Assurance collaborative, théorie des graphes et actuariat

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Colloque SCOR & Institut des Actuaires, 2022

SCOR FONDATION POUR LA SCIENCE en partenariat avec Institut des ACTUAIRES

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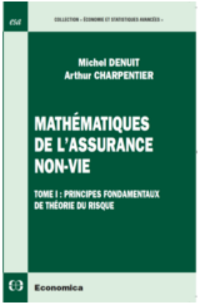
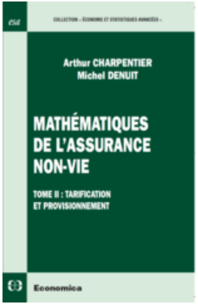
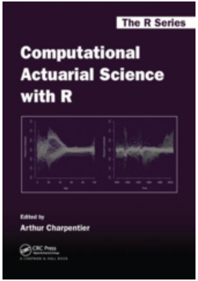
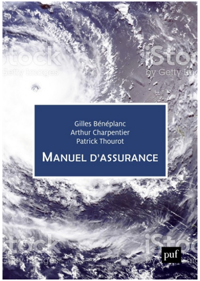
freakonometrics & freakonometrics.hypotheses.org

Modélisation prédictive, Science actuarielle,
Économie mathématique, Risque, Inégalités,
Économétrie, statistiques, apprentissage automatique
Modélisation du climat, Extrêmes, Équité

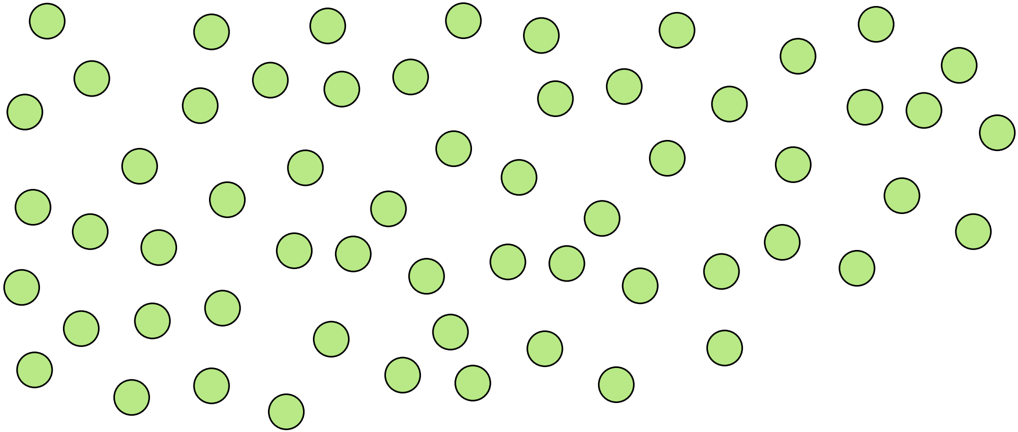






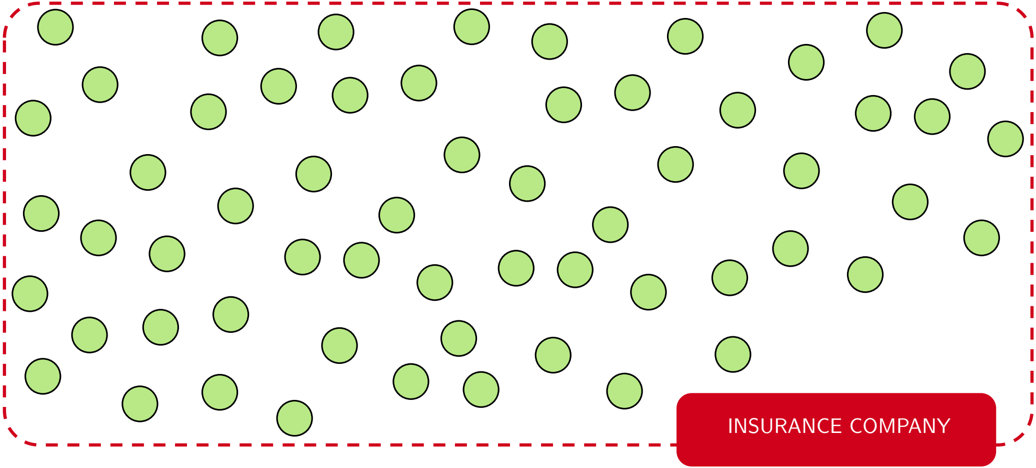


Risk Transfert



"Insurance is the contribution of the many to the misfortune of the few"

Risk Transfert



Risk Aversion

Following Hardy et al. (1929, 1934), and Marshall and Olkin (1979)

Def Consider two sorted vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ ($x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$) such that $\displaystyle\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, then $\boldsymbol{x} \preceq_M \boldsymbol{y}$ (majorization order) if $\displaystyle\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$, $\forall k$.

For example,

$$\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n},\frac{1}{n}\right) \prec_M \left(\frac{1}{n-1},\frac{1}{n}-1,\cdots,\frac{1}{n-1},0\right) \prec_M (1,0,\cdots,0,0)\,.$$

$$\iff \sum_{i=1}^n h(x_i) \leq \sum_{i=1}^n h(y_i) \text{ for any convex function}$$

$$\iff \boldsymbol{x} = D\boldsymbol{y} \text{ for some doubly stochastic matrix } D, \text{ i.e. } \sum_{k=1}^n D_{i,k} = \sum_{k=1}^n D_{k,j} = 1,\ \forall i,j$$

$$\iff \boldsymbol{x} = P_1\cdots P_k\boldsymbol{y} \text{ for finitely some Pigou-Dalton transfert matrices } P_j$$

$$(P_j = \alpha\mathbb{I} + (1-\alpha)T \text{ for some } \alpha\in(0,1) \text{ and } T = 0 \text{ except } T_{i,j} = T_{j,i} = 1)$$

Risk Aversion and Risk Sharing

Def Consider two random variables X and Y, $X \preceq_{CX} Y$ if $\mathbb{E}[h(X)] \leq \mathbb{E}[h(Y)]$ for any convex function h

$\Longleftrightarrow$ Y is a mean-preserving spread of X, i.e. $Y \overset{\mathcal{L}}{=} X + Z$, where $\mathbb{E}[Z|X] = 0$.

$\Longleftrightarrow$ $\mathbb{E}[(X-s)_+] \leq \mathbb{E}[(Y-s)_+]$ for all $s \in \mathbb{R}$.

$\Longrightarrow$ $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\text{Var}[X] \preceq \text{Var}[Y]$.

$\Longleftrightarrow$ Pigou-Dalton transfert, majorization order, etc

Following Denuit and Dhaene (2012) and Carlier et al. (2012),

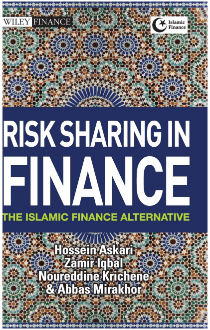
Def Consider two random vectors $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ and $\boldsymbol{X} = (X_1, \cdots, X_n)$ on $\mathbb{R}_+^n$.
$\boldsymbol{\xi}$ is a risk-sharing scheme of $\boldsymbol{X}$ if $X_1 + \cdots + X_n = \xi_1 + \cdots + \xi_n$ almost surely.

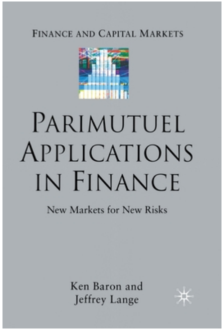
Def Consider two random vectors $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ and $\boldsymbol{X} = (X_1, \cdots, X_n)$ on $\mathbb{R}_+^n$.
$\boldsymbol{\xi} \preceq_{CCX} \boldsymbol{X}$ if $\xi_i \preceq_{CX} X_i$.

