + 5\delta y_1 + \delta y_2 - 2\delta y_3 - 2\delta y_4 \\
\delta x_1 - 2\delta x_2 + 5\delta x_3 - 2\delta x_4 + \delta y_1 + 5\delta y_2 - 2\delta y_3 - 2\delta y_4 \\
\delta x_1 - 2\delta x_2 + 5\delta x_3 + 14\delta x_4 + \delta y_1 + 5\delta y_2 - 2\delta y_3 - 2\delta y_4 \\
-3\delta x_1 - 2\delta x_2 - 7\delta x_3 - 2\delta x_4 + 5\delta y_1 + \delta y_2 + 14\delta y_3 - 2\delta y_4 \\
\delta x_1 - 2\delta x_2 + 5\delta x_3 - 2\delta x_4 + \delta y_1 + 5\delta y_2 - 2\delta y_3 + 14\delta y_4
\end{pmatrix}.
\end{aligned} \tag{3.9}
$$

3.4 Reconstruction is direct, not iterative

Define the zone matrix as

$$
Z = (A^T A)^+ A^T \tag{3.10}
$$

which describes the geometry of the sampling zones. This matrix is determined by the construction of the Hartmann plate and needs to be computed just once. It can then be stored and reconstruction then becomes a matrix multiplication problem:

$$
\Phi = Z\Delta \tag{3.11}
$$

where Z is precomputed, Δ contains the gradient measurements and Φ is the solution vector giving the value of the field ϕ at the zone vertices.

3.5 Idle musings

The matrix $A^T A$ has rank 7,, that is $A^T A \in \mathbb{R}_7^{8\times 8}$. The rank deficiency is 1.

What does the product $(A^TA)^+A^TA$ look like How close is it to the identity matrix?

$$I^+ = (A^TA)^+A^TA = -\frac{1}{8}\begin{pmatrix} -7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -7 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -7 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -7 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -7 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -7 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -7 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -7 \end{pmatrix}. \tag{3.12}$$

The matrix norm of the difference is

$$\|I^+ - I_8\| = 1. \tag{3.13}$$

4 Full coupling

4.1 Functional definition

The full coupling model is the method of choice. In this case, instead of computing the field ϕ at three of the zone vertices, we use all four. This amounts to averaging two differences. That is

$$\delta x_\mu(p) = \frac{1}{2}\left(\phi(p+\hat{\mu}) - \phi(p) + \phi(p+\hat{\mu}+\hat{\nu}) - \phi(p+\hat{\nu})\right) \tag{4.1}$$

here $\hat{\nu}$ is the unit vector complement to $\hat{\mu}$. So if $\mu = (1,0)$ then $\nu = (0,1)$; if $\mu = (0,1)$, $\nu = (1,0)$. The difference measurement is more thoroughly spread across the measurement zone.

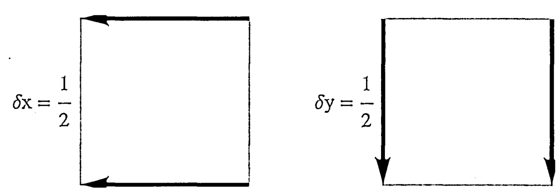

Figure 6: The x measurement is interpreted as the difference between field values which are horizontal neighbors. Similarly, the y measurement is the difference between vertical neighbors.

Explicitly the measurements encode the following relationships:

$$\begin{aligned}
\delta x_1 &= \phi(1,0) - \phi(0,0) + \phi(1,1) - \phi(0,1) \\
\delta x_2 &= \phi(2,0) - \phi(1,0) + \phi(2,1) - \phi(1,1) \\
\delta x_3 &= \phi(1,1) - \phi(0,1) + \phi(1,2) - \phi(0,2) \\
\delta x_4 &= \phi(2,1) - \phi(1,1) + \phi(2,2) - \phi(1,2) \\
\delta y_1 &= \phi(0,1) - \phi(0,0) + \phi(1,1) - \phi(1,0) \\
\delta y_2 &= \phi(1,1) - \phi(1,0) + \phi(2,1) - \phi(2,0) \\
\delta y_3 &= \phi(0,2) - \phi(0,1) + \phi(1,2) - \phi(1,1) \\
\delta y_4 &= \phi(1,2) - \phi(1,1) + \phi(2,2) - \phi(2,1)
\end{aligned} \tag{4.2}$$

This time all the vertices are active as shown below in figure (7).

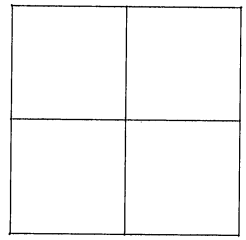

Figure 7: Now all the vertices are in play.

