Actuarial Science: Econometrics vs. Machine Learning

Arthur Charpentier (Université de Rennes 1)

Generalli, Paris, June 2016

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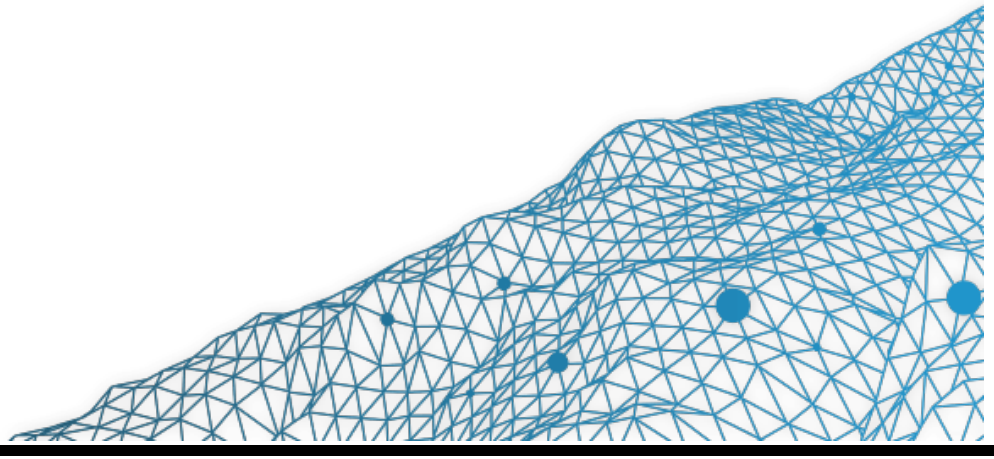

Actuarial Science: Econometrics vs. Machine Learning

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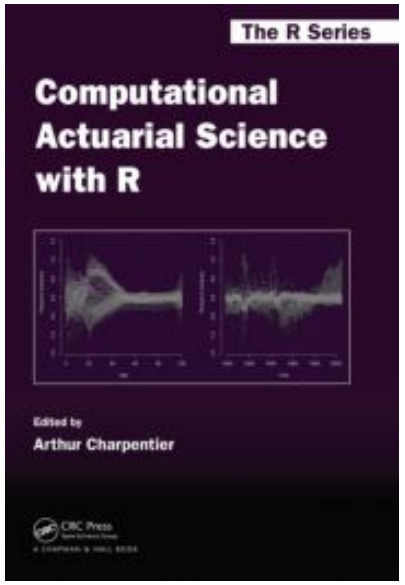


Agenda

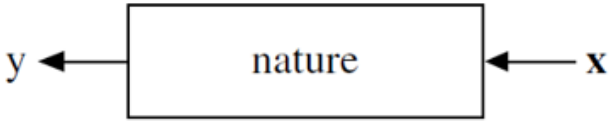
"*the numbers have no way of speaking for themselves. We speak for them.* [· · ·] *Before we demand more of our data, we need to demand more of ourselves* " from Silver (2012).

- **(big) data**
- **econometrics & probabilistic modeling**
- **algorithmics & statistical learning**
- **model competition in insurance**

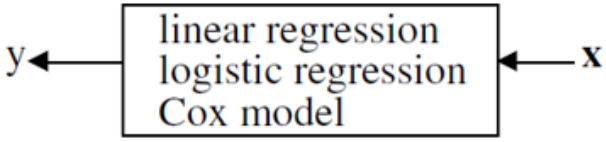
see Berk (2008), Hastie, Tibshirani & Friedman (2009), but also Breiman (2001)

Statistical Science
2001, Vol. 16, No. 3, 199–231

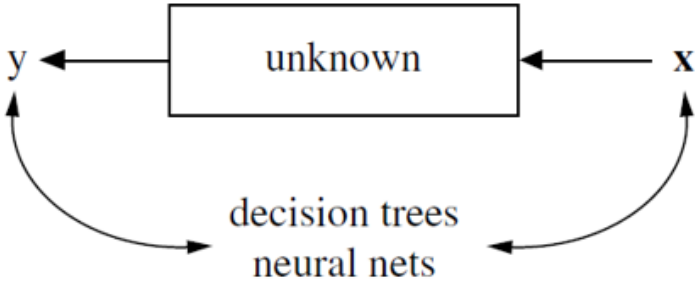
Statistical Modeling: The Two Cultures



The Data Modeling Culture



The Algorithmic Modeling Culture



Data and Models

From $\{(y_i, \boldsymbol{x}_i)\}$, there are different stories behind, see Freedman (2005)

- the **causal story** : $x_{j,i}$ is usually considered as independent of the other covariates $x_{k,i}$. For all possible $\boldsymbol{x}$, that value is mapped to $m(\boldsymbol{x})$ and a noise is attached, ε. The goal is to recover $m(\cdot)$, and the residuals are just the difference between the response value and $m(\boldsymbol{x})$.
- the **conditional distribution story** : for a linear model, we usually say that Y given $\boldsymbol{X} = \boldsymbol{x}$ is a $\mathcal{N}(m(\boldsymbol{x}), \sigma^2)$ distribution. $m(\boldsymbol{x})$ is then the conditional mean. Here $m(\cdot)$ is assumed to really exist, but no causal assumption is made, only a conditional one.
- the **explanatory data story** : there is no model, just data. We simply want to summarize information contained in $\boldsymbol{x}$'s to get an accurate summary, close to the response (i.e. $\min\{\ell(\boldsymbol{y}, m(\boldsymbol{x}))\}$) for some loss function ℓ.

See also Varian (2014)

Data, Models & Causal Inference

We cannot differentiate data and model that easily..

After an operation, should I stay at hospital, or go back home ?

as in Angrist & Pischke (2008),

$$(\text{health} \mid \text{hospital}) - (\text{health} \mid \text{stayed home}) \quad \text{[observed]}$$

should be written

$$(\text{health} \mid \text{hospital}) - (\text{health} \mid \textit{had} \text{ stayed home}) \quad \text{[treatment effect]}$$

$$+ (\text{health} \mid \textit{had} \text{ stayed home}) - (\text{health} \mid \text{stayed home}) \quad \text{[selection bias]}$$

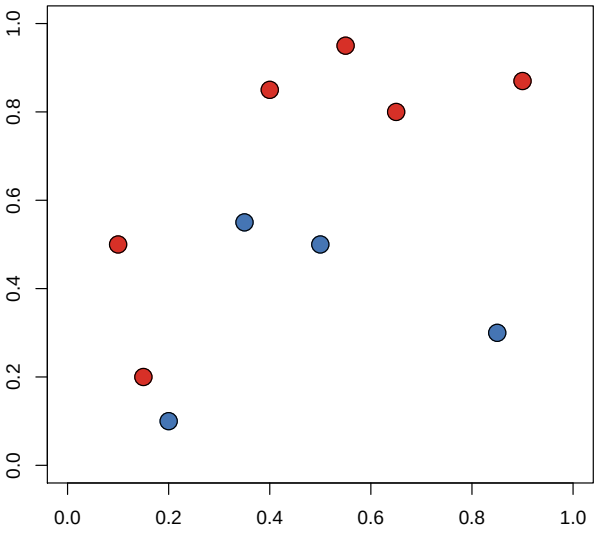
Need randomization to solve selection bias.

Econometric Modeling

Data $\{(y_i, \boldsymbol{x}_i)\}$, for $i = 1, \cdots, n$, with $\boldsymbol{x}_i \in \mathcal{X} \subset \mathbb{R}^p$ and $y_i \in \mathcal{Y}$.

A model is a $m : \mathcal{X} \mapsto \mathcal{Y}$ **mapping**

- regression, $\mathcal{Y} = \mathbb{R}$ (but also $\mathcal{Y} = \mathbb{N}$)
- classification, $\mathcal{Y} = \{0, 1\}$, $\{-1, +1\}$, $\{\bullet, \bullet\}$ (binary, or more)



Classification models are based on two steps,

- **score** function, $s(\boldsymbol{x}) = \mathbb{P}(Y = 1 | \boldsymbol{X} = \boldsymbol{x}) \in [0, 1]$

- **classifier** $s(\boldsymbol{x}) \to \widehat{y} \in \{0, 1\}$.

High Dimensional Data (not to say 'Big Data')

See Bühlmann & van de Geer (2011) or Koch (2013), $\boldsymbol{X}$ is a $n \times p$ matrix

Portnoy (1988) proved that maximum likelihood estimators are asymptotically normal when $p^2/n \to 0$ as $n, p \to \infty$. Hence, **massive data**, when $p > \sqrt{n}$.

More intersting is the **sparcity** concept, based not on p, but on the effective size. Hence one can have $p > n$ and convergent estimators.

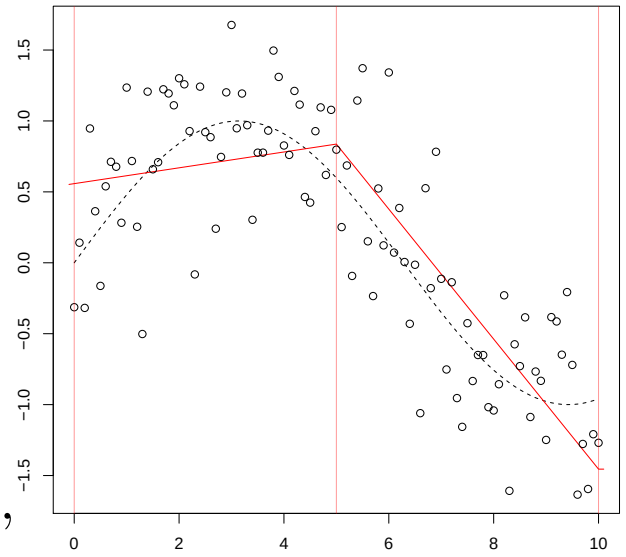
High dimension might be scary because of **curse of dimensionality**, see Bellman (1957). The volume of the unit sphere in $\mathbb{R}^p$ tends to 0 as $p \to \infty$, i.e.space is sparse.

Computational & Nonparametric Econometrics

Linear Econometrics: estimate $g : \boldsymbol{x} \mapsto \mathbb{E}[Y|\boldsymbol{X} = \boldsymbol{x}]$ by a linear function.

Nonlinear Econometrics: consider the approximation for some **functional basis**

$$g(\boldsymbol{x}) = \sum_{j=0}^{\infty} \omega_j g_j(\boldsymbol{x}) \text{ and } \widehat{g}(\boldsymbol{x}) = \sum_{j=0}^{h} \omega_j g_j(\boldsymbol{x})$$



or consider a **local model**, on the neighborhood of $\boldsymbol{x}$,

$$\widehat{g}(x) = \frac{1}{n_{\boldsymbol{x}}} \sum_{i \in \mathcal{I}_{\boldsymbol{x}}} y_i, \text{ with } \mathcal{I}_{\boldsymbol{x}} = \{\boldsymbol{x} \in \mathbb{R}^p : \|\boldsymbol{x}_i - \boldsymbol{x}\| \leq h\},$$

see Nadaraya (1964) and Watson (1964).

Here h is some **tunning parameter**: not estimated, but chosen (optimaly).

