Data Science for Actuaries (ACT6100)

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Rappels # 3.4 (Gaussian Vectors)

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https://github.com/freakonometrics/ACT6100/

Random Vectors

Soient $\boldsymbol{X}$ un vecteur aléatoire de dimension d

- L'espérance de $\boldsymbol{X}$, notée $\mathbb{E}(\boldsymbol{X})$ est définie (si elle existe) par le vecteur de dimension d $\mathbb{E}(\boldsymbol{X}) = (\mathbb{E}(\boldsymbol{X}_1), \dots, \mathbb{E}(\boldsymbol{X}_d))^\top$.
- La matrice de covariance (appelée aussi matrice de variance-covariance de $\boldsymbol{X}$) est définie (si elle existe) par la matrice de taille (d, d)

$$\text{Var}(\boldsymbol{X}) = \mathbb{E}\left((\boldsymbol{X} - \mathbb{E}(\boldsymbol{X}))(\boldsymbol{X} - \mathbb{E}(\boldsymbol{X}))^\top\right).$$

Ainsi le terme ij de cette matrice représente la covariance entre X_i et X_j,

$$\text{Cov}(X_i, X_j) = \mathbb{E}\left[(X_i - \mathbb{E}(X_i))(X_j - \mathbb{E}(X_j))\right].$$

Random Vectors

Soit $\boldsymbol{X}$ un vecteur aléatoire de dimension d, de moyenne $\boldsymbol{\mu}$ et de matrice de covariance $\boldsymbol{\Sigma}$.
Soient $\boldsymbol{A}$ et $\boldsymbol{B}$ deux matrices réeeles de taille (d, p) et (d, q) et enfin soit $\mathbf{a} \in \mathbb{R}^p$ alors

- $\text{Var}(\boldsymbol{X}) = \mathbb{E}\left((\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^\top\right) = \mathbb{E}(\boldsymbol{X}\boldsymbol{X}^\top) - \boldsymbol{\mu}\boldsymbol{\mu}^\top$.
- $\mathbb{E}\left(\boldsymbol{A}^\top\boldsymbol{X} + \mathbf{a}\right) = \boldsymbol{A}^\top\boldsymbol{\mu} + \mathbf{a}$.
- $\text{Var}\left(\boldsymbol{A}^\top\boldsymbol{X} + \mathbf{a}\right) = \boldsymbol{A}^\top\boldsymbol{\Sigma}\boldsymbol{A}$.
- $\text{Cov}\left(\boldsymbol{A}^\top\boldsymbol{X}, \boldsymbol{B}^\top\boldsymbol{X}\right) = \boldsymbol{A}^\top\boldsymbol{\Sigma}\boldsymbol{B}$.

The Gaussian Distribution

A Gaussian variable, with distribution $\mathcal{N}(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ and $\sigma > 0$, has density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \text{ for all } x \in \mathbb{R}.$$

Then $\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \sigma^2$.
Observe that if $Z \sim \mathcal{N}(0,1)$, $X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$.
The Gaussian **vector** $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: $\boldsymbol{X} = (X_1, ..., X_n)$ is a Gaussian vector with mean $\mathbb{E}(\boldsymbol{X}) = \boldsymbol{\mu}$ and covariance matrix $\text{Var}(\boldsymbol{X}) = \boldsymbol{\Sigma} = \mathbb{E}\left((\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^\top\right)$ non-degenerated ($\boldsymbol{\Sigma}$ is invertible) if its density is

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \boldsymbol{\Sigma}}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\right), \ \boldsymbol{x} \in \mathbb{R}^n,$$

see multivariate Gaussian distribution

Gaussian (multivariate) distribution

$\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with density

$$f_{\mathbf{X}}(x_1, \ldots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)$$

where $\mathbb{E}(\boldsymbol{X}) = \boldsymbol{\mu}$ and $\text{Var}(\boldsymbol{X}) = \boldsymbol{\Sigma}$.

