Risk Sharing on Irregular Networks

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Preferences and Dispersion

Following Hardy et al. (1929, 1934), and Marshall and Olkin (1979)

Def Consider two sorted vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ ($x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$) such that $\displaystyle\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, then $\boldsymbol{x} \preceq_M \boldsymbol{y}$ (majorization order) if $\displaystyle\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$, $\forall k$.

$$\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}, \frac{1}{n}\right) \prec_M \left(\frac{1}{n-1}, \frac{1}{n} - 1, \cdots, \frac{1}{n-1}, 0\right) \prec_M (1, 0, \cdots, 0, 0).$$

Def Consider two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\displaystyle\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, then $\boldsymbol{x} \preceq_M \boldsymbol{y}$ if

$$\text{either } \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]},\ \forall k \quad \text{or } \sum_{i=1}^k x_{(i)} \geq \sum_{i=1}^k y_{(i)},\ \forall k$$

where $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ (increasing) while $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ (decreasing).

Preferences and Dispersion

If $\boldsymbol{x} \in \mathcal{S}_n$, $\dfrac{1}{n}\mathbf{1} \prec_M \boldsymbol{x} \prec_M e_i$, $\forall i$, where $e_i = (0, \cdots, 0, 1, , 0 \cdots, 0)$.

Prop $\boldsymbol{x} \preceq_M \boldsymbol{y}$

$\Longleftrightarrow$ $\displaystyle\sum_{i=1}^n h(x_i) \leq \sum_{i=1}^n h(y_i)$ for any convex function

$\Longleftrightarrow$ $\displaystyle\sum_{i=1}^n (x_i - d)_+ \leq \sum_{i=1}^n (y_i - d)_+$ for $d \in \mathbb{R}$

$\Longleftrightarrow$ $\boldsymbol{x} = D\boldsymbol{y}$ for some doubly stochastic matrix D, i.e. $\displaystyle\sum_{k=1}^n D_{i,k} = \sum_{k=1}^n D_{k,j} = 1$, $\forall i, j$

$\Longleftrightarrow$ $\boldsymbol{x} = P_1 \cdots P_k \boldsymbol{y}$ for finitely some (Muirhead)-Pigou-Dalton transfert matrices P_j ($P_j = \alpha\mathbb{I} + (1-\alpha)T$ for some $\alpha \in (0,1)$ and $T = 0$ except $T_{i,j} = T_{j,i} = 1$)

Risk Aversion and Risk Sharing

Def Consider two random variables X and Y, $X \preceq_{CX} Y$ if $\mathbb{E}[h(X)] \leq \mathbb{E}[h(Y)]$ for any convex function h

$\Longleftrightarrow$ Y is a mean-preserving spread of X, i.e. $Y \overset{\mathcal{L}}{=} X + Z$, where $\mathbb{E}[Z|X] = 0$.

$\Longleftrightarrow$ $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $s \in \mathbb{R}$.

$\Longrightarrow$ $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\text{Var}[X] \preceq \text{Var}[Y]$.

Prop (6 in Kaas et al. (2002)) Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ denote a collection of variables and $\boldsymbol{X}^+$ a comonotonic version, then $\boldsymbol{a}^\top \boldsymbol{X} \preceq_{CX} \boldsymbol{a}^\top \boldsymbol{X}^+$.

Prop (3.4.48 in Denuit et al. (2005)) Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ denote a collection of i.i.d. variables, if $\boldsymbol{a} \prec \boldsymbol{b}$ for the majorization order, $\boldsymbol{a}^\top \boldsymbol{X} \preceq_{CX} \boldsymbol{b}^\top \boldsymbol{X}$.

As a consequence

Prop Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ denote a collection of i.i.d. variables, and $\boldsymbol{p}$ some n-dimensional probability vector. Then $\boldsymbol{p}^\top \boldsymbol{X} \preceq_{CX} X_i$ for any i.

Risk Aversion and Risk Sharing

Def Consider two random vectors $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ and $\boldsymbol{X} = (X_1, \cdots, X_n)$ on $\mathbb{R}_+^n$.
$\boldsymbol{\xi}$ is a risk-sharing scheme of $\boldsymbol{X}$ if $X_1 + \cdots + X_n = \xi_1 + \cdots + \xi_n$ almost surely.

Following Denuit and Dhaene (2012) and Carlier et al. (2012),

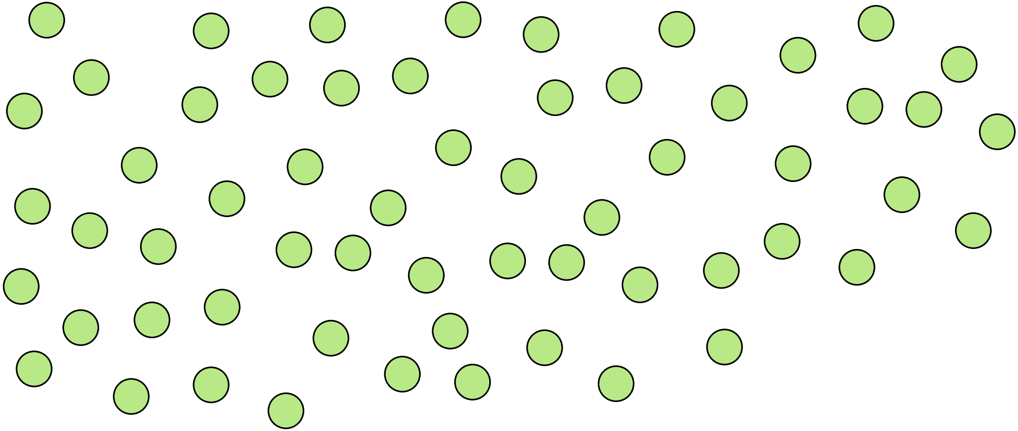
Def Consider two random vectors $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ and $\boldsymbol{X} = (X_1, \cdots, X_n)$ on $\mathbb{R}_+^n$.
$\boldsymbol{\xi} \preceq_{CCX} \boldsymbol{X}$ if $\xi_i \preceq_{CX} X_i$.

Def A risk sharing scheme $\boldsymbol{\xi}$ of $\boldsymbol{X}$ is desirable if $\boldsymbol{\xi} \preceq_{CCX} \boldsymbol{X}$.

Let $\xi_j = \overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i, \;\forall j$

- Risk sharing : $\xi_1 + \cdots + \xi_n = X_1 + \cdots + X_n$
- Desirable (componentwise convex-order) : $\xi_j \preceq_{CX} X_j, \;\forall j$

Risk Sharing



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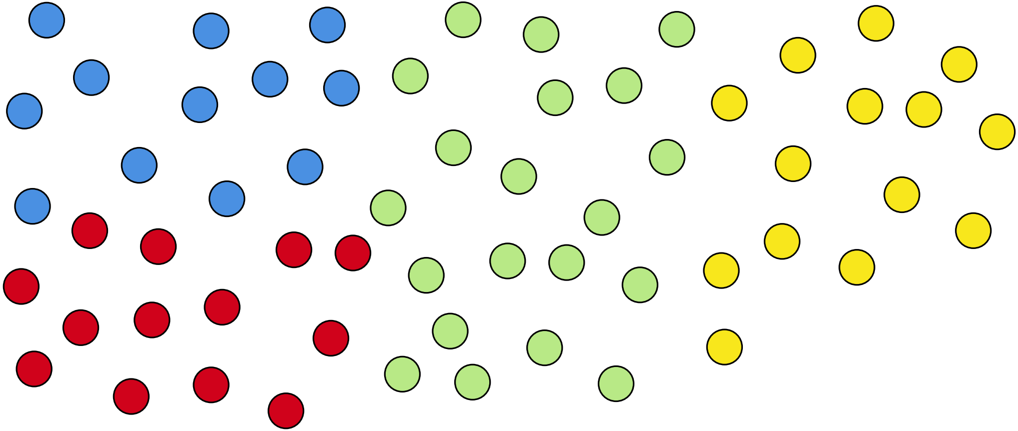
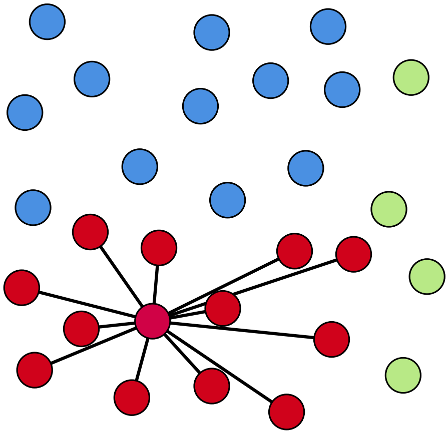


Risk Sharing



Let $\xi_j = \frac{1}{n}\sum_{i=1}^{n} X_i, \ \forall j$

- Risk sharing

$\xi_1 + \cdots + \xi_n = X_1 + \cdots + X_n$

- Componentwise convex-order

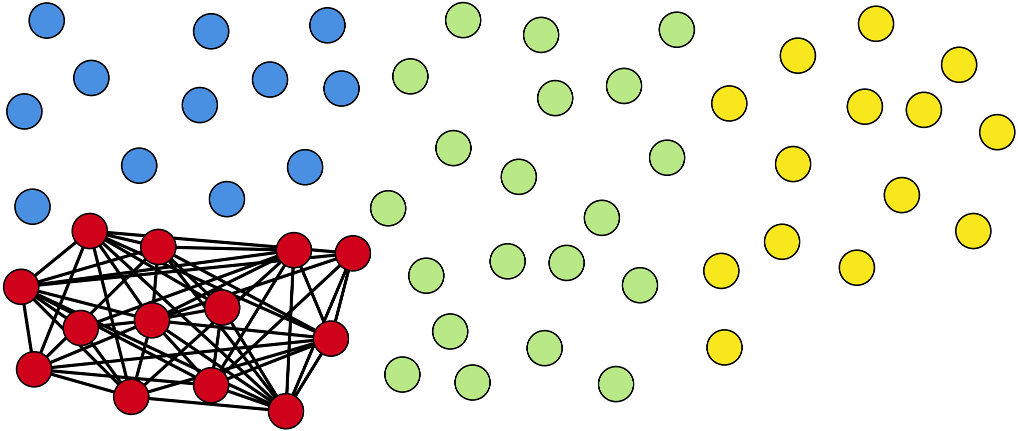
$\xi_j \preceq_{CX} X_j, \ \forall j$

More generally, consider some linear risk sharing $\boldsymbol{\xi} = M\boldsymbol{X}$, for some $n \times n$ matrix

$$M = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{M}_k \end{bmatrix}, \ \mathbf{M}_k = \frac{1}{n_k}\mathbf{1}_k$$

where $\mathbf{1}_k$ is the $n_k \times n_k$ matrix full of 1's.