Again the goal is to find the value of the scalar field ϕ at every red dot. Begin by assembling the linear systems. The horizontal components are

$$\begin{pmatrix} -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \phi(0,0) \\ \phi(1,0) \\ \phi(2,0) \\ \phi(0,1) \\ \phi(1,1) \\ \phi(2,1) \\ \phi(0,2) \\ \phi(1,2) \\ \phi(2,2) \end{pmatrix} = \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \end{pmatrix}; \tag{4.3}$$

the vertical components are

$$\begin{pmatrix} -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \phi(0,0) \\ \phi(1,0) \\ \phi(2,0) \\ \phi(0,1) \\ \phi(1,1) \\ \phi(2,1) \\ \phi(0,2) \\ \phi(1,2) \\ \phi(2,2) \end{pmatrix} = \begin{pmatrix} \delta y_1 \\ \delta y_2 \\ \delta y_3 \\ \delta y_4 \end{pmatrix}. \tag{4.4}$$

The coupled system is

$$A\Phi = \begin{pmatrix} -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \phi(0,0) \\ \phi(1,0) \\ \phi(2,0) \\ \phi(0,1) \\ \phi(1,1) \\ \phi(2,1) \\ \phi(0,2) \\ \phi(1,2) \\ \phi(2,2) \end{pmatrix} = \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \\ \delta y_1 \\ \delta y_2 \\ \delta y_3 \\ \delta y_4 \end{pmatrix} = \Delta \tag{4.5}$$

4.2 Normal equations

Now to 'square-up' the system by multiplication of the transpose of A. This process will form the normal equations:

$$\begin{aligned}
A^T A\Phi &= \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(0,0) \\ \phi(1,0) \\ \phi(2,0) \\ \phi(0,1) \\ \phi(1,1) \\ \phi(2,1) \\ \phi(0,2) \\ \phi(1,2) \\ \phi(2,2) \end{pmatrix} \\
&= \begin{pmatrix} -\delta x_1 - \delta y_1 \\ \delta x_1 - \delta x_2 - \delta y_1 - \delta y_2 \\ \delta x_2 - \delta y_2 \\ -\delta x_1 - \delta x_3 + \delta y_1 - \delta y_3 \\ \delta x_1 - \delta x_2 + \delta x_3 - \delta x_4 + \delta y_1 + \delta y_2 - \delta y_3 - \delta y_4 \\ \delta x_2 + \delta x_4 + \delta y_2 - \delta y_4 \\ \delta y_3 - \delta x_3 \\ \delta x_3 - \delta x_4 + \delta y_3 + \delta y_4 \\ \delta x_4 + \delta y_4 \end{pmatrix} \\
&= A^T \Delta
\end{aligned} \tag{4.6}$$

4.3 Solution

The solution of the linear system is

$$\Phi = \left(A^T A\right)^+ A^T \Delta. \tag{4.7}$$

Because *Mathematica* generates the LaTeX code it's easy to look at the components:

$$A^T A = \begin{pmatrix} 2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & -2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 4 & 0 & 0 & 0 & -2 & 0 \\ -2 & 0 & -2 & 0 & 8 & 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 & 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 2 \end{pmatrix}, \tag{4.8}$$

and

$$\left(A^TA\right)^+ = \frac{1}{800}\begin{pmatrix} 304 & 0 & -96 & 0 & -16 & 0 & -96 & 0 & -96 \\ 0 & 125 & 0 & -25 & 0 & -25 & 0 & -75 & 0 \\ -96 & 0 & 304 & 0 & -16 & 0 & -96 & 0 & -96 \\ 0 & -25 & 0 & 125 & 0 & -75 & 0 & -25 & 0 \\ -16 & 0 & -16 & 0 & 64 & 0 & -16 & 0 & -16 \\ 0 & -25 & 0 & -75 & 0 & 125 & 0 & -25 & 0 \\ -96 & 0 & -96 & 0 & -16 & 0 & 304 & 0 & -96 \\ 0 & -75 & 0 & -25 & 0 & -25 & 0 & 125 & 0 \\ -96 & 0 & -96 & 0 & -16 & 0 & -96 & 0 & 304 \end{pmatrix}. \tag{4.9}$$

