

# Autocalibration & Insurance Pricing

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Université de Sherbrooke, 2021

# Agenda

- ▶ On insurance pricing
- ▶ What is a model ?
- ▶ Bias of a model
- ▶ Correcting from bias
- ▶ Application
- ▶ Theoretical properties



**Demetri** @PhDemetri · 13h

All this talk about XGboost prompted me to try it again on some toy datasets I have laying around.

The long and the short of it is: XGBoost results in a better AUC than my logistic regression (99.7 v 87) but XGB is so poorly calibrated it doesn't make sense to trust the probs

7

2

42



**Demetri** @PhDemetri · 13h

Calibration is a thing I never see people talk about in data science. We should really care about calibration more than we do.

2

1

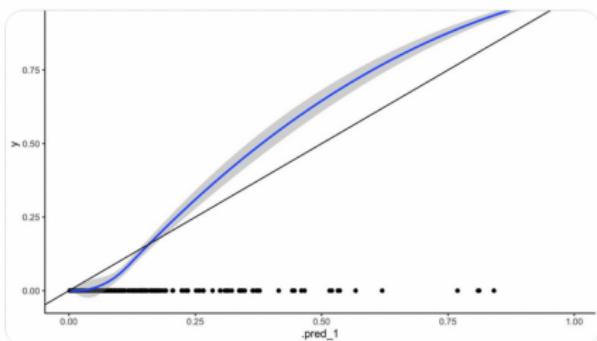
24



**Demetri** @PhDemetri · 13h

Calibration for the xgboost model.

Yuck



3

2

11



# Calibration

Dawid, A.P. (1982). The Well-Calibrated Bayesian, (JASA)

*“Suppose that a forecaster sequentially assigns probabilities to events. He is **well calibrated** if, for example, of those events to which he assigns a probability 30 percent, the long-run proportion that actually occurs turns out to be 30 percent. ”*

Silver, N. (2012). The signal and the noise, (Penguin Group)

*“Out of all the times you said there was a 40 percent chance of rain, how often did rain actually occur? If, over the long run, it really did rain about 40 percent of the time, that means your forecasts were **well calibrated** ”*

Kuhn, M., & Johnson, K.(2013). Applied Predictive Modeling  
(Springer)

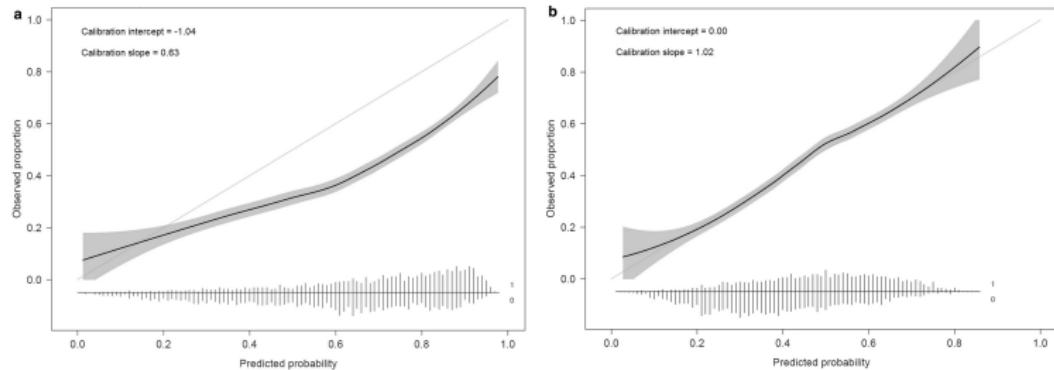
*“we desire that the estimated class probabilities are reflective of the true underlying probability of the sample ”*

# Calibration & Auto-calibration

Kruger, F. & Ziegel, J.F. (2020). Generic conditions for forecast dominance, (JBES)

**Definition 3.1** “*the forecast  $X$  of  $Y$  is an auto-calibrated forecast of  $Y$  if  $\mathbb{E}(Y|X) = X$  almost surely*”, or  $\mathbb{E}(Y|\hat{Y} = y) = y, \forall y$

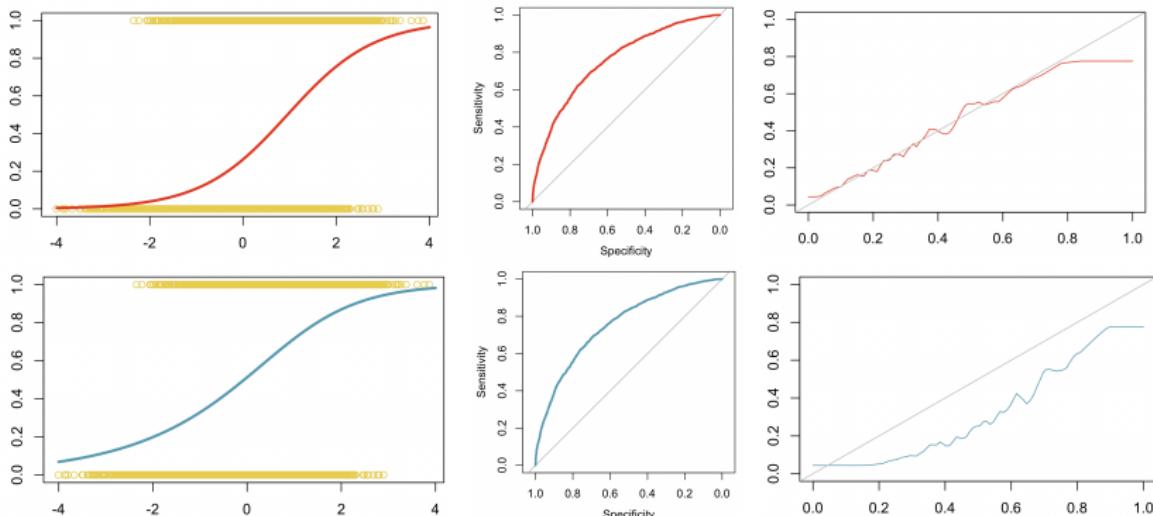
Van Calster, B. et al. (2019). Calibration: the Achilles heel of predictive analytics, (BMC Medecine)



# Machine Learning & Accuracy Measures

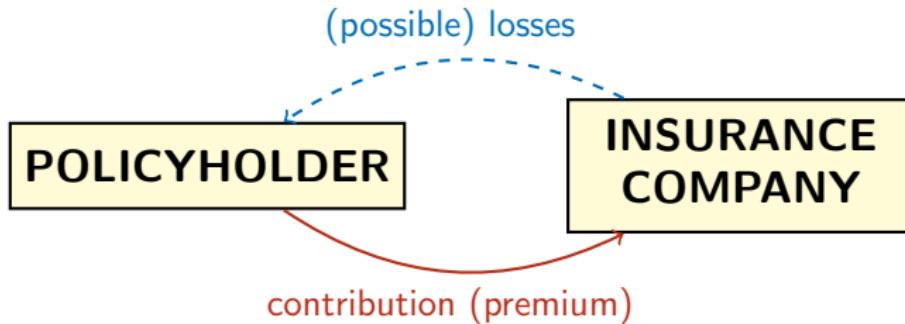
"XGBoost results in a better AUC than my logistic regression but XGBoost is so poorly calibrated it doesn't make sense to trust the probs "