Risk Sharing



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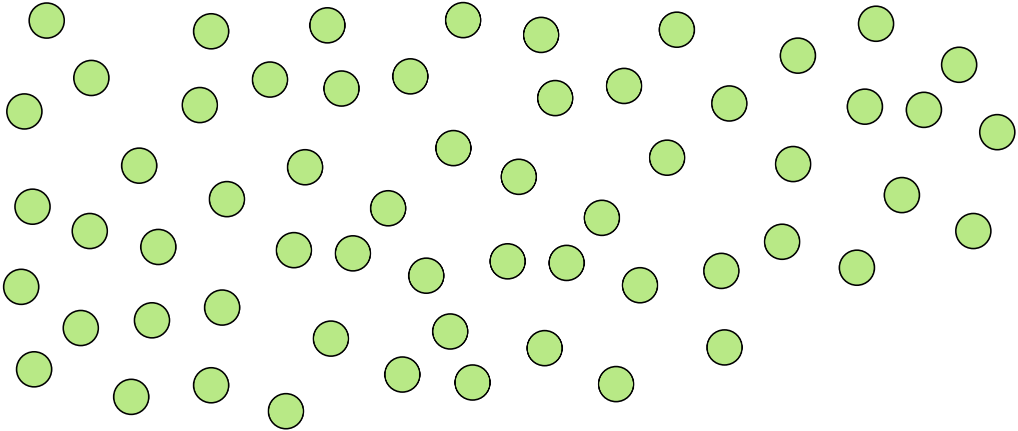
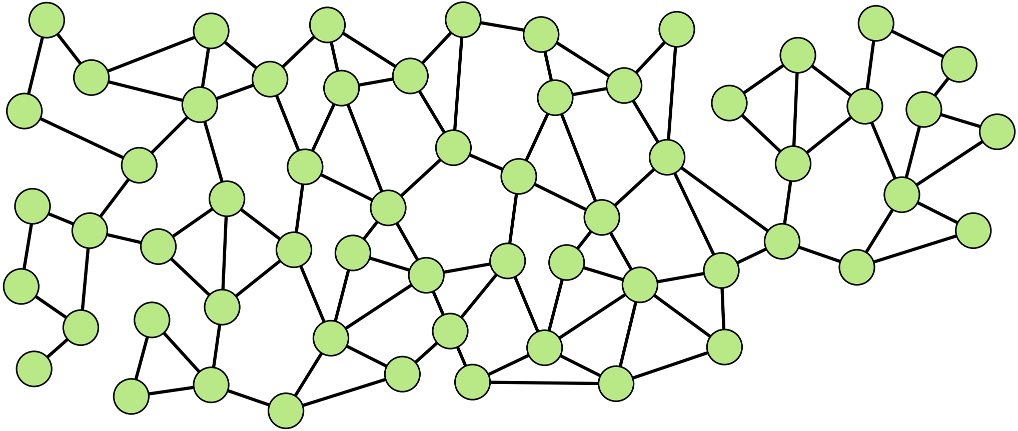


Risk Sharing on a Network



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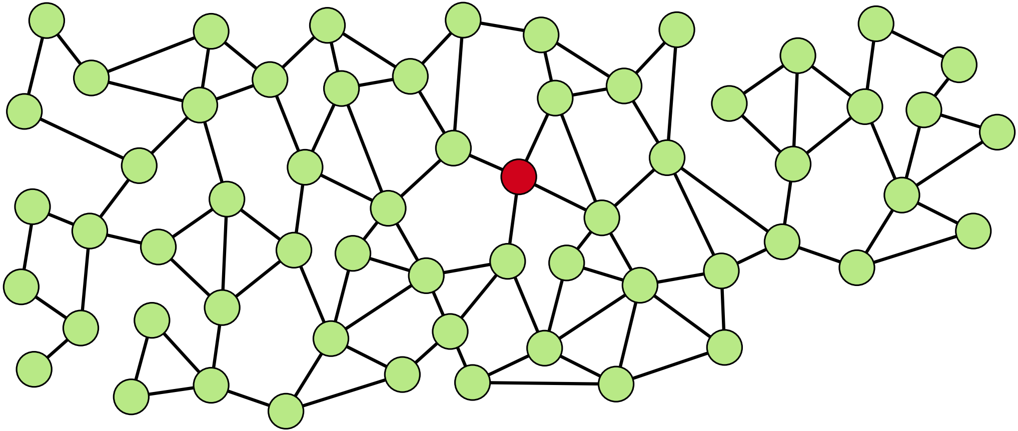

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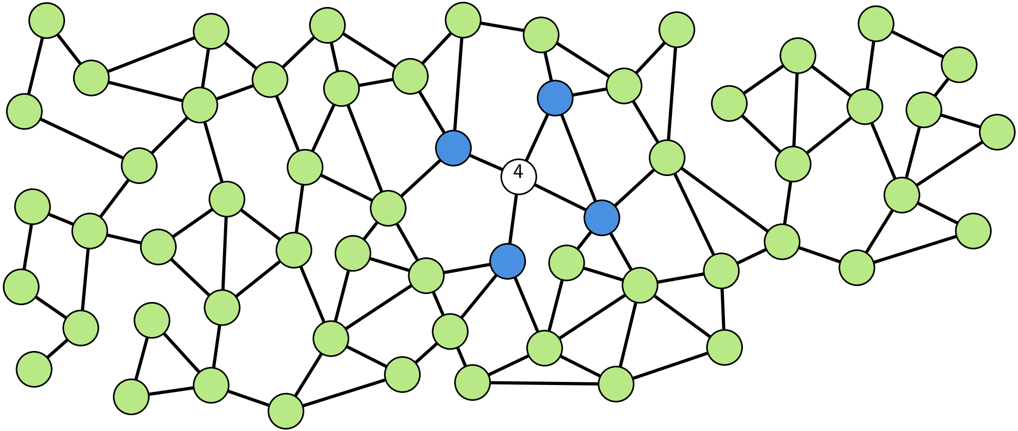

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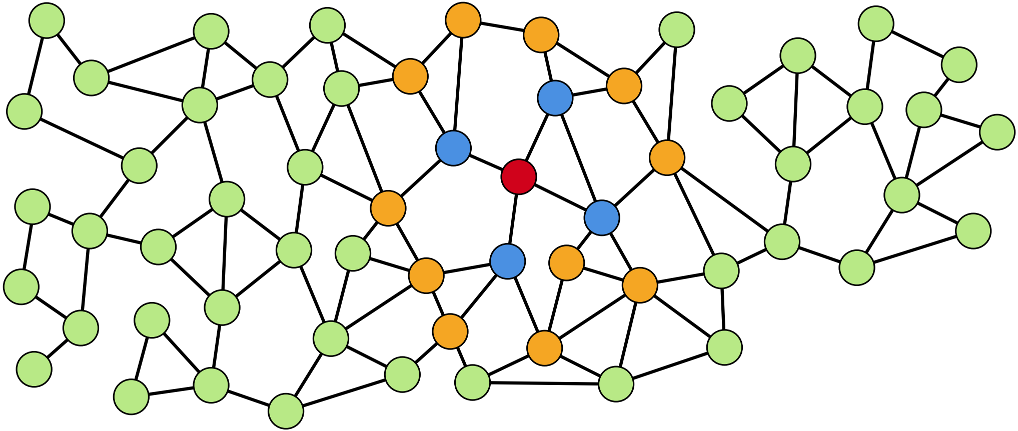

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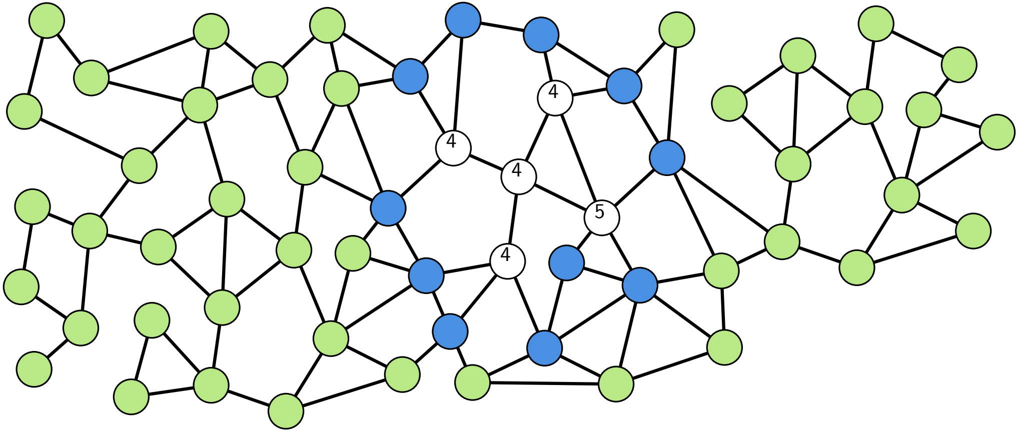

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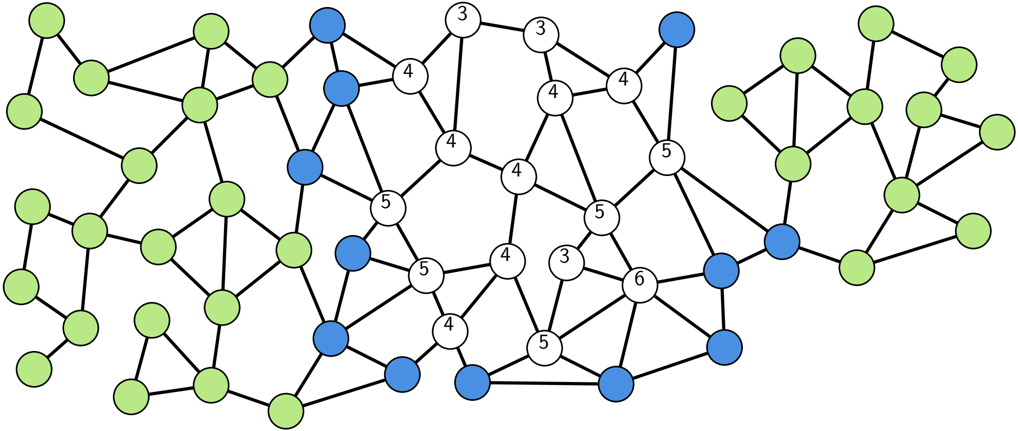

