Data Science for Actuaries (ACT6100)

Arthur Charpentier

Rappels # 4.2 (Convex Optimization)

automne 2020

https://github.com/freakonometrics/ACT6100/

Convex Optimization Problem

$$\min_{\boldsymbol{x}\in D}\{f(\boldsymbol{x})\}$$

$$\text{subject to } \begin{cases} g_i(\boldsymbol{x}) \leq 0, \ \forall i = 1, \cdots, m \\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{cases}$$

for some convex function $f, g_1, \cdots, g_m$

- f is the objective function
- g_i's are inequality constraint functions
- if $\boldsymbol{x} \in D$ satisfies $g_i(\boldsymbol{x}) \leq 0, \ \forall i = 1, \cdots, m$ and $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, then $\boldsymbol{x}$ is a feasible point
- the minimum values for f is $f^\star$
- if $\boldsymbol{x}^\star$ is a feasible point such $f(\boldsymbol{x}^\star = f^\star$ is a solution
- if $\boldsymbol{x}^\star_\epsilon$ is a feasible point such $f(\boldsymbol{x}^\star_\epsilon \leq f^\star + \epsilon$ is ϵ-suboptimal
- if $\boldsymbol{x}$ is a feasible point and $g_i(\boldsymbol{x}) = 0$, then g_i is saturated

Convex Optimization Problem

Proposition: the set of solutions of a convex problem is convex

Proposition: if f is strictly convex, the solution is unique

Let C denote the feasible set. The problem can be written

$$\min_{\boldsymbol{x}}\{f(\boldsymbol{x})\}\text{subject to } \boldsymbol{x} \in C$$

Proposition: if $\boldsymbol{x}^\star \in C$, it is optimal if and only if

$$\nabla f\left(\boldsymbol{x}^\star\right)^\top\left(\boldsymbol{y}-\boldsymbol{x}^\star\right) \geq 0 \text{ for all } \boldsymbol{y} \in C.$$

If $C = \mathbb{R}^n$, the condition is simply $\nabla f\left(\boldsymbol{x}^\star\right) = \mathbf{0}$.

Slack Variables

We can transform the problem

$$\min_{\boldsymbol{x}}\{f(\boldsymbol{x})\}$$

$$\text{subject to } \begin{cases} g_i(\boldsymbol{x}) \leq 0, \ \forall i = 1, \cdots, m \\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{cases}$$

into

$$\min_{\boldsymbol{x},\boldsymbol{s}}\{f(\boldsymbol{x})\}$$

$$\text{subject to } \begin{cases} s_i \geq 0, \ \forall i = 1, \cdots, m \\ g_i(\boldsymbol{x}) + s_i = 0, \ \forall i = 1, \cdots, m \\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{cases}$$

(no longer convex, unless g_i's are affine)

Linear Programming

$$\min_{\boldsymbol{x}}\{\boldsymbol{c}^\top \boldsymbol{x}\}$$

$$\text{subject to } \begin{cases} \boldsymbol{D}\boldsymbol{x} \geq \boldsymbol{d} \\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{cases}$$

or, in the standard form

$$\min_{\boldsymbol{x}}\{\boldsymbol{c}^\top \boldsymbol{x}\}$$

$$\text{subject to } \begin{cases} \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{cases}$$

Example:

Median computation, given $\{y_1, \cdots, y_n\}$, $\min\left\{\sum_{i=1}^n |y_i - m|\right\}$, or

$$\min_{m,\boldsymbol{a}\boldsymbol{b}}\left\{\sum_{i=1}^n \boldsymbol{a}_i + \boldsymbol{b}_i\right\} \text{ s.t. } \begin{cases} \boldsymbol{a}, \boldsymbol{b} \geq \boldsymbol{0} \\ \boldsymbol{y} - m\boldsymbol{1} = \boldsymbol{a} - \boldsymbol{b} \end{cases}$$

Quadratic Programming

$$\min_{\boldsymbol{x}} \left\{ \boldsymbol{c}^\top \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} \right\}$$
$$\text{subject to } \begin{cases} \boldsymbol{D}\boldsymbol{x} \geq \boldsymbol{d} \\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{cases}$$

where $\boldsymbol{Q}$ is positive semidefinite.

Example: portfolio allocation (percentage)

$$\min_{\boldsymbol{x}} \left\{ \boldsymbol{\mu}^\top \boldsymbol{x} - \frac{\gamma}{2}\boldsymbol{x}^\top \boldsymbol{\Sigma}\boldsymbol{x} \right\}$$
$$\text{subject to } \begin{cases} \boldsymbol{x} \geq \boldsymbol{0} \\ \boldsymbol{1}^\top \boldsymbol{x} = 1 \end{cases}$$

(this will be discussed soon)