4 Classification & Neural Nets

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Machine Learning & Econometrics

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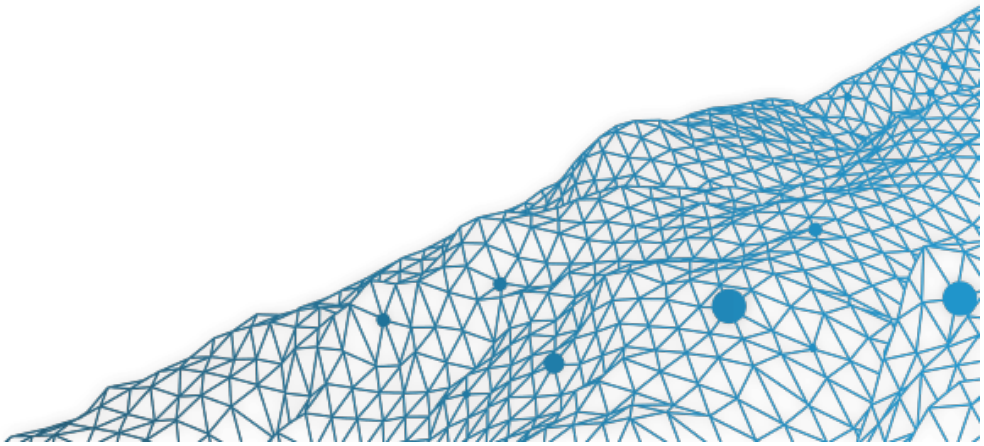


Modeling a 0/1 random variable

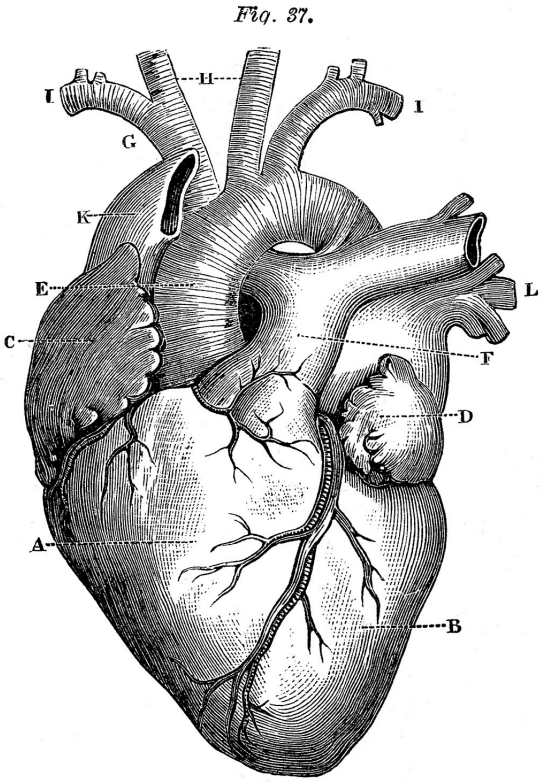
Myocardial infarction of patients admited in E.R.

```
> myocarde=read.table("http://freakonometrics.free.fr/myocarde.csv",
    head=TRUE,sep=";")
```

dataset $\{(\boldsymbol{x}_i, y_i)\}$ where $i = 1, \ldots, n$, $n = 73$.

```
FRCAR
INCAR
INSYS
PRDIA
PAPUL
PVENT
REPUL
PRONO
```

- heart rate x_1
- heart index x_2
- stroke index x_3
- diastolic pressure x_4
- pulmonary arterial pressure x_5
- ventricular pressure x_6
- lung resistance x_7
- death or survival y



Classification : (Linear) Fisher's Discrimination

Consider the follwing naive classification rule

$$m^\star(\boldsymbol{x}) = \text{argmin}_y\{\mathbb{P}[Y = y|\boldsymbol{X} = \boldsymbol{x}]\}$$

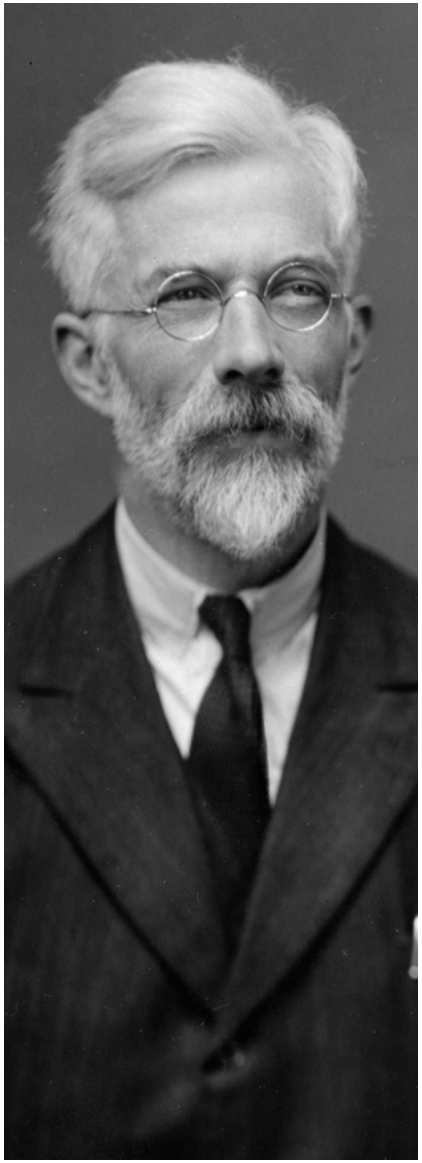
or

$$m^\star(\boldsymbol{x}) = \text{argmin}_y \left\{ \frac{\mathbb{P}[\boldsymbol{X} = \boldsymbol{x}|Y = y]}{\mathbb{P}[\boldsymbol{X} = \boldsymbol{x}]} \right\}$$

(where $\mathbb{P}[\boldsymbol{X} = \boldsymbol{x}]$ is the density in the continuous case).

In the case where y takes two values, that will be standard $\{0, 1\}$ here, one can rewrite the later as

$$m^\star(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \mathbb{E}(Y|\boldsymbol{X} = \boldsymbol{x}) > \dfrac{1}{2} \\ 0 \text{ otherwise} \end{cases}$$



Classification : (Linear) Fisher's Discrimination

the set

$$\mathcal{D}_S = \left\{ \boldsymbol{x}, \mathbb{E}(Y|\boldsymbol{X}=\boldsymbol{x}) = \frac{1}{2} \right\}$$

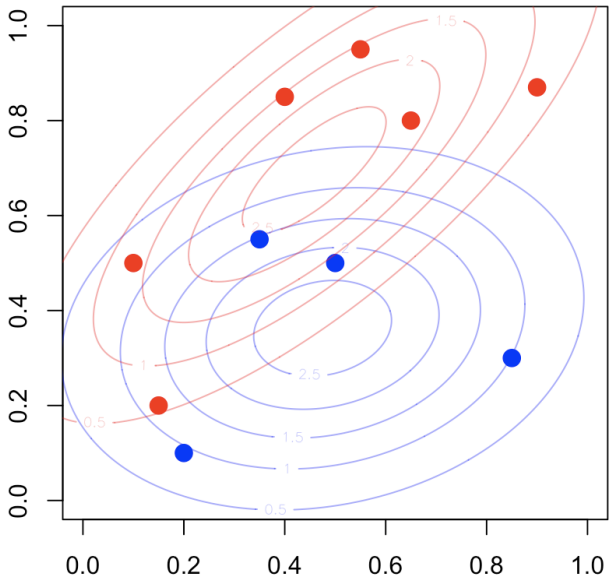
is called the decision boundary.

Assume that $\boldsymbol{X}|Y=0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})$ and $\boldsymbol{X}|Y=1 \sim \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$, then explicit expressions can be derived.

$$m^\star(\boldsymbol{x}) = \begin{cases} 1 \text{ if } r_1^2 < r_0^2 + 2\log\dfrac{\mathbb{P}(Y=1)}{\mathbb{P}(Y=0)} + \log\dfrac{|\boldsymbol{\Sigma}_0|}{|\boldsymbol{\Sigma}_1|} \\ 0 \text{ otherwise} \end{cases}$$

where r_y^2 is the Mahalanobis distance,

$$r_y^2 = [\boldsymbol{X} - \boldsymbol{\mu}_y]^\mathsf{T} \boldsymbol{\Sigma}_y^{-1} [\boldsymbol{X} - \boldsymbol{\mu}_y]$$



Classification : (Linear) Fisher's Discrimination

Let $\delta_y(\boldsymbol{x})$ be defined as

$$-\frac{1}{2}\log|\boldsymbol{\Sigma}_y| - \frac{1}{2}[\boldsymbol{x}-\boldsymbol{\mu}_y]^\mathsf{T}\boldsymbol{\Sigma}_y^{-1}[\boldsymbol{x}-\boldsymbol{\mu}_y] + \log\mathbb{P}(Y=y)$$

the decision boundary of this classifier is

$$\{\boldsymbol{x} \text{ such that } \delta_0(\boldsymbol{x}) = \delta_1(\boldsymbol{x})\}$$

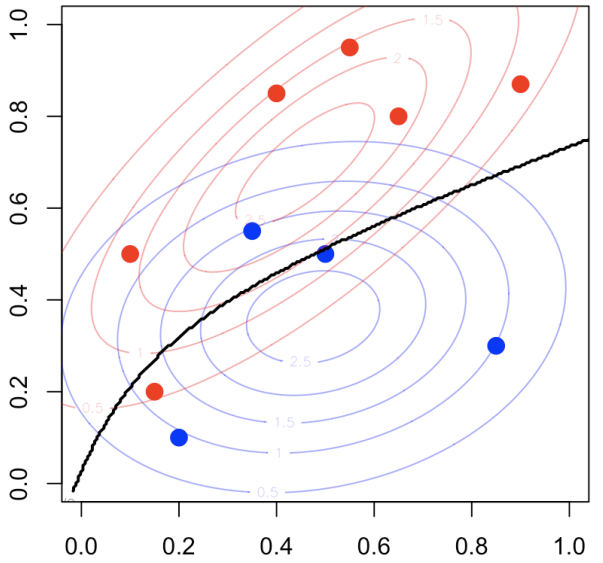
which is quadratic in $\boldsymbol{x}$. This is the quadratic discriminant analysis.