- ▶  $(x_i, y_i)$  (where  $y_i \in \{0, 1\}$ ) with  $\hat{\pi}_1(x)$  or  $\hat{\pi}_2(x)$  ( $= \sqrt{\hat{\pi}_1(x)}$ )
- ▶ ROC curve of  $\hat{\pi}_1$  or  $\hat{\pi}_2$  are identical (and same AUC)
- ▶  $\mathbb{E}[Y|\hat{\pi}_1(X) = s]$  or  $\mathbb{E}[Y|\hat{\pi}_2(X) = s]$ ,  
ie. regression of  $y_i$  on  $\pi(x_i)$  (expected?  $\mathbb{E}[Y|\hat{\pi}(X) = s] \sim s$ )



# Insurance

*“Insurance is the contribution of the many to the misfortune of the few”*



The “*contribution*” is obtained using predictive [models](#)  
(interpretability / black box / etc issues)

# A model, $m : \mathbf{x} \mapsto y$

To train / estimate a model  $m$ ,  
we need a dataset, i.e. a collection  
of observations  $(\mathbf{x}_i, y_i)$

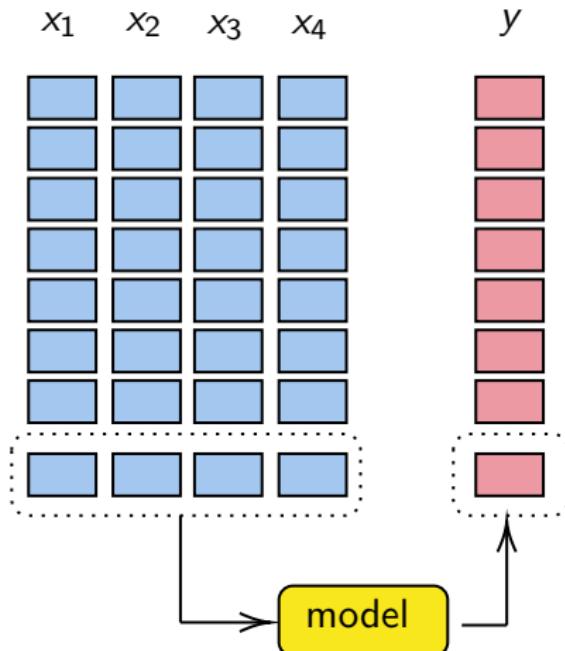
Usually  $y_i$  denotes the annual loss

$$y_i = \sum_{j=1}^{n_i} z_{i,j}$$

$n$  is the annual frequency

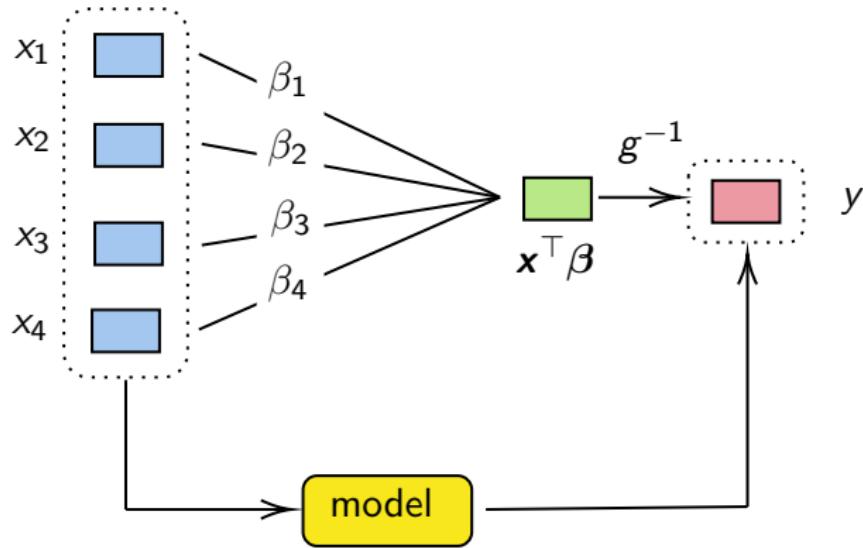
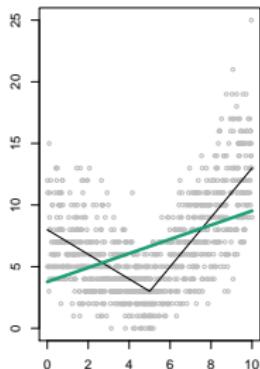
$z$  is the cost of a single claim  
(see Tweedie models)

To illustrate,  $y$  is a counting variable



# A model, $m : \mathbf{x} \mapsto y$ (GLM)

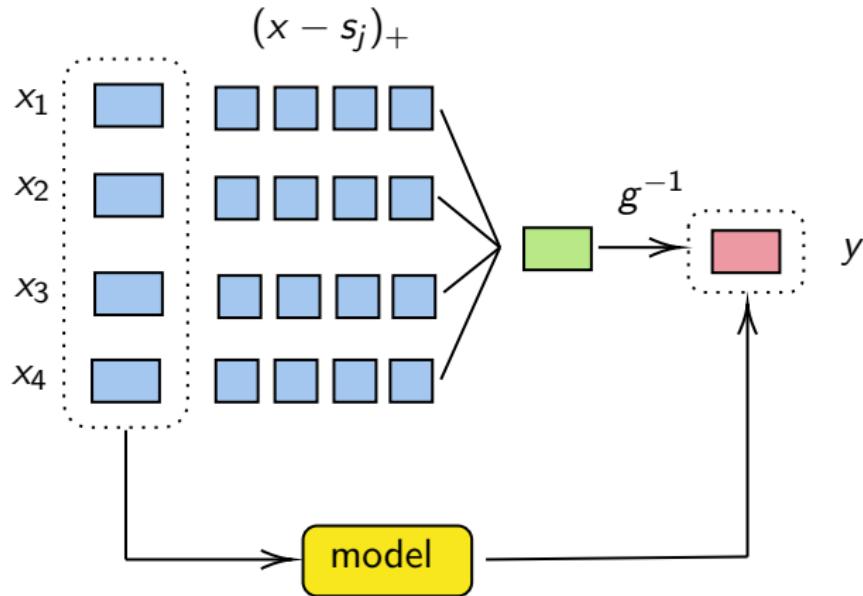
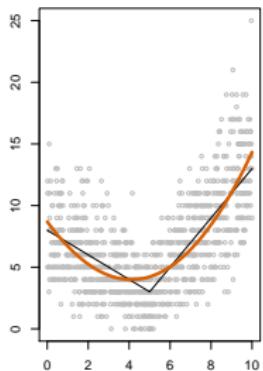
Poisson regression



$$m(\mathbf{x}) = g^{-1}(\mathbf{x}^\top \boldsymbol{\beta}) = g^{-1} \left( \sum_{j=1}^p \beta_j x_j \right)$$

