

# Assurance collaborative, théorie des graphes et actuariat

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Matthias Löwe, Philipp Ratz, Franck Vermet

Colloque SCOR & Institut des Actuaires, 2022

**SCOR**  
FONDATION POUR LA SCIENCE

en partenariat avec

Institut des  
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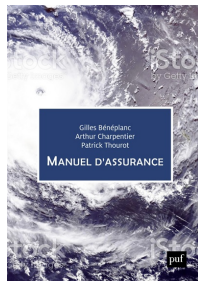
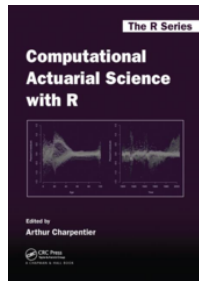
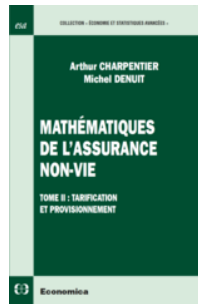
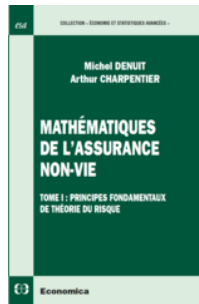
[freakonometrics.hypotheses.org](https://freakonometrics.hypotheses.org) - Arthur Charpentier, 2022

# Arthur Charpentier

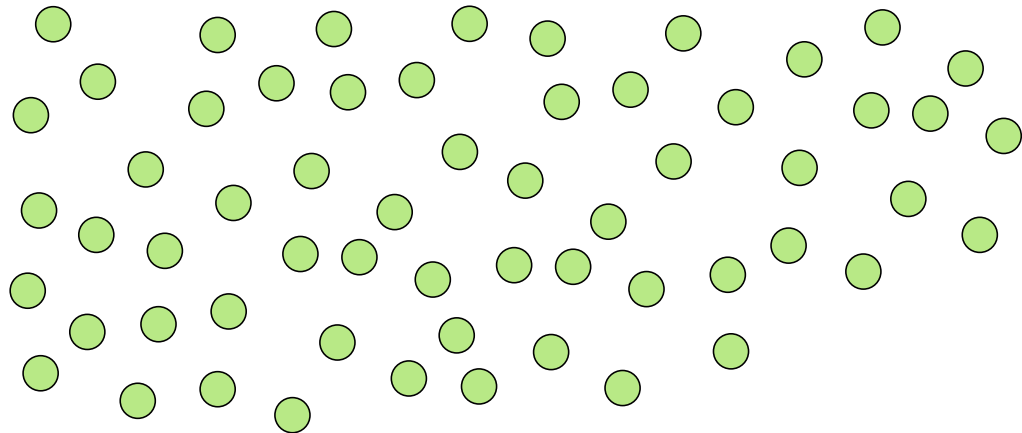
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Modélisation prédictive, Science actuarielle,  
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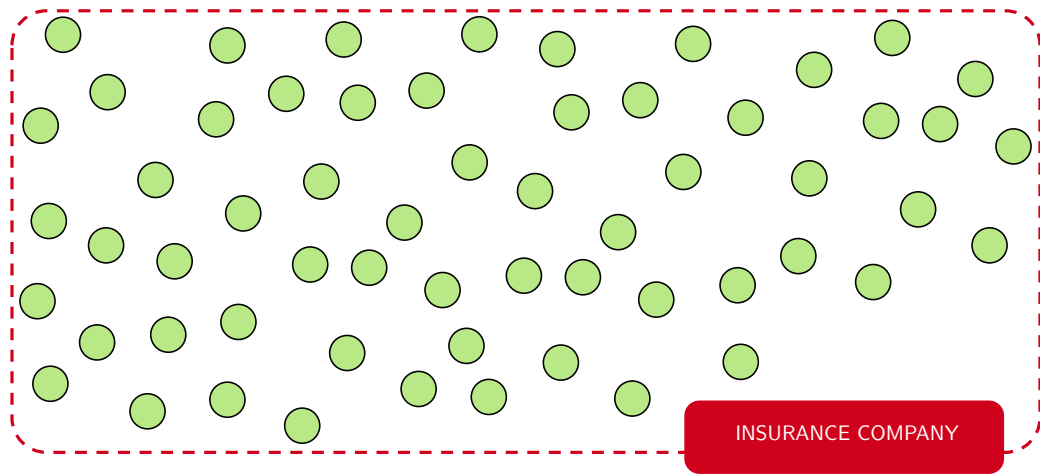


# Risk Transfert



"Insurance is the contribution of the many to the misfortune of the few"

# Risk Transfert



# Risk Aversion

Following [Hardy et al. \(1929, 1934\)](#), and [Marshall and Olkin \(1979\)](#)

**Def** Consider two sorted vectors  $\mathbf{x}$  and  $\mathbf{y}$  ( $x_1 \geq x_2 \geq \dots \geq x_n$  and  $y_1 \geq y_2 \geq \dots \geq y_n$ )

such that  $\sum_{i=1}^n x_i \leq \sum_{i=1}^n y_i$ , then  $\mathbf{x} \preceq_M \mathbf{y}$  (majorization order) if  $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i, \forall k$ .

For example,

$$\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}, \frac{1}{n}\right) \prec_M \left(\frac{1}{n-1}, \frac{1}{n} - 1, \dots, \frac{1}{n-1}, 0\right) \prec_M (1, 0, \dots, 0, 0).$$

$$\iff \sum_{i=1}^n h(x_i) \leq \sum_{i=1}^n h(y_i) \text{ for any convex function}$$

$$\iff \mathbf{x} = D\mathbf{y} \text{ for some doubly stochastic matrix } D, \text{ i.e. } \sum_{k=1}^n D_{i,k} = \sum_{k=1}^n D_{k,j} = 1, \forall i, j$$

$$\iff \mathbf{x} = P_1 \cdots P_k \mathbf{y} \text{ for finitely some Pigou-Dalton transfert matrices } P_j \\ (P_j = \alpha \mathbb{I} + (1 - \alpha)T \text{ for some } \alpha \in (0, 1) \text{ and } T = 0 \text{ except } T_{i,j} = T_{j,i} = 1)$$

# Risk Aversion and Risk Sharing

**Def** Consider two random variables  $X$  and  $Y$ ,  $X \preceq_{CX} Y$  if  $\mathbb{E}[h(X)] \leq \mathbb{E}[h(Y)]$  for any convex function  $h$

$\iff Y$  is a mean-preserving spread of  $X$ , i.e.  $Y \stackrel{\mathcal{L}}{=} X + Z$ , where  $\mathbb{E}[Z|X] = 0$ .

$\iff \mathbb{E}[(X - s)_+] \leq \mathbb{E}[(Y - s)_+]$  for all  $s \in \mathbb{R}$ .

$\implies \mathbb{E}[X] = \mathbb{E}[Y]$  and  $\text{Var}[X] \preceq \text{Var}[Y]$ .

$\iff$  Pigou-Dalton transfert, majorization order, etc

Following [Denuit and Dhaene \(2012\)](#) and [Carlier et al. \(2012\)](#),

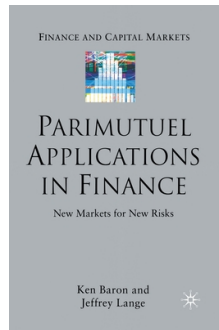
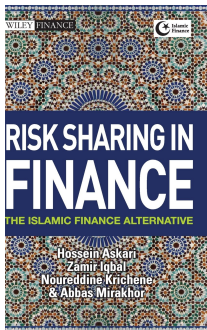
**Def** Consider two random vectors  $\xi = (\xi_1, \dots, \xi_n)$  and  $\mathbf{X} = (X_1, \dots, X_n)$  on  $\mathbb{R}_+^n$ .  $\xi$  is a risk-sharing scheme of  $\mathbf{X}$  if  $X_1 + \dots + X_n = \xi_1 + \dots + \xi_n$  almost surely.

