Data Science for Actuaries (ACT6100)

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Rappels # 3.1 (Matrices & Eigen-Values/Vectors)

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https://github.com/freakonometrics/ACT6100/

Spectral Decomposition

A real symmetric $n \times n$ $\boldsymbol{M}$ is positive semidefinite - denoted $\boldsymbol{M} \geq 0$ - if $\boldsymbol{z}^\top \boldsymbol{M}\boldsymbol{z} \geq 0$ for all $\boldsymbol{z} \in \mathbb{R}^d$
$\boldsymbol{M}$ is positive definite - denoted $\boldsymbol{M} > 0$ - if $\boldsymbol{z}^\top \boldsymbol{M}\boldsymbol{z} > 0$ for all $\boldsymbol{z} \in \mathbb{R}^d$

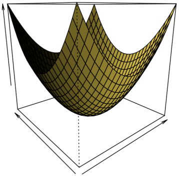
Note: $\boldsymbol{M}$ is positive definite if all its eigenvalues λ_i are > 0.

Quadratic Forms

On peut tracer la surface $S(\boldsymbol{z}) = \boldsymbol{z}^\top \boldsymbol{M}\boldsymbol{z}$, i.e.

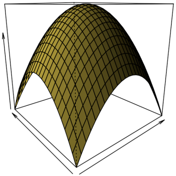
$$S : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x & y \end{pmatrix} \boldsymbol{M} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\boldsymbol{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



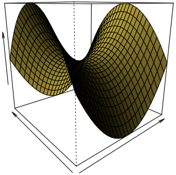
positive definite

$\boldsymbol{M} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$



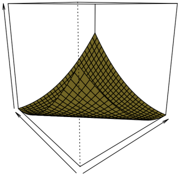
negative definite

$\boldsymbol{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



indefinite

$\boldsymbol{M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$



positive semi-definite

Eigenvalues & Eigenvectors for Squared Matrices

Let $\boldsymbol{M}$ denote some real $n \times n$ matrix. λ is an eigenvalue, associated to eigenvector $\vec{\boldsymbol{u}}$ if one of the following holds

- $\boldsymbol{M}\vec{\boldsymbol{u}} = \lambda \ \vec{\boldsymbol{u}}$
- $(\boldsymbol{M} - \lambda\mathbb{I})$ cannot be inverted, or $\det(\boldsymbol{M} - \lambda\mathbb{I}) = 0$

Example Let $\boldsymbol{M}$ be a real symmetric $n \times n$ matrix. Then $\boldsymbol{M}$ has n real eigenvalues (not necessarily distinct).
Furthermore, there is a set of n corresponding eigenvectors $\{\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2, \cdots, \vec{\boldsymbol{u}}_n\}$, that constitute an orthonormal basis of $\mathbb{R}^n$, that is $\langle \vec{\boldsymbol{u}}_i, \vec{\boldsymbol{u}}_j \rangle = \delta_{i,j}$.
Example Let $\boldsymbol{M}$ be a real symmetric $n \times n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then

$$\lambda_1 = \max_{\boldsymbol{z}:\|\boldsymbol{z}\|=1} \left\{\boldsymbol{z}^\top \boldsymbol{M}\boldsymbol{z}\right\} \text{ and } \lambda_n = \min_{\boldsymbol{z}:\|\boldsymbol{z}\|=1} \left\{\boldsymbol{z}^\top \boldsymbol{M}\boldsymbol{z}\right\}$$

and the optimum is obtained when $\boldsymbol{z} \propto \vec{\boldsymbol{u}}_1$ and $\boldsymbol{z} \propto \vec{\boldsymbol{u}}_n$

Example

```
> M=c(1,2,3,4)
> dim(M)=c(2,2)
> eigen(M)
$values
[1]  5.3722813 -0.3722813

$vectors
           [,1]       [,2]
[1,] -0.5657675 -0.9093767
[2,] -0.8245648  0.4159736
> L=eigen(M)$values
> P=eigen(M)$vector
> M %*% P[,1]
          [,1]
[1,] -3.039462
[2,] -4.429794
> M %*% P[,2]
          [,1]
[1,]  0.338544
[2,] -0.154859
```

```
> L[1] * P[,1]
[1] -3.039462 -4.429794
> L[2] * P[,2]
[1]  0.338544 -0.154859
> t(P) %*% P
     [,1] [,2]
[1,]  1.0 -0.4
[2,] -0.4  1.0
```

hence

$$\boldsymbol{M}\vec{\boldsymbol{u}}_1 = \lambda_1 \ \vec{\boldsymbol{u}}_1$$

$$\boldsymbol{M}\vec{\boldsymbol{u}}_2 = \lambda_1 \ \vec{\boldsymbol{u}}_2$$

with $\|\vec{\boldsymbol{u}}_1\| = \|\vec{\boldsymbol{u}}_2\| = 1$, $\vec{\boldsymbol{u}}_1 \not\perp \vec{\boldsymbol{u}}_2$

