Introduction to Machine Learning for Social Scientists

Class 10: Unsupervised learning/Distance Metrics

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Summer 2018

Homework 4. Due Wednesday August 8th at midnight

Group Project materials.
Due Tuesday August 14th at midnight

Homework 5.
Available on Wednesday August 8th,
Due Wednesday August 15th

Plan for the day

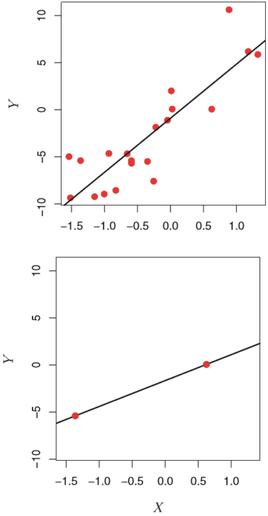
1. Loose ends: General review of concepts
2. Introducing unsupervised learning.
3. Text similarity and distance.

General concepts:

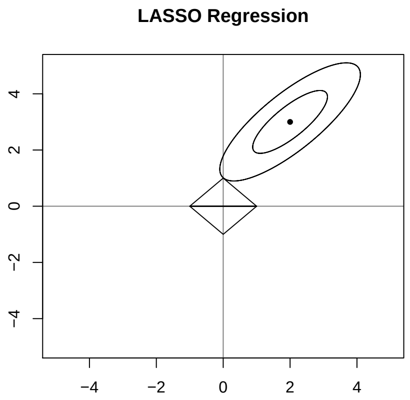
- Inference vs. Prediction
- Supervised vs. Unsupervised
- Regression vs. Classification
- Training, test and validation sets
- Overfitting
- Performance metrics

LASSO, intuition:

- High dimensionality: $n \leq p$, or $n \approx p$
- Regularization: is a process of introducing additional information in order to prevent overfitting (λ).
- Shrinkage



LASSO Penalty: Geometry



Bias and Variance

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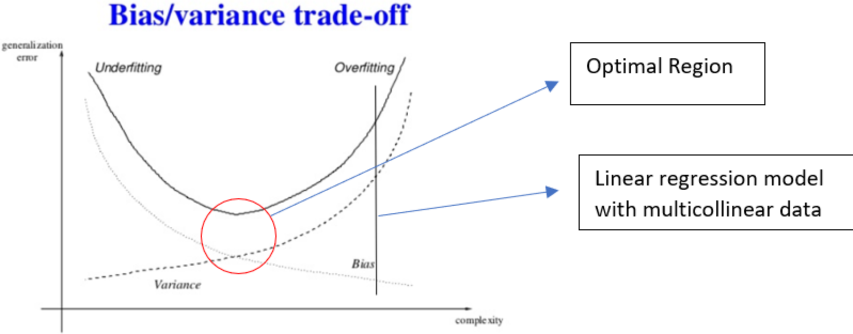
- Generally, more flexible methods result in less bias.

Bias-Variance



Other supervised learning tools:

- Linear Discriminant Analysis
- Quadratic Discriminant Analysis
- K-Nearest neighbors
- Ridge regression
- Principal Component Regression
- Tree based methods
- Support Vector Machines

Logistics

Supervised vs Unsupervised Learning

Clustering

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Supervised learning and Unsupervised learning are not competitors!

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⇝Cluster analysis / Clustering

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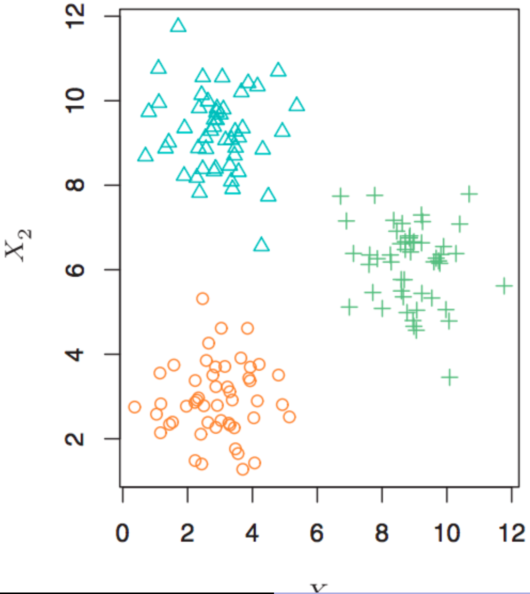
- These groups are interesting because the may correspond to some category or quantity of interest.