# A model, $m : \mathbf{x} \mapsto y$ (GAM)

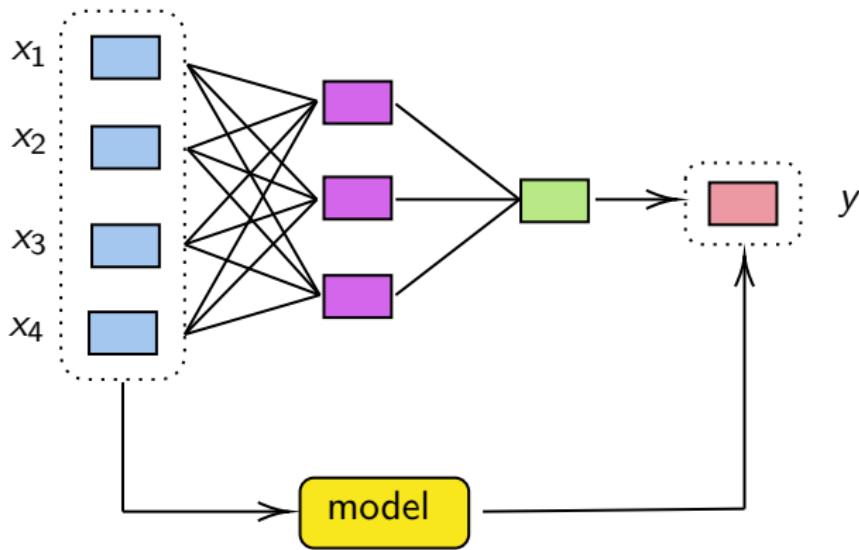
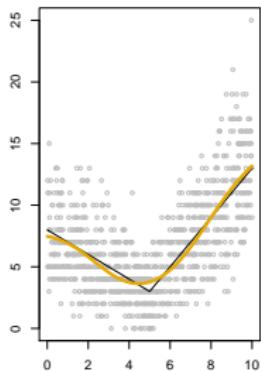
Poisson regression



$$m(\mathbf{x}) = g^{-1} \left( \sum_{j=1}^p \beta_j \psi_j(x_j) \right), \text{ where } \psi_j(x) = \sum_{k=1}^5 \alpha_{j,k} (x - s_{j,k})_+$$

# A model, $m : \mathbf{x} \mapsto y$ (neural nets)

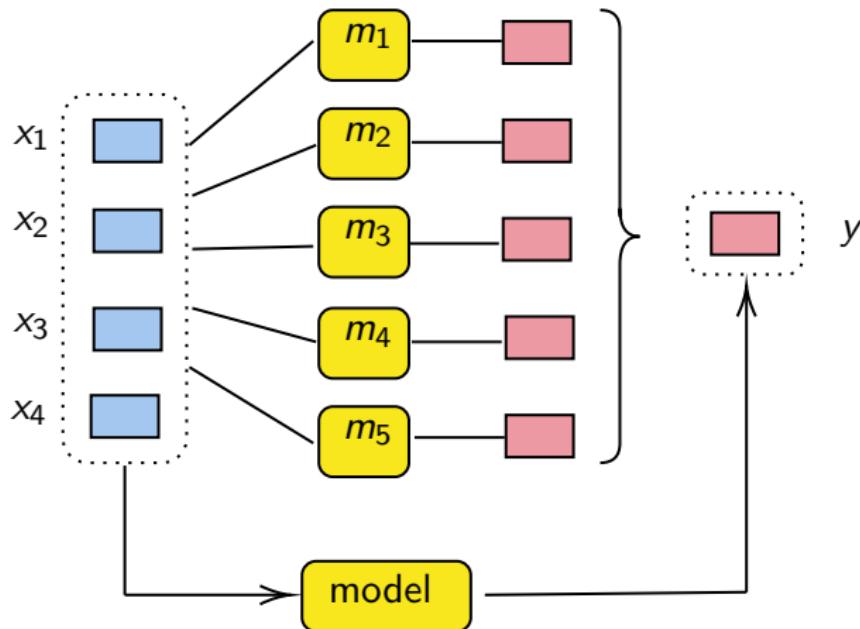
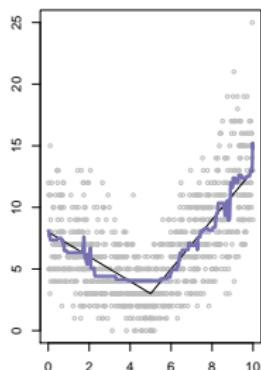
Poisson loss



$$\text{E.g. } m(\mathbf{x}) = \sum_{j=1}^3 \omega_{1:j} h(\mathbf{x}^\top \omega_{2:j})$$

# A model, $m : \mathbf{x} \mapsto y$ (ensemble parallel, bagging)

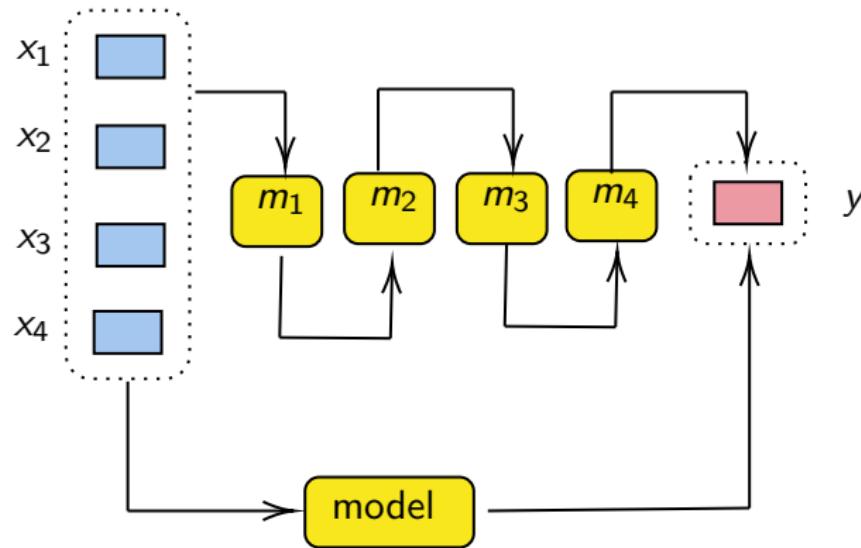
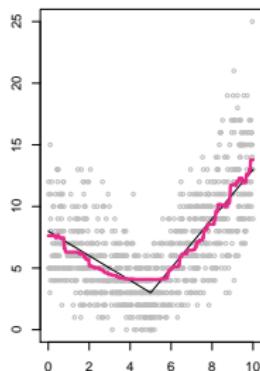
Random forest



$$\text{E.g. } m(\mathbf{x}) = \frac{1}{B} \sum_{b=1}^B m_b(\mathbf{x}) \text{ where } m_b(\mathbf{x}) = \sum_{k=1}^K \alpha_{b,k} \mathbf{1}(\mathbf{x} \in \mathcal{X}_{b,k})$$

# A model, $m : \mathbf{x} \mapsto y$ (ensemble sequential, boosting)

Boosting



$$m(\mathbf{x}) = \sum_{t=1}^T m_t(\mathbf{x}) \text{ where } m_t \text{ is a (weak) model on } y_i - m_{t-1}(\mathbf{x}_i)$$

(residuals from the previous step) such as (not too deep) trees

Not a model,  $\hat{m} : \mathbf{x} \mapsto y$  (local regression)

To approximate  $\mathbb{E}[Y]$  use

$$\hat{m} = \underset{\mu \in \mathbb{R}}{\operatorname{argmin}} \left\{ \sum_{i=1}^n \frac{1}{n} [y_i - \mu]^2 \right\}$$

To approximate  $\mathbb{E}[Y | \mathbf{X} = \mathbf{x}]$ , use

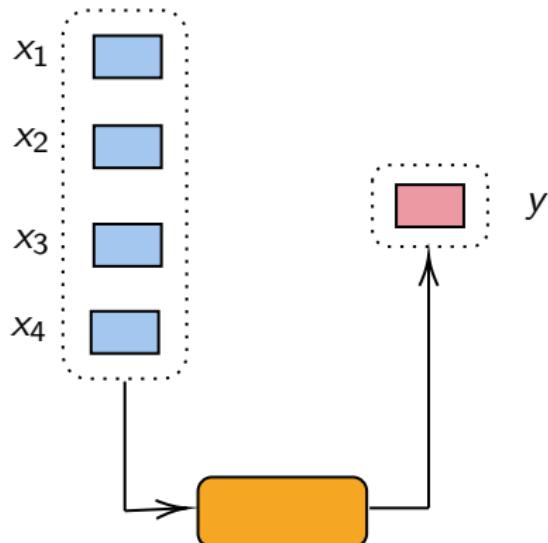
$$\hat{m}(\mathbf{x}) = \underset{\mu \in \mathbb{R}}{\operatorname{argmin}} \left\{ \sum_{i=1}^n \omega_i(\mathbf{x}) [y_i - \mu]^2 \right\}$$

where, see [Loader \(1999\)](#),

$$\omega_i(\mathbf{x}) \propto k_\alpha (\|\mathbf{x} - \mathbf{x}_i\|)$$

or  $k$  nearest neighbors indicator...