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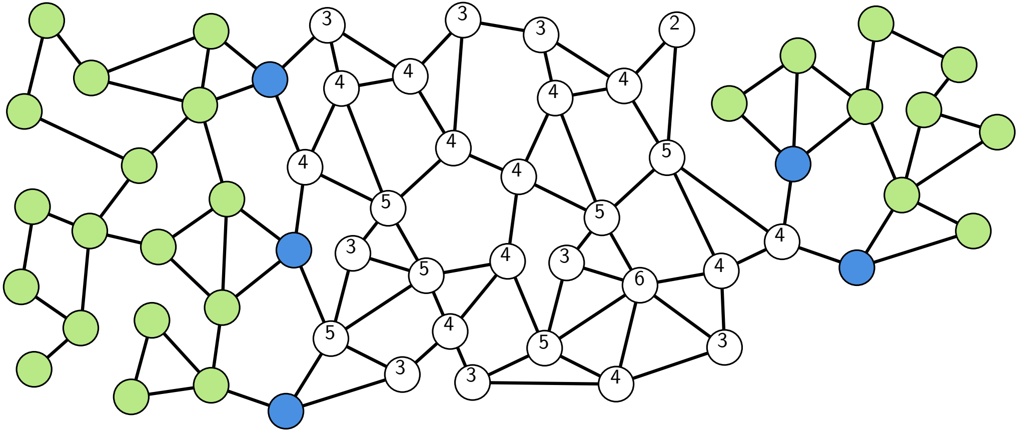

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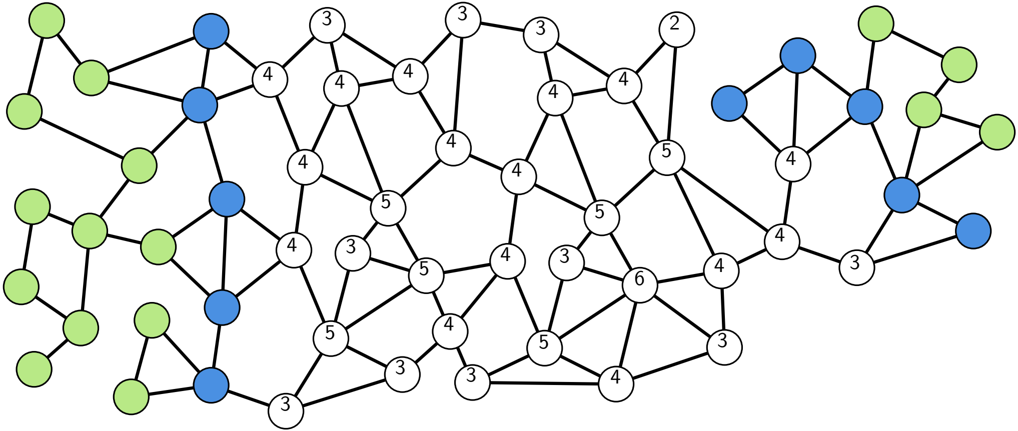

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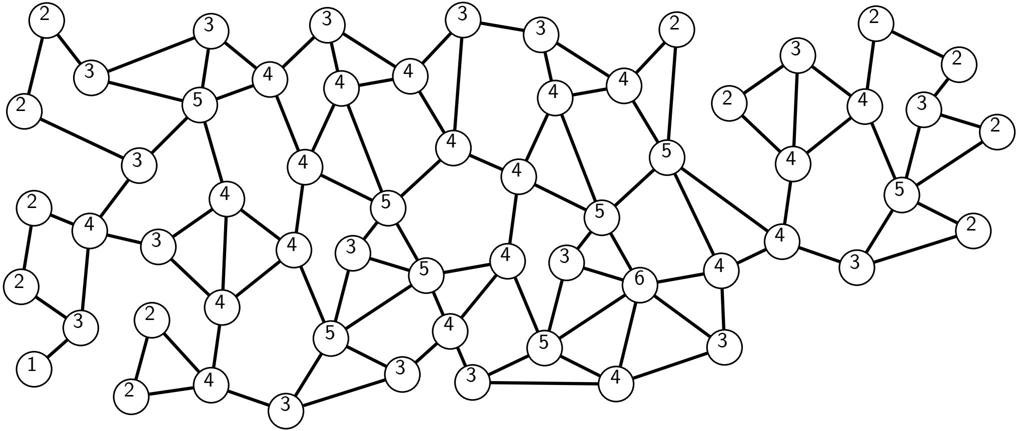

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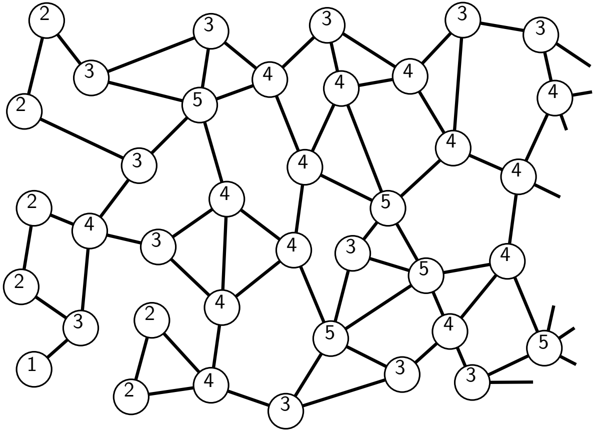

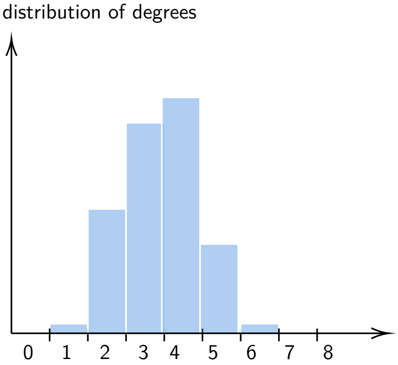

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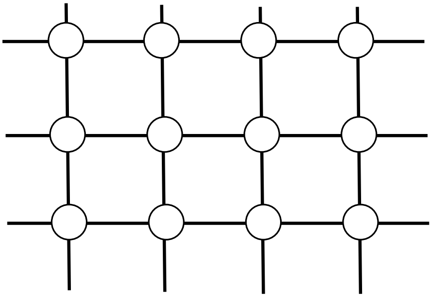
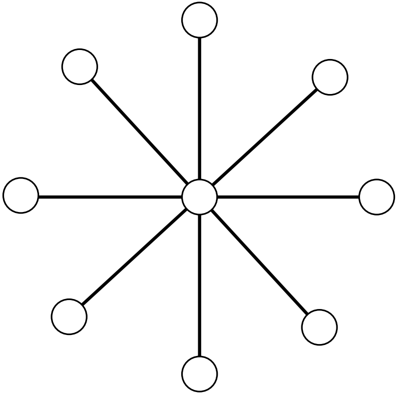




Regular graph vs. star shaped graph
(low variance vs. large variance on D)

Linear Risk Sharings

Def Consider two random vectors $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$ and $\boldsymbol{X} = (X_1, \cdots, X_n)$ on $\mathbb{R}^n_+$, such that $\boldsymbol{\xi}$ is a risk-sharing scheme of $\boldsymbol{X}$, i.e. $\boldsymbol{\xi}^\top \mathbf{1} = \boldsymbol{X}^\top \mathbf{1}$. It is said to be a linear risk sharing scheme if there exists a matrix M, $n \times n$, with positive entries, such that $\boldsymbol{\xi} = M\boldsymbol{X}$, almost surely.

Ex $\bar{M} = [\bar{M}_{i,j}]$ with $\bar{M}_{i,j} = 1/n$.

Ex $M_\alpha = \alpha\mathbb{I} + (1-\alpha)\bar{M}$.

Ex M is a doubly stochastic matrix D, (row and column conditions on D are called "*zero-balance conservation*" in Feng et al. (2021)).

Prop Consider two linear risk sharing schemes $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ of $\boldsymbol{X}$, such that there is a doubly stochastic matrix D, $n \times n$ such that $\boldsymbol{\xi}_2 = D\boldsymbol{\xi}_1$. Then $\boldsymbol{\xi}_2 \preceq_{CCX} \boldsymbol{\xi}_1$.

Linear Risk Sharings within cliques

Def A network (or undirected graph) is a collection of vertices and edges, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Def A clique $\mathcal{C}_i$ within an network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a subset of vertices $\mathcal{C}_i \subseteq \mathcal{V}$ such that every two distinct vertices are adjacent.

Thus, the induced subgraph of $\mathcal{G}$ by $\mathcal{C}_i$ is complete.

Def A clique cover is a partition of the graph $\mathcal{G}$ into a set of cliques.

Assume that policyholders face random losses $\boldsymbol{X} = (X_1, \cdots, X_k, X_{k+1}, \cdots, X_{k+m})$ and consider the following risk sharing

$$\xi_i = \begin{cases} \displaystyle\frac{1}{k}\sum_{j=1}^{k} X_j \text{ if } i \in \{1, 2, \cdots, k\} \\ \displaystyle\frac{1}{m}\sum_{j=k+1}^{k+m} X_j \text{ if } i \in \{k+1, k+2, \cdots, k+m\} \end{cases}$$

Linear Risk Sharings within cliques

This is a linear risk sharing, with sharing matrix $D_{k,m}$, so that $\boldsymbol{\xi} = D_{k,m}\boldsymbol{X}$, defined as

$$D_{k,m} = \begin{array}{c c} & \begin{array}{ccccccc} 1 & \cdots & k & k+1 & k+2 & \cdots & k+m \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ k \\ k+1 \\ k+2 \\ \vdots \\ k+m \end{array} & \begin{pmatrix} k^{-1} & \cdots & k^{-1} & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ k^{-1} & \cdots & k^{-1} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & m^{-1} & m^{-1} & \cdots & m^{-1} \\ 0 & \cdots & 0 & m^{-1} & m^{-1} & \cdots & m^{-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & m^{-1} & m^{-1} & \cdots & m^{-1} \end{pmatrix} \end{array} = \begin{bmatrix} \mathbf{M}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_m \end{bmatrix},$$

where $\mathbf{M}_k = \dfrac{1}{k}\mathbf{1}_k$. Since $D_{k,m}$ is a doubly-stochastic matrix, $\boldsymbol{\xi} \prec_{CCX} \boldsymbol{X}$.

Linear Risk Sharings within cliques

Pick someone at random (let I denote a uniform variable over $\{1, 2, \cdots, n\}$, and $\xi' = \xi_I$), with i.i.d. losses X_i

$$\mathbb{E}[\xi'] = \mathbb{E}[\mathbb{E}[\xi_I|I]] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[\xi_i] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \mathbb{E}[X]$$

$$\begin{aligned}\text{Var}[\xi'] = \mathbb{E}[\text{Var}[\xi_I|I]] = \frac{1}{n}\sum_{i=1}^{n}\text{Var}[\xi_i] &= \frac{1}{n}\left(k\frac{\text{Var}[X]}{k^2} + m\frac{\text{Var}[X]}{m^2}\right)\\ &= \frac{\text{Var}[X]}{n}\left(\frac{1}{k} + \frac{1}{m}\right) = \frac{\text{Var}[X]}{k(n-k)} < \text{Var}[X]\end{aligned}$$

$k \mapsto k(n-k)$ is maximal when $k = \lfloor n/2 \rfloor$, which means that risk sharing benefit is maximal (socially maximal, for a randomly chosen representative policyholder) when the two cliques have the same size.