**Def** Consider two random vectors  $\xi = (\xi_1, \dots, \xi_n)$  and  $\mathbf{X} = (X_1, \dots, X_n)$  on  $\mathbb{R}_+^n$ .  $\xi \preceq_{CCX} \mathbf{X}$  if  $\xi_i \preceq_{CX} X_i$ .

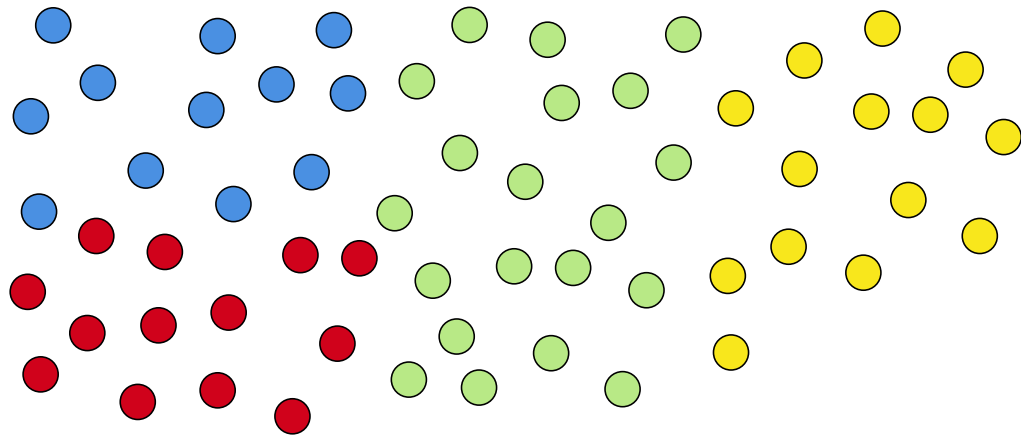
# Risk Sharing

Peer-to-peer insurance is a risk sharing network where a group of individuals pool their premiums together to insure against a risk. Peer-to-Peer Insurance mitigates the conflict that inherently arises between a traditional insurer and a policyholder when an insurer keeps the premiums that it doesn't pay out in claims

- ▶ Takaful التكافل
- ▶ Wakalah وَكَالَة
- ▶ Musharakah مشاركة
- ▶ Xiang Hu Bao 相互保
- ▶ Parimutuel

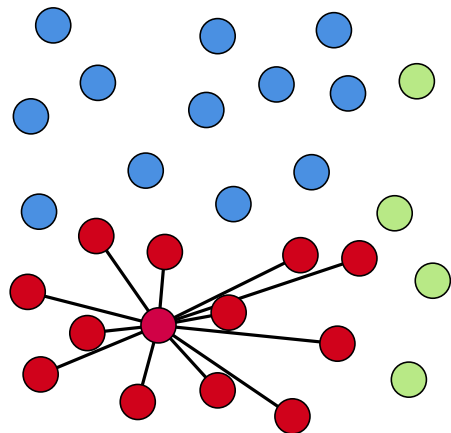


## Risk Sharing





# Risk Sharing



$$\text{Let } \xi_j = \frac{1}{n} \sum_{i=1}^n X_i, \forall j$$

► Risk sharing

$$\xi_1 + \dots + \xi_n = X_1 + \dots + X_n$$

► Componentwise convex-order

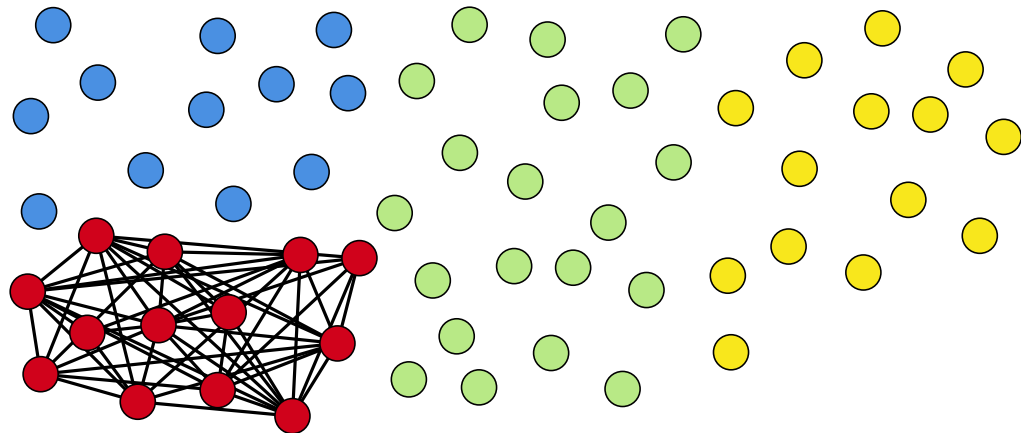
$$\xi_j \preceq_{CX} X_j, \forall j$$

More generally, consider some linear risk sharing  $\xi = M\mathbf{X}$ , for some  $n \times n$  matrix

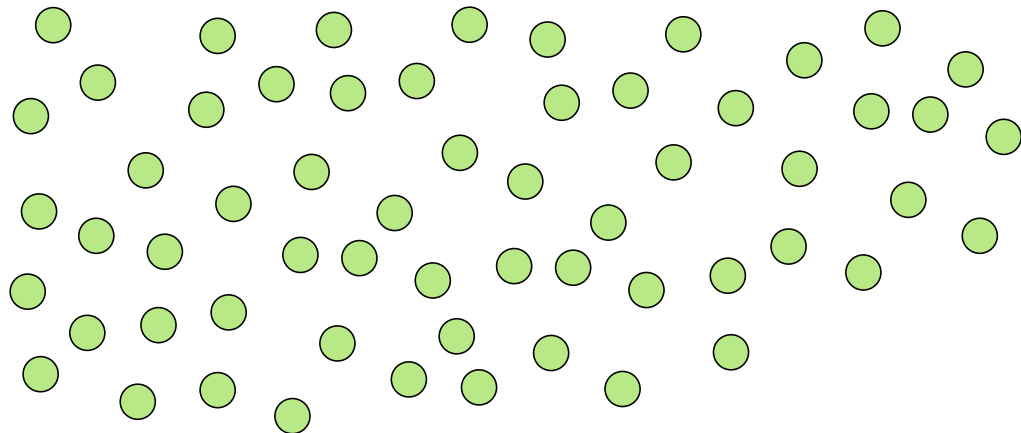
$$M = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{M}_k \end{bmatrix}, \quad \mathbf{M}_k = \frac{1}{n_k} \mathbf{1}_k$$

where  $\mathbf{1}_k$  is the  $n_k \times n_k$  matrix full of 1's.

