Multiparty Session Types *as* Coherence Proofs

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Joint work with

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In a nutshell

A Curry-Howard correspondence between Multiparty Session Types and Linear Logic.

From Linear Logic to Session Types, and back again

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- Proofs
- Propositions

$$\frac{\vdots}{\vdash \Delta}$$

where $\Delta = A_1, \ldots, A_n$

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$$\dfrac{\vdots}{P \vdash \Delta}$$

where $\Delta = x_1 : A_1, \ldots, x_n : A_n$

Read "Process P uses each channel x_i following protocol A_i"

Some rules (adapted from [Wadler, 12])

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$$\frac{R \vdash \Sigma, y : A, x : B}{\overline{x}\ (y); R \vdash \Sigma, x : A \parr B} \parr$$

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$$\frac{P \vdash \Gamma, y{:}A \qquad Q \vdash \Delta, x{:}B}{x\,(y);(P \mid Q) \vdash \Gamma, \Delta, x{:}A \otimes B}\ \otimes \qquad \frac{R \vdash \Sigma, y{:}A, x{:}B}{\overline{x}\,(y); R \vdash \Sigma, x{:}A \parr B}\ \parr$$

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$$\frac{P \vdash \Gamma, x{:}A \qquad Q \vdash \Delta, x{:}A^{\perp}}{(\boldsymbol{\nu} x : A)\,(P \mid Q) \vdash \Gamma, \Delta}\ \mathsf{Cut}$$

where $A^{\perp}$ is the "dual" of A, e.g., $(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$.

An example

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$$\frac{P \vdash \Gamma, y{:}A \qquad Q \vdash \Delta, x{:}B}{x(y);(P \mid Q) \vdash \Gamma, \Delta, x{:}A \otimes B}\ \otimes$$

An example

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{x\,(y); (P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\otimes \qquad \frac{R \vdash \Sigma, x : A^{\perp}, y : B^{\perp}}{\overline{x}\,(y); R \vdash \Sigma, x : A^{\perp} \parr B^{\perp}}\parr$$

An example

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\dfrac{
\dfrac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{x\,(y); (P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\otimes
\qquad
\dfrac{R \vdash \Sigma, y : A^{\perp}, x : B^{\perp}}{\overline{x}\,(y); R \vdash \Sigma, x : A^{\perp} \parr B^{\perp}}\parr
}{
(\boldsymbol{\nu} x : A \otimes B)\big(x\,(y); (P \mid Q) \mid \overline{x}\,(y); R\big) \vdash \Gamma, \Delta, \Sigma
}\,\mathsf{Cut}
$$

because $(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$.

Cut Elimination

In linear logic, cuts can always be eliminated from proofs.

Cut Elimination

$$\dfrac{\dfrac{P \vdash \Gamma, y{:}A \qquad Q \vdash \Delta, x{:}B}{x\,(y);(P \mid Q) \vdash \Gamma, \Delta, x{:}A \otimes B}\otimes \qquad \dfrac{R \vdash \Sigma, y{:}A^{\perp}, x{:}B^{\perp}}{\overline{x}\,(y);R \vdash \Sigma, x{:}A^{\perp} \mathbin{\rotatebox[origin=c]{180}{$\&$}} B^{\perp}}\mathbin{\rotatebox[origin=c]{180}{$\&$}}}{(\boldsymbol{\nu} x)\left(x\,(y);(P \mid Q) \mid \overline{x}\,(y);R\right) \vdash \Gamma, \Delta, \Sigma}\,\mathsf{Cut}$$

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$$Q \vdash \Delta, x{:}B$$

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$$

which corresponds to the typical reduction

$$
(\boldsymbol{\nu} x : A \otimes B)\,\big(x\,(y);(P \mid Q) \;\mid\; \overline{x}\,(y);R\big) \;\to\; (\boldsymbol{\nu} y : A)\,\big(P \mid (\boldsymbol{\nu} x : B)\,(Q \mid R)\big)
$$

