Data Science for Actuaries (ACT6100)

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Rappels # 4.4 (Convex Optimization & Portfolio Optimization)

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https://github.com/freakonometrics/ACT6100/

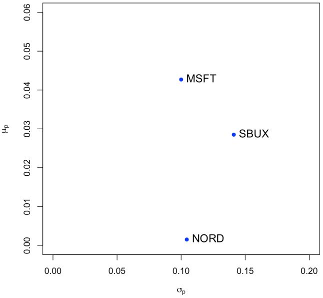
Portfolio Optimization – Markowitz (1952)

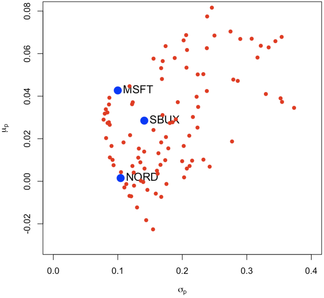
Consider n assets, with expected return $\boldsymbol{\mu} = (\mu_1 \cdots, \mu_n)$ and covariance matrix $\boldsymbol{\Sigma}$

```
> asset.names = c("MSFT", "NORD", "
    SBUX")
> mu.vec = c(0.0427, 0.0015,
    0.0285)
> names(mu.vec) = asset.names
> sigma.mat = matrix(c(0.0100,
    0.0018, 0.0113, 0.0018, 0.0109,
     0.0026, 0.0113, 0.0026,
    0.0199), nrow=3, ncol=3)
> dimnames(sigma.mat) = list(asset.
    names, asset.names)
```

A portfolio is $\boldsymbol{\omega} \in \mathbb{R}^n$ with $\boldsymbol{\omega}^\top \mathbf{1} = 1$,

$$\mathbb{E}(P) = \boldsymbol{\omega}^\top \boldsymbol{\mu}, \ \mathrm{Var}(P) = \boldsymbol{\omega}^\top \boldsymbol{\Sigma} \boldsymbol{\omega}$$





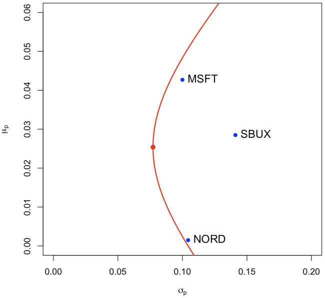
Portfolio Optimization – Markowitz (1952)

Efficient portfolio: $\omega_r^\star$,

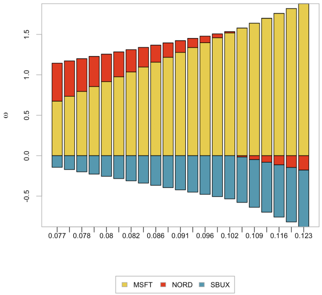
$$\omega_r^\star = \text{argmin}\{\omega^\top \boldsymbol{\Sigma}\omega\} \text{ s.t. } \begin{cases} \omega^\top \mathbf{1} = 1 \\ \omega^\top \boldsymbol{\mu} = r \end{cases}$$

or minimum variance portfolio

$$\omega^\star = \text{argmin}\{\omega^\top \boldsymbol{\Sigma}\omega\} \text{ s.t. } \omega^\top \mathbf{1} = 1$$




```
> one.vec = rep(1, 3)
> sigma.inv.mat = solve(sigma.mat)
> top.mat = sigma.inv.mat%*%one.vec
> bot.val = as.numeric((t(one.vec)
    %*%sigma.inv.mat%*%one.vec))
> m.mat = top.mat/bot.val
> m.mat[, 1]
[1]  0.674  0.470 -0.144
```



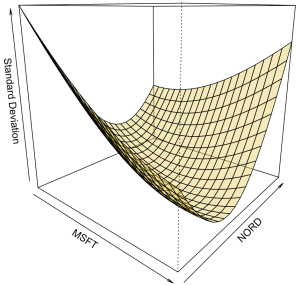
Possible only if short sells are possible,