X_2
12
10
8
6
4
2
0
2
4
6
8
10
12
X_1

Today (and Tuesday): Cluster press releases
Goal: partition documents such that:

- similar documents are together
- dissimilar documents are apart

Method: Clustering methods
Game Plan:

1) What makes two data points (i.e. documents) similar?
2) How do we find a good partition?
3) How do we interpret the clusters?

Key Terms:

- (Multidimensional) Space
- Distance
- Euclidean Distance
- Cosine Distance
- Cluster Analysis / Clustering
- K-means
- Centroid

What makes two documents similar?

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Similar = Geometrically Close Dissimilar = Geometrically Distant

Texts and Geometry

Consider a document-term matrix

$$\boldsymbol{X} = \begin{pmatrix} 1 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 3 \end{pmatrix}$$

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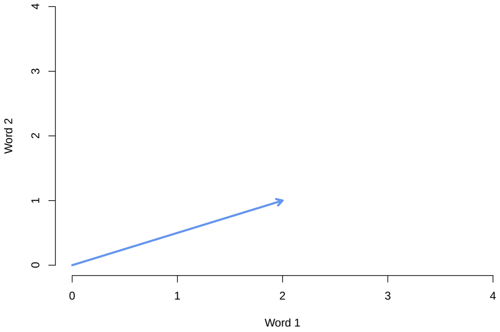
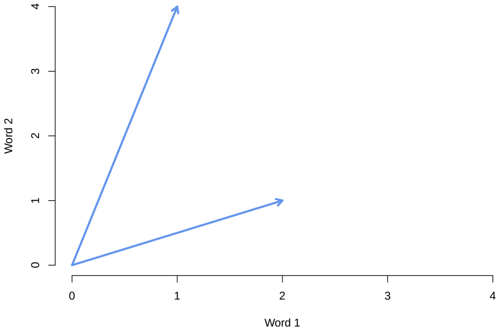
- Provides a geometry
- Natural notions of distance and similarity
- Tools from linear algebra to calculate distances mathematically.

Texts in Space



Doc1 = "Wait? No wait." $\rightsquigarrow (2, 1)$

Texts in Space



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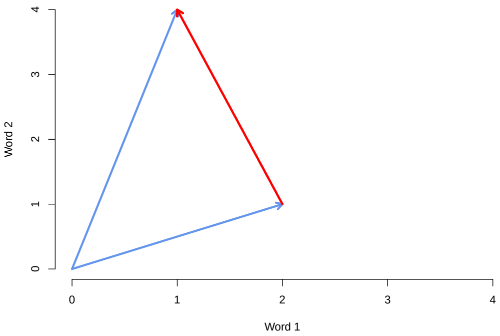

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Suppose $\boldsymbol{X}_1 = (1, 4)$ and $\boldsymbol{X}_2 = (2, 1)$.