Risk Sharing

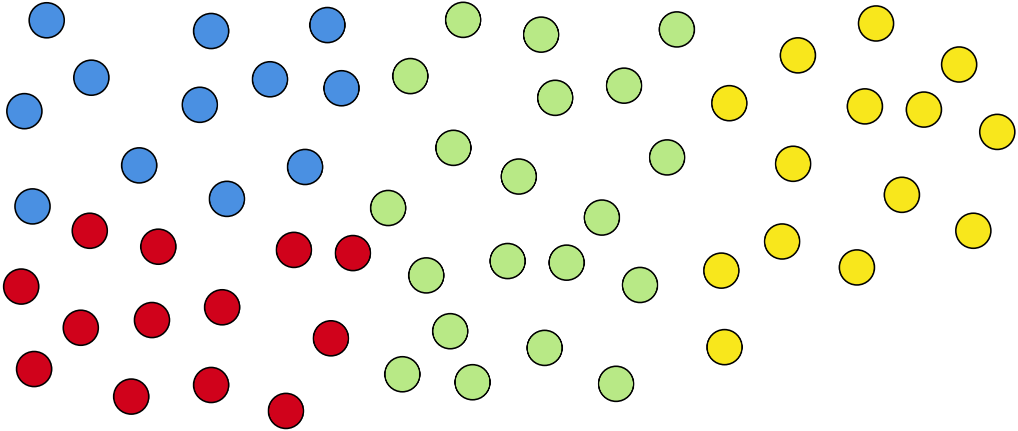
Peer-to-peer insurance is a risk sharing network where a group of individuals pool their premiums together to insure against a risk. Peer-to-Peer Insurance mitigates the conflict that inherently arises between a traditional insurer and a policyholder when an insurer keeps the premiums that it doesn't pay out in claims

- Takaful التكافل
- Wakalah وَكَالَة
- Musharakah مشاركة
- Xiang Hu Bao 相互保
- Parimutuel

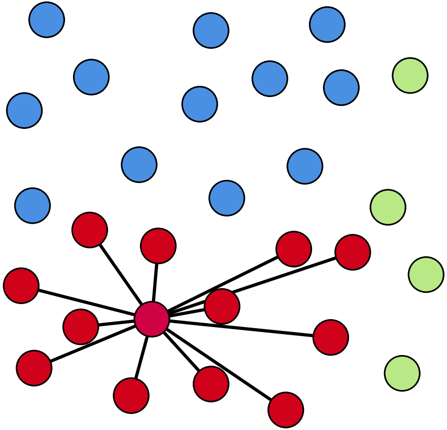




Risk Sharing



Risk Sharing



Let $\xi_j = \dfrac{1}{n}\sum_{i=1}^{n} X_i, \;\forall j$

- Risk sharing

$\xi_1 + \cdots + \xi_n = X_1 + \cdots + X_n$

- Componentwise convex-order

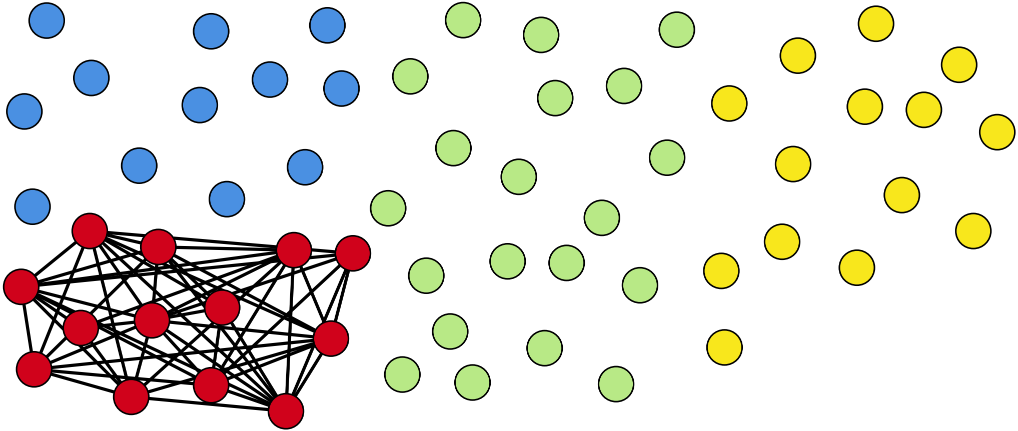
$\xi_j \preceq_{CX} X_j, \;\forall j$

More generally, consider some linear risk sharing $\boldsymbol{\xi} = M\boldsymbol{X}$, for some $n \times n$ matrix

$$M = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{M}_k \end{bmatrix}, \;\; \mathbf{M}_k = \frac{1}{n_k}\mathbf{1}_k$$

where $\mathbf{1}_k$ is the $n_k \times n_k$ matrix full of 1's.

Risk Sharing



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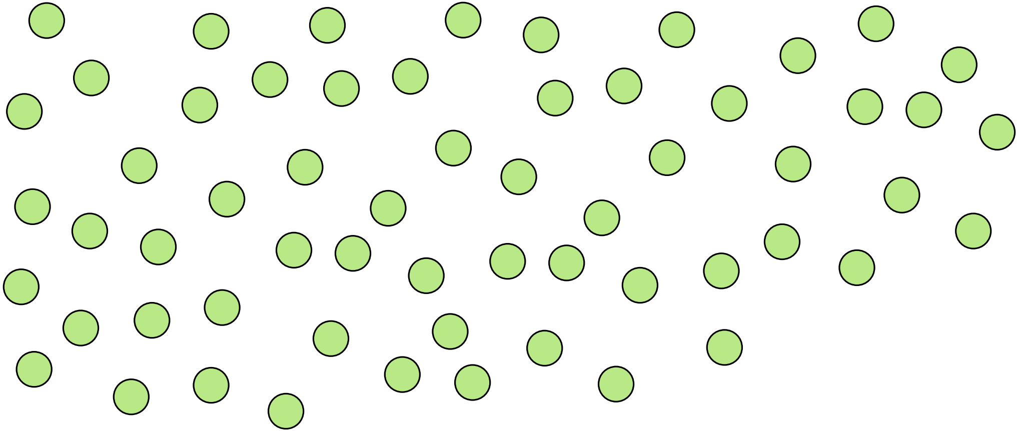
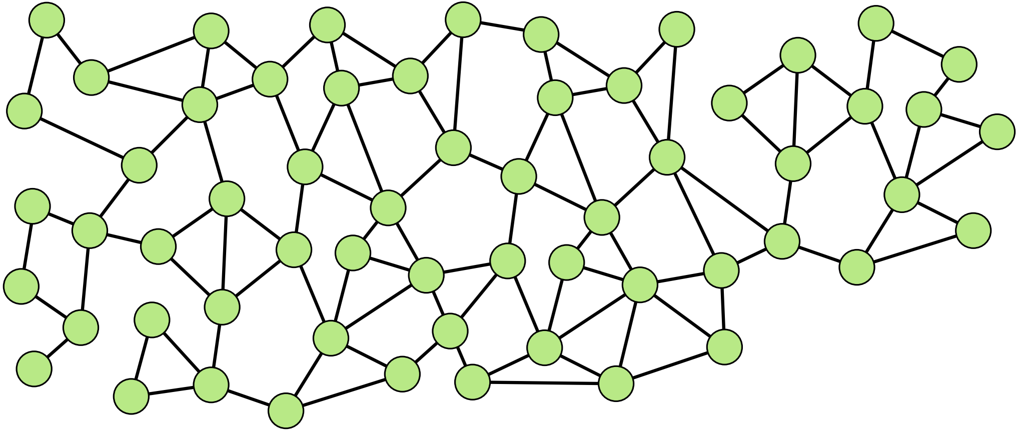
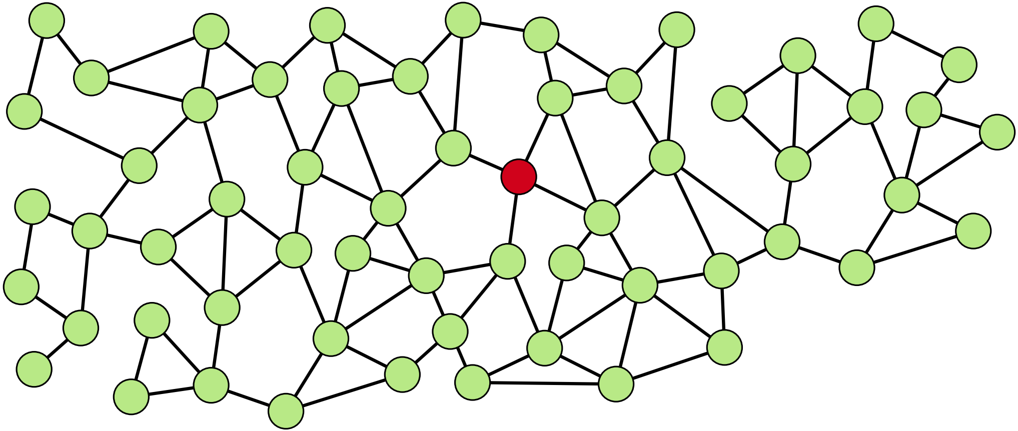