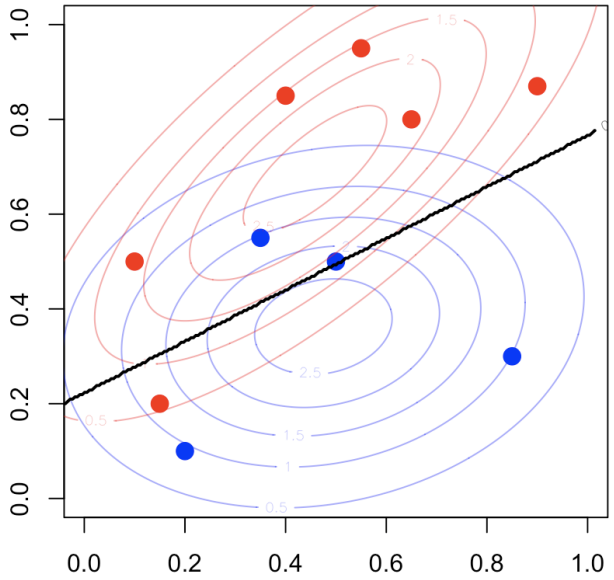
In Fisher (1936, The use of multile measurements in taxinomic problems , it was assumed that $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}_1$

In that case, actually,

$$\delta_y(\boldsymbol{x}) = \boldsymbol{x}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_y - \frac{1}{2}\boldsymbol{\mu}_y^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_y + \log\mathbb{P}(Y=y)$$

and the decision frontier is now linear in $\boldsymbol{x}$.





Neural Nets & Deep Learning in a Nutshell

Very popular for pictures...

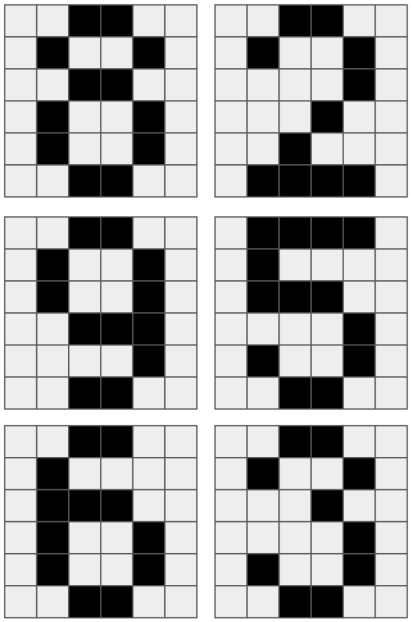
Picture $\boldsymbol{x}_i$ is

- a $n \times n$ matrix in $\{0,1\}^{n^2}$ for black & white
- a $n \times n$ matrix in $[0,1]^{n^2}$ for grey-scale
- a $3 \times n \times n$ array in $([0,1]^3)^{n^2}$ for color
- a $T \times 3 \times n \times n$ tensor in $(([0,1]^3)^T)^{n^2}$ for video

y here is the label ("8", "9", "6", etc)

Suppose we want to recognize a "6" on a picture

$$m(\boldsymbol{x}) = \begin{cases} +1 \text{ if } \boldsymbol{x} \text{ is a "6"} \\ -1 \text{ otherwise} \end{cases}$$



Neural Nets & Deep Learning in a Nutshell

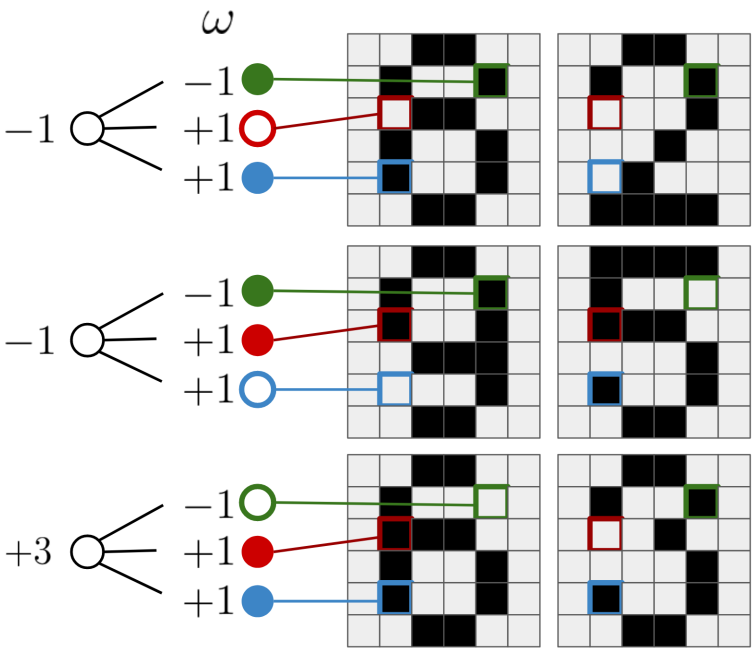
Consider some specifics pixels, and associate weights ω such that

$$\widehat{m}(\boldsymbol{x}) = \text{sign}\left(\sum_{i,j} \omega_{i,j} x_{i,j}\right)$$

where

$$x_{i,j} = \begin{cases} +1 \text{ if pixel } x_{i,j} \text{ is black} & \blacksquare \\ -1 \text{ if pixel } x_{i,j} \text{ is white} & \square \end{cases}$$

for some weights $\omega_{i,j}$ (that can be negative...)

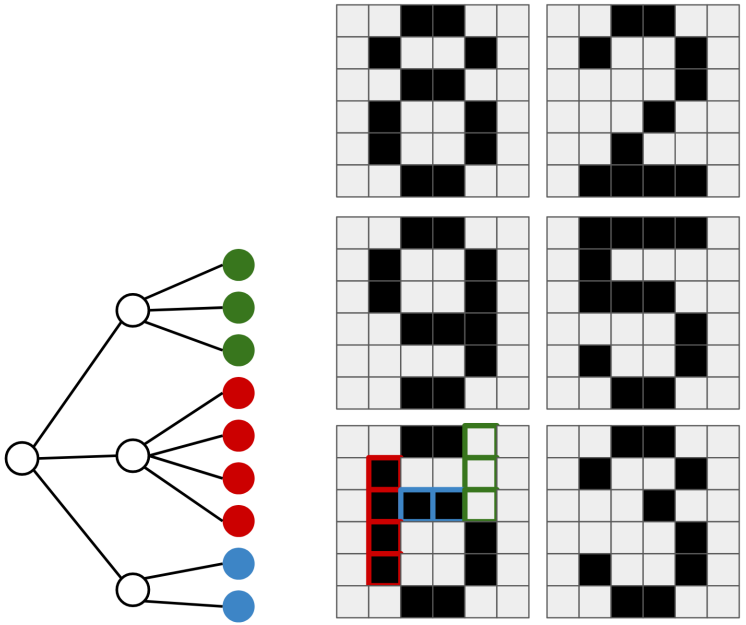


Neural Nets & Deep Learning in a Nutshell

A deep network is a network with a lot of layers

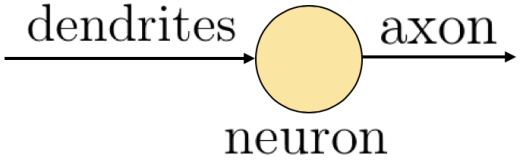
$$\widehat{m}(\boldsymbol{x}) = \text{sign}\left(\sum_i \omega_i \widehat{m}_i(\boldsymbol{x})\right)$$

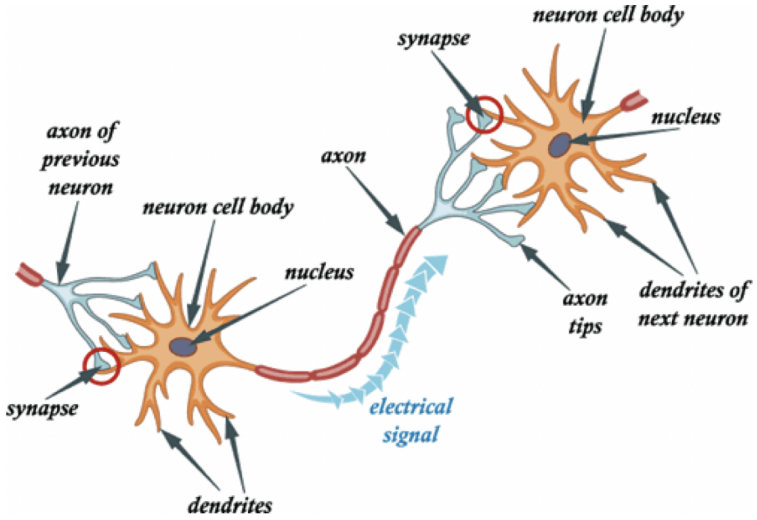
where $\widehat{m}_i$'s are outputs of previous neural nets
Those layers can capture shapes in some areas
nonlinearities, cross-dependence, etc



Classification : Neural Nets

A neuron is simply :





Human body has ~ 100 billion neuron cells

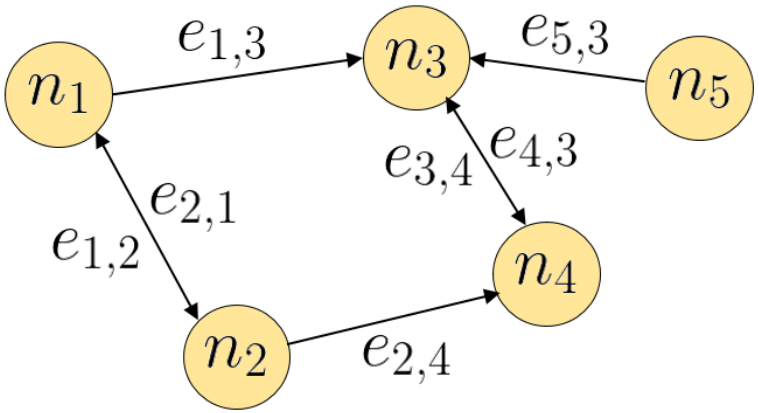
Good at :

vision, speech, driving, playing games, etc

The most interesting part is not 'neural' but 'network'.