A Weaker Linear Risk Sharing

Following Martínez Pería et al. (2005), a weaker ordering can be considered with row-stochastic matrices, instead of doubly stochastic matrices. A row-stochastic matrix (or left-stochastic matrix) C satisfies

$$R = [R_{i,j}] \text{ where } R_{i,j} \geq 0, \text{ and } \sum_{i=1}^{n} R_{i,j} = 1 \ \forall j.$$

Prop Let R be some $n \times n$ row-stochastic matrix, and given $\boldsymbol{X}$, a positive vector in $\mathbb{R}^{+}$, define $\xi = R\boldsymbol{X}$. Then ξ is a linear risk sharing of $\boldsymbol{X}$.

Following Martínez Pería et al. (2005), we can define an ordering based those row-stochastic matrices

Def Consider two linear risk sharing schemes ξ_1 and ξ_2 of $\boldsymbol{X}$. ξ_1 weakly dominates ξ_2, denoted $\xi_2 \preceq_{wCX} \xi_1$ if and only if there is a column-stochastic matrix R, $n \times n$ such that $\xi_2 = R\xi_1$.

Linear Risk Sharings with Friends

Def Given $\boldsymbol{b} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$,

$$\boldsymbol{b} \otimes A = \begin{bmatrix} b_1 \mathbf{a}_{1\cdot}^\top \\ b_2 \mathbf{a}_{2\cdot}^\top \\ \vdots \\ b_m \mathbf{a}_{m\cdot}^\top \end{bmatrix} = \begin{bmatrix} b_1 a_{11} & b_1 a_{12} & \cdots & b_1 a_{1n} \\ b_2 a_{21} & b_2 a_{22} & \cdots & b_2 a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m a_{m1} & b_m a_{m2} & \cdots & b_m a_{mn} \end{bmatrix}$$

With cliques, we had

$$M = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{M}_k \end{bmatrix}, \ \mathbf{M}_k = \frac{1}{n_k}\mathbf{1}_k \text{ or } M = \begin{bmatrix} \boldsymbol{n}_1^{-1} \\ \boldsymbol{n}_2^{-1} \\ \vdots \\ \boldsymbol{n}_k^{-1} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{1}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_k \end{bmatrix},$$

where $\mathbf{1}_k$ is the $n_k \times n_k$ matrix full of 1's, $\boldsymbol{n}_k = (n_k, \cdots, n_k)$.

Linear Risk Sharings with Friends

But we can consider risk sharings based on more general networks,

Lemma Given an adjacency matrix A of an undirected graph, if $\boldsymbol{d}$ denotes the vector of degrees $\boldsymbol{d} = A\mathbf{1}$, then $M = \boldsymbol{d}^{-1} \otimes A$ is a row-stochastic matrix.

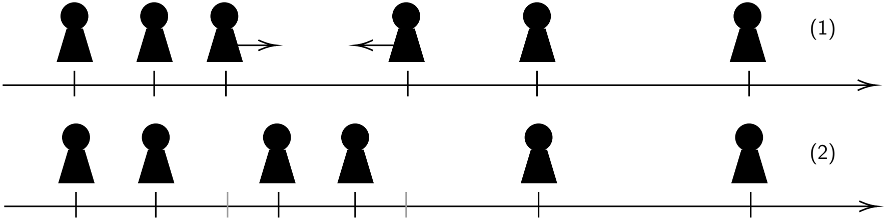
Ex $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. and $M = \boldsymbol{d}^{-1} \otimes A = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$.

Def Define $\mathcal{I}_n = \{1, 2, \cdots, n\}$, let $\mathcal{G}_A$ denote the network $\mathcal{G}_A = (\mathcal{I}_n, \mathcal{E}_A)$, associated with adjacency matrix A.

Def Given a network $\mathcal{G}_A = (\mathcal{I}_n, \mathcal{E}_A)$, the linear risk sharing $\boldsymbol{\xi}$ of $\boldsymbol{x} \in \mathbb{R}_+^n$, through $\mathcal{G}_A$ is $\boldsymbol{\xi}_A = \boldsymbol{d}^{-1} \otimes A\boldsymbol{x}$,

$$\boldsymbol{\xi}_A = \boldsymbol{d}^{-1} \otimes A\boldsymbol{x} = \text{ i.e. } \xi_j = \frac{1}{d_j} A_j^\top \boldsymbol{x} = \frac{1}{d_j} \sum_{i \in \mathcal{V}_j} x_i.$$

Stochastic Dominance and Networks



(Muirhead)-Pigou-Dalton transferts (Muirhead (1903)Dalton (1920))
see also Atkinson (2015),

$$\boldsymbol{y}^{(2)} \preceq_M \boldsymbol{y}^{(1)} \longleftarrow \begin{cases} y_i^{(2)} = y_i^{(1)}, \ \forall i \neq j, k \\ y_j^{(2)} = y_j^{(1)} + h, \\ y_k^{(2)} = y_k^{(1)} - h, \ y_j^{(2)} > y_j^{(1)} \end{cases}$$

see martingale property of mean-preserving spread, $Y^{(1)} \overset{\mathcal{L}}{=} Y^{(2)} + Z$, where $\mathbb{E}[Z|Y^{(2)}] = 0$ (convex order is a dispersion order)

Stochastic Dominance and Networks

From Marshall and Olkin (1979),

B.2. Theorem (Hardy, Littlewood, and Pólya, 1929). A necessary and sufficient condition that $x \prec y$ is that there exist a doubly stochastic matrix P such that $x = yP$.

The lemma involves a special kind of linear transformation called a *T-transformation*, or more briefly a *T-transform*. The matrix of a T-transform has the form

$$T = \lambda I + (1-\lambda)Q,$$

where $0 \leq \lambda \leq 1$ and Q is a permutation matrix that just interchanges two coordinates. Thus xT has the form

$$\begin{aligned} xT = (x_1, \ldots, x_{j-1}, \lambda x_j + (1-\lambda)x_k, x_{j+1}, \ldots, x_{k-1}, \\ \lambda x_k + (1-\lambda)x_j, x_{k+1}, \ldots, x_n). \end{aligned}$$

B.1. Lemma (Muirhead, 1903; Hardy, Littlewood, and Pólya, 1934, 1952, p. 47). If $x \prec y$, then x can be derived from y by successive applications of a finite number of T-transforms.

D Majorization in Integers

Consider the basic Lemma 2.B.1, which states that if $x \prec y$, then x can be derived from y by successive applications of a finite number of "T-transforms." Recall that a T-transform leaves all but two components of a vector unchanged, and replaces these two components by averages. If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are integers and $a \prec b$, can a be derived from b by successive applications of a finite number of T-transforms in such a way that after the application of each T-transform a vector with integer components is obtained? An affirmative answer was given by Muirhead (1903) and by Folkman and Fulkerson (1969). Using the same term as Dalton (1920), Folkman and

Fulkerson (1969) define an operation called a *transfer*. If $b_1 \geq \cdots \geq b_n$ are integers and $b_i > b_j$, then the transformation

$$\begin{aligned} b_i' &= b_i - 1, \\ b_j' &= b_j + 1, \\ b_k' &= b_k, \qquad k \neq i, j, \end{aligned}$$

is called a *transfer from i to j*. This transfer is a T-transform, because

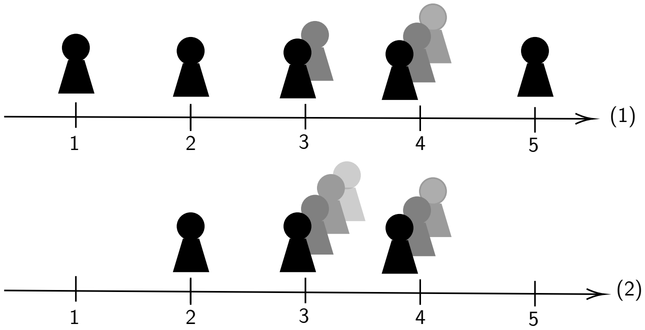
$$b_i' = \alpha b_i + (1-\alpha)b_j, \qquad b_j' = (1-\alpha)b_i + \alpha b_j,$$

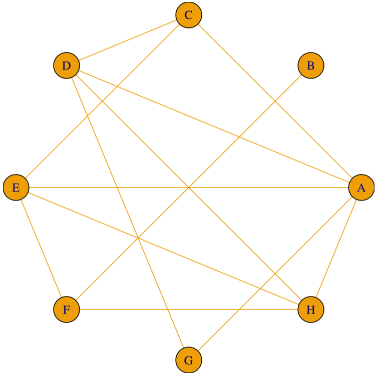
where $\alpha = (b_i - b_j - 1)/(b_i - b_j)$.

D.1. Lemma (Muirhead, 1903). If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are integers and $a \prec b$, then a can be derived from b by successive applications of a finite number of transfers.

Stochastic Dominance and Networks

$$\boldsymbol{y}^{(1)} = (5,4,4,4,3,3,2,1) \rightarrow (4,4,4,3,3,3,3,2) = \boldsymbol{y}^{(2)}$$





Stochastic Dominance and Networks

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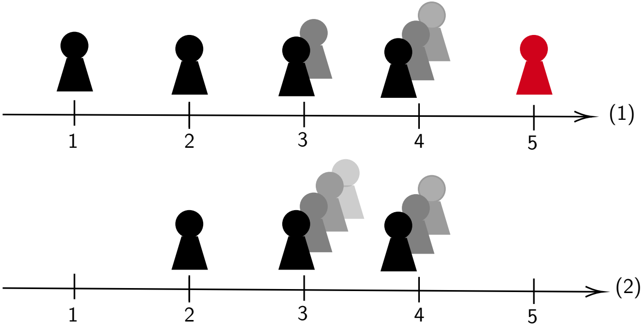
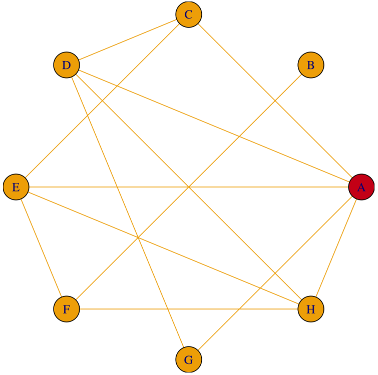
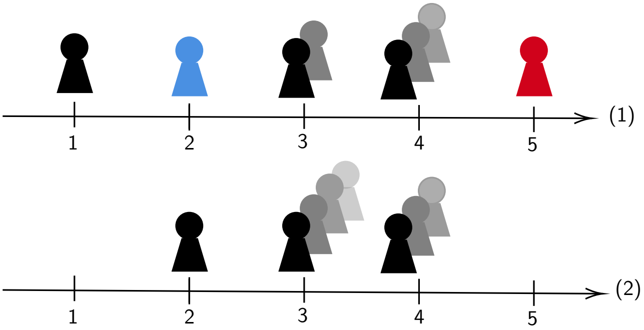