Risk Sharing on a Network



Risk Sharing on a Network



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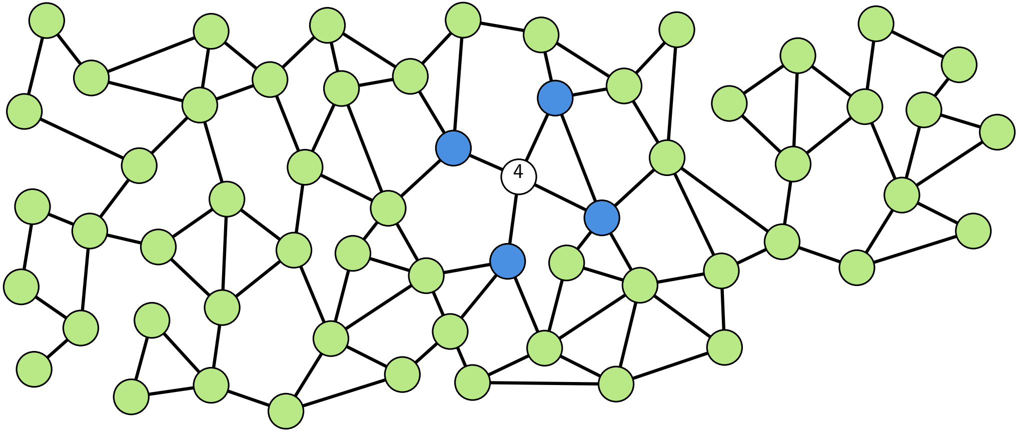

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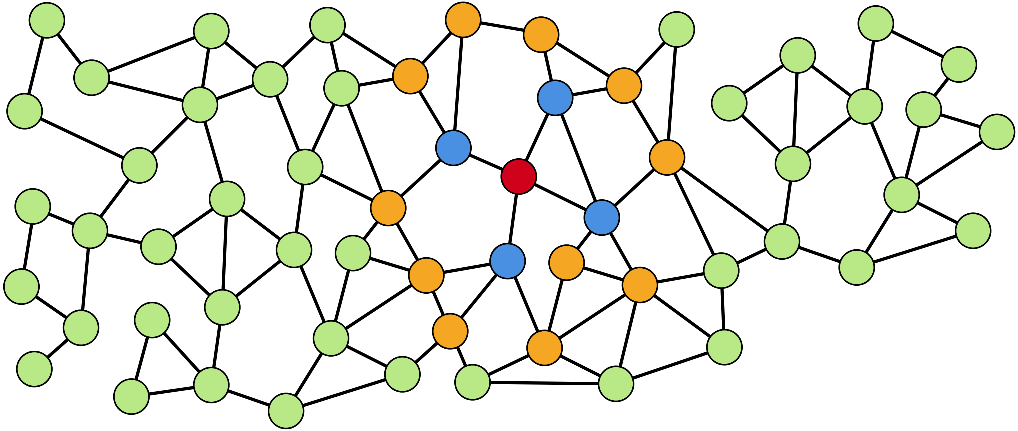

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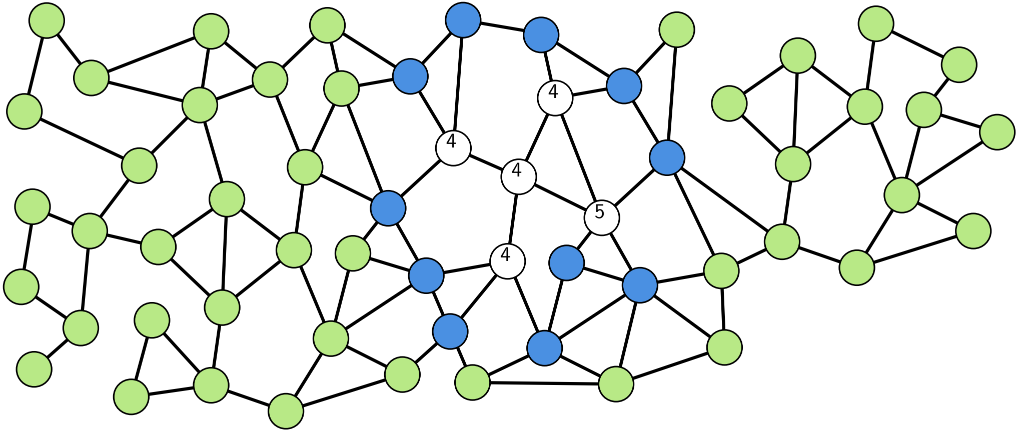

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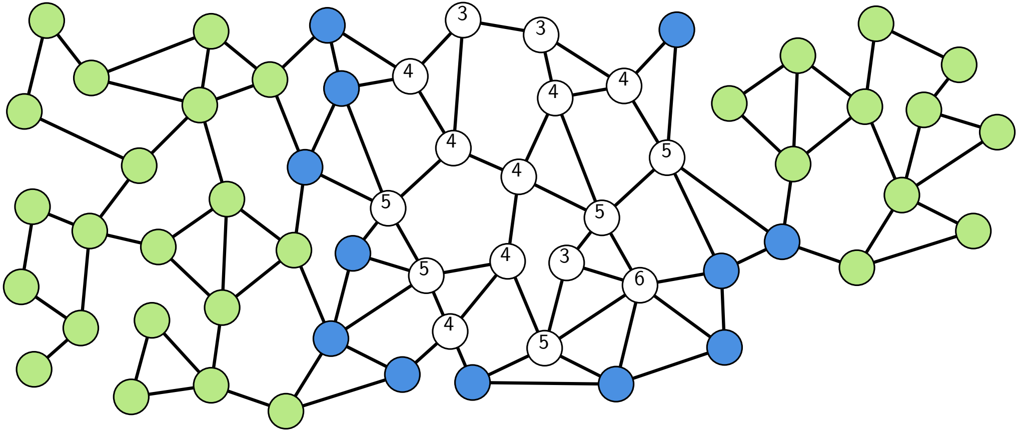