Econometrics & Probabilistic Model

> The primary goal in a regression analysis is to understand, as far as possible with the available data, how the conditional distribution of the response y varies across subpopulations determined by the possible values of the predictor or predictors. Since this is the central idea, it will be helpful to have a conve-

from Cook & Weisberg (1999), see also Haavelmo (1965).

$$(Y|\boldsymbol{X}=\boldsymbol{x}) \sim \mathcal{N}(\mu(\boldsymbol{x}), \sigma^2) \text{ with } \mu(\boldsymbol{x}) = \beta_0 + \boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}, \text{ and } \boldsymbol{\beta} \in \mathbb{R}^p.$$

Linear Model: $\mathbb{E}[Y|\boldsymbol{X}=\boldsymbol{x}] = \beta_0 + \boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}$

Homoscedasticity: $\text{Var}[Y|\boldsymbol{X}=\boldsymbol{x}] = \sigma^2$.

Conditional Distribution and Likelihood

$$(Y|\boldsymbol{X}=\boldsymbol{x}) \sim \mathcal{N}(\mu(\boldsymbol{x}), \sigma^2) \text{ with } \mu(\boldsymbol{x}) = \beta_0 + \boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}, \text{ et } \boldsymbol{\beta} \in \mathbb{R}^p$$

The log-likelihood is

$$\log \mathcal{L}(\beta_0, \boldsymbol{\beta}, \sigma^2|\boldsymbol{y}, \boldsymbol{x}) = -\frac{n}{2}\log[2\pi\sigma^2] - \frac{1}{2\sigma^2}\underbrace{\sum_{i=1}^{n}(y_i - \beta_0 - \boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta})^2}.$$

Set

$$(\widehat{\beta}_0, \widehat{\boldsymbol{\beta}}, \widehat{\sigma}^2) = \text{argmax}\left\{\log \mathcal{L}(\beta_0, \boldsymbol{\beta}, \sigma^2|\boldsymbol{y}, \boldsymbol{x})\right\}.$$

First order condition $\boldsymbol{X}^\mathsf{T}[\boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}] = \boldsymbol{0}$. If matrix $\boldsymbol{X}$ is a full rank matrix

$$\widehat{\boldsymbol{\beta}} = (\boldsymbol{X}^\mathsf{T}\boldsymbol{X})^{-1}\boldsymbol{X}^\mathsf{T}\boldsymbol{y} = \boldsymbol{\beta} + (\boldsymbol{X}^\mathsf{T}\boldsymbol{X})^{-1}\boldsymbol{X}^\mathsf{T}\boldsymbol{\varepsilon}.$$

Asymptotic properties of $\widehat{\boldsymbol{\beta}}$,

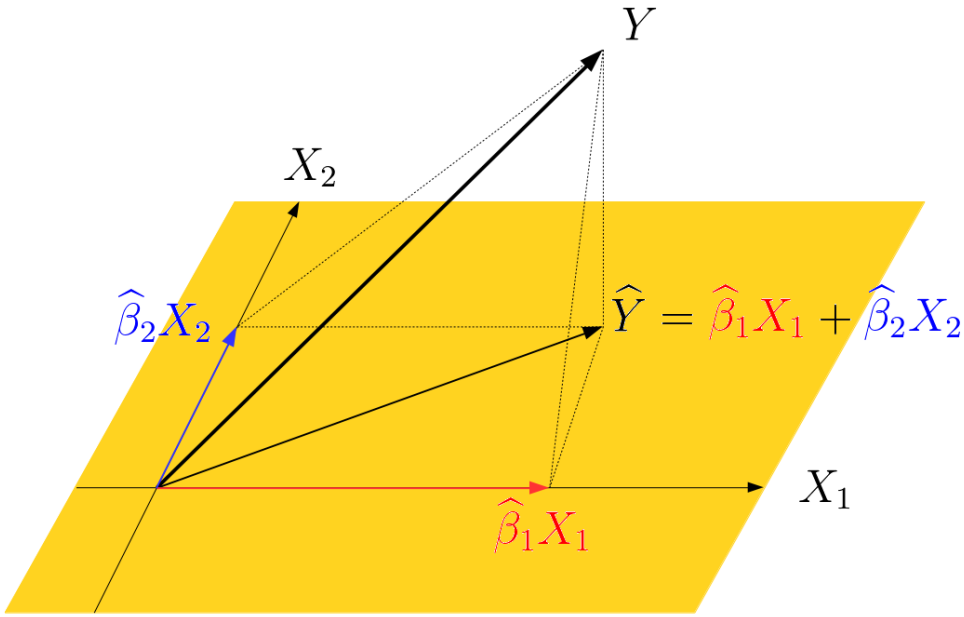
$$\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{\mathcal{L}} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}) \text{ as } n \to \infty$$

Geometric Perspective

Define the orthogonal projection on $\mathcal{X}$,

$$\Pi_{\boldsymbol{X}} = \boldsymbol{X}[\boldsymbol{X}^\top\boldsymbol{X}]^{-1}\boldsymbol{X}^\top$$

$$\widehat{\boldsymbol{y}} = \underbrace{\boldsymbol{X}[\boldsymbol{X}^\top\boldsymbol{X}]^{-1}\boldsymbol{X}^\top}_{\Pi_{\boldsymbol{X}}}\boldsymbol{y} = \Pi_{\boldsymbol{X}}\boldsymbol{y}.$$



Pythagoras' theorem can be writen

$$\|\boldsymbol{y}\|^2 = \|\Pi_{\boldsymbol{X}}\boldsymbol{y}\|^2 + \|\Pi_{\boldsymbol{X}^\perp}\boldsymbol{y}\|^2 = \|\Pi_{\boldsymbol{X}}\boldsymbol{y}\|^2 + \|\boldsymbol{y} - \Pi_{\boldsymbol{X}}\boldsymbol{y}\|^2$$

which can be expressed as

$$\underbrace{\sum_{i=1}^n y_i^2}_{n\times\text{total variance}} = \underbrace{\sum_{i=1}^n \widehat{y}_i^2}_{n\times\text{explained variance}} + \underbrace{\sum_{i=1}^n (y_i - \widehat{y}_i)^2}_{n\times\text{residual variance}}$$

Geometric Perspective

Define the angle θ between $\boldsymbol{y}$ and $\Pi_{\mathcal{X}}\boldsymbol{y}$,

$$R^2 = \frac{\|\Pi_{\mathcal{X}}\boldsymbol{y}\|^2}{\|\boldsymbol{y}\|^2} = 1 - \frac{\|\Pi_{\mathcal{X}^\perp}\boldsymbol{y}\|^2}{\|\boldsymbol{y}\|^2} = \cos^2(\theta)$$

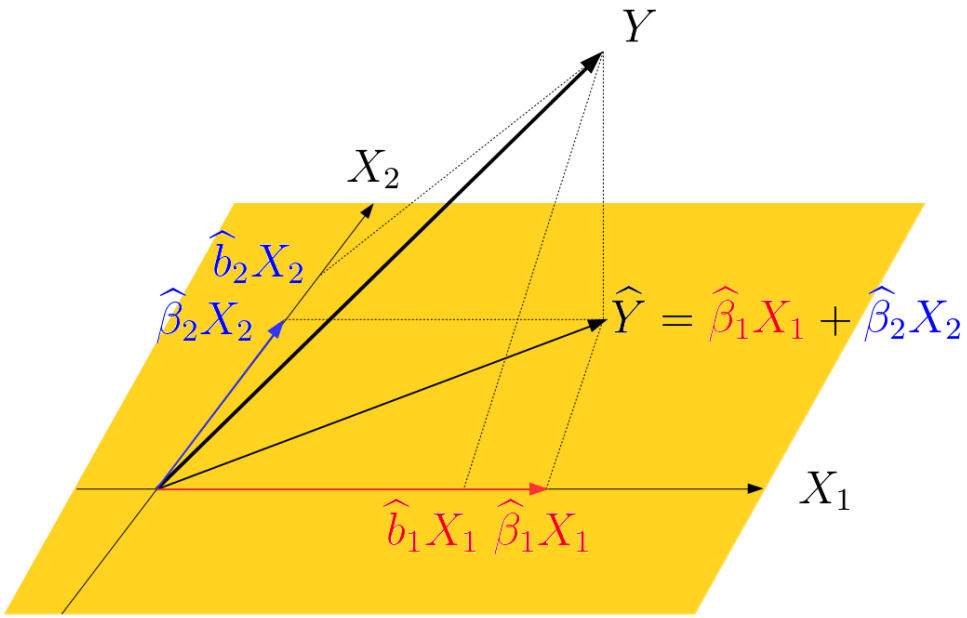
see Davidson & MacKinnon (2003)

$$\boldsymbol{y} = \beta_0 + \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \varepsilon$$

If $\boldsymbol{y}_2^\star = \Pi_{\mathcal{X}_1^\perp}\boldsymbol{y}$ and $\boldsymbol{X}_2^\star = \Pi_{\mathcal{X}_1^\perp}\boldsymbol{X}_2$, then

$$\widehat{\boldsymbol{\beta}}_2 = [\boldsymbol{X}_2^{\star\mathsf{T}}\boldsymbol{X}_2^\star]^{-1}\boldsymbol{X}_2^{\star\mathsf{T}}\boldsymbol{y}_2^\star$$

$\boldsymbol{X}_2^\star = \boldsymbol{X}_2$ if $\boldsymbol{X}_1 \perp \boldsymbol{X}_2$,
Frisch-Waugh theorem.



From Linear to Non-Linear

$$\widehat{\boldsymbol{y}} = \boldsymbol{X}\widehat{\boldsymbol{\beta}} = \underbrace{\boldsymbol{X}[\boldsymbol{X}^\mathsf{T}\boldsymbol{X}]^{-1}\boldsymbol{X}^\mathsf{T}}_{\boldsymbol{H}}\boldsymbol{y} \text{ i.e. } \widehat{y}_i = \boldsymbol{h}_{\boldsymbol{x}_i}^\mathsf{T}\boldsymbol{y},$$

with - for the linear regression - $\boldsymbol{h}_{\boldsymbol{x}} = \boldsymbol{X}[\boldsymbol{X}^\mathsf{T}\boldsymbol{X}]^{-1}\boldsymbol{x}$.

One can consider some smoothed regression, see Nadaraya (1964) and Watson (1964), with some smoothing matrix $\boldsymbol{S}$

$$\widehat{m}_h(x) = \boldsymbol{s}_x^\mathsf{T}\boldsymbol{y} = \sum_{i=1}^n s_{x,i}y_i \text{ withs } s_{x,i} = \frac{K_h(x-x_i)}{K_h(x-x_1)+\cdots+K_h(x-x_n)}$$

for some kernel $K(\cdot)$ and some bandwidth $h > 0$.

