Autocalibration & Insurance Pricing

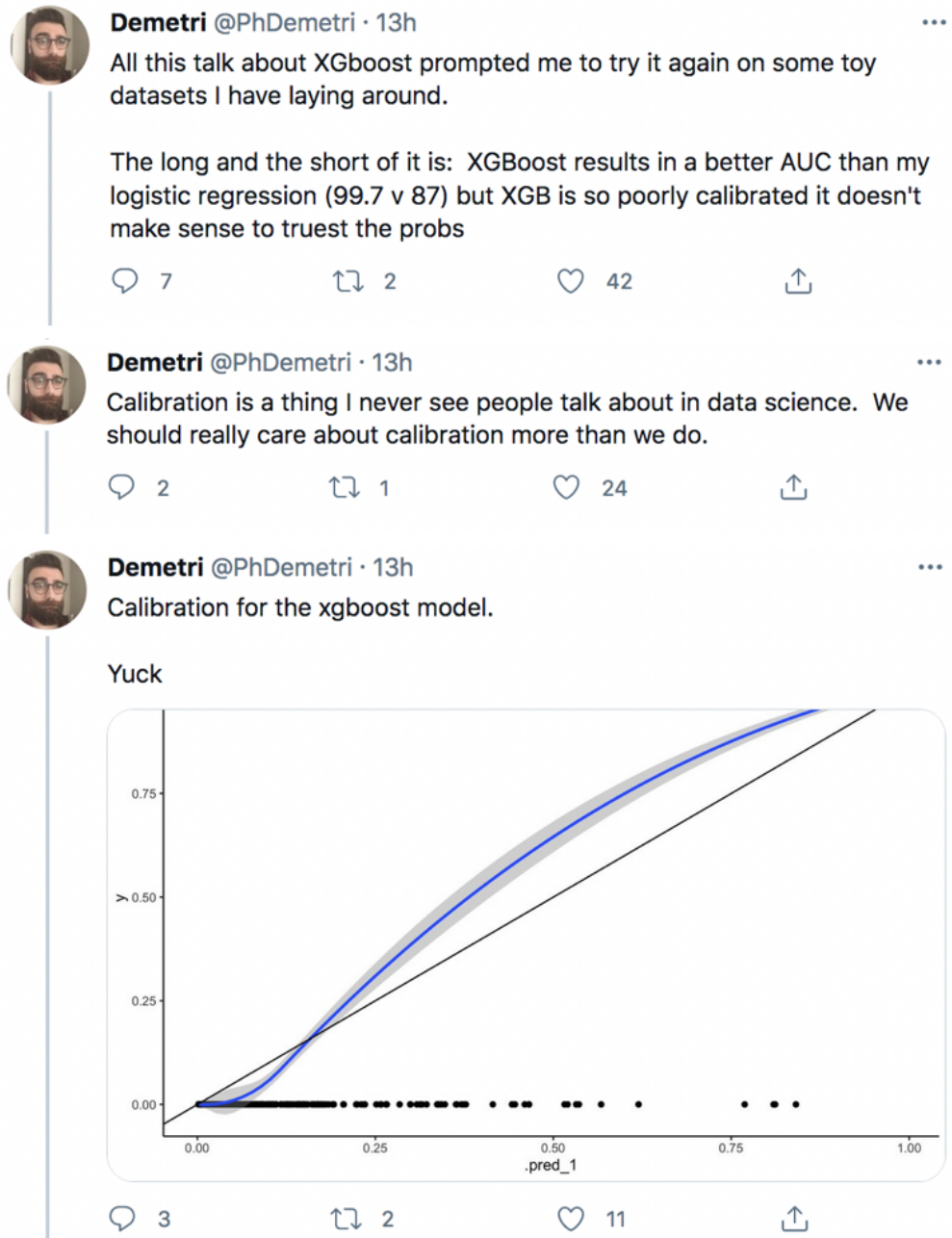
Arthur Charpentier (UQAM)

with Michel Denuit (UCL) & Julien Trufin (UCB)

Université de Sherbrooke, 2021

Agenda

- On insurance pricing
- What is a model ?
- Bias of a model
- Correcting from bias
- Application
- Theoretical properties



Accuracy, Calibration & Bias

For classication problems, **calibration** measures how well your model's scores can be interpreted as probabilities. **Accuracy** measures how often your model produces correct answers.

"**Accuracy** *is a qualitative term referring to whether there is agreement between a measurement made on an object and its true (target or reference) value.* **Bias** *is a quantitative term describing the difference between the average of measurements made on the same object and its true value..* " Handbook of Statistical Methods

"**Well calibrated** *classifiers are probabilistic classifiers for which the output can be directly interpreted as a confidence level. For instance, a well calibrated (binary) classifier should classify the samples such that among the samples to which it gave a*[predicted probability] *value close to 0.8, approximately 80% actually belong to the positive class.* " scikit learn: Probability calibration

Calibration

Dawid, A.P. (1982). The Well-Calibrated Bayesian, (JASA)
"*Suppose that a forecaster sequentially assigns probabilities to events. He is* **well calibrated** *if, for example, of those events to which he assigns a probability 30 percent, the long-run proportion that actually occurs turns out to be 30 percent.* "

Silver, N. (2012). The signal and the noise,
"*Out of all the times you said there was a 40 percent chance of rain, how often did rain actually occur? If, over the long run, it really did rain about 40 percent of the time, that means your forecasts were* **well calibrated** "

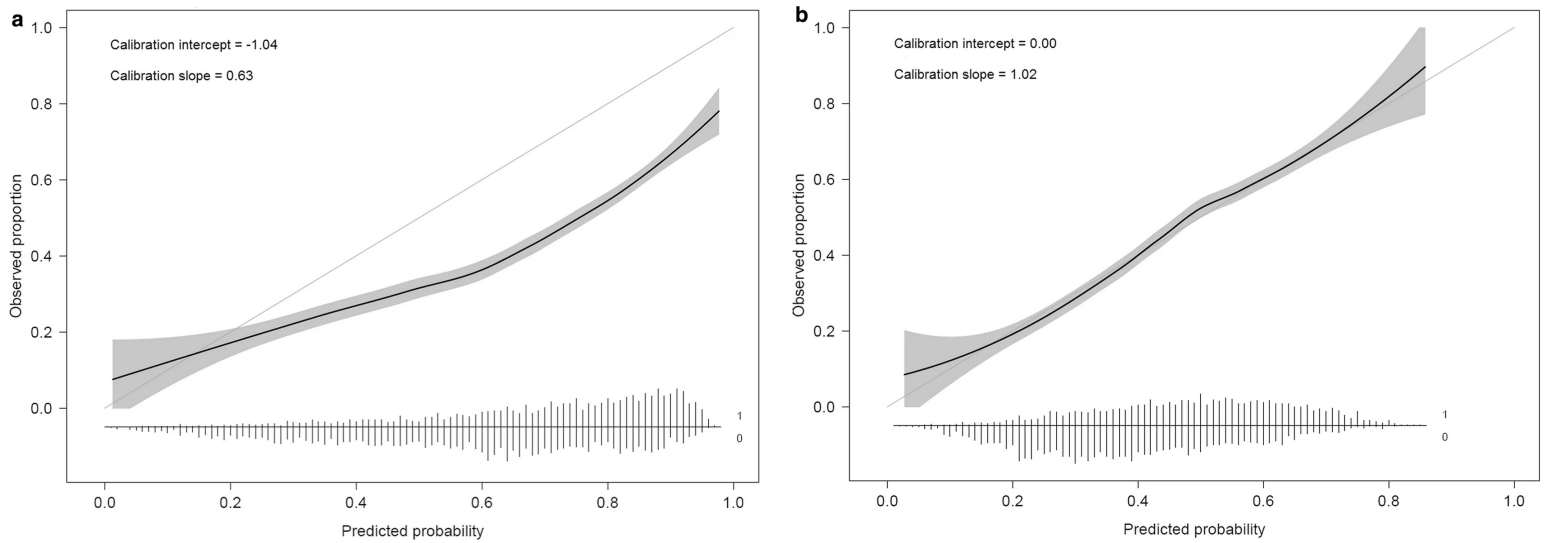
Kuhn, M., & Johnson, K.(2013). Applied Predictive Modeling
"*we desire that the estimated class probabilities are reflective of the true underlying probability of the sample* "

Calibration & Auto-calibration

Kruger, F. & Ziegel, J.F. (2020). Generic conditions for forecast dominance, (JBES)

Definition 3.1 "*the forecast X of Y is an auto-calibrated forecast of Y if $\mathbb{E}(Y|X) = X$ almost surely*", or $\mathbb{E}(Y|\widehat{Y} = y) = y, \ \forall y$

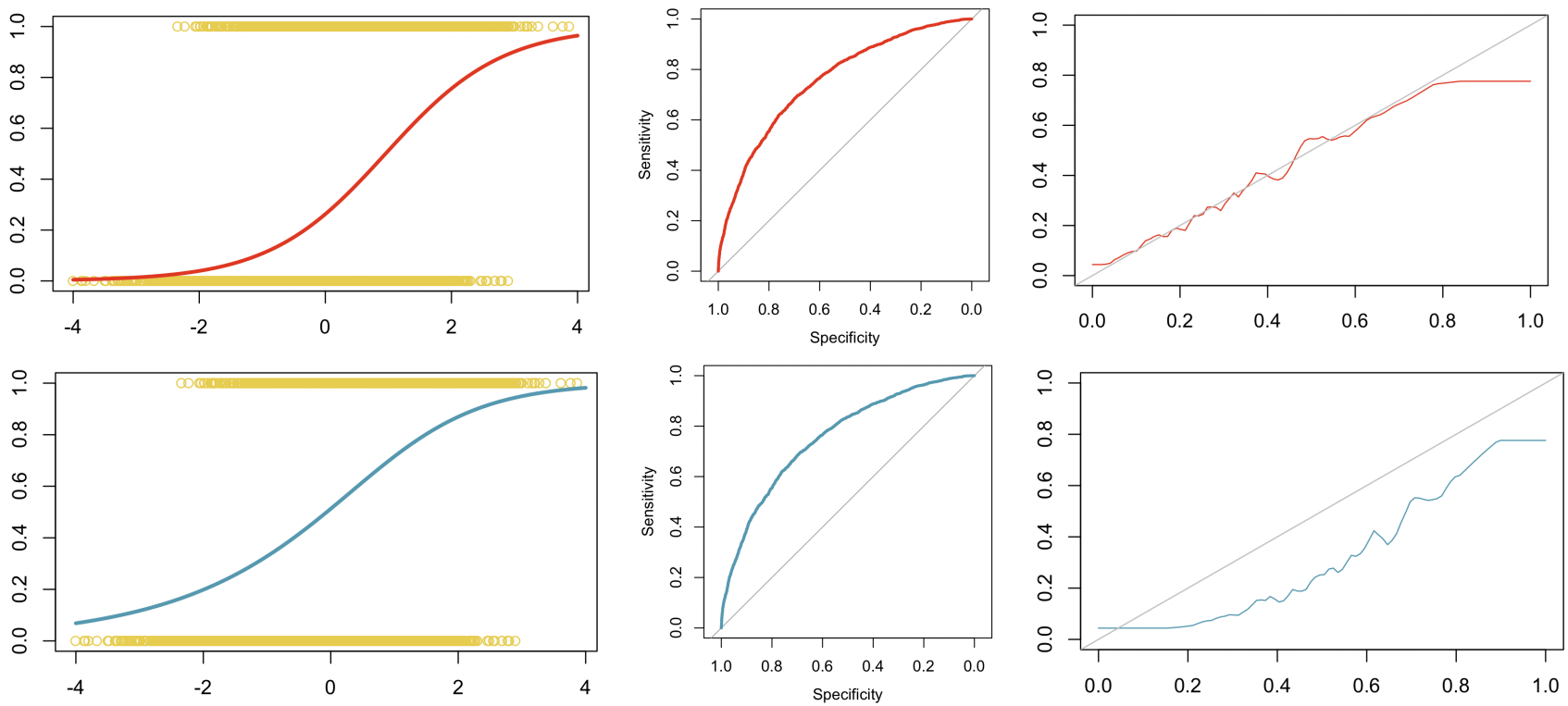
Van Calster, B. *et al.* (2019). Calibration: the Achilles heel of predictive analytics, (BMC Medecine)



Machine Learning & Accuracy Measures

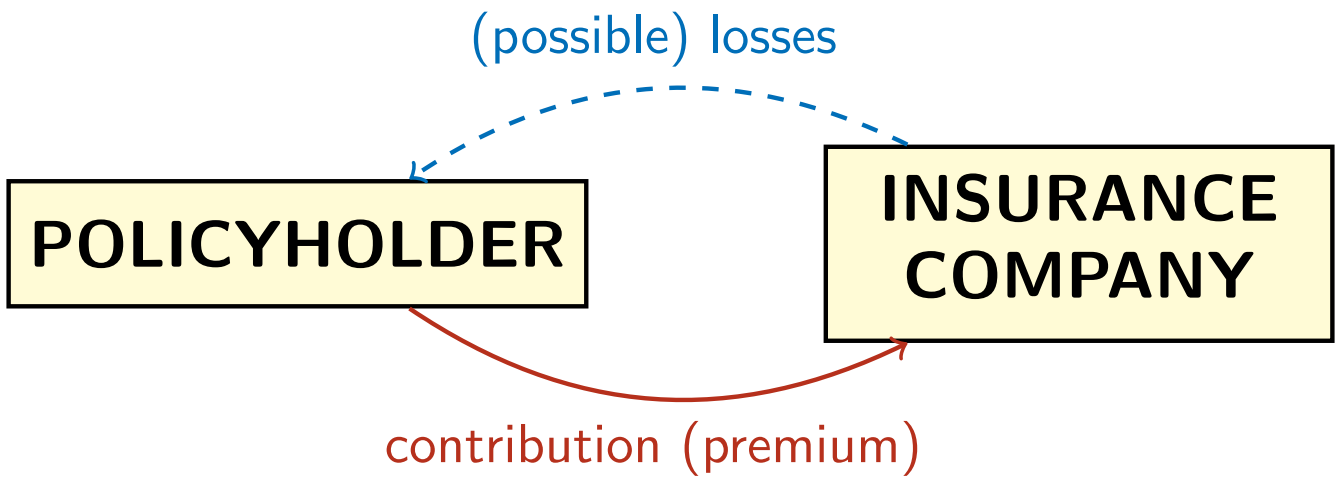
"*XGBoost results in a better AUC than my logistic regression but XGBoost is so poorly calibrated it doesn't make sense to trust the probs*"

- (x_i, y_i) (where $y_i \in \{0, 1\}$) with $\widehat{\pi}_1(x)$ or $\widehat{\pi}_2(x)$ $(= \sqrt{\widehat{\pi}_1(x)})$
- ROC curve of $\widehat{\pi}_1$ or $\widehat{\pi}_2$ are identical (and same AUC)
- $\mathbb{E}[Y|\widehat{\pi}_1(X) = s]$ or $\mathbb{E}[Y|\widehat{\pi}_2(X) = s]$,
  ie. regression of y_i on $\pi(x_i)$ (expected? $\mathbb{E}[Y|\widehat{\pi}(X) = s] \sim s$)



