Data Science for Actuaries (ACT6100)

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Rappels # 3.2 (Singular Value Decomposition)

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https://github.com/freakonometrics/ACT6100/

Spectral vs. Singular Value Decomposition

Example We've seen before the spectral decomposition

$$\boldsymbol{M}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\underbrace{\begin{bmatrix}-0.416 & -0.825\\ -0.909 & 0.566\end{bmatrix}}_{\boldsymbol{P}}\underbrace{\begin{bmatrix}5.372 & 0\\ 0 & -0.372\end{bmatrix}}_{\boldsymbol{D}}\underbrace{\begin{bmatrix}-0.416 & -0.825\\ -0.909 & 0.566\end{bmatrix}^{-1}}_{\boldsymbol{P}^{-1}}$$

where

$$\underbrace{\begin{bmatrix}-0.416 & -0.825\\ -0.909 & 0.566\end{bmatrix}^{-1}}_{\boldsymbol{P}^{-1}}=\begin{bmatrix}-0.574 & -0.837\\ -0.923 & 0.422\end{bmatrix}$$

Alternatively, there are $\boldsymbol{U}$ and $\boldsymbol{V}$ two rotation matrices, and $\boldsymbol{\Delta}$ some diagonal matrix such that

$$\boldsymbol{M}=\boldsymbol{U}\boldsymbol{\Delta}\boldsymbol{V}^\top$$

Example

$$\boldsymbol{M}=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}=\underbrace{\begin{bmatrix}-0.404 & -0.914\\ -0.914 & 0.404\end{bmatrix}}_{\boldsymbol{U}}\underbrace{\begin{bmatrix}5.465 & 0\\ 0 & 0.366\end{bmatrix}}_{\boldsymbol{\Delta}}\underbrace{\begin{bmatrix}-0.576 & 0.817\\ -0.817 & -0.576\end{bmatrix}}_{\boldsymbol{V}^\top}$$

Spectral vs. Singular Value Decomposition

```r
> M=c(1,3,2,4)
> dim(M)=c(2,2)
> M
     [,1] [,2]
[1,]    1    2
[2,]    3    4
> eigen(M)
$values
[1]  5.3722813 -0.3722813

$vectors
           [,1]       [,2]
[1,] -0.4159736 -0.8245648
[2,] -0.9093767  0.5657675
```

```r
> svd(M)
$d
[1] 5.4649857 0.3659662

$u
           [,1]       [,2]
[1,] -0.4045536 -0.9145143
[2,] -0.9145143  0.4045536

$v
           [,1]       [,2]
[1,] -0.5760484  0.8174156
[2,] -0.8174156 -0.5760484
>
```

$$\boldsymbol{D} = \begin{bmatrix} 5.372 & 0 \\ 0 & -0.372 \end{bmatrix} \text{ and } \boldsymbol{\Delta} = \begin{bmatrix} 5.465 & 0 \\ 0 & 0.366 \end{bmatrix}$$

$$\boldsymbol{M}^\top \boldsymbol{M} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.576 & -0.817 \\ 0.817 & 0.576 \end{bmatrix}}_{\boldsymbol{P}'} \underbrace{\begin{bmatrix} 29.866 & 0 \\ 0 & 0.134 \end{bmatrix}}_{\boldsymbol{D}'=\boldsymbol{\Delta}^2} \underbrace{\begin{bmatrix} 0.576 & -0.817 \\ 0.817 & 0.576 \end{bmatrix}^{-1}}_{\boldsymbol{P}'^{-1}}$$

Spectral vs. Singular Value Decomposition

$$\boldsymbol{M} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} \text{ or } \boldsymbol{U}\boldsymbol{\Delta}\boldsymbol{V}^\top \text{ ?}$$

with

-
- $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal, $\boldsymbol{U}\boldsymbol{U}^\top = \mathbb{I}$ and $\boldsymbol{V}\boldsymbol{V}^\top = \mathbb{I}$

λ is a singular value for $\boldsymbol{M}$ if

$$\exists \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \text{ such that } \boldsymbol{M}\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{u}} \text{ and } \boldsymbol{M}^\top\vec{\boldsymbol{u}} = \lambda\vec{\boldsymbol{v}}$$

$\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ are left-singular and right-singular vectors.

Note that if $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Delta}\boldsymbol{V}^\top$,

$$\boldsymbol{M}\boldsymbol{M}^\top = \boldsymbol{U}\boldsymbol{\Delta}\underbrace{\boldsymbol{V}^\top\boldsymbol{V}}\boldsymbol{\Delta}^\top\boldsymbol{U} = \boldsymbol{U}\underbrace{\boldsymbol{\Delta}\boldsymbol{\Delta}^\top}\boldsymbol{U}^\top$$

$$\boldsymbol{M}^\top\boldsymbol{M} = \boldsymbol{V}\boldsymbol{\Delta}^\top\underbrace{\boldsymbol{U}^\top\boldsymbol{U}}\boldsymbol{\Delta}\boldsymbol{V}^\top = \boldsymbol{V}\underbrace{\boldsymbol{\Delta}^\top\boldsymbol{\Delta}}\boldsymbol{V}^\top$$

Spectral vs. Singular Value Decomposition

$$\boldsymbol{M} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{D}^{-1} \text{ or } \boldsymbol{U}\boldsymbol{\Delta}\boldsymbol{V}^{\top} \text{ ?}$$