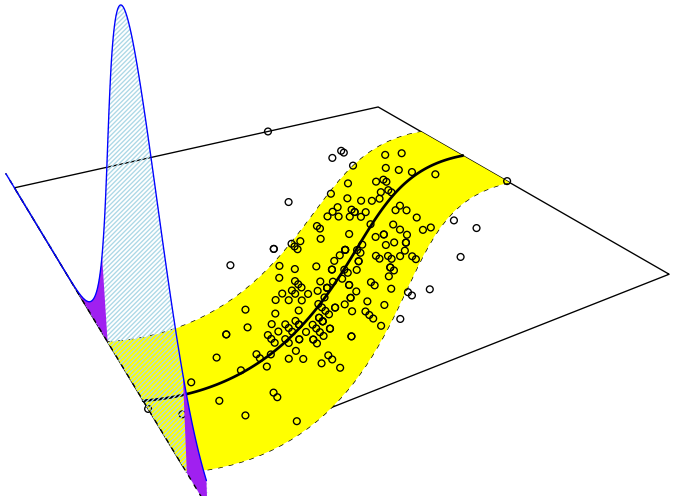
From Linear to Non-Linear

$$T = \frac{\|\boldsymbol{S}\boldsymbol{y} - \boldsymbol{H}\boldsymbol{y}\|}{\text{trace}([\boldsymbol{S} - \boldsymbol{H}]^{\mathsf{T}}[\boldsymbol{S} - \boldsymbol{H}])}$$

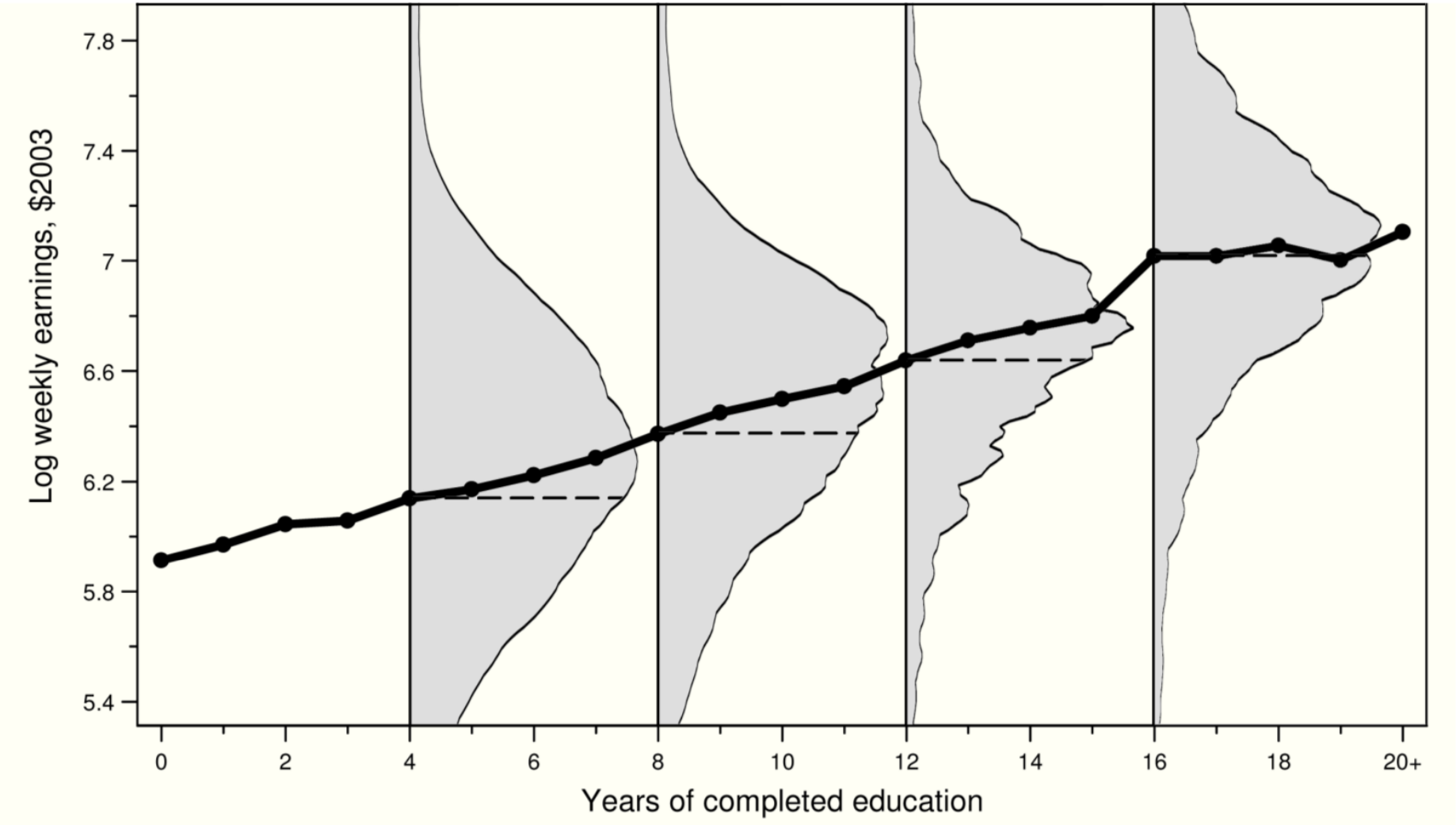
can be used to test for linearity, Simonoff (1996). $\text{trace}(\boldsymbol{S})$ is the equivalent number of parameters, and $n - \text{trace}(\boldsymbol{S})$ the degrees of freedom, Ruppert *et al.* (2003).

Nonlinear Model, but Homoscedastic - Gaussian

- $(Y|\boldsymbol{X} = \boldsymbol{x}) \sim \mathcal{N}(\mu(\boldsymbol{x}), \sigma^2)$
- $\mathbb{E}[Y|\boldsymbol{X} = \boldsymbol{x}] = \mu(\boldsymbol{x})$



Conditional Expectation



from Angrist & Pischke (2008), $\boldsymbol{x} \mapsto \mathbb{E}[Y|\boldsymbol{X}=\boldsymbol{x}]$.

Exponential Distributions and Linear Models

$$f(y_i|\theta_i,\phi)=\exp\left(\frac{y_i\theta_i-b(\theta_i)}{a(\phi)}+c(y_i,\phi)\right) \text{ with } \theta_i=h(\boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta})$$

Log likelihood is expressed as

$$\log\mathcal{L}(\boldsymbol{\theta},\phi|\boldsymbol{y})=\sum_{i=1}^n\log f(y_i|\theta_i,\phi)=\frac{\sum_{i=1}^n y_i\theta_i-\sum_{i=1}^n b(\theta_i)}{a(\phi)}+\sum_{i=1}^n c(y_i,\phi)$$

and first order conditions

$$\frac{\partial\log\mathcal{L}(\boldsymbol{\theta},\phi|\boldsymbol{y})}{\partial\boldsymbol{\beta}}=\boldsymbol{X}^\mathsf{T}\boldsymbol{W}^{-1}[\boldsymbol{y}-\boldsymbol{\mu}]=\boldsymbol{0}$$

as in Müller (2001), where $\boldsymbol{W}$ is a weight matrix, function of $\boldsymbol{\beta}$.

We usually specify the **link** function $g(\cdot)$ defined as

$$\widehat{y}=m(\boldsymbol{x})=\mathbb{E}[Y|\boldsymbol{X}=\boldsymbol{x}]=g^{-1}(\boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}).$$

Exponential Distributions and Linear Models

Note that $\boldsymbol{W} = \text{diag}(\nabla g(\widehat{\boldsymbol{y}}) \cdot \text{Var}[\boldsymbol{y}])$, and set

$$\boldsymbol{z} = g(\widehat{\boldsymbol{y}}) + (\boldsymbol{y} - \widehat{\boldsymbol{y}}) \cdot \nabla g(\widehat{\boldsymbol{y}})$$

the the maximum likelihood estimator is obtained iteratively

$$\widehat{\boldsymbol{\beta}}_{k+1} = [\boldsymbol{X}^\mathsf{T} \boldsymbol{W}_k^{-1} \boldsymbol{X}]^{-1} \boldsymbol{X}^\mathsf{T} \boldsymbol{W}_k^{-1} \boldsymbol{z}_k$$

Set $\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}_\infty$, so that

$$\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{\mathcal{L}} \mathcal{N}(\boldsymbol{0}, I(\boldsymbol{\beta})^{-1})$$

with $I(\boldsymbol{\beta}) = \phi \cdot [\boldsymbol{X}^\mathsf{T} \boldsymbol{W}_\infty^{-1} \boldsymbol{X}]$.

Note that $[\boldsymbol{X}^\mathsf{T} \boldsymbol{W}_k^{-1} \boldsymbol{X}]$ is a $p \times p$ matrix.

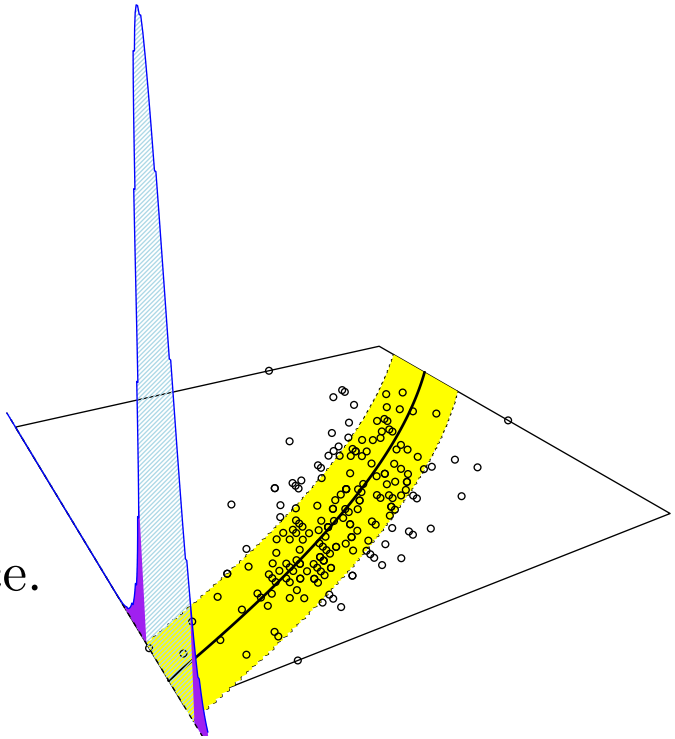
Exponential Distributions and Linear Models

Generalized Linear Model:

- $(Y|\boldsymbol{X}=\boldsymbol{x}) \sim \mathcal{L}(\theta_{\boldsymbol{x}}, \varphi)$
- $\mathbb{E}[Y|\boldsymbol{X}=\boldsymbol{x}] = h^{-1}(\theta_{\boldsymbol{x}}) = g^{-1}(\boldsymbol{x}^{\mathsf{T}}\boldsymbol{\beta})$

e.g. $(Y|\boldsymbol{X}=\boldsymbol{x}) \sim \mathcal{P}(\exp[\boldsymbol{x}^{\mathsf{T}}\boldsymbol{\beta}])$.

Use of maximum likelihood techniques for inference.



Actually, more a moment condition than a distribution assumption.

