Using Transformations of Variables to Improve Inference

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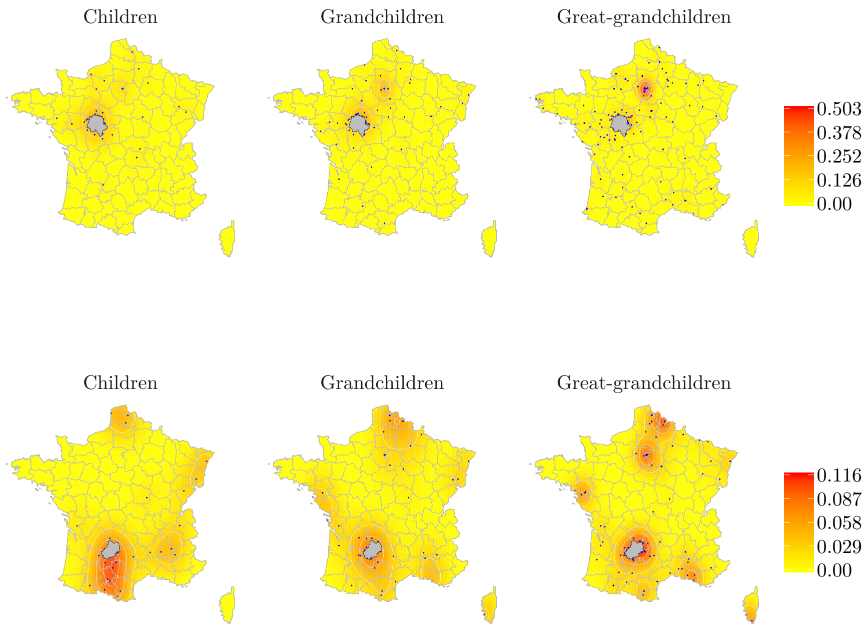
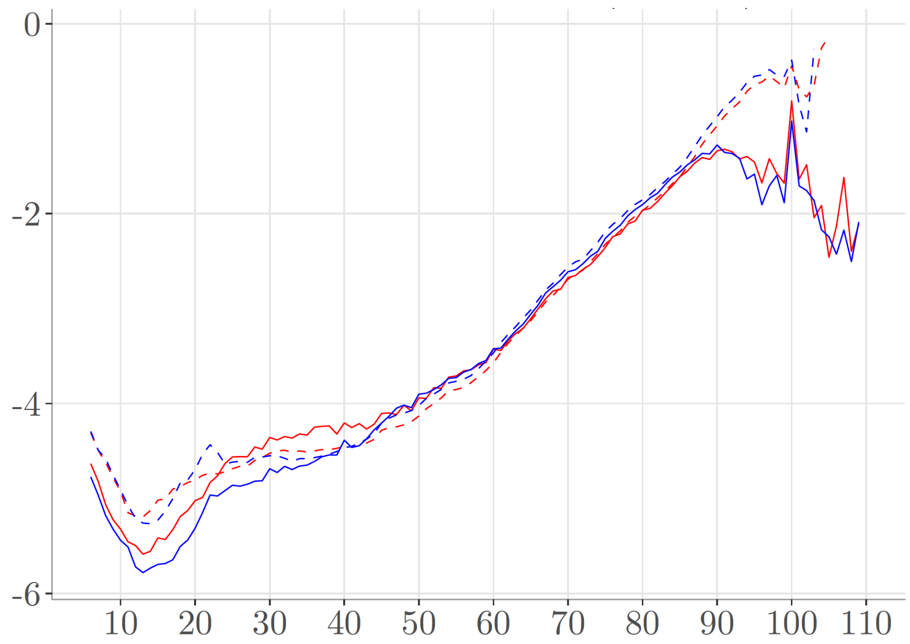
Side Note

Most of the contents here is based on old results (revised, and continued)

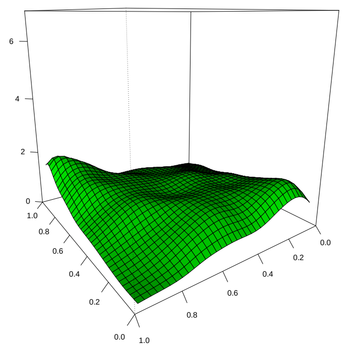
Work on genealogical trees, Étude de la démographie française du XIXe siècle à partir de données collaboratives de généalogie and Internal Migrations in France in the Nineteenth Century with E. Gallic.

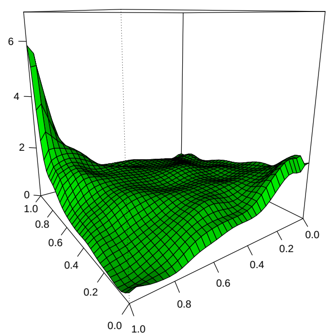


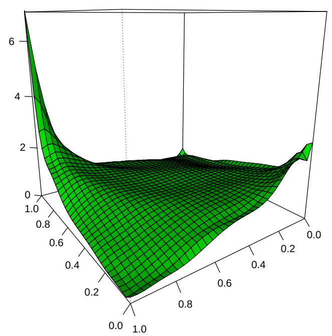


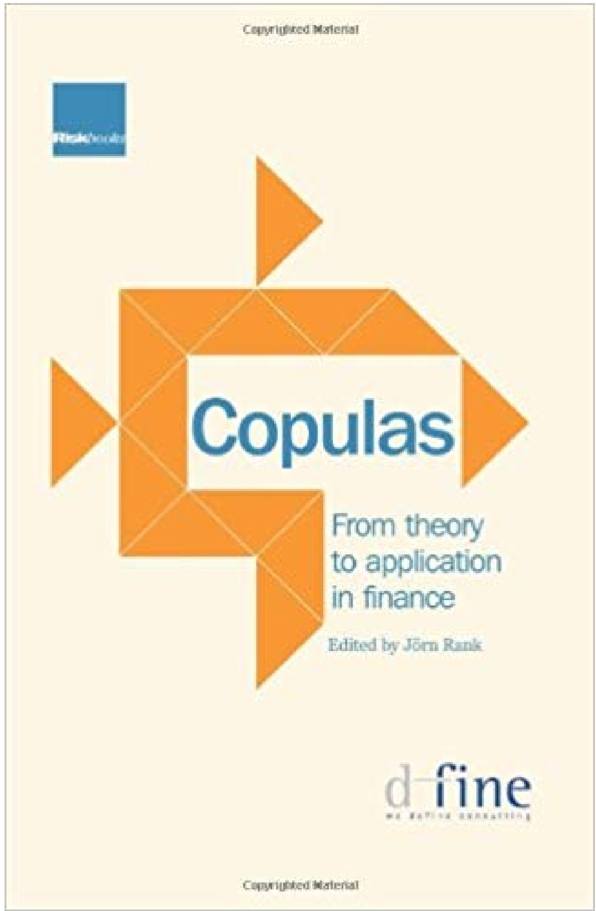
Motivation

2005, The Estimation of Copulas: Theory and Practice
with J.-D. Fermanian and O. Scaillet
survey on non-parametric techniques
(kernel base) to visualize
the estimator of a copula density









Idea : beta kernels and transformed kernels

More recently, mix those techniques in the univariate case

Motivation

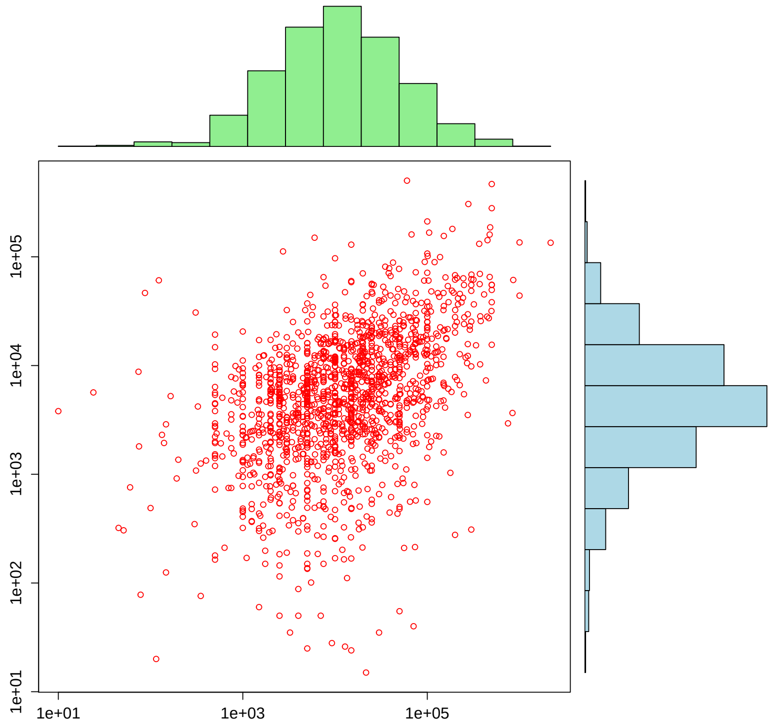
Consider some n-i.i.d. sample $\{(X_i, Y_i)\}$ with cumulative distribution function F_{XY} and joint density f_{XY}. Let F_X and F_Y denote the marginal distributions, and C the copula,

$$F_{XY}(x, y) = C(F_X(x), F_Y(y))$$

so that

$$f_{XY}(x, y) = f_X(x) f_Y(y) c(F_X(x), F_Y(y))$$

We want a nonparametric estimate of c on $[0, 1]^2$.



Notations

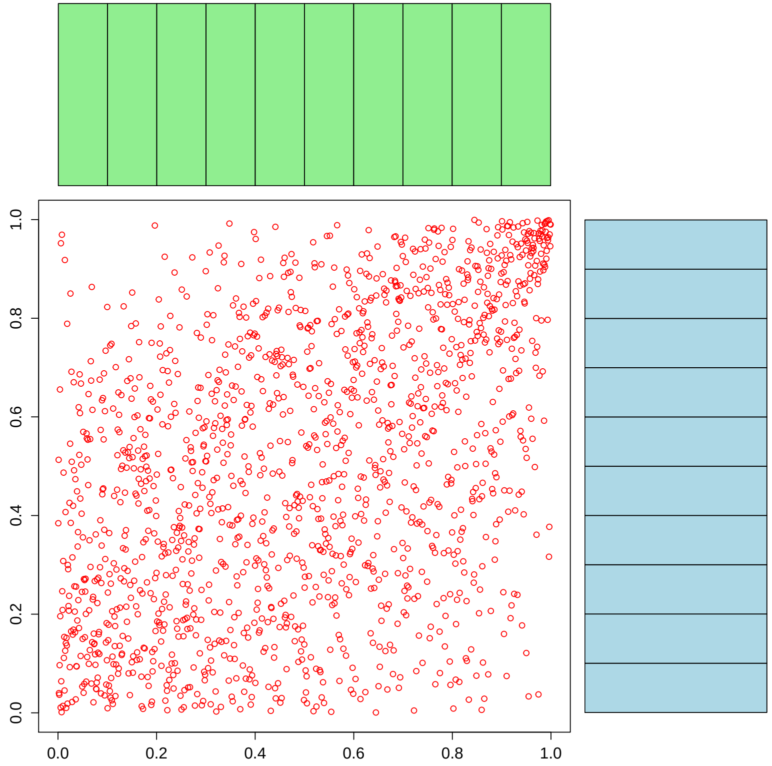
Define uniformized n-i.i.d. sample $\{(U_i, V_i)\}$

$$U_i = F_X(X_i) \text{ and } V_i = F_Y(Y_i)$$

or uniformized n-i.i.d. pseudo-sample $\{(\hat{U}_i, \hat{V}_i)\}$

$$\hat{U}_i = \frac{n}{n+1}\hat{F}_{Xn}(X_i) \text{ and } \hat{V}_i = \frac{n}{n+1}\hat{F}_{Yn}(Y_i)$$

where $\hat{F}_{Xn}$ and $\hat{F}_{Yn}$ denote empirical c.d.f.



Standard Kernel Estimate

The standard kernel estimator for c, say $\hat{c}^*$, at $(u,v)\in\mathcal{I}$ would be (see Wand & Jones (1995))

$$\hat{c}^*(u,v)=\frac{1}{n|\mathbf{H}_{UV}|^{1/2}}\sum_{i=1}^{n}\mathbf{K}\left(\mathbf{H}_{UV}^{-1/2}\begin{pmatrix}u-U_i\\v-V_i\end{pmatrix}\right),\tag{1}$$

where $\mathbf{K}:\mathbb{R}^2\to\mathbb{R}$ is a kernel function and $\mathbf{H}_{UV}$ is a bandwidth matrix.

Standard Kernel Estimate

However, this estimator is not consistent along boundaries of $[0,1]^2$

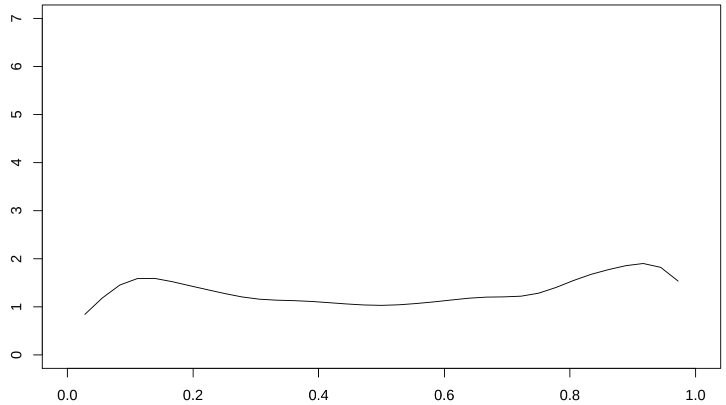
$$\mathbb{E}(\hat{c}^*(u,v)) = \frac{1}{4}c(u,v) + O(h) \text{ at corners}$$

$$\mathbb{E}(\hat{c}^*(u,v)) = \frac{1}{2}c(u,v) + O(h) \text{ on the borders}$$

if $\mathbf{K}$ is symmetric and $\mathbf{H}_{UV}$ symmetric.

