Data Science for Actuaries (ACT6100)

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Rappels # 4.1 (Convex Optimization)

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https://github.com/freakonometrics/ACT6100/

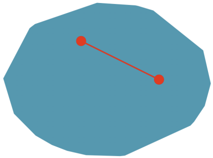
Convex Sets

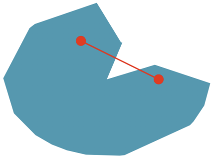
$C \subset \mathbb{R}^n$ is convex if

$$\boldsymbol{x}, \boldsymbol{y} \in C \Longrightarrow t\boldsymbol{x} + (1-t)\boldsymbol{y} \in C$$

for all $t \in [0,1]$

- unit ball $\{\boldsymbol{x} : \|\boldsymbol{x}\| \leq 1\}$ is convex,
- hyperplane $\{\boldsymbol{x} : \boldsymbol{x}^\top \boldsymbol{\beta} = y_0\}$ is convex,
- half-space $\{\boldsymbol{x} : \boldsymbol{x}^\top \boldsymbol{\beta} \leq y_0\}$ is convex,
- polyhedron $\{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq a\}$ is convex,
- simplex $\{\boldsymbol{x} \in \mathbb{R}_+ : \boldsymbol{1}^\top \boldsymbol{x} = 1\}$ is convex,
- if C_1 and C_2 are convex, so is $C_1 \cap C_2$





Convex Hull

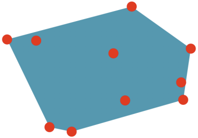
$f : \mathbb{R}^n \to R$ is convex if $\text{dom}(f)$ is convex and

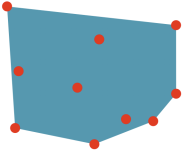
$$f\big(t\boldsymbol{x} + (1-t)\boldsymbol{y}\big) \leq tf\big(\boldsymbol{x}\big) + (1-t)f\big(\boldsymbol{y}\big)$$

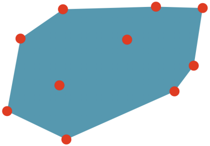
for all $\boldsymbol{x}, \boldsymbol{y} \in \text{dom}(f)$.
Let $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k \in \mathbb{R}^n$, then a convex combination is any linear combination $\omega_1\boldsymbol{x}_1 + \cdots + \omega_k\boldsymbol{x}_k$ with $(\omega_1, \cdots, \omega_k) \in \mathcal{S}_k$, i.e.

$$\omega_1, \cdots, \omega_k \in \mathbb{R}_+ \text{ such that } \sum_{j=1}^{k} \omega_j = 1$$

The convex hull of a set C is the set of all convex combinations of elements of C







Separating Theorem

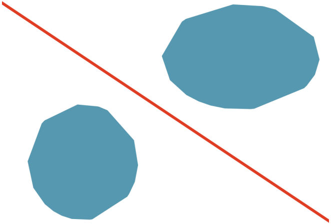
Theorem Consider two disjoint convex sets have a separating hyperplane between them

i.e. if C and D are convex, disjoint, then there are $\boldsymbol{a}$ and b such that

$$C \subset \{\boldsymbol{x} : \boldsymbol{x}^\top \boldsymbol{a} \leq b\}$$

and

$$D \subset \{\boldsymbol{x} : \boldsymbol{x}^\top \boldsymbol{a} \geq b\}$$



Separating Theorem

Application : section 4.3.1 in Financial Asset Pricing Theory

See also Farkas lemma:
if $\boldsymbol{A}$ is a $m \times n$ real matrix, and $\boldsymbol{b} \in \mathbb{R}^m$, then exactly one of the two is true

- $\boldsymbol{Ax} = \boldsymbol{b}$ for some $\boldsymbol{x} \in \mathbb{R}^n_+$
- $\boldsymbol{y}^\top \boldsymbol{A} \in \mathbb{R}^n_+$ and $\boldsymbol{y}^\top \boldsymbol{b} \in \mathbb{R}_-$ for some $\boldsymbol{y} \in \mathbb{R}^m$

Theorem 4.5 *A state-price deflator exists if and only if prices admit no arbitrage.*