Insurance

"*Insurance is the contribution of the many to the misfortune of the few*"



The "*contribution*" is obtained using predictive models (interpretability / black box / etc issues)

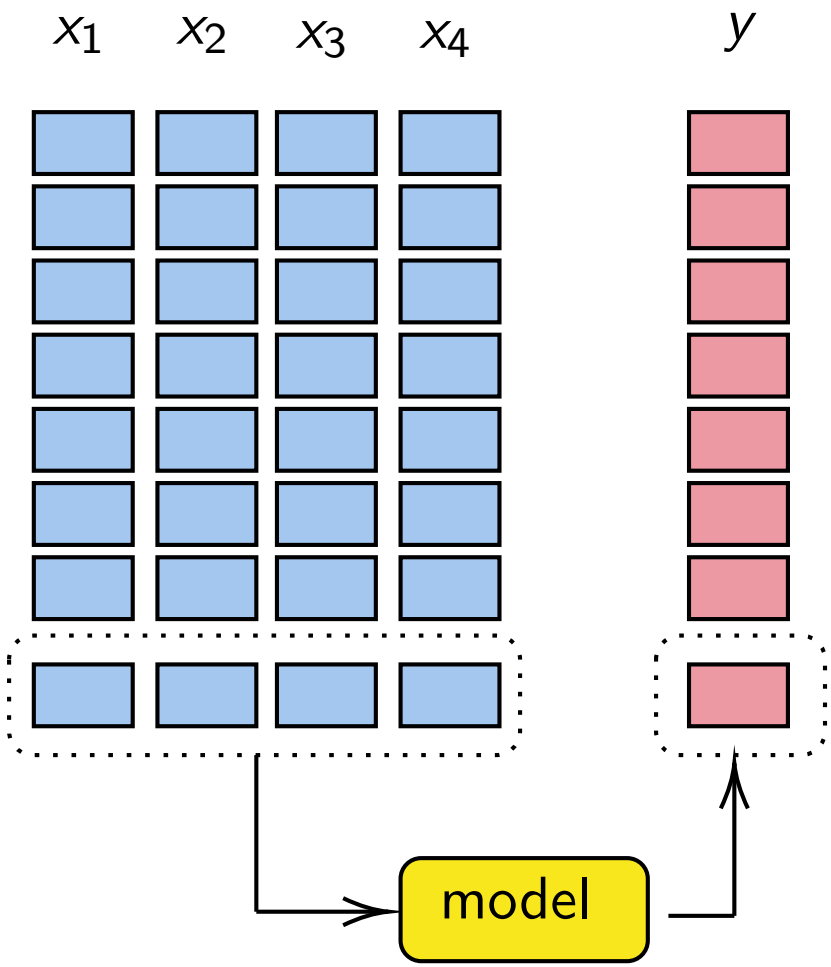
A model, $m : \boldsymbol{x} \mapsto y$

To train / estimate a model m,
we need a dataset, i.e. a collection
of observations $(\boldsymbol{x}_i, y_i)$

Usually y_i denotes the annual loss

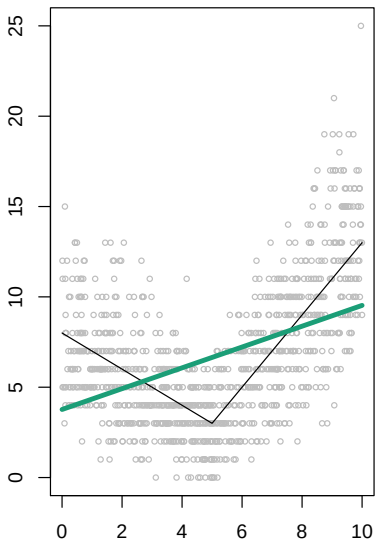
$$y_i = \sum_{j=1}^{n_i} z_{i,j}$$

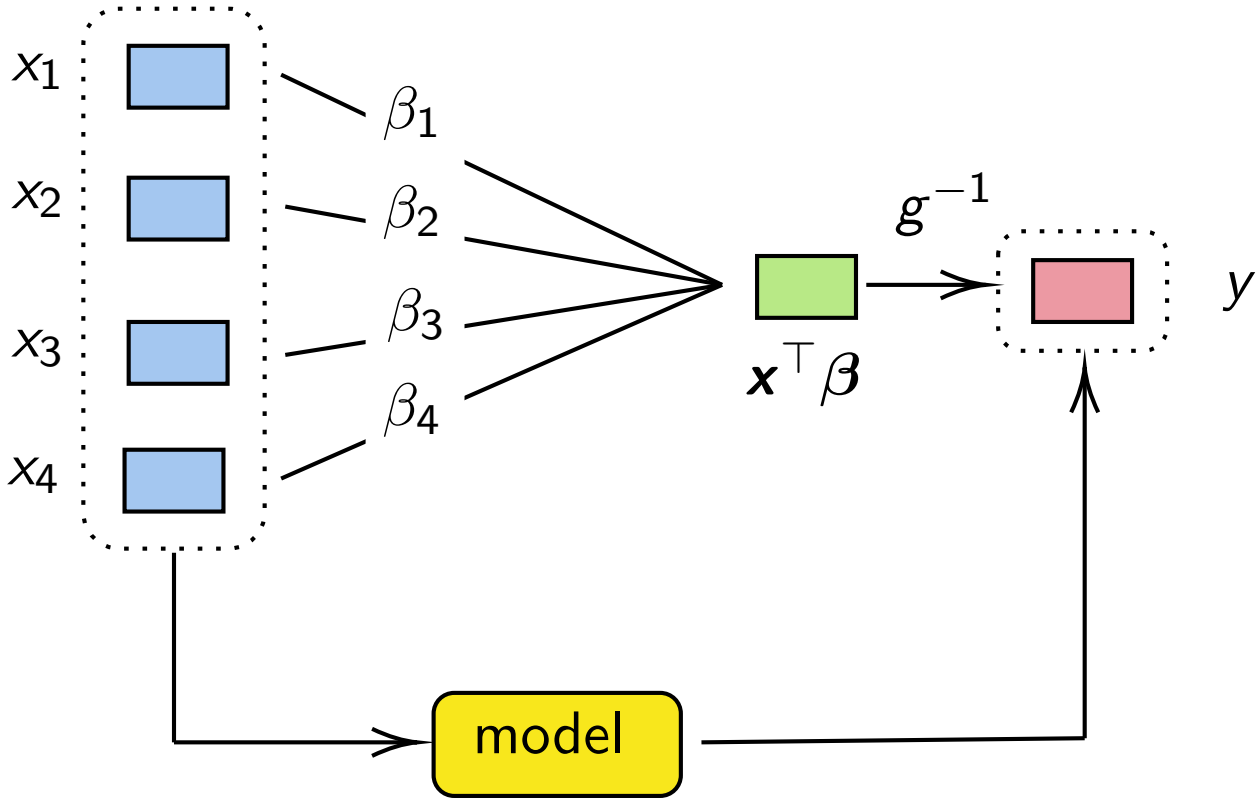
n is the annual frequency
z is the cost of a single claim
(see Tweedie models)
To illustrate, y is a counting variable



A model, $m : \boldsymbol{x} \mapsto y$ (GLM)

Poisson regression

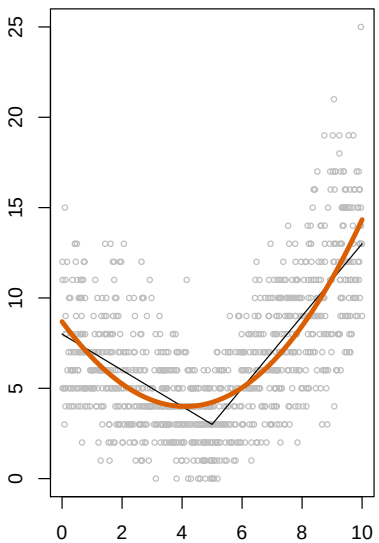


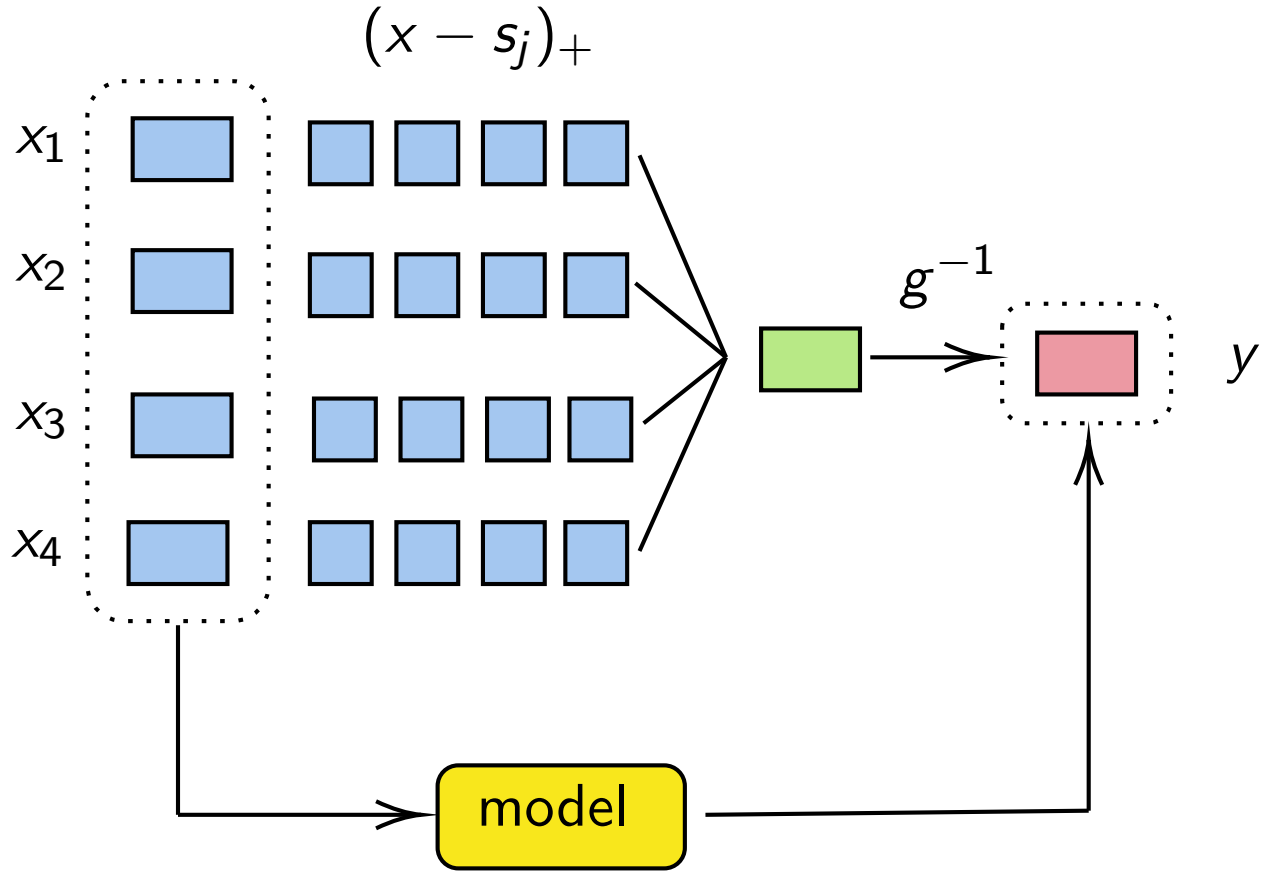


$$m(\boldsymbol{x}) = g^{-1}(\boldsymbol{x}^\top\boldsymbol{\beta}) = g^{-1}\left(\sum_{j=1}^{p}\beta_j x_j\right)$$

A model, $m : \boldsymbol{x} \mapsto y$ (GAM)

Poisson regression

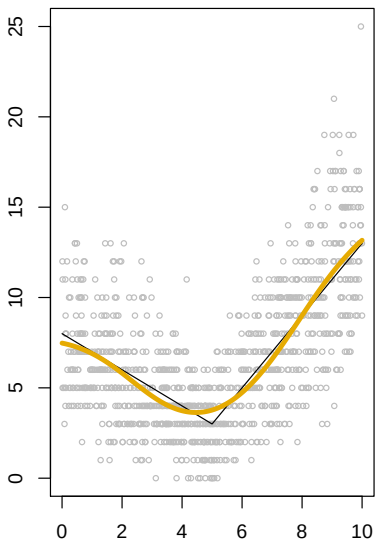


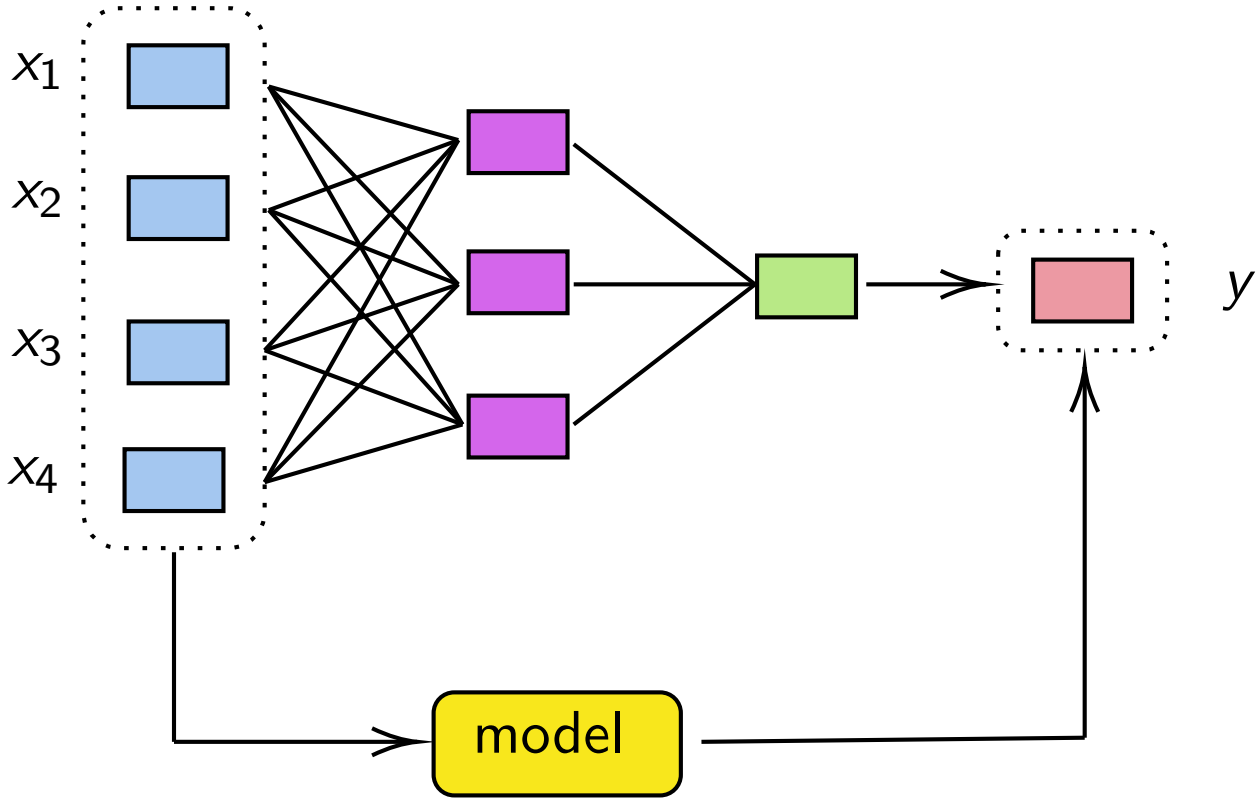


$$m(\boldsymbol{x}) = g^{-1}\left(\sum_{j=1}^{p} \beta_j \psi_j(x_j)\right), \text{ where } \psi_j(x) = \sum_{k=1}^{5} \alpha_{j,k}(x - s_{j,k})_+$$