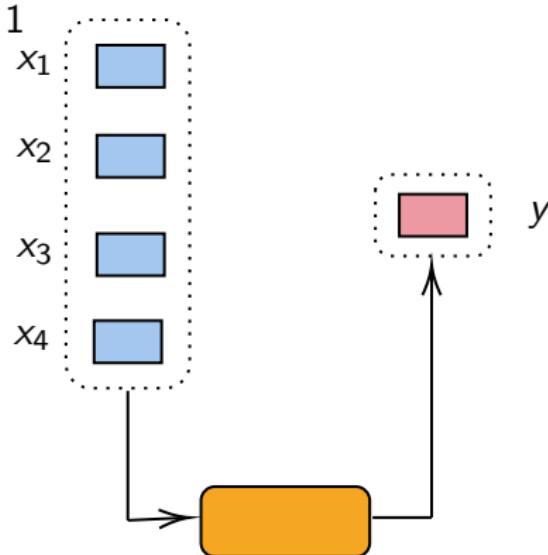
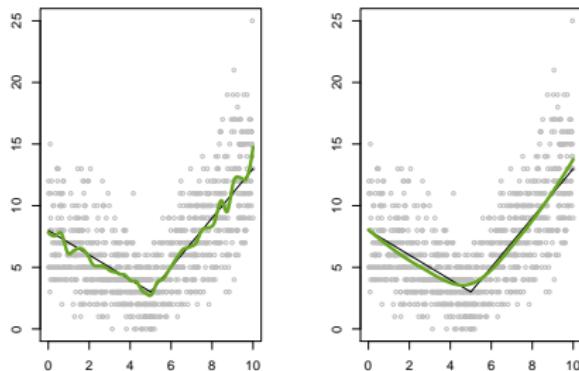
Behaves poorly in high dimension (but efficient in dimension 1)



## Not a model, $\hat{m} : x \mapsto y$ (local regression)

$$\hat{m}(x) = \sum_{i=1}^n \omega_i(x) y_i, \text{ with } \sum_{i=1}^n \omega_i(x) = 1$$

see `locfit` in R



(depends on a bandwidth parameter  $\alpha$ )

Provides a local estimate of  $\mathbb{E}[Y|X = x]$  on a neighborhood of  $x$

## GLM, Bias, & Economic Interpretation

For GLMs,  $f(y_i) = \exp\left(\frac{y_i\theta_i - b(\theta_i)}{\varphi} + c(y_i, \varphi)\right)$

The natural parameter for  $y_i$  :  $\theta_i$

Prediction for  $y_i$  :  $\hat{y}_i = \mu_i = \mathbb{E}(Y_i) = b'(\theta_i)$

Score associated with  $y_i$  :  $\eta_i = \mathbf{x}_i^\top \beta$

Link function :  $g$  such that  $\eta_i = g(\mu_i) = g(b'(\theta_i))$

$$\log \mathcal{L}(\boldsymbol{\theta}, \varphi, \mathbf{y}) = \sum_{i=1}^n \log \mathcal{L}_i = \sum_{i=1}^n \left[ \frac{y_i\theta_i - b(\theta_i)}{\varphi} + c(y_i, \varphi) \right]$$

First order conditions:  $\frac{\partial \log \mathcal{L}_i}{\partial \beta_j} = \frac{\partial \log \mathcal{L}_i}{\partial \theta_i} \cdot \frac{\partial \theta_i}{\partial \mu_i} \cdot \frac{\partial \mu_i}{\partial \eta_i} \cdot \frac{\partial \eta_i}{\partial \beta_j}$

$$\frac{\partial \log \mathcal{L}_i}{\partial \theta_i} = \frac{y_i - \mu_i}{\varphi}, \quad \frac{\partial \theta_i}{\partial \mu_i} = \left( \frac{\partial \mu_i}{\partial \theta_i} \right)^{-1} = \frac{1}{b''(\theta_i)} = \frac{1}{V(\mu_i)}$$

$$\frac{\partial \eta_i}{\partial \beta_j} = \frac{\partial \mathbf{x}_i^\top \beta}{\partial \beta_j} = x_{i,j}, \quad \frac{\partial \mu_i}{\partial \eta_i} = \left( \frac{\partial \eta_i}{\partial \mu_i} \right)^{-1} = (g'(\mu_i))^{-1}$$

# GLM, Bias, & Economic Interpretation

With canonical link  $g_\star = b'^{-1}$ , i.e.  $\eta_i = \theta_i$ ,

$$\mathbf{X}^\top(\mathbf{y} - \boldsymbol{\mu}) = \mathbf{0}, \text{ where } \hat{\mathbf{y}} = \boldsymbol{\mu}$$

so, if there is an intercept,  $\mathbf{1}^\top(\mathbf{y} - \boldsymbol{\mu}) = \mathbf{0}$ , i.e.  $\sum_{i=1}^n y_i = \sum_{i=1}^n \hat{y}_i$

which is the empirical version (training dataset) of  $\mathbb{E}[Y] = \mathbb{E}[\hat{Y}]$   
(see logistic regression or Poisson with log-link)

**But** more generally, the first order condition is

$$\mathbf{X}^\top \mathbf{W} \Delta (\mathbf{y} - \boldsymbol{\mu}) = \mathbf{0},$$

where  $\mathbf{W} = \text{diag}((V(\mu_i)g'(\mu_i)^2)^{-1})$  and  $\Delta = \text{diag}(g'(\mu_i))$ .

**But** usually not an important issue in ML classification problems

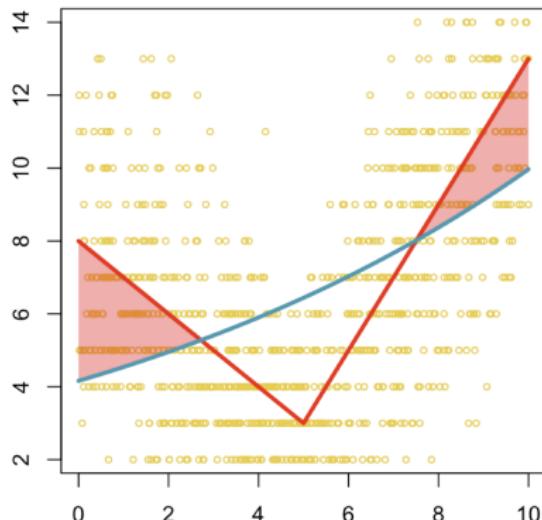
# GLM, Bias, & Economic Interpretation

Model  $\hat{\pi}$  is **globally unbiased** if  $\mathbb{E}[\hat{\pi}(\mathbf{X})] = \mathbb{E}[Y]$ ,  $\sum_{i=1}^n \hat{y}_i = \sum_{i=1}^n y_i$