Example

```
> M = matrix(c(1,3,1,4),2,2)
> eigen(M)
$values
[1] 4.7912878 0.2087122

$vectors
           [,1]       [,2]
[1,] -0.2550401 -0.7841904
[2,] -0.9669305  0.6205203
```

```
> u=c(cos(pi/6),sin(pi/6))
> X = matrix(u,2,1)
> M = X %*% solve(t(X)%*%X) %*% t(X)
> eigen(M)
$values
[1] 1 0
```

```
$vectors
           [,1]       [,2]
[1,] -0.8660254  0.5000000
[2,] -0.5000000 -0.8660254
```

```
> M=c(1,3,3,1)
> dim(M)=c(2,2)
> eigen(M)
$values
[1]  4 -2

$vectors
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068
> P = eigen(M)$vectors
> t(P) %*% P
     [,1] [,2]
[1,]    1    0
[2,]    0    1
```

Spectral Decomposition

As a special case, for every $n \times n$ real symmetric matrix, the eigenvalues are real and the eigenvectors can be chosen such that they are orthogonal to each other. Thus a real symmetric matrix $\boldsymbol{M}$ can be decomposed as

$$\boldsymbol{M} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$$

where $\boldsymbol{P}$ is an orthonormal matrix whose columns are the eigenvectors of $\boldsymbol{M}$, and $\boldsymbol{D}$ is a diagonal matrix whose entries are the eigenvalues of $\boldsymbol{M}$.

Let $\boldsymbol{P} = [\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2, \cdots, \vec{\boldsymbol{u}}_n]$, then $\langle \vec{\boldsymbol{u}}_i, \vec{\boldsymbol{u}}_j \rangle = \delta_{i,j}$.

$$\boldsymbol{M} = \sum_{i=1}^{n} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^\top$$

Example

```
> M=c(1,2,3,4)
> dim(M)=c(2,2)
> eigen(M)
$values
[1]  5.3722813 -0.3722813

$vectors
           [,1]       [,2]
[1,] -0.5657675 -0.9093767
[2,] -0.8245648  0.4159736

> P = eigen(M)$vectors
> D = diag(eigen(M)$values)
> P %*% D %*% solve(P)
     [,1] [,2]
[1,]    1    3
[2,]    2    4
```

```
> M=c(1,3,3,1)
> dim(M)=c(2,2)
> eigen(M)
$values
[1]  4 -2

$vectors
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068

> P = eigen(M)$vectors
> D = diag(eigen(M)$values)
> P %*% D %*% solve(P)
     [,1] [,2]
[1,]    1    3
[2,]    3    1
```

Approximation

$$\boldsymbol{M} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 5 \\ 5 & 3 & 7 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.19 & -0.32 & 0.74 \\ 0.75 & -0.59 & 0.16 \\ 0.64 & 0.74 & -0.65 \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} 12.04 & 0 & 0 \\ 0 & 2.41 & 0 \\ 0 & 0 & 0.55 \end{bmatrix}}_{\boldsymbol{D}} \boldsymbol{P}^{-1}$$

$$\underbrace{\begin{bmatrix} 0.19 & -0.32 & 0.74 \\ 0.75 & -0.59 & 0.16 \\ 0.64 & 0.74 & -0.65 \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} 12.04 & 0 & 0 \\ 0 & 2.41 & 0 \\ 0 & 0 & 0.00 \end{bmatrix}}_{\boldsymbol{D}'} \boldsymbol{P}^{-1} = \begin{bmatrix} 0.31 & 2.26 & 0.91 \\ 2.85 & 7.05 & 4.98 \\ 5.60 & 2.78 & 7.08 \end{bmatrix} = \boldsymbol{M}'$$