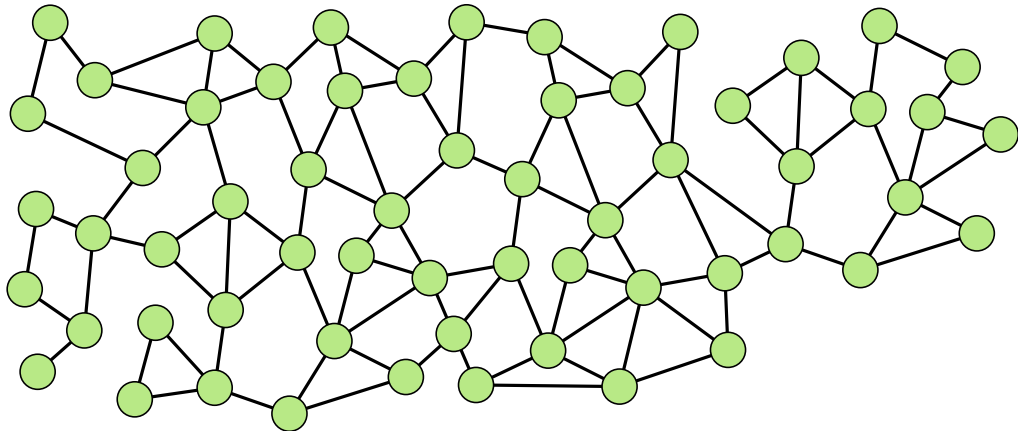
# Risk Sharing



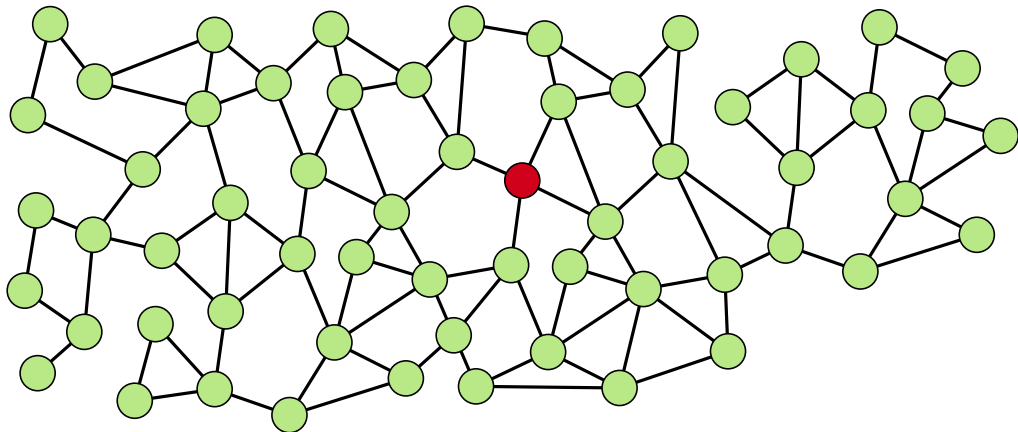
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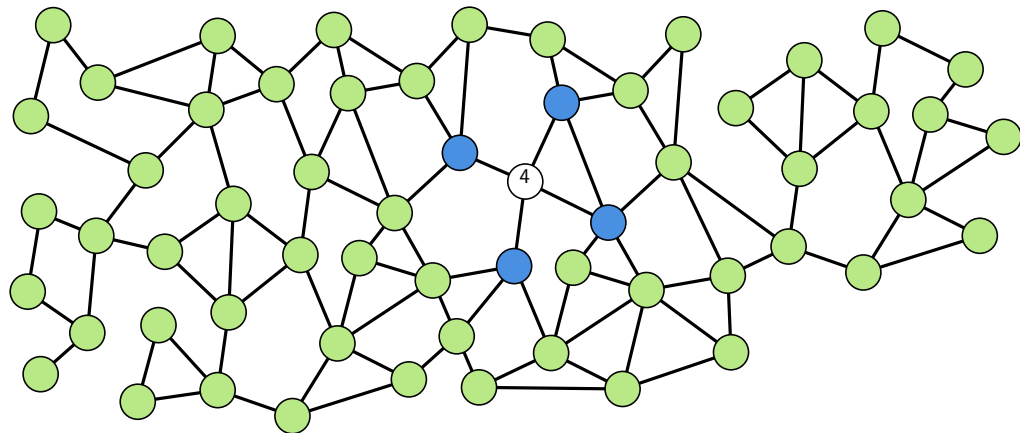
## Risk Sharing on a Network



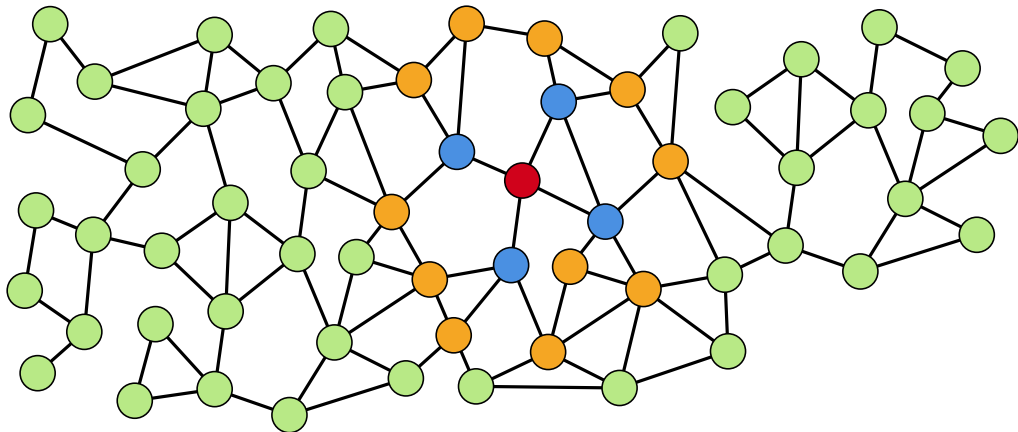
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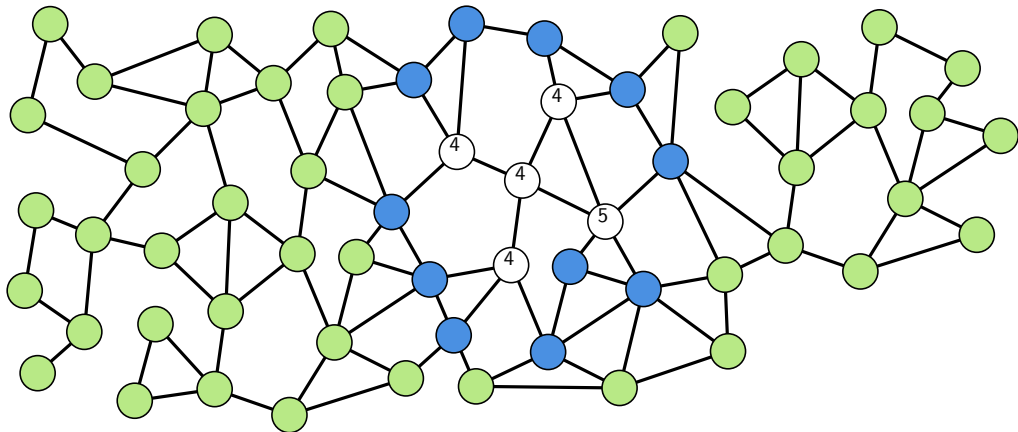
## Risk Sharing on a Network