Model  $\hat{\pi}$  is **locally unbiased** if  $\mathbb{E}[Y|\hat{\pi}(\mathbf{X}) = s] = s$

GLM  $\hat{\pi}$  globally unbiased,  
but possibly **locally biased**  
Major economic impact

- ▶  $\hat{\pi}(\mathbf{x}) < \mu(\mathbf{x})$   
attractive, but underpriced
- ▶  $\hat{\pi}(\mathbf{x}) > \mu(\mathbf{x})$   
not attractive

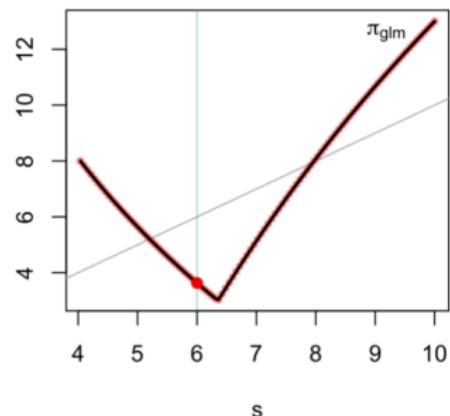
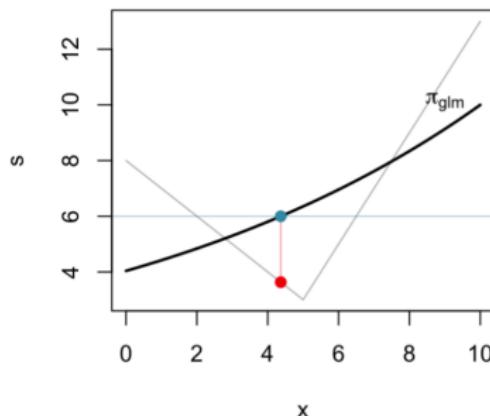


Natural idea: plot  $s \mapsto \mathbb{E}[Y|\hat{\pi}(\mathbf{X}) = s]$  “ $= \mu(\hat{\pi}^{-1}(s))$ ”

## Computing $\mathbb{E}[Y|\hat{\pi}(\mathbf{X}) = s]$

True model  $\mu(x)$  and GLM model  $\hat{\pi}(x)$

- ▶ select  $s$ , e.g.  $s = 6$
- ▶ compute  $x = \hat{\pi}^{-1}(s)$  ( $\hat{\pi}$  is strictly increasing), here  $x = 4.2$
- ▶ compute  $\mu(\hat{\pi}^{-1}(s))$ , here 3.8
- ▶ plot  $(s, \mu(\hat{\pi}^{-1}(s)))$

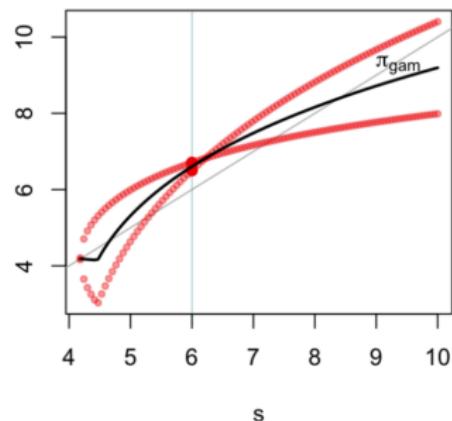
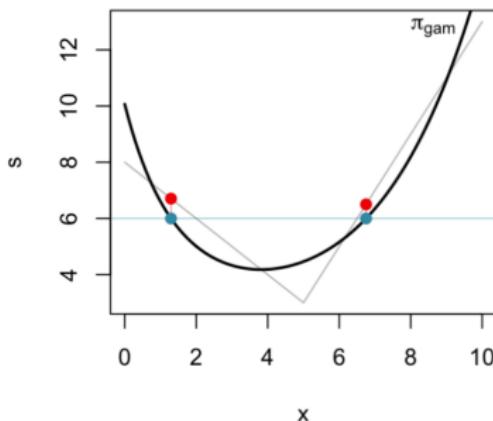


$$s \mapsto \mathbb{E}[Y|\hat{\pi}(X) = s] = \mu(\hat{\pi}^{-1}(s)) \text{ since } \mu(x) = \mathbb{E}[Y|X = x].$$

# Computing $\mathbb{E}[Y|\hat{\pi}(\mathbf{X}) = s]$

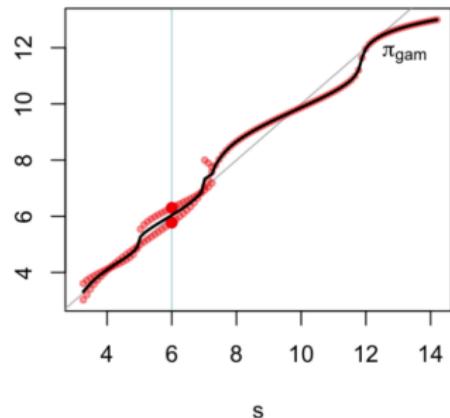
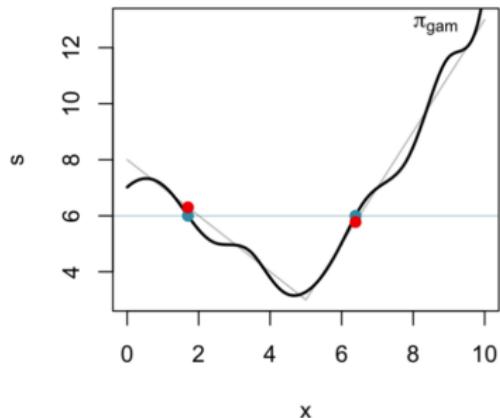
True model  $\mu(x)$  and GAM model  $\hat{\pi}(x)$

- ▶ select  $s$ , e.g.  $s = 6$
- ▶ compute  $\mathcal{X}_s^{\hat{\pi}} = \{\mathbf{x} \in \mathcal{X}, \hat{\pi}(\mathbf{x}) = s\}$ , here  $\{1.6; 6.7\}$
- ▶ compute  $\mu(\hat{\pi}^{-1}(s))$ ,  $\{6.4, 6.2\}$  and its mean  $\bar{\mu}(\hat{\pi}^{-1}(s))$ , 6.3
- ▶ plot  $(s, \bar{\mu}(\hat{\pi}^{-1}(s)))$



## Computing $\mathbb{E}[Y|\hat{\pi}(X) = s]$

True model  $\mu(x)$  and GAM model  $\hat{\pi}(x)$  (more degrees of freedom)  
Plot  $s \mapsto \mathbb{E}[Y|\hat{\pi}(X) = s]$ , should be close to the first diagonal



Seems locally unbiased...

- ▶ impossible to get the figure on the left in higher dimension
- ▶ we used here  $\mu$  but in practice,  $\mu$  is unknown !