Corrections have been proposed, e.g. mirror reflection Gijbels (1990) or the usage of boundary kernels Chen (2007), but with mixed results.

Remark : the graph on the bottom is $\hat{c}^*$ on the (first) diagonal.

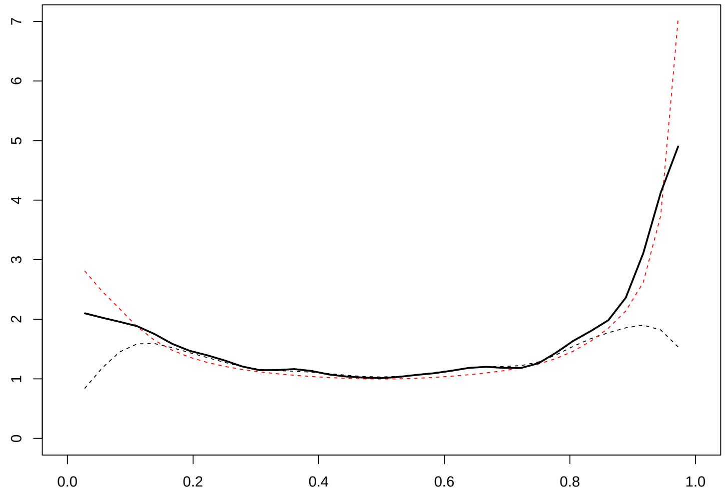


Mirror Kernel Estimate

Use an enlarged sample : instead of only $\{(\hat{U}_i, \hat{V}_i)\}$, add $\{(-\hat{U}_i, \hat{V}_i)\}$, $\{(\hat{U}_i, -\hat{V}_i)\}$, $\{(-\hat{U}_i, -\hat{V}_i)\}$, $\{(\hat{U}_i, 2 - \hat{V}_i)\}$, $\{(2 - \hat{U}_i, \hat{V}_i)\}$,$\{(-\hat{U}_i, 2 - \hat{V}_i)\}$, $\{(2 - \hat{U}_i, -\hat{V}_i)\}$ and $\{(2 - \hat{U}_i, 2 - \hat{V}_i)\}$.

See Gijbels & Mielniczuk (1990).

That estimator will be used as a benchmark in the simulation study.

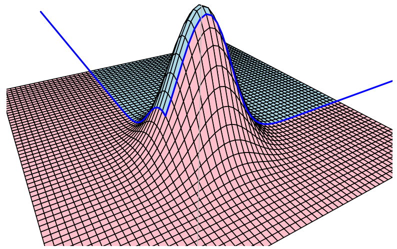


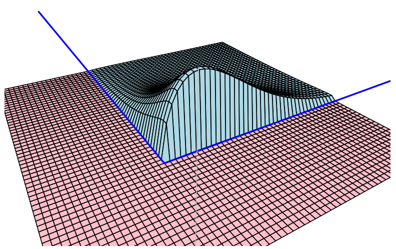
Using Beta Kernels

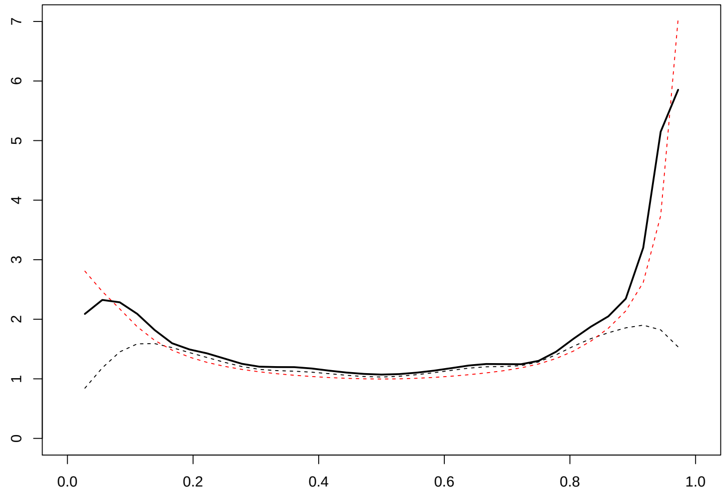
Use a Kernel which is a product of beta kernels

$$\mathbf{K}_{\boldsymbol{x}_i}(\boldsymbol{u}) \propto \left(x_{1,i}^{\frac{u_1}{b}} [1 - x_{1,i}]^{\frac{u_1}{b}} \right) \cdot \left(x_{2,i}^{\frac{u_2}{b}} [1 - x_{2,i}]^{\frac{u_2}{b}} \right)$$

for some $b > 0$, see Chen (1999).

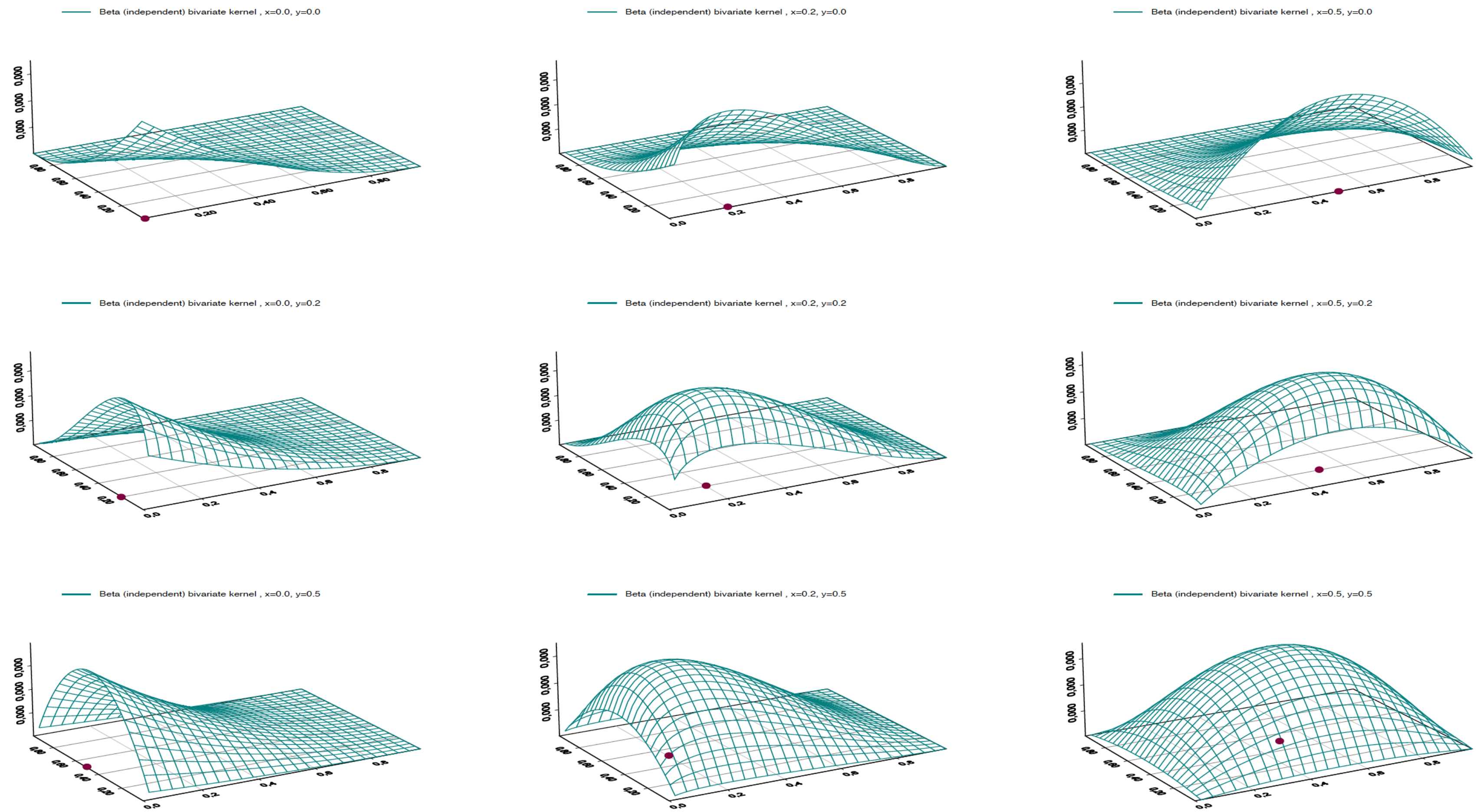






for some observation $\boldsymbol{x}_i$ in the lower left corner

Using Beta Kernels



Probit Transformation

See Devroye & Györfi (1985) and Marron & Ruppert (1994).

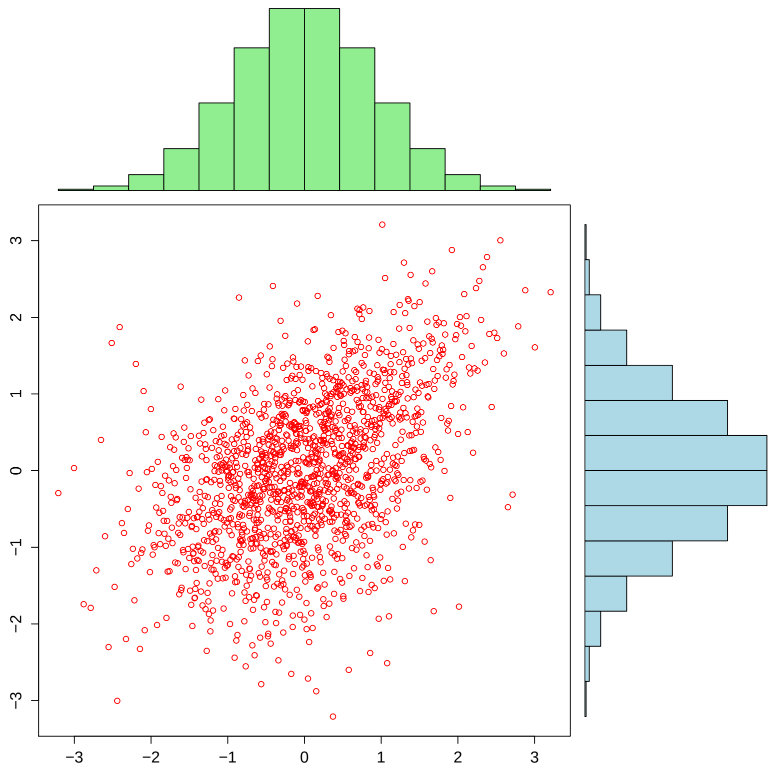
Define normalized n-i.i.d. sample $\{(S_i, T_i)\}$

$$S_i = \Phi^{-1}(U_i) \text{ and } T_i = \Phi^{-1}(V_i)$$

or normalized n-i.i.d. pseudo-sample $\{(\hat{S}_i, \hat{T}_i)\}$

$$\hat{U}_i = \Phi^{-1}(\hat{U}_i) \text{ and } \hat{V}_i = \Phi^{-1}(\hat{V}_i)$$

where Φ^{-1} is the quantile function of $\mathcal{N}(0,1)$ (probit transformation).



Probit Transformation

$$F_{ST}(x,y) = C(\Phi(x),\Phi(y))$$

so that

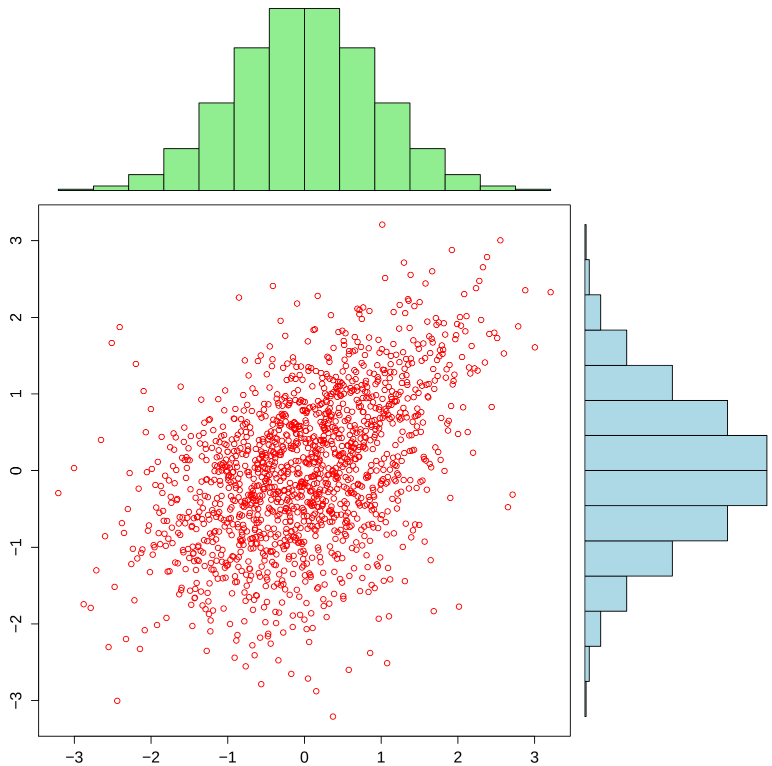
$$f_{ST}(x,y) = \phi(x)\phi(y)c(\Phi(x),\Phi(y))$$