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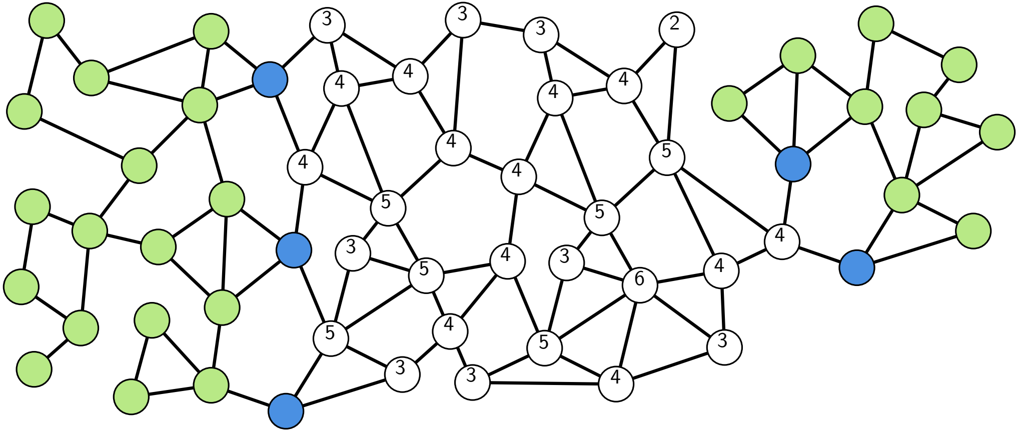

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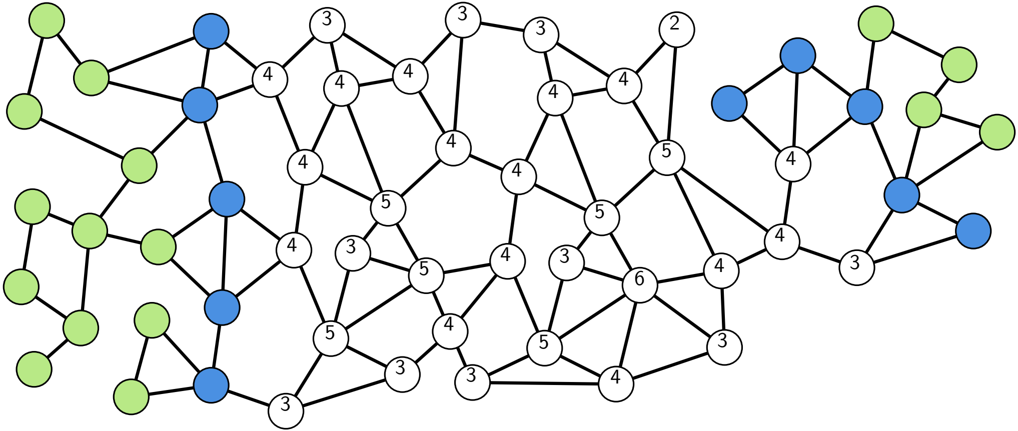

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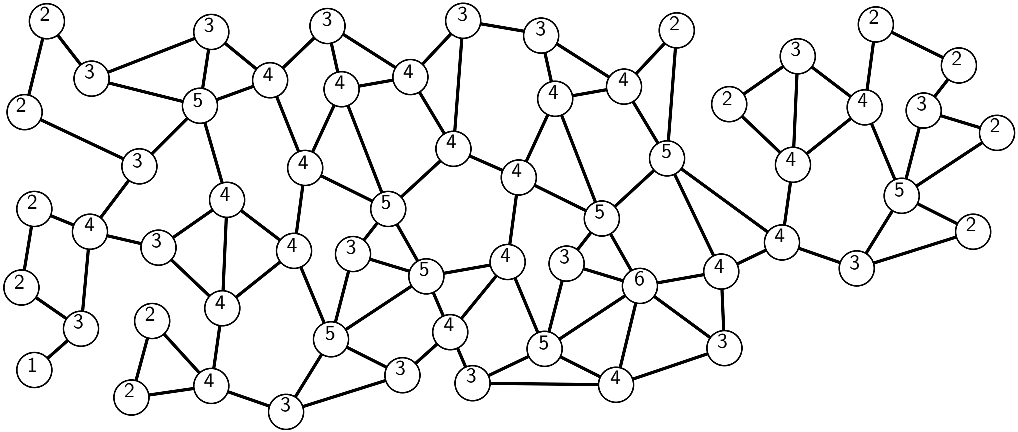

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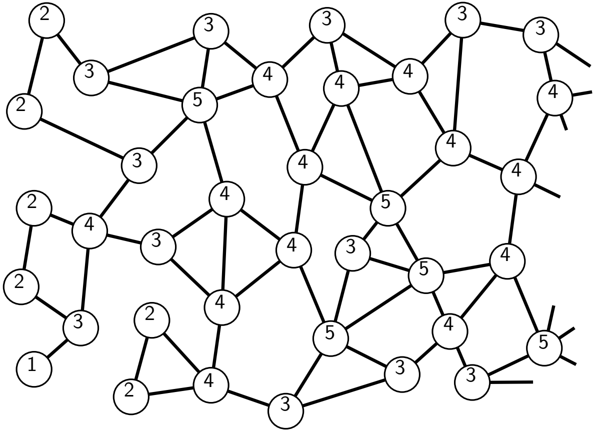
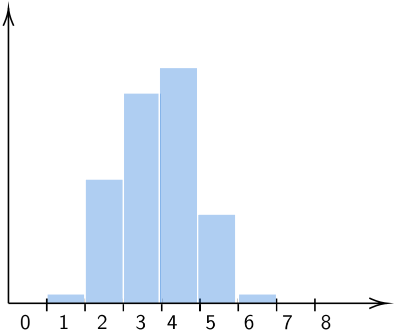


distribution of degrees



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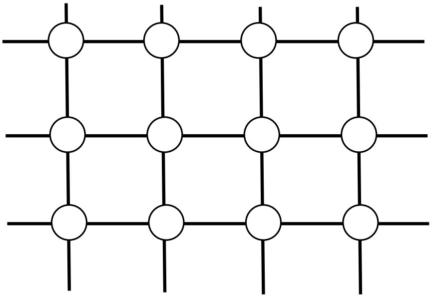
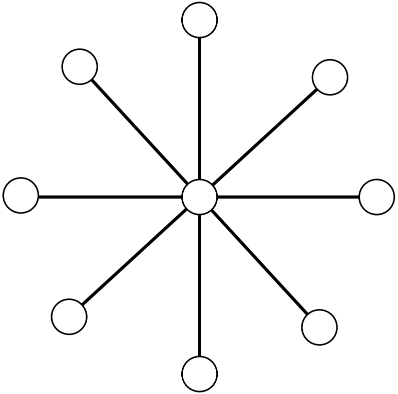


Regular graph vs. star shaped graph
(low variance vs. large variance on D)



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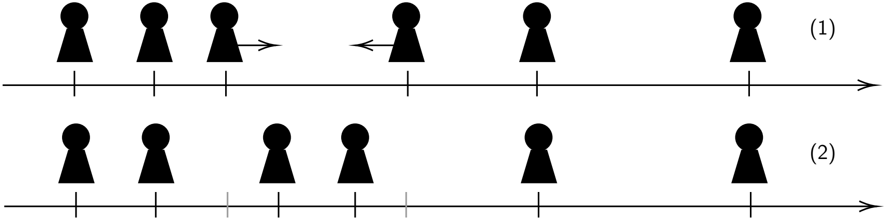
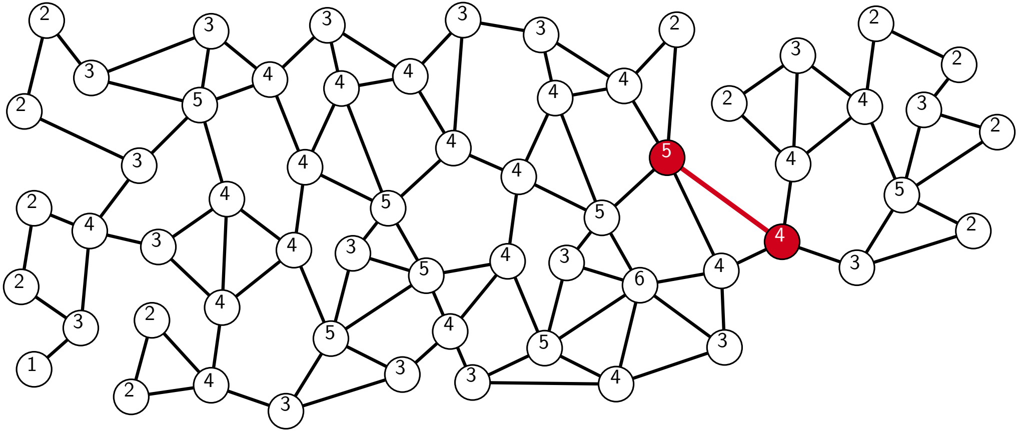