A model, $m : \boldsymbol{x} \mapsto y$ (neural nets)

Poisson loss

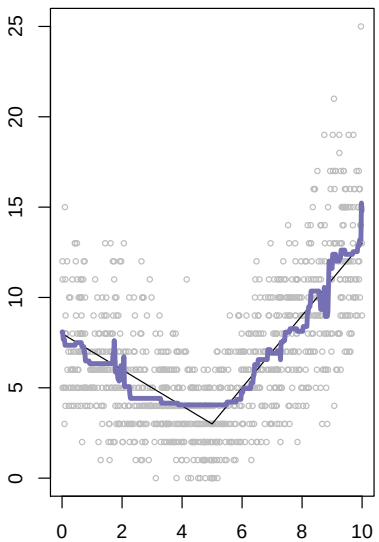


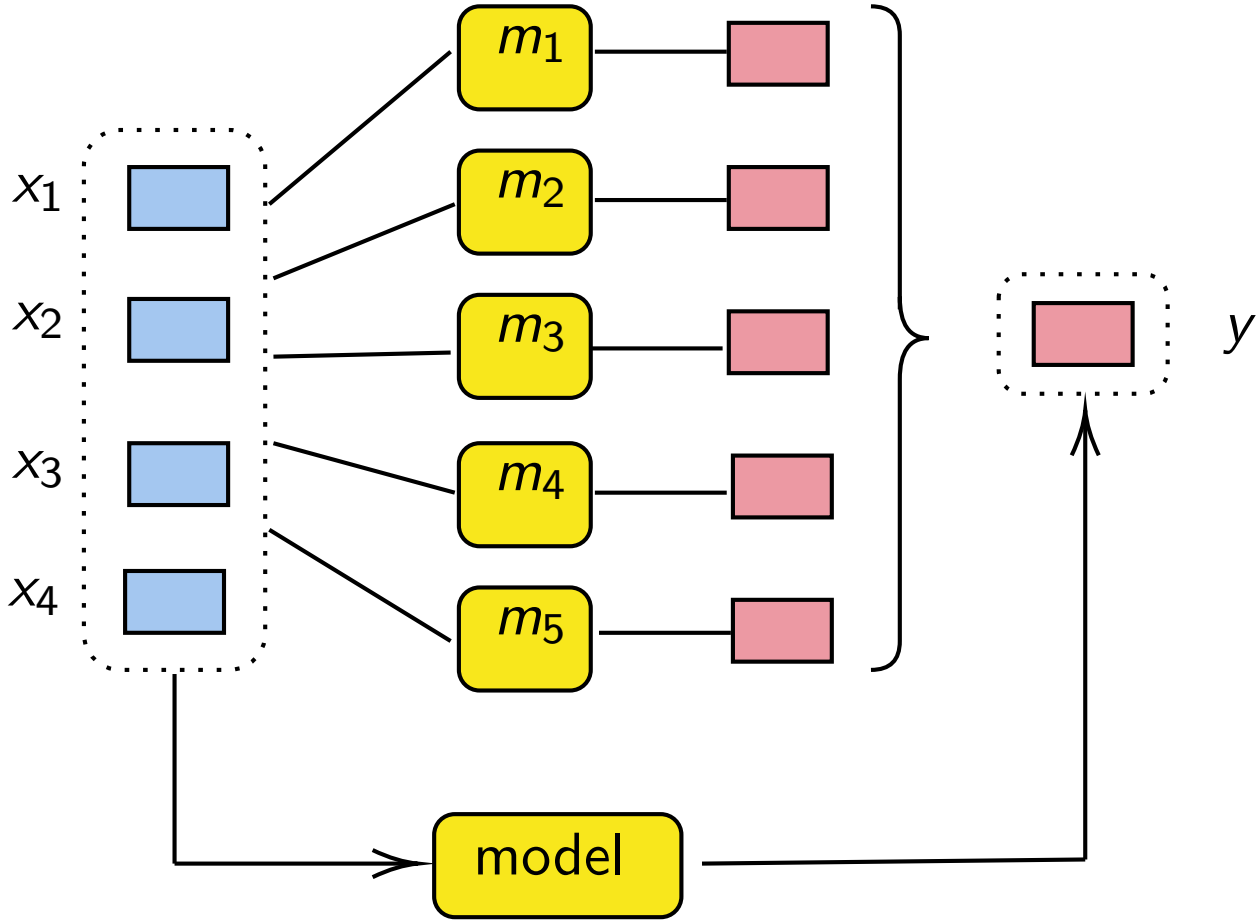


E.g. $m(\boldsymbol{x}) = \sum_{j=1}^{3} \omega_{1:j} \; h(\boldsymbol{x}^\top \boldsymbol{\omega}_{2:j})$

A model, $m : \boldsymbol{x} \mapsto y$ (ensemble parallel, bagging)

Random forest

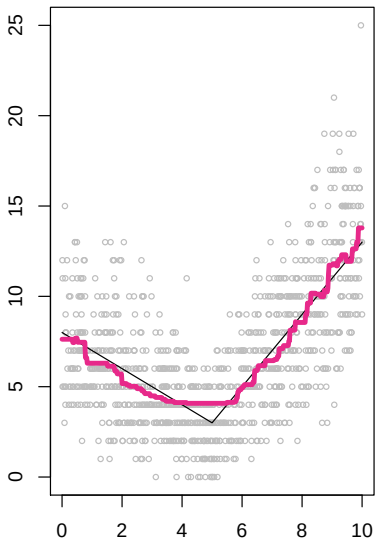


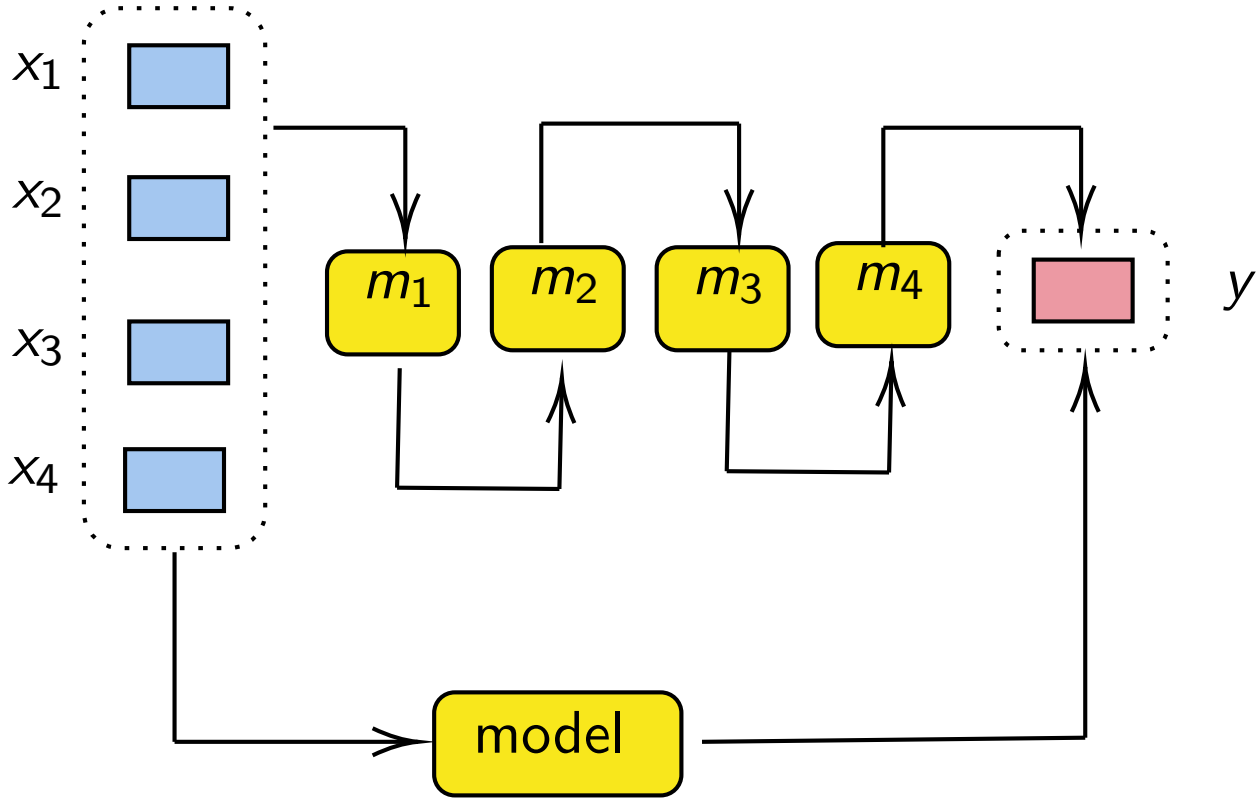


E.g. $m(\boldsymbol{x}) = \frac{1}{B}\sum_{b=1}^{B} m_b(\boldsymbol{x})$ where $m_b(\boldsymbol{x}) = \sum_{k=1}^{K} \alpha_{b,k} 1(\boldsymbol{x} \in \mathcal{X}_{b,k})$

A model, $m : \boldsymbol{x} \mapsto y$ (ensemble sequential, boosting)

Boosting





$m(\boldsymbol{x}) = \sum_{t=1}^{T} m_t(\boldsymbol{x})$ where m_t is a (weak) model on $y_i - m_{t-1}(\boldsymbol{x}_i)$ (residuals from the previous step) such as (not too deep) trees

Not a model, $\widehat{m} : \boldsymbol{x} \mapsto y$ (local regression)

To approximate $\mathbb{E}[Y]$ use

$$\widehat{m} = \underset{\mu \in \mathbb{R}}{\text{argmin}} \left\{ \sum_{i=1}^{n} \frac{1}{n} [y_i - \mu]^2 \right\}$$

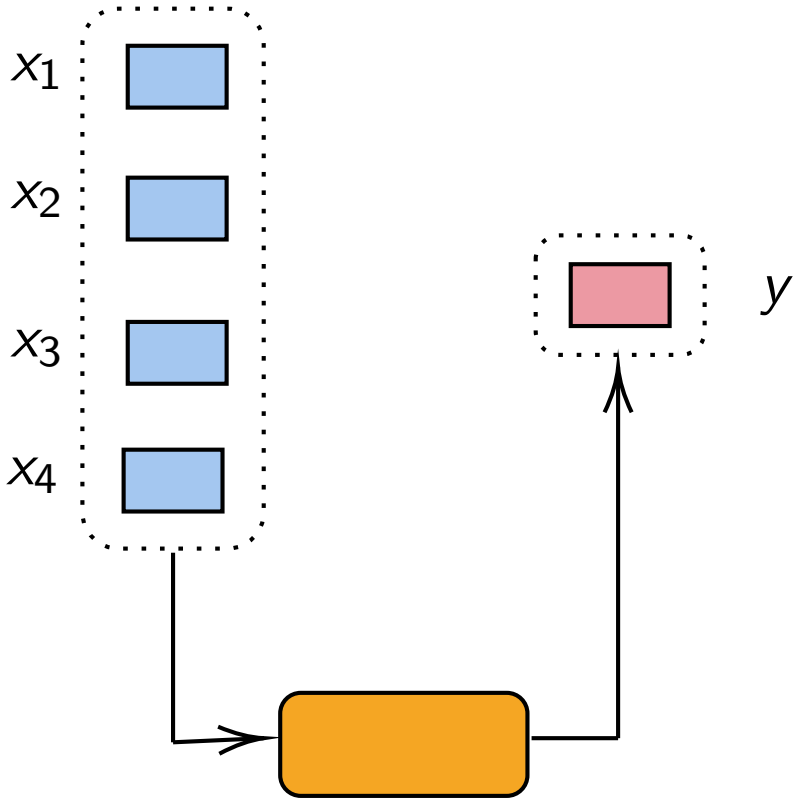
To approximate $\mathbb{E}[Y|\boldsymbol{X} = \boldsymbol{x}]$, use

$$\widehat{m}(\boldsymbol{x}) = \underset{\mu \in \mathbb{R}}{\text{argmin}} \left\{ \sum_{i=1}^{n} \omega_i(\boldsymbol{x})[y_i - \mu]^2 \right\}$$

where, see Loader (1999),

$$\omega_i(\boldsymbol{x}) \propto k_\alpha \left(\|\boldsymbol{x} - \boldsymbol{x}_i\| \right)$$

or k nearest neighbors indicator...