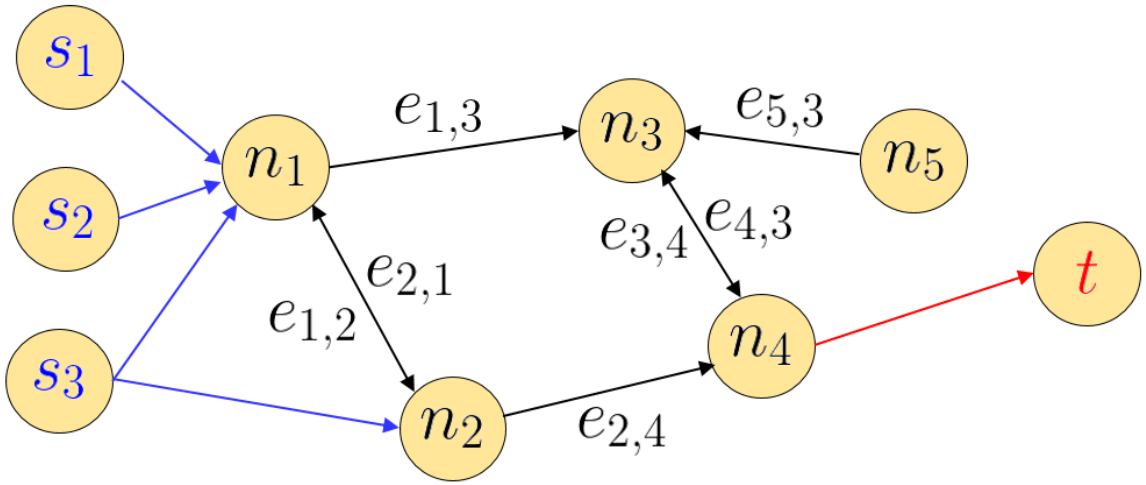
Classification : Neural Nets

Networks have nodes, and edges (possibly directed) that connect nodes,



or maybe, to more specific some sort of flow network,

Classification : Neural Nets



In such a network, we usually have sources (here multiple) sources (here $\{s_1, s_2, s_3\}$), on the left, on a sink (here $\{t\}$), on the right. To continue with this metaphorical introduction, information from the sources should reach the sink. An usually, sources are explanatory variables, $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_p\}$, and the sink is our variable of interest y.

The output here is a binary variable $y \in \{0, 1\}$

Classification : the Binary Threshold Neuron & the Perceptron

If $\boldsymbol{x} \in \{0,1\}^p$, McCulloch & Pitts (1943, A logical calculus of the ideas immanent in nervous activity) suggested a simple model, with threshold b

$$y_i = f\left(\sum_{j=1}^{p} x_{j,i}\right) \text{ where } f(x) = \mathbf{1}(x \geq b)$$

where $\mathbf{1}(x \geq b) = +1$ if $x \geq b$ and 0 otherwise, or (equivalently)

$$y_i = f\left(\omega + \sum_{j=1}^{p} x_{j,i}\right) \text{ where } f(x) = \mathbf{1}(x \geq 0)$$

with weight $\omega = -b$. The trick of adding 1 as an input was very important !

$\omega = -1$ is the or logical operator : $y_i = 1$ if $\exists j$ such that $x_{j,i} = 1$

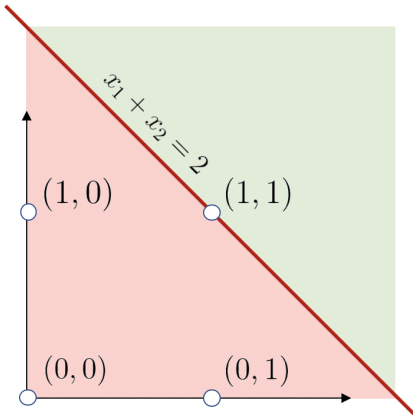
$\omega = -p$, is the and logical operator : $y_i = 1$ if $\forall j$, $x_{j,i} = 1$

Classification : the Binary Threshold Neuron & the Perceptron

but not possible for the xor logical operator : $y_i = 1$ if $x_{1,i} \neq x_{2,i}$

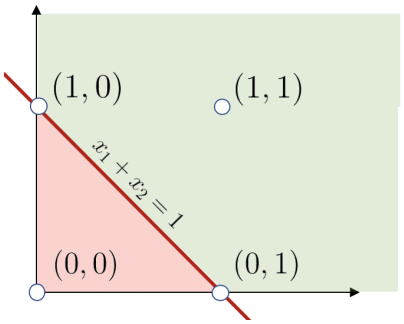
Rosenblatt (1961, Principles of Neurodynamics: Perceptron & Theory of Brain Mechanism) considered the extension where x's are real-valued, with weight $\boldsymbol{\omega} \in \mathbb{R}^p$

$$y_i = f\left(\sum_{j=1}^{p} \omega_j x_{j,i}\right) \text{ where } f(x) = \mathbf{1}(x \geq b)$$



where $\mathbf{1}(x \geq b) = +1$ if $x \geq b$ and 0 otherwise, or (equivalently)

$$y_i = f\left(\omega_0 + \sum_{j=1}^{p} \omega_j x_{j,i}\right) \text{ where } f(x) = \mathbf{1}(x \geq 0)$$



with weights $\boldsymbol{\omega} \in \mathbb{R}^{p+1}$

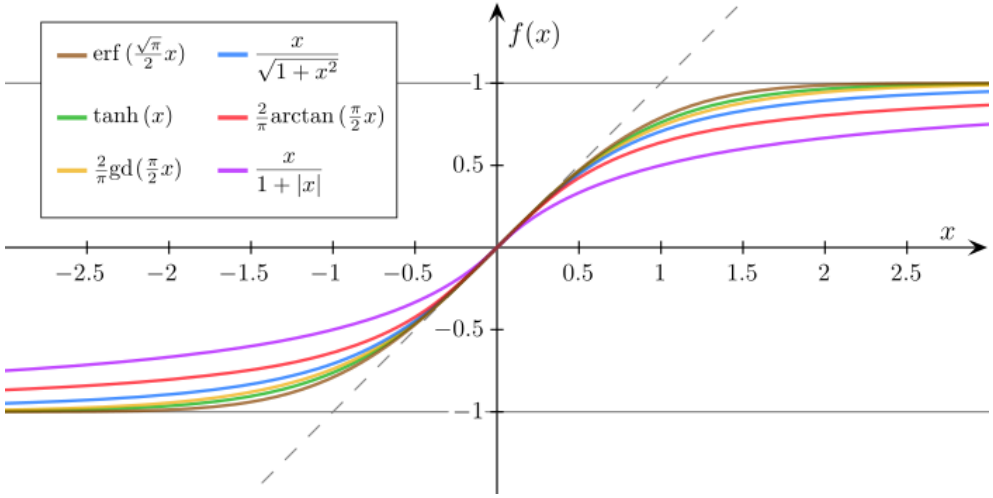
Classification : the Binary Threshold Neuron & the Perceptron

Minsky & Papert (1969, Perceptrons: an Introduction to Computational Geometry) proved that perceptron were a linear separator, not very powerful

Define the sigmoid function $f(x) = \dfrac{1}{1+e^{-x}} = \dfrac{e^x}{e^x+1}$ (= logistic function).

This function f is called the activation function.

If $y \in \{-1, +1\}$, one can consider the hyperbolic tangent $f(x)) = \dfrac{(e^x - e^{-x})}{(e^x + e^{-x})}$ or the inverse tangent function (see wikipedia).



Classification : Neural Nets

So here, for a classication problem,

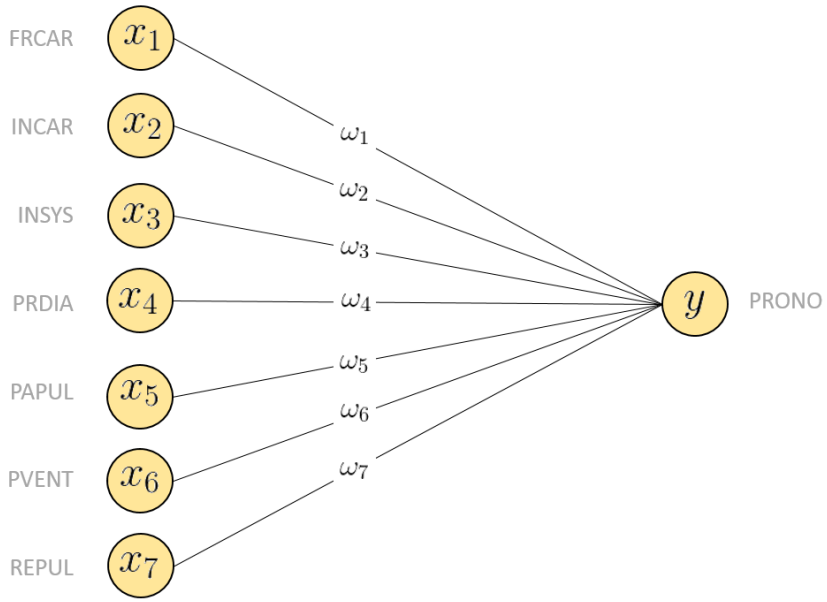
$$y_i = f\left(\omega_0 + \sum_{j=1}^{p} \omega_j x_{j,i}\right) = p(\boldsymbol{x}_i$$