Curry-Howard Linear Logic $\leftrightarrow$ π-calculus

A *deep* correspondence:

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A *deep* correspondence:

- Proofs *as* Processes

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- Proofs *as* Processes
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- Cut Elimination *as* Communication

Benefits of the correspondence

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- **Canonicity**, from the underlying re-appearing structure.
- **Free results**, e.g., deadlock-freedom from cut elimination.
- **Reuse** of well-understood logical tools. Examples:
  - Proof-carrying code [Pfenning et al., 11]
  - Typed translation from Functions to Processes [Toninho et al., 12]
  - Logical relations [Pérez et al., 12]
  - ...

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- But session types have a very active community (20 years).

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Can we import results from Session Types?
Session Types $\rightarrow$ *Logic*

Session Types does not check for taxes

$$
\begin{array}{ll}
\textsf{buyer} & \overline{x}\ (money);\ x\,(receipt);\ P \\
 & \quad | \\
\textsf{seller} & x\,(money);\ \overline{y}\ (taxes);\ \overline{x}\ (receipt);\ Q \\
 & \quad | \\
\textsf{tax off.} & y\,(taxes);\ R
\end{array}
$$

Session Types does not check for taxes

I can forget paying my taxes!

$$\begin{array}{ll} \textsf{buyer} & \overline{x}\,(money);\ x\,(receipt);\ P \\ & \quad | \\ \textsf{seller} & x\,(money);\ \overline{x}\,(receipt);\ Q \end{array}$$

Multiparty Session Types [Honda et al., 08]

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Multiparty Session Types [Honda et al., 08]

buyer	$\overline{x}^{\,\mathsf{B\,S}}(money);\ x^{\mathsf{B\,S}}(receipt);\ P$
	$\mid$
seller	$x^{\mathsf{S\,B}}(money);\ \overline{x}^{\,\mathsf{S\,T}}(taxes);\ \overline{x}^{\,\mathsf{S\,B}}(receipt);\ Q$
	$\mid$
tax off.	$x^{\mathsf{T\,S}}(taxes);\ R$

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The type of x is a *global type*:

$$\mathsf{B} \texttt{->} \mathsf{S} : \langle\rangle;\; \mathsf{S} \texttt{->} \mathsf{T} : \langle\rangle;\; \mathsf{S} \texttt{->} \mathsf{B} : \langle\rangle$$

Type checking in MPSTs

From the global type

$$\mathsf{B} \texttt{ -> } \mathsf{S} : \langle\rangle;\ \mathsf{S} \texttt{ -> } \mathsf{T} : \langle\rangle;\ \mathsf{S} \texttt{ -> } \mathsf{B} : \langle\rangle$$

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project the *local type* for each role:

$$\begin{aligned} \text{role } \mathsf{B} :&\quad \mathsf{send}\,\mathsf{S}; \mathsf{recv}\,\mathsf{S} \\ \text{role } \mathsf{S} :&\quad \mathsf{recv}\,\mathsf{B}; \mathsf{send}\,\mathsf{T}; \mathsf{send}\,\mathsf{B} \\ \text{role } \mathsf{T} :&\quad \mathsf{recv}\,\mathsf{S} \end{aligned}$$

- So Multiparty Session Types are not based on duality!
- Rather, the compositionality principle is called *coherence*:

Definition (Coherence)

A set of local types is coherent if they can all be projected from one global type.

Can we really adapt linear logic to this radical change?

From Duality to Coherence

Local typing just requires repainting the correspondence!

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From Duality to Coherence

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Typing Buyer-Seller-Taxes

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\begin{array}{ll}
\text{buyer} & \overline{x}^{\,\mathsf{B\,S}}(money);\ x^{\mathsf{B\,S}}(receipt);\ P \\
 & \ \mid \\
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\hline
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\hline
\text{buyer} & \vdash \Gamma, x^{\mathsf{B}} : \bot \parr^{\mathsf{S}} (1 \otimes^{\mathsf{S}} A)
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Composing Multiparty Processes

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How can we compose them?