Thus

$$c(u,v) = \frac{f_{ST}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}.$$

So use

$$\hat{c}^{(\tau)}(u,v) = \frac{\hat{f}_{ST}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$



The naive estimator

Since we cannot use

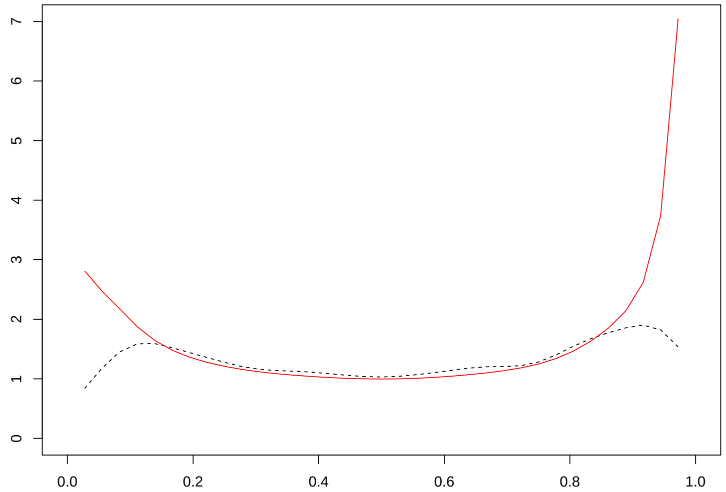
$$\hat{f}^*_{ST}(s,t) = \frac{1}{n|\mathbf{H}_{ST}|^{1/2}} \sum_{i=1}^{n} \mathbf{K}\left(\mathbf{H}_{ST}^{-1/2}\begin{pmatrix} s-S_i \\ t-T_i \end{pmatrix}\right),$$

where $\mathbf{K}$ is a kernel function and $\mathbf{H}_{ST}$ is a bandwidth matrix, use

$$\hat{f}_{ST}(s,t) = \frac{1}{n|\mathbf{H}_{ST}|^{1/2}} \sum_{i=1}^{n} \mathbf{K}\left(\mathbf{H}_{ST}^{-1/2}\begin{pmatrix} s-\hat{S}_i \\ t-\hat{T}_i \end{pmatrix}\right).$$

and the copula density is

$$\hat{c}^{(\tau)}(u,v) = \frac{\hat{f}_{ST}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$



The naive estimator

$$\hat{c}^{(\tau)}(u,v) = \frac{1}{n|\mathbf{H}_{ST}|^{1/2}\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}\sum_{i=1}^{n}\mathbf{K}\left(\mathbf{H}_{ST}^{-1/2}\begin{pmatrix}\Phi^{-1}(u)-\Phi^{-1}(\hat{U}_i)\\ \Phi^{-1}(v)-\Phi^{-1}(\hat{V}_i)\end{pmatrix}\right)$$

as suggested in C., Fermanian & Scaillet (2007) and Lopez-Paz . *et al.* (2013).

Note that Omelka . *et al.* (2009) obtained theoretical properties on the convergence of $\hat{C}^{(\tau)}(u,v)$ (not c).

In Probit transformation for nonparametric kernel estimation of the copula density with G. Geenens and D. Paindaveine, we extended that estimator.

See also kdecopula R package by T. Nagler

Improved probit-transformation copula density estimators

When estimating a density from **pseudo-sample**, Loader (1996) and Hjort & Jones (1996) define a local likelihood estimator

Around $(s,t)\in\mathbb{R}^2$, use a polynomial approximation of order p for $\log f_{ST}$

$$\log f_{ST}(\check{s},\check{t}) \simeq a_{1,0}(s,t) + a_{1,1}(s,t)(\check{s}-s) + a_{1,2}(s,t)(\check{t}-t) \doteq P_{\mathbf{a}_1}(\check{s}-s,\check{t}-t)$$

$$\begin{aligned}\log f_{ST}(\check{s},\check{t}) \simeq\; & a_{2,0}(s,t) + a_{2,1}(s,t)(\check{s}-s) + a_{2,2}(s,t)(\check{t}-t)\\ & + a_{2,3}(s,t)(\check{s}-s)^2 + a_{2,4}(s,t)(\check{t}-t)^2 + a_{2,5}(s,t)(\check{s}-s)(\check{t}-t)\\ & \doteq P_{\mathbf{a}_2}(\check{s}-s,\check{t}-t).\end{aligned}$$

Improved probit-transformation copula density estimators

Remark Vectors $\mathbf{a}_1(s,t) = (a_{1,0}(s,t), a_{1,1}(s,t), a_{1,2}(s,t))$ and $\mathbf{a}_2(s,t) \doteq (a_{2,0}(s,t), \ldots, a_{2,5}(s,t))$ are then estimated by solving a weighted maximum likelihood problem.

$$\tilde{\mathbf{a}}_p(s,t) = \arg\max_{\mathbf{a}_p} \left\{ \sum_{i=1}^n \mathbf{K}\left(\mathbf{H}_{ST}^{-1/2}\begin{pmatrix} s-\hat{S}_i \\ t-\hat{T}_i \end{pmatrix}\right) P_{\mathbf{a}_p}(\hat{S}_i - s, \hat{T}_i - t) \right.$$
$$\left. -n \iint_{\mathbb{R}^2} \mathbf{K}\left(\mathbf{H}_{ST}^{-1/2}\begin{pmatrix} s-\check{s} \\ t-\check{t} \end{pmatrix}\right) \exp\left(P_{\mathbf{a}_p}(\check{s}-s, \check{t}-t)\right) d\check{s}\, d\check{t} \right\},$$

The estimate of f_{ST} at (s,t) is then $\tilde{f}_{ST}^{(p)}(s,t) = \exp(\tilde{a}_{p,0}(s,t))$, for $p = 1, 2$.

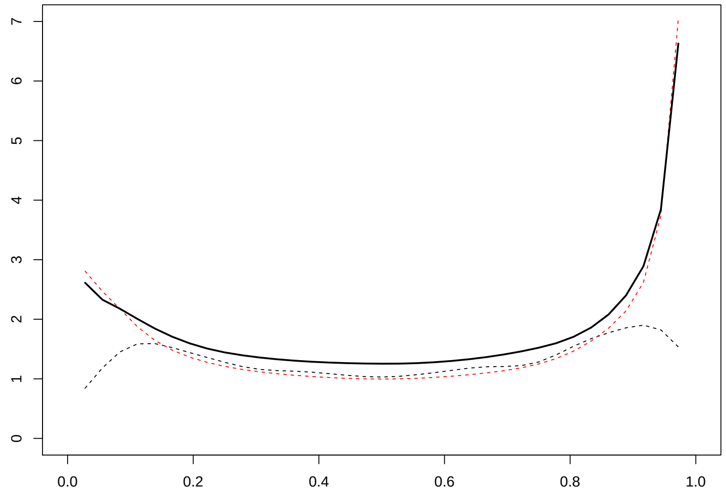
The Improved probit-transformation kernel copula density estimators are

$$\tilde{c}^{(\tau,p)}(u,v) = \frac{\tilde{f}_{ST}^{(p)}(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$

Improved probit-transformation copula density estimators

For the local log-linear ($p = 1$) approximation

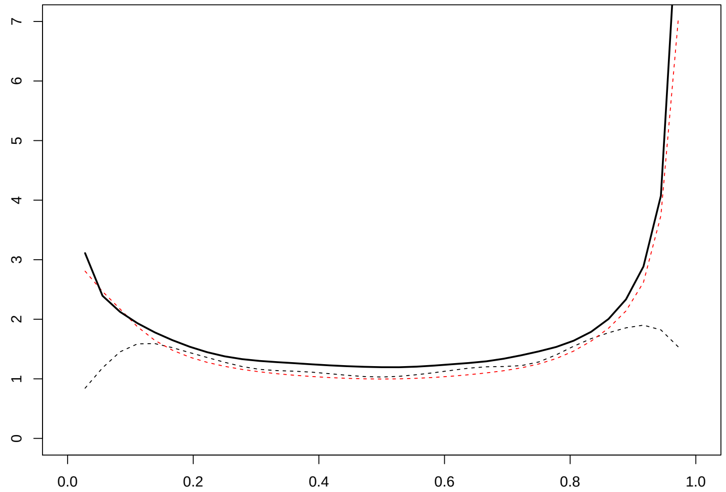
$$\tilde{c}^{(\tau,1)}(u,v) = \frac{\exp(\tilde{a}_{1,0}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$



Improved probit-transformation copula density estimators

For the local log-quadratic ($p = 2$) approximation

$$\tilde{c}^{(\tau,2)}(u,v) = \frac{\exp(\tilde{a}_{2,0}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$



Asymptotic properties

A1. The sample $\{(X_i, Y_i)\}$ is a n- i.i.d. sample from the joint distribution F_{XY}, an absolutely continuous distribution with marginals F_X and F_Y strictly increasing on their support ;

(uniqueness of the copula)

Asymptotic properties

A2. The copula C of F_{XY} is such that $(\partial C/\partial u)(u,v)$ and $(\partial^2 C/\partial u^2)(u,v)$ exist and are continuous on $\{(u,v) : u \in (0,1), v \in [0,1]\}$, and $(\partial C/\partial v)(u,v)$ and $(\partial^2 C/\partial v^2)(u,v)$ exist and are continuous on $\{(u,v) : u \in [0,1], v \in (0,1)\}$. In addition, there are constants K_1 and K_2 such that

$$\begin{cases} \left|\dfrac{\partial^2 C}{\partial u^2}(u,v)\right| \leq \dfrac{K_1}{u(1-u)} & \text{for } (u,v) \in (0,1)\times[0,1]; \\ \left|\dfrac{\partial^2 C}{\partial v^2}(u,v)\right| \leq \dfrac{K_2}{v(1-v)} & \text{for } (u,v) \in [0,1]\times(0,1); \end{cases}$$

A3. The density c of C exists, is positive and admits continuous second-order partial derivatives on the interior of the unit square $\mathcal{I}$. In addition, there is a constant K_{00} such that

$$c(u,v) \leq K_{00} \min\left(\frac{1}{u(1-u)}, \frac{1}{v(1-v)}\right) \qquad \forall (u,v) \in (0,1)^2.$$

see Segers (2012).

Asymptotic properties

Assume that $\mathbf{K}(z_1, z_2) = \phi(z_1)\phi(z_2)$ and $\mathbf{H}_{ST} = h^2\mathbf{I}$ with $h \sim n^{-a}$ for some $a \in (0, 1/4)$. Under Assumptions A1-A3, the 'naive' probit transformation kernel copula density estimator at any $(u, v) \in (0, 1)^2$ is such that

$$\sqrt{nh^2}\left(\hat{c}^{(\tau)}(u,v) - c(u,v) - h^2\frac{b(u,v)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^2(u,v)\right),$$

$$\begin{aligned}\text{where } b(u,v) = \frac{1}{2}\Bigg\{ & \frac{\partial^2 c}{\partial u^2}(u,v)\phi^2(\Phi^{-1}(u)) + \frac{\partial^2 c}{\partial v^2}(u,v)\phi^2(\Phi^{-1}(v)) \\ & - 3\left(\frac{\partial c}{\partial u}(u,v)\Phi^{-1}(u)\phi(\Phi^{-1}(u)) + \frac{\partial c}{\partial v}(u,v)\Phi^{-1}(v)\phi(\Phi^{-1}(v))\right) \\ & + c(u,v)\left(\{\Phi^{-1}(u)\}^2 + \{\Phi^{-1}(v)\}^2 - 2\right)\Bigg\} \qquad (2)\end{aligned}$$

and $\sigma^2(u,v) = \dfrac{c(u,v)}{4\pi\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$.

The Amended version

The last unbounded term in b be easily adjusted.

$$\hat{c}^{(\tau\mathrm{am})}(u,v)=\frac{\hat{f}_{ST}(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}\times\frac{1}{1+\frac{1}{2}h^2\left(\{\Phi^{-1}(u)\}^2+\{\Phi^{-1}(v)\}^2-2\right)}.$$

The asymptotic bias becomes proportional to

$$b^{(\mathrm{am})}(u,v)=\frac{1}{2}\left\{\frac{\partial^2 c}{\partial u^2}(u,v)\phi^2(\Phi^{-1}(u))+\frac{\partial^2 c}{\partial v^2}(u,v)\phi^2(\Phi^{-1}(v))\right.$$
$$\left.-3\left(\frac{\partial c}{\partial u}(u,v)\Phi^{-1}(u)\phi(\Phi^{-1}(u))+\frac{\partial c}{\partial v}(u,v)\Phi^{-1}(v)\phi(\Phi^{-1}(v))\right)\right\}.$$

A local log-linear probit-transformation kernel estimator

$$\tilde{c}^{*(\tau,1)}(u,v) = \tilde{f}^{*(1)}_{ST}(\Phi^{-1}(u),\Phi^{-1}(v))/\left(\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))\right)$$

Then

$$\sqrt{nh^2}\left(\tilde{c}^{*(\tau,1)}(u,v) - c(u,v) - h^2\frac{b^{(1)}(u,v)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0,\sigma^{(1)\,2}(u,v)\right),$$

$$\text{where}\quad b^{(1)}(u,v) = \frac{1}{2}\left\{\frac{\partial^2 c}{\partial u^2}(u,v)\phi^2(\Phi^{-1}(u)) + \frac{\partial^2 c}{\partial v^2}(u,v)\phi^2(\Phi^{-1}(v))\right.$$
$$-\frac{1}{c(u,v)}\left(\left\{\frac{\partial c}{\partial u}(u,v)\right\}^2\phi^2(\Phi^{-1}(u)) + \left\{\frac{\partial c}{\partial v}(u,v)\right\}^2\phi^2(\Phi^{-1}(v))\right)$$
$$\left.-\left(\frac{\partial c}{\partial u}(u,v)\Phi^{-1}(u)\phi(\Phi^{-1}(u)) + \frac{\partial c}{\partial v}(u,v)\Phi^{-1}(v)\phi(\Phi^{-1}(v))\right) - 2c(u,v)\right\}$$

Using a higher order polynomial approximation

Locally fitting a polynomial of a higher degree is known to reduce the asymptotic bias of the estimator, here from order $O(h^2)$ to order $O(h^4)$, see Loader (1996) or Hjort (1996), under sufficient smoothness conditions.