Pigou-Dalton transferts (Dalton (1920)) see also Atkinson (2015),

$$\boldsymbol{y}^{(2)} \preceq_M \boldsymbol{y}^{(1)} \longleftarrow \begin{cases} y_i^{(2)} = y_i^{(1)}, \ \forall i \neq j, k \\ y_j^{(2)} = y_j^{(1)} + h, \\ y_k^{(2)} = y_k^{(1)} - h, \ y_j^{(2)} > y_j^{(1)} \end{cases}$$

see martingale property of mean-preserving spread, $Y^{(1)} \overset{\mathcal{L}}{=} Y^{(2)} + Z$, where $\mathbb{E}[Z|Y^{(1)}] = 0$ (convex order is a dispersion order)

Risk Sharing on a Network



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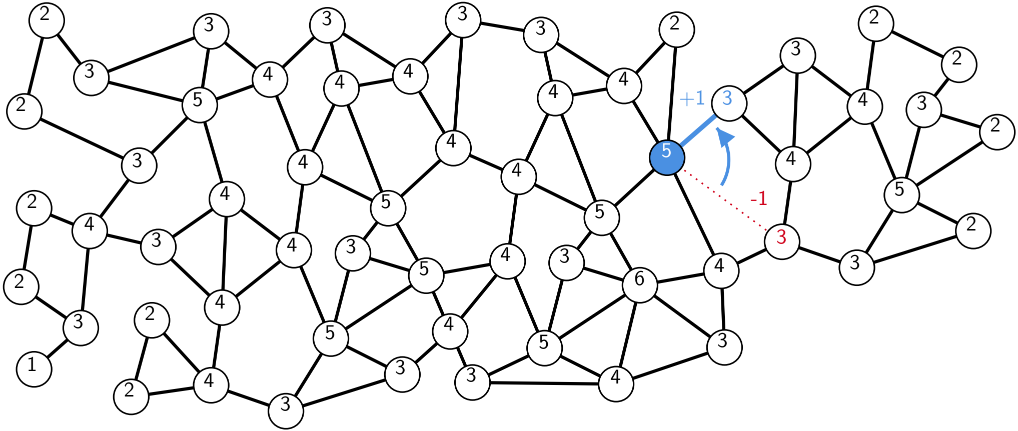
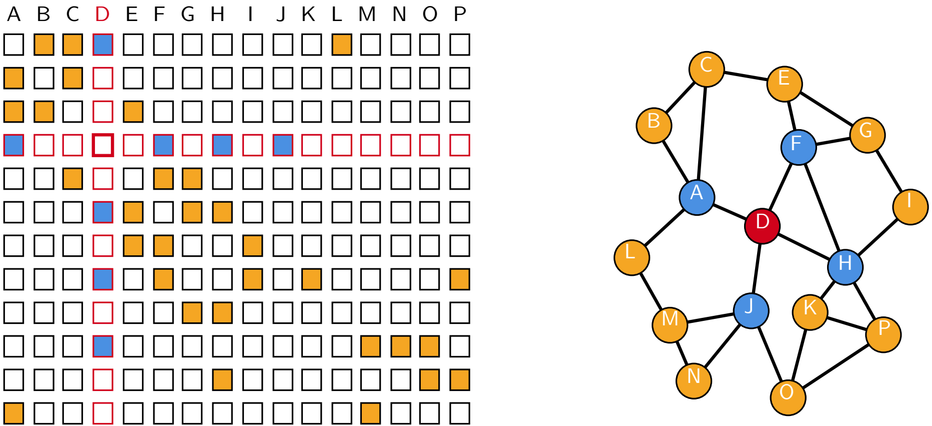


Sharing Risks with Friends



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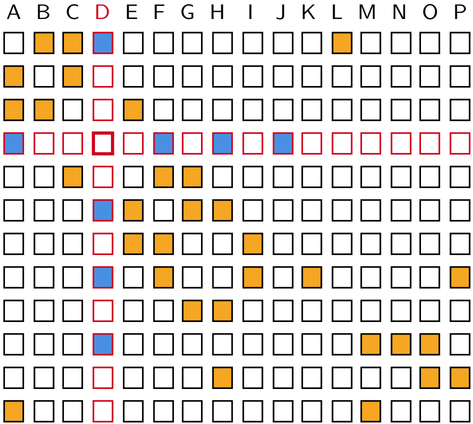

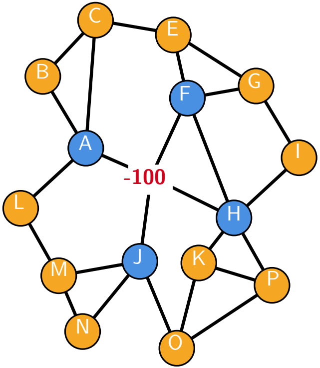

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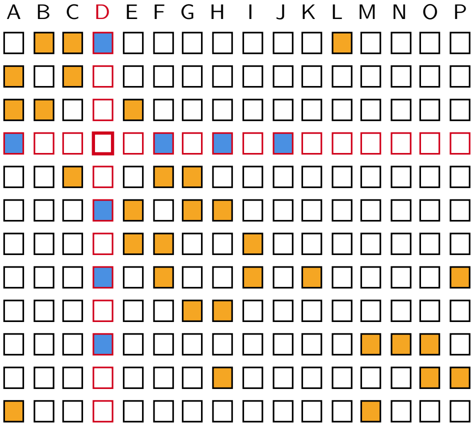

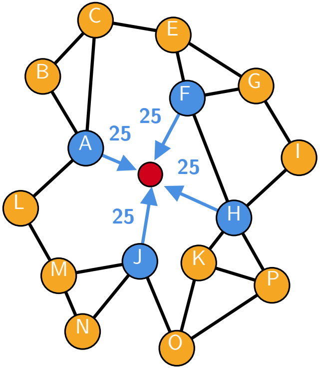

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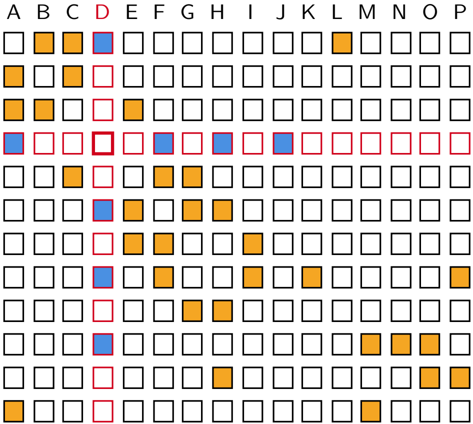