The error of that model can be computed used the quadratic loss function,

$$\sum_{i=1}^{n} (y_i - p(\boldsymbol{x}_i))^2$$

or cross-entropy

$$\sum_{i=1}^{n} (y_i \log p(\boldsymbol{x}_i) + [1 - y_i] \log[1 - p(\boldsymbol{x}_i)])$$



Classification : Neural Nets

If

$$p(\boldsymbol{x}) = f\left(\omega_0 + \sum_{h=1}^{3} \omega_h f_h\left(\omega_{h,0} + \boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_h\right)\right)$$

we want to solve, with back propagation,

$$\boldsymbol{\omega}^\star = \text{argmin}\left\{\frac{1}{n}\sum_{i=1}^{n} \ell\left(y_i, p(\boldsymbol{x}_i)\right)\right\}$$

To visualize a neural network, use

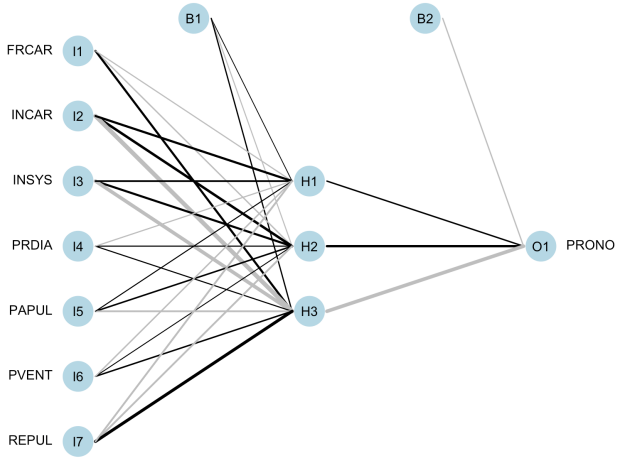
```
library(devtools)
source_url('https://freakonometrics.free.fr/nnet_plot.R')
```

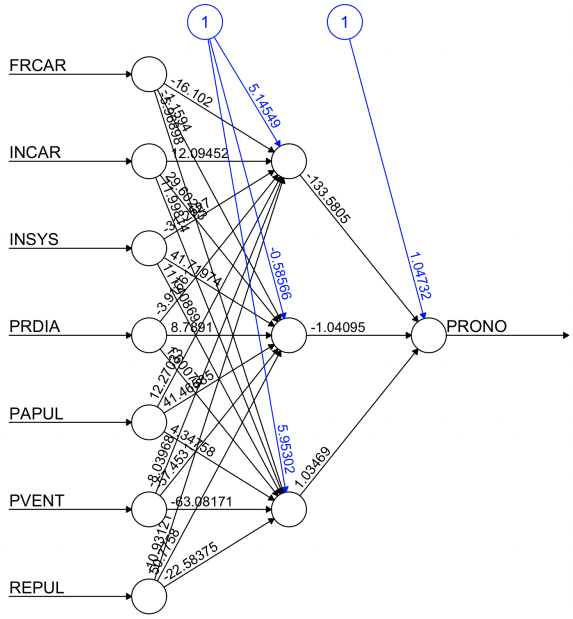
Classification : Neural Nets

```
1 library(nnet)
2 mix = function(z) (z-min(z))/(max(z)-min(z))
3 myocarde = apply(myocarde, 1, mix)
4 fit_nn = nnet(PRONO~., data=myocarde, size=3)
5 plot.nnet(fit_nn)
```

or

```
1 library(neuralnet)
2 fit_nn = neuralnet(formula(glm(PRONO~., data =
      myocarde)), myocarde, hidden = 3, act.fct
      = sigmoid)
3 plot(fit_nn)
```





Classification : Neural Nets

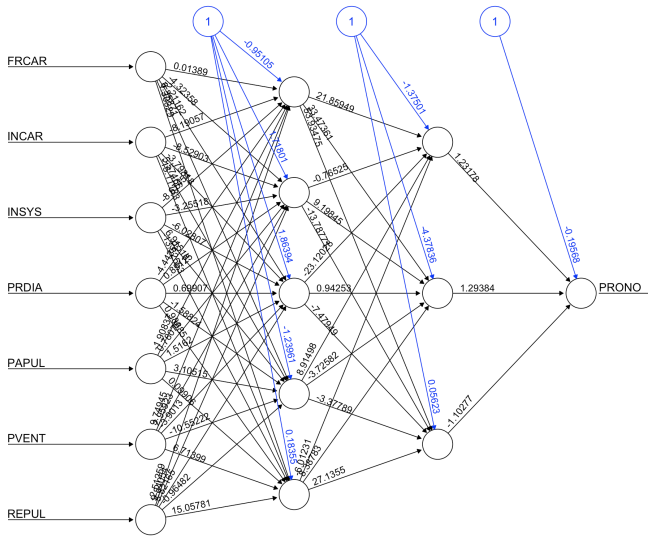
Of course, one can consider a multiple layer network,

$$p(\boldsymbol{x}) = f\left(\omega_0 + \sum_{h=1}^{3} \omega_h f_h\left(\omega_{h,0} + \boldsymbol{z}_h{}^{\mathsf{T}}\boldsymbol{\omega}_h\right)\right)$$

where

$$\boldsymbol{z}_h = f\left(\omega_{h,0} + \sum_{j=1}^{k_h} \omega_{h,j} f_{h,j}\left(\omega_{h,j,0} + \boldsymbol{x}^{\mathsf{T}}\boldsymbol{\omega}_{h,j}\right)\right)$$

```
1 library(neuralnet)
2 model_nnet = neuralnet(formula(glm(PRONO~.,
      data = myocarde)), myocarde, hidden = 3,
      act.fct = sigmoid)
3 plot(model_nnet)
```



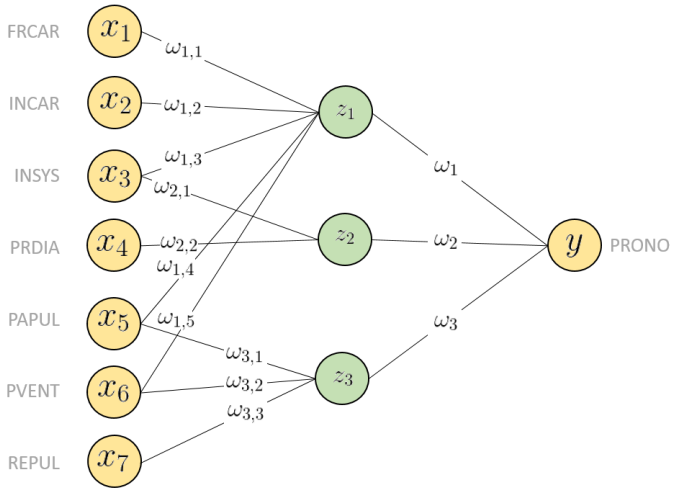
Classification : Neural Nets

Consider a single hidden layer, with 3 different neurons, so that

$$p(\boldsymbol{x}) = f\left(\omega_0 + \sum_{h=1}^{3} \omega_h f_h\left(\omega_{h,0} + \sum_{j=1}^{p} \omega_{h,j} x_j\right)\right)$$

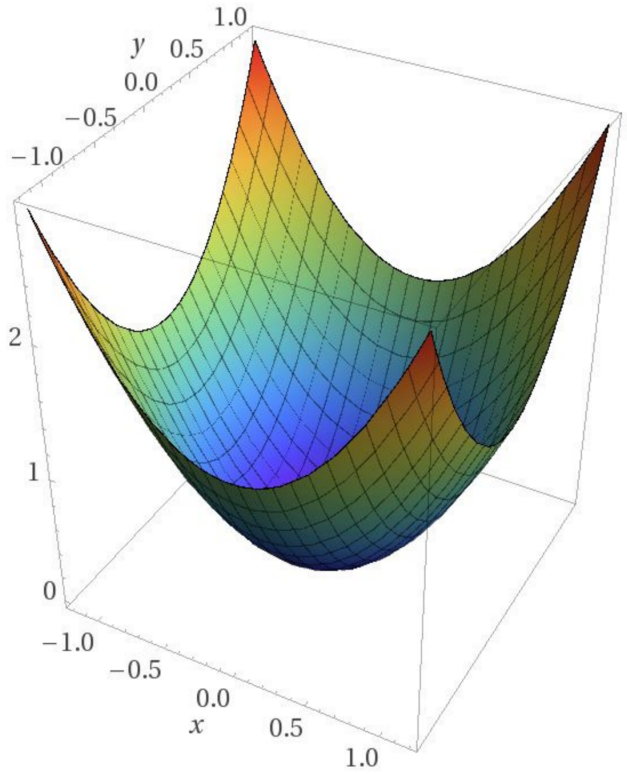
or

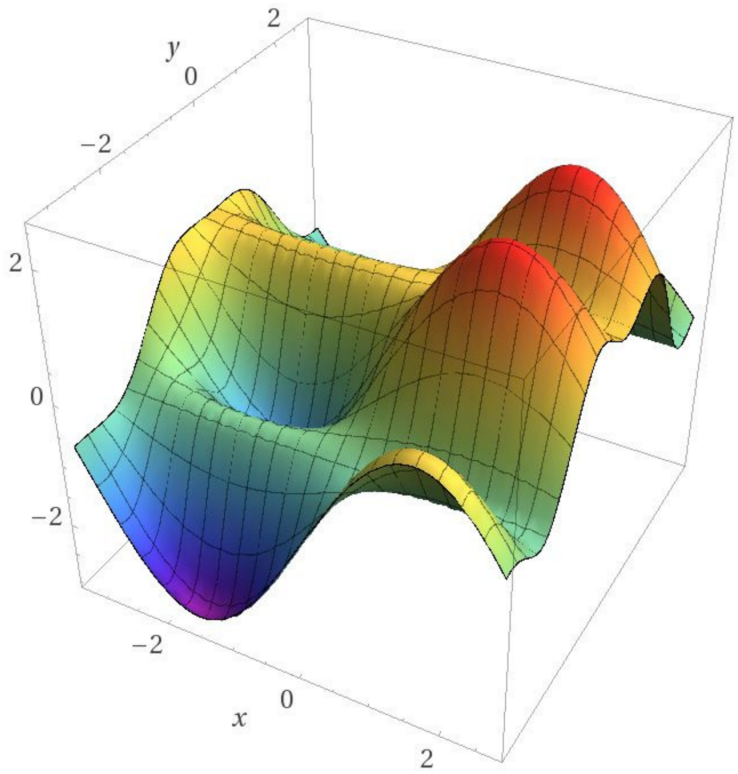
$$p(\boldsymbol{x}) = f\left(\omega_0 + \sum_{h=1}^{3} \omega_h f_h\left(\omega_{h,0} + \boldsymbol{x}^{\mathsf{T}}\boldsymbol{\omega}_h\right)\right)$$