If f_{ST} admits continuous fourth-order partial derivatives and is positive at (s,t), then

$$\sqrt{nh^2}\left(\tilde{f}^{*(2)}_{ST}(s,t) - f_{ST}(s,t) - h^4 b^{(2)}_{ST}(s,t)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, {\sigma^{(2)}_{ST}}^2(s,t)\right),$$

where ${\sigma^{(2)}_{ST}}^2(s,t) = \dfrac{5}{2}\,\dfrac{f_{ST}(s,t)}{4\pi}$ and

$$b^{(2)}_{ST}(s,t) = -\frac{1}{8} f_{ST}(s,t)$$

$$\times\left\{\left(\frac{\partial^4 g}{\partial s^4} + \frac{\partial^4 g}{\partial t^4}\right) + 4\left(\frac{\partial^3 g}{\partial s^3}\frac{\partial g}{\partial s} + \frac{\partial^3 g}{\partial t^3}\frac{\partial g}{\partial t} + \frac{\partial^3 g}{\partial s^2 \partial t}\frac{\partial g}{\partial t} + \frac{\partial^3 g}{\partial s \partial t^2}\frac{\partial g}{\partial s}\right) + 2\frac{\partial^4 g}{\partial s^2 \partial t^2}\right\}(s,t),$$

with $g(s,t) = \log f_{ST}(s,t)$.

Using a higher order polynomial approximation

A4. The copula density $c(u,v) = (\partial^2 C/\partial u \partial v)(u,v)$ admits continuous fourth-order partial derivatives on the interior of the unit square $[0,1]^2$.

Then

$$\sqrt{nh^2}\left(\tilde{c}^{*(\tau,2)}(u,v) - c(u,v) - h^4 \frac{b^{(2)}(u,v)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, {\sigma^{(2)}}^2(u,v)\right)$$

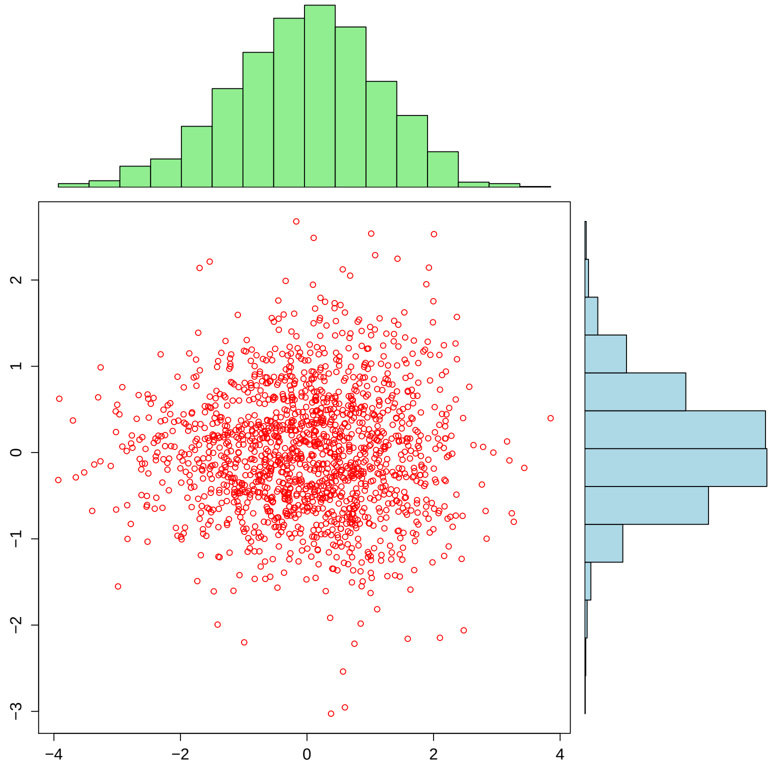
where ${\sigma^{(2)}}^2(u,v) = \dfrac{5}{2}\dfrac{c(u,v)}{4\pi\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$

Improving Bandwidth choice

Consider the principal components decomposition of the $(n \times 2)$ matrix $[\hat{\boldsymbol{S}}, \hat{\boldsymbol{T}}] = \boldsymbol{M}$.

Let $W_1 = (W_{11}, W_{12})^\top$ and $W_2 = (W_{21}, W_{22})^\top$ be the eigenvectors of $\boldsymbol{M}^\top \boldsymbol{M}$. Set

$$\begin{pmatrix} Q \\ R \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} S \\ T \end{pmatrix} = \mathbf{W} \begin{pmatrix} S \\ T \end{pmatrix}$$



which is only a linear reparametrization of $\mathbb{R}^2$, so an estimate of f_{ST} can be readily obtained from an estimate of the density of (Q, R)

Since $\{\hat{Q}_i\}$ and $\{\hat{R}_i\}$ are empirically uncorrelated, consider a diagonal bandwidth matrix $\mathbf{H}_{QR} = \text{diag}(h_Q^2, h_R^2)$.

Improving Bandwidth choice

Use univariate procedures to select h_Q and h_R independently

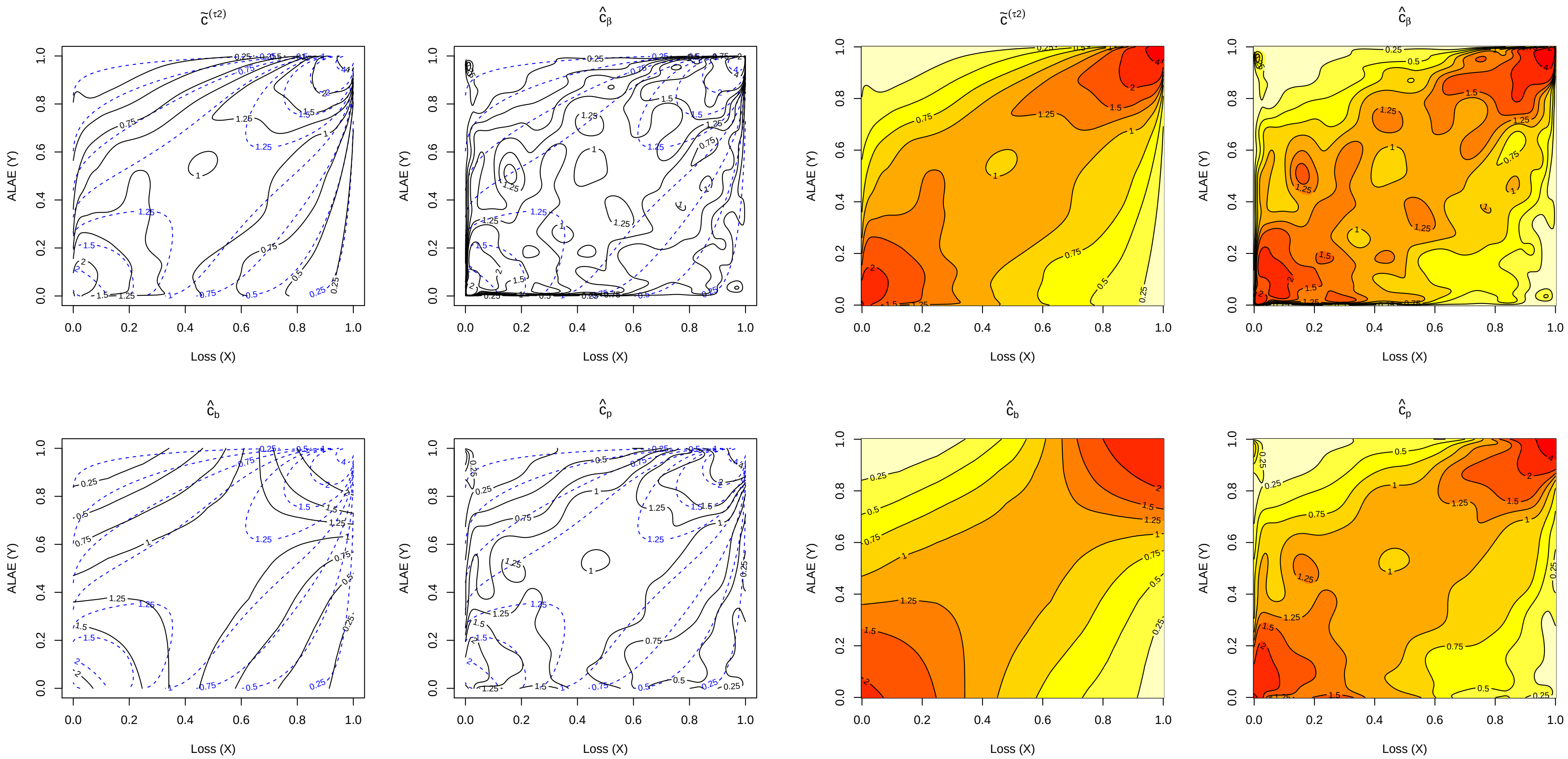
Denote $\tilde{f}_Q^{(p)}$ and $\tilde{f}_R^{(p)}$ $(p = 1, 2)$, the local log-polynomial estimators for the densities

h_Q can be selected via cross-validation (see Section 5.3.3 in Loader (1999))

$$h_Q = \arg\min_{h>0} \left\{ \int_{-\infty}^{\infty} \left\{ \tilde{f}_Q^{(p)}(q) \right\}^2 dq - \frac{2}{n} \sum_{i=1}^{n} \tilde{f}_{Q(-i)}^{(p)}(\hat{Q}_i) \right\},$$

where $\tilde{f}_{Q(-i)}^{(p)}$ is the 'leave-one-out' version of $\tilde{f}_Q^{(p)}$.

Graphical Comparison (loss ALAE dataset)



Simulation Study

$M = 1,000$ independent random samples $\{(U_i, V_i)\}_{i=1}^n$ of sizes $n = 200$, $n = 500$ and $n = 1000$ were generated from each of the following copulas :

- the independence copula (i.e., U_i's and V_i's drawn independently) ;
- the Gaussian copula, with parameters $\rho = 0.31$, $\rho = 0.59$ and $\rho = 0.81$;
- the Student t-copula with 4 degrees of freedom, with parameters $\rho = 0.31$, $\rho = 0.59$ and $\rho = 0.81$;
- the Frank copula, with parameter $\theta = 1.86$, $\theta = 4.16$ and $\theta = 7.93$;
- the Gumbel copula, with parameter $\theta = 1.25$, $\theta = 1.67$ and $\theta = 2.5$;
- the Clayton copula, with parameter $\theta = 0.5$, $\theta = 1.67$ and $\theta = 2.5$.

(approximated) MISE relative to the MISE of the mirror-reflection estimator (last column), $n = 1000$. Bold values show the minimum MISE for the corresponding copula (non-significantly different values are highlighted as well).

$\boxed{n = 1000}$	$\hat{c}^{(\tau)}$	$\hat{c}^{(\tau am)}$	$\tilde{c}^{(\tau,1)}$	$\tilde{c}^{(\tau,2)}$	$\hat{c}_1^{(\beta)}$	$\hat{c}_2^{(\beta)}$	$\hat{c}_1^{(B)}$	$\hat{c}_2^{(B)}$	$\hat{c}_1^{(p)}$	$\hat{c}_2^{(p)}$	$\hat{c}_3^{(p)}$
Indep	3.57	2.80	2.89	1.40	7.96	11.65	1.69	3.43	1.62	0.50	**0.14**
Gauss2	2.03	1.52	1.60	0.76	4.63	6.06	1.10	1.82	0.98	**0.66**	0.89
Gauss4	0.63	0.49	0.44	**0.21**	1.72	1.60	0.75	0.58	0.62	0.99	2.93
Gauss6	0.21	0.20	0.11	**0.05**	0.74	0.33	0.77	0.37	0.72	1.21	2.83
Std(4)2	0.61	0.56	0.50	**0.40**	1.57	1.80	0.78	0.67	0.75	1.01	1.88
Std(4)4	0.21	0.27	**0.17**	**0.15**	0.88	0.51	0.75	0.42	0.75	1.12	2.07
Std(4)6	**0.09**	0.17	**0.08**	**0.09**	0.70	0.19	0.82	0.47	0.90	1.17	1.90
Frank2	3.31	2.42	2.57	1.35	7.16	9.63	1.70	2.95	1.31	**0.45**	**0.49**
Frank4	2.35	1.45	1.51	0.99	4.42	4.89	1.49	1.65	**0.60**	0.72	6.14
Frank6	0.96	0.52	**0.45**	**0.44**	1.51	1.19	1.35	0.76	0.65	1.58	7.25
Gumbel2	0.65	0.62	0.56	**0.43**	1.77	1.97	0.82	0.75	0.83	1.03	1.52
Gumbel4	**0.18**	0.28	**0.16**	**0.19**	0.89	0.41	0.78	0.47	0.81	1.10	1.78
Gumbel6	**0.09**	0.21	**0.10**	0.15	0.78	0.29	0.85	0.58	0.94	1.12	1.63
Clayton2	0.63	0.60	0.51	**0.34**	1.78	1.99	0.78	0.70	0.79	1.04	1.79
Clayton4	**0.11**	0.26	**0.10**	**0.15**	0.79	0.27	0.83	0.56	0.90	1.10	1.50
Clayton6	**0.11**	0.28	**0.08**	0.15	0.82	0.35	0.88	0.67	0.96	1.09	1.36

Probit Transform in the Univariate Case : the log transform

See Log-Transform Kernel Density Estimation of Income Distribution with E. Flachaire

The Gaussian kernel estimator of a density is

$$\widehat{f}_Z(z) = \frac{1}{n}\sum_{i=1}^{n}\varphi(z; z_i, h)$$

where $\varphi(\cdot;\mu,\sigma)$ is the density of the normal distribution. Use a Gaussian kernel estimation of the density using a logarithmic transformation of data x_i's

Density Estimation for Statistics and Data Analysis

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Another possible approach is to transform the data, for example by taking logarithms as in the example given in Section 2.9 above. If the density estimated from the logarithms of the data is $\hat{g}$, then standard arguments lead to

$$\hat{f}(x) = \frac{1}{x}\,\hat{g}(\log x) \qquad \text{for } x > 0.$$

It is of course the presence of the multiplier $1/x$ that gives rise to the spike in Fig. 2.13; notwithstanding difficulties of this kind, Copas and Fryer (1980) did find estimates based on logarithmic transforms to be very useful with some other data sets.

$$\widehat{f}_X(x) = \frac{\widehat{f}_Z(\log(x))}{x} = \frac{1}{n}\sum_{i=1}^{n} h(x; \log x_i, h)$$

where $h(\cdot;\mu,\sigma)$ is the density of the lognormal distribution.

