

Data Science for Actuaries (ACT6100)

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Rappels # 3.2 (QR & SVD - Singular Value Decomposition)

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 <https://github.com/freakonometrics/ACT6100/>

QR Factorization

Let \mathbf{A} be some $n \times k$ matrix, with k linearly independent vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$ in \mathbb{R}^n , $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_k]$.

From **Gram-Schmidt** decomposition, there is $\mathbf{Q} = [\mathbf{q}_1, \dots, \mathbf{q}_k]$ such that $\mathbf{Q}^\top \mathbf{Q} = \mathbb{I}$ and \mathbf{R} upper triangular, such that $\mathbf{A} = \mathbf{QR}$.

Note: QR factorization is interesting to inverse matrices,

$$\mathbf{A}^{-1} = (\mathbf{QR})^{-1} = \mathbf{R}^{-1} \mathbf{Q}^\top$$

\mathbf{R}^{-1} appears when solving linear systems

Spectral vs. Singular Value Decomposition

Example We've seen before the spectral decomposition

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.416 & -0.825 \\ -0.909 & 0.566 \end{bmatrix}}_P \underbrace{\begin{bmatrix} 5.372 & 0 \\ 0 & -0.372 \end{bmatrix}}_D \underbrace{\begin{bmatrix} -0.416 & -0.825 \\ -0.909 & 0.566 \end{bmatrix}^{-1}}_{P^{-1}}$$

where

$$\underbrace{\begin{bmatrix} -0.416 & -0.825 \\ -0.909 & 0.566 \end{bmatrix}^{-1}}_{P^{-1}} = \begin{bmatrix} -0.574 & -0.837 \\ -0.923 & 0.422 \end{bmatrix}$$

Alternatively, there are **U** and **V** two rotation matrices, and **Δ** some diagonal matrix such that

$$M = U\Delta V^T$$

Example

$$M = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.404 & -0.914 \\ -0.914 & 0.404 \end{bmatrix}}_U \underbrace{\begin{bmatrix} 5.465 & 0 \\ 0 & 0.366 \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} -0.576 & 0.817 \\ -0.817 & -0.576 \end{bmatrix}}_{V^T}$$

Spectral vs. Singular Value Decomposition

```

1 > M=c(1,3,2,4)
2 > dim(M)=c(2,2)
3 > M
4      [,1] [,2]
5 [1,]    1    2
6 [2,]    3    4
7 > eigen(M)
8 $values
9 [1]  5.3722813 -0.3722813
10
11 $vectors
12      [,1] [,2]
13 [1,] -0.4159736 -0.8245648
14 [2,] -0.9093767  0.5657675
    
```

```

1 > svd(M)
2 $d
3 [1]  5.4649857  0.3659662
4
5 $u
6      [,1] [,2]
7 [1,] -0.4045536 -0.9145143
8 [2,] -0.9145143  0.4045536
9
10 $v
11      [,1] [,2]
12 [1,] -0.5760484  0.8174156
13 [2,] -0.8174156 -0.5760484
14 >
    
```

$$D = \begin{bmatrix} 5.372 & 0 \\ 0 & -0.372 \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} 5.465 & 0 \\ 0 & 0.366 \end{bmatrix}$$

$$M^T M = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.576 & -0.817 \\ 0.817 & 0.576 \end{bmatrix}}_{P'} \underbrace{\begin{bmatrix} 29.866 & 0 \\ 0 & 0.134 \end{bmatrix}}_{D'=\Delta^2} \underbrace{\begin{bmatrix} 0.576 & -0.817 \\ 0.817 & 0.576 \end{bmatrix}^{-1}}_{P'^{-1}}$$

Spectral vs. Singular Value Decomposition

$$\mathbf{M} = \mathbf{PDP}^{-1} \text{ or } \mathbf{U}\mathbf{\Delta}\mathbf{V}^{\top} ?$$

with



► \mathbf{U} and \mathbf{V} are orthogonal, $\mathbf{UU}^{\top} = \mathbb{I}$ and $\mathbf{VV}^{\top} = \mathbb{I}$

λ is a **singular value** for \mathbf{M} if

$$\exists \vec{u}, \vec{v} \text{ such that } \mathbf{M}\vec{v} = \lambda\vec{u} \text{ and } \mathbf{M}^{\top}\vec{u} = \lambda\vec{v}$$

\vec{u} and \vec{v} are left-singular and right-singular vectors.

Note that if $\mathbf{M} = \mathbf{U}\mathbf{\Delta}\mathbf{V}^{\top}$,

$$\mathbf{M}\mathbf{M}^{\top} = \mathbf{U}\mathbf{\Delta}\underbrace{\mathbf{V}^{\top}\mathbf{V}}_{\mathbb{I}}\mathbf{\Delta}^{\top}\mathbf{U} = \mathbf{U}\underbrace{\mathbf{\Delta}\mathbf{\Delta}^{\top}}_{\mathbb{I}}\mathbf{U}^{\top}$$

$$\mathbf{M}^{\top}\mathbf{M} = \mathbf{V}\mathbf{\Delta}^{\top}\underbrace{\mathbf{U}^{\top}\mathbf{U}}_{\mathbb{I}}\mathbf{\Delta}\mathbf{V}^{\top} = \mathbf{V}\underbrace{\mathbf{\Delta}^{\top}\mathbf{\Delta}}_{\mathbb{I}}\mathbf{V}^{\top}$$