The results are

$$\begin{aligned}\Phi &= \left(A^TA\right)^+ A^T\Delta = Z\Delta \\ &= \frac{1}{80}\begin{pmatrix} -32 & -8 & 8 & -8 & -32 & 8 & -8 & -8 \\ 15 & -15 & -5 & 5 & -15 & -15 & -5 & -5 \\ 8 & 32 & 8 & -8 & 8 & -32 & -8 & -8 \\ -15 & -5 & -15 & -5 & 15 & -5 & -15 & 5 \\ 8 & -8 & 8 & -8 & 8 & 8 & -8 & -8 \\ 5 & 15 & 5 & 15 & -5 & 15 & 5 & -15 \\ 8 & -8 & -32 & -8 & 8 & 8 & 32 & -8 \\ -5 & 5 & 15 & -15 & 5 & 5 & 15 & 15 \\ 8 & -8 & 8 & 32 & 8 & 8 & -8 & 32 \end{pmatrix}\begin{pmatrix}\delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \\ \delta y_1 \\ \delta y_2 \\ \delta y_3 \\ \delta y_4\end{pmatrix} \\ &= \frac{1}{80}\begin{pmatrix} -32x_1 - 8x_2 + 8x_3 - 8x_4 - 32y_1 + 8y_2 - 8y_3 - 8y_4 \\ 15x_1 - 15x_2 - 5x_3 + 5x_4 - 15y_1 - 15y_2 - 5y_3 - 5y_4 \\ 8x_1 + 32x_2 + 8x_3 - 8x_4 + 8y_1 - 32y_2 - 8y_3 - 8y_4 \\ -15x_1 - 5x_2 - 15x_3 - 5x_4 + 15y_1 - 5y_2 - 15y_3 + 5y_4 \\ 8x_1 - 8x_2 + 8x_3 - 8x_4 + 8y_1 + 8y_2 - 8y_3 - 8y_4 \\ 5x_1 + 15x_2 + 5x_3 + 15x_4 - 5y_1 + 15y_2 + 5y_3 - 15y_4 \\ 8x_1 - 8x_2 - 32x_3 - 8x_4 + 8y_1 + 8y_2 + 32y_3 - 8y_4 \\ -5x_1 + 5x_2 + 15x_3 - 15x_4 + 5y_1 + 5y_2 + 15y_3 + 15y_4 \\ 8x_1 - 8x_2 + 8x_3 + 32x_4 + 8y_1 + 8y_2 - 8y_3 + 32y_4 \end{pmatrix}.\end{aligned} \tag{4.10}$$

4.4 Idle musings

The matrix A^TA has rank 7, that is $A^TA \in \mathbb{R}^{9\times 9}_7$. The rank deficiency is 2.

What does the product $(A^TA)^+A^TA$ look like How close is it to the identity matrix?

$$I^+ = (A^TA)^+A^TA = -\frac{1}{20}\begin{pmatrix} 16 & 0 & -4 & 0 & -4 & 0 & -4 & 0 & -4 \\ 0 & 15 & 0 & -5 & 0 & -5 & 0 & -5 & 0 \\ -4 & 0 & 16 & 0 & -4 & 0 & -4 & 0 & -4 \\ 0 & -5 & 0 & 15 & 0 & -5 & 0 & -5 & 0 \\ -4 & 0 & -4 & 0 & 16 & 0 & -4 & 0 & -4 \\ 0 & -5 & 0 & -5 & 0 & 15 & 0 & -5 & 0 \\ -4 & 0 & -4 & 0 & -4 & 0 & 16 & 0 & -4 \\ 0 & -5 & 0 & -5 & 0 & -5 & 0 & 15 & 0 \\ -4 & 0 & -4 & 0 & -4 & 0 & -4 & 0 & 16 \end{pmatrix}. \tag{4.11}$$

The matrix norm of the difference is still unity:

$$\|I^{+} - I_9\| = 1. \tag{4.12}$$

5 Southwell

This section is recasting of the method describes in Southwell's influential paper.[1] Referring to his equations (9) and (10) (with a typographical error corrected) he considers the difference in gradient measurements between neighboring zones to be the average of the gradient in those cells.

For a new model appropriate for Hartmann sensor measurements, we consider the grid geometry of Fig. 1(A) and return to Eqs. (1) and (2) for a representation of the phase between mesh points. This time we have two measurements per interval, a slope at each end. This enables the determination of both c_1 and c_2 in Eq. (2).

This produces the relationships

$$\frac{(S^x_{i+1,j} + S^x_{ij})}{2} = \frac{(\phi_{i+1,j} - \phi_{ij})}{h} \quad \begin{matrix} i = 1, N-1 \\ j = 1, N \end{matrix}, \tag{9}$$

$$\frac{(S^y_{i+1,j} + S^y_{ij})}{2} = \frac{(\phi_{i,j+1} - \phi_{ij})}{h} \quad \begin{matrix} i = 1, N \\ j = 1, N-1 \end{matrix}. \tag{10}$$

Figure 8: An excerpt from Southwell's paper showing his basic model.

6 Reconstruction from the average of the gradient

This is the classic Shack-Hartmann problem and the device provides a convenient background to the theory. The heart of the instrument is the array of microlenslets which partition the incident wavefront. The lenslets form a continuous array. What we will see ...

6.1 Eliminate the antiderivatives

For a modal reconstruction the antiderivatives

[1] W.H. Southwell, *Wave-front estimation from wave-front slope measurements*, J. Opt. Soc. Am., Vol. 70, No. 8, August 1980