## Risk Sharing on a Network

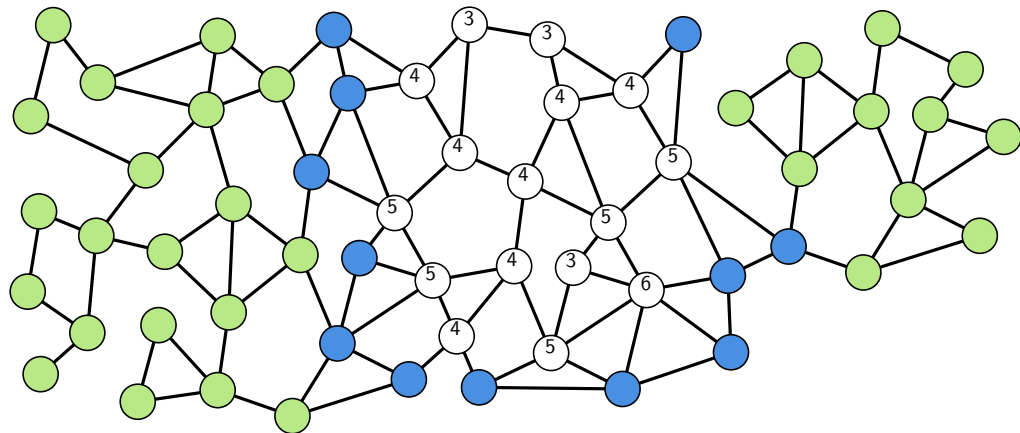


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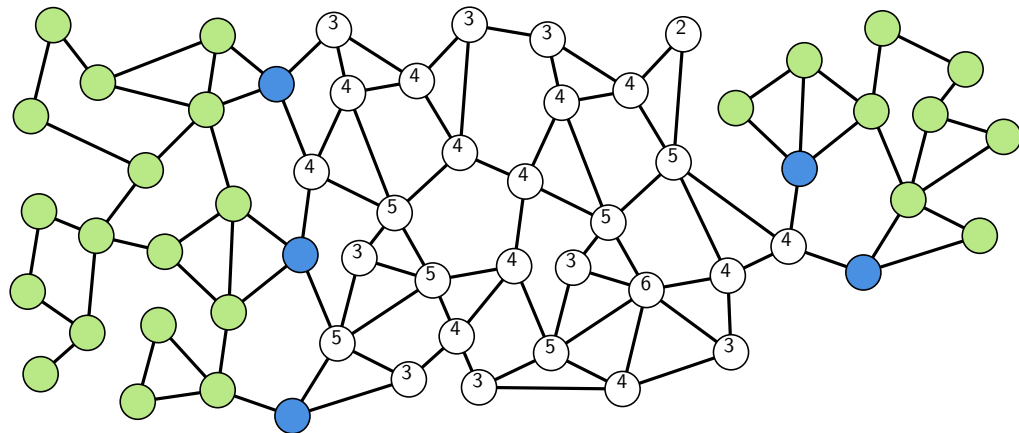




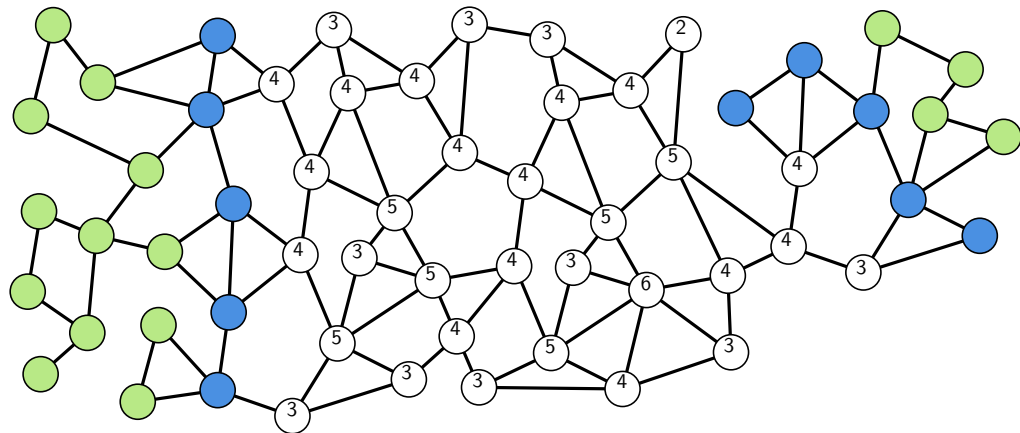
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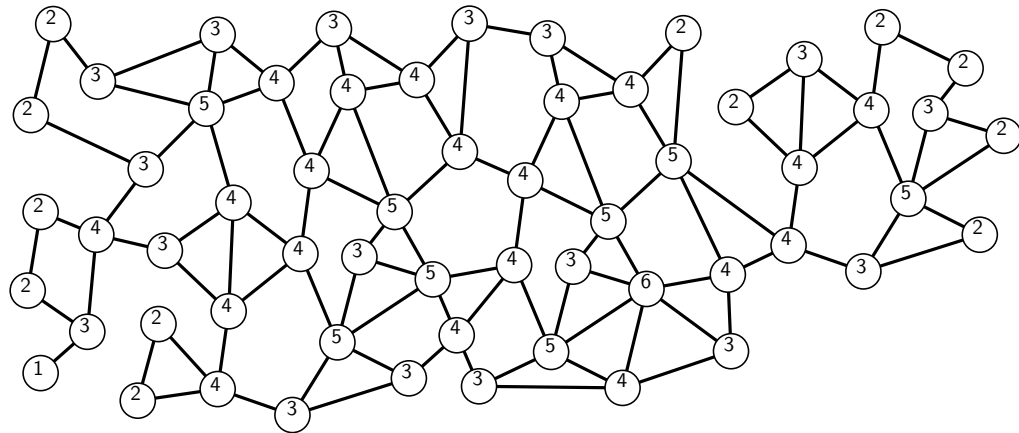
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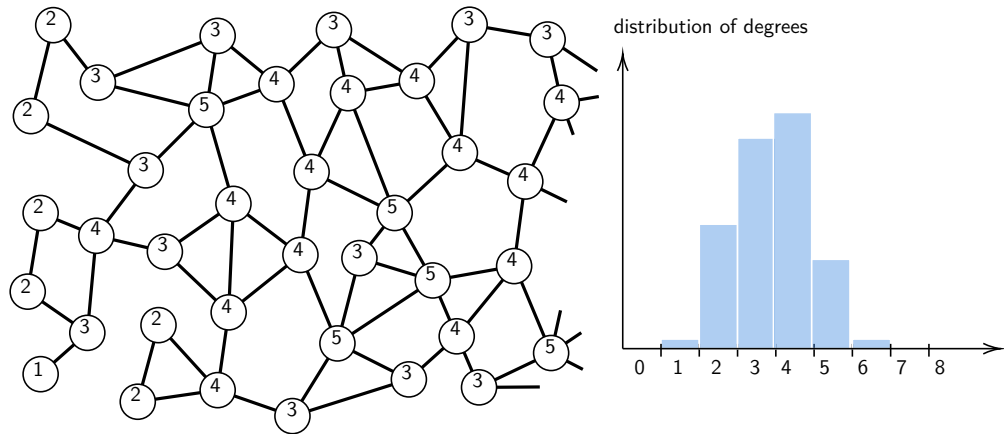
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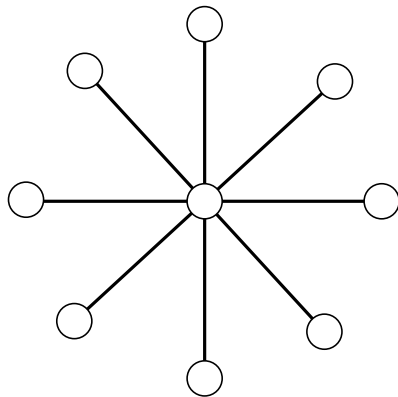
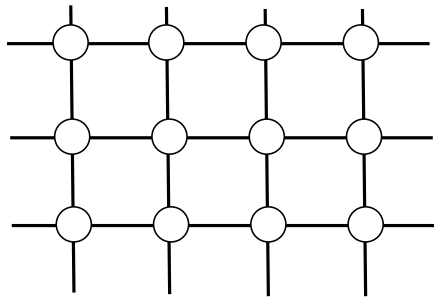
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# Risk Sharing on a Network

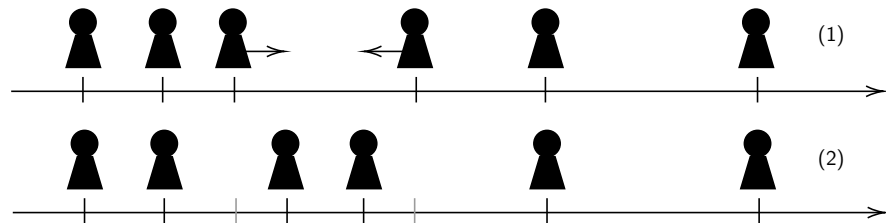


## Risk Sharing on a Network



Regular graph vs. star shaped graph  
(low variance vs. large variance on  $D$ )