Probit Transform in the Univariate Case : the log transform

Recall that classically $\text{bias}[\widehat{f}_Z(z)] \sim \dfrac{h^2}{2} f''_Z(z)$ and $\text{Var}[\widehat{f}_Z(z)] \sim \dfrac{0.2821}{nh} f_Z(z)$

Here, in the neighborhood of 0,

$$\text{bias}[\widehat{f}_X(x)] \sim \frac{h^2}{2}\left(f_X(x) + 3x \cdot f'_X(x) + x^2 \cdot f''_X(x)\right)$$

which is positive if $f_X(0) > 0$, while

$$\text{Var}[\widehat{f}_X(x)] \sim \frac{0.2821}{nhx} f_X(x)$$

The log-transform kernel may perform poorly when $f_X(0) > 0$, see Silverman (1986).

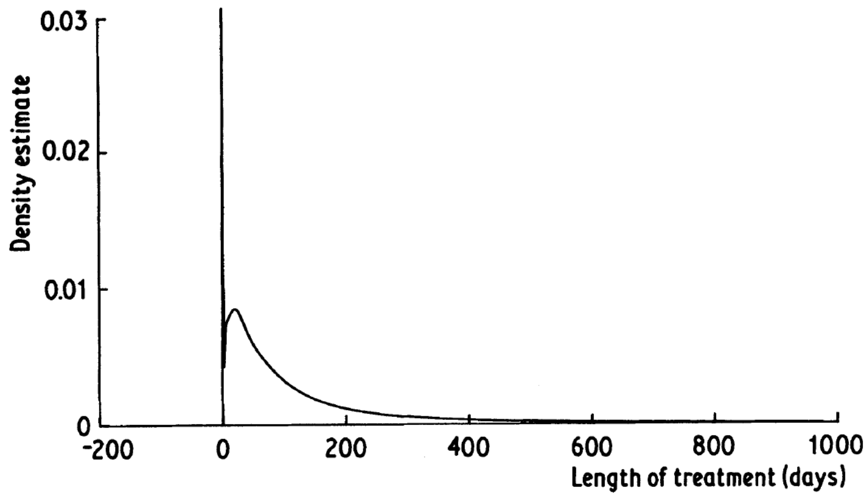


Fig. 2.13 *Log-normal weight function estimate for suicide study data, obtained by transformation of Fig. 2.12. Note that the vertical scale differs from that used in previous figures for this data set.*

Back on the Transformed Kernel

See Devroye & Györfi (1985), and Devroye & Lugosi (2001)

CHAPTER 9

The Transformed Kernel Estimate

... use the transformed kernel the other way, $\mathbb{R} \to [0,1] \to \mathbb{R}$

The *transformed kernel estimate* (Devroye et al., 1983) is based upon a transformation T: $R^1 \to [0,1]$ which is strictly monotonically increasing, continuously differentiable, one-to-one and onto, and which has a continuously differentiable inverse. The transformed data sequence is $Y_1, \ldots, Y_n$, where $Y_i = T(X_i)$. Note that Y_1 has density

$$g(x) = f(T^{-1}(x))T^{-1\prime}(x).$$

Now, g is estimated by g_n from $Y_1, \ldots, Y_n$, and f is estimated by

$$f_n(x) = g_n(T(x))T'(x). \tag{2}$$

Back on the Transformed Kernel

Interesting point, the optimal T should be F,

The only unknown in the design at this moment is our transformation T. We point out that for a transformed histogram estimate, the optimal T gives a uniform [0, 1] density and should therefore be equal to $T(x) = F(x)$, all x. The h to be used in the histogram estimate is $(2\pi n)^{-1/3}$ (Table 5.1).

thus, T can be $F_{\widehat{\theta}}$

The key observation is that if g_n is a density on [0, 1], the f_n is a density on R^1, and furthermore,

$$\int |f_n - f| = \int |g_n - g|.$$

Consistency 251

For variable transformations T, we must worry about the consistency of the resulting estimate.

The transformation $Y_i = T(X_i)$ is usually of the form

$$Y_i = T_n(X_i; X_1, \ldots, X_n),$$

Heavy Tailed distribution

Let X denote a (heavy-tailed) random variable with tail index $\alpha \in (0, \infty)$, i.e.

$$\mathbb{P}(X > x) = x^{-\alpha}\mathcal{L}_1(x)$$

where $\mathcal{L}_1$ is some regularly varying function.

Let T denote a $\mathbb{R} \to [0, 1]$ function, such that $1 - T$ is regularly varying at infinity, with tail index $\beta \in (0, \infty)$.

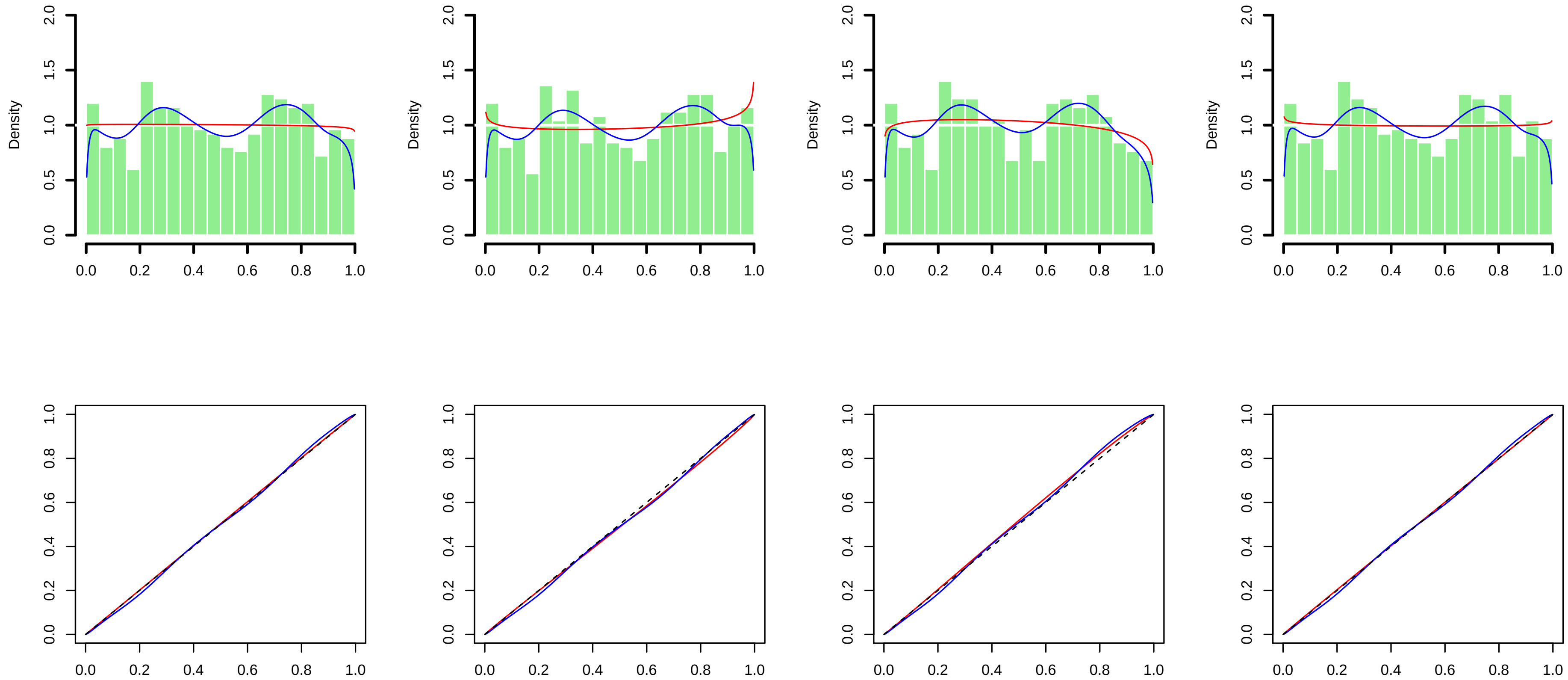
Define $Q(x) = T^{-1}(1 - x^{-1})$ the associated tail quantile function, then $Q(x) = x^{1/\beta}\mathcal{L}_2^\star(1/x)$, where $\mathcal{L}_2^\star$ is some regularly varying function (the de Bruyn conjugate of the regular variation function associated with T). Assume here that $Q(x) = bx^{1/\beta}$

Let $U = T(X)$. Then, as $u \to 1$

$$\mathbb{P}(U > u) \sim (1 - u)^{\alpha/\beta}.$$

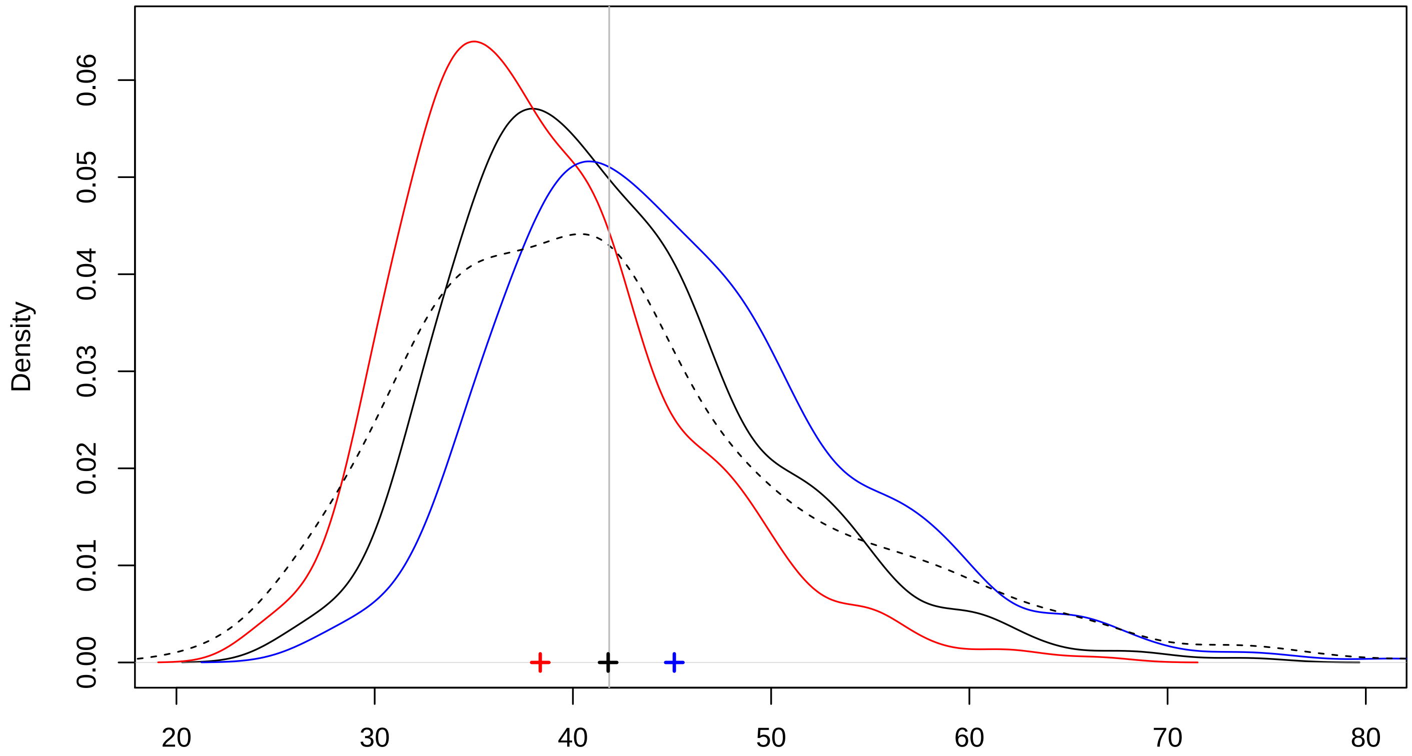
Heavy Tailed distribution

see C. & Oulidi (2010), $\alpha = 0.75^{-1}$, $T_{0.75^{-1}}$, $\underbrace{T_{0.65^{-1}}}_{\text{lighter}}$, $\underbrace{T_{0.85^{-1}}}_{\text{heavier}}$ and $T_{\hat{\alpha}}$