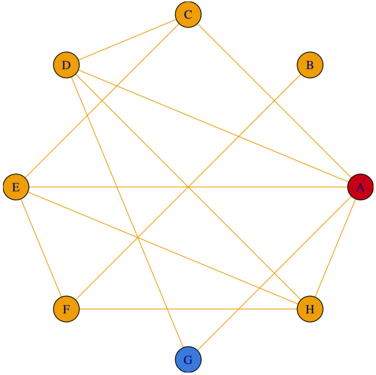


Find largest j such that $\displaystyle\sum_{i=1}^{j} y_i^{(1)} < \sum_{i=1}^{j} y_i^{(2)}$

Stochastic Dominance and Networks

$$\boldsymbol{y}^{(1)} = (5,4,4,4,3,3,2,1) \to (4,4,4,3,3,3,3,2) = \boldsymbol{y}^{(2)}$$

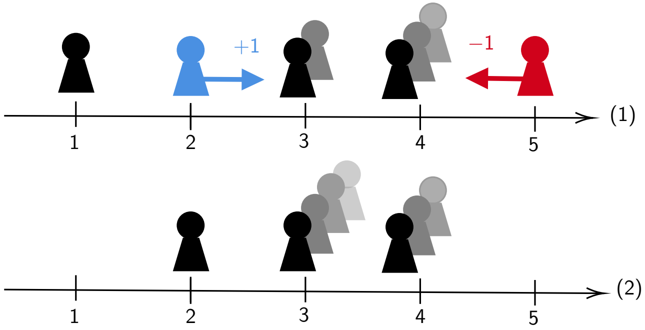


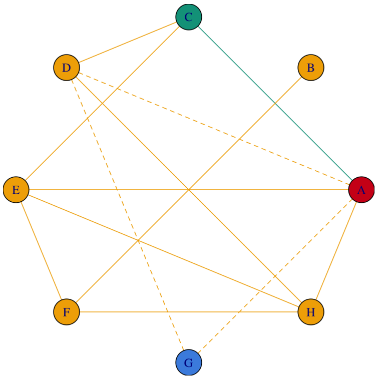


Find largest $k < j$ such that $y^{(2)}_k > y^{(1)}_k > y^{(2)}_{j+1} > y^{(1)}_{j+1}$

Stochastic Dominance and Networks

$$\boldsymbol{y}^{(1)} = (5,4,4,4,3,3,2,1) \rightarrow (4,4,4,3,3,3,3,2) = \boldsymbol{y}^{(2)}$$

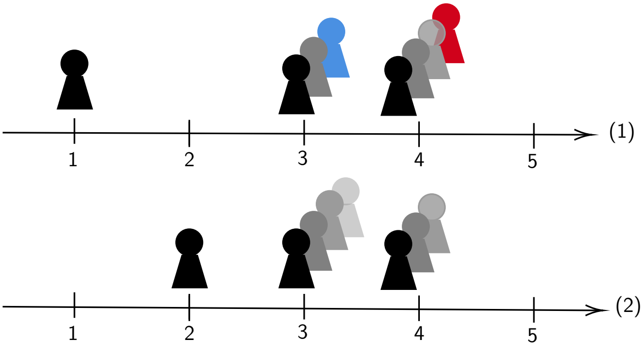


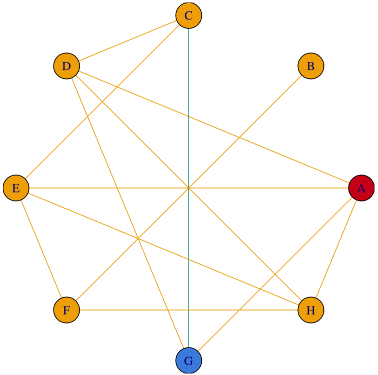


To get a transfert, we need to select ℓ

Stochastic Dominance and Networks

$$\boldsymbol{y}^{(1)} = (5,4,4,4,3,3,2,1) \rightarrow (4,4,4,3,3,3,3,2) = \boldsymbol{y}^{(2)}$$

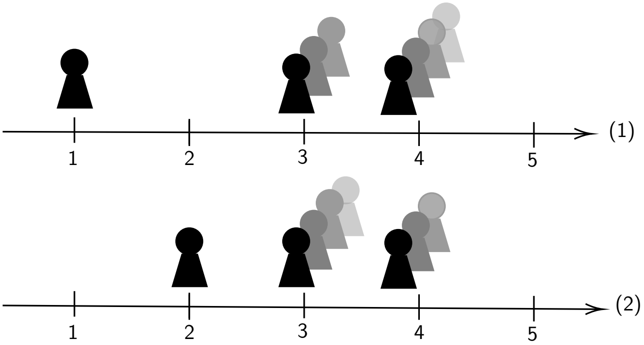


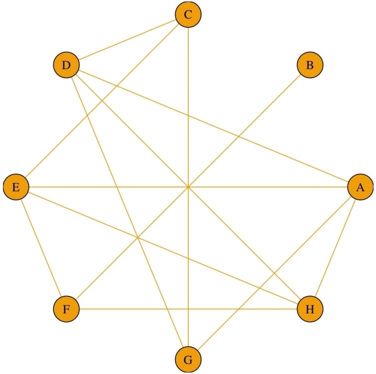


Remove edge (ℓ, j) and create (ℓ, k)

Stochastic Dominance and Networks

$$\boldsymbol{y}^{(1)} = (5,4,4,4,3,3,2,1) \rightarrow (4,4,4,3,3,3,3,2) = \boldsymbol{y}^{(2)}$$

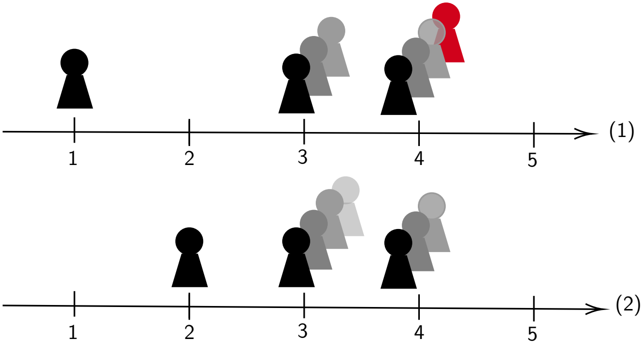


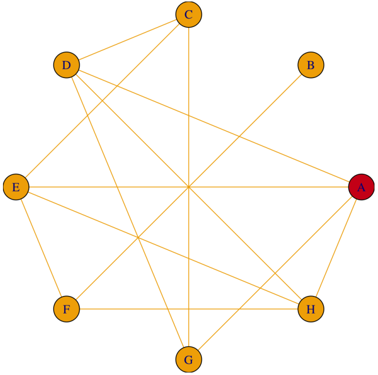


... we're back at stage 1...

Stochastic Dominance and Networks

$$\boldsymbol{y}^{(1)} = (5, 4, 4, 4, 3, 3, 2, 1) \to (4, 4, 4, 3, 3, 3, 3, 2) = \boldsymbol{y}^{(2)}$$

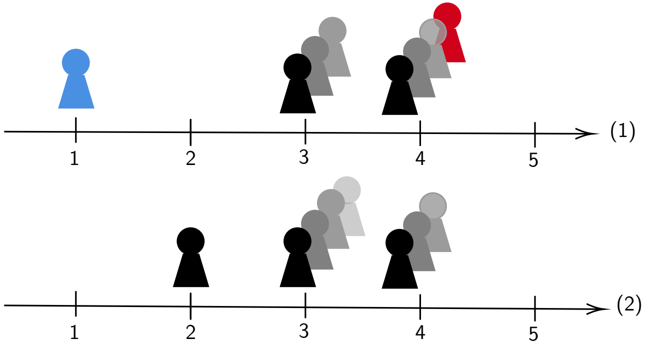


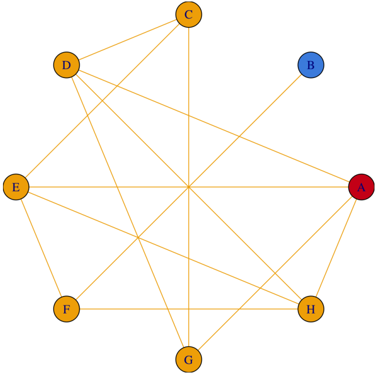


Find largest j such that $\sum_{i=1}^{j} y_i^{(1)} < \sum_{i=1}^{j} y_i^{(2)}$

Stochastic Dominance and Networks

$$\boldsymbol{y}^{(1)} = (5,4,4,4,3,3,2,1) \to (4,4,4,3,3,3,3,2) = \boldsymbol{y}^{(2)}$$

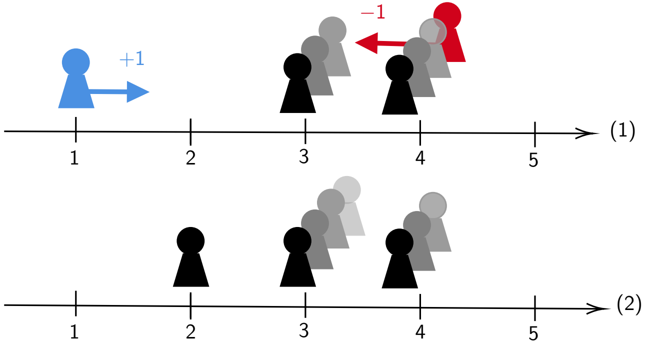


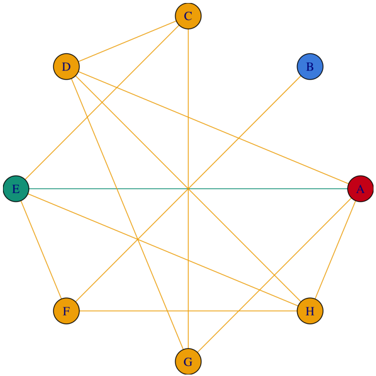


Find largest $k < j$ such that $y_k^{(2)} > y_k^{(1)} > y_{j+1}^{(2)} > y_{j+1}^{(1)}$

Stochastic Dominance and Networks

$$\boldsymbol{y}^{(1)} = (5,4,4,4,3,3,2,1) \rightarrow (4,4,4,3,3,3,3,2) = \boldsymbol{y}^{(2)}$$