# Risk Sharing on a Network

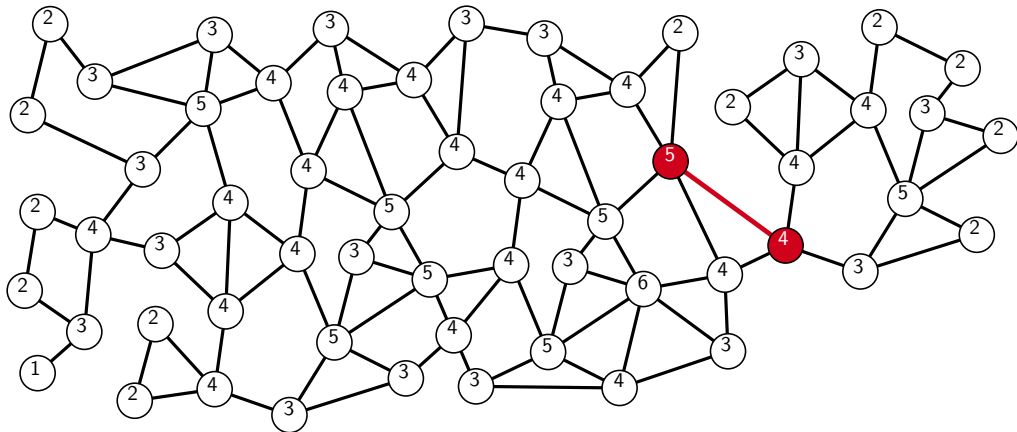


Pigou-Dalton transferts ([Dalton \(1920\)](#)) see also [Atkinson \(2015\)](#),

$$\mathbf{y}^{(2)} \preceq_M \mathbf{y}^{(1)} \leftarrow \begin{cases} y_i^{(2)} = y_i^{(1)}, \forall i \neq j, k \\ y_j^{(2)} = y_j^{(1)} + h, \\ y_k^{(2)} = y_k^{(1)} - h, y_j^{(2)} > y_j^{(1)} \end{cases}$$

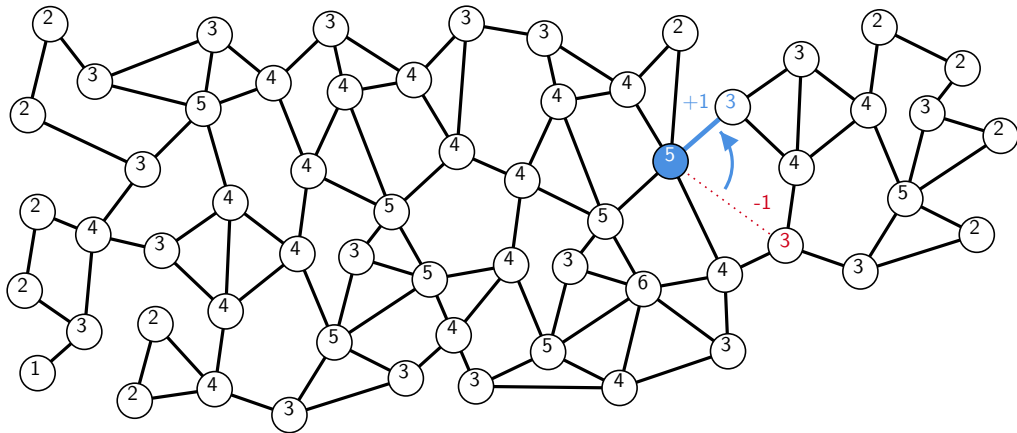
see martingale property of mean-preserving spread,  $Y^{(1)} \stackrel{\mathcal{L}}{=} Y^{(2)} + Z$ , where  $\mathbb{E}[Z|Y^{(1)}] = 0$  (convex order is a dispersion order)

# Risk Sharing on a Network

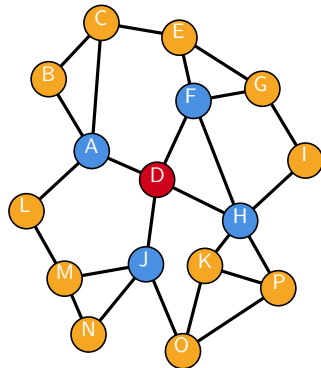
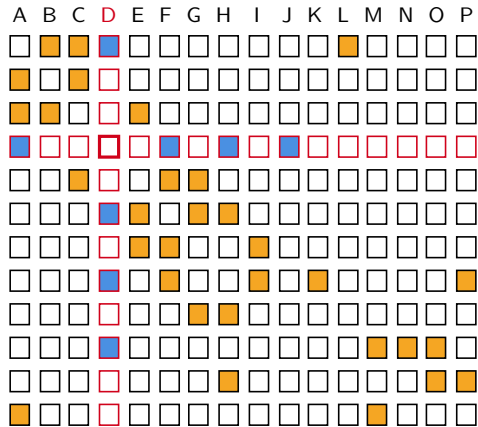




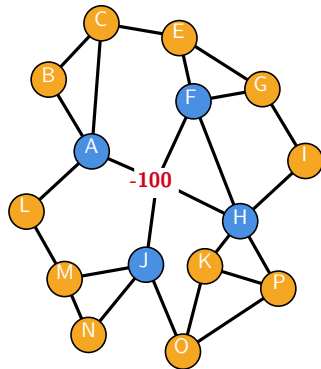
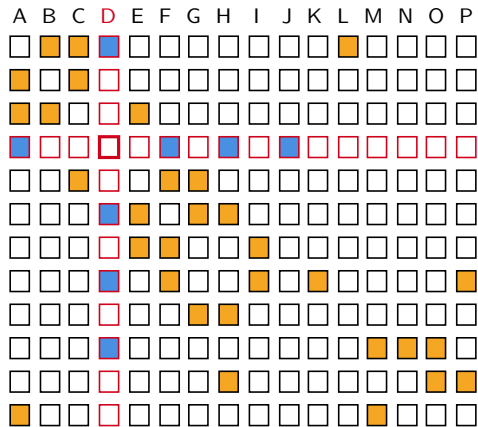
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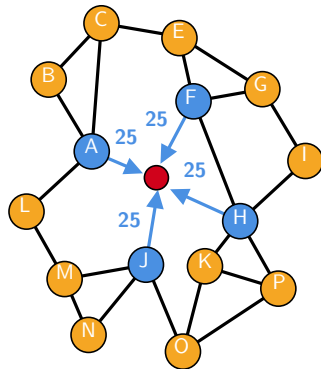
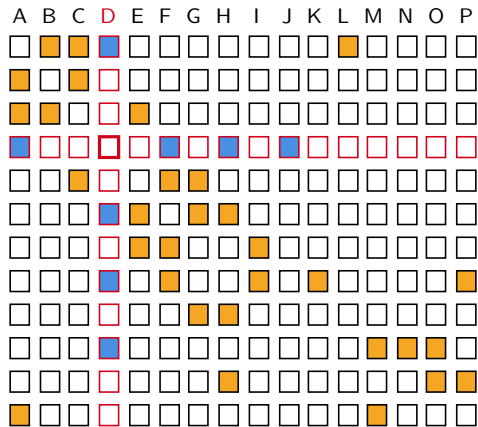
# Sharing Risks with Friends