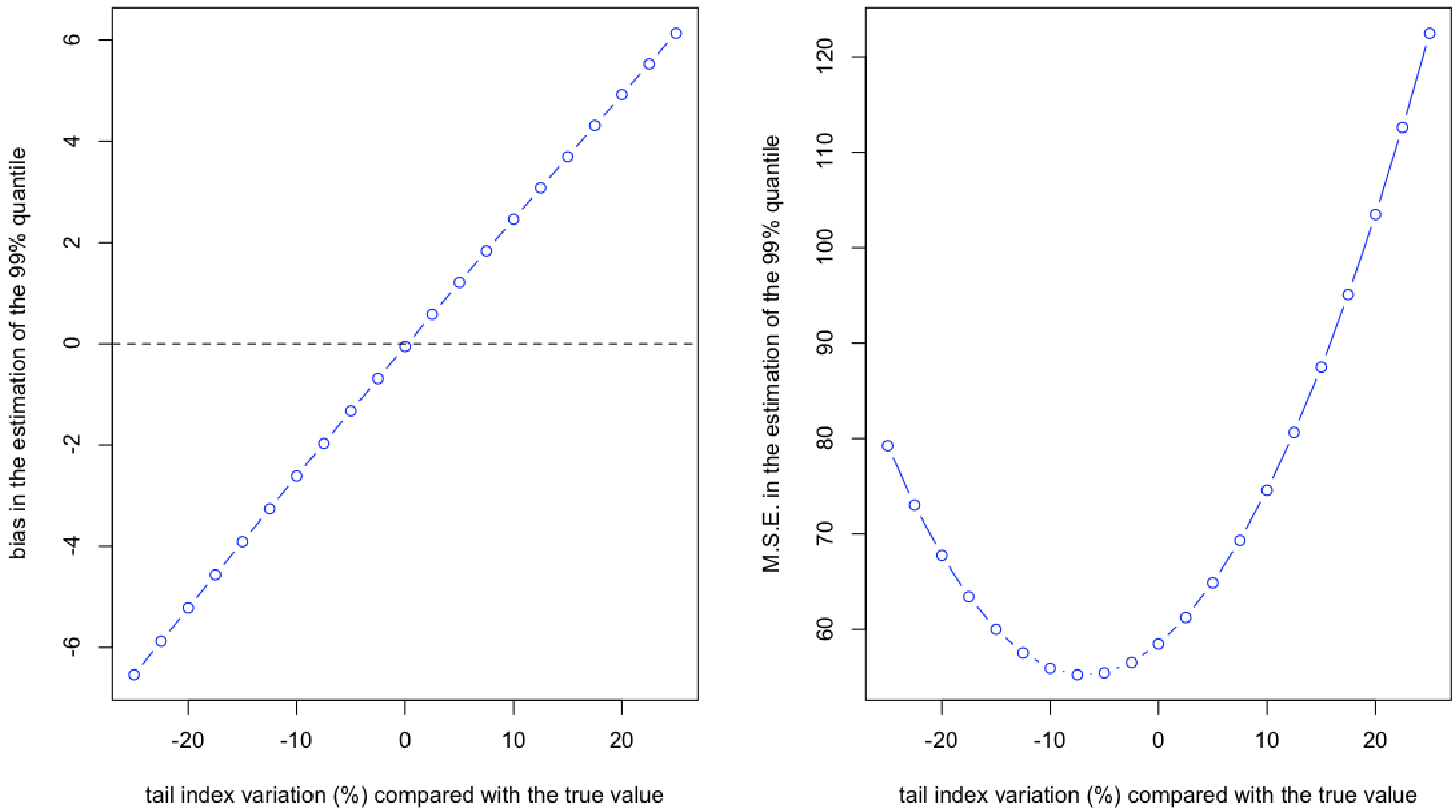
Heavy Tailed distribution

see C. & Oulidi (2007) Beta kernel quantile estimators of heavy-tailed loss distributions, impact on quantile estimation ?



Heavy Tailed distribution

see C. & Oulidi (2007), impact on quantile estimation ? bias ? m.s.e. ?



Bimodal distribution

Let X denote a bimodal distribution, obtained from a mixture

$$X \sim F_\Theta \begin{cases} F_0 \text{ if } \Theta = 0, & \text{(probability } p_0) \\ F_1 \text{ if } \Theta = 1, & \text{(probability } p_1) \end{cases}$$

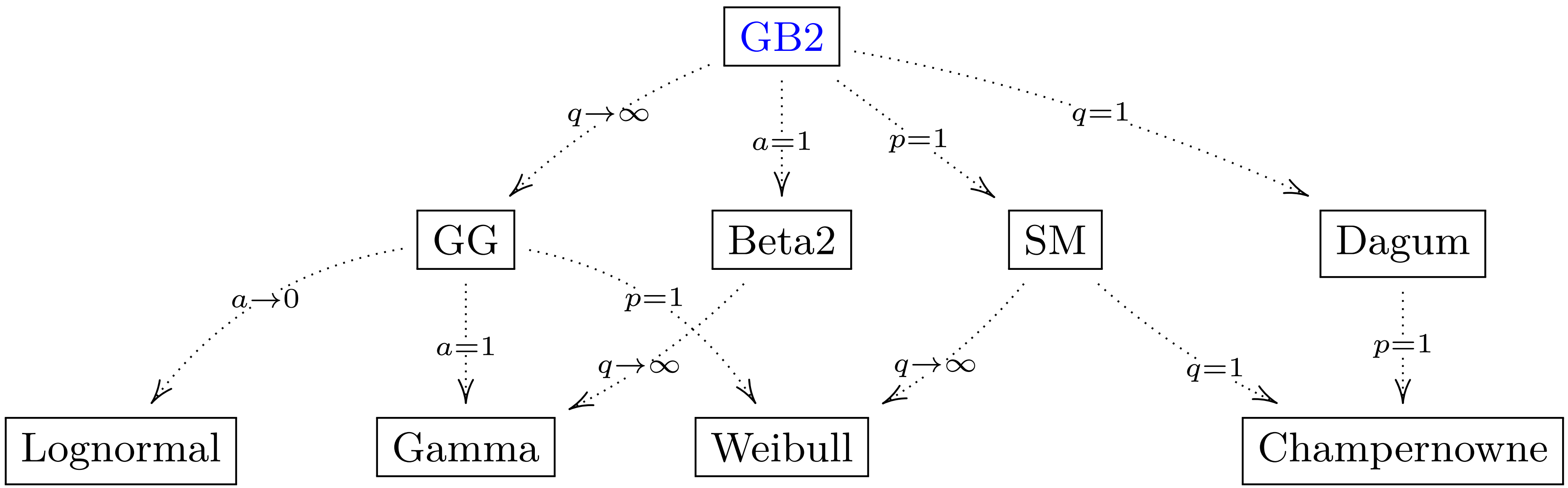
Idea : $T(X)$ can be obtained as transformation of two distributions on $[0,1]$,

$$T(X) \sim G_\Theta \begin{cases} G_0 \text{ if } \Theta = 0, & \text{(probability } p_0) \\ G_1 \text{ if } \Theta = 1, & \text{(probability } p_1) \end{cases}$$

$\rightarrow$ standard for income observations...

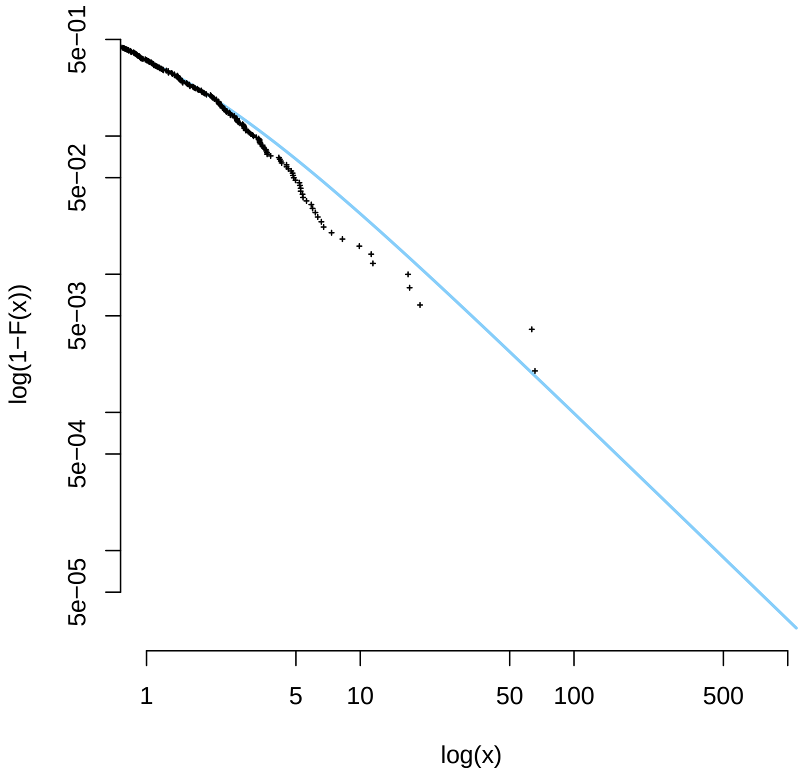
Which transformation ?

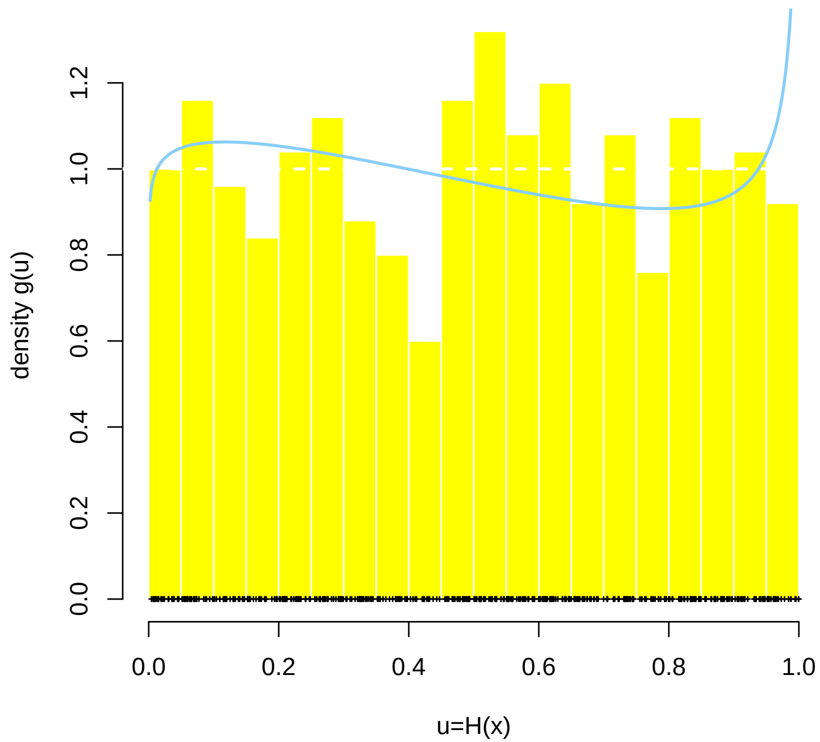
$$\text{GB2}: \quad t(y;a,b,p,q) = \frac{|a|y^{ap-1}}{b^{ap}\mathrm{B}(p,q)[1+(y/b)^a]^{p+q}}, \qquad \text{for } y>0,$$



Example, $X \sim$ Pareto

Pareto plot ($\log \overline{F}(x)$ vs. $\log x$), and histogram of $\{U_1, \cdots, U_n\}$, $U_i = T_{\hat{\theta}}(X_i)$