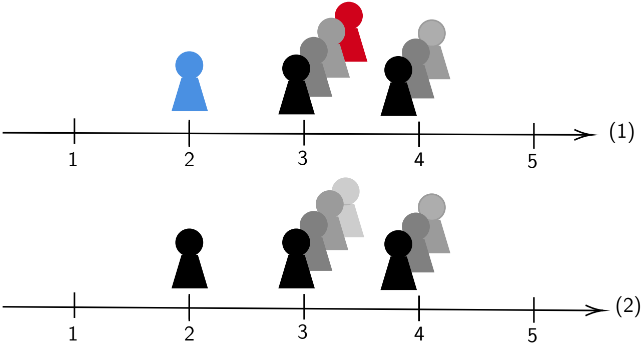


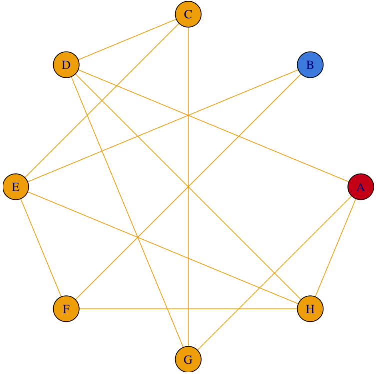


To get a transfert, we need to select ℓ

Stochastic Dominance and Networks

$$\boldsymbol{y}^{(1)} = (5, 4, 4, 4, 3, 3, 2, 1) \to (4, 4, 4, 3, 3, 3, 3, 2) = \boldsymbol{y}^{(2)}$$





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Stochastic Dominance and Networks

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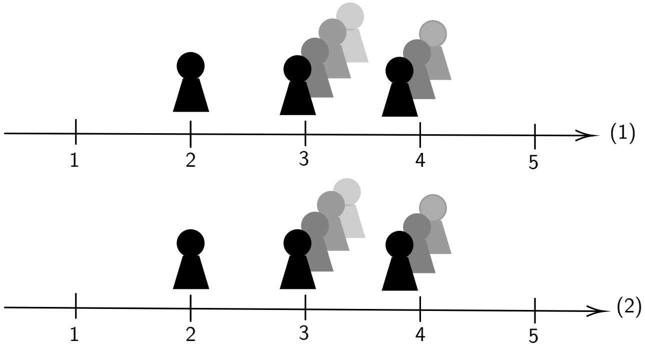
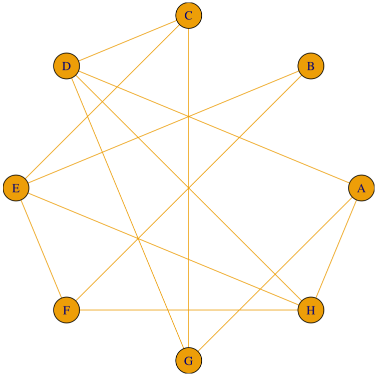




Remove edge (ℓ, j) and create (ℓ, k)

Stochastic Dominance and Networks

$$A_1 = \begin{bmatrix} . & . & 1 & 1 & 1 & . & 1 & 1 \\ . & . & . & . & . & 1 & . & . \\ 1 & . & . & 1 & 1 & . & . & . \\ 1 & . & 1 & . & . & . & 1 & 1 \\ 1 & . & 1 & . & . & 1 & . & 1 \\ . & 1 & . & . & 1 & . & . & 1 \\ 1 & . & . & 1 & . & . & . & . \\ 1 & . & . & 1 & 1 & 1 & . & . \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \\ 4 \\ 4 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} . & . & . & 1 & . & . & 1 & 1 \\ . & . & . & . & 1 & 1 & . & . \\ . & . & . & 1 & 1 & . & 1 & . \\ 1 & . & 1 & . & . & . & 1 & 1 \\ . & 1 & 1 & . & . & 1 & . & 1 \\ . & 1 & . & . & 1 & . & . & 1 \\ 1 & . & 1 & 1 & . & . & . & . \\ 1 & . & . & 1 & 1 & 1 & . & . \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \\ 4 \\ 4 \\ 3 \\ 3 \\ 4 \end{bmatrix}$$

Note that $\boldsymbol{d}_2 = \boldsymbol{y}^{(2)} \preceq \boldsymbol{y}^{(1)} = \boldsymbol{d}_1$ and $A_2 \preceq A_1$.

Linear Sharing Risks with Friends

Define $\mathcal{I}_n = \{1, 2, \cdots, n\}$, consider two networks

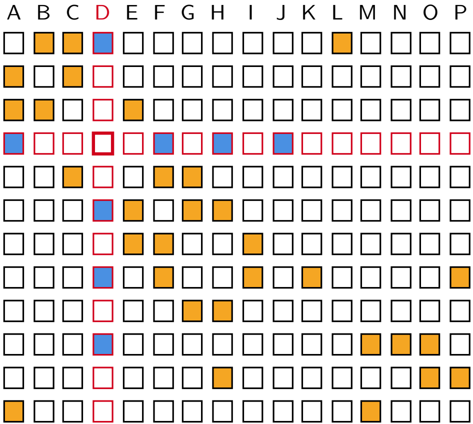
$$\begin{cases} \mathcal{G}_1 = (\mathcal{I}_n, \mathcal{E}_1), \text{ with adjacency matrix } A_1, \text{ with degree vectors } \boldsymbol{d}_1 \\ \mathcal{G}_2 = (\mathcal{I}_n, \mathcal{E}_2), \text{ with adjacency matrix } A_2, \text{ with degree vectors } \boldsymbol{d}_2 \end{cases}$$

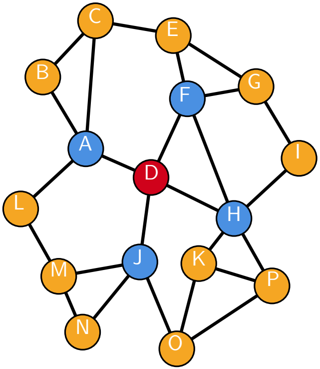
Def Consider two networks $\mathcal{G}_1 = (\mathcal{I}_n, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{I}_n, \mathcal{E}_2)$, write $\mathcal{G}_1 \preceq_D \mathcal{G}_2$ if $\boldsymbol{d}_1 \preceq \boldsymbol{d}_2$

Def Consider two networks $\mathcal{G}_1 = (\mathcal{I}_n, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{I}_n, \mathcal{E}_2)$, write $\mathcal{G}_1 \preceq_A \mathcal{G}_2$ if $A_1 \preceq A_2$

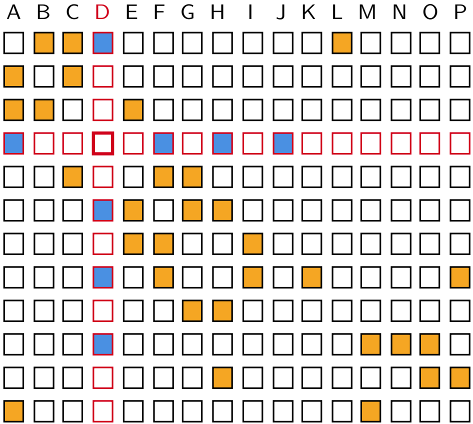
Prop If $\mathcal{G}_1 \preceq_A \mathcal{G}_2$, then $\boldsymbol{\xi}_{A_1} \preceq \boldsymbol{\xi}_{A_2} \preceq \boldsymbol{x}$

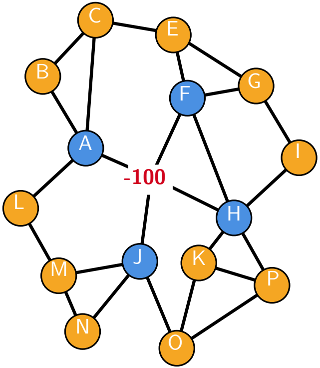
Sharing Risks with Friends



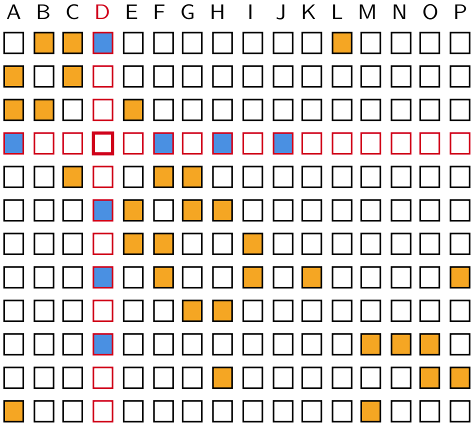


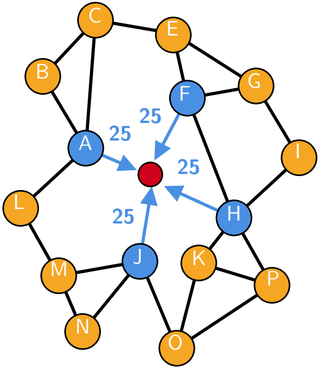
Sharing Risks with Friends





Sharing Risks with Friends





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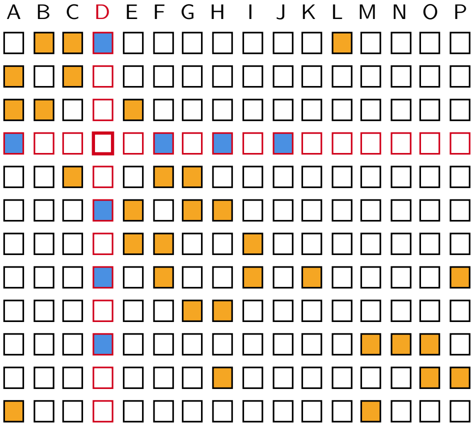

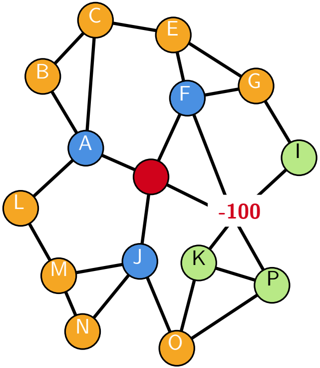

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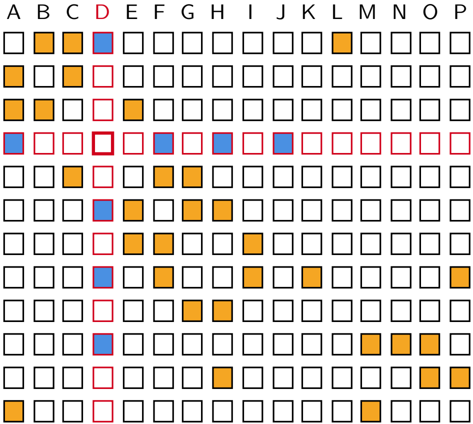

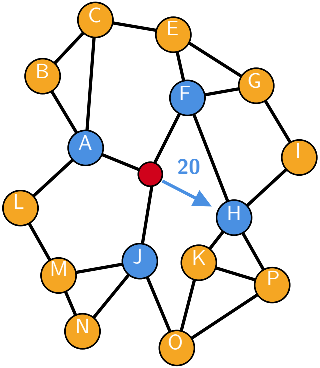

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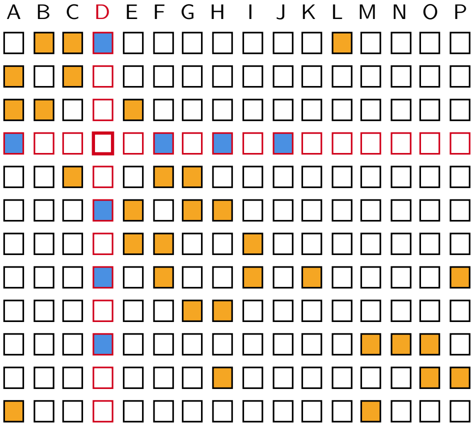


Looks like a linear risk sharing mechanism,

$$\boldsymbol{\xi} = B\boldsymbol{X} \text{ a.s., where } B_{i,j} = A_{i,j}/d_i,$$

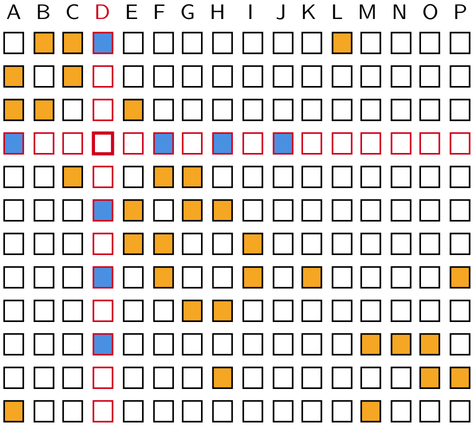
A being the adjacency matrix of the network. Here, B is a row-stochastic matrix.

