

Data Science for Actuaries (ACT6100)

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Rappels # 4.1 (Convex Optimization)

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 <https://github.com/freakonometrics/ACT6100/>

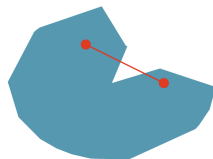
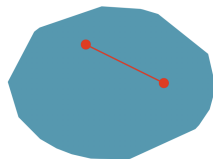
Convex Sets

$C \subset \mathbb{R}^n$ is convex if

$$\mathbf{x}, \mathbf{y} \in C \implies t\mathbf{x} + (1-t)\mathbf{y} \in C$$

for all $t \in [0, 1]$

- ▶ unit ball $\{\mathbf{x} : \|\mathbf{x}\| \leq 1\}$ is convex,
- ▶ hyperplane $\{\mathbf{x} : \mathbf{x}^\top \boldsymbol{\beta} = y_0\}$ is convex,
- ▶ half-space $\{\mathbf{x} : \mathbf{x}^\top \boldsymbol{\beta} \leq y_0\}$ is convex,
- ▶ polyhedron $\{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{a}\}$ is convex,
- ▶ simplex $\{\mathbf{x} \in \mathbb{R}_+ : \mathbf{1}^\top \mathbf{x} = 1\}$ is convex,
- ▶ if C_1 and C_2 are convex, so is $C_1 \cap C_2$



Convex Hull

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\text{dom}(f)$ is convex and

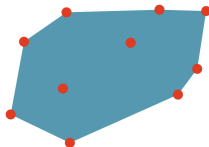
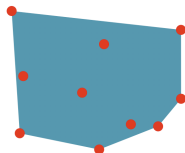
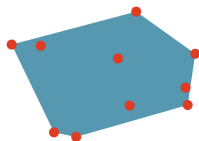
$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \text{dom}(f)$.

Let $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$, then a convex combination is any linear combination $\omega_1\mathbf{x}_1 + \dots + \omega_k\mathbf{x}_k$ with $(\omega_1, \dots, \omega_k) \in \mathcal{S}_k$, i.e.

$$\omega_1, \dots, \omega_k \in \mathbb{R}_+ \text{ such that } \sum_{j=1}^k \omega_j = 1$$

The convex hull of a set C is the set of all convex combinations of elements of C



Separating Theorem

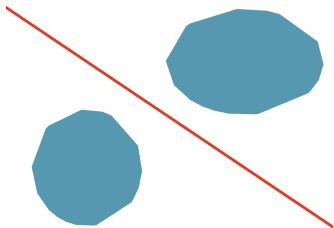
Theorem Consider two disjoint convex sets have a separating hyperplane between them

i.e. if C and D are convex, disjoint, then there are \mathbf{a} and b such that

$$C \subset \{\mathbf{x} : \mathbf{x}^T \mathbf{a} \leq b\}$$

and

$$D \subset \{\mathbf{x} : \mathbf{x}^T \mathbf{a} \geq b\}$$



Separating Theorem

Application : section 4.3.1 in Financial Asset Pricing Theory

See also Farkas lemma:

if \mathbf{A} is a $m \times n$ real matrix, and $\mathbf{b} \in \mathbb{R}^m$, then exactly one of the two is true

- ▶ $\mathbf{Ax} = \mathbf{b}$ for some $\mathbf{x} \in \mathbb{R}_+^n$
- ▶ $\mathbf{y}^\top \mathbf{A} \in \mathbb{R}_+^n$ and $\mathbf{y}^\top \mathbf{b} \in \mathbb{R}_-$ for some $\mathbf{y} \in \mathbb{R}^m$

Theorem 4.5 A state-price deflator exists if and only if prices admit no arbitrage.