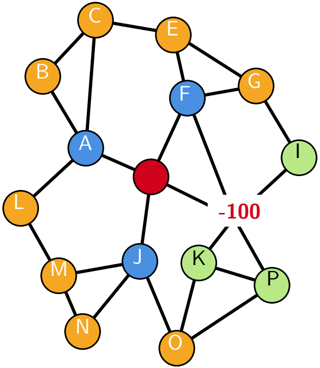

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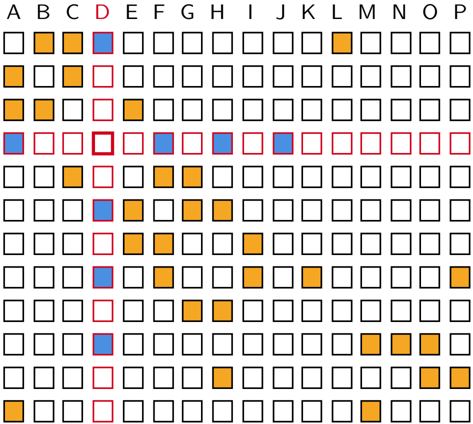

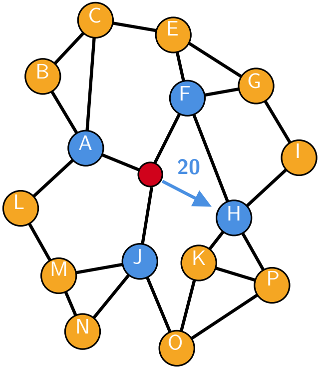

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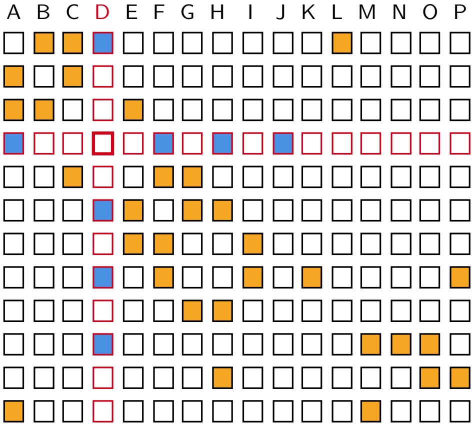


Looks like a linear risk sharing mechanism,

$$\xi = B\boldsymbol{X} \text{ a.s., where } B_{i,j} = A_{i,j}/d_i,$$

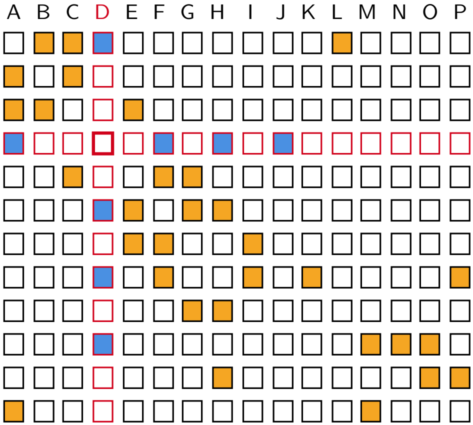
A being the adjacency matrix of the network
Here, B is a doubly stochastic matrix.

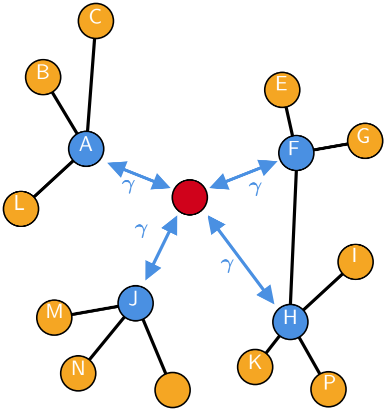
But it suffers some drawbacks...

- need an upper bound
- unfairness ($B_{i,i} = 0, \ \forall i$)

(no longer "linear" risk sharing mechanism)

Sharing Risks with Friends





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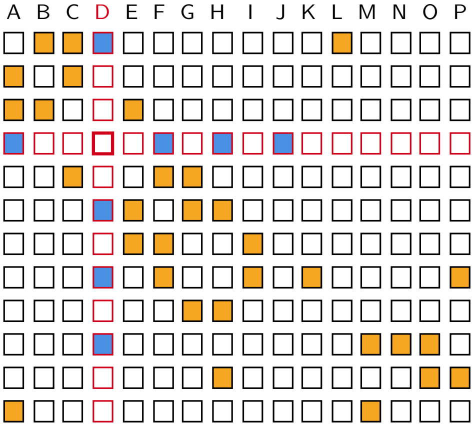

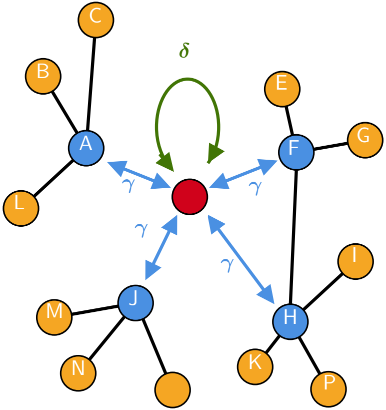

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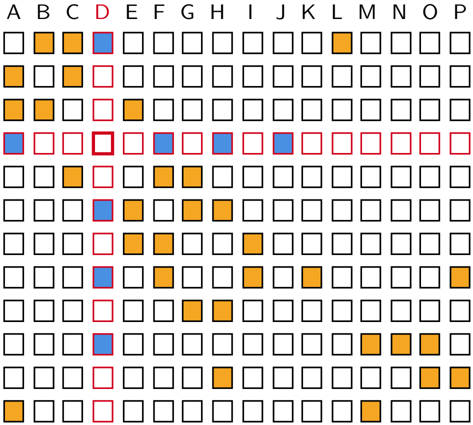

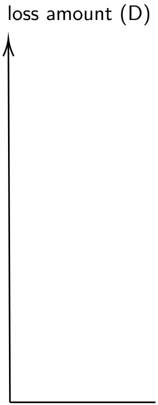

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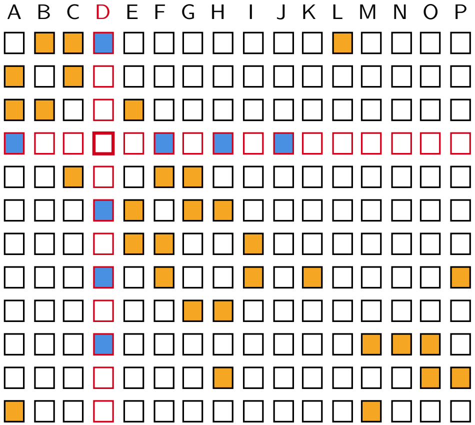

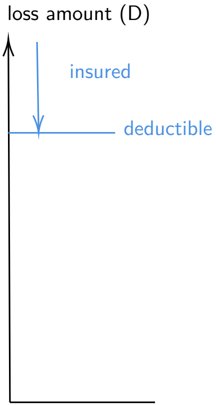

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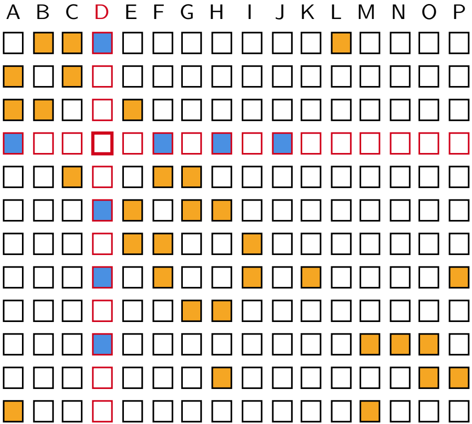