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Test your knowledge

The Euclidean distance between any documents $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ is:

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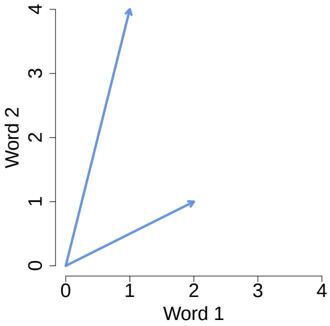
Problem(?) with Euclidean Distance



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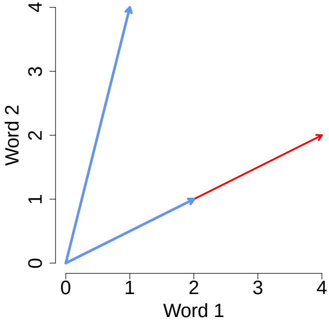

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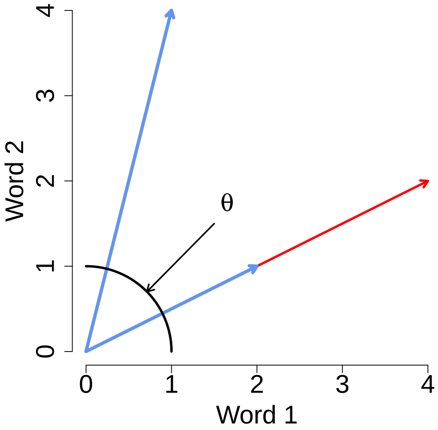
\boldsymbol{X}_3 &= 2\boldsymbol{X}_1 = (4,2) \\
d(\boldsymbol{X}_3, \boldsymbol{X}_2) &= \sqrt{(4-1)^2 + (2-4)^2} \\
&= \sqrt{13}
\end{aligned}
$$

Euclidean distance depends on document-length.

Cosine Similarity



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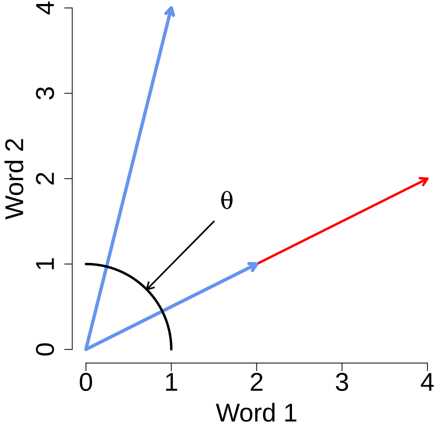

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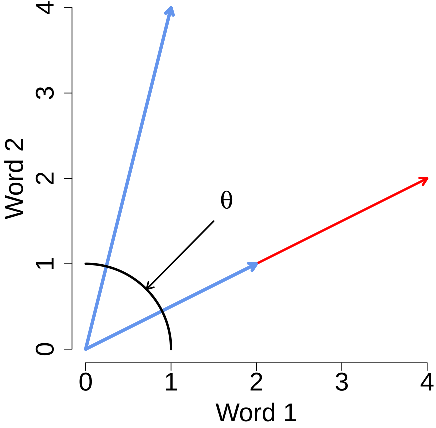

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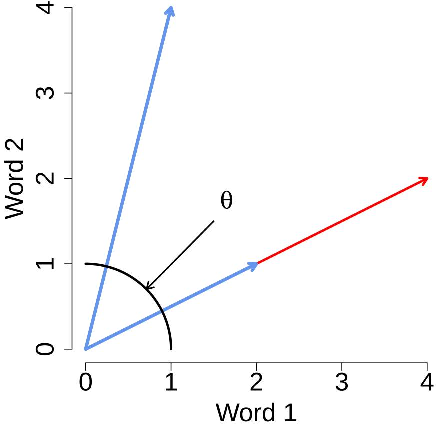

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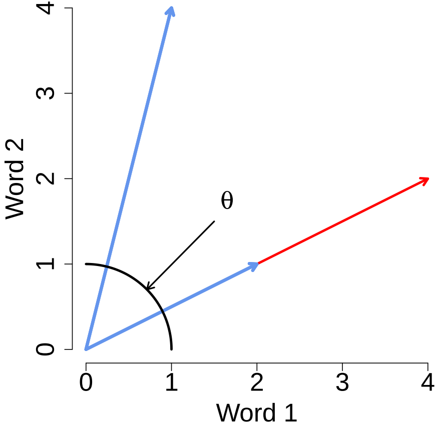

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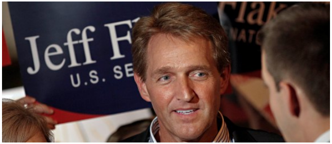
Wednesday

- How do we find a good partition?
- How do we interpret the clusters?