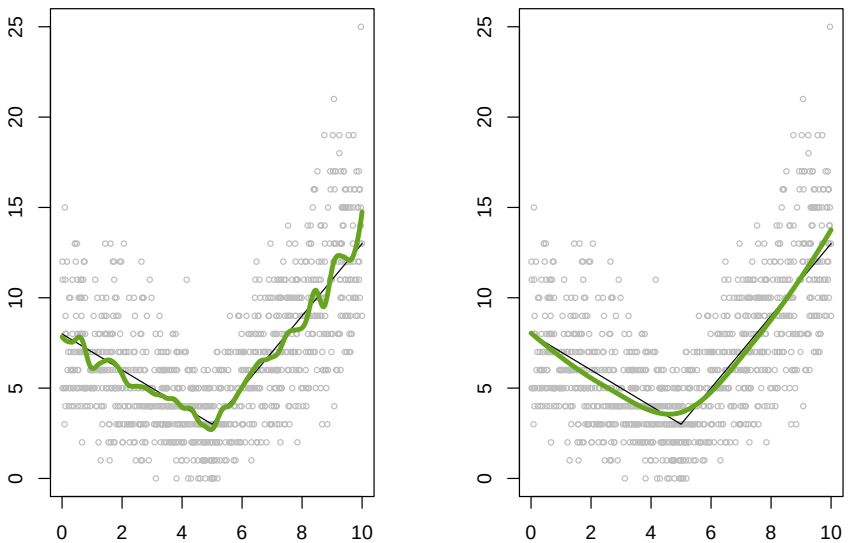


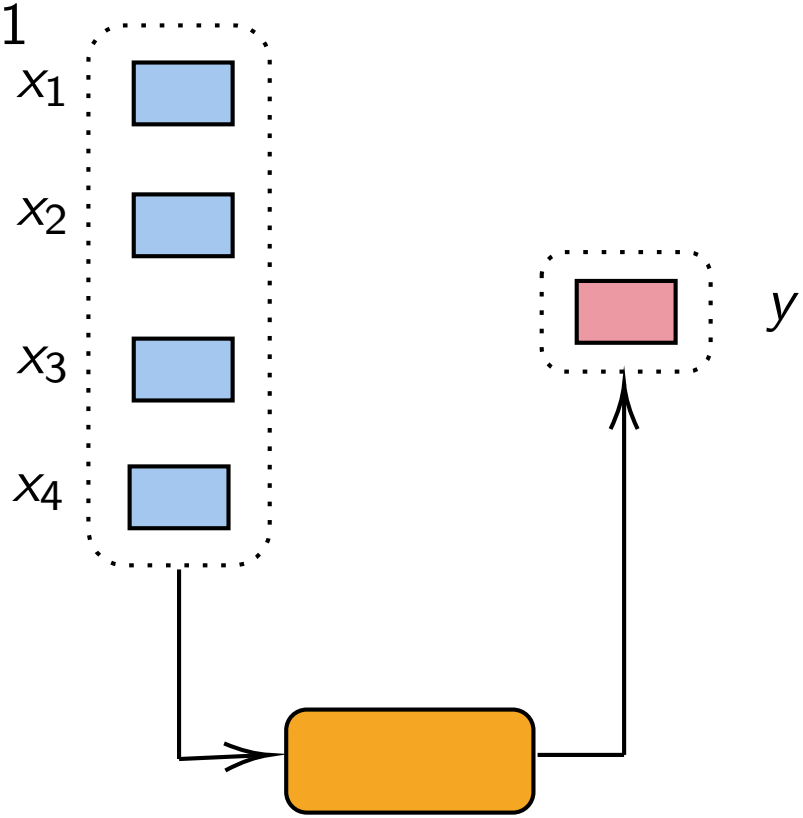
Behaves poorly in high dimension (but efficient in dimension 1)

Not a model, $\widehat{m} : \boldsymbol{x} \mapsto y$ (local regression)

$$\widehat{m}(\boldsymbol{x}) = \sum_{i=1}^{n} \omega_i(\boldsymbol{x})\ y_i, \text{ with } \sum_{i=1}^{n} \omega_i(\boldsymbol{x}) = 1$$

see `locfit` in R

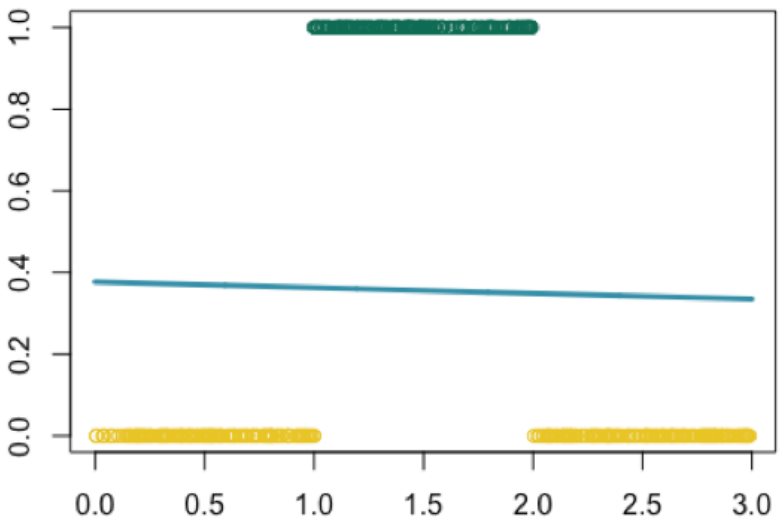


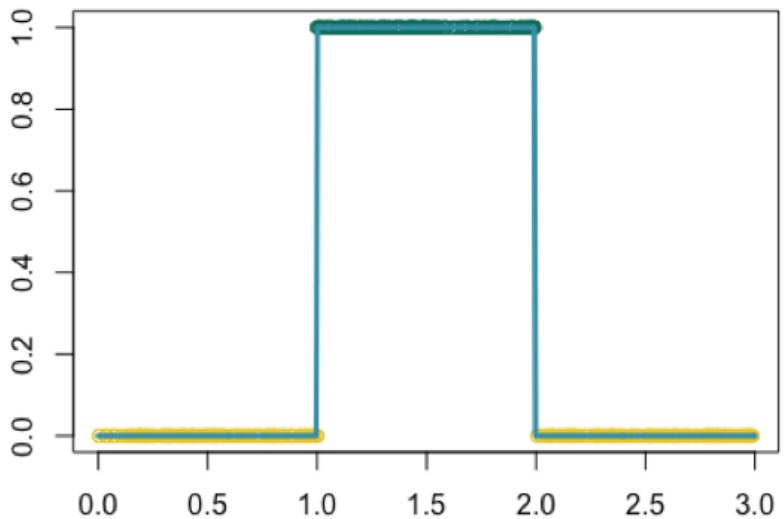


(depends on a bandwidth parameter α)
Provides a local estimate of $\mathbb{E}[Y|X = x]$ on a neighborhood of x

GLM, GAM & Neural Nets

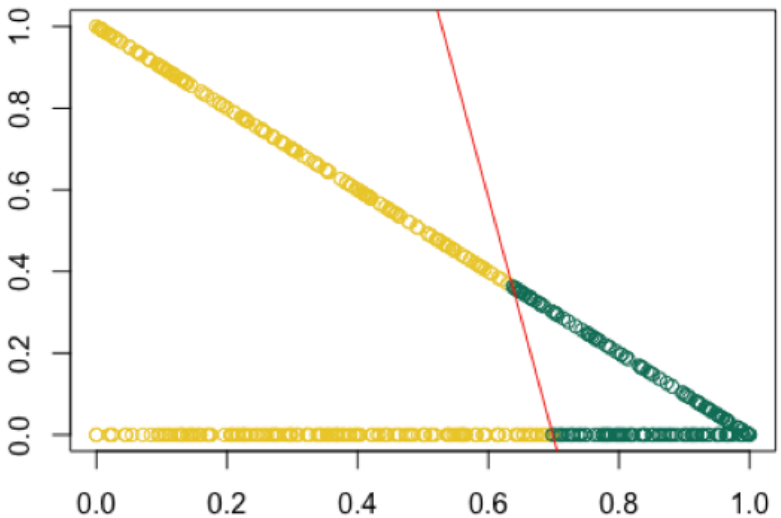
Consider some simple dataset $\{(x_i, y_i)\}$.
Fit a GLM and a GAM (linear, 3 knots)

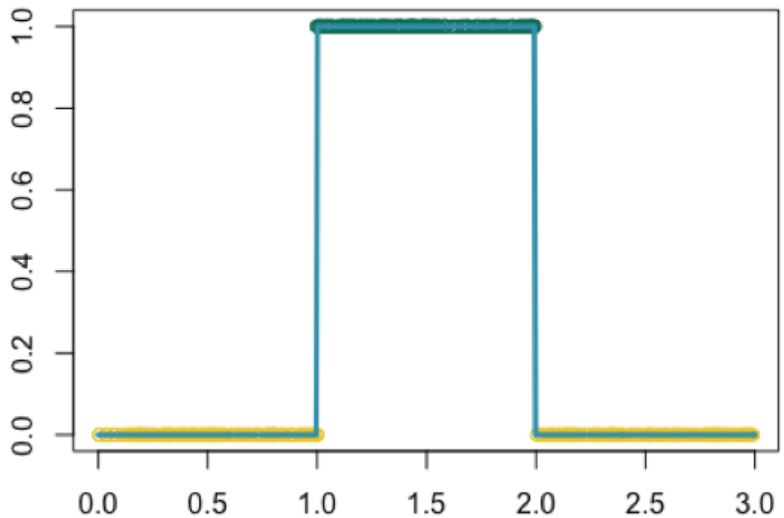




GLM, GAM & Neural Nets

Consider some simple dataset $\{(x_i, y_i)\}$.
Fit a GLM and a GAM (linear, 3 knots)

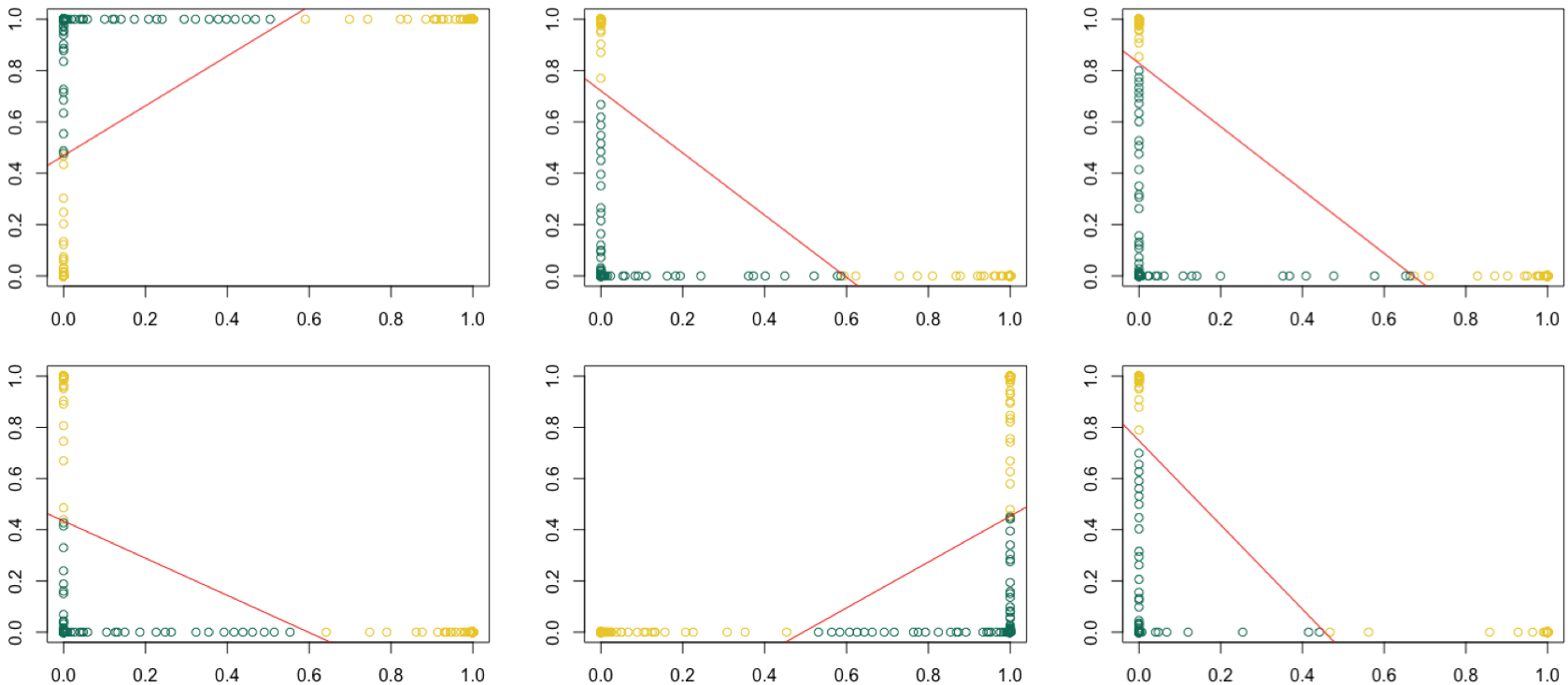




Two variables, $x_1 = x$ and $x_2 = (x - s)_+$

GLM, GAM & Neural Nets

Consider some simple dataset $\{(x_i, y_i)\}$.
Fit a Neural Net



Identifiability issues here...

GLM, Bias, & Economic Interpretation

For GLMs, $f(y_i) = \exp\left(\frac{y_i\theta_i - b(\theta_i)}{\varphi} + c(y_i, \varphi)\right)$

The natural parameter for y_i : θ_i
Prediction for y_i : $\widehat{y}_i = \mu_i = \mathbb{E}(Y_i) = b'(\theta_i)$
Score associated with y_i : $\eta_i = \boldsymbol{x}_i^\top \boldsymbol{\beta}$
Link function : g such that $\eta_i = g(\mu_i) = g(b'(\theta_i))$

$$\log \mathcal{L}(\boldsymbol{\theta}, \varphi, \boldsymbol{y}) = \sum_{i=1}^{n} \log \mathcal{L}_i = \sum_{i=1}^{n} \left[\frac{y_i\theta_i - b(\theta_i)}{\varphi} + c(y_i, \varphi)\right]$$

First order conditions: $\frac{\partial \log \mathcal{L}_i}{\partial \beta_j} = \frac{\partial \log \mathcal{L}_i}{\partial \theta_i} \cdot \frac{\partial \theta_i}{\partial \mu_i} \cdot \frac{\partial \mu_i}{\partial \eta_i} \cdot \frac{\partial \eta_i}{\partial \beta_j}$

$$\frac{\partial \log \mathcal{L}_i}{\partial \theta_i} = \frac{y_i - \mu_i}{\varphi}, \quad \frac{\partial \theta_i}{\partial \mu_i} = \left(\frac{\partial \mu_i}{\partial \theta_i}\right)^{-1} = \frac{1}{b''(\theta_i)} = \frac{1}{V(\mu_i)}$$

$$\frac{\partial \eta_i}{\partial \beta_j} = \frac{\partial \boldsymbol{x}_i^\top \boldsymbol{\beta}}{\partial \beta_j} = x_{i,j}, \quad \frac{\partial \mu_i}{\partial \eta_i} = (\frac{\partial \eta_i}{\partial \mu_i})^{-1} = (g'(\mu_i))^{-1}$$

GLM, Bias, & Economic Interpretation

With canonical link $g_\star = b'^{-1}$, i.e. $\eta_i = \theta_i$,

$$\boldsymbol{X}^\top(\boldsymbol{y} - \boldsymbol{\mu}) = 0, \quad \text{where } \widehat{\boldsymbol{y}} = \boldsymbol{\mu}$$

so, if there is an intercept, $1^\top(\boldsymbol{y} - \boldsymbol{\mu}) = 0$, i.e. $\sum_{i=1}^n y_i = \sum_{i=1}^n \widehat{y}_i$

which is the empirical version (training dataset) of $\mathbb{E}[Y] = \mathbb{E}[\widehat{Y}]$

(see logistic regression or Poisson with log-link)

But more generally, the first order condition is

$$\boldsymbol{X}^\top \boldsymbol{W} \Delta(\boldsymbol{y} - \boldsymbol{\mu}) = 0,$$

where $\boldsymbol{W} = \text{diag}\big((V(\mu_i)g'(\mu_i)^2)^{-1}\big)$ and $\Delta = \text{diag}\big(g'(\mu_i)\big)$.