Optimization Issues

If classical econometrics are based on convex optimization problems (most the time), (deep) Neural Networks are usually not convex





Optimization Issues

Consider optimization problem

$$\min_{\boldsymbol{x}\in\mathcal{X}}\left\{f(\boldsymbol{x})\right\}$$

solved by gradient descent,

$$\boldsymbol{x}_{n+1}=\boldsymbol{x}_n-\gamma_n\nabla f(\boldsymbol{x}_n),\ n\geq 0$$

with starting point $\boldsymbol{x}_0$
γ_n can be call learning rate

Sometime the dimension of the dataset can be really big: we can't pass all the data to the computer at once to compute $f(\boldsymbol{x})$

we need to divide the data into smaller sizes and give it to our computer one by one

Optimization Issues

Batches

The Batch size is a hyperparameter that defines the number of sub-samples we use to compute quantities (e.g. the gradient).

Epoch

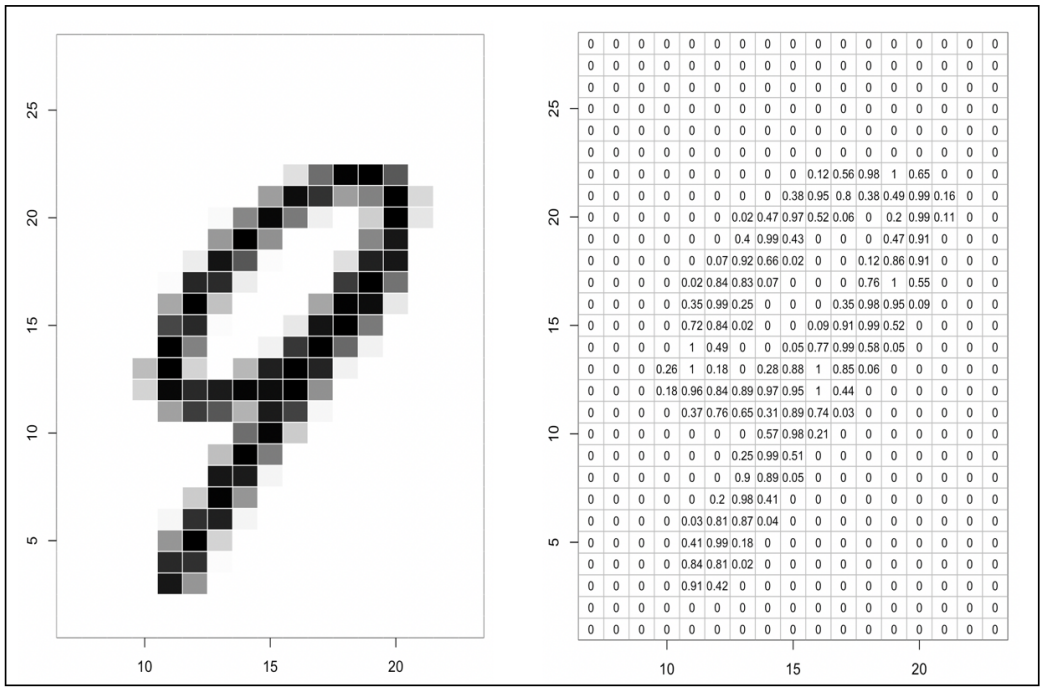
One Epoch is when the dataset is passed forward and backward through the neural network (only once).

We can divide the dataset of $2,000$ examples into batches of 400 then it will take 5 iterations to complete 1 epoch.

The number of epochs is the hyperparameter that defines the number times that the learning algorithm will work through the entire training dataset.

Classification : Image Recognition with Neural Networks

Classical $\{0, 1, \cdots, 9\}$ handwritten digit recognition



$$\boldsymbol{x}_i \in \mathcal{M}_{28,28} \qquad y_i \in \{0, 1, \cdots, 9\}$$

see tensorflow (Google) and keras.

Classification : Image Recognition with Logistic Regression?

See https://www.tensorflow.org/install

```
library(tensorflow)
install_tensorflow(method = "conda")
library(keras)
install_keras(method = "conda")
```

```
mnist <- dataset_mnist()
idx123 = which(mnist$train$y %in% c(1,2,3))
V <- mnist$train$x[idx123[1:800],,]
MV = NULL
for(i in 1:800) MV=cbind(MV,as.vector(V[i,,]))
MV=t(MV)
df=data.frame(y=mnist$train$y[idx123][1:800],x=MV)
reg=glm((y==1)~.,data=df,family=binomial)
```

Here $\boldsymbol{x}_i \in \mathbb{R}^{784}$ (for a small greyscale picture).

Using Principal Components to Reduce Dimension

Instead of using $\boldsymbol{X}$ as regressors, use $\tilde{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X}$ with less covariates

Note that $\boldsymbol{y} = \beta_0 + \tilde{\boldsymbol{X}}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ become $\boldsymbol{y} = \beta_0 + \boldsymbol{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{\varepsilon}$ where $\tilde{\boldsymbol{\beta}} = \boldsymbol{A}\boldsymbol{\beta}$.

We can use either principal components or PLS (partial least squares), as introduced in Wold (1966, Estimation of principal components and related models by iterative least square).

For both, we need to normalize the data (to have mean 0 and variance 1).

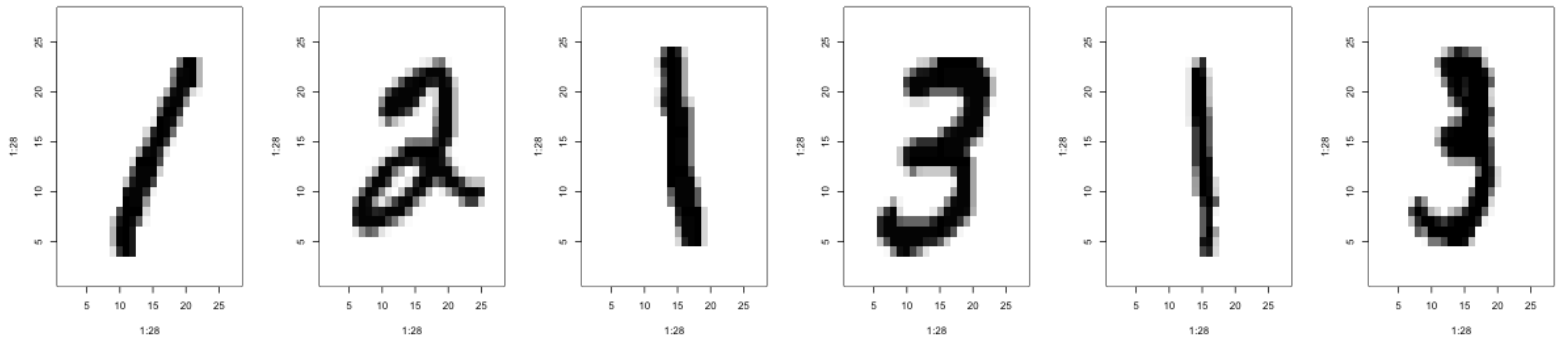
With PCA, the first component $\boldsymbol{z}_1 = \boldsymbol{X}\boldsymbol{\omega}_1$ is obtained as solution of

$$\boldsymbol{\omega}_1 = \underset{\|\boldsymbol{\omega}\|=1}{\operatorname{argmax}} \{\operatorname{Var}[\boldsymbol{X}\boldsymbol{\omega}]\} = \underset{\|\boldsymbol{\omega}\|=1}{\operatorname{argmax}} \{< \boldsymbol{X}\boldsymbol{\omega}, \boldsymbol{X}\boldsymbol{\omega} >\}$$

With PLS, the first component $\boldsymbol{z}_1 = \boldsymbol{X}\boldsymbol{\omega}_1$ is obtained as solution of

$$\boldsymbol{\omega}_1 = \underset{\|\boldsymbol{\omega}\|=1}{\operatorname{argmax}} \{\operatorname{Cov}[\boldsymbol{y}, \boldsymbol{X}\boldsymbol{\omega}]\} = \underset{\|\boldsymbol{\omega}\|=1}{\operatorname{argmax}} \{< \boldsymbol{y}, \boldsymbol{X}\boldsymbol{\omega} >\}$$

Classification : Image Recognition and Convolutional Neural Network



Classical strategy : reduce dimension via layers $\boldsymbol{z}$ (e.g. PCA, principal component analysis)

and then consider a classifier $\boldsymbol{z} \mapsto y$

CNN : Convolutional Neural Network

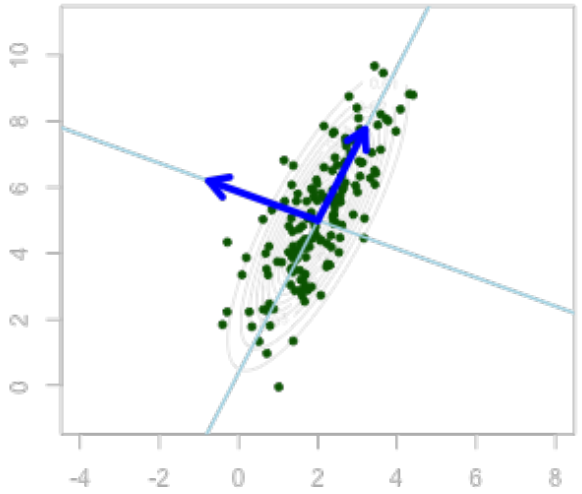
A convolutional neural network is a fully connected networks : each neuron in one layer is connected to all neurons in the next layer.