Flake press releases

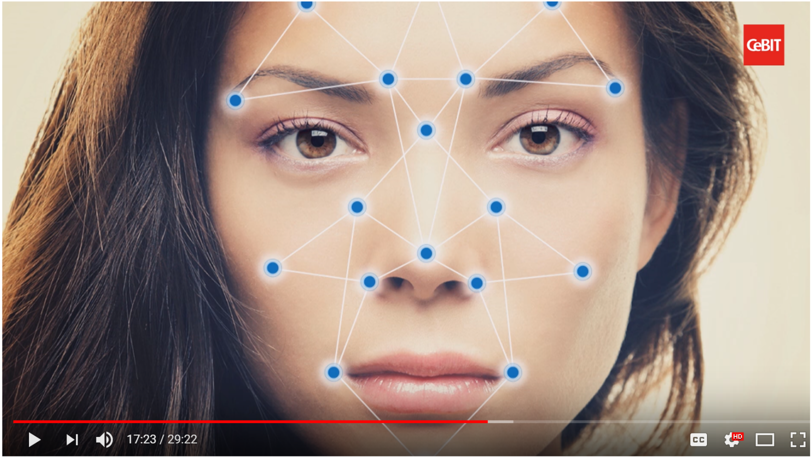
- Arizona senator Jeff Flake
- We already have the files preprocessed and available in ‘FlakeMatrix.RData‘



R!

Michal Kosinski

The End of Privacy

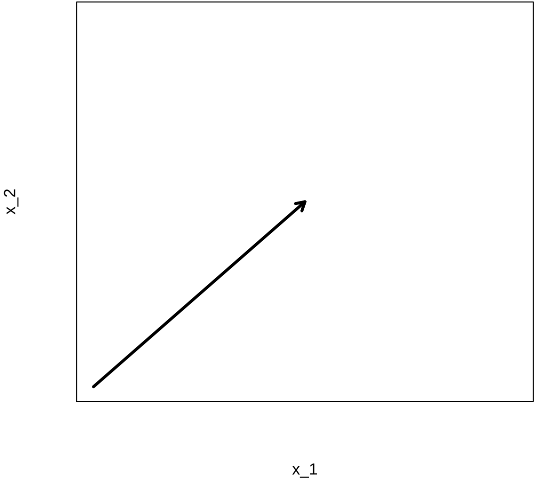


Keynote "The End of Privacy", Dr. Michal Kosinski

Bonus Slides

For those who heart math.

Vector Length



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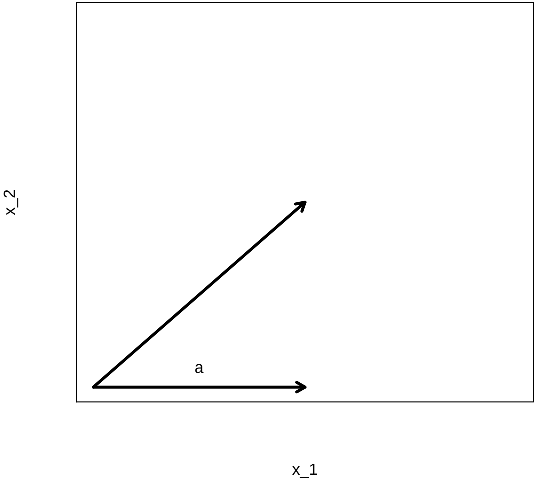
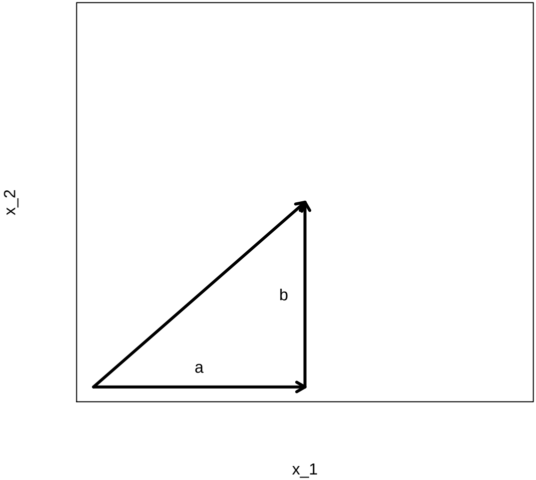