But usually not an important issue in ML classification problems

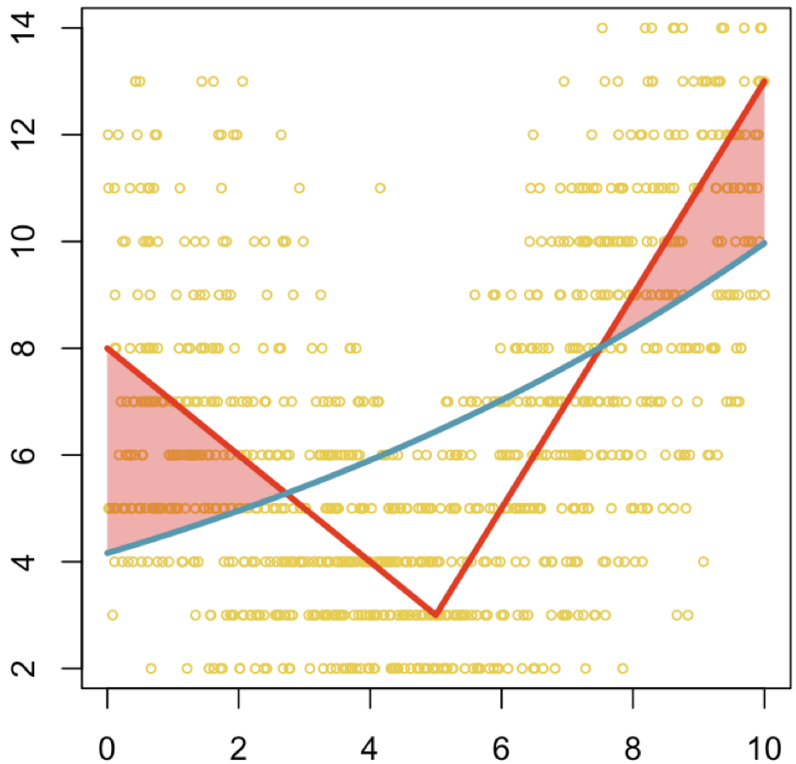
GLM, Bias, & Economic Interpretation

Model $\widehat{\pi}$ is globally unbiased if $\mathbb{E}[\widehat{\pi}(\boldsymbol{X})] = \mathbb{E}[Y]$, $\sum_{i=1}^{n} \widehat{y}_i = \sum_{i=1}^{n} y_i$

Model $\widehat{\pi}$ is locally unbiased if $\mathbb{E}[Y|\widehat{\pi}(\boldsymbol{X}) = s] = s$

GLM $\widehat{\pi}$ globally unbiased,
but possibly locally biased
Major economic impact

- $\widehat{\pi}(\boldsymbol{x}) < \mu(\boldsymbol{x})$
  attractive, but underpriced
- $\widehat{\pi}(\boldsymbol{x}) > \mu(\boldsymbol{x})$
  not attractive

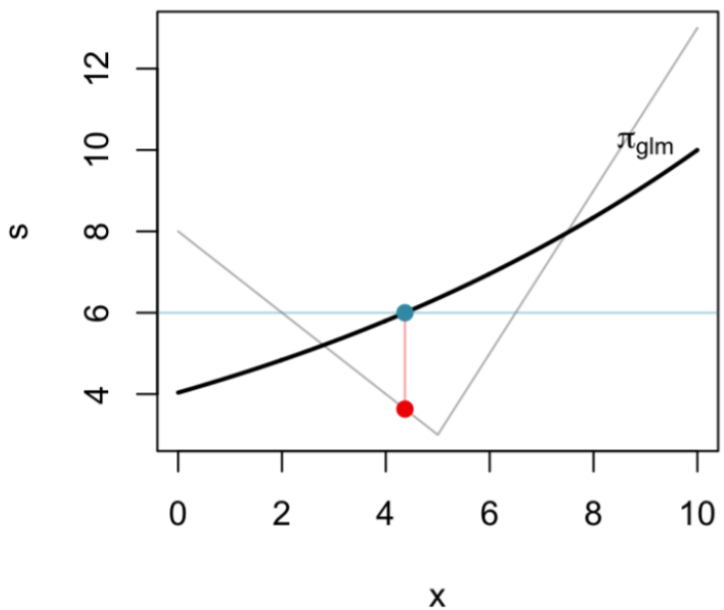


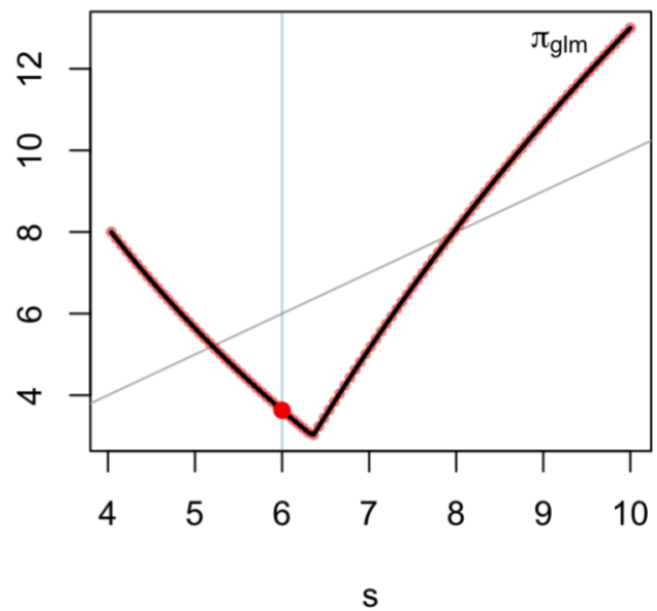
Natural idea: plot $s \mapsto \mathbb{E}[Y|\widehat{\pi}(\boldsymbol{X}) = s]$ " $= \mu(\widehat{\pi}^{-1}(s))$ "

Computing $\mathbb{E}[Y|\widehat{\pi}(\boldsymbol{X}) = s]$

True model $\mu(x)$ and GLM model $\widehat{\pi}(x)$

- select s, e.g. $s = 6$
- compute $x = \widehat{\pi}^{-1}(s)$ ($\widehat{\pi}$ is strictly increasing), here $x = 4.2$
- compute $\mu(\widehat{\pi}^{-1}(s))$, here 3.8
- plot $(s, \mu(\widehat{\pi}^{-1}(s)))$



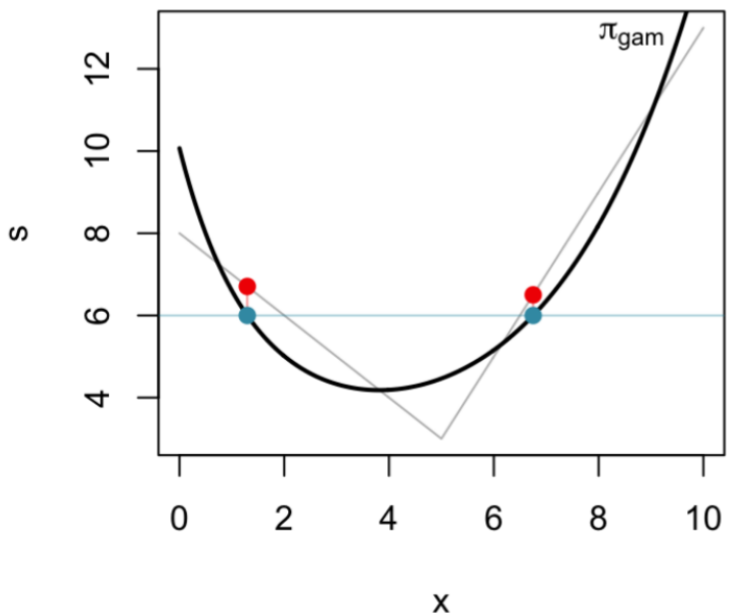


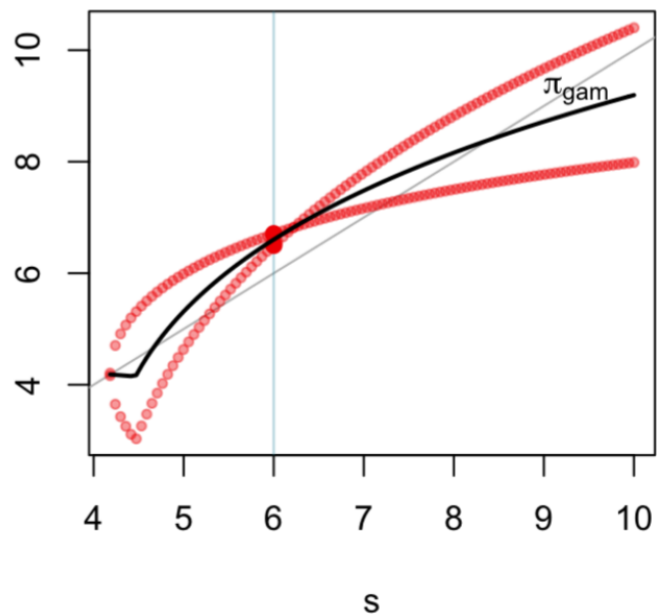
$s \mapsto \mathbb{E}[Y|\widehat{\pi}(X) = s] = \mu(\widehat{\pi}^{-1}(s))$ since $\mu(x) = \mathbb{E}[Y|X = x]$.

Computing $\mathbb{E}[Y|\widehat{\pi}(\boldsymbol{X}) = s]$

True model $\mu(x)$ and GAM model $\widehat{\pi}(x)$

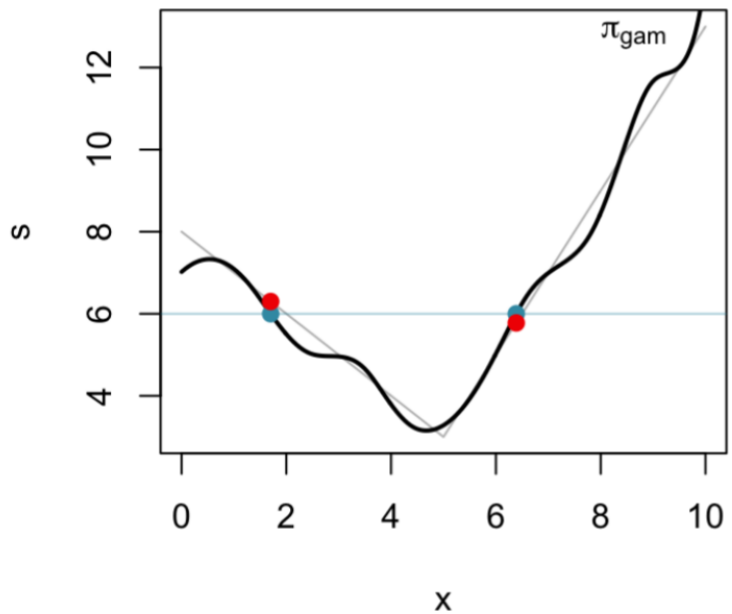
- select s, e.g. $s = 6$
- compute $\mathcal{X}_s^{\widehat{\pi}} = \{\boldsymbol{x} \in \mathcal{X}, \widehat{\pi}(\boldsymbol{x}) = s\}$, here $\{1.6; 6.7\}$
- compute $\mu(\widehat{\pi}^{-1}(s))$, $\{6.4, 6.2\}$ and its mean $\overline{\mu}(\widehat{\pi}^{-1}(s))$, 6.3
- plot $(s, \overline{\mu}(\widehat{\pi}^{-1}(s)))$

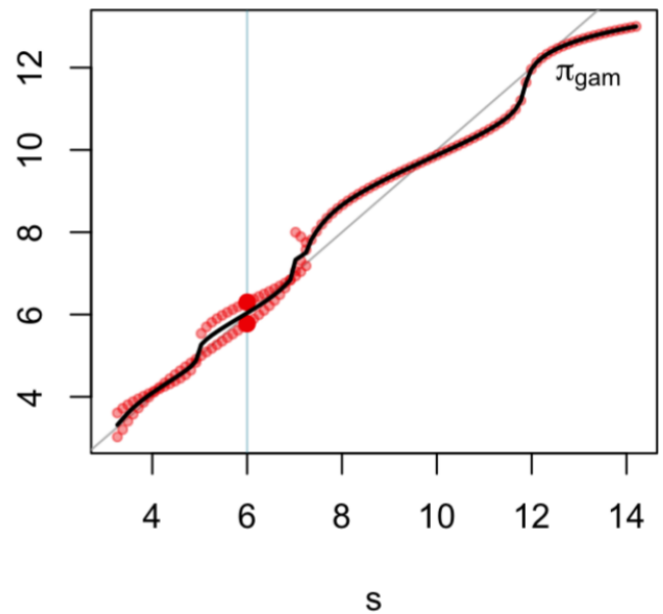




Computing $\mathbb{E}[Y|\widehat{\pi}(\boldsymbol{X}) = s]$

True model $\mu(x)$ and GAM model $\widehat{\pi}(x)$ (more degrees of freedom)
Plot $s \mapsto \mathbb{E}[Y|\widehat{\pi}(X) = s]$, should be close to the first diagonal





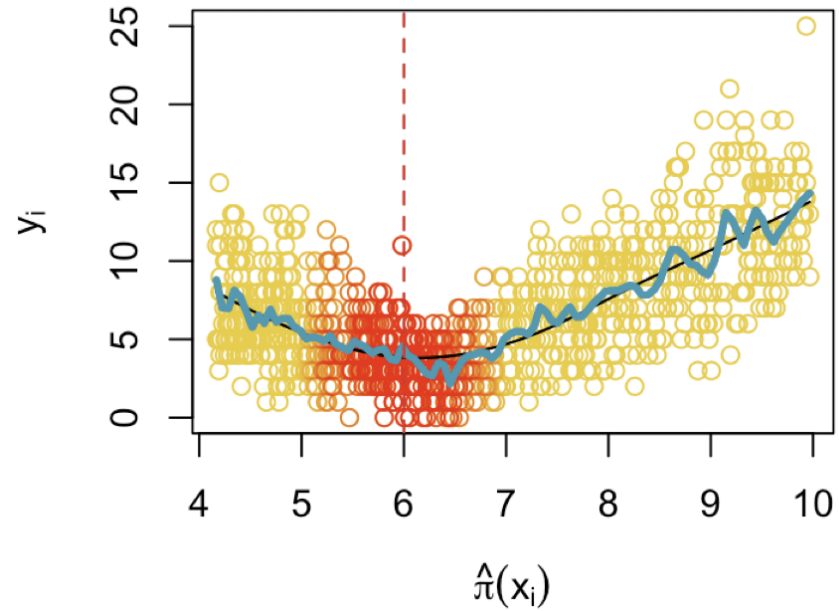
Seems locally unbiased...

