

# Data Science for Actuaries (ACT6100)

Arthur Charpentier

Rappels # 3.1 (Matrices & Eigen-Values/Vectors)

automne 2020

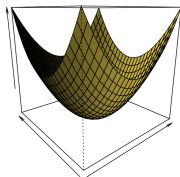
 <https://github.com/freakonometrics/ACT6100/>

# Positive Matrices

- ▶ A real symmetric  $n \times n$   $\mathbf{M}$  is positive semidefinite (denoted  $\mathbf{M} \geq 0$ ) if  $\mathbf{z}^\top \mathbf{M} \mathbf{z} \geq 0$  for all  $\mathbf{z} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$
- ▶  $\mathbf{M}$  is positive definite (denoted  $\mathbf{M} > 0$ ) if  $\mathbf{z}^\top \mathbf{M} \mathbf{z} > 0$  for all  $\mathbf{z} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$

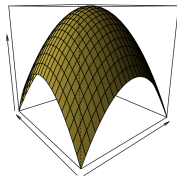
**Note:**  $\mathbf{M}$  is positive definite if all its eigenvalues  $\lambda_i$  are  $> 0$ .

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



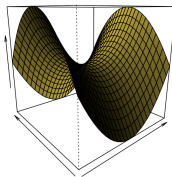
positive  
definite

$$\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$



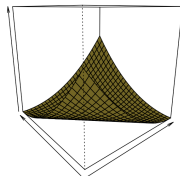
negative  
definite

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



indefinite

$$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$



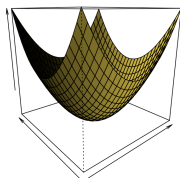
positive  
semi-definite

# Quadratic Forms

On peut tracer la surface  $S(\mathbf{z}) = \mathbf{z}^\top \mathbf{M} \mathbf{z}$ , i.e.

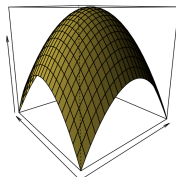
$$S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto (x \ y) \mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



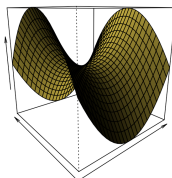
positive  
definite

$$\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$



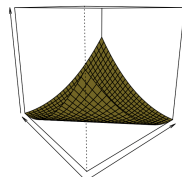
negative  
definite

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



indefinite

$$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$



positive  
semi-definite

# Eigenvalues & Eigenvectors for Squared Matrices

Let  $\mathbf{M}$  denote some real  $n \times n$  matrix.  $\lambda$  is an eigenvalue, associated to eigenvector  $\vec{\mathbf{u}}$  if one of the following holds

- ▶  $\mathbf{M}\vec{\mathbf{u}} = \lambda \vec{\mathbf{u}}$
- ▶  $(\mathbf{M} - \lambda\mathbb{I})$  cannot be inverted, or  $\det(\mathbf{M} - \lambda\mathbb{I}) = 0$

**Example** Let  $\mathbf{M}$  be a real symmetric  $n \times n$  matrix. Then  $\mathbf{M}$  has  $n$  real eigenvalues (not necessarily distinct).

Furthermore, there is a set of  $n$  corresponding eigenvectors  $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \dots, \vec{\mathbf{u}}_n\}$ , that constitute an orthonormal basis of  $\mathbb{R}^n$ , that is  $\langle \vec{\mathbf{u}}_i, \vec{\mathbf{u}}_j \rangle = \delta_{ij}$ .

**Example** Let  $\mathbf{M}$  be a real symmetric  $n \times n$  with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ , then

$$\lambda_1 = \max_{\mathbf{z}: \|\mathbf{z}\|=1} \{\mathbf{z}^\top \mathbf{M} \mathbf{z}\} \text{ and } \lambda_n = \min_{\mathbf{z}: \|\mathbf{z}\|=1} \{\mathbf{z}^\top \mathbf{M} \mathbf{z}\}$$

and the optimum is obtained when  $\mathbf{z} \propto \vec{\mathbf{u}}_1$  and  $\mathbf{z} \propto \vec{\mathbf{u}}_n$