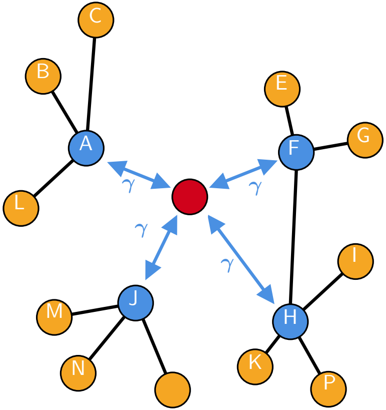
But it suffers some drawbacks...

- need an upper bound
- unfairness ($B_{i,i} = 0, \ \forall i$)

(no longer "linear" risk sharing mechanism)

Sharing Risks with Friends





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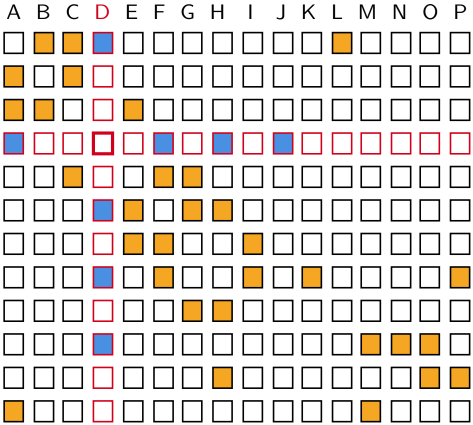

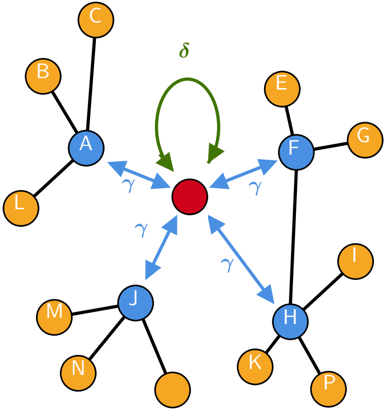

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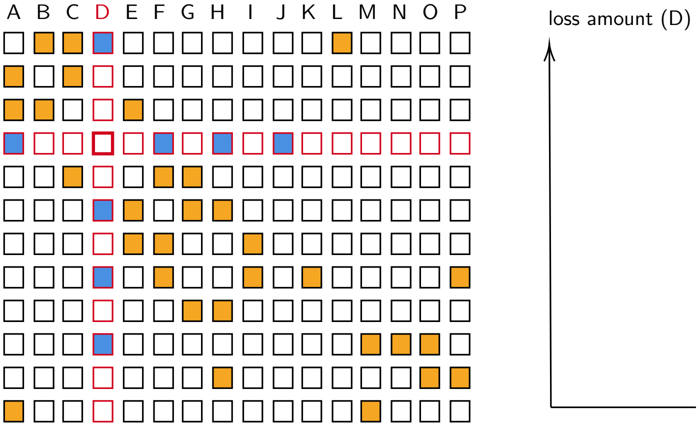

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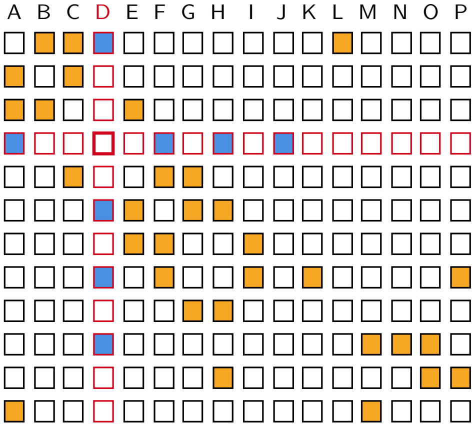

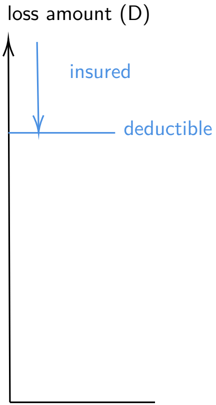

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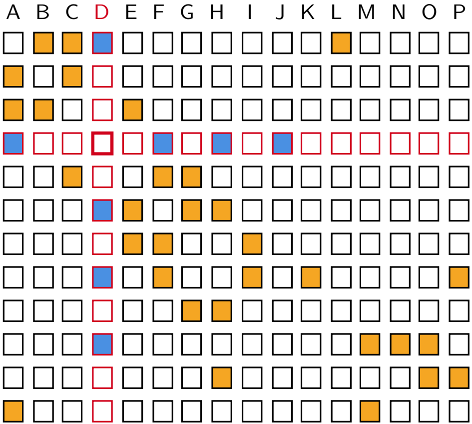

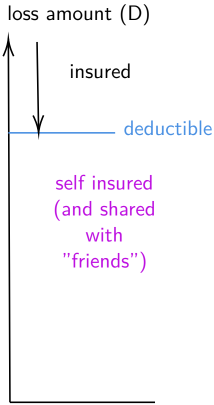

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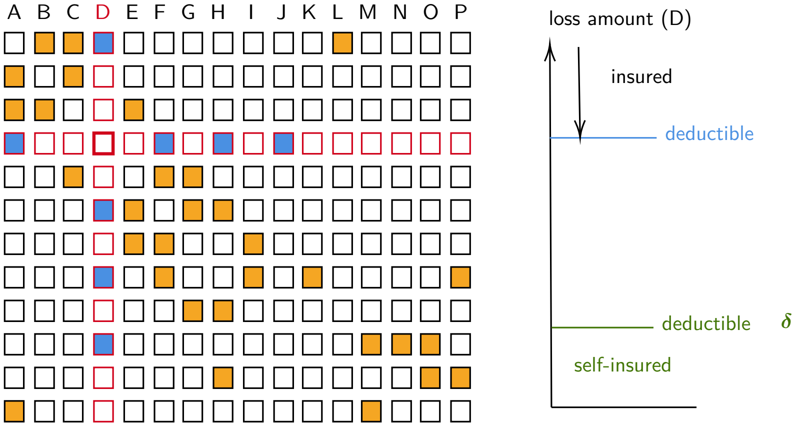

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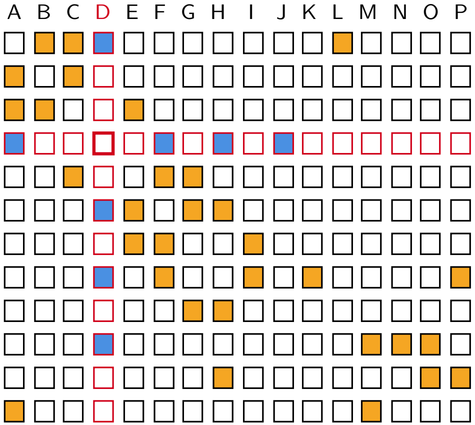

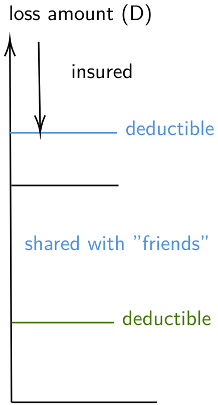

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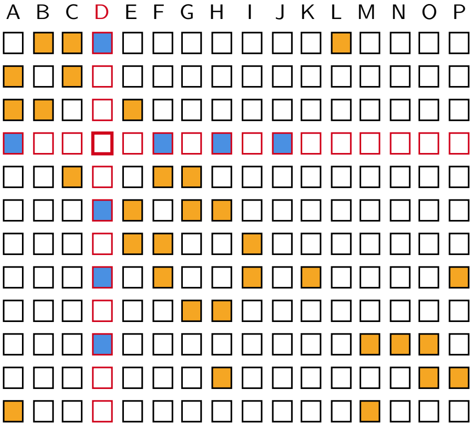

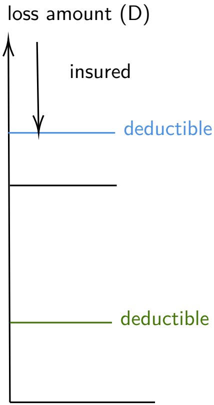

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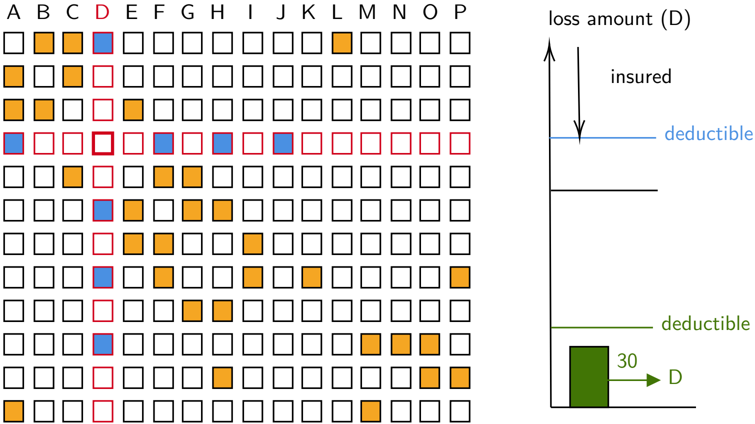

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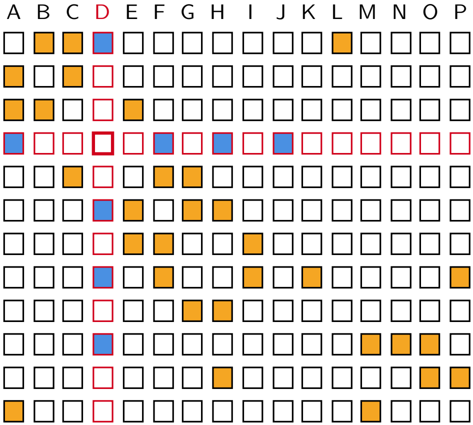