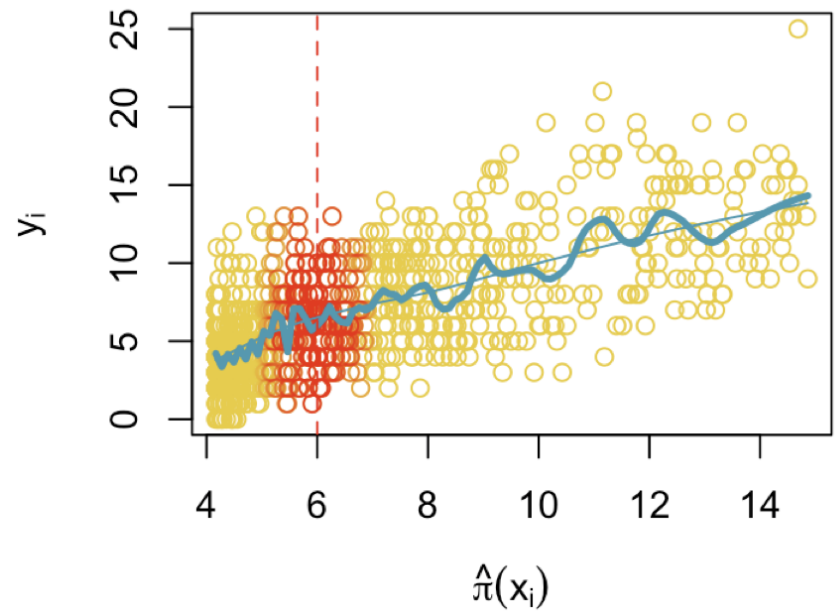
- impossible to get the figure on the left in higher dimension
- we used here μ but in practice, μ is unknown !

$\mathbb{E}[Y|\widehat{\pi}(\boldsymbol{X}) = s]$: empirical version

How to plot $s \mapsto \mathbb{E}[Y|\widehat{\pi}(X) = s]$?
but in real life, μ is unknown
Consider $\{(\widehat{\pi}(\boldsymbol{x}_i), y_i)\}$
and fit a local regression

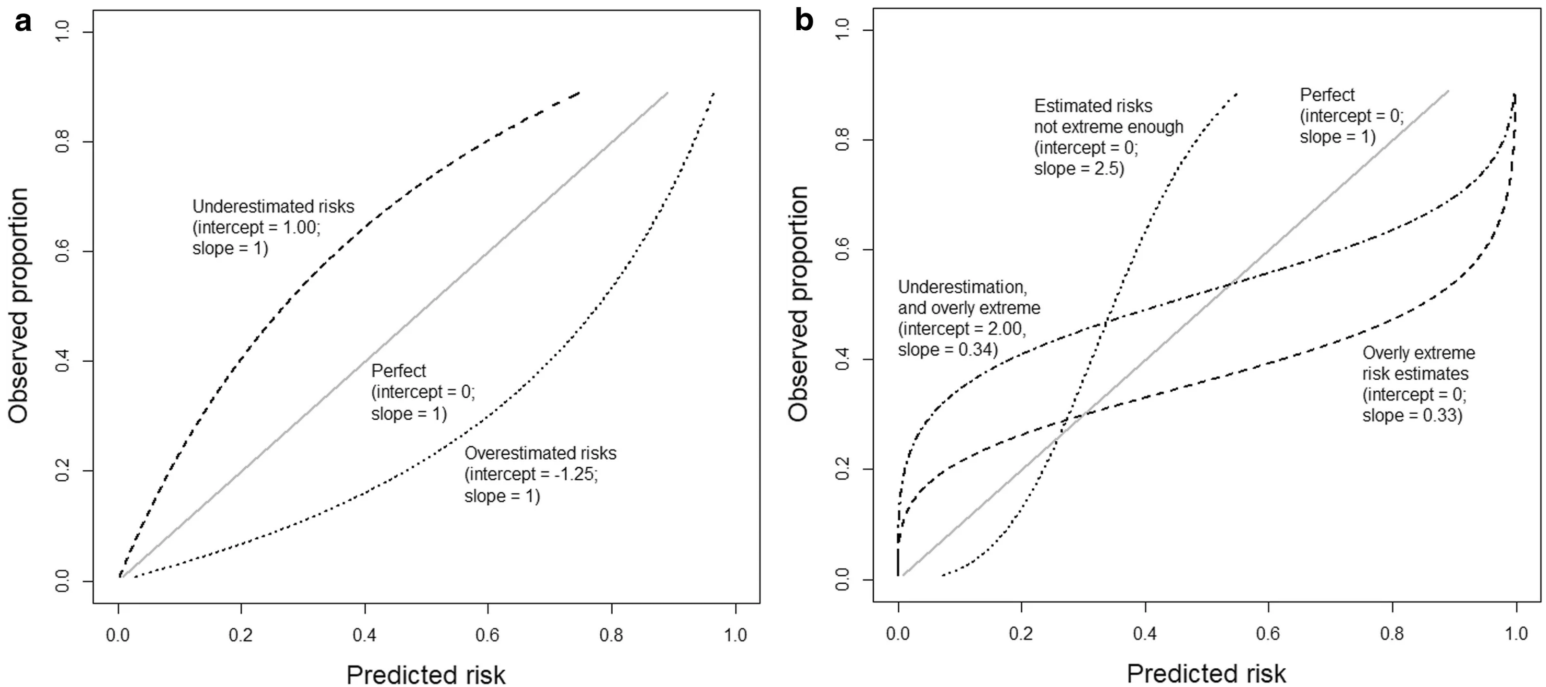
- fit a model $\widehat{\pi}$
- estimate $\mathbb{E}[Y|\widehat{\pi}(X) = s]$
  local regression on $\{(\widehat{\pi}(\boldsymbol{x}_i), y_i)\}$
- local (multiplicative) correction
  $$\lambda_\alpha(s) = \frac{\mathbb{E}[Y|\widehat{\pi}(X) = s]}{s}$$
- correct $\widehat{\pi}$
  $$\widehat{\pi}_{BC}(\boldsymbol{x}) = \lambda_\alpha(\widehat{\pi}(\boldsymbol{x})) \cdot \widehat{\pi}(\boldsymbol{x})$$





Calibration & Auto-calibration

Van Calster, B. *et al.* (2019). Calibration: the Achilles heel of predictive analytics, (BMC Medecine)
In the context of binary classification, we get the following graphs

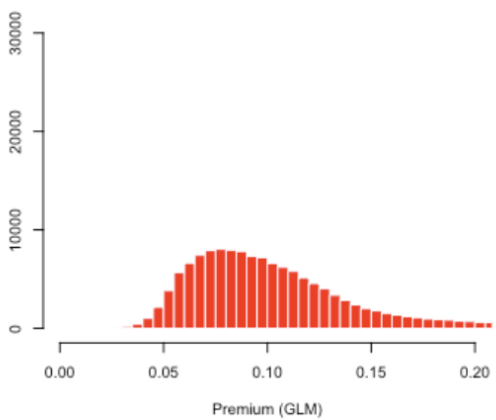


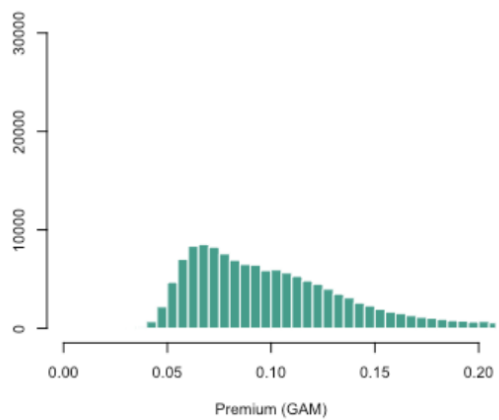
Application of a motor-insurance dataset

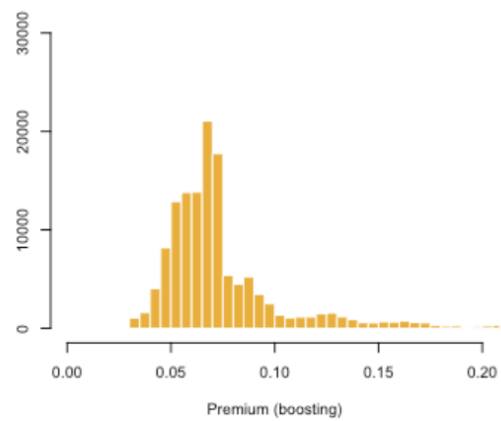
Here we focus only on claims (annual) frequency, corrected from the exposure, `freMTPL2freq` from `CASDataset` package,

	π^{glm}	π^{gam}	π^{bst}
average $\overline{\pi}$	0.1087	0.1092	0.0820
10% quantile	0.0605	0.0598	0.0498
90% quantile	0.1682	0.1713	0.1244

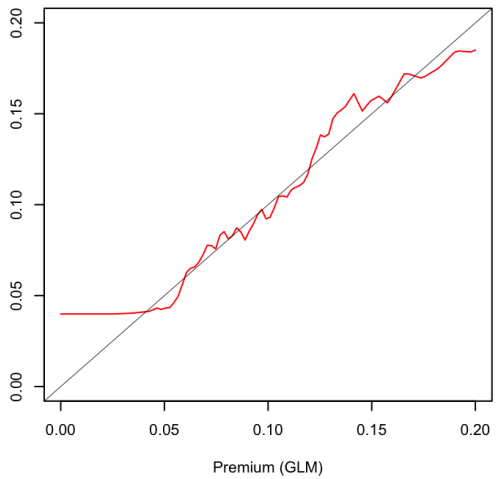
Table 1: Summary statistics on $\{\pi(\boldsymbol{x}_1), \cdots, \pi(\boldsymbol{x}_n)\}$, on the validation dataset (assuming an exposure of 1 to provide annualized predictions).

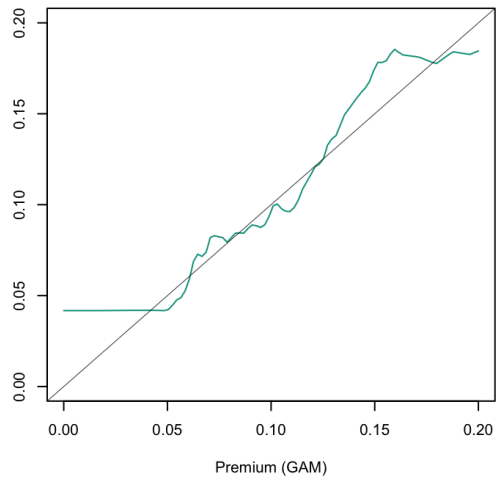


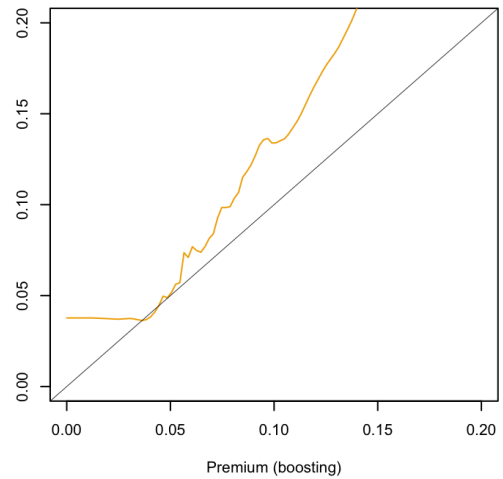




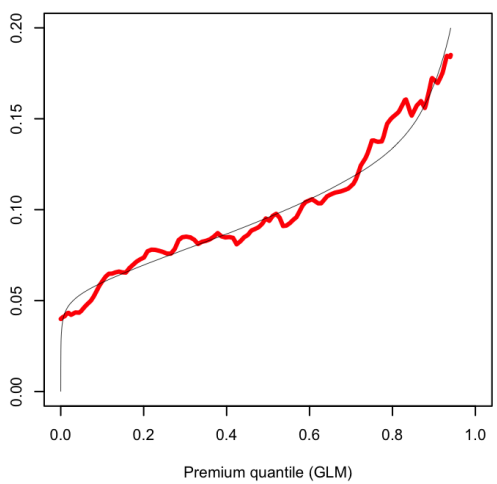
Application of a motor-insurance dataset

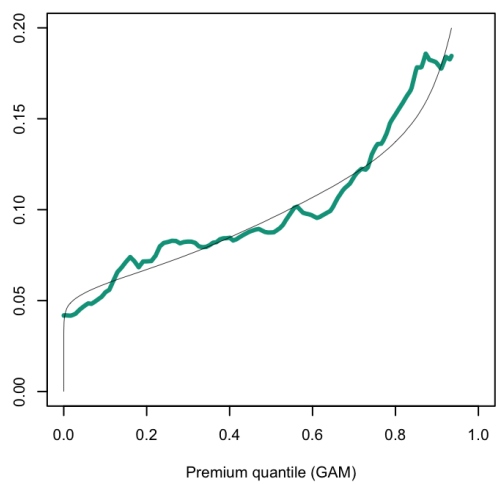


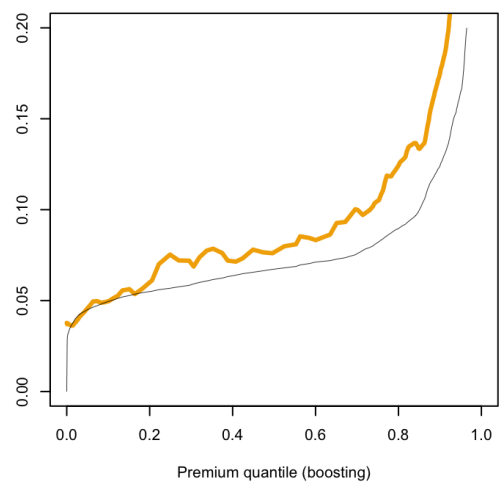




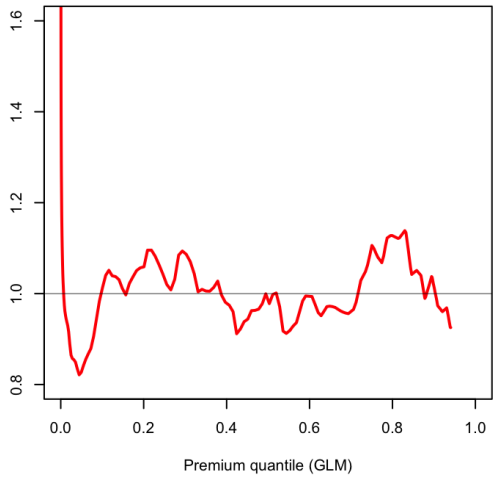
Evolution of $s \mapsto \mathbb{E}[Y|\widehat{\pi}(\boldsymbol{X}) = s]$ and $u \mapsto \mathbb{E}[Y|\widehat{\pi}(\boldsymbol{X}) = F_{\pi}^{-1}(u)]$