## Example

```
1 > M=c(1,2,3,4)
2 > dim(M)=c(2,2)
3 > eigen(M)
4 $values
5 [1] 5.3722813 -0.3722813
6
7 $vectors
8           [,1]      [,2]
9 [1,] -0.5657675 -0.9093767
10 [2,] -0.8245648  0.4159736
11 > L=eigen(M)$values
12 > P=eigen(M)$vector
13 > M %*% P[,1]
14           [,1]
15 [1,] -3.039462
16 [2,] -4.429794
17 > M %*% P[,2]
18           [,1]
19 [1,] 0.338544
20 [2,] -0.154859
```

```
1 > L[1] * P[,1]
2 [1] -3.039462 -4.429794
3 > L[2] * P[,2]
4 [1] 0.338544 -0.154859
5 > t(P) %*% P
6           [,1] [,2]
7 [1,] 1.0 -0.4
8 [2,] -0.4 1.0
```

hence

$$M\vec{u}_1 = \lambda_1 \vec{u}_1$$

$$M\vec{u}_2 = \lambda_2 \vec{u}_2$$

with  $\|\vec{u}_1\| = \|\vec{u}_2\| = 1$ ,  $\vec{u}_1 \not\propto \vec{u}_2$

# Example

```
1 > u=c(cos(pi/6),sin(pi/6))
2 > X = matrix(u,2,1)
3 > M = X %%% solve(t(X)%%X)
      %%% t(X)
4 > eigen(M)
5 $values
6 [1] 1 0
7 $vectors
8           [,1]      [,2]
9 [1,] -0.8660254  0.5000000
10 [2,] -0.5000000 -0.8660254
```

```
1 > M=c(1,3,3,1)
2 > dim(M)=c(2,2)
3 > eigen(M)
4 $values
5 [1] 4 -2
6
7 $vectors
8           [,1]      [,2]
9 [1,] 0.7071068 -0.7071068
10 [2,] 0.7071068  0.7071068
11 > P = eigen(M)$vectors
12 > t(P) %%% P
13           [,1] [,2]
14 [1,] 1 0
15 [2,] 0 1
```

# Spectral Decomposition

As a special case, for every  $n \times n$  real symmetric matrix, the eigenvalues are real and the eigenvectors can be chosen such that they are orthogonal to each other. Thus a real symmetric matrix  $\mathbf{M}$  can be decomposed as

$$\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

where  $\mathbf{P}$  is an orthonormal matrix whose columns are the eigenvectors of  $\mathbf{M}$ , and  $\mathbf{D}$  is a diagonal matrix whose entries are the eigenvalues of  $\mathbf{M}$ .

Let  $\mathbf{P} = [\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n]$ , then  $\langle \vec{u}_i, \vec{u}_j \rangle = \delta_{i,j}$ .

$$\mathbf{M} = \sum_{i=1}^n \lambda_i \mathbf{u}_i \mathbf{u}_i^{\top}$$

# Example

```
1 > M=c(1,2,3,4)
2 > dim(M)=c(2,2)
3 > eigen(M)
4 $values
5 [1] 5.3722813 -0.3722813
6
7 $vectors
8           [,1]      [,2]
9 [1,] -0.5657675 -0.9093767
10 [2,] -0.8245648  0.4159736
11
12 > P = eigen(M)$vectors
13 > D = diag(eigen(M)$values)
14 > P %*% D %*% solve(P)
15           [,1] [,2]
16 [1,]      1    3
17 [2,]      2    4
```

```
1 > M=c(1,3,3,1)
2 > dim(M)=c(2,2)
3 > eigen(M)
4 $values
5 [1] 4 -2
6
7 $vectors
8           [,1]      [,2]
9 [1,] 0.7071068 -0.7071068
10 [2,] 0.7071068  0.7071068
11
12 > P = eigen(M)$vectors
13 > D = diag(eigen(M)$values)
14 > P %*% D %*% solve(P)
15           [,1] [,2]
16 [1,]      1    3
17 [2,]      3    1
```



# Approximation

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 5 \\ 5 & 3 & 7 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.19 & -0.32 & 0.74 \\ 0.75 & -0.59 & 0.16 \\ 0.64 & 0.74 & -0.65 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} 12.04 & 0 & 0 \\ 0 & 2.41 & 0 \\ 0 & 0 & 0.55 \end{bmatrix}}_{\mathbf{D}} \mathbf{P}^{-1}$$

$$\underbrace{\begin{bmatrix} 0.19 & -0.32 & 0.74 \\ 0.75 & -0.59 & 0.16 \\ 0.64 & 0.74 & -0.65 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} 12.04 & 0 & 0 \\ 0 & 2.41 & 0 \\ 0 & 0 & 0.00 \end{bmatrix}}_{\mathbf{D}'} \mathbf{P}^{-1} = \begin{bmatrix} 0.31 & 2.26 & 0.91 \\ 2.85 & 7.05 & 4.98 \\ 5.60 & 2.78 & 7.08 \end{bmatrix} = \mathbf{M}'$$