Goodness of Fit & Model Choice

From the variance decomposition

$$\underbrace{\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2}_{\text{total variance}}=\underbrace{\frac{1}{n}\sum_{i=1}^{n}(y_i-\widehat{y}_i)^2}_{\text{residual variance}}+\underbrace{\frac{1}{n}\sum_{i=1}^{n}(\widehat{y}_i-\bar{y})^2}_{\text{explained variance}}$$

and define

$$R^2=\frac{\sum_{i=1}^{n}(y_i-\bar{y})^2-\sum_{i=1}^{n}(y_i-\widehat{y}_i)^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2}$$

More generally

$$\text{Deviance}(\boldsymbol{\beta})=-2\log[\mathcal{L}]=2\sum_{i=1}^{n}(y_i-\widehat{y}_i)^2=\text{Deviance}(\widehat{\boldsymbol{y}})$$

The null deviance is obtained using $\widehat{y}_i=\overline{y}$, so that

$$R^2=\frac{\text{Deviance}(\overline{y})-\text{Deviance}(\widehat{\boldsymbol{y}})}{\text{Deviance}(\overline{y})}=1-\frac{\text{Deviance}(\widehat{\boldsymbol{y}})}{\text{Deviance}(\overline{y})}=1-\frac{\text{D}}{\text{D}_0}$$

Goodness of Fit & Model Choice

One usually prefers a penalized version

$$\bar{R}^2 = 1 - (1 - R^2)\frac{n-1}{n-p} = R^2 - \underbrace{(1 - R^2)\frac{p-1}{n-p}}_{\text{penalty}}$$

See also **Akaike** criteria $AIC = \text{Deviance} + 2 \cdot p$

or **Schwarz**, $BIC = \text{Deviance} + \log(n) \cdot p$

In high dimension, consider a corrected version

$$AICc = \text{Deviance} + 2 \cdot p \cdot \frac{n}{n-p-1}$$

Stepwise Procedures

Forward algorithm

1. set $j_1^\star = \underset{j\in\{\emptyset,1,\cdots,n\}}{\text{argmin}} \{AIC(\{j\})\}$
2. set $j_2^\star = \underset{j\in\{\emptyset,1,\cdots,n\}\backslash\{j_1^\star\}}{\text{argmin}} \{AIC(\{j_1^\star, j\})\}$
3. ... until $j^\star = \emptyset$

Backward algorithm

1. set $j_1^\star = \underset{j\in\{\emptyset,1,\cdots,n\}}{\text{argmin}} \{AIC(\{1,\cdots,n\}\backslash\{j\})\}$
2. set $j_2^\star = \underset{j\in\{\emptyset,1,\cdots,n\}\backslash\{j_1^\star\}}{\text{argmin}} \{AIC(\{1,\cdots,n\}\backslash\{j_1^\star, j\})\}$
3. ... until $j^\star = \emptyset$

Econometrics & Statistical Testing

Standard test for $H_0 : \beta_k = 0$ against $H_1 : \beta_k \neq 0$ is **Student-t** est $t_k = \widehat{\beta}_k / \text{se}_{\widehat{\beta}_k}$,

Use the **p-value** $\mathbb{P}[|T| > |t_k|]$ with $T \sim t_\nu$ (and $\nu = \text{trace}(\boldsymbol{H})$).

In high dimension, consider the FDR (False Discovery Ratio).

With $\alpha = 5\%$, 5% variables are wrongly significant.

If $p = 100$ with only 5 significant variables, one should expect also 5 false positive, i.e. 50% FDR, see Benjamini & Hochberg (1995).

Under & Over-Identification

Under-identification is obtained when the true model is $y = \beta_0 + \boldsymbol{x}_1^\mathsf{T}\boldsymbol{\beta}_1 + \boldsymbol{x}_2^\mathsf{T}\boldsymbol{\beta}_2 + \varepsilon$, but we estimate $y = \beta_0 + \boldsymbol{x}_1^\mathsf{T}\boldsymbol{b}_1 + \eta$.

Maximum likelihood estimator for $\boldsymbol{b}_1$ is

$$\begin{aligned}
\widehat{\boldsymbol{b}}_1 &= (\boldsymbol{X}_1^\mathsf{T}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1^\mathsf{T}\boldsymbol{y} \\
&= (\boldsymbol{X}_1^\mathsf{T}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1^\mathsf{T}[\boldsymbol{X}_{1,i}\boldsymbol{\beta}_1 + \boldsymbol{X}_{2,i}\boldsymbol{\beta}_2 + \varepsilon] \\
&= \boldsymbol{\beta}_1 + \underbrace{(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1^\mathsf{T}\boldsymbol{X}_2\boldsymbol{\beta}_2}_{\boldsymbol{\beta}_{12}} + \underbrace{(\boldsymbol{X}_1^\mathsf{T}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1^\mathsf{T}\varepsilon}_{\nu_i}
\end{aligned}$$

so that $\mathbb{E}[\widehat{\boldsymbol{b}}_1] = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_{12}$, and the bias is null when $\boldsymbol{X}_1^\mathsf{T}\boldsymbol{X}_2 = \boldsymbol{0}$ i.e. $\boldsymbol{X}_1 \perp \boldsymbol{X}_2$, see Frisch-Waugh).

Over-identification is obtained when the true model is $y = \beta_0 + \boldsymbol{x}_1^\mathsf{T}\boldsymbol{\beta}_1\varepsilon$, but we fit $y = \beta_0 + \boldsymbol{x}_1^\mathsf{T}\boldsymbol{b}_1 + \boldsymbol{x}_2^\mathsf{T}\boldsymbol{b}_2 + \eta$.

Inference is unbiased since $\mathbb{E}(\boldsymbol{b}_1) = \boldsymbol{\beta}_1$ but the estimator is not efficient.

Statistical Learning & Loss Function

Here, no probabilistic model, but a **loss function**, ℓ. For some set of functions $\mathcal{M}$, $\mathcal{X} \to \mathcal{Y}$, define

$$m^\star = \underset{m \in \mathcal{M}}{\text{argmin}} \left\{ \sum_{i=1}^{n} \ell(y_i, m(\boldsymbol{x}_i)) \right\}$$

Quadratic loss functions are interesting since

$$\overline{y} = \underset{m \in \mathbb{R}}{\text{argmin}} \left\{ \sum_{i=1}^{n} \frac{1}{n} [y_i - m]^2 \right\}$$

which can be writen, with some underlying probabilistic model

$$\mathbb{E}(Y) = \underset{m \in \mathbb{R}}{\text{argmin}} \left\{ \|Y - m\|_{\ell_2}^2 \right\} = \underset{m \in \mathbb{R}}{\text{argmin}} \left\{ \mathbb{E}\left([Y - m]^2 \right) \right\}$$

For $\tau \in (0, 1)$, we obtain the **quantile regression** (see Koenker (2005))

$$m^\star = \underset{m \in \mathcal{M}_0}{\text{argmin}} \left\{ \sum_{i=1}^{n} \ell_\tau(y_i, m(\boldsymbol{x}_i)) \right\} \text{ avec } \ell_\tau(x, y) = |(x - y)(\tau - \mathbf{1}_{x \leq y})|$$

Boosting & Weak Learning

$$m^\star = \underset{m\in\mathcal{M}}{\text{argmin}} \left\{ \sum_{i=1}^{n} \ell(y_i, m(\boldsymbol{x}_i)) \right\}$$

is hard to solve for some very large and general space $\mathcal{M}$ of $\mathcal{X} \to \mathcal{Y}$ functions.

Consider some iterative procedure, where we learn from the errors,

$$m^{(k)}(\cdot) = \underbrace{m_1(\cdot)}_{\sim \boldsymbol{y}} + \underbrace{m_2(\cdot)}_{\sim \boldsymbol{\varepsilon}_1} + \underbrace{m_3(\cdot)}_{\sim \boldsymbol{\varepsilon}_2} + \cdots + \underbrace{m_k(\cdot)}_{\sim \boldsymbol{\varepsilon}_{k-1}} = m^{(k-1)}(\cdot) + m_k(\cdot).$$

Formely $\boldsymbol{\varepsilon}$ can be seen as $\nabla \ell$, the gradient of the loss.

Boosting & Weak Learning

It is possible to see this algorithm as a gradient descent. Not

$$\underbrace{f(\boldsymbol{x}_k)}_{\langle f,\boldsymbol{x}_k\rangle} \sim \underbrace{f(\boldsymbol{x}_{k-1})}_{\langle f,\boldsymbol{x}_{k-1}\rangle} + \underbrace{(\boldsymbol{x}_k-\boldsymbol{x}_{k-1})}_{\alpha_k}\underbrace{\nabla f(\boldsymbol{x}_{k-1})}_{\langle \nabla f,\boldsymbol{x}_{k-1}\rangle}$$

but some kind of dual version

$$\underbrace{f_k(\boldsymbol{x})}_{\langle f_k,\boldsymbol{x}\rangle} \sim \underbrace{f_{k-1}(\boldsymbol{x})}_{\langle f_{k-1},\boldsymbol{x}\rangle} + \underbrace{(f_k-f_{k-1})}_{a_k}\underbrace{\star}_{\langle f_{k-1},\nabla \boldsymbol{x}\rangle}$$

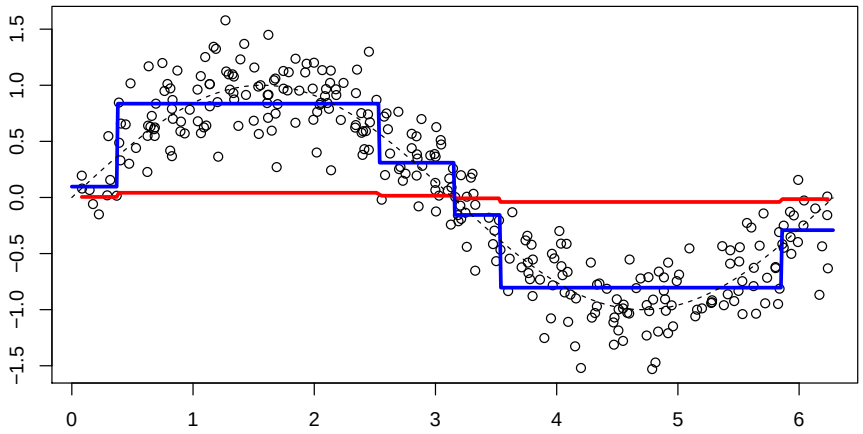
where $\star$ is a gradient is some functional space.

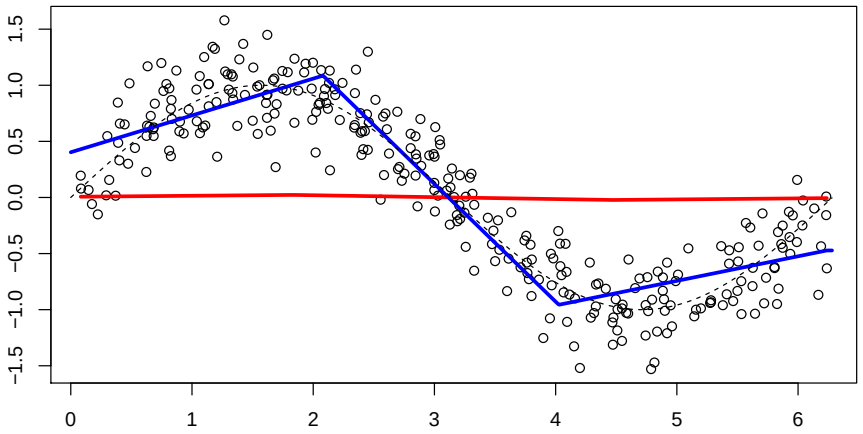
$$m^{(k)}(\boldsymbol{x}) = m^{(k-1)}(\boldsymbol{x}) + \underset{f\in\mathcal{F}}{\text{argmin}}\left\{\sum_{i=1}^{n}\ell(y_i, m^{(k-1)}(\boldsymbol{x})+f(\boldsymbol{x}))\right\}$$

for some simple space $\mathcal{F}$ so that we define some **weak learner**, e.g. step functions (so called stumps)

Boosting & Weak Learning

Standard set $\mathcal{F}$ are stumps functions but one can also consider splines (with non-fixed knots).





One might add a **shrinkage** parameter to learn even more weakly, i.e. set $\varepsilon_1 = y - \alpha \cdot m_1(\boldsymbol{x})$ with $\alpha \in (0, 1)$, etc.

Big Data & Linear Model

Consider some linear model $y_i = \boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta} + \varepsilon_i$ for all $i = 1, \cdots, n$.