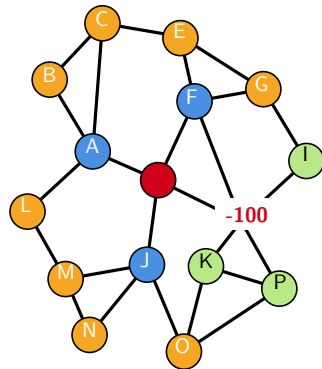
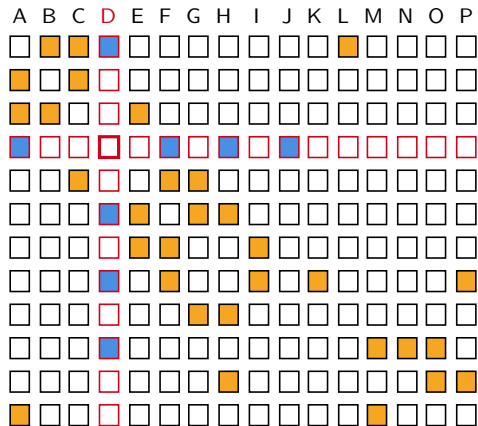
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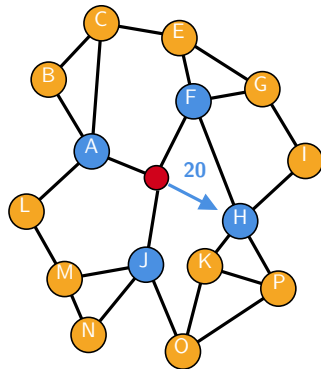
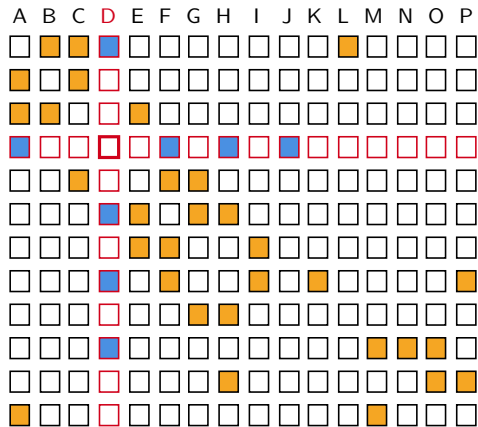
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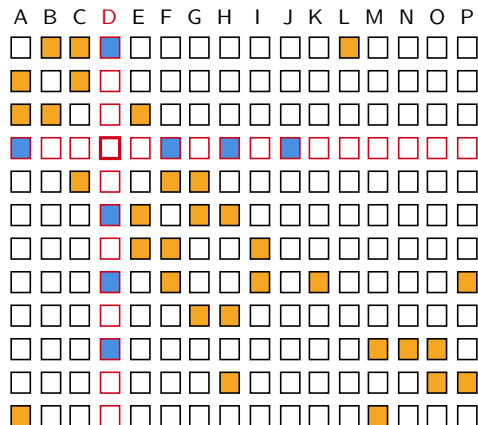
# Sharing Risks with Friends



# Sharing Risks with Friends



# Sharing Risks with Friends



Looks like a linear risk sharing mechanism,

$$\xi = B\mathbf{X} \text{ a.s., where } B_{i,j} = A_{i,j}/d_i,$$

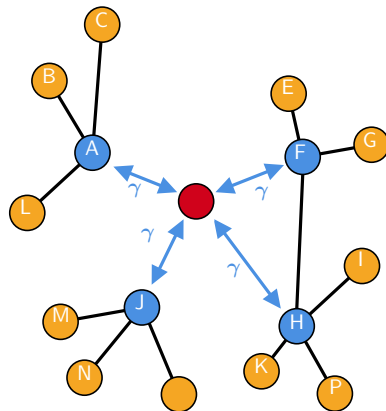
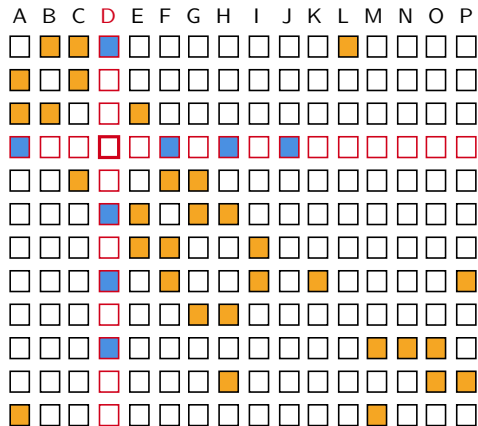
$A$  being the adjacency matrix of the network  
Here,  $B$  is a doubly stochastic matrix.

But it suffers some drawbacks...

- ▶ need an upper bound
- ▶ unfairness ( $B_{i,i} = 0, \forall i$ )

(no longer "linear" risk sharing mechanism)

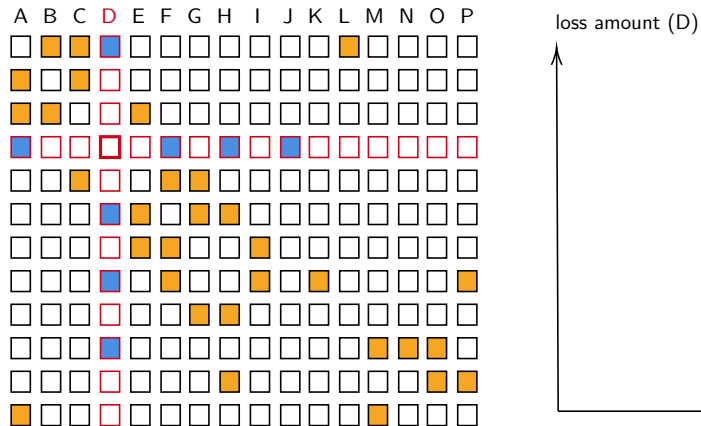
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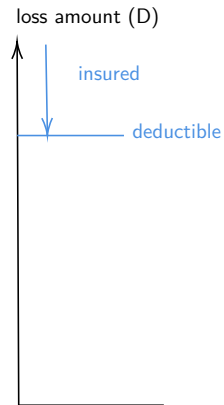
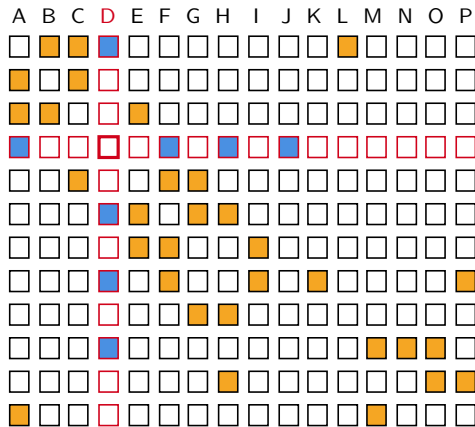