Non-supervised : PCA

Projection method, used to reduce dimensionality

PCA : Principal Component Analysis

find a small number of directions in input space that explain variation in input data represent data by projecting along those directions



Consider (joint) n observations of d variables, stored in matrix $\boldsymbol{X} \in \mathcal{M}_{n \times d}$

We want to project (multiply by a projection matrix) into (much) lower dimensional space $k < d$

Search for orthogonal directions in space with highest variance

Non-supervised : PCA

Information is stored in the covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{X}^\top \boldsymbol{X}$ (variables are centered)

The k largest eigenvectors of $\boldsymbol{\Sigma}$ are the principal components

Assemble these eigenvectors into a $d \times k$ matrix $\tilde{\boldsymbol{\Sigma}}$

Express d-dimensional vectors $\boldsymbol{x}$ by projecting them to m-dimensional $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{\Sigma}}^\top \boldsymbol{x}$

- Heuristics for the *first* principal component

We have data $\boldsymbol{X} \in \mathcal{M}_{n\times d}$, i.e. n vectors $\boldsymbol{x} \in \mathbb{R}^d$,

Let $\boldsymbol{\omega}_1 \in \mathbb{R}^d$ denote weights.

We want to maximize the variance of $\boldsymbol{z}_1 = \boldsymbol{\omega}_1^\top \boldsymbol{x}$

$$\max\left\{\frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{\omega}_1^\top \boldsymbol{x}_i - \boldsymbol{\omega}_1^\top \overline{x})^2\right\} = \max\left\{\boldsymbol{\omega}_1^\top \boldsymbol{\Sigma}\boldsymbol{\omega}_1\right\}$$

subject to $\|\boldsymbol{\omega}_1\| = 1$.

Non-supervised : PCA

We can use Lagrange multiplier to solve this constraint optimization problem

$$\max_{\boldsymbol{\omega}_1,\lambda_1} \left\{\boldsymbol{\omega}_1^\top \boldsymbol{\Sigma}\boldsymbol{\omega}_1 + \lambda(1-\boldsymbol{\omega}_1^\top\boldsymbol{\omega}_1)\right\}$$

If we differentiate

$$\boldsymbol{\Sigma}\boldsymbol{\omega}_1 - \lambda_1\boldsymbol{\omega}_1 = \mathbf{0}, \text{ i.e. } \boldsymbol{\Sigma}\boldsymbol{\omega}_1 = \lambda_1\boldsymbol{\omega}_1$$

i.e. $\boldsymbol{\omega}_1$ is an eigenvector of $\boldsymbol{\Sigma}$, with eigenvalue the Lagrange multiplier. It should be the one with the largest eigenvalue.

A nonlinear version can be obtained using reproducing kernel Hilbert spaces, and kernels, see Schölkopf *al.* (1996, Nonlinear Component Analysis as a Kernel Eigenvalue Problem).

For the second one, we should solve a similar problem, with $\boldsymbol{\omega}_2 \perp \boldsymbol{\omega}_1$, i.e. $\boldsymbol{\omega}_2^\top\boldsymbol{\omega}_1 = 0$

Non-supervised : PCA

- Heuristics for the *second* principal component

$$\max \left\{ \boldsymbol{\omega}_2^\top \boldsymbol{\Sigma} \boldsymbol{\omega}_2 \right\}$$
$$\text{subject to } \|\boldsymbol{\omega}_2\| = 1$$
$$\boldsymbol{\omega}_2^\top \boldsymbol{\omega}_1 = 0$$

The Lagragian is

$$\boldsymbol{\omega}_2^\top \boldsymbol{\Sigma} \boldsymbol{\omega}_2 + \lambda_2 (1 - \boldsymbol{\omega}_2^\top \boldsymbol{\omega}_2) - \lambda_1 \boldsymbol{\omega}_2^\top \boldsymbol{\omega}_1$$

when differentiation with respect to $\boldsymbol{\omega}_2$, we get $\boldsymbol{\Sigma} \boldsymbol{\omega}_2 = \lambda_2 \boldsymbol{\omega}_2$

hence, λ_2 is the second largest eigenvalue, etc.

Non-supervised : PLS

Partial Least Squares

(i) set $\boldsymbol{x}_j^{(0)} = \boldsymbol{x}_j$

(ii) at step k

- fit regressions marginally, $\widehat{y} = \widehat{\theta}_j \boldsymbol{x}_j^{(k-1)}$, and set $\boldsymbol{z}_k = \widehat{\theta}_1 \boldsymbol{x}_1^{(k-1)} + \cdots + \widehat{\theta}_p \boldsymbol{x}_p^{(k-1)}$
  - regress $\boldsymbol{y}$ on $\boldsymbol{z}_k$ and let $\widehat{\boldsymbol{y}}^{(k)}$ be the prediction (of $\boldsymbol{y}$)
  - orthogonalize each $\boldsymbol{x}_j^{(k-1)}$ by regressing on $\boldsymbol{z}_k$ i.e. $\boldsymbol{x}_j^{(k)} = \boldsymbol{x}_j^{(k-1)} + \boldsymbol{z}_k \widehat{\tau}_j$

where $\widehat{\tau}_j = (\boldsymbol{z}_k^\top \boldsymbol{z}_k)^{-1} \boldsymbol{z}_k^\top \boldsymbol{x}_j^{(k-1)}$

(iii) loop step (ii) and set finally, $\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{y}}^{(1)} + \widehat{\boldsymbol{y}}^{(2)} + \cdots$

In practice, PLS and PCA have similar results (especially with low R^2)

see `pls` library (on partial least squares), with `pls::pslr` for PLS regression, or `pls::pcr` for partial components. One can use `stats::prcomp` for PCA

Non-supervised : Principal Components on Handwritten Pictures

Consider some handwritten pictures, from the mnist dataset. Here $\{(y_i, \boldsymbol{x}_i)\}$ with $y_i =$ "3" and $\boldsymbol{x}_i \in [0,1]^{28\times 28}$

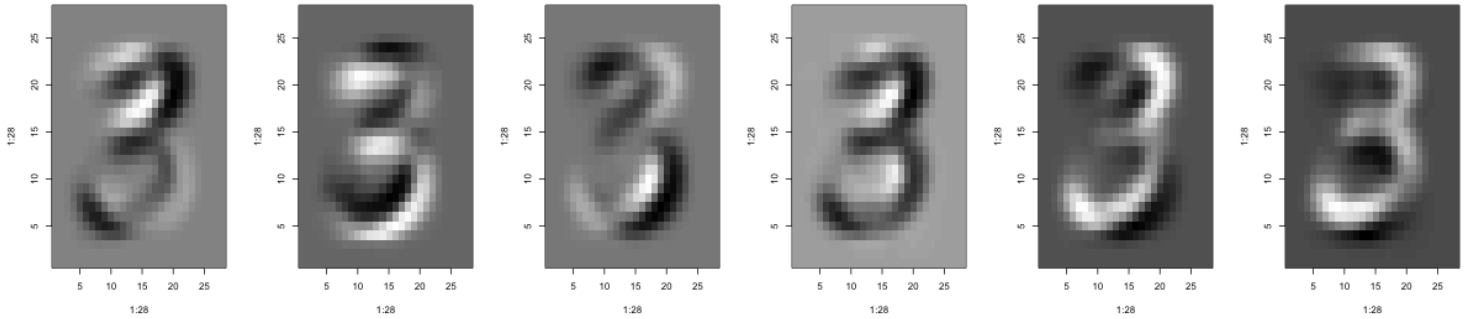
A *variable* is a pixel here...

Matrix $\boldsymbol{X}$ is a $n \times 784$ matrix, where n is the number of pictures in the dataset.

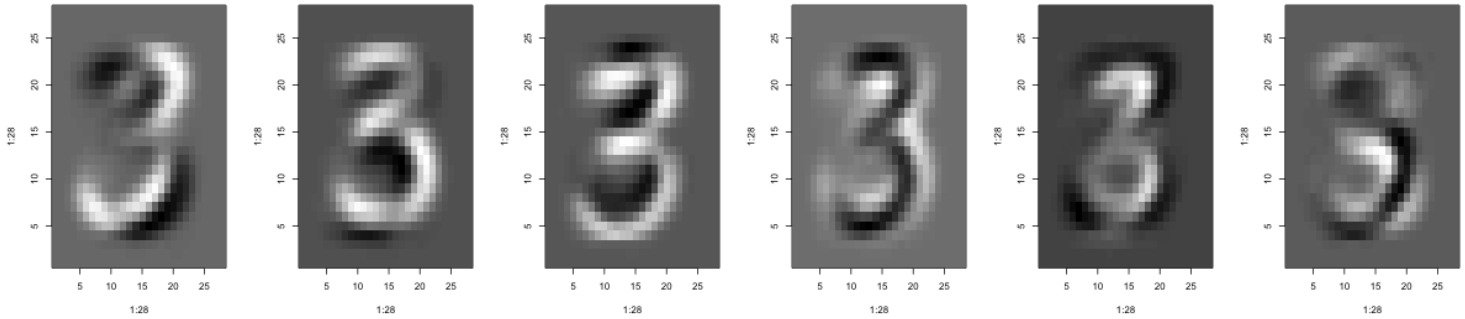
Non-supervised : PCA on Handwritten Pictures

Consider the PCA of matrix $\boldsymbol{X}$, then $\boldsymbol{\omega}_j \in \mathbb{R}^{784}$ can be interpreted as a picture (after rescaling).

