Take Home Test #1

1. a) $\vec{v} = \vec{\omega} \times \vec{r}$ $\vec{\omega} = \omega \vec{k}$ find $\vec{\nabla} \times \vec{v}$

$$\omega \vec{k} \times (x\hat{i} + y\hat{j} + z\vec{k}) = \omega(\hat{j}x - \hat{i}y)$$

$$\left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} \vec{k}\right) \times \omega(\vec{j}x - \vec{i}y) = \omega\left(\vec{k}\frac{\partial x}{\partial x} + \vec{k}\frac{\partial y}{\partial y}\right) = 2\omega\vec{k}$$

b) $$\frac{\oint \vec{v} \cdot dr}{\oint ds} \xrightarrow{\text{by Stokes}} \frac{\int(\vec{\nabla} \times \vec{v}) \cdot ds}{\oint ds} = \frac{2\omega \vec{k} \cdot \int d\vec{s}}{\oint ds} = 2\omega k \cos\theta$$

c) $\vec{B} = \vec{\nabla} \times \vec{A}$ $\vec{A} = \frac{\vec{B}}{2} \times \vec{r}$ parallel with

$2\omega\vec{k} = \vec{\nabla} \times \vec{v}$

2. a) $A = \sum_s C_s u_s(x, y, z) e^{-i\omega t}$ $\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 \vec{J}$

ansatz

$$\sum_s C_s u_s(x, y, z) e^{-i\omega t}\left(-K_s^2 + \frac{\omega}{c^2}\right) = -\mu_0 I_0 \delta(x - \tfrac{a}{2})\delta(z - \tfrac{c}{12}) e^{-i\omega t} \vec{J}$$

integrate

$$C_{s'} n_{s'} (-K_{s'}^2 + \omega^2 k^2) = \mu_0 I_0 u_{s'}(a/2, c/12)$$

$$C_{TE103} = \frac{2\sqrt{2}}{ac} \frac{\mu_0 I_0}{K^2 - \omega^2/c^2} \sqrt{1 + 9\,a^2/c^2}$$

b) As the driving frequency $\omega \to cK_s$ the natural frequency of the mode, the amplitude $\to \infty$ (resonance). Wall losses prevent infinite amplitude

c) Since $\frac{\partial}{\partial t}\nabla\phi \to 0$ $\phi = \int \frac{\rho(r')}{r - r'} d^3 r'$

$E = -\frac{\partial A}{\partial t} - \nabla\phi$ $A = c_s \cos\omega t\, u_s(a/2, c/12)$

$\nabla^2 \phi = \frac{\rho}{\epsilon_0}$

$E = -\frac{\partial A}{\partial t} = c_s \sin\omega t\, u_s(a/2, c/12)$ $V = \int_0^b E \cdot dy = Eb$

$$V = b c_s \sin\omega t\, u_s(a/2, c/12)$$

$$\downarrow$$

$$\frac{1}{K_s^2 - (\omega K)^2}$$