Now assume that prices do not admit arbitrage. For simplicity, take a one-period framework with a finite state space $\Omega = \{1, 2, \dots, S\}$ and assume that none of the I basic assets are redundant (the proof can easily be extended to the case with redundant assets). Define the sets A and B as

$$A = \{\kappa \mathbf{P} \mid \kappa > 0\}, \quad B = \{\underline{D}\psi \mid \psi \in \mathbb{R}_{++}^S\},$$

where \mathbb{R}_{++}^S is the set of S -dimensional real vectors with strictly positive components. As before, \mathbf{P} is the I -dimensional vector of prices of the basic assets, and \underline{D} is the $I \times S$ matrix of dividends. Both A and B are convex subsets¹ of \mathbb{R}^I . A is convex since $\alpha\kappa_1\mathbf{P} + (1-\alpha)\kappa_2\mathbf{P} = (\alpha\kappa_1 + (1-\alpha)\kappa_2)\mathbf{P} \in A$ for any $\alpha \in [0, 1]$ and any $\kappa_1, \kappa_2 > 0$. Likewise, B is convex because, given $\psi_1, \psi_2 \in \mathbb{R}_{++}^S$ and $\alpha \in [0, 1]$, then $\alpha\underline{D}\psi_1 + (1-\alpha)\underline{D}\psi_2 = \underline{D}(\alpha\psi_1 + (1-\alpha)\psi_2)$ which belongs to B .

Suppose now that no state-price vector exists. Then we would have $A \cap B = \emptyset$. By the Separating Hyperplane Theorem,² a non-zero vector $\theta \in \mathbb{R}^I$ exists so that

$$\kappa \theta^\top \mathbf{P} \leq \theta^\top \underline{D}\psi \quad \text{for all } \kappa > 0, \psi \in \mathbb{R}_{++}^S.$$

This implies that $\theta^\top \mathbf{P} \leq 0$ and $\theta^\top \theta \geq 0$. Since we have assumed that none of the basic assets are redundant, the dividend matrix \underline{D} has full rank so there is no θ for which $\underline{D}^\top \theta = 0$. Hence, θ must satisfy $\theta^\top \mathbf{P} \leq 0$ and $\theta^\top \theta > 0$, and therefore θ is an arbitrage. This contradicts our assumption. Hence, a state-price vector must exist. \square

¹ A set $X \subseteq \mathbb{R}^n$ is convex if, for all $x_1, x_2 \in X$ and all $\alpha \in [0, 1]$, we have $\alpha x_1 + (1-\alpha)x_2 \in X$.

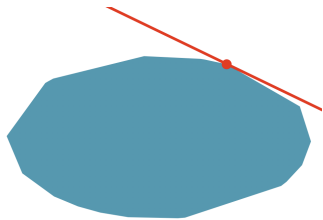
² The version of the Separating Hyperplane Theorem applied here is the following: suppose that A and B are non-empty, disjoint, convex subsets of \mathbb{R}^n for some integer $n \geq 1$. Then there is a non-zero vector $\theta \in \mathbb{R}^n$ and a scalar ξ so that $\theta^\top a \leq \xi \leq \theta^\top b$ for all $a \in A$ and all $b \in B$. Hence, A and B are separated by the hyperplane $H = \{x \in \mathbb{R}^n \mid a^\top x = \xi\}$. See, for example, Sydsæter *et al.* (2005, Thm. 13.6.3).

Supporting Theorem

Theorem Any boundary point of a convex set has a supporting hyperplane passing through it

i.e. C is convex, disjoint, $\mathbf{x}_0 \in \partial C$ then there is \mathbf{a} such that

$$C \subset \{\mathbf{x} : \mathbf{x}^\top \mathbf{a} \leq \mathbf{x}_0^\top \mathbf{a}\}$$



Cones

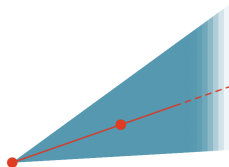
$C \subset \mathbb{R}^n$ is a cone if

$$\mathbf{x} \in C \implies t\mathbf{x} \in C$$

for all $t \geq 0$

Example: quadrant $(\mathbb{R}_+)^n$ is a cone

Example: $\{\mathbf{x} \in \mathbb{R}^n : x_1 \leq \dots \leq x_n\}$



Convex Functions

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\text{dom}(f)$ is convex and