We can also consider the approximation of $\boldsymbol{x}_i$ using only the first k principal components (here $k = 3$)



Non-supervised : PCA on Tensor Data



1D Tensor (vector)

$\boldsymbol{A} = [A_i] \in \mathbb{R}^d$ - i = individual

2D Tensor (matrix)

$\boldsymbol{A} = [A_{i,j}] \in \mathcal{M}_{d_1,d_2}$, j = feature

3D Tensor (array)

$\boldsymbol{A} = [A_{i,j,k}] = \boldsymbol{A}_i$'s

$\boldsymbol{A}_i \in \mathcal{M}_{d_1,d_2}$

e.g. (black & white) picture

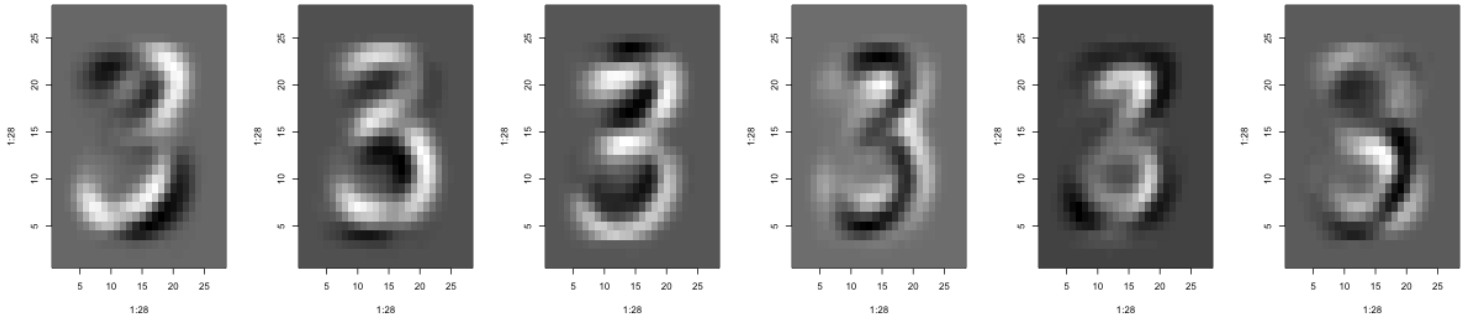
4D Tensor (arrays)

$\boldsymbol{A} = [A_{i,j,k,l}] \in \mathcal{M}_{d_1,d_2}$

e.g. color picture,

$l \in \{$red, green, blue $\}$

Non-supervised : PCA on Tensor Data

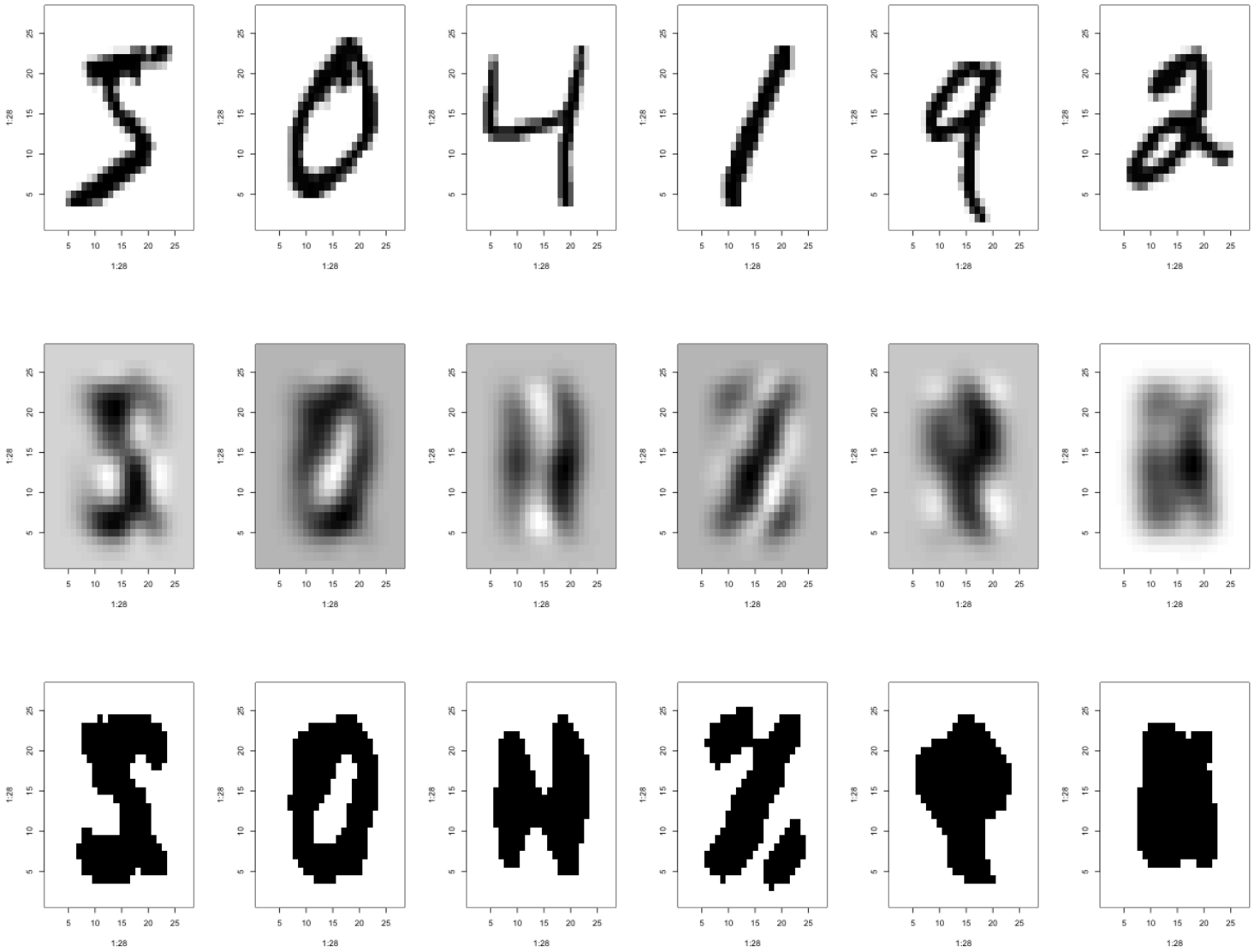


One can define a multilinear principal component analysis

see also Tucker decomposition, from Hitchcock (1927, The expression of a tensor or a polyadic as a sum of products)

We can use those decompositions to derive a simple classifier.

Non-supervised : Principal Components on Handwritten Pictures



Non-supervised : Classification of Handwritten Pictures

Consider a dataset with only three labels, $y_i \in \{\text{“1”}, \text{“2”}, \text{“3”}\}$ and the k principal components obtained by PCA,

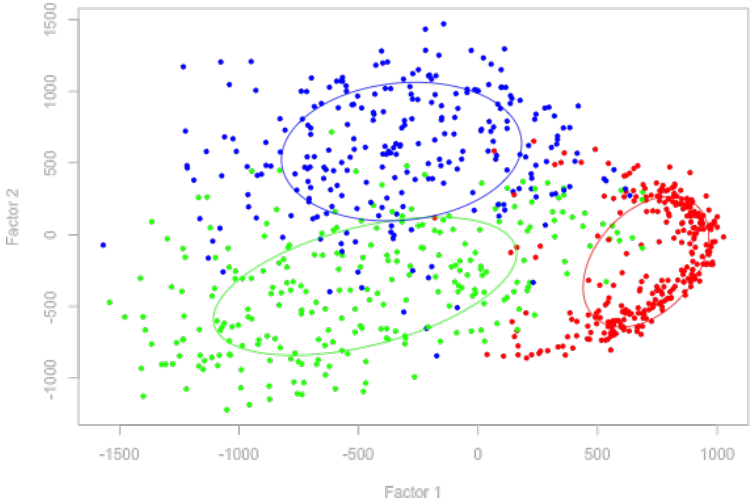
$\{x_1, \cdots, x_{784}\} \rightarrow \{\tilde{x}_1, \cdots, \tilde{x}_k\}$

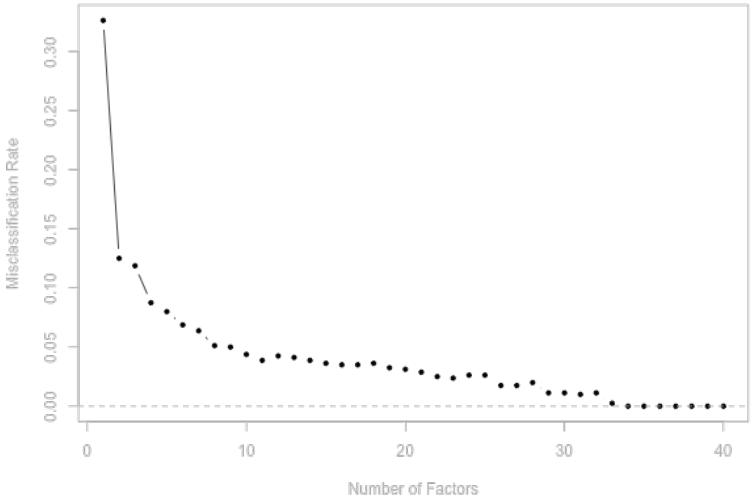
E.g. scatterplot $\{\tilde{x}_{1,i}, \tilde{x}_{2,i}\}$

with • when $y_i = \text{“1”}$, • $y_i = \text{“2”}$ and • $y_i = \text{“3”}$,

Let m_k denote the multinomial logistic regression on $\tilde{x}_1, \cdots, \tilde{x}_k$ and visualize the misclassification rate