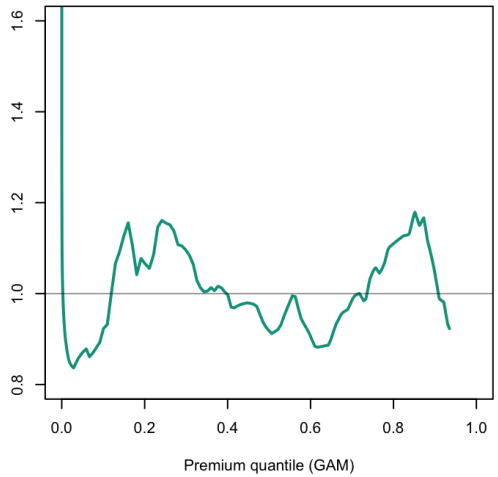


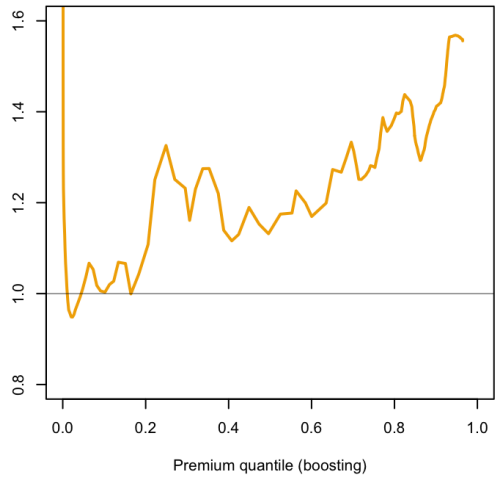




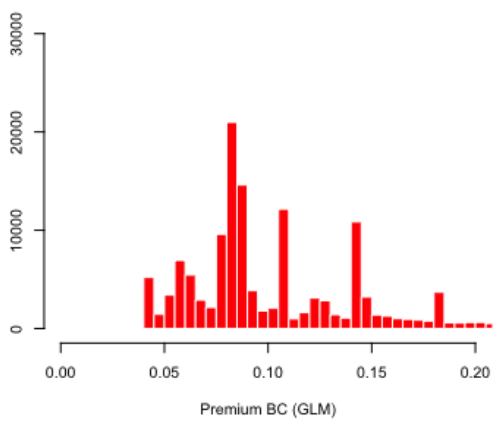
Application of a motor-insurance dataset

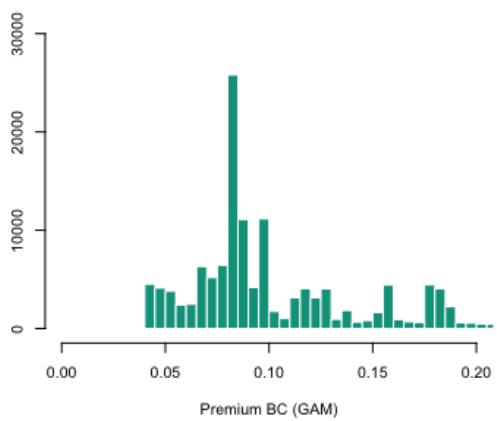


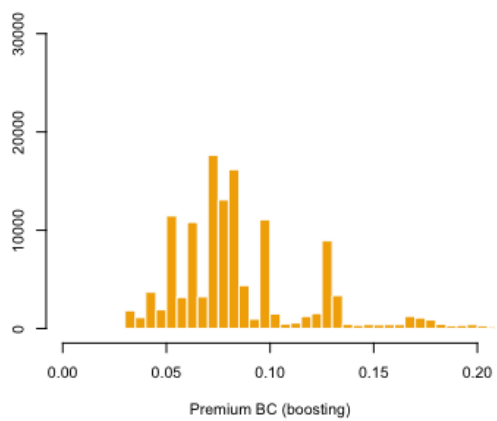




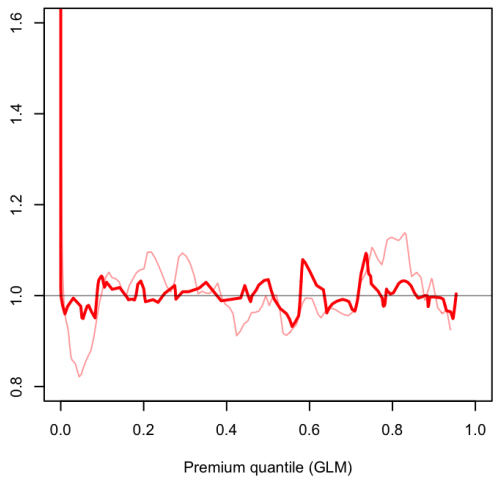
Multificative correction $\lambda_\alpha(u) = \mathbb{E}[Y|\widehat{\pi}(\boldsymbol{X}) = F^{-1}_{\widehat{\pi}}(u)]/F^{-1}_{\widehat{\pi}}(u)$

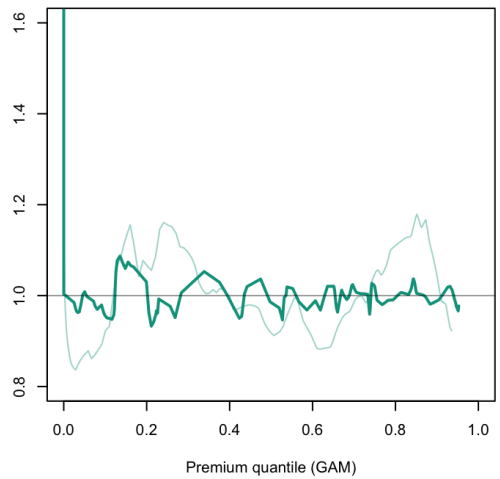


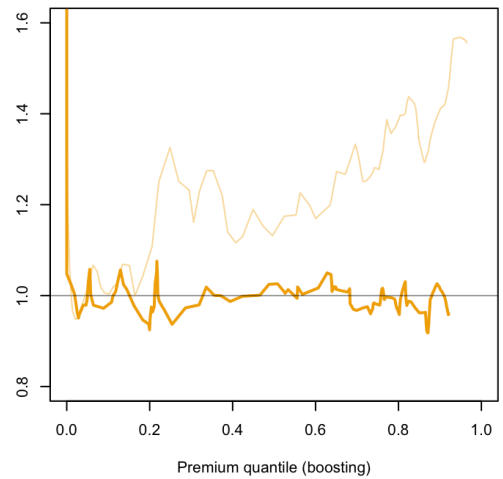




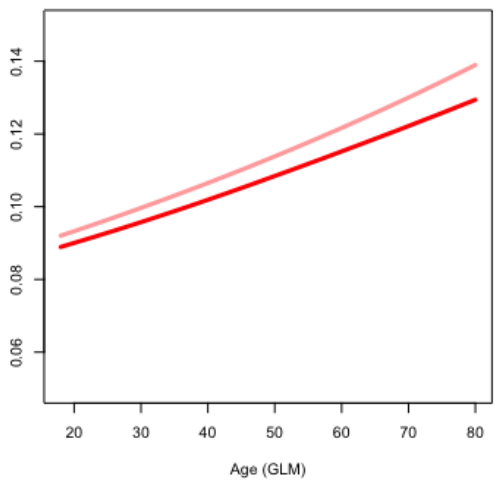
Application of a motor-insurance dataset

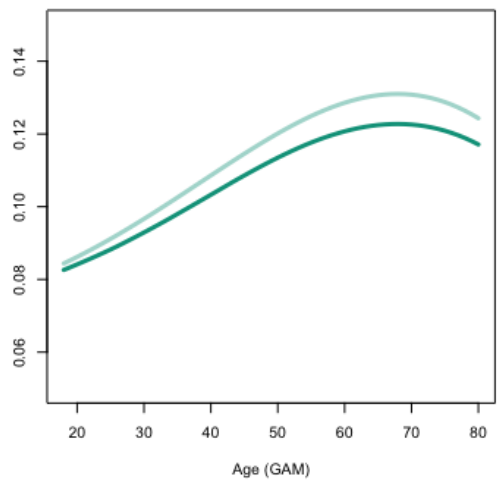


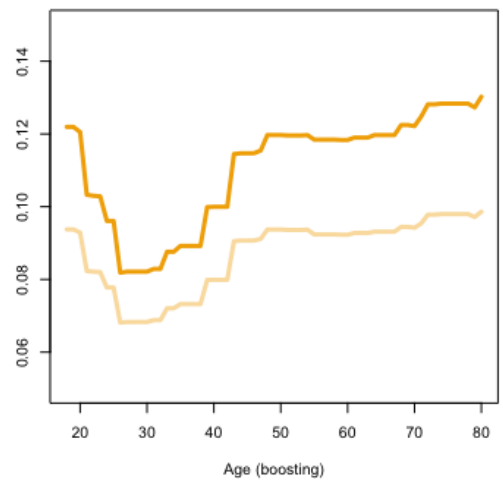




λ_α on the corrected model $\widehat{\pi}_{BC}$, and some partial dependence plot (on the age of the driver)

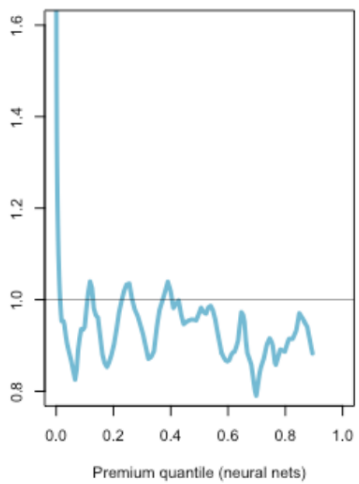


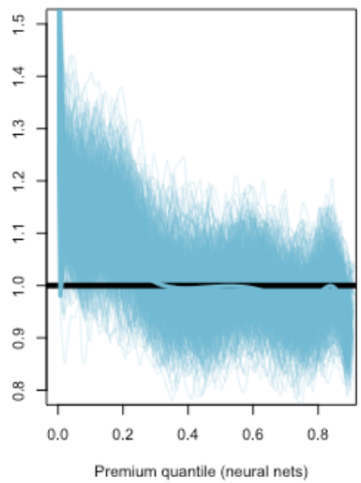


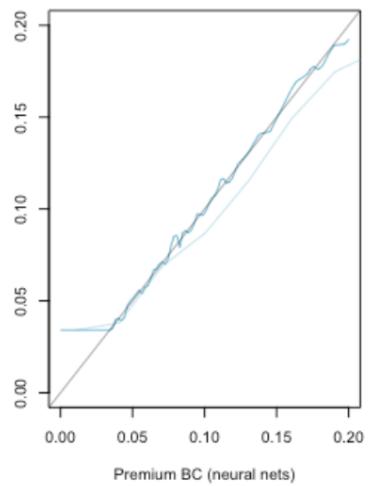


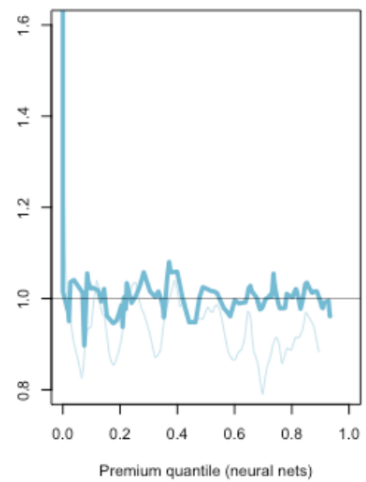
Using Neural Nets on motor-insurance dataset