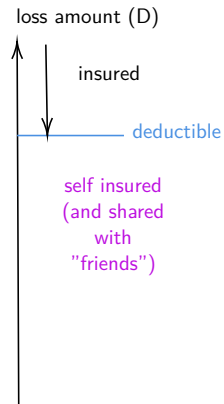
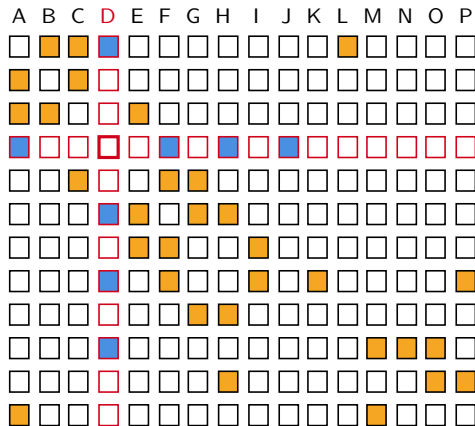
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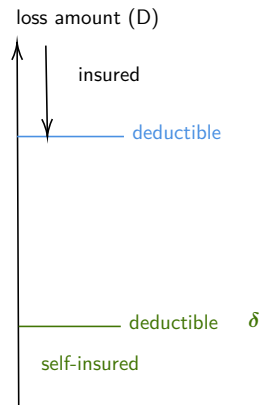
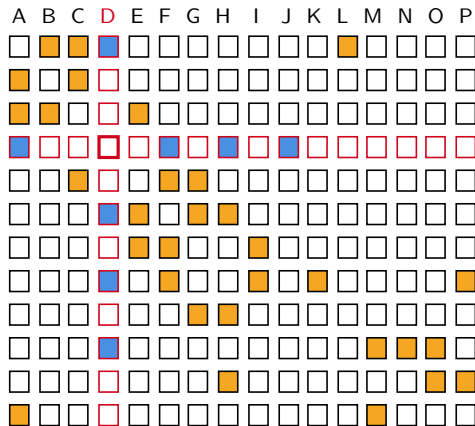
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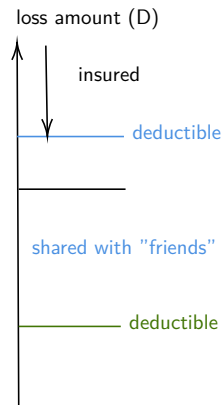
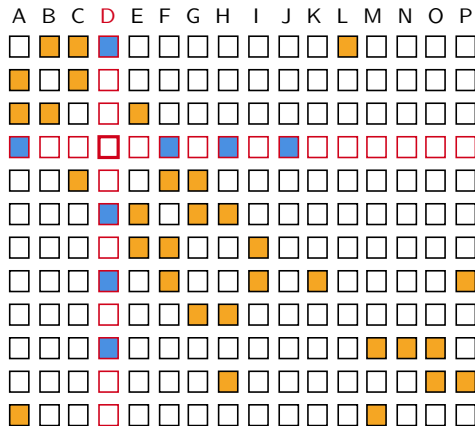
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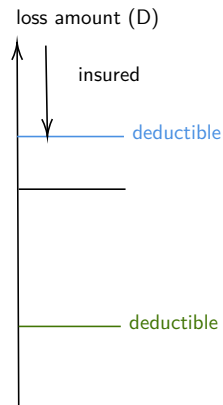
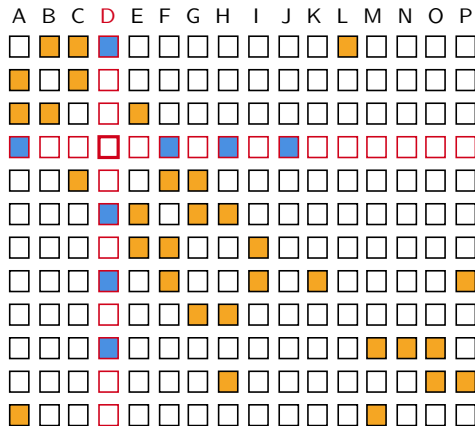
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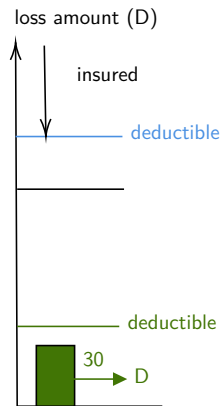
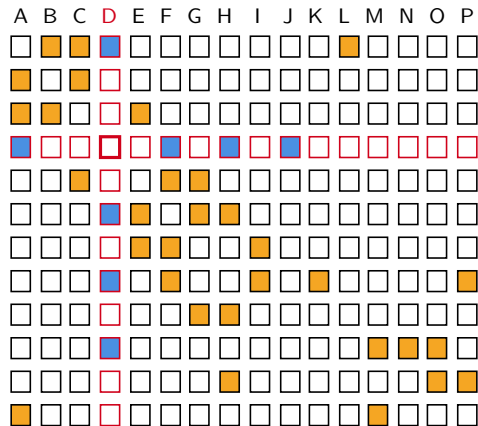
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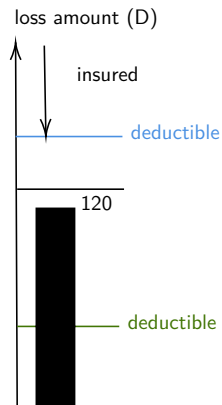
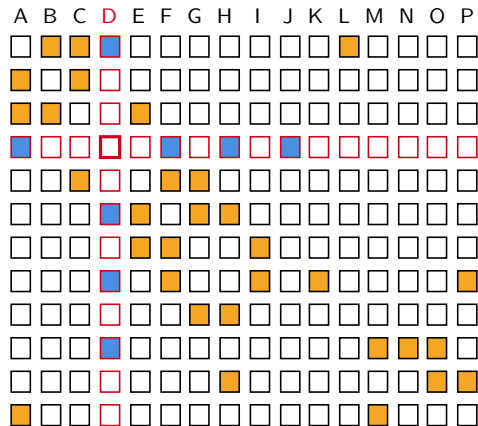


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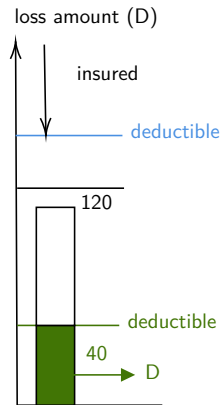
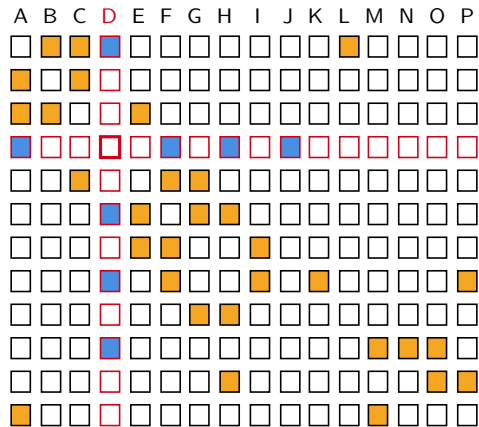




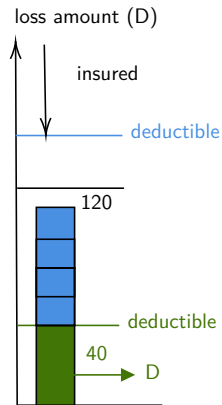
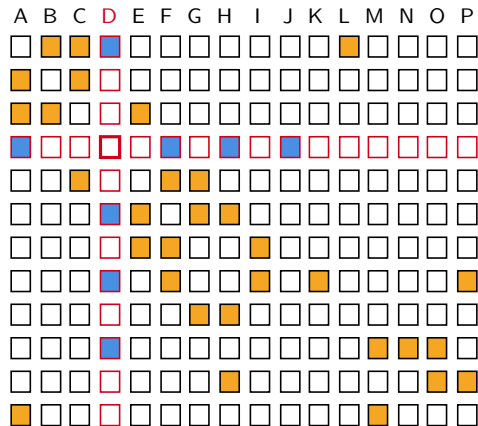
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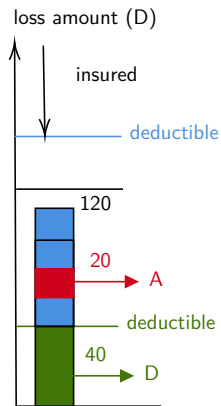
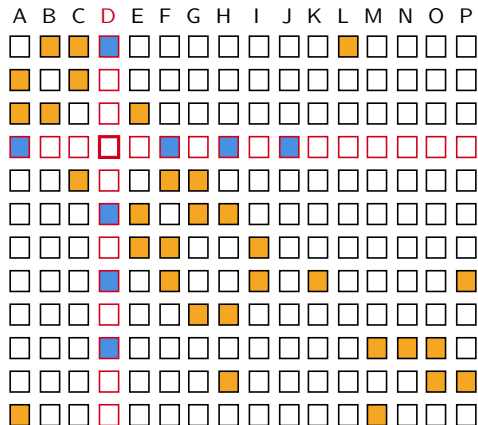
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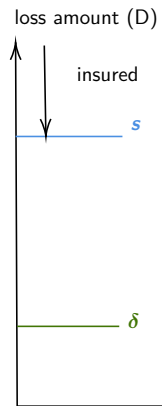
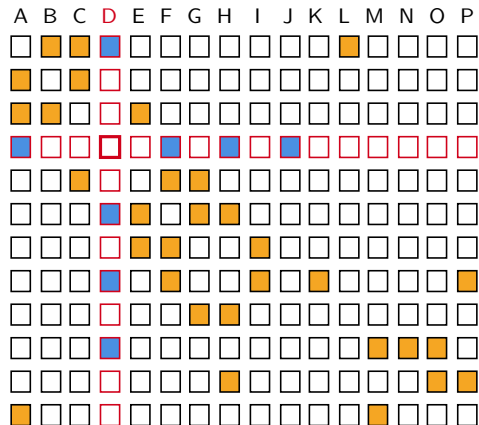
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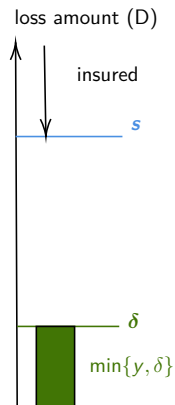
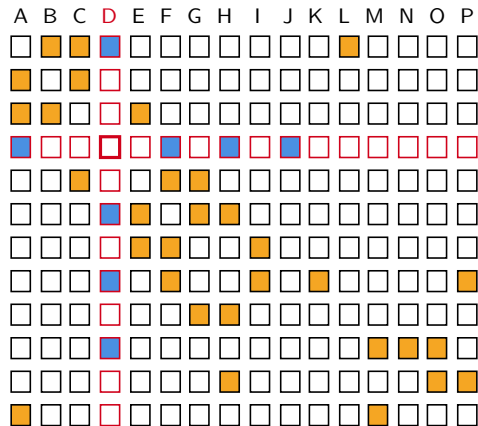
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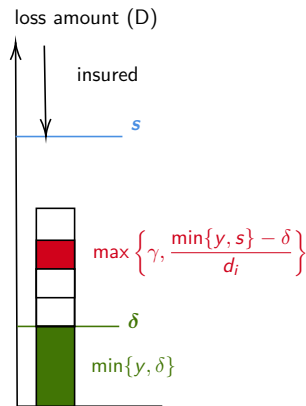
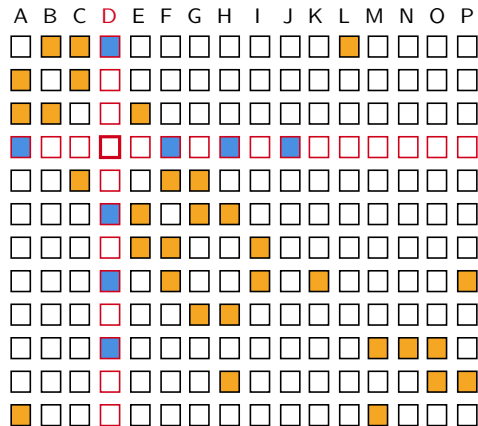
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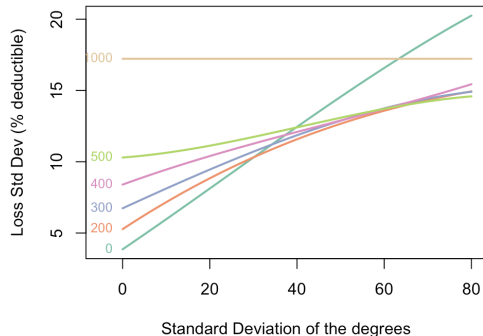
# Sharing Risks with Friends