Assume that ε_i are i.i.d. with $\mathbb{E}(\varepsilon) = 0$ (and finite variance). Write

$$\underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}_{\boldsymbol{y}, n\times 1} = \underbrace{\begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n,1} & \cdots & x_{n,p} \end{pmatrix}}_{\boldsymbol{X}, n\times(p+1)} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}}_{\boldsymbol{\beta}, (p+1)\times 1} + \underbrace{\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}}_{\boldsymbol{\varepsilon}, n\times 1}.$$

Assuming $\boldsymbol{\varepsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2\mathbb{I})$, the maximum likelihood estimator of $\boldsymbol{\beta}$ is

$$\widehat{\boldsymbol{\beta}} = \text{argmin}\{\|\boldsymbol{y} - \boldsymbol{X}^\mathsf{T}\boldsymbol{\beta}\|_{\ell_2}\} = (\boldsymbol{X}^\mathsf{T}\boldsymbol{X})^{-1}\boldsymbol{X}^\mathsf{T}\boldsymbol{y}$$

... under the assumtption that $\boldsymbol{X}^\mathsf{T}\boldsymbol{X}$ is a full-rank matrix.

What if $\boldsymbol{X}^\mathsf{T}\boldsymbol{X}$ cannot be inverted? Then $\widehat{\boldsymbol{\beta}} = [\boldsymbol{X}^\mathsf{T}\boldsymbol{X}]^{-1}\boldsymbol{X}^\mathsf{T}\boldsymbol{y}$ does not exist, but $\widehat{\boldsymbol{\beta}}_\lambda = [\boldsymbol{X}^\mathsf{T}\boldsymbol{X} + \lambda\mathbb{I}]^{-1}\boldsymbol{X}^\mathsf{T}\boldsymbol{y}$ always exist if $\lambda > 0$.

Ridge Regression & Regularization

The estimator $\widehat{\boldsymbol{\beta}} = [\boldsymbol{X}^\mathsf{T}\boldsymbol{X} + \lambda\mathbb{I}]^{-1}\boldsymbol{X}^\mathsf{T}\boldsymbol{y}$ is the **Ridge** estimate obtained as solution of

$$\widehat{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}}{\text{argmin}} \left\{ \sum_{i=1}^{n} [y_i - \beta_0 - \boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta}]^2 + \lambda \underbrace{\|\boldsymbol{\beta}\|_{\ell_2}^2}_{\mathbf{1}^\mathsf{T}\boldsymbol{\beta}^2} \right\}$$

for some tuning parameter λ. One can also write

$$\widehat{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta};\|\boldsymbol{\beta}\|_{\ell_2}\leq s}{\text{argmin}} \{\|\boldsymbol{Y} - \boldsymbol{X}^\mathsf{T}\boldsymbol{\beta}\|_{\ell_2}\}$$

There is a Bayesian interpretation of that regularization, when $\boldsymbol{\beta}$ has some prior $\mathcal{N}(\boldsymbol{\beta}_0, \tau\mathbb{I})$.

Over-Fitting & Penalization

Solve here, for some norm $\|\cdot\|$,

$$\min\left\{\sum_{i=1}^{n}\ell(y_i,\beta_0+\boldsymbol{x}^\mathsf{T}\boldsymbol{\beta})+\lambda\|\boldsymbol{\beta}\|\right\}=\min\Big\{\text{objective}(\boldsymbol{\beta})+\text{penality}(\boldsymbol{\beta})\Big\}.$$

Estimators are **no longer unbiased**, but might have a smaller mse.

Consider some i.id. sample $\{y_1,\cdots,y_n\}$ from $\mathcal{N}(\theta,\sigma^2)$, and consider some estimator proportional to $\overline{y}$, i.e. $\widehat{\theta}=\alpha\overline{y}$. $\alpha=1$ is the maximum likelihood estimator.

Note that

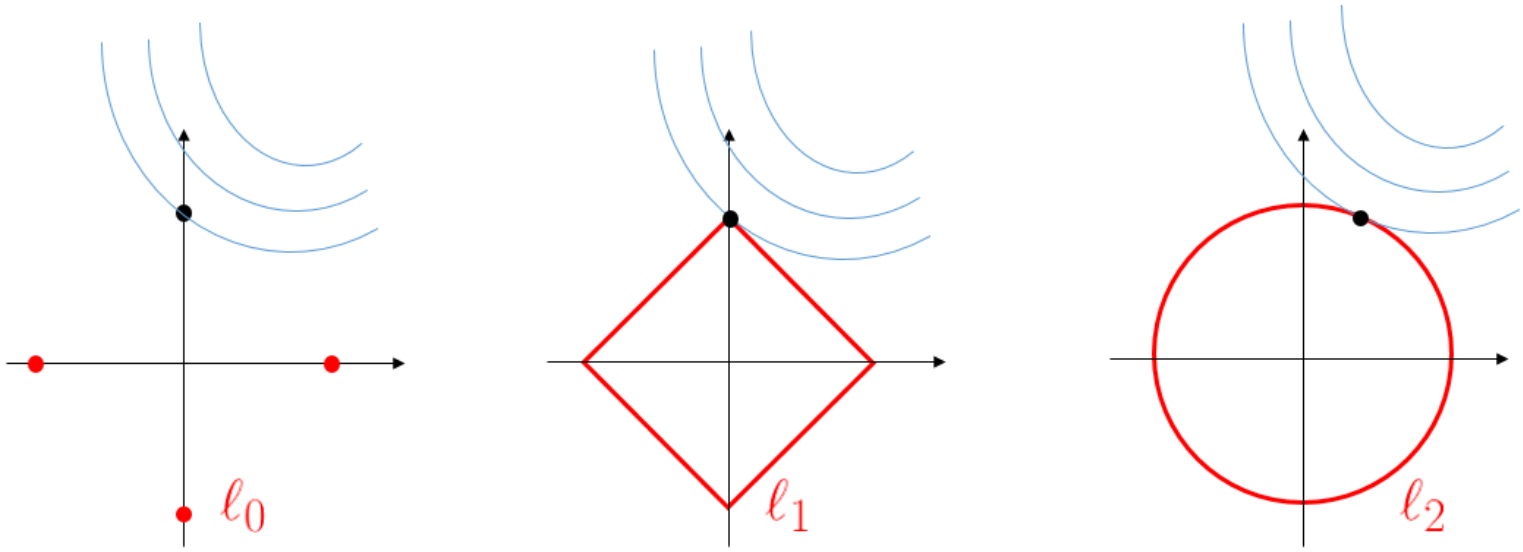
$$\text{mse}[\widehat{\theta}]=\underbrace{(\alpha-1)^2\mu^2}_{\text{bias}[\widehat{\theta}]^2}+\underbrace{\frac{\alpha^2\sigma^2}{n}}_{\text{Var}[\widehat{\theta}]}$$

and $\alpha^\star=\mu^2\cdot\left(\mu^2+\dfrac{\sigma^2}{n}\right)^{-1}<1.$

$$(\widehat{\beta}_0, \widehat{\boldsymbol{\beta}}) = \text{argmin} \left\{ \sum_{i=1}^{n} \ell(y_i, \beta_0 + \boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}) + \lambda \|\boldsymbol{\beta}\| \right\},$$

can be seen as a **Lagrangian** minimization problem

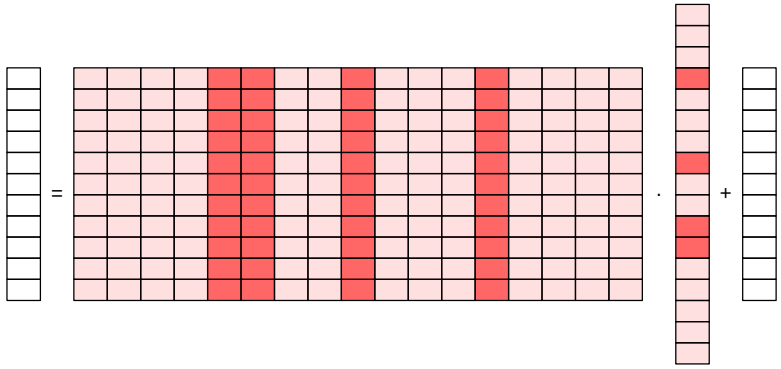
$$(\widehat{\beta}_0, \widehat{\boldsymbol{\beta}}) = \underset{\boldsymbol{\beta}; \|\boldsymbol{\beta}\| \leq s}{\text{argmin}} \left\{ \sum_{i=1}^{n} \ell(y_i, \beta_0 + \boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}) \right\}$$



LASSO & Sparcity

In severall applications, p can be (very) large, but a lot of features are just noise: $\beta_j = 0$ for many j's. Let s denote the number of **relevent features**, with $s << p$, cf Hastie, Tibshirani & Wainwright (2015),

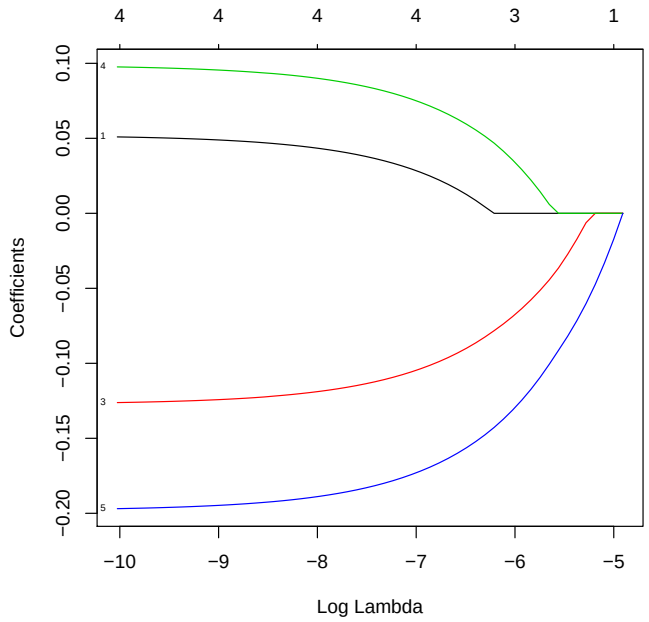
$$s = \text{card}\{\mathcal{S}\} \text{ where } \mathcal{S} = \{j; \beta_j \neq 0\}$$

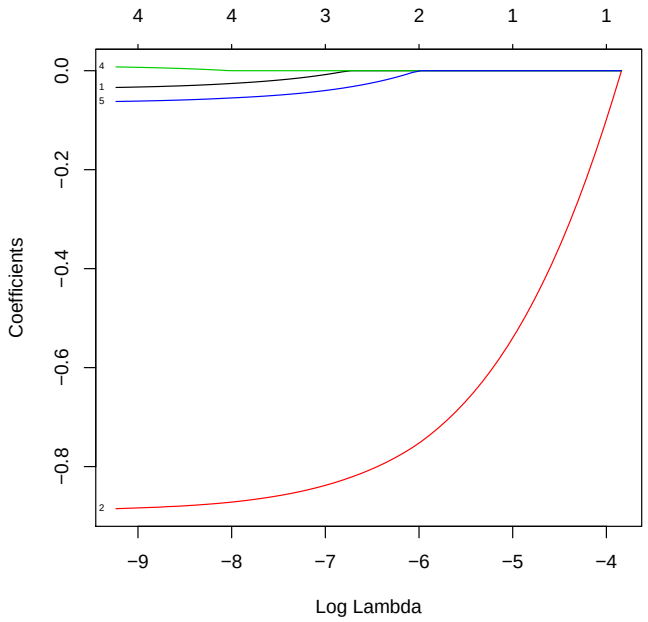


The true model is now $y = \boldsymbol{X}_{\mathcal{S}}^{\mathsf{T}}\boldsymbol{\beta}_{\mathcal{S}} + \varepsilon$, where $\boldsymbol{X}_{\mathcal{S}}^{\mathsf{T}}\boldsymbol{X}_{\mathcal{S}}$ is a full rank matrix.