We can also look at neural networks performance



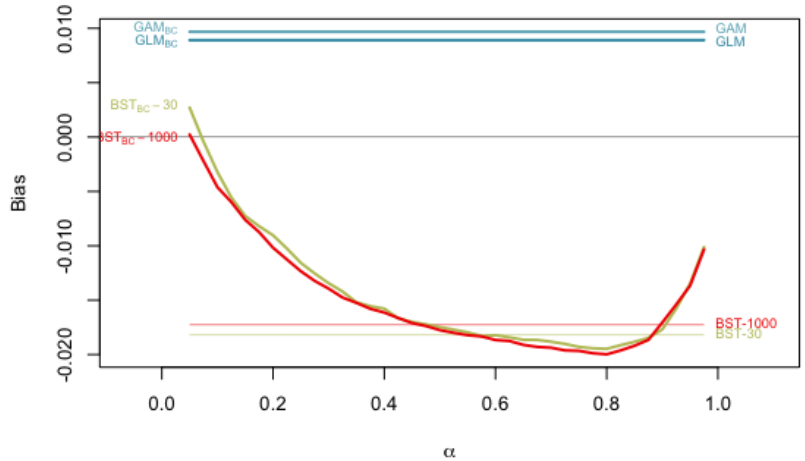


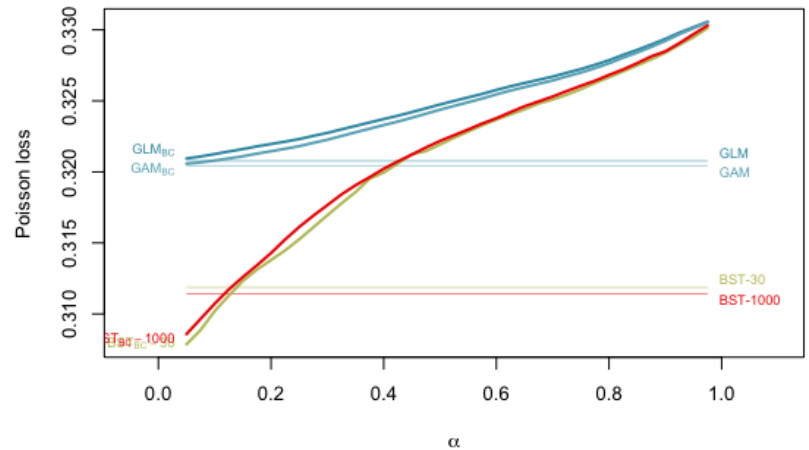


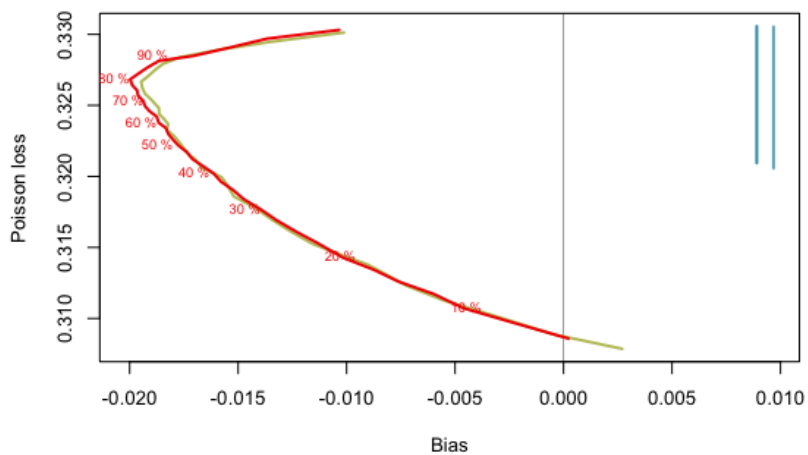


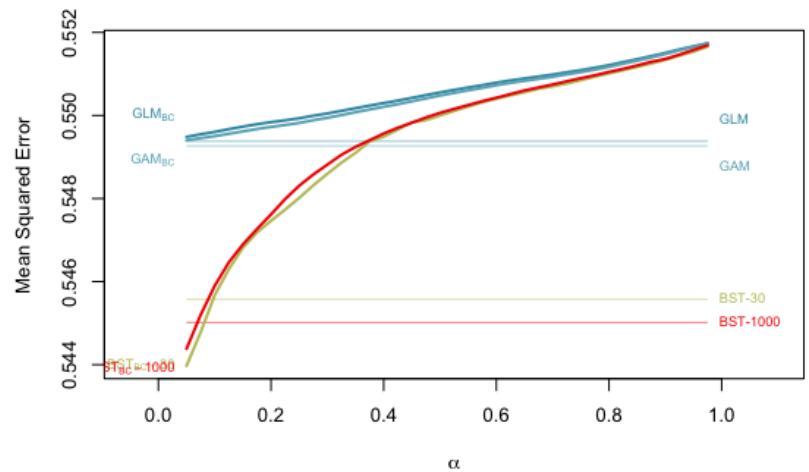
Reproducibility issues here

Choice of α

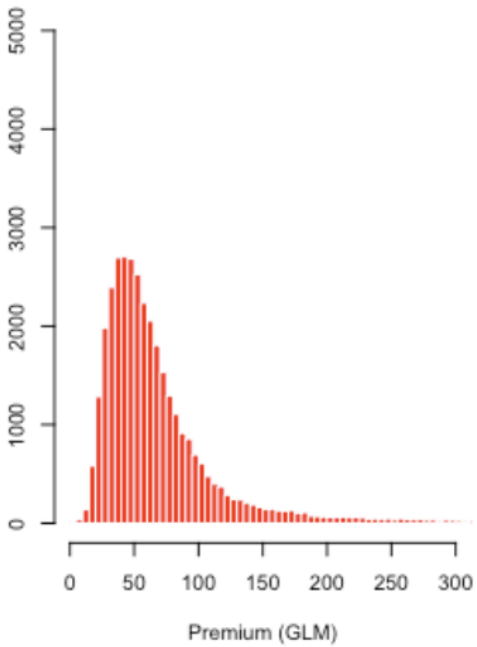


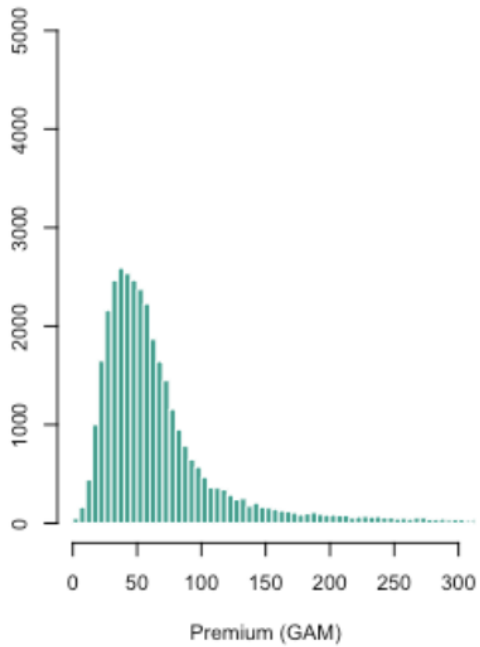


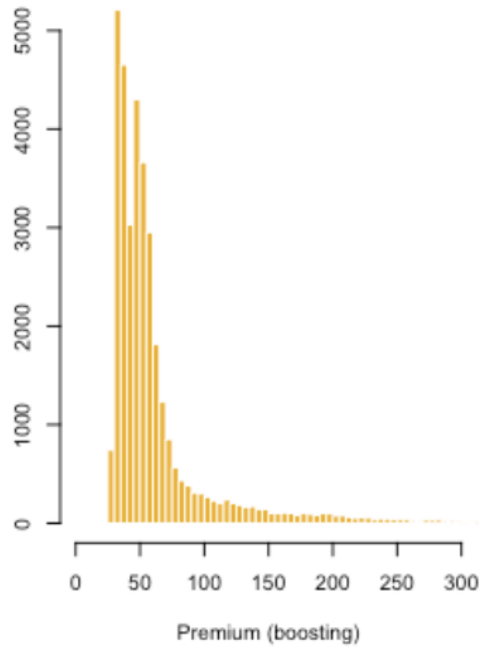




Tweedie model on motor-insurance dataset







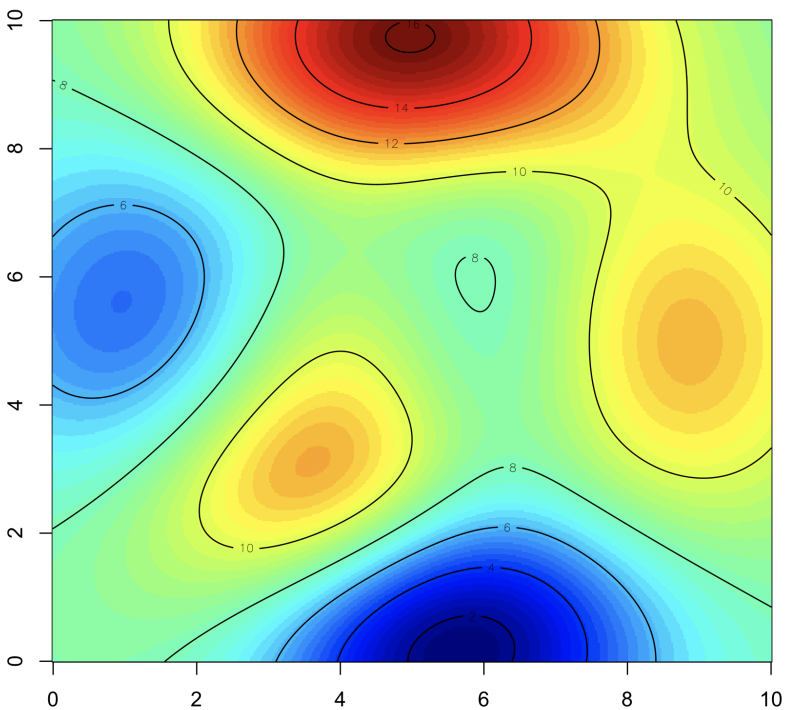
Wrap-Up

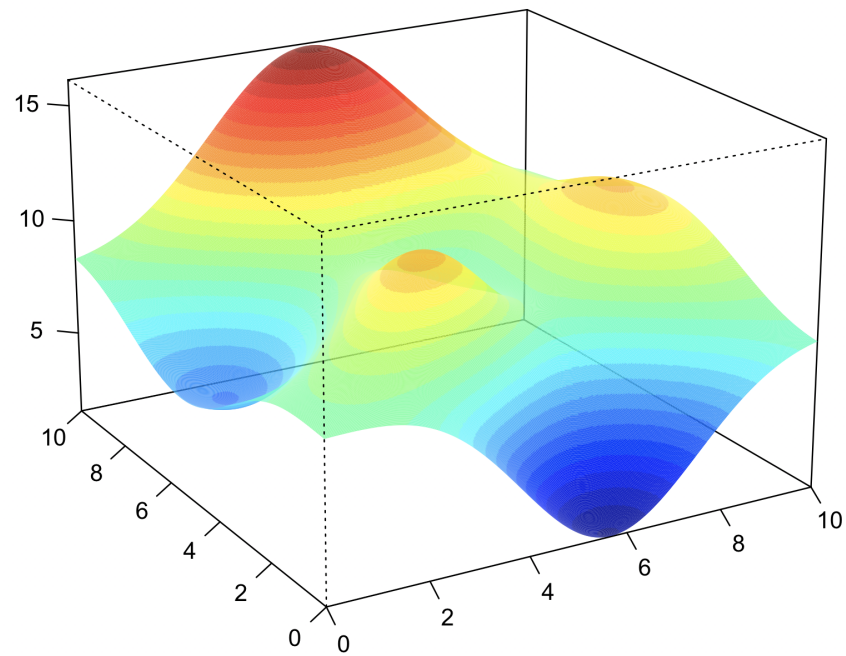
- fit a model $\widehat{\pi}$ using an ML algorithm
- estimate $\mathbb{E}[Y|\widehat{\pi}(X)]$ with local regression on $\{(\widehat{\pi}(\boldsymbol{x}_i), y_i)\}$
- local (multiplicative) correction $\lambda_\alpha(s) = \dfrac{\mathbb{E}[Y|\widehat{\pi}(X) = s]}{s}$
- correct $\widehat{\pi}$ by setting $\widehat{\pi}_{BC}(\boldsymbol{x}) = \lambda_\alpha(\widehat{\pi}(\boldsymbol{x})) \cdot \widehat{\pi}(\boldsymbol{x})$

Denuit, M., Charpentier, A. & Trufin, J. (2021). Autocalibration and Tweedie-dominance for Insurance Pricing with Machine Learning, (IME)

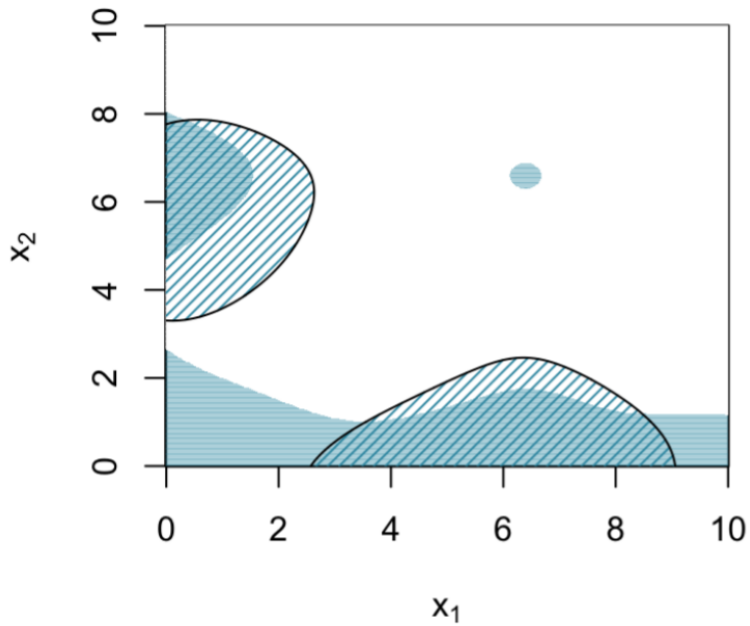
Appendix: Bivariate simulated data

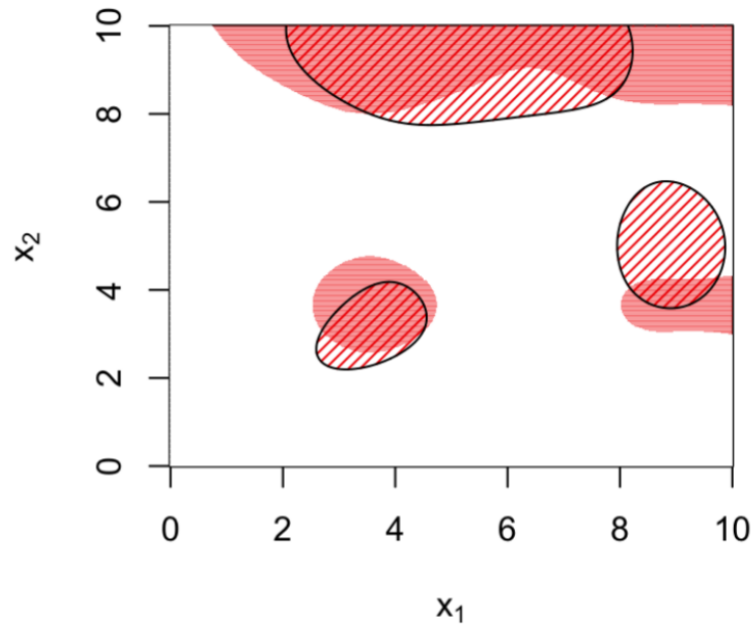
Generate some data $(x_{1,i}, x_{2,i}, y_i)$ where $Y|\boldsymbol{X}$ has some distribution with mean $\mathbb{E}[Y|\boldsymbol{x}] = \mu(\boldsymbol{x})$

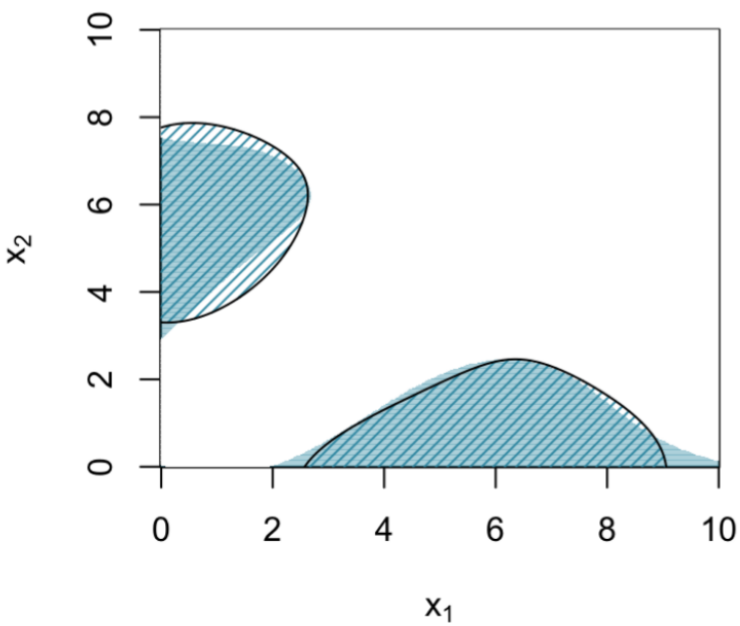


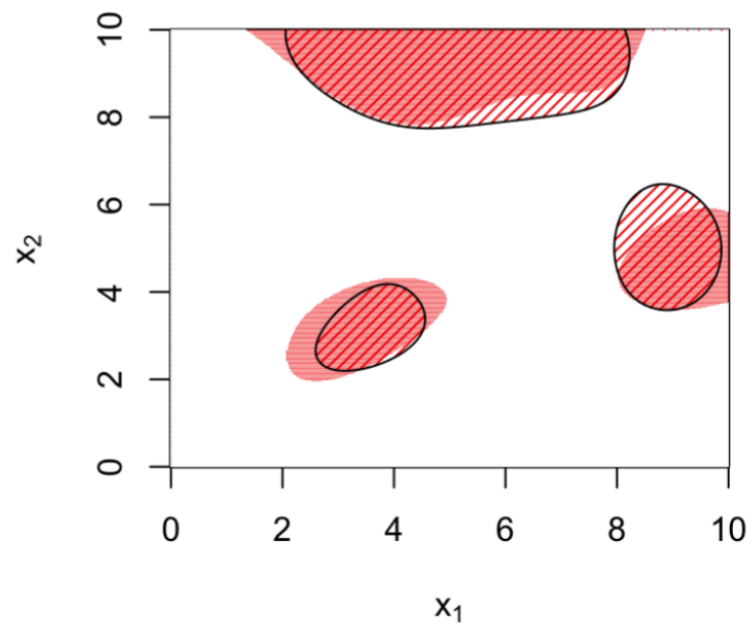


Model inversion, $\mathcal{X}_s^{\widehat{\pi}} = \{\boldsymbol{x} \in \mathcal{X}, \widehat{\pi}(\boldsymbol{x}) \leqslant s\}$









$\mathcal{X}_s^{\widehat{\pi}}$, $Y|\boldsymbol{X} \in \mathcal{X}_s^{\widehat{\pi}}$ and $\mathbb{E}[Y|\widehat{\pi}(\boldsymbol{X}) = s]$: $s = 10$