- eigenvalues $\boldsymbol{D}_{i,i}$ can be negative
- entries $\boldsymbol{\Delta}_{i,i}$ are all positive

```
> M=c(1,3,1,4)
> dim(M)=c(2,2)
> M
     [,1] [,2]
[1,]    1    1
[2,]    3    4
> eigen(M)
$values
[1] 4.7912878 0.2087122

$vectors
           [,1]       [,2]
[1,] -0.2550401 -0.7841904
[2,] -0.9669305  0.6205203
```

```
> svd(M)
$d
[1] 5.1925824 0.1925824

$u
           [,1]       [,2]
[1,] -0.2700013 -0.9628599
[2,] -0.9628599  0.2700013

$v
           [,1]       [,2]
[1,] -0.6082872 -0.7937170
[2,] -0.7937170  0.6082872
>
```

Spectral vs. Singular Value Decomposition

$$\boldsymbol{M} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{D}^{-1} \text{ or } \boldsymbol{U}\boldsymbol{\Delta}\boldsymbol{V}^\top \text{ ?}$$

If $\boldsymbol{M}$ is a $n \times n$ symmetric matrix, the two coincide (up to signs),

```
> M=c(1,3,3,1)
> dim(M)=c(2,2)
> M
     [,1] [,2]
[1,]    1    3
[2,]    3    1
> eigen(M)
$values
[1]  4 -2

$vectors
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068
```

```
> svd(M)
$d
[1] 4 2

$u
           [,1]       [,2]
[1,] -0.7071068 -0.7071068
[2,] -0.7071068  0.7071068

$v
           [,1]       [,2]
[1,] -0.7071068  0.7071068
[2,] -0.7071068 -0.7071068
>
```

Singular Value Decomposition $2 \times 2 \ \rightarrow \times 2 \times 3$

More generally

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-0.386 & -0.922\\ -0.922 & 0.386\end{bmatrix}}_{\boldsymbol{U}} \underbrace{\begin{bmatrix}9.508 & 0\\ 0 & 0.773\end{bmatrix}}_{\boldsymbol{\Delta}} \underbrace{\begin{bmatrix}-0.429 & -0.566 & -0.704\\ 0.806 & 0.112 & -0.581\end{bmatrix}}_{\boldsymbol{V}^\top}$$

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-0.39 & -0.92\\ -0.92 & 0.39\end{bmatrix}}_{\boldsymbol{U}} \underbrace{\begin{bmatrix}9.51 & 0 & 0\\ 0 & 0.77 & 0\end{bmatrix}}_{\boldsymbol{\Delta}} \underbrace{\begin{bmatrix}-0.42 & -0.57 & -0.70\\ 0.81 & 0.11 & -0.58\\ \star & \star & \star\end{bmatrix}}_{\boldsymbol{V}^\top}$$

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-0.39 & -0.92\\ -0.92 & 0.39\end{bmatrix}}_{\boldsymbol{U}} \underbrace{\begin{bmatrix}9.51 & 0 & 0\\ 0 & 0.77 & 0\end{bmatrix}}_{\boldsymbol{\Delta}} \underbrace{\begin{bmatrix}-0.42 & -0.57 & -0.70\\ 0.81 & 0.11 & -0.58\\ 0.41 & -0.82 & 0.41\end{bmatrix}}_{\boldsymbol{V}^\top}$$

where

- $\boldsymbol{U}$ and $\boldsymbol{V}$ are (square) rotation matrices (orthogonal)
- $\boldsymbol{\Delta}$ is a “diagonal” matrix

Singular Value Decomposition

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-0.39 & -0.92\\ -0.92 & 0.39\end{bmatrix}}_{\boldsymbol{U}} \underbrace{\begin{bmatrix}9.51 & 0 & 0\\ 0 & 0.77 & 0\end{bmatrix}}_{\boldsymbol{\Delta}} \underbrace{\begin{bmatrix}-0.42 & -0.57 & -0.70\\ 0.81 & 0.11 & -0.58\\ 0.41 & -0.82 & 0.41\end{bmatrix}}_{\boldsymbol{V}^\top}$$

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-3.67 & -0.71 & 0\\ -8.8 & 0.30 & 0\end{bmatrix}}_{\boldsymbol{U\Delta}} \underbrace{\begin{bmatrix}-0.42 & -0.57 & -0.70\\ 0.81 & 0.11 & -0.58\\ 0.41 & -0.82 & 0.41\end{bmatrix}}_{\boldsymbol{V}^\top}$$

Spectral vs. Singular Value Decomposition

Let $\boldsymbol{M}$ be a $m\times n$ matrix with $\text{rank}(\boldsymbol{M})=r$, then

$$\boldsymbol{M}=\boldsymbol{U}\boldsymbol{\Delta}\boldsymbol{V}^\top$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are $m\times m$ and $n\times n$ orthogonal matrices

$$\boldsymbol{U}^\top\boldsymbol{U}=\mathbb{I}_m \text{ and } \boldsymbol{V}^\top\boldsymbol{V}=\mathbb{I}_n$$

and $\boldsymbol{\Delta}$ is a $m\times n$ matrix with 0's everywhere, except on the m main diagonal, with entries $\boldsymbol{\Delta}_{ii}$ such that

$$\boldsymbol{\Delta}_{11}\geq\boldsymbol{\Delta}_{22}\geq\boldsymbol{\Delta}_{rr}>0 \text{ and } \boldsymbol{\Delta}_{ii}=0,\ \forall i>r$$

. Hence

$$\boldsymbol{M}=\sum_{i=1}^r\boldsymbol{\Delta}_{ii}\boldsymbol{U}_{\cdot i}\boldsymbol{V}_{\cdot i}^\top$$

Note: $\boldsymbol{M}'_k=\sum_{i=1}^k\boldsymbol{\Delta}_{ii}\boldsymbol{U}_{\cdot i}\boldsymbol{V}_{\cdot i}^\top$ is an approximation of $\boldsymbol{M}$ of rank k. It is the best rank-k approximation (Schmidt–Mirsky–Eckart–Young).