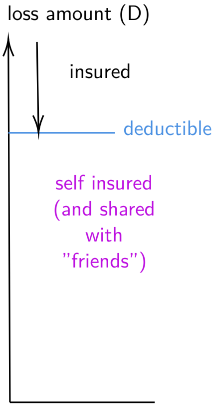

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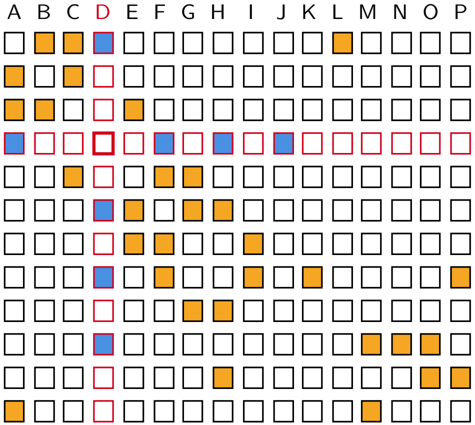

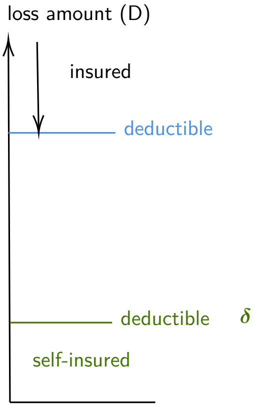

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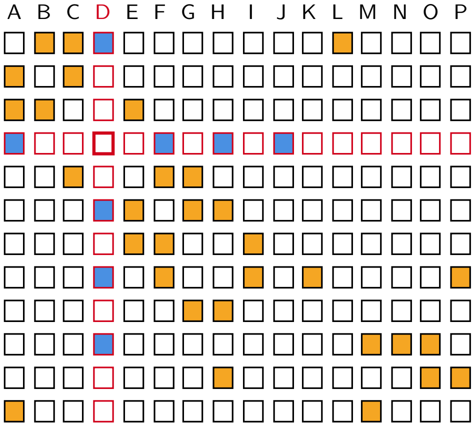

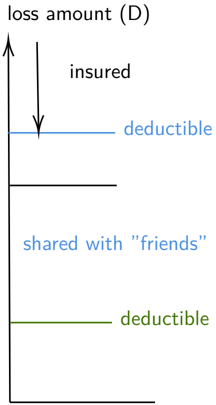

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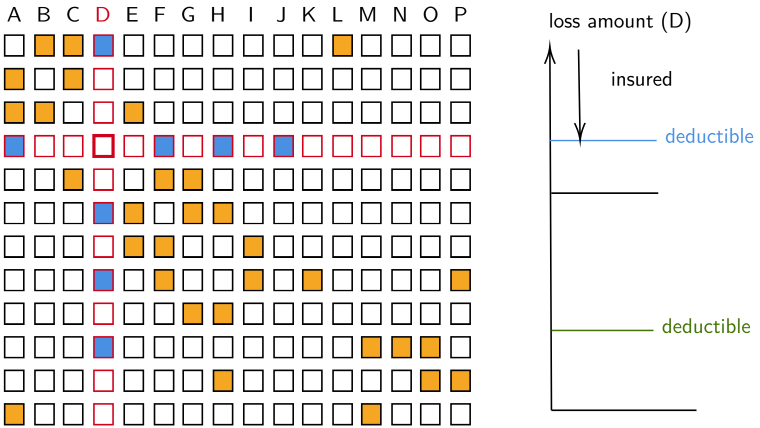

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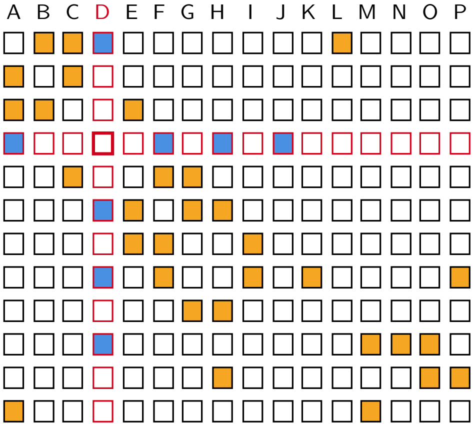

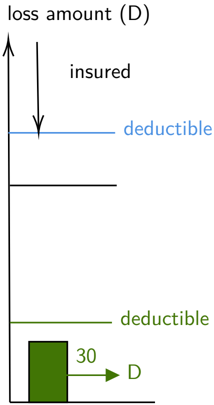

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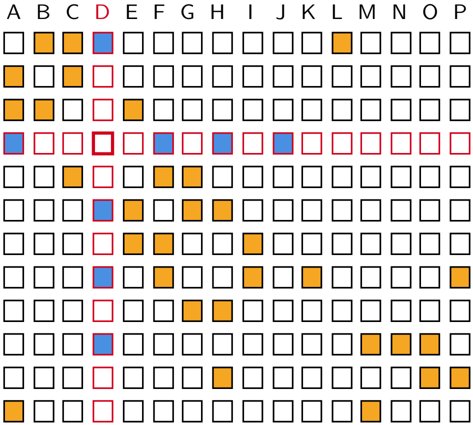

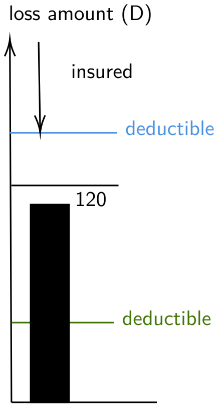

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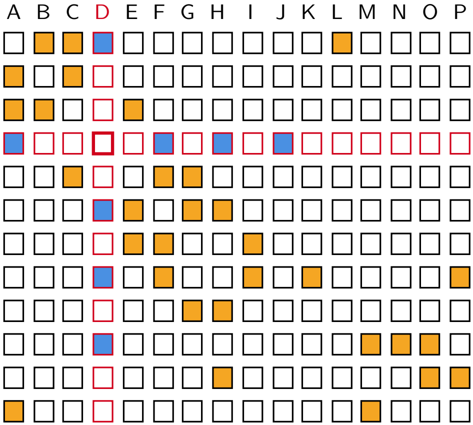

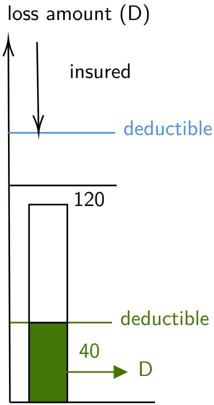

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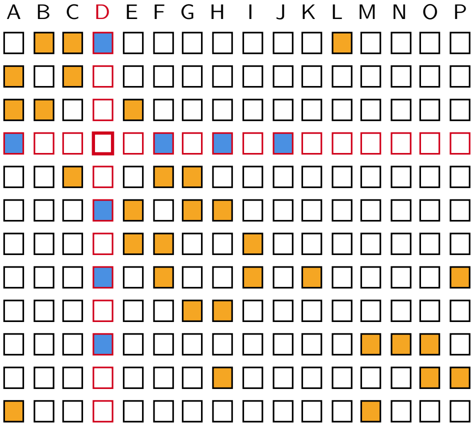

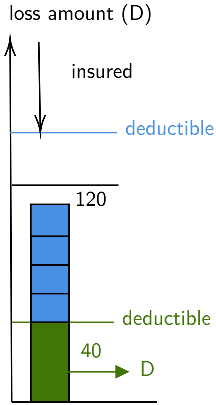

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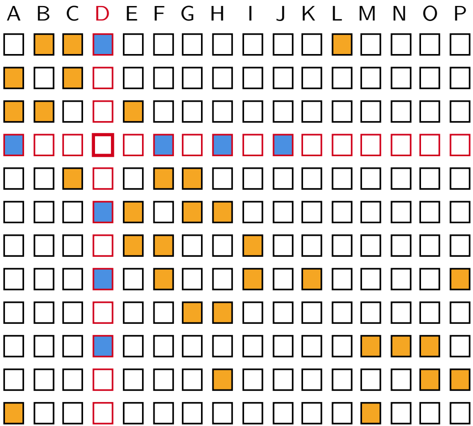