# Sharing Risks with Friends

- ▶  $Y_i$  loss of insured  $i$ ,  $Z_i = \mathbf{1}(Y_i > 0)$
- ▶  $\mathcal{V}_i$  is the set of friends of insured  $i$ ,  $d_i = \text{Card}(\mathcal{V}_i)$
- ▶  $s$  deductible of insurance contracts
- ▶  $\gamma$  is the maximum amount shared between  $i$  and  $j$  (reciprocal contracts)

$$\begin{aligned} \xi_i = & Z_i \cdot \min\{s, Y_i\} \\ & + \sum_{j \in \mathcal{V}_i} Z_j \min \left\{ \gamma, \frac{\min\{s, Y_j\} - \delta}{d_j} \right\} \\ & - Z_i \cdot \min\{d_i \gamma, \min\{s, Y_i\} - \delta\} \end{aligned}$$





## Optimization\* of the Risk Sharing Mechanism

$$\begin{cases} \max \left\{ \sum_{(i,j) \in \mathcal{E}} \gamma_{(i,j)} \right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0, \gamma], \forall (i,j) \in \mathcal{E} \\ \sum_{j \in \mathcal{V}_i} \gamma_{(i,j)} \leq s, \forall i \in \mathcal{V} \end{cases}$$

Given losses  $\mathbf{X} = (X_1, \dots, X_n)$ , define contributions  $C_{i \rightarrow j}^* = \min \left\{ \frac{\gamma_{(i,j)}^*}{\sum_{i \in \mathcal{V}_j} \gamma_{(i,j)}^*} \cdot X_j, \gamma_{(i,j)}^* \right\}$ ,

and  $\xi_i^* = X_i + \sum_{j \in \mathcal{V}_i} [Z_j C_{i \rightarrow j}^* - Z_i C_{j \rightarrow i}^*]$  is a risk sharing, called optimal risk sharing.

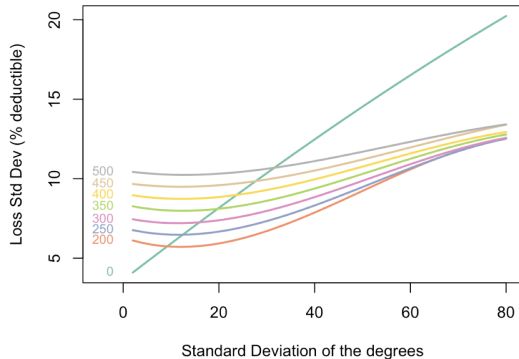
\* from a welfare (social planner) perspective

# Sharing Risks with Friends, and Friends of Friends

We can also consider friends of friends

$$\left\{ \begin{array}{l} \gamma_1^* = \operatorname{argmax} \left\{ \sum_{(i,j) \in \mathcal{E}^{(1)}} \gamma_{(i,j)} \right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0, \gamma_1], \forall (i,j) \in \mathcal{E}^{(1)} \\ \sum_{j \in \mathcal{V}_i^{(1)}} \gamma_{(i,j)} \leq s, \forall i \end{array} \right.$$
  

$$\left\{ \begin{array}{l} \gamma_2^* = \operatorname{argmax} \left\{ \sum_{(i,j) \in \mathcal{E}^{(2)}} \gamma_{(i,j)} \right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0, \gamma_2], \forall (i,j) \in \mathcal{E}^{(2)} \\ \sum_{j \in \mathcal{V}_i^{(1)}} \gamma_{1:(i,j)}^* + \sum_{j \in \mathcal{V}_i^{(2)}} \gamma_{(i,j)} \leq s, \forall i \end{array} \right.$$



# Take-away

- ▶ Back to the roots of insurance with risk sharing,
- ▶ Important to better model interactions
- ▶ Nice mathematical properties of linear risk sharing (connexions with convex ordering)
- ▶ More complex to derive a more realistic insurance product (with lower and upper limits)
- ▶ ... ongoing work



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# References

- Atkinson, A. B. (2015). Inequality. In *Inequality*. Harvard University Press.
- Barabási, A. and Albert, R. (1999). Emergence of scaling in random networks. *Science*, 286(5439):509–512.
- Carlier, G., Dana, R.-A., and Galichon, A. (2012). Pareto efficiency for the concave order and multivariate comonotonicity. *Journal of Economic Theory*, 147(1):207–229.
- Charpentier, A., Kouakou, L., Löwe, M., Ratz, P., and Vermet, F. (2021). Collaborative insurance sustainability and network structure. *arXiv*, 2107.02764.
- Dalton, H. (1920). The measurement of the inequality of incomes. *The Economic Journal*, 30(119):348–361.
- Denuit, M. and Dhaene, J. (2012). Convex order and comonotonic conditional mean risk sharing. *Insurance: Mathematics and Economics*, 51(2):265–270.
- Hardy, G., Littlewood, J., and G. Polya, . (1929). Some simple inequalities satisfied by convex functions. *British Mathematics Journal*, 38:145–152.
- Hardy, G., Littlewood, J., and Polya, G. (1934). *Inequalities*. Cambridge University Press.
- Marshall, A. W. and Olkin, I. (1979). *Inequalities: Theory of Majorization and its Applications*, volume 143. Academic Press.
- Watts, D. and Strogatz, S. (1998). Collective dynamics of ‘small-world’ networks. *Nature*, 398:440–442.