LASSO & Sparcity

Evoluation of $\widehat{\boldsymbol{\beta}}_\lambda$ as a function of $\log \lambda$ in various applications





In-Sample & Out-Sample

Write $\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}}((\boldsymbol{x}_1, y_1), \cdots, (\boldsymbol{x}_n, y_n))$. Then (for the linear model)

$$\text{Deviance}_{\text{IS}}(\widehat{\boldsymbol{\beta}}) = \sum_{i=1}^{n} [y_i - \boldsymbol{x}_i^{\mathsf{T}} \widehat{\boldsymbol{\beta}}((\boldsymbol{x}_1, y_1), \cdots, (\boldsymbol{x}_n, y_n))]^2$$

Withe this "in-sample" deviance, we cannot use the central limit theorem

$$\frac{\text{Deviance}_{\text{IS}}(\widehat{\boldsymbol{\beta}})}{n} \not\longrightarrow \mathbb{E}\left([Y - \boldsymbol{X}^{\mathsf{T}}\boldsymbol{\beta}]\right)$$

Hence, we can compute some "out-of-sample" deviance

$$\text{Deviance}_{\text{OS}}(\widehat{\boldsymbol{\beta}}) = \sum_{i=n+1}^{m+n} [y_i - \boldsymbol{x}_i^{\mathsf{T}} \widehat{\boldsymbol{\beta}}((\boldsymbol{x}_1, y_1), \cdots, (\boldsymbol{x}_n, y_n)]^2$$

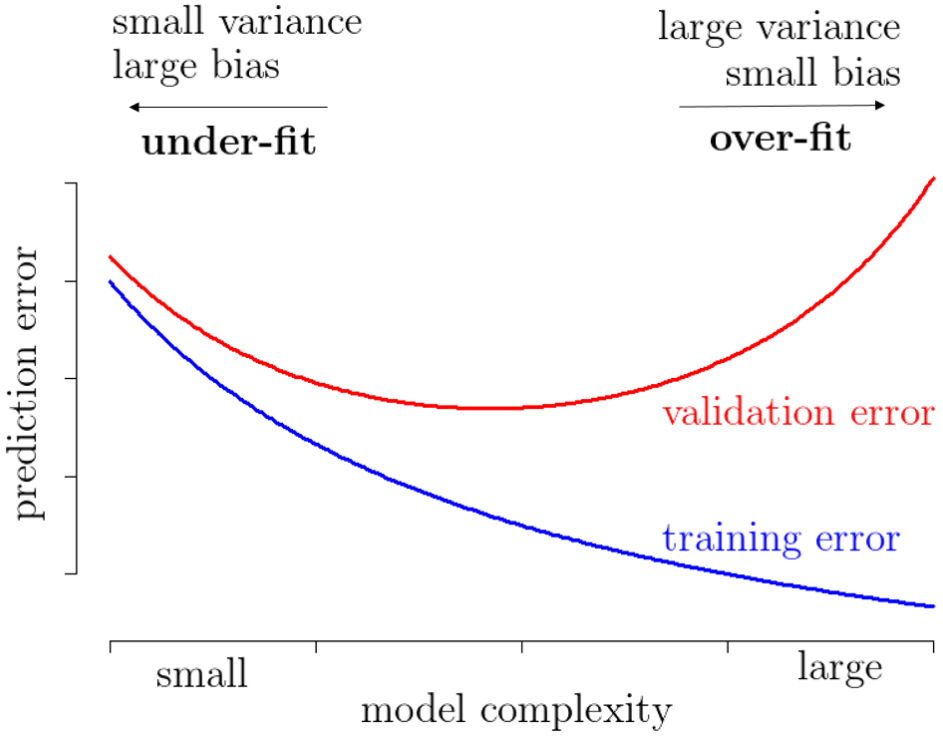
In-Sample & Out-Sample

Observe that there are connexions with Akaike penaly function

$$\text{Deviance}_{\text{IS}}(\widehat{\boldsymbol{\beta}}) - \text{Deviance}_{\text{OS}}(\widehat{\boldsymbol{\beta}}) \approx 2 \cdot \text{degrees of freedom}$$

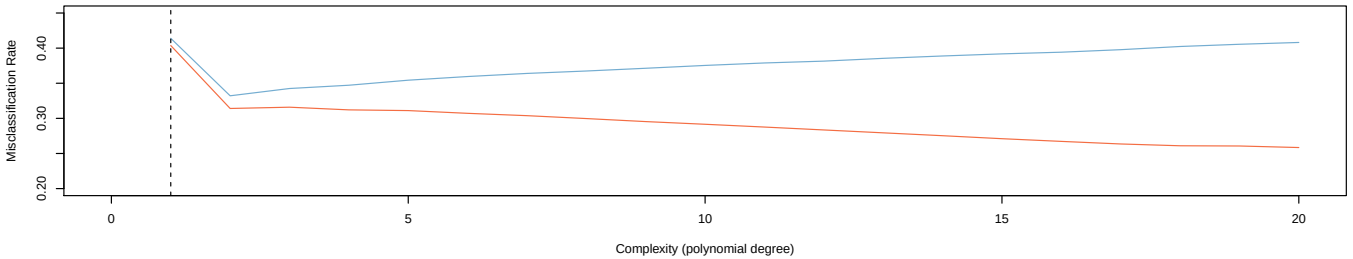
From Stone (1977), minimizing AIC is closed to cross validation,

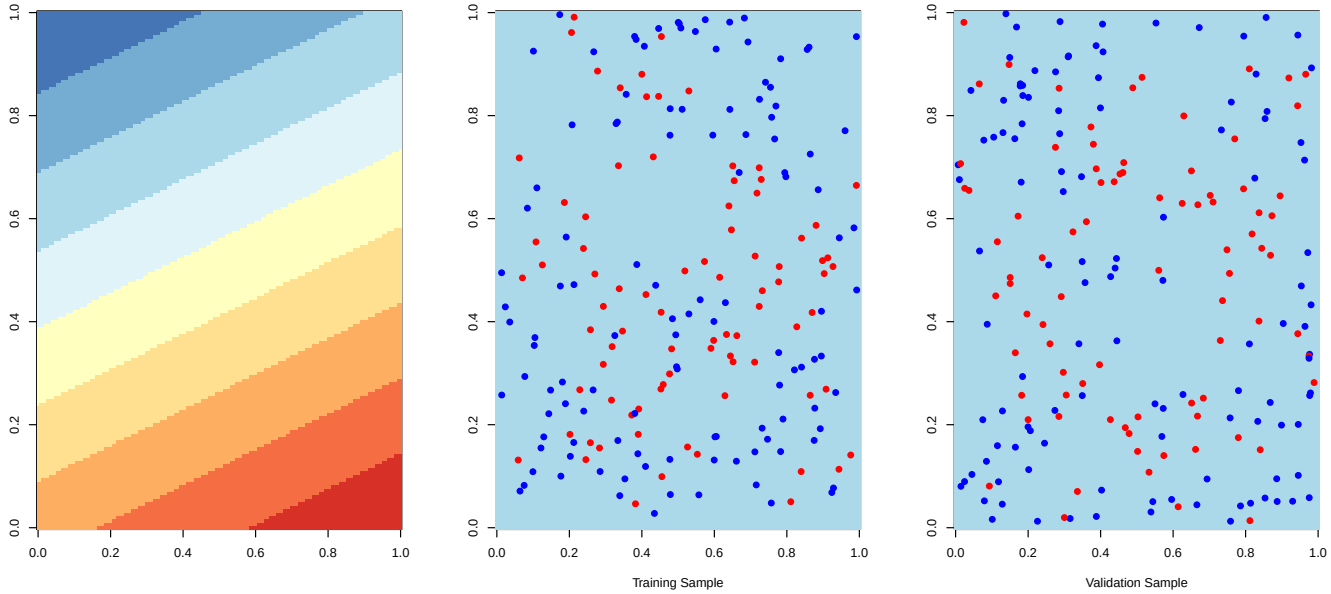
From Shao (1997) minimizing BIC is closed to k-fold cross validation with $k = n/\log n$.



Overfit, Generalization & Model Complexity

Complexity of the model is the degree of the polynomial function





Cross-Validation

See Jacknife technique Quenouille (1956) or Tukey (1958) to reduce the bias.

If $\{y_1, \cdots, y_n\}$ is an i.id. sample from F_θ, with estimator $T_n(\boldsymbol{y}) = T_n(y_1, \cdots, y_n)$, such that $\mathbb{E}[T_n(\boldsymbol{Y})] = \theta + O\left(n^{-1}\right)$, consider

$$\widetilde{T}_n(\boldsymbol{y}) = \frac{1}{n}\sum_{i=1}^{n} T_{n-1}(\boldsymbol{y}_{(i)}) \text{ avec } \boldsymbol{y}_{(i)} = (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n).$$

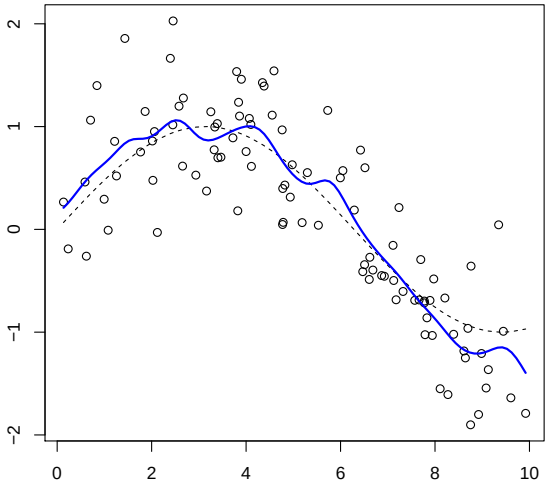
Then $\mathbb{E}[\widetilde{T}_n(\boldsymbol{Y})] = \theta + O\left(n^{-2}\right)$.