# $\mathbb{E}[Y|\hat{\pi}(X) = s]$ : empirical version

How to plot  $s \mapsto \mathbb{E}[Y|\hat{\pi}(X) = s]$  ?

but in real life,  $\mu$  is unknown

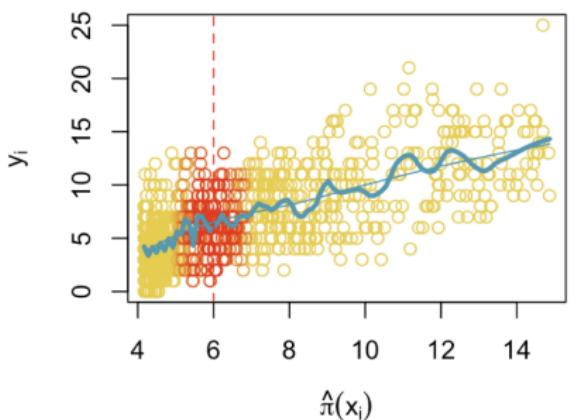
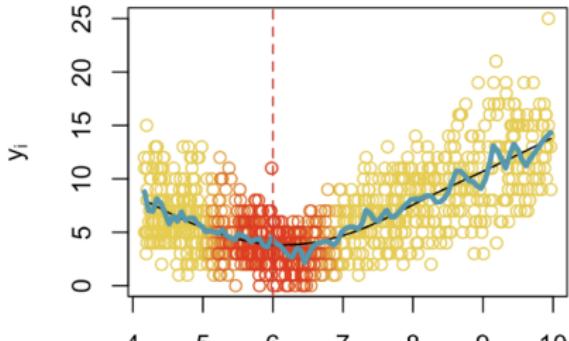
Consider  $\{(\hat{\pi}(x_i), y_i)\}$

and fit a local regression

- ▶ fit a model  $\hat{\pi}$
- ▶ estimate  $\mathbb{E}[Y|\hat{\pi}(X) = s]$   
local regression on  $\{(\hat{\pi}(x_i), y_i)\}$
- ▶ local (multiplicative) correction

$$\lambda_\alpha(s) = \frac{\mathbb{E}[Y|\hat{\pi}(X) = s]}{s}$$

- ▶ correct  $\hat{\pi}$
- $$\hat{\pi}_{BC}(x) = \lambda_\alpha(\hat{\pi}(x)) \cdot \hat{\pi}(x)$$

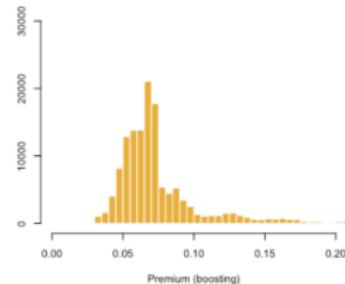
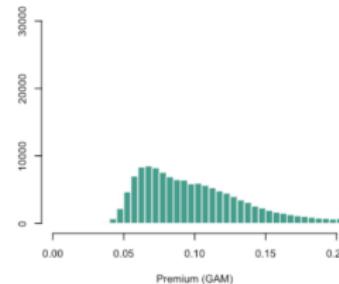
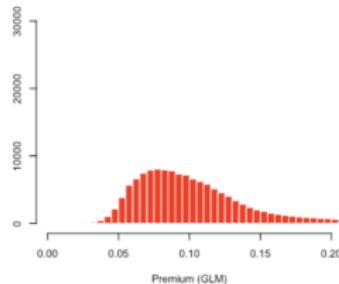


# Application of a motor-insurance dataset

Here we focus only on claims (annual) frequency, corrected from the exposure, `freMTPL2freq` from `CASDataset` package, 

	$\pi^{\text{glm}}$	$\pi^{\text{gam}}$	$\pi^{\text{bst}}$
average $\bar{\pi}$	0.1087	0.1092	0.0820
10% quantile	0.0605	0.0598	0.0498
90% quantile	0.1682	0.1713	0.1244

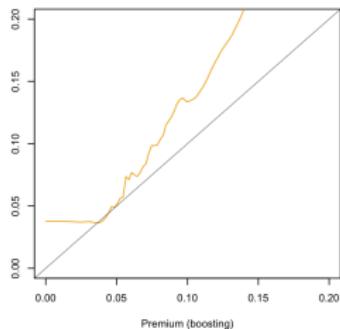
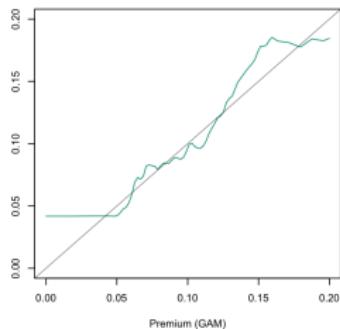
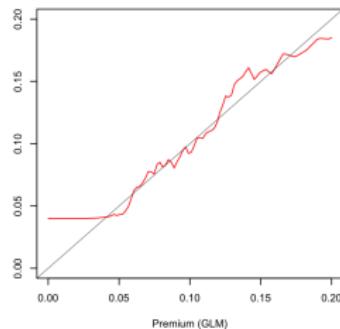
Table 1: Summary statistics on  $\{\pi(\mathbf{x}_1), \dots, \pi(\mathbf{x}_n)\}$ , on the validation dataset (assuming an exposure of 1 to provide annualized predictions).



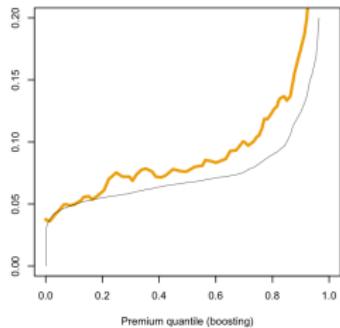
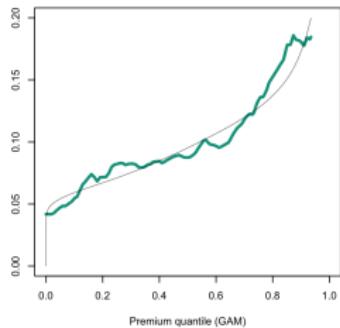
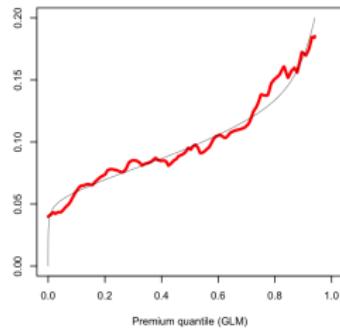
freakonometrics

freakonometrics.hypotheses.org

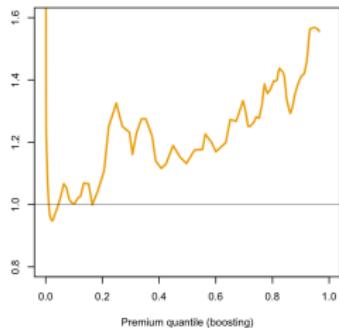
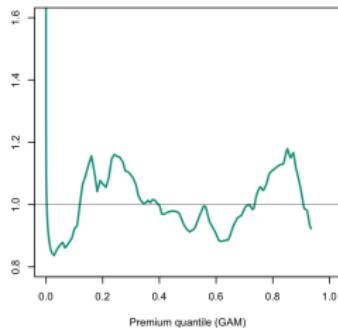
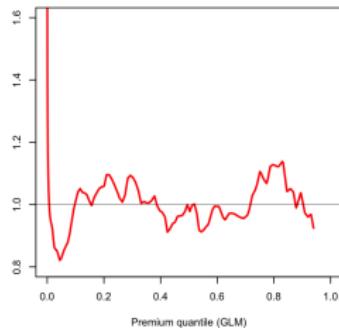
# Application of a motor-insurance dataset