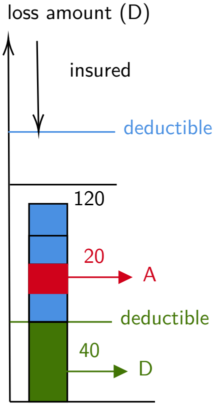

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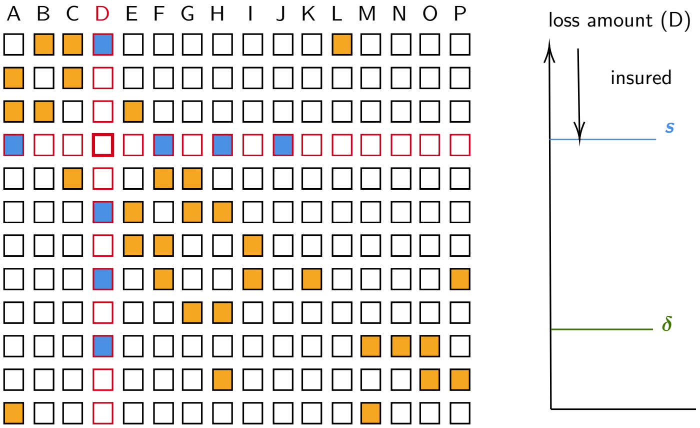

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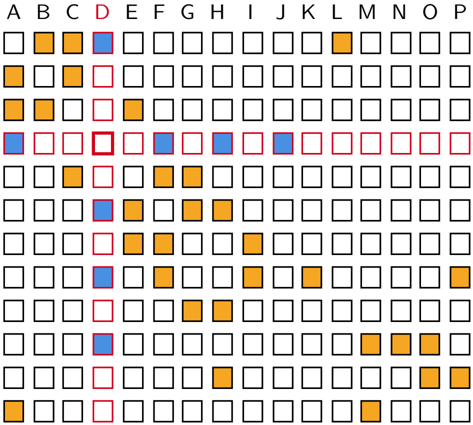

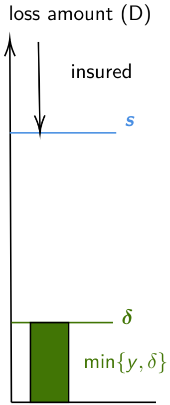

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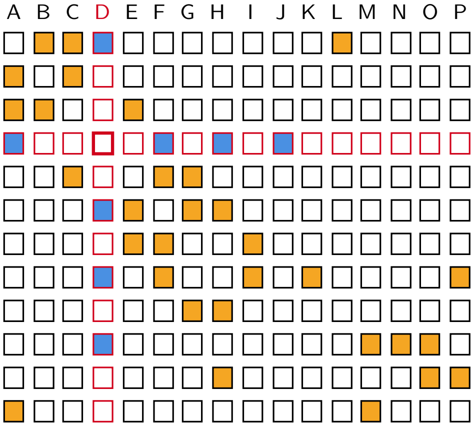
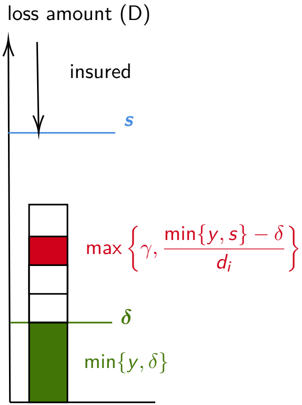
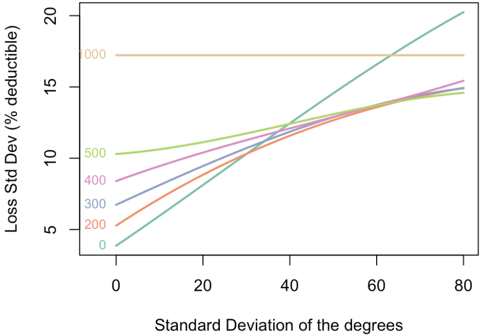




Sharing Risks with Friends

- Y_i loss of insured i, $Z_i = \mathbf{1}(Y_i > 0)$
- $\mathcal{V}_i$ is the set of friends of insured i, $d_i = \text{Card}(\mathcal{V}_i)$
- s deductible of insurance contracts
- γ is the maximum amount shared between i and j (reciprocal contracts)

$$\begin{aligned}\xi_i \quad = \quad & Z_i \cdot \min\{s, Y_i\} \\ & + \sum_{j \in \mathcal{V}_i} Z_j \min\left\{\gamma, \frac{\min\{s, Y_j\} - \delta}{d_j}\right\} \\ & - Z_i \cdot \min\{d_i\gamma, \min\{s, Y_i\} - \delta\}\end{aligned}$$



Optimization* of the Risk Sharing Mechanism

$$\begin{cases} \max \left\{ \displaystyle\sum_{(i,j)\in\mathcal{E}} \gamma_{(i,j)} \right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0,\gamma],\ \forall (i,j)\in\mathcal{E} \\ \qquad \displaystyle\sum_{j\in\mathcal{V}_i} \gamma_{(i,j)} \leq s,\ \forall i \in \mathcal{V} \end{cases}$$

Given losses $\boldsymbol{X} = (X_1, \cdots, X_n)$, define contributions $C^\star_{i\to j} = \min\left\{ \dfrac{\gamma^\star_{(i,j)}}{\displaystyle\sum_{i\in\mathcal{V}_j} \gamma^\star_{(i,j)}} \cdot X_j, \gamma^\star_{(i,j)} \right\}$,

and $\xi^\star_i = X_i + \displaystyle\sum_{j\in\mathcal{V}_i} [Z_j C^\star_{i\to j} - Z_i C^\star_{j\to i}]$ is a risk sharing, called optimal risk sharing.

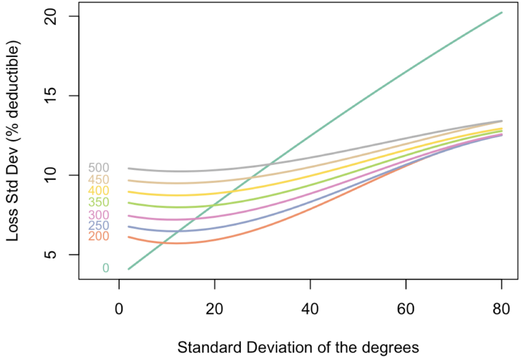
* from a welfare (social planner) perspective

Sharing Risks with Friends, and Friends of Friends

We can also consider friends of friends

$$\begin{cases} \gamma_1^\star = \text{argmax}\left\{ \sum_{(i,j)\in\mathcal{E}^{(1)}} \gamma_{(i,j)} \right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0,\gamma_1],\ \forall(i,j)\in\mathcal{E}^{(1)} \\ \qquad \sum_{j\in\mathcal{V}_i^{(1)}} \gamma_{(i,j)} \leq s,\ \forall i \end{cases}$$

$$\begin{cases} \gamma_2^\star = \text{argmax}\left\{ \sum_{(i,j)\in\mathcal{E}^{(2)}} \gamma_{(i,j)} \right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0,\gamma_2],\ \forall(i,j)\in\mathcal{E}^{(2)}_{\gamma_1^\star} \\ \qquad \sum_{j\in\mathcal{V}_i^{(1)}} \gamma^\star_{1:(i,j)} + \sum_{j\in\mathcal{V}_i^{(2)}} \gamma_{(i,j)} \leq s,\ \forall i \end{cases}$$



Take-away

- Back to the roots of insurance with risk sharing,
- Important to better model interactions
- Nice mathematical properties of linear risk sharing (connexions with convex ordering)
- More complex to derive a more realistic insurance product (with lower and upper limits)
- ... ongoing work...

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