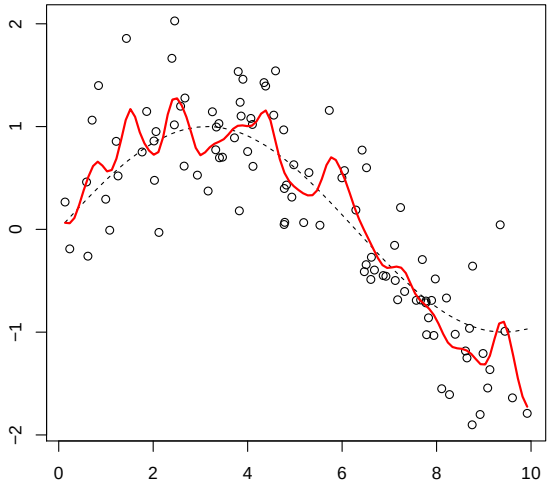
Similar idea in **leave-one-out cross validation**

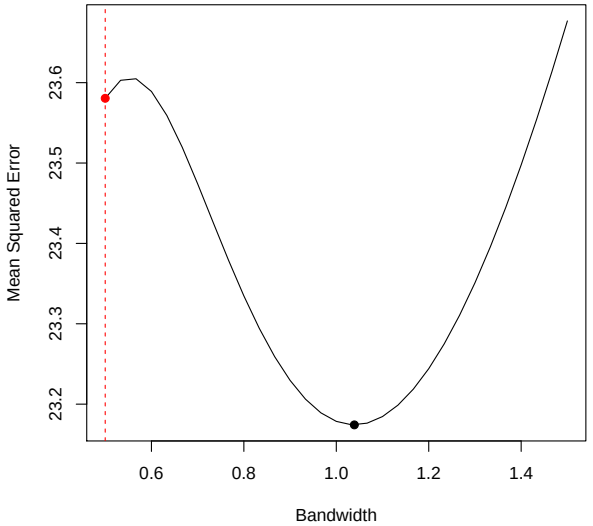
$$\text{Risk} = \frac{1}{n}\sum_{i=1}^{n} \ell(y_i, \widehat{m}_{(i)}(\boldsymbol{x}_i))$$

Rule of Thumb vs. Cross Validation

$$\widehat{m}^{[h^\star]}(x) = \widehat{\beta}_0^{[x]} + \widehat{\beta}_1^{[x]} x \text{ with } (\widehat{\beta}_0^{[x]}, \widehat{\beta}_1^{[x]}) = \underset{(\beta_0,\beta_1)}{\text{argmin}} \left\{ \sum_{i=1}^{n} \omega_{h^\star}^{[x]} [y_i - (\beta_0 + \beta_1 x_i)]^2 \right\}$$







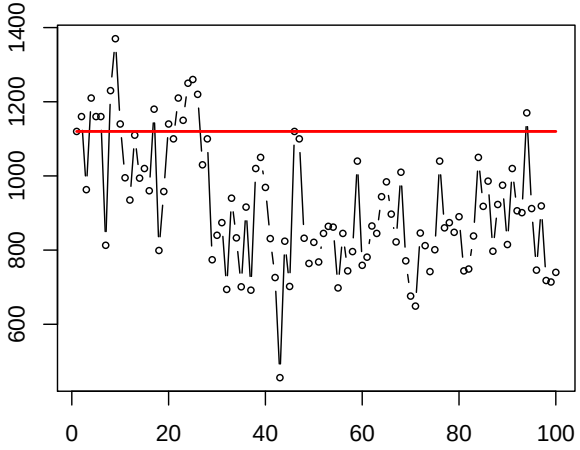
$$\text{set } h^\star = \text{argmin}\big\{\text{mse}(h)\big\} \text{ with } \text{mse}(h) = \frac{1}{n}\sum_{i=1}^{n} \big[y_i - \widehat{m}_{(i)}^{[h]}(x_i)\big]^2$$

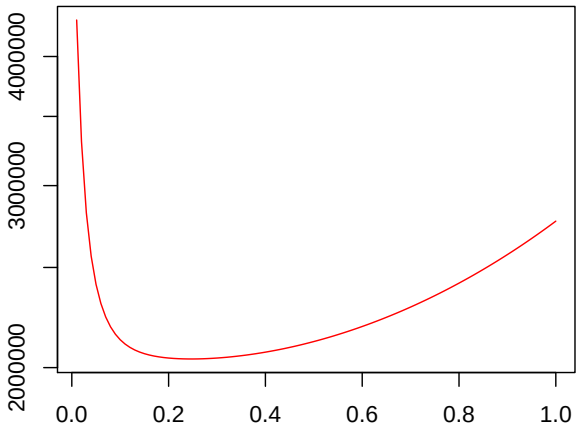
Exponential Smoothing for Time Series

Consider some exponential smoothing filter, on a time series (x_t), $\widehat{y}_{t+1} = \alpha\widehat{y}_t + (1-\alpha)y_t$, then consider

$$\alpha^\star = \text{argmin}\left\{\sum_{t=2}^{T} \ell(\widehat{y}_t, y_t)\right\},$$

see Hyndman *et al.* (2003).

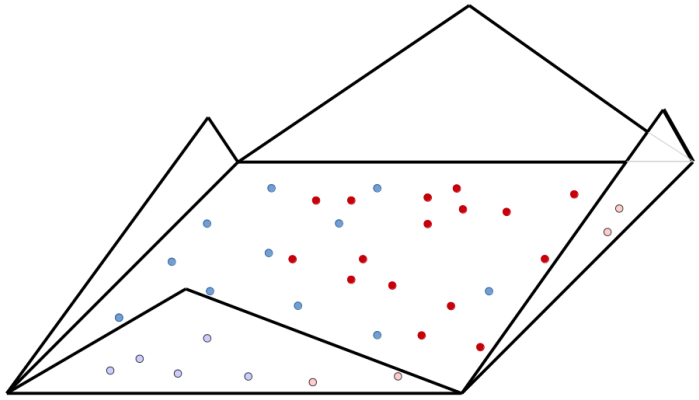




Cross-Validation

Consider a partition of $\{1, \cdots, n\}$ in k groups with the same size, $\mathcal{I}_1, \cdots, \mathcal{I}_k$, and set $\mathcal{I}_{\overline{j}} = \{1, \cdots, n\} \backslash \mathcal{I}_j$. Fit $\widehat{m}_{(j)}$ on $\mathcal{I}_{\overline{j}}$, and

$$\text{Risk} = \frac{1}{k} \sum_{j=1}^{k} \text{Risk}_j \text{ where } \text{Risk}_j = \frac{k}{n} \sum_{i \in \mathcal{I}_j} \ell(y_i, \widehat{m}_{(j)}(\boldsymbol{x}_i))$$



Randomization is too important to be left to chance!

Consider some **bootstraped** sample, $\mathcal{I}_b = \{i_{1,b}, \cdots, i_{n,b}\}$, with $i_{k,b} \in \{1, \cdots, n\}$

Set $n_i = \mathbf{1}_{i \notin \mathcal{I}_1} + \cdots + \mathbf{1}_{i \notin v_B}$, and fit $\widehat{m}_b$ on $\mathcal{I}_b$

$$\text{Risk} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n_i} \sum_{b: i \notin I_b} \ell(y_i, \widehat{m}_b(\boldsymbol{x}_i))$$

Probability that ith obs. is not selection $(1 - n^{-1})^n \to e^{-1} \sim 36.8\%$,

see training / validation samples (2/3-1/3).

Bootstrap

From Efron (1987), generate samples from $(\Omega, \mathcal{F}, \mathbb{P}_n)$

$$\widehat{F}_n(y) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(y_i \leq y) \text{ and } \widehat{F}_n(y_i) = \frac{\text{rank}(y_i)}{n}.$$

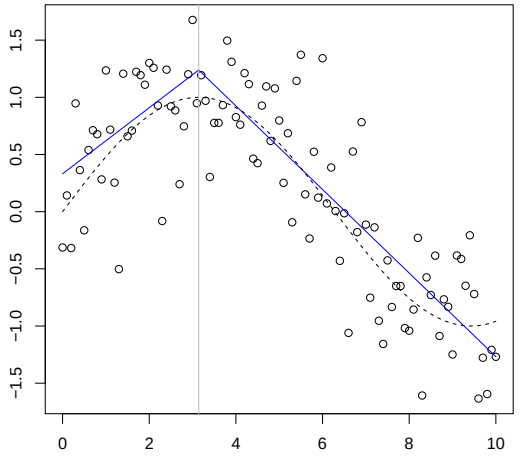
If $U \sim \mathcal{U}([0,1])$, $F^{-1}(U) \sim F$

If $U \sim \mathcal{U}([0,1])$, $\widehat{F}_n^{-1}(U)$ is uniform

on $\left\{\frac{1}{n}, \cdots, \frac{n-1}{n}, 1\right\}$.

Consider some **boostraped sample**,

- either $(y_{i_k}, \boldsymbol{x}_{i_k})$, $i_k \in \{1, \cdots, n\}$
- or $(\widehat{y}_k + \widehat{\varepsilon}_{i_k}, \boldsymbol{x}_k)$, $i_k \in \{1, \cdots, n\}$



Classification & Logistic Regression

Generalized Linear Model when Y has a **Bernoulli distribution**, $y_i \in \{0, 1\}$,

$$m(\boldsymbol{x}) = \mathbb{E}[Y|\boldsymbol{X} = \boldsymbol{x}] = \frac{e^{\beta_0+\boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}}}{1+e^{\beta_0+\boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}}} = H(\beta_0+\boldsymbol{x}^\mathsf{T}\boldsymbol{\beta})$$

Estimate $(\beta_0, \boldsymbol{\beta})$ using maximum likelihood techniques

$$\mathcal{L} = \prod_{i=1}^{n} \left(\frac{e^{\boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta}}}{1+e^{\boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta}}}\right)^{y_i} \left(\frac{1}{1+e^{\boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta}}}\right)^{1-y_i}$$

$$\text{Deviance} \propto \sum_{i=1}^{n} \left[\log(1+e^{\boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta}}) - y_i\boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta}\right]$$

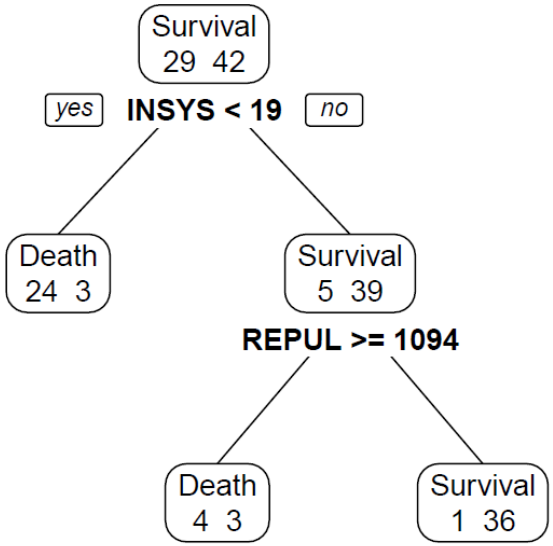
Observe that

$$\mathrm{D}_0 \propto \sum_{i=1}^{n} \left[y_i \log(\overline{y}) + (1-y_i)\log(1-\overline{y})\right]$$

Classification Trees

To split $\{N\}$ into two $\{N_L, N_R\}$, consider

$$\mathcal{I}(N_L, N_R) = \sum_{x\in\{L,R\}} \frac{n_x}{n}\mathcal{I}(N_x)$$



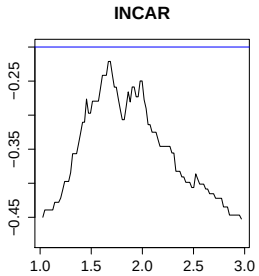
e.g. **Gini index** (used originally in CART, see Breiman *et al.* (1984))

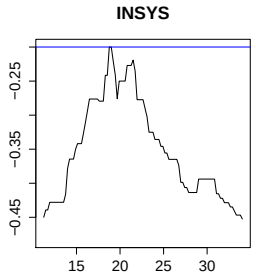
$$\text{gini}(N_L, N_R) = -\sum_{x\in\{L,R\}} \frac{n_x}{n} \sum_{y\in\{0,1\}} \frac{n_{x,y}}{n_x}\left(1-\frac{n_{x,y}}{n_x}\right)$$

and the **cross-entropy** (used in C4.5 and C5.0)

$$\text{entropy}(N_L, N_R) = -\sum_{x\in\{L,R\}} \frac{n_x}{n} \sum_{y\in\{0,1\}} \frac{n_{x,y}}{n_x}\log\left(\frac{n_{x,y}}{n_x}\right)$$