Spectral vs. Singular Value Decomposition

$$M = P D D^{-1} \text{ or } U \Delta V^T ?$$

- ▶ eigenvalues $D_{i,i}$ can be negative
- ▶ entries $\Delta_{i,i}$ are all positive

```
1 > M=c(1,3,1,4)
2 > dim(M)=c(2,2)
3 > M
      [,1] [,2]
4 [1,]    1    1
5 [2,]    3    4
6 > eigen(M)
7 $values
8 [1] 4.7912878 0.2087122
9
10 $vectors
11      [,1] [,2]
12 [1,] -0.2550401 -0.7841904
13 [2,] -0.9669305  0.6205203
```

```
1 > svd(M)
2 $d
3 [1] 5.1925824 0.1925824
4
5 $u
6      [,1] [,2]
7 [1,] -0.2700013 -0.9628599
8 [2,] -0.9628599  0.2700013
9
10 $v
11      [,1] [,2]
12 [1,] -0.6082872 -0.7937170
13 [2,] -0.7937170  0.6082872
14 >
```

Spectral vs. Singular Value Decomposition

$$M = P D D^{-1} \text{ or } U \Delta V^T ?$$

If M is a $n \times n$ symmetric matrix, the two coincide (up to signs),

```
1 > M=c(1,3,3,1)
2 > dim(M)=c(2,2)
3 > M
4      [,1] [,2]
5 [1,]    1    3
6 [2,]    3    1
7 > eigen(M)
8 $values
9 [1]  4 -2
10
11 $vectors
12      [,1] [,2]
13 [1,] 0.7071068 -0.7071068
14 [2,] 0.7071068  0.7071068
```

```
1 > svd(M)
2 $d
3 [1]  4  2
4
5 $u
6      [,1] [,2]
7 [1,] -0.7071068 -0.7071068
8 [2,] -0.7071068  0.7071068
9
10 $v
11      [,1] [,2]
12 [1,] -0.7071068  0.7071068
13 [2,] -0.7071068 -0.7071068
14 >
```

Singular Value Decomposition $2 \times 2 \rightarrow 2 \times 3$

More generally

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.386 & -0.922 \\ -0.922 & 0.386 \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} 9.508 & 0 \\ 0 & 0.773 \end{bmatrix}}_{\mathbf{\Delta}} \underbrace{\begin{bmatrix} -0.429 & -0.566 & -0.704 \\ 0.806 & 0.112 & -0.581 \end{bmatrix}}_{\mathbf{V}^T}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.39 & -0.92 \\ -0.92 & 0.39 \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & 0.77 & 0 \end{bmatrix}}_{\mathbf{\Delta}} \underbrace{\begin{bmatrix} -0.42 & -0.57 & -0.70 \\ 0.81 & 0.11 & -0.58 \\ * & * & * \end{bmatrix}}_{\mathbf{V}^T}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.39 & -0.92 \\ -0.92 & 0.39 \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & 0.77 & 0 \end{bmatrix}}_{\mathbf{\Delta}} \underbrace{\begin{bmatrix} -0.42 & -0.57 & -0.70 \\ 0.81 & 0.11 & -0.58 \\ 0.41 & -0.82 & 0.41 \end{bmatrix}}_{\mathbf{V}^T}$$

where

- ▶ \mathbf{U} and \mathbf{V} are (square) rotation matrices (orthogonal)
- ▶ $\mathbf{\Delta}$ is a “diagonal” matrix

Singular Value Decomposition

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.39 & -0.92 \\ -0.92 & 0.39 \end{bmatrix}}_U \underbrace{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & 0.77 & 0 \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} -0.42 & -0.57 & -0.70 \\ 0.81 & 0.11 & -0.58 \\ 0.41 & -0.82 & 0.41 \end{bmatrix}}_{V^T}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -3.67 & -0.71 & 0 \\ -8.8 & 0.30 & 0 \end{bmatrix}}_{U\Delta} \underbrace{\begin{bmatrix} -0.42 & -0.57 & -0.70 \\ 0.81 & 0.11 & -0.58 \\ 0.41 & -0.82 & 0.41 \end{bmatrix}}_{V^T}$$

Spectral vs. Singular Value Decomposition

Let \mathbf{M} be a $m \times n$ matrix with $\text{rank}(\mathbf{M}) = r$, then

$$\mathbf{M} = \mathbf{U} \mathbf{\Delta} \mathbf{V}^\top$$

where \mathbf{U} and \mathbf{V} are $m \times m$ and $n \times n$ orthogonal matrices

$$\mathbf{U}^\top \mathbf{U} = \mathbb{I}_m \text{ and } \mathbf{V}^\top \mathbf{V} = \mathbb{I}_n$$

and $\mathbf{\Delta}$ is a $m \times n$ matrix with 0's everywhere, except on the m main diagonal, with entries Δ_{ii} such that

$$\Delta_{11} \geq \Delta_{22} \geq \Delta_{rr} > 0 \text{ and } \Delta_{ii} = 0, \forall i > r$$

. Hence

$$\mathbf{M} = \sum_{i=1}^r \Delta_{ii} \mathbf{U}_{\cdot i} \mathbf{V}_{\cdot i}^\top$$

Note: $\mathbf{M}'_k = \sum_{i=1}^k \Delta_{ii} \mathbf{U}_{\cdot i} \mathbf{V}_{\cdot i}^\top$ is an approximation of \mathbf{M} of rank k . It is the best rank- k approximation (Schmidt–Mirsky–Eckart–Young).