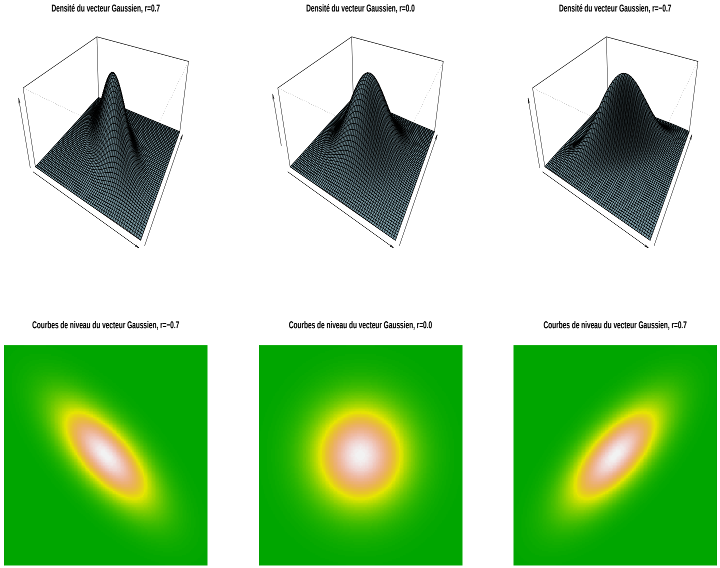
Estimates are $\overline{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$ and $\widehat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})^\top$

In dimension 2, $f(x, y)$ is proportional to

$$\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right)$$

levels curves (isodensities) are ellipses.

Gaussian (multivariate) distribution

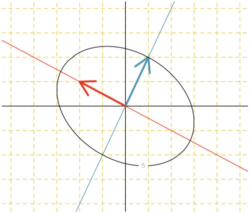


Quadratic Forms

Consider $\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$,

and function $\boldsymbol{z} \mapsto \boldsymbol{z}^\top \boldsymbol{M} \boldsymbol{z}$, i.e.

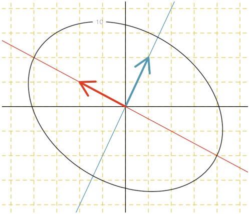
$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



or $ax^2 + 2bxy + cy^2$ is a quadratic form.

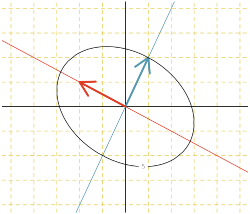
If $\boldsymbol{M} > 0$, points $\boldsymbol{z} = (x, y)$ such that $\boldsymbol{z}^\top \boldsymbol{M} \boldsymbol{z} = \gamma$, for some $\gamma > 0$, are on an ellipse (centered on $\boldsymbol{0}$)

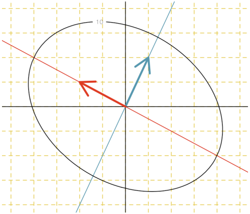
Let $\lambda_1 \geq \lambda_2 > 0$ denote the eigenvalues of $\boldsymbol{M}$ and $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ denote the eigenvectors.



Quadratic Forms

On the picture, $\boldsymbol{M} = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.9 \end{pmatrix}$





```
> M=matrix(c(.6,.2,.2,.9),2,2)
> eigen(M)
eigen() decomposition
$values
[1] 1.0 0.5
$vectors
          [,1]       [,2]
[1,] 0.4472136 -0.8944272
[2,] 0.8944272  0.4472136
```

i.e. $\lambda_1 = 1$ and $\lambda_2 = 1/2$, and

$$\vec{\boldsymbol{v}}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ \vec{\boldsymbol{v}}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Note that $\|\vec{\boldsymbol{v}}_1\| = \|\vec{\boldsymbol{v}}_2\| = 1$ and $\vec{\boldsymbol{v}}_1 \perp \vec{\boldsymbol{v}}_2$

The Gaussian Distribution

If $\boldsymbol{X}$ is a Gaussian vector, then for any i, X_i has a (univariate) Gaussian distribution, but its converse it not necessarily true.

Let $\boldsymbol{X} = (X_1, ..., X_n)$ be a random vector with mean $\mathbb{E}(\boldsymbol{X}) = \boldsymbol{\mu}$ and with covariance matrix $\boldsymbol{\Sigma}$, if $\boldsymbol{A}$ is a $k \times n$ matrix, and $\boldsymbol{b} \in \mathbb{R}^k$, then $\boldsymbol{Y} = \boldsymbol{AX} + \boldsymbol{b}$ is a Gaussian vector $\mathbb{R}^k$, with distribution $\mathcal{N}\left(\boldsymbol{A\mu} + \boldsymbol{b}, \boldsymbol{A\Sigma A}^\top\right)$.

Observe that if (X_1, X_2) is a Gaussian vector, X_1 and X_2 are independent if and only if

$$\text{Cov}(X_1, X_2) = \mathbb{E}\left((X_1 - \mathbb{E}(X_1))(X_2 - \mathbb{E}(X_2))\right) = 0.$$