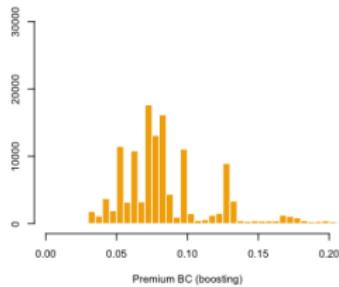
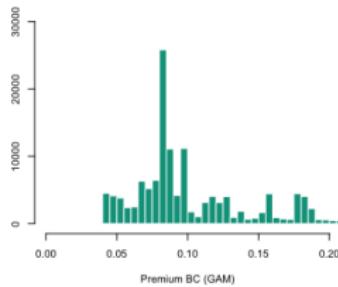
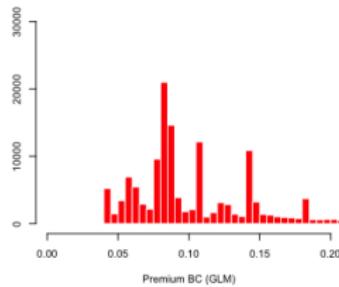
Evolution of  $s \mapsto \mathbb{E}[Y|\hat{\pi}(\mathbf{X}) = s]$  and  $u \mapsto \mathbb{E}[Y|\hat{\pi}(\mathbf{X}) = F_\pi^{-1}(u)]$



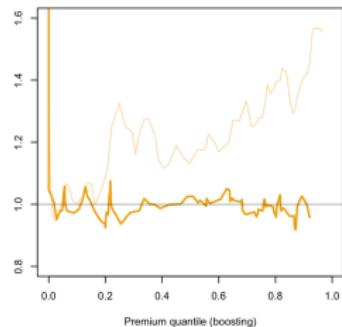
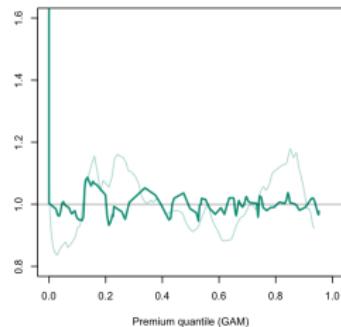
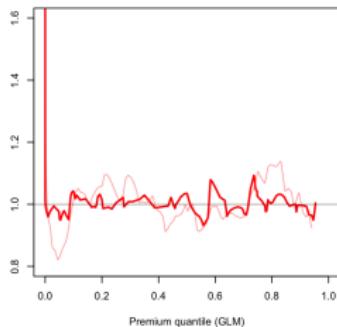
# Application of a motor-insurance dataset



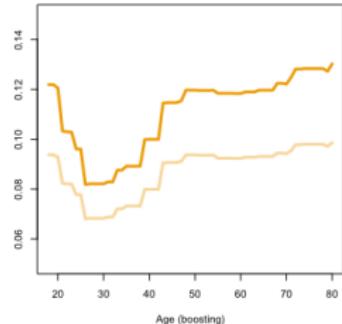
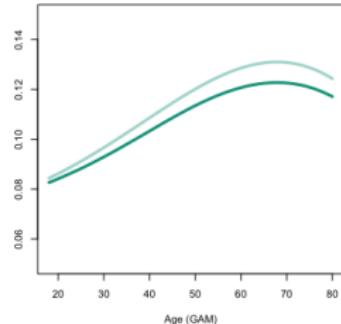
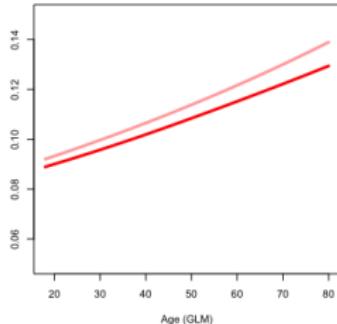
Multiplicative correction  $\lambda_\alpha(u) = \mathbb{E}[Y|\hat{\pi}(\mathbf{X}) = F_{\hat{\pi}}^{-1}(u)]/F_{\hat{\pi}}^{-1}(u)$



# Application of a motor-insurance dataset

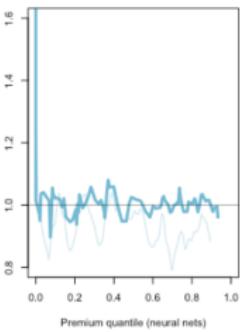
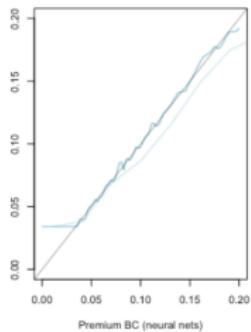
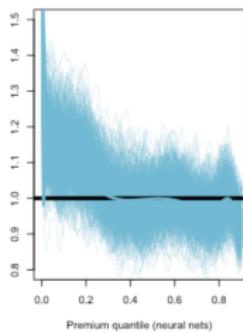
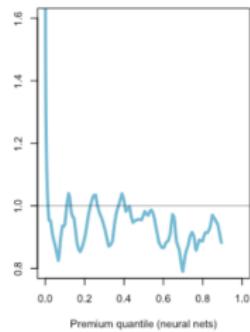


$\lambda_\alpha$  on the corrected model  $\hat{\pi}_{BC}$ , and some partial dependence plot  
(on the age of the driver)



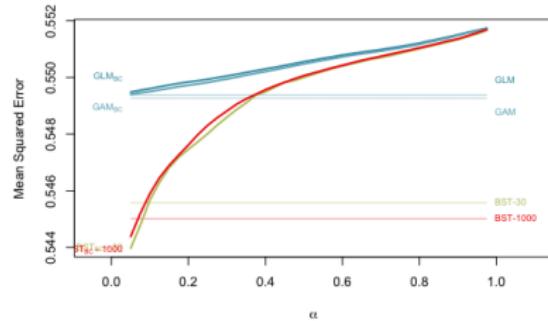
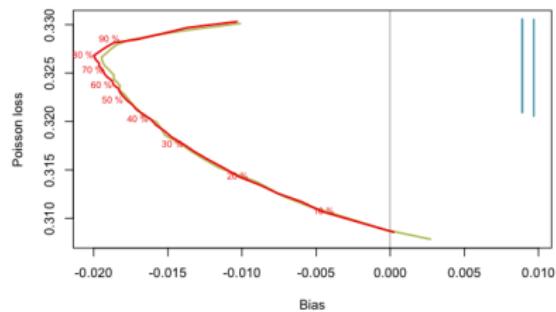
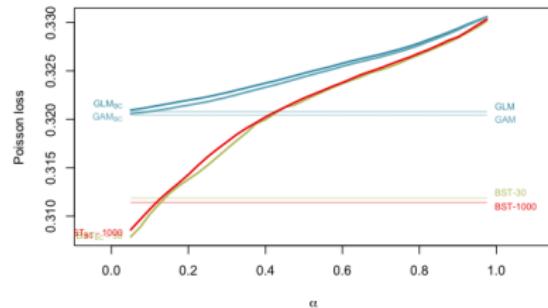
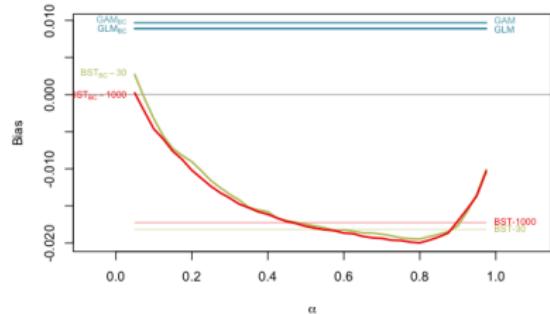
# Using Neural Nets on motor-insurance dataset