Now assume that prices do not admit arbitrage. For simplicity, take a one-period framework with a finite state space $\Omega = \{1, 2, \ldots, S\}$ and assume that none of the I basic assets are redundant (the proof can easily be extended to the case with redundant assets). Define the sets A and B as

$$A = \{\kappa \boldsymbol{P} \,|\, \kappa > 0\}, \quad B = \{\underline{\underline{D}}\boldsymbol{\psi} \,|\, \boldsymbol{\psi} \in \mathbb{R}^S_{++}\},$$

where R^S_{++} is the set of S-dimensional real vectors with strictly positive components. As before, $\boldsymbol{P}$ is the I-dimensional vector of prices of the basic assets, and $\underline{\underline{D}}$ is the $I \times S$ matrix of dividends. Both A and B are convex subsets[1] of $\mathbb{R}^I$. A is convex since $\alpha\kappa_1\boldsymbol{P} + (1-\alpha)\kappa_2\boldsymbol{P} = (\alpha\kappa_1 + (1-\alpha)\kappa_2)\,\boldsymbol{P} \in A$ for any $\alpha \in [0,1]$ and any $\kappa_1, \kappa_2 > 0$. Likewise, B is convex because, given $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in \mathbb{R}^S_{++}$ and $\alpha \in [0,1]$, then $\alpha\underline{\underline{D}}\boldsymbol{\psi}_1 + (1-\alpha)\underline{\underline{D}}\boldsymbol{\psi}_2 = \underline{\underline{D}}\left(\alpha\boldsymbol{\psi}_1 + (1-\alpha)\boldsymbol{\psi}_2\right)$ which belongs to B.

Suppose now that no state-price vector exists. Then we would have $A \cap B = \emptyset$. By the Separating Hyperplane Theorem,[2] a non-zero vector $\boldsymbol{\theta} \in \mathbb{R}^I$ exists so that

$$\kappa\boldsymbol{\theta}^\top\boldsymbol{P} \leq \boldsymbol{\theta}^\top\underline{\underline{D}}\boldsymbol{\psi} = (\underline{\underline{D}}^\top\boldsymbol{\theta})^\top\boldsymbol{\psi}, \quad \text{for all } \kappa > 0, \boldsymbol{\psi} \in \mathbb{R}^S_{++}.$$

This implies that $\boldsymbol{\theta}^\top\boldsymbol{P} \leq 0$ and $\underline{\underline{D}}^\top\boldsymbol{\theta} \geq 0$. Since we have assumed that none of the basic assets are redundant, the dividend matrix $\underline{\underline{D}}$ has full rank so there is no $\boldsymbol{\theta}$ for which $\underline{\underline{D}}^\top\boldsymbol{\theta} = 0$. Hence, $\boldsymbol{\theta}$ must satisfy $\boldsymbol{\theta}^\top\boldsymbol{P} \leq 0$ and $\underline{\underline{D}}^\top\boldsymbol{\theta} > 0$, and therefore $\boldsymbol{\theta}$ is an arbitrage. This contradicts our assumption. Hence, a state-price vector must exist. □

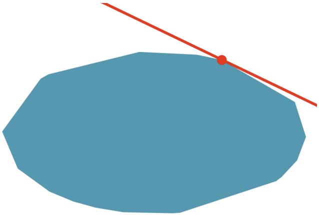
[1] A set $X \subseteq \mathbb{R}^n$ is convex if, for all $x_1, x_2 \in X$ and all $\alpha \in [0,1]$, we have $\alpha x_1 + (1-\alpha)x_2 \in X$.

[2] The version of the Separating Hyperplane Theorem applied here is the following: suppose that A and B are non-empty, disjoint, convex subsets of $\mathbb{R}^n$ for some integer $n \geq 1$. Then there is a non-zero vector $\boldsymbol{\theta} \in \mathbb{R}^n$ and a scalar ξ so that $\boldsymbol{\theta}^\top\boldsymbol{a} \leq \xi \leq \boldsymbol{\theta}^\top\boldsymbol{b}$ for all $\boldsymbol{a} \in A$ and all $\boldsymbol{b} \in B$. Hence, A and B are separated by the hyperplane $H = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{a}^\top\boldsymbol{x} = \xi\}$. See, for example, Sydsaeter *et al.* (2005, Them. 13.6.3).