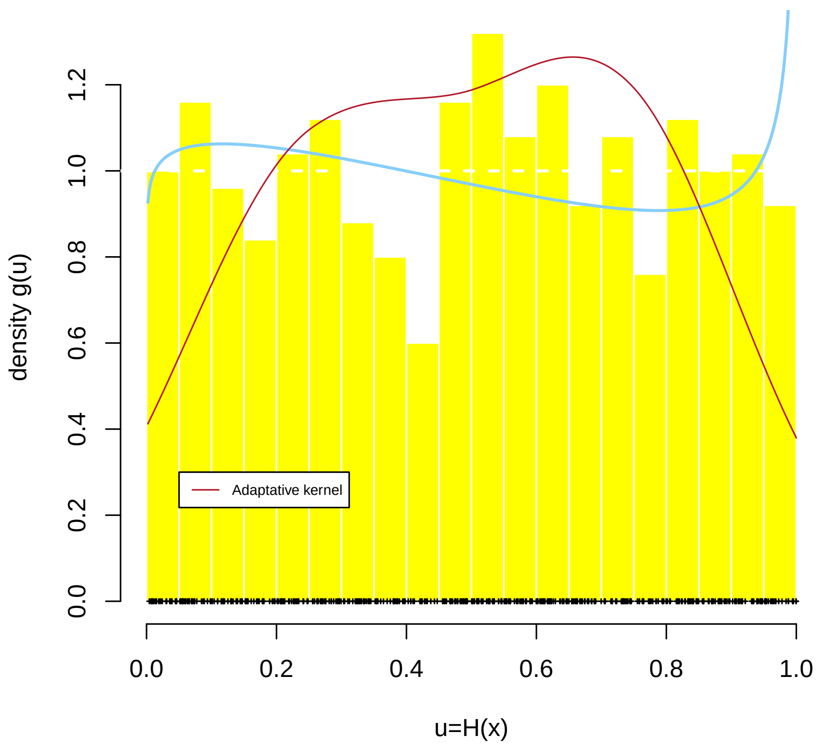


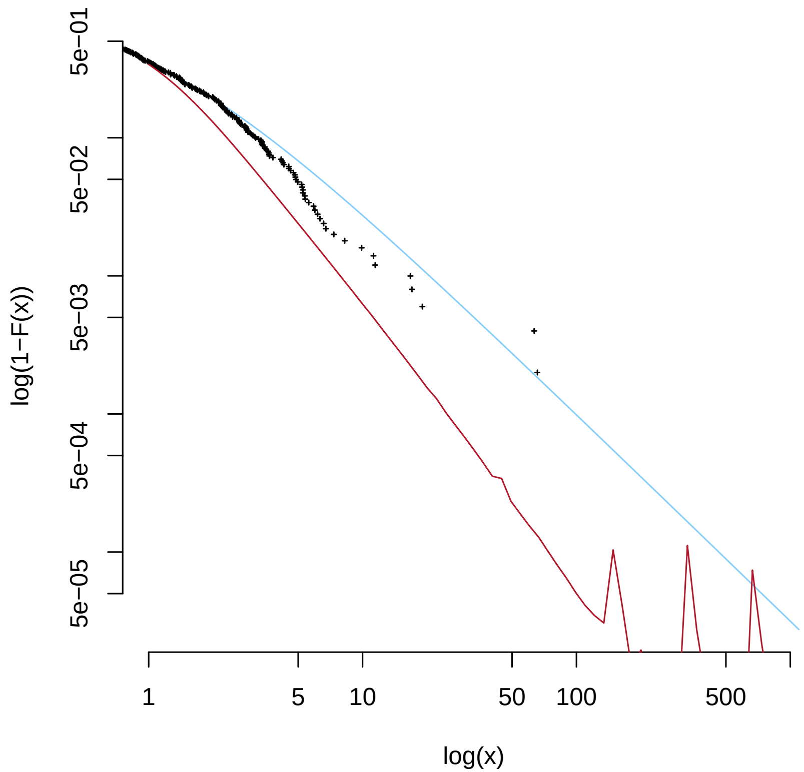


Example, $X \sim$ Pareto

Estimation of the density g of $U = T_{\widehat{\theta}}(X)$, and estimated c.d.f of X,

$$\widehat{F}_n(x) = \int_0^x \widehat{f}_n(y)dy \text{ where } \widehat{f}_n(y) = \widehat{g}_n(T_{\hat{\theta}}(y)) \cdot t_{\hat{\theta}}(y)$$





Beta kernel

$$\widehat{g}(u) = \sum_{i=1}^{n} \frac{1}{n} \cdot b\left(u; \frac{U_i}{h}, \frac{1-U_i}{h}\right) \quad u \in [0,1].$$

with some possible boundary correction, as suggested in Chen (1999),

$$\frac{u}{h} \to \rho(u,h) = 2h^2 + 2.5 - (4h^4 + 6h^2 + 2.25 - u^2 - u/h)^{1/2}$$

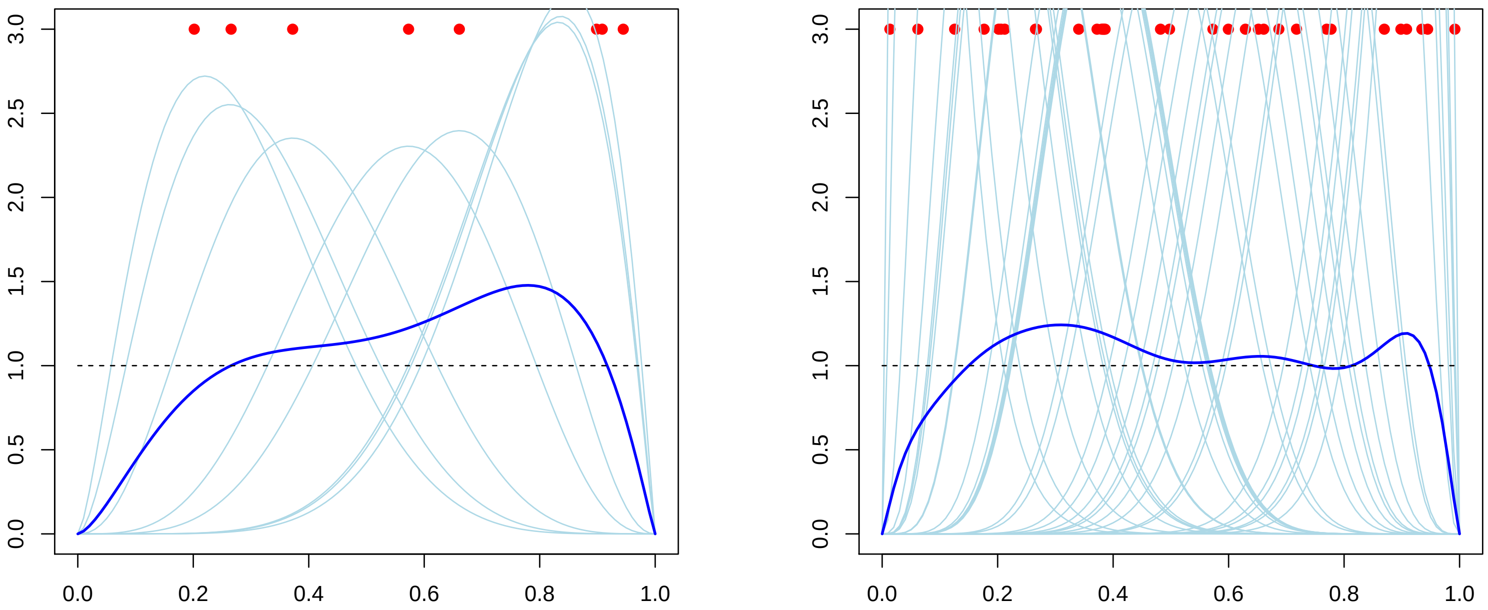
Problem : choice of the bandwidth $h^\star$? Standard loss function

$$L(h) = \int [\widehat{g}_n(u) - g(u)]^2 du = \underbrace{\int [\widehat{g}_n(u)]^2 du - 2\int \widehat{g}_n(u) \cdot g(u) du}_{CV(h)} + \int [g(u)]^2 du$$

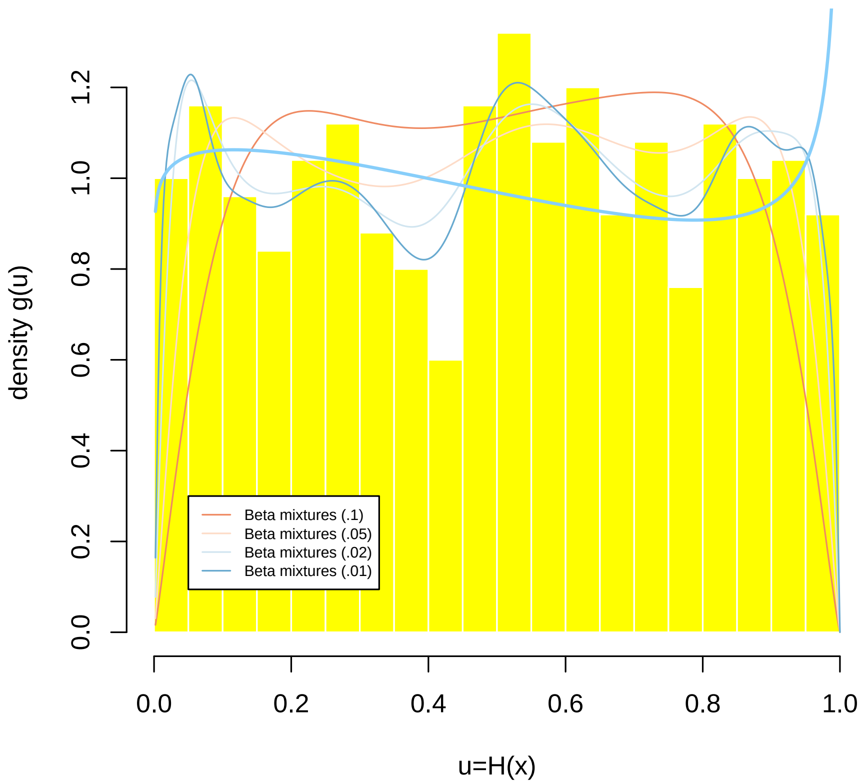
where

$$\widehat{CV}(h) = \left(\int \widehat{g}_n(u) du\right)^2 - \frac{2}{n}\sum_{i=1}^{n} \widehat{g}_{(-i)}(U_i)$$

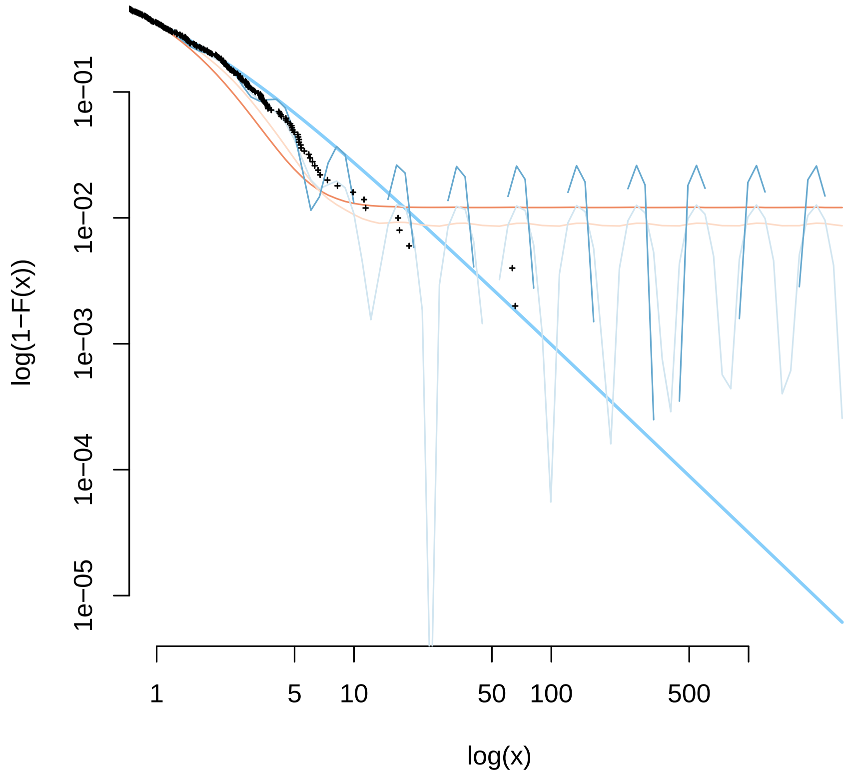
Beta kernel



Beta kernel



density g(u)
u=H(x)
0.0
0.2
0.4
0.6
0.8
1.0
1.2
Beta mixtures (.1)
Beta mixtures (.05)
Beta mixtures (.02)
Beta mixtures (.01)



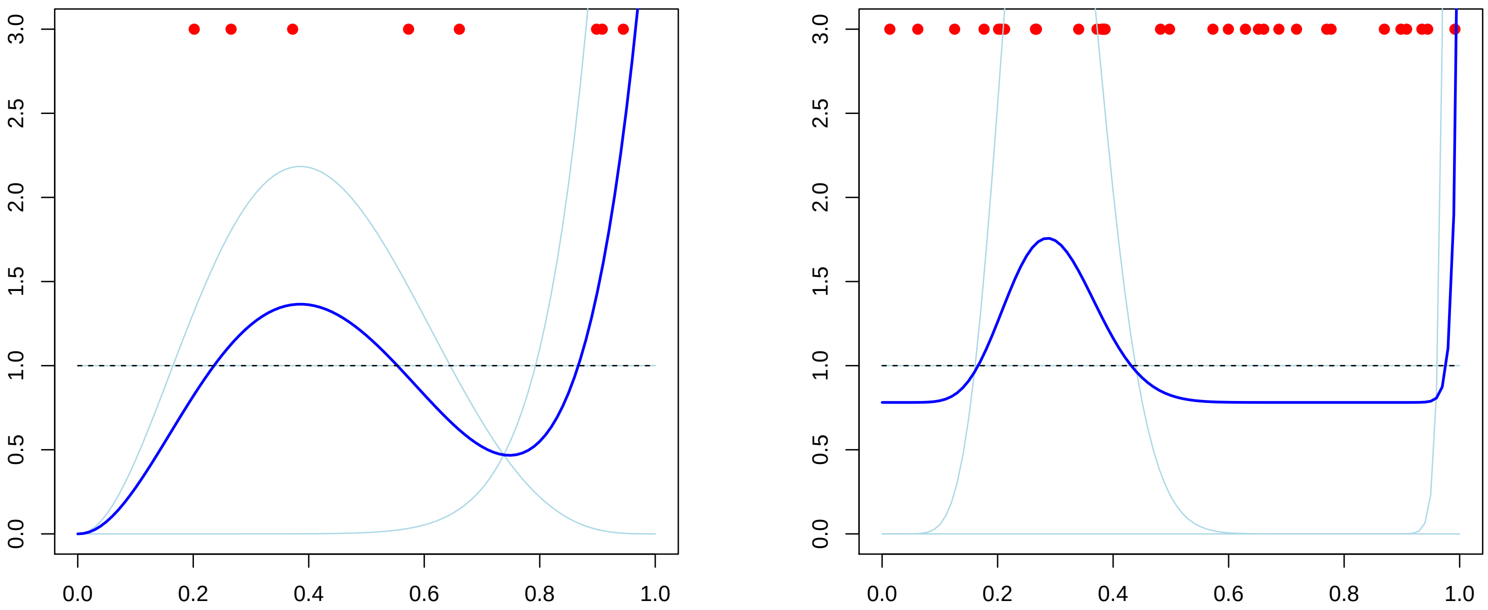
log(1−F(x))
log(x)
1e−01
1e−02
1e−03
1e−04
1e−05
1
5
10
50
100
500