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y})$$

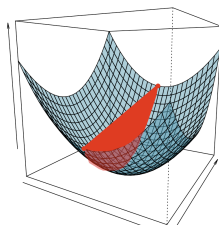
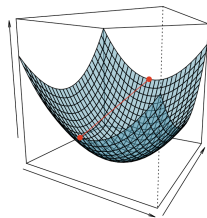
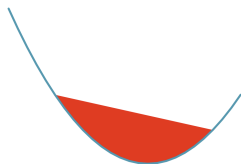
for all $\mathbf{x}, \mathbf{y} \in \text{dom}(f)$.

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave if $\text{dom}(f)$ is convex and

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq tf(\mathbf{x}) + (1-t)f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \text{dom}(f)$.

- ▶ $x \mapsto e^{ax}$ is convex on \mathbb{R} (for any a)
- ▶ $x \mapsto x^a$ is convex on \mathbb{R} , for $a \geq 1$
- ▶ $\mathbf{x} \mapsto \mathbf{x}^\top \mathbf{Q} \mathbf{x}$ is convex if \mathbf{Q} is positive semidefinite



Convex Functions

- ▶ $\mathbf{x} \mapsto \|\mathbf{x}\|$ is convex for any norm
- ▶ $\mathbf{x} \mapsto \min_{\mathbf{y} \in S} \|\mathbf{x} - \mathbf{y}\|$ (minimum distance to some set S) is convex for any norm (and so is the maximum)

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly-convex if $\text{dom}(f)$ is convex and for all $\mathbf{x}, \mathbf{y} \in \text{dom}(f)$ and $t \in (0, 1)$,

$$f(t\mathbf{x} + (1 - t)\mathbf{y}) < tf(\mathbf{x}) + (1 - t)f(\mathbf{y})$$

First & Second Order Characterization

Proposition If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable, then f is convex if $\text{dom}(f)$ is convex and

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x})$$

for all $\mathbf{x}, \mathbf{y} \in \text{dom}(f)$.

Proposition If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice-differentiable, then f is convex if $\text{dom}(f)$ is convex and

$$\nabla^2 f(\mathbf{x}) \text{ is positive semi-definite}$$

for all $\mathbf{x} \in \text{dom}(f)$.

Taylor's Expansion

The so called **fundamental theorem of calculus** states that

$$f(x+h) = f(x) + \int_0^h f'(x+a)da$$

by substituting

$$f(x+h) = f(x) + \int_0^h \left(f'(x) + \int_0^a f''(x+b)db \right) da$$

$$f(x+h) = f(x) + f'(x)h + \int_0^h \int_0^a f''(x+b)dbda$$

by substituting

$$f(x+h) = f(x) + f'(x)h + \int_0^h \int_0^a \left(f''(x) + \int_0^b f'''(x+c)dc \right) dbda$$

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2} + \int_0^h \int_0^a \int_0^b f'''(x+c)dcdbda$$

Taylor's Expansion

For $f : \mathbb{R} \rightarrow \mathbb{R}$ functions,

$$f(x + h) \sim f(x) + f'(x)h + f''(x)\frac{h^2}{2}$$

and for $f : \mathbb{R}^n \rightarrow \mathbb{R}$ functions,

$$f(\mathbf{x} + \mathbf{h}) \sim f(\mathbf{x}) + \nabla f(\mathbf{x})^\top \mathbf{h} + \frac{1}{2} \mathbf{h}^\top \nabla^2 f(\mathbf{x}) \mathbf{h}$$

where $\nabla^2 f(\mathbf{x})$ is the Hessian matrix H .

If f is convex, H is positive and

$$f(\mathbf{x} + \mathbf{h}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top \mathbf{h}$$