- Pythogorean Theorem: Side with length *a*

Vector Length



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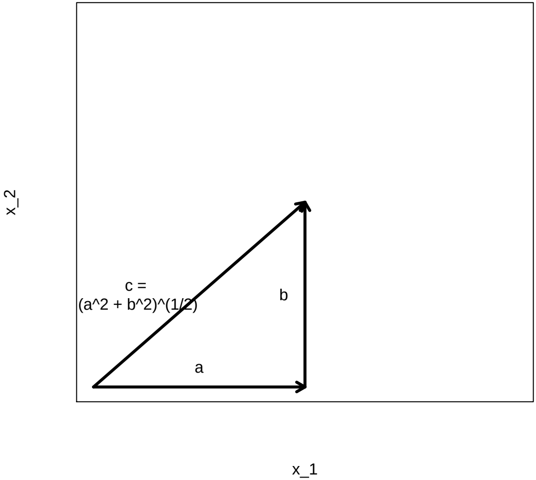

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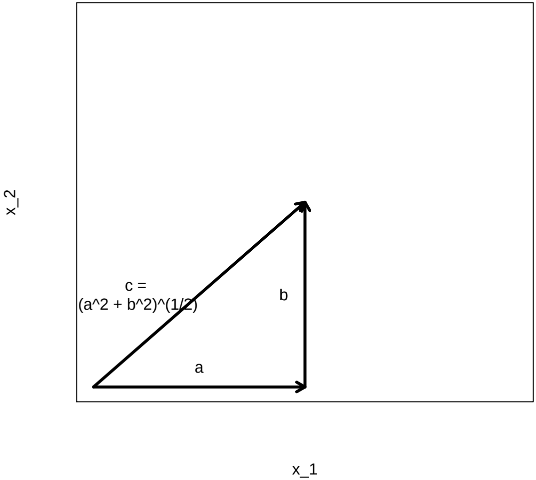
- $c = \sqrt{a^2 + b^2}$

Vector Length



- Pythogorean Theorem: Side with length a
- Side with length b and right triangle
- $c = \sqrt{a^2 + b^2}$
- Extends beyond 2 dimensions

Vector (Euclidean) Length

Suppose $\boldsymbol{X}_i$ is a document (row from an $N \times K$ document-term matrix).

Then, we will define its length as

$$
\begin{aligned}
||\boldsymbol{X}_i|| &= \sqrt{(\boldsymbol{X}_i \cdot \boldsymbol{X}_i)} \\
&= \sqrt{(X_{i1}^2 + X_{i2}^2 + X_{i3}^2 + \ldots + X_{iK}^2)} \\
&= \sqrt{\sum_{k=1}^{K} X_{ik}^2}
\end{aligned}
$$

Cosine Similarity

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$$\cos\theta \quad = \quad \left(\frac{X_1}{||X_1||}\right) \cdot \left(\frac{X_2}{||X_2||}\right)$$

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\cos\theta &= \left(\frac{X_1}{||X_1||}\right)\cdot\left(\frac{X_2}{||X_2||}\right)\\
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(0.89, 0.45)\cdot(0.24, 0.97) &= 0.65
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\text{cos dissimilarity} &= 1 - \cos\theta
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