We can also look at neural networks performance

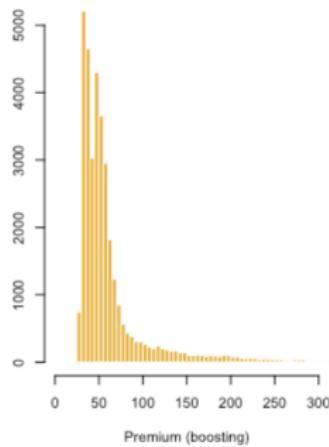
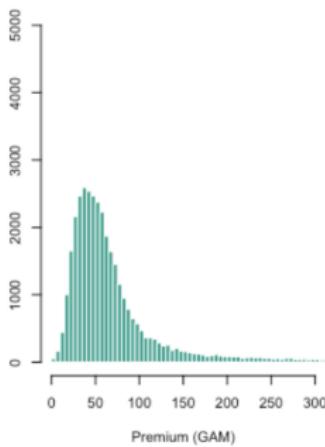
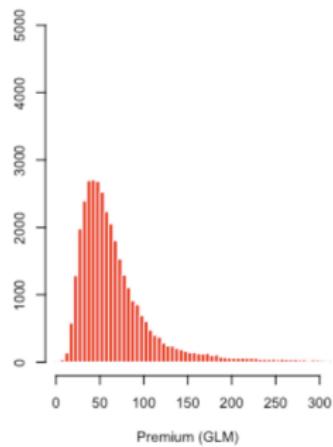


Reproducibility issues here

# Choice of $\alpha$



# Tweedie model on motor-insurance dataset



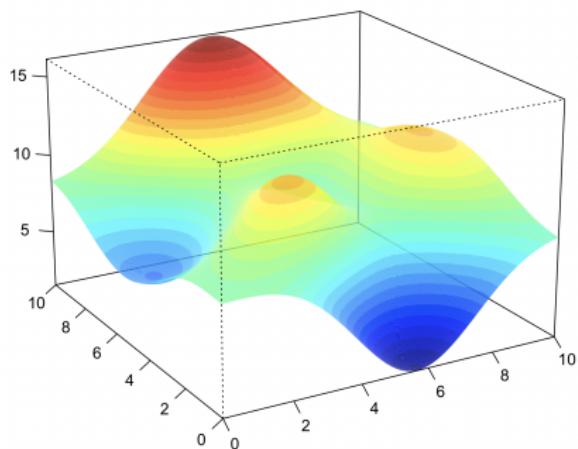
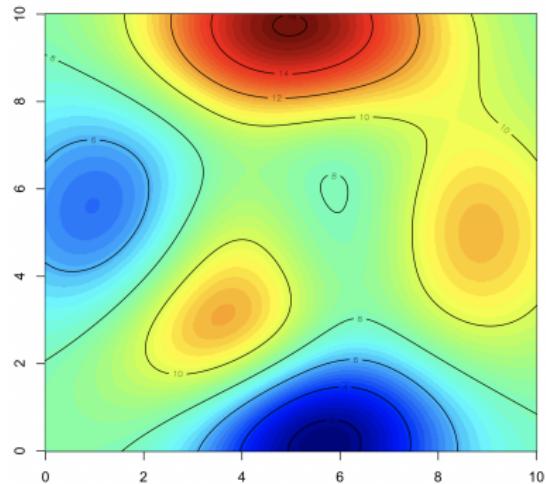
## Wrap-Up

- ▶ fit a model  $\hat{\pi}$  using an ML algorithm
- ▶ estimate  $\mathbb{E}[Y|\hat{\pi}(X)]$  with local regression on  $\{(\hat{\pi}(\mathbf{x}_i), y_i)\}$
- ▶ local (multiplicative) correction  $\lambda_\alpha(s) = \frac{\mathbb{E}[Y|\hat{\pi}(X) = s]}{s}$
- ▶ correct  $\hat{\pi}$  by setting  $\hat{\pi}_{BC}(\mathbf{x}) = \lambda_\alpha(\hat{\pi}(\mathbf{x})) \cdot \hat{\pi}(\mathbf{x})$

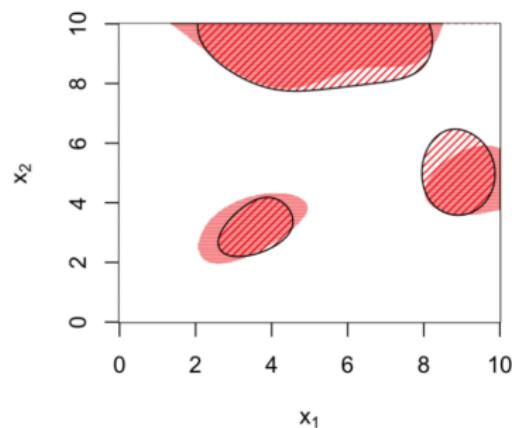
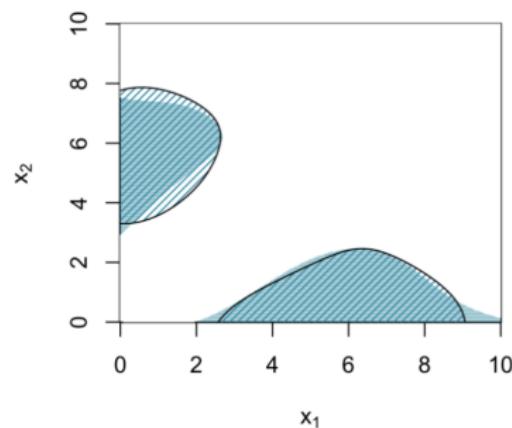
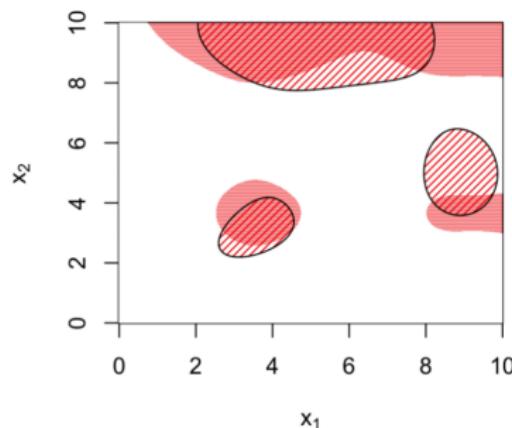
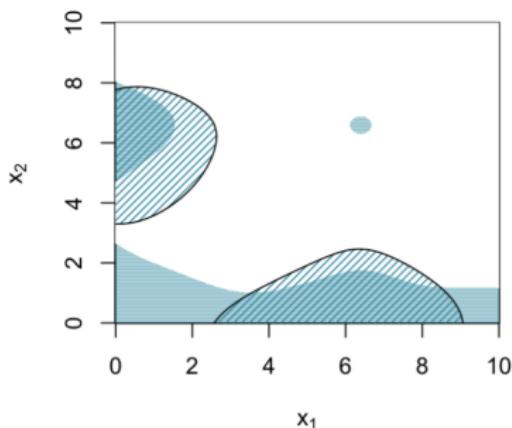
Denuit, M., Charpentier, A. & Trufin, J. (2021). Autocalibration and Tweedie-dominance for Insurance Pricing with Machine Learning, (IME)

## Appendix: Bivariate simulated data

Generate some data  $(x_{1,i}, x_{2,i}, y_i)$  where  $Y|\mathbf{X}$  has some distribution with mean  $\mathbb{E}[Y|\mathbf{x}] = \mu(\mathbf{x})$



Model inversion,  $\mathcal{X}_s^{\hat{\pi}} = \{x \in \mathcal{X}, \hat{\pi}(x) \leq s\}$



$\mathcal{X}_s^{\widehat{\pi}}$ ,  $Y | \mathbf{X} \in \mathcal{X}_s^{\widehat{\pi}}$  and  $\mathbb{E}[Y | \widehat{\pi}(\mathbf{X}) = s]$ :  $s = 10$