Supporting Theorem

Theorem Any boundary point of a convex set has a supporting hyperplane passing through it
i.e. C is convex, disjoint, $\boldsymbol{x}_0 \in \partial C$ then there is $\boldsymbol{a}$ such that

$$C \subset \{\boldsymbol{x} : \boldsymbol{x}^\top \boldsymbol{a} \leq \boldsymbol{x}_0^\top \boldsymbol{a}\}$$



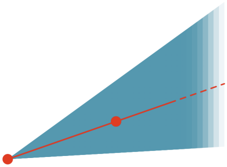
Cones

$C \subset \mathbb{R}^n$ is a cone if

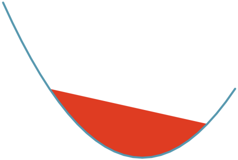
$$\boldsymbol{x} \in C \Longrightarrow t\boldsymbol{x} \in C$$

for all $t \geq 0$

Exemple: quadrant $(\mathbb{R}_+)^n$ is a cone
Example: $\{\boldsymbol{x} \in \mathbb{R}^n : x_1 \leqslant \cdots \leqslant x_n\}$



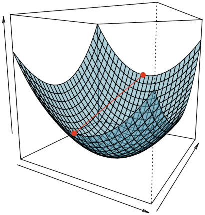
Convex Functions



$f : \mathbb{R}^n \rightarrow \; R$ is convex if $\text{dom}(f)$ is convex and

$$f\big(t\boldsymbol{x} + (1-t)\boldsymbol{y}\big) \leq tf\big(\boldsymbol{x}\big) + (1-t)f\big(\boldsymbol{y}\big)$$

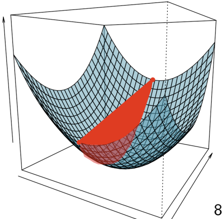
for all $\boldsymbol{x}, \boldsymbol{y} \in \text{dom}(f)$.
$f : \mathbb{R}^n \rightarrow \; R$ is concave if $\text{dom}(f)$ is convex and



$$f\big(t\boldsymbol{x} + (1-t)\boldsymbol{y}\big) \geq tf\big(\boldsymbol{x}\big) + (1-t)f\big(\boldsymbol{y}\big)$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \text{dom}(f)$.

- $x \mapsto e^{ax}$ is convex on $\mathbb{R}$ (for any a)
- $x \mapsto x^a$ is convex on $\mathbb{R}$, for $a \geq 1$
- $\boldsymbol{x} \mapsto \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x}$ is convex if $\boldsymbol{Q}$ is positive semidefinite



Convex Functions

- $\boldsymbol{x} \mapsto \|\boldsymbol{x}\|$ is convex for any norm
- $\boldsymbol{x} \mapsto \min_{\boldsymbol{y} \in S} \|\boldsymbol{x} - \boldsymbol{y}\|$ (minimum distance to some set S) is convex for any norm (and so is the maximum)

$f : \mathbb{R}^n \to R$ is strictly-convex if $\text{dom}(f)$ is convex and for all $\boldsymbol{x}, \boldsymbol{y} \in \text{dom}(f)$ and $t \in (0,1)$,

$$f\big(t\boldsymbol{x} + (1-t)\boldsymbol{y}\big) < tf\big(\boldsymbol{x}\big) + (1-t)f\big(\boldsymbol{y}\big)$$

First & Second Order Characterization

Proposition If $f : \mathbb{R}^n \to\ R$ is differentiable, then f is convex if dom(f) is convex and

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{y} - \boldsymbol{x})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in$ dom(f).

Proposition If $f : \mathbb{R}^n \to\ R$ is twice-differentiable, them f is convex if dom(f) is convex and

$$\nabla^2 f(\boldsymbol{x}) \text{ is positive semi-definite}$$

for all $\boldsymbol{x} \in$ dom(f).

Taylor's Expansion

The so called fundamental theorem of calculus states that

$$f(x+h) = f(x) + \int_0^h f'(x+a)da$$

by substituting

$$f(x+h) = f(x) + \int_0^h \left(f'(x) + \int_0^a f''(x+b)db \right) da$$

$$f(x+h) = f(x) + f'(x)h + \int_0^h \int_0^a f''(x+b)dbda$$

by substituting

$$f(x+h) = f(x)+f'(x)h+\int_0^h \int_0^a \left(f''(x) + \int_0^b f''(x+c)dc \right) dbda$$

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2}+\int_0^h \int_0^a \int_0^b f'''(x+c)dcdbda$$

Taylor's Expansion

For $f : \mathbb{R} \to \mathbb{R}$ functions,

$$f(x+h) \sim f(x) + f'(x)h + f''(x)\frac{h^2}{2}$$

and for $f : \mathbb{R}^n \to \mathbb{R}$ functions,

$$f(\boldsymbol{x}+\boldsymbol{h}) \sim f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top \boldsymbol{h} + \frac{1}{2}\boldsymbol{h}^\top \nabla^2 f(\boldsymbol{x})\boldsymbol{h}$$

where $\nabla^2 f(\boldsymbol{x})$ is the Hessian matrix H.
If f is convex, H is positive and

$$f(\boldsymbol{x}+\boldsymbol{h}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top \boldsymbol{h}$$