Minimum Variance Portfolio – Markowitz (1952)

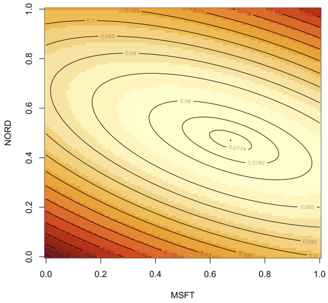
minimum variance portfolio

$$\boldsymbol{\omega}^\star = \text{argmin}\{\boldsymbol{\omega}^\top \boldsymbol{\Sigma} \boldsymbol{\omega}\} \text{ s.t. } \boldsymbol{\omega}^\top \mathbf{1} = 1$$

$$\begin{pmatrix} \omega_M \\ \omega_N \\ 1-\omega_M-\omega_N \end{pmatrix}^\top \boldsymbol{\Sigma} \begin{pmatrix} \omega_M \\ \omega_N \\ 1-\omega_M-\omega_N \end{pmatrix}$$




```
> var_poids = function(z){
+    z.vec = cbind(z[1],z[2],1-z[1]-z[2])
+    sqrt((z.vec)%*%sigma.mat%*%t(z.vec))
+ }
> optim(par=c(1,1),var_poids)
$par
[1] 0.675 0.470
```



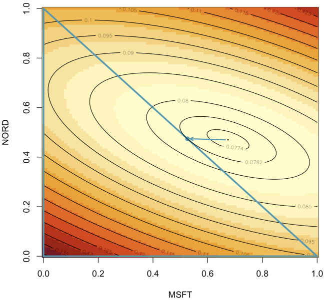
i.e. $\boldsymbol{\omega}^\star = (0.675, 0.470, -0.145)$

Minimum Variance Portfolio – Markowitz (1952)

with no-short sales,

$$\omega^\star_{NSS} = \text{argmin}\{\omega^\top \boldsymbol{\Sigma}\omega\} \text{ s.t. } \begin{cases} \omega^\top \mathbf{1} = 1 \\ \omega \geq \mathbf{0} \\ \omega \leq \mathbf{1} \end{cases}$$

```
> grd_var_poids= function(z){
+   h=1e-5
+   c((var_poids(c(z[1]+h,z[2]))-
    var_poids(c(z[1]-h,z[2])))/(2*h
    ), (var_poids(c(z[1],z[2]+h))-
    var_poids(c(z[1],z[2]-h)))/(2*h
    )) }
```

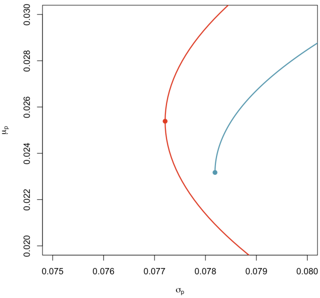



```
> constrOptim(c(.2,.2),var_poids,grd_var_poids, ui=
    matrix(c(1,0,0,1,-1,0,0,-1,-1,-1),5,2,byrow = TRUE
    ), ci=c(0,0,-1,-1,-1))
$par
[1] 0.526 0.474
```

Portfolio Optimization – Markowitz (1952)

Efficient portfolio: $\omega^\star_{r:NSS}$,

$$\omega^\star_{r:NSS} = \text{argmin}\{\omega^\top \boldsymbol{\Sigma}\omega\} \text{ s.t. } \begin{cases} \omega^\top \mathbf{1} = 1 \\ \omega^\top \boldsymbol{\mu} = r \\ \omega \geq \mathbf{0} \\ \omega \leq \mathbf{1} \end{cases}$$



Duality

Primal problem,

$$\min_{(x,y)\in\mathbb{R}^2} \{f(x,y)\}$$

subject to

$$g(x,y) = 0$$

Dual problem,

$$\max_{(x,y)\in\mathbb{R}^2} \{g(x,y)\}$$

subject to

$$f(x,y) = f^\star$$