Mixture of Beta distributions

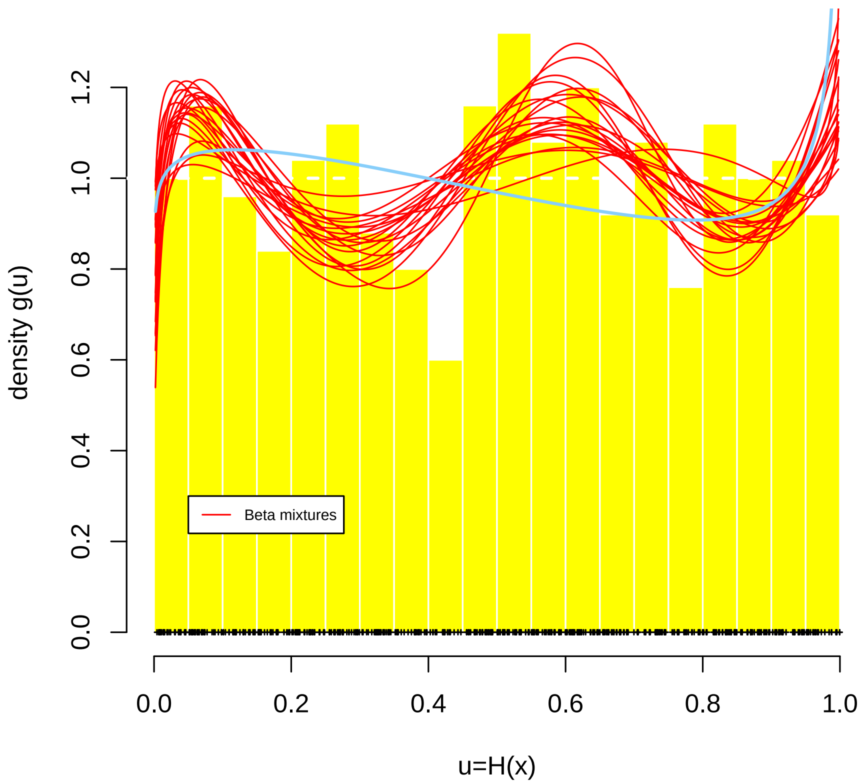
$$\widehat{g}(u)=\sum_{j=1}^{k}\widehat{\pi}_j\cdot b\left(u;\widehat{\alpha}_j,\widehat{\beta}_j\right)\quad u\in[0,1].$$

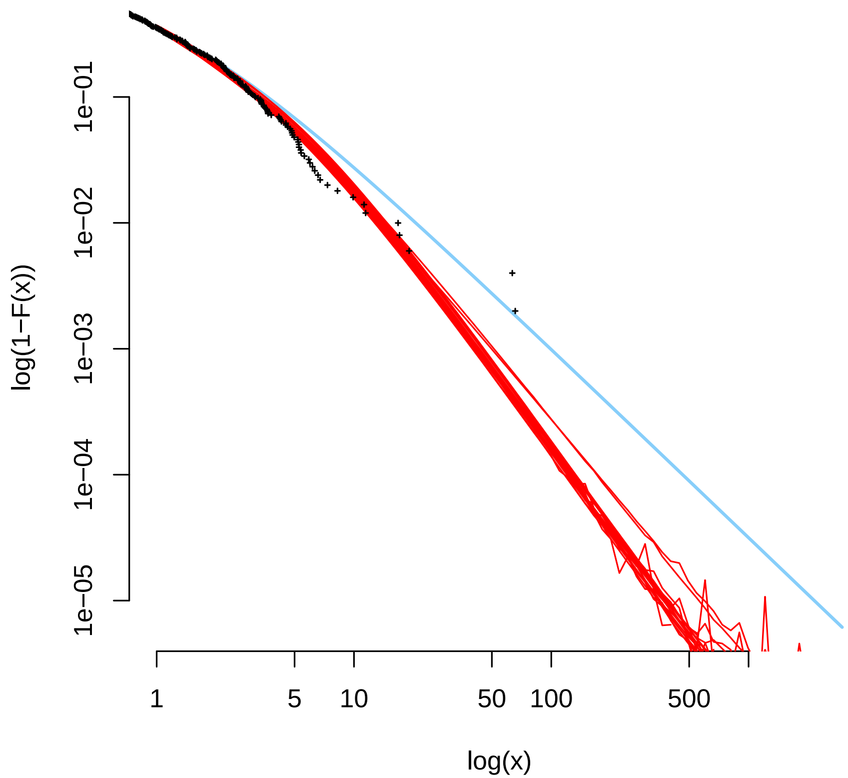
Problem : choice the number of components k (and estimation...). Use of stochastic EM algorithm (or sort of) see Celeux & Diebolt (1985).

Mixture of Beta distributions



Mixture of Beta distributions



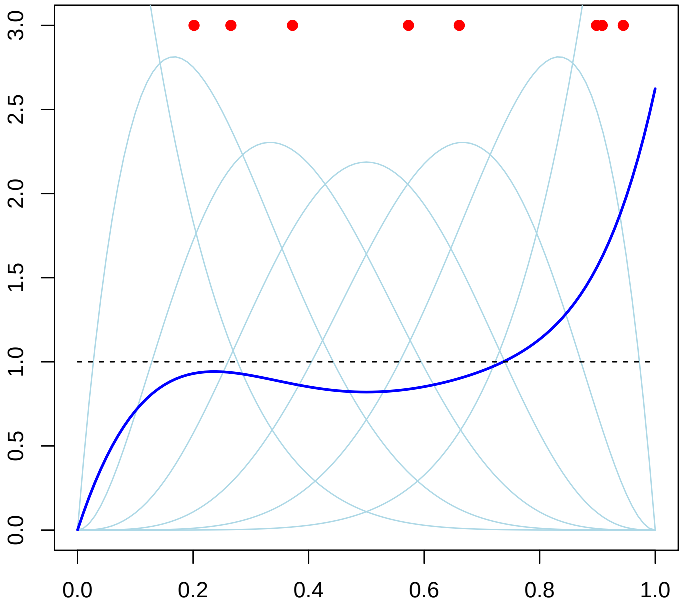


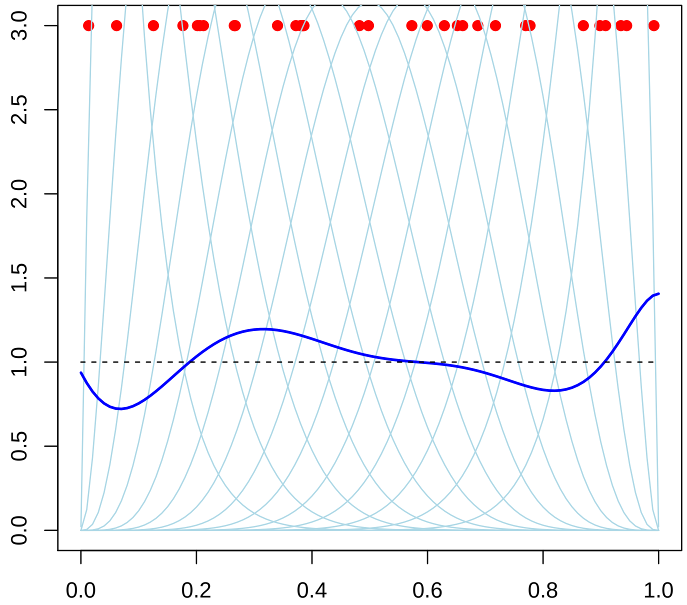
Bernstein approximation

$$\widehat{g}(u)=\sum_{k=1}^{m}[m\omega_k]\cdot b\,(u;k,m-k)\ \ u\in[0,1].$$

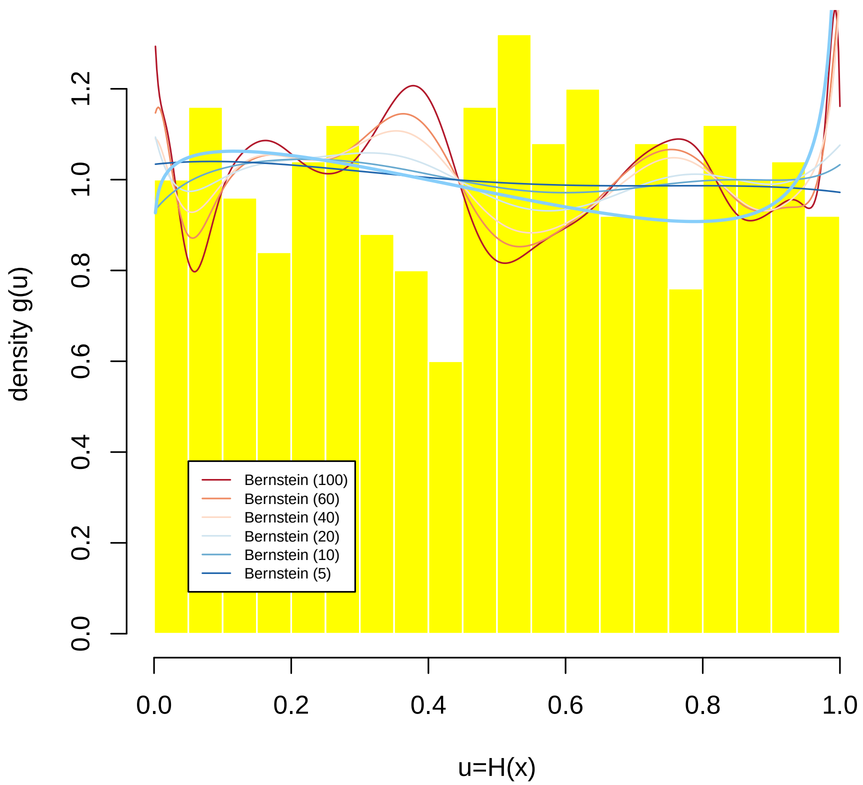
where $\omega_k=\widehat{G}\left(\frac{k}{m}\right)-\widehat{G}\left(\frac{k-1}{m}\right)$.

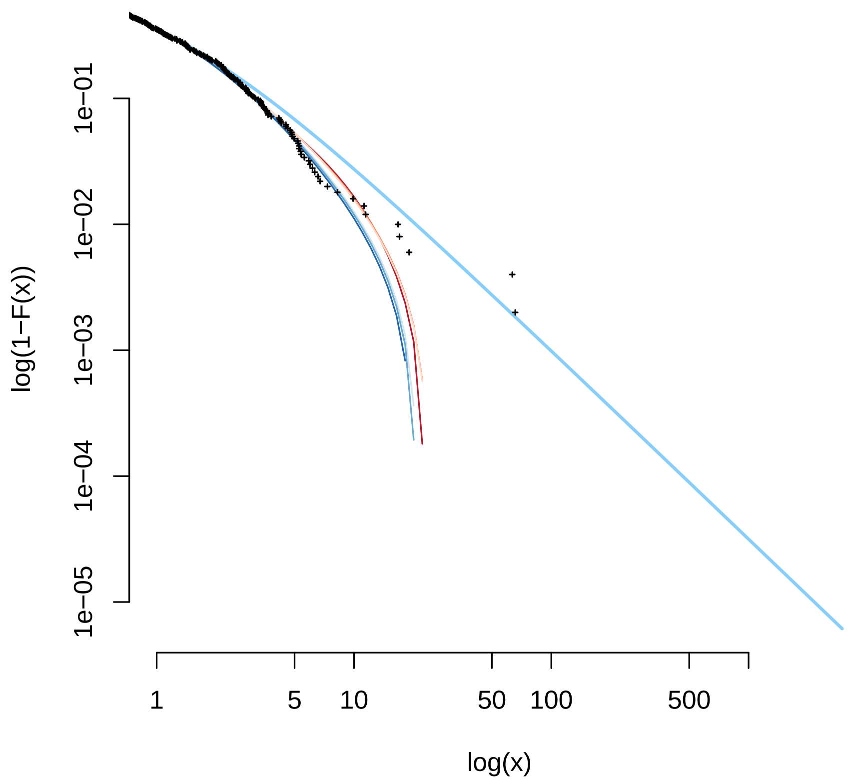
Bernstein approximation





Bernstein approximation





Quantities of interest

Standard statistical quantities

- miae, $\left(\int_0^\infty \left|\widehat{f}_n(x) - f(x)\right| \,\mathrm{d}x\right)$
- mise, $\left(\int_0^\infty \left[\widehat{f}_n(x) - f(x)\right]^2 \,\mathrm{d}x\right)$

Inequality indices and risk measures, based on $F(x) = \displaystyle\int_0^x f(t)dt$,

- Gini, $\displaystyle\frac{1}{\mu}\int_0^\infty F(t)[1 - F(t)]dt$
- Theil, $\displaystyle\int_0^\infty \frac{t}{\mu}\log\left(\frac{t}{\mu}\right) f(t)dt$
- VaR-quantile, x such that $F(x) = \mathbb{P}(X \leq x) = \alpha$, i.e. $F^{-1}(\alpha)$
- TVaR-expected shorfall, $\mathbb{E}[X|X > F^{-1}(\alpha)]$

where $\mu = \displaystyle\int_0^\infty [1 - F(x)]dx$.