Composing Multiparty Processes

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How can we compose them? First attempt:

$$
\frac{P_i \vdash \Gamma_i, x^{p_i} : A_i \qquad \exists G \text{ s.t. } \mathsf{proj}(G) = \{p_i : A_i\}_i}{(\boldsymbol{\nu} x : G) \left(\prod_i P_i \right) \vdash \{\Gamma_i\}_i} \; \mathsf{MCut}
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Multicut

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Two problems with that condition:

Multicut

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Two problems with that condition:

- it does not tell us how to prove it;
- it does not tell us why the composition is safe.

Coherence Proofs

We propose to treat coherence as a proof system:

$$\dfrac{P_i \vdash \Gamma_i, x^{p_i} : A_i \qquad G \vDash \{p_i : A_i\}_i}{(\boldsymbol{\nu} x : G)\left(\prod_i P_i\right) \vdash \{\Gamma_i\}_i} \;\textsf{MCut}$$

Coherence is simple

Here are all the four rules:

$$\frac{G \ \vDash\ \Theta,\ p{:}B,\ \{q_i{:}D_i\}_i \qquad G' \ \vDash\ p{:}A,\ \{q_i{:}C_i\}_i}{p \text{ -> } \tilde{q} : \langle G' \rangle; G \ \ \vDash\ \ \Theta,\ \ p{:}A \parr^{\tilde{q}} B,\ \{q_i{:}C_i \otimes^p D_i\}_i}\ \otimes\parr$$

$$\frac{}{\mathsf{end}^{p\tilde{q}} \vDash p{:}\bot, q_1{:}1, \ldots, q_n{:}1}\ 1\bot$$

$$\frac{G_1 \vDash \Theta, p{:}A, \{q_i{:}C_i\}_i \qquad G_2 \vDash \Theta, p{:}B, \{q_i{:}D_i\}_i}{p \text{ -> } \tilde{q} : \&(G_1, G_2) \vDash \Theta, p{:}A \oplus^{\tilde{q}} B, \{q_i{:}C_i \&^p D_i\}_i}\ \oplus\&$$

$$\frac{G \vDash p{:}A, \{q_i{:}B_i\}_i}{?p \text{ -> } !\tilde{q} : \langle G \rangle \vDash p{:}?A, \{q_i{:}!B_i\}_i}\ !?$$

Coherence looks right

- Isomorphism between well-formed global types and coherence proofs: Global Types as Coherence Proofs!
- We know how to prove the condition $G \vDash \{p_i : A_i\}_i$ now: just do a proof.

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- But most importantly...
  - We know why it works: Cut Elimination!

A communication

$$(\boldsymbol{\nu} x : p \texttt{->} \tilde{q} : \langle G' \rangle ; G) \left(\prod_i x^{q_i\, p}(y); (P_i \mid Q_i) \quad \mid \quad \overline{x}^{p\tilde{q}}(y); R \quad \mid \quad \prod_j P_j \right)$$

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$$\rightarrow \qquad (\boldsymbol{\nu} y : G') \left(\prod_i P_i \;\mid\; (\boldsymbol{\nu} x : G) \left(\prod_i Q_i \;\mid\; R \;\mid\; \prod_j P_j\right)\right)$$

Results

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- **Session fidelity**: reductions follow the protocols.

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- **Cut Elimination**, and hence deadlock-freedom.

More on coherence

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- **Projection**: a global type yields a set of corresponding local types, by isomorphism with coherence proofs.

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- **Projection**: a global type yields a set of corresponding local types, by isomorphism with coherence proofs.
- **Extraction**: a proof search for coherence extracts the global type that some local types follow.

In the paper

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- All the rules.

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- More nice properties.

In the paper

- All the rules.
- More nice properties.
- Examples with multiple sessions.

Conclusions

- Curry-Howard goes both ways.

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- What new things will arise?

Thank you!

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Questions?