See Nielsen (2018, Neural Networks and Deep Learning)



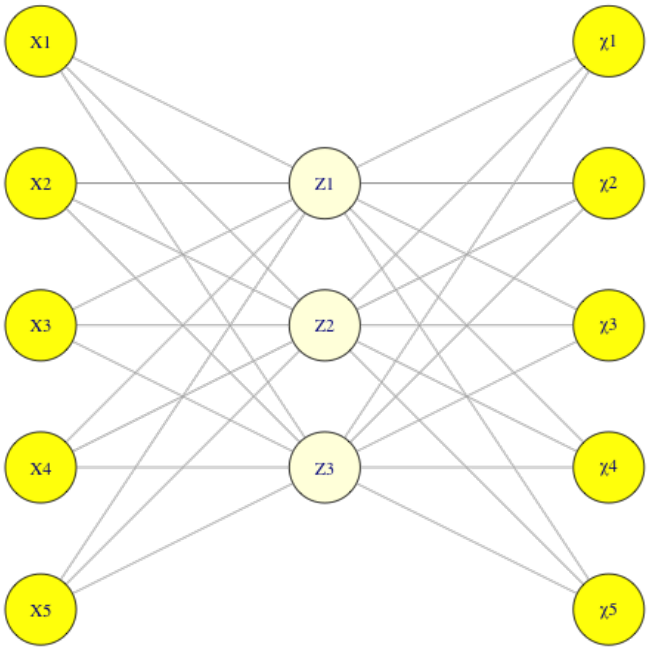


Non-supervised : Autoencoders

PCA is closely related to a neural network

Autoencoder

An autoencoder is a neural network whose outputs are its own inputs

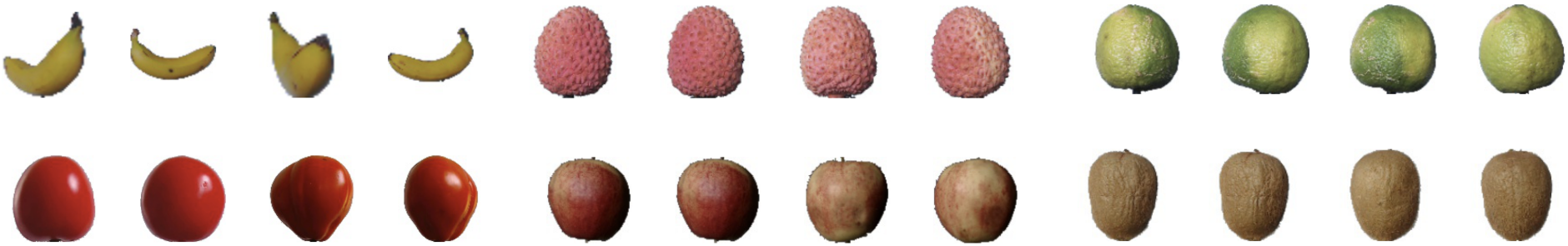


Here $\boldsymbol{z} = f(W\boldsymbol{x})$ and $\boldsymbol{\chi} = g(M\boldsymbol{z})$.
Given f and g solve

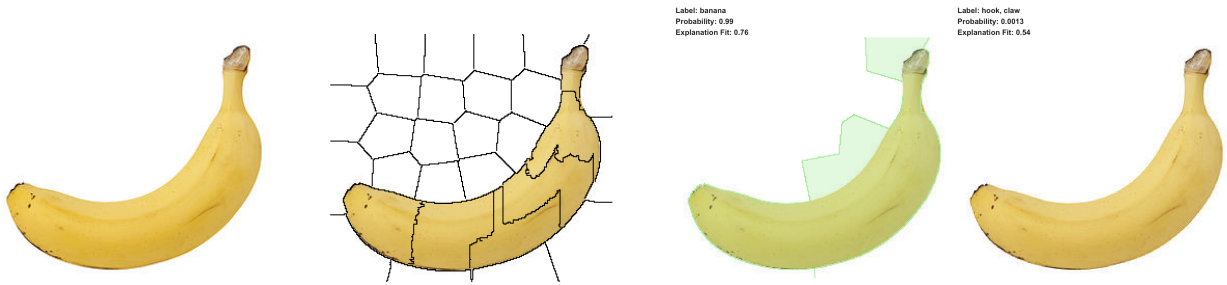
$$\min_{W,M}\left\{\sum_{i=1}^{n}\ell(x_i,\chi_i)\right\} = \min_{W,M}\left\{\sum_{i=1}^{n}\ell(x_i,MWx_i)\right\} \text{ if } f \text{ and } g \text{ are linear}$$

Neural Networks for More Complex Pictures

The algorithm is trained on tagged picture, i.e. $(\boldsymbol{x}_i, y_i)$ where here y_i is some class of fruits (banana, apple, kiwi, tomatoe, lime, cherry, pear, lychee, papaya, etc)



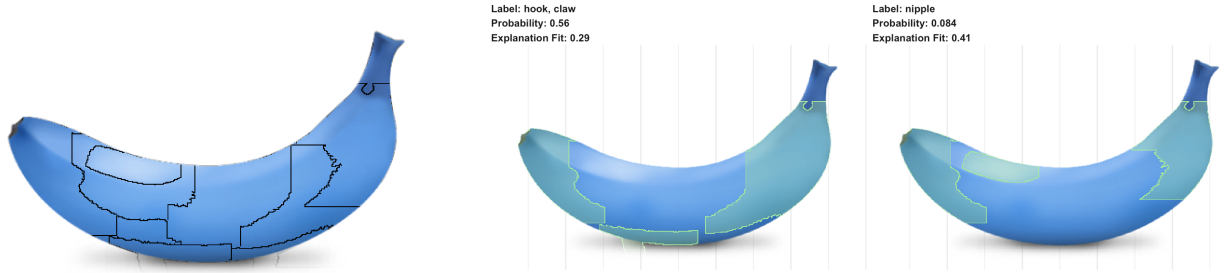
The challenge is to provide a new picture ($\boldsymbol{x}_{n+1}$ and to see the prediction $\widehat{y}_{n+1}$)



Neural Networks for More Complex Pictures

Note that ($\boldsymbol{x}_{n+1}$ can be something that was not in the training dataset (here yellow zucchini)

or some invented one (here a blue banana)



see also pyimagesearch for food/no food classifier

Neural Networks for Text and Go (Alpha Go)

Neural nets are also popular for text classification...

Neural network architecture. The input to the policy network is a $19 \times 19 \times 48$ image stack consisting of 48 feature planes. The first hidden layer zero pads the input into a 23×23 image, then convolves k filters of kernel size 5×5 with stride 1 with the input image and applies a rectifier nonlinearity. Each of the subsequent hidden layers 2 to 12 zero pads the respective previous hidden layer into a 21×21 image, then convolves k filters of kernel size 3×3 with stride 1, again followed by a rectifier nonlinearity. The final layer convolves 1 filter of kernel size 1×1 with stride 1, with a different bias for each position, and applies a softmax function. The match version of AlphaGo used $k = 192$ filters; Fig. 2b and Extended Data Table 3 additionally show the results of training with $k = 128$, 256 and 384 filters.

RNN : Recurrent Neural Networks

Introduced in Williams, Hinton & Rumelhart (1986, Learning representations by back-propagating errors)

RNN : Recurrent Neural Networks

NN where connections between nodes form a directed graph along a temporal sequence
Allows it to exhibit temporal dynamic behavior.

They will be discussed in the part on time series and dynamic modeling.

Neural Networks for Pictures Classification

See http://junma5.weebly.com/ for more advanced models

We should preprocess the data by removing near zero variance columns (and scaling by max)

```
train = xxxxxxx
nzv <- nearZeroVar(train)
nzv.nolabel <- nzv-1
```

The data is also splitted into two for cross validation

```
inTrain <- createDataPartition(y=train$label, p=0.7, list=FALSE)
training <- train[inTrain, ]
CV <- train[-inTrain, ]

X <- as.matrix(training[, -1]) # data matrix (each row = single
    example)
N <- nrow(X) # number of examples
y <- training[, 1] # class labels
```

```
8
9  K <- length(unique(y)) # number of classes
10 X.proc <- X[, -nzv.nolabel]/max(X) # scale
11 D <- ncol(X.proc) # dimensionality
12
13 Xcv <- as.matrix(CV[, -1]) # data matrix (each row = single example)
14 ycv <- CV[, 1] # class labels
15 Xcv.proc <- Xcv[, -nzv.nolabel]/max(X) # scale CV data
16
17 Y <- matrix(0, N, K)
18
19 for (i in 1:N){
20   Y[i, y[i]+1] <- 1
21 }
```

We can train the model

```
1 nnet.mnist <- nnet(X.proc, Y, step_size = 0.3,
2                    reg = 0.0001, niteration = 3500)
```

```
## [1] "iteration 0 : loss 2.30265553844748"
## [1] "iteration 1000 : loss 0.303718250939774"
## [1] "iteration 2000 : loss 0.271780096710725"
## [1] "iteration 3000 : loss 0.252415244824614"
## [1] "iteration 3500 : loss 0.250350279456443"
predicted_class <- nnetPred(X.proc, nnet.mnist)
print(paste('training set accuracy:',
            mean(predicted_class == (y+1))))
## [1] "training set accuracy: 0.930891405638888"
predicted_class <- nnetPred(Xcv.proc, nnet.mnist)
print(paste('CV accuracy:',
            mean(predicted_class == (ycv+1))))
## [1] "CV accuracy: 0.912360085734699"
```

```
test <- Xcv[sample(1:nrow(Xcv), 1), ]
Xtest.proc <- as.matrix(Xtest[-nzv.nolabel], nrow = 1)
predicted_test <- nnetPred(t(Xtest.proc), nnet.mnist)
print(paste('The predicted digit is:',predicted_test-1 ))
## [1] "The predicted digit is: 3"
```

```
6 displayDigit(Xtest)
```