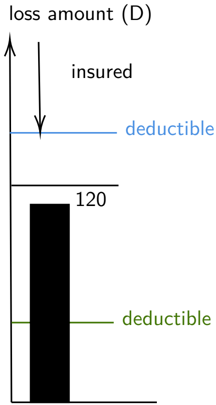

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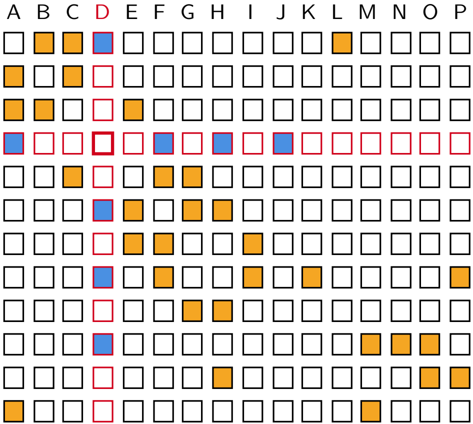

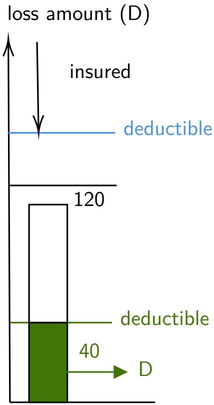

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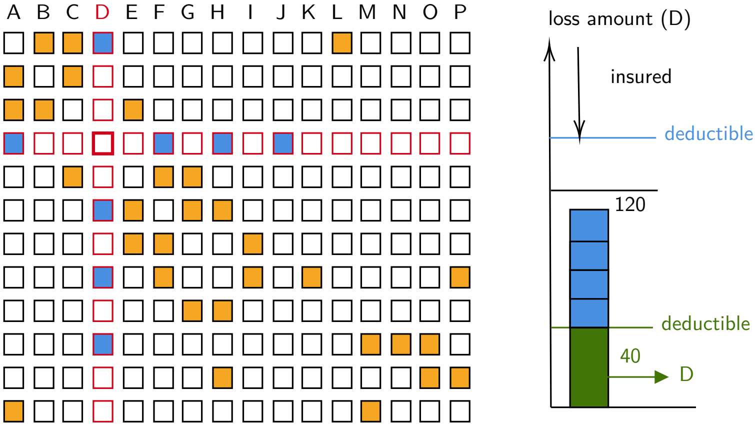

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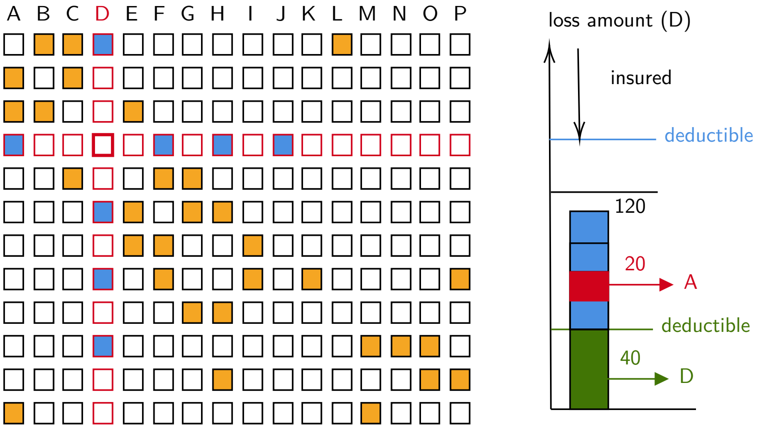

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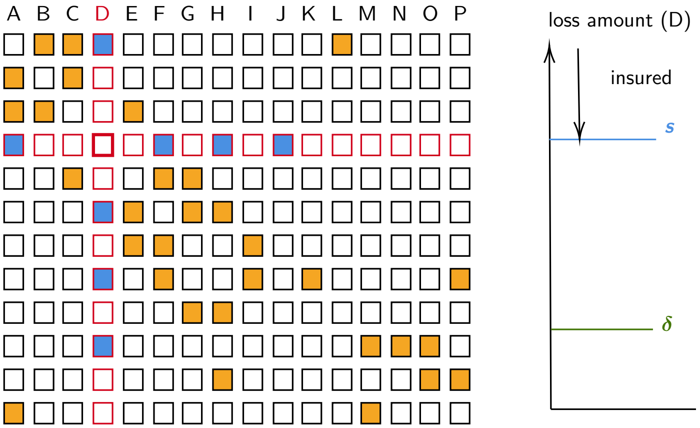

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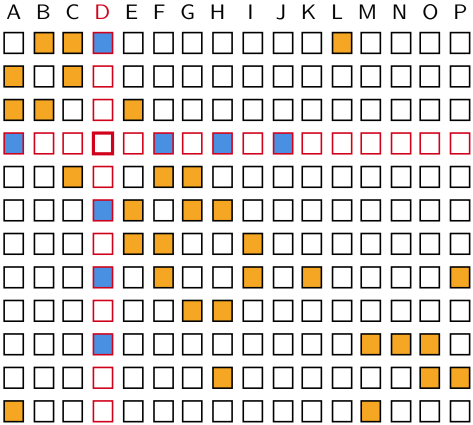

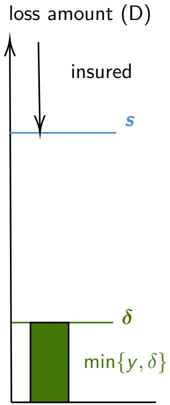

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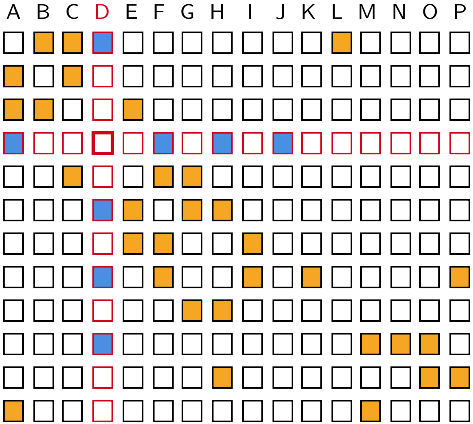
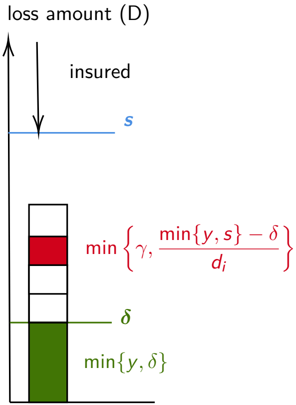
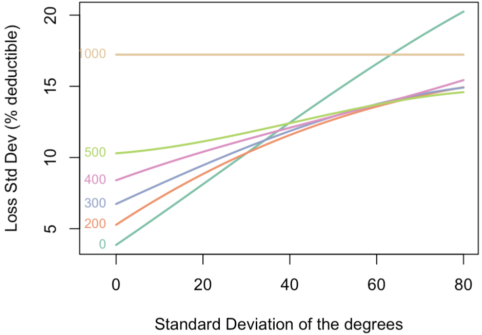




Sharing Risks with Friends

- Y_i loss of insured i, $Z_i = \mathbf{1}(Y_i > 0)$
- $\mathcal{V}_i$ is the set of friends of insured i, $d_i = \text{Card}(\mathcal{V}_i)$
- s deductible of insurance contracts
- γ is the maximum amount shared between i and j (reciprocal contracts)

$$\begin{aligned}\xi_i \quad = \quad & Z_i \cdot \min\{s, Y_i\} \\ & + \sum_{j \in \mathcal{V}_i} Z_j \min\left\{\gamma, \frac{\min\{s, Y_j\} - \delta}{d_j}\right\} \\ & - Z_i \cdot \min\{d_i\gamma, \min\{s, Y_i\} - \delta\}\end{aligned}$$



Optimization* of the Risk Sharing Mechanism

$$\begin{cases} \max \left\{ \displaystyle\sum_{(i,j)\in\mathcal{E}} \gamma_{(i,j)} \right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0,\gamma],\ \forall (i,j)\in\mathcal{E} \\ \qquad \displaystyle\sum_{j\in\mathcal{V}_i} \gamma_{(i,j)} \leq s,\ \forall i \in \mathcal{V} \end{cases}$$

Given losses $\boldsymbol{X} = (X_1, \cdots, X_n)$, define contributions $C^\star_{i\to j} = \min\left\{ \dfrac{\gamma^\star_{(i,j)}}{\displaystyle\sum_{i\in\mathcal{V}_j} \gamma^\star_{(i,j)}} \cdot X_j, \gamma^\star_{(i,j)} \right\}$,

and $\xi^\star_i = X_i + \displaystyle\sum_{j\in\mathcal{V}_i} [Z_j C^\star_{i\to j} - Z_i C^\star_{j\to i}]$ is a risk sharing, called optimal risk sharing.

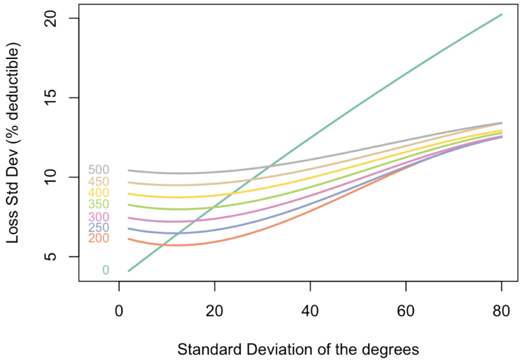
* from a welfare (social planner) perspective

Sharing Risks with Friends, and Friends of Friends

We can also consider friends of friends

$$\begin{cases} \gamma_1^\star = \text{argmax}\left\{\displaystyle\sum_{(i,j)\in\mathcal{E}^{(1)}} \gamma_{(i,j)}\right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0,\gamma_1],\ \forall (i,j)\in\mathcal{E}^{(1)} \\ \qquad \displaystyle\sum_{j\in\mathcal{V}_i^{(1)}} \gamma_{(i,j)} \leq s,\ \forall i \end{cases}$$

$$\begin{cases} \gamma_2^\star = \text{argmax}\left\{\displaystyle\sum_{(i,j)\in\mathcal{E}^{(2)}} \gamma_{(i,j)}\right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0,\gamma_2],\ \forall (i,j)\in\mathcal{E}^{(2)}_{\gamma_1^\star} \\ \qquad \displaystyle\sum_{j\in\mathcal{V}_i^{(1)}} \gamma^\star_{1:(i,j)} + \sum_{j\in\mathcal{V}_i^{(2)}} \gamma_{(i,j)} \leq s,\ \forall i \end{cases}$$



Take-away

- Back to the roots of insurance with risk sharing,
- Important to better model interactions
- Nice mathematical properties of linear risk sharing (connexions with convex ordering)
- More complex to derive a more realistic insurance product (with lower and upper limits)
- ... ongoing work...

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