Classification Trees

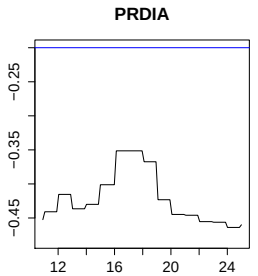


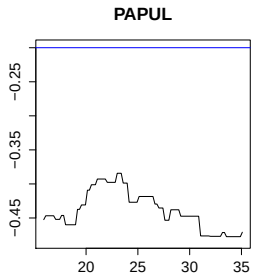


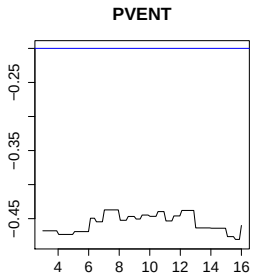
$$N_L\text{: } \{x_{i,j} \leq s\} \quad N_R\text{: } \{x_{i,j} > s\}$$

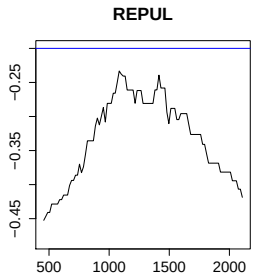
$$\text{solve} \max_{j \in \{1,\cdots,k\}, s} \{\mathcal{I}(N_L, N_R)\}$$

$\longleftarrow$ first split

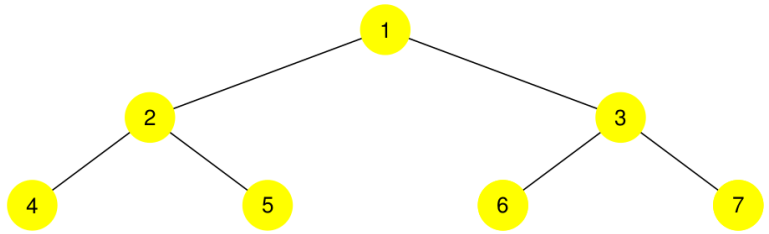


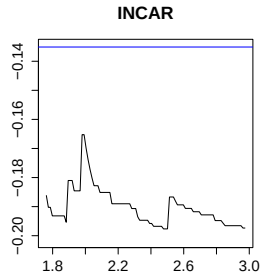


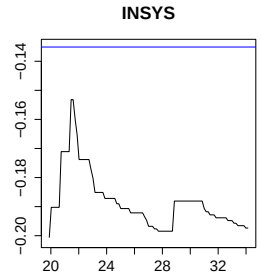


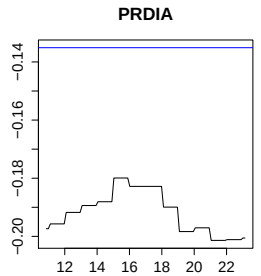


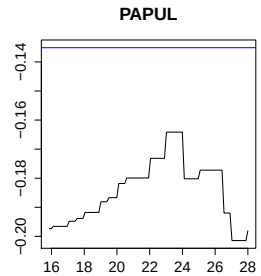
second split $\longrightarrow$

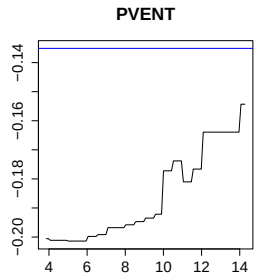


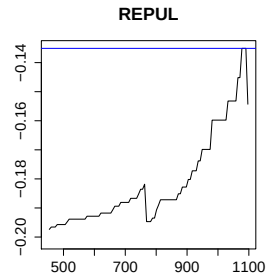












Trees & Forests

Boostrap can be used to define the concept of **margin**,

$$\text{margin}_i = \frac{1}{B}\sum_{b=1}^{B}\mathbf{1}(\widehat{y}_i^{(b)} = y_i) - \frac{1}{B}\sum_{b=1}^{B}\mathbf{1}(\widehat{y}_i^{(b)} \neq y_i)$$

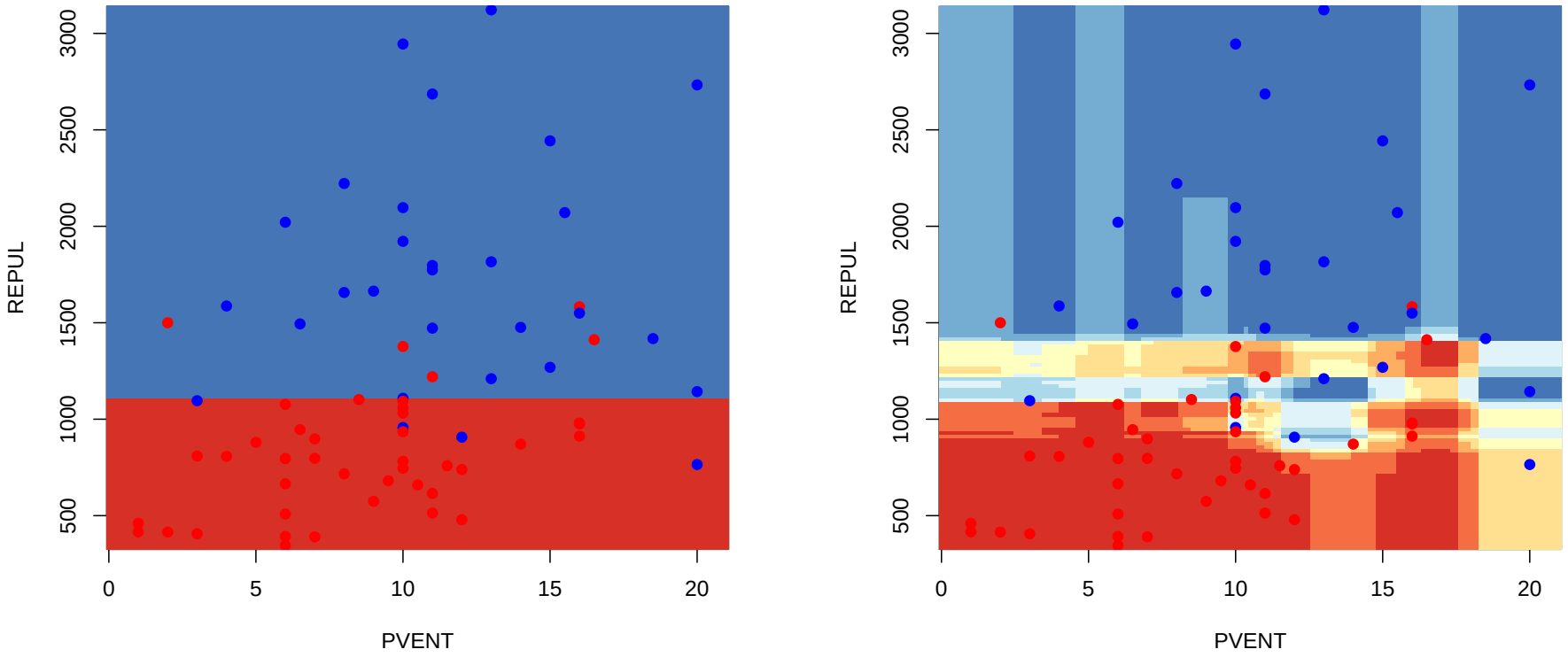
Subsampling of variable, at each knot (e.g. $\sqrt{k}$ out of k)

Concept of **variable importance**: given some random forest with M trees,

$$\text{importance of variable } k \quad I(X_k) = \frac{1}{M}\sum_{m}\sum_{t}\frac{N_t}{N}\Delta\mathcal{I}(t)$$

where the first sum is over all trees, and the second one is over all nodes where the split is done based on variable X_k.

Trees & Forests



See also discriminant analysis, SVM, neural networks, etc.

Model Selection & ROC Curves

Given a scoring function $m(\cdot)$, with $m(\boldsymbol{x}) = \mathbb{E}[Y|\boldsymbol{X} = \boldsymbol{x}]$, and a threshold $s \in (0, 1)$, set

$$\widehat{Y}^{(s)} = \mathbf{1}[m(\boldsymbol{x}) > s] = \begin{cases} 1 \text{ if } m(\boldsymbol{x}) > s \\ 0 \text{ if } m(\boldsymbol{x}) \leq s \end{cases}$$

Define the confusion matrix as $\boldsymbol{N} = [N_{u,v}]$

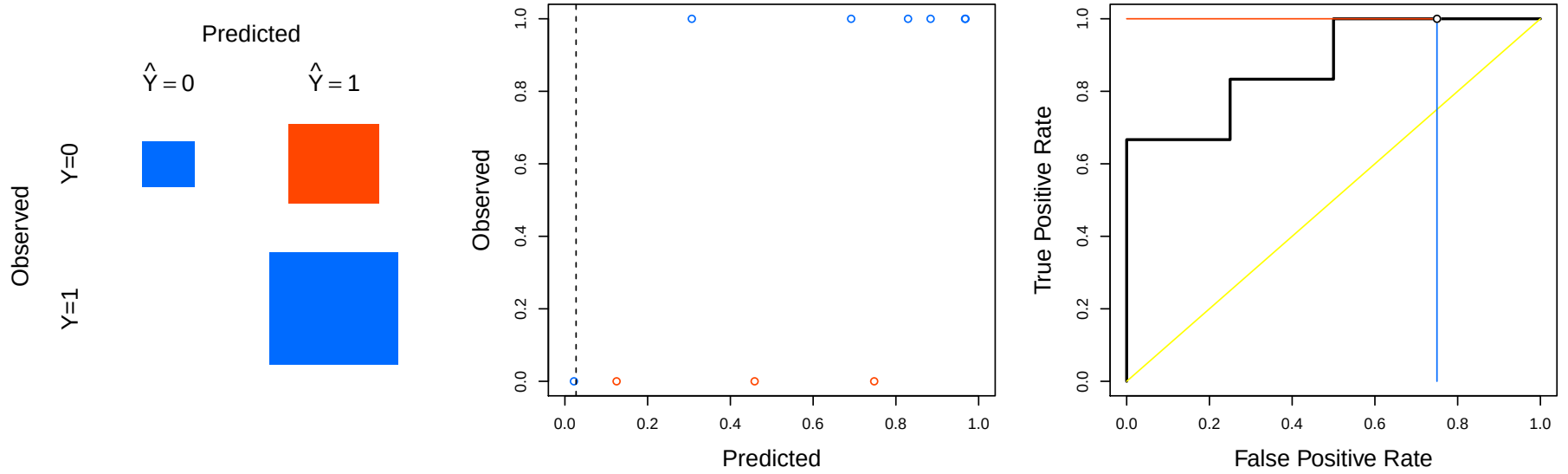
$$N_{u,v}^{(s)} = \sum_{i=1}^{n} \mathbf{1}(\widehat{y}_i^{(s)} = u, y_j = v) \text{ for } (u, v) \in \{0, 1\}.$$

	$Y = 0$	$Y = 1$	
$\widehat{Y}_s = 0$	TN_s	FN_s	TN_s+FN_s
$\widehat{Y}_s = 1$	FP_s	TP_s	FP_s+TP_s
	TN_s+FP_s	FN_s+TP_s	n

Model Selection & ROC Curves

ROC curve is

$$\text{ROC}_s = \left(\frac{\text{FP}_s}{\text{FP}_s + \text{TN}_s}, \frac{\text{TP}_s}{\text{TP}_s + \text{FN}_s} \right) \text{ with } s \in (0,1)$$



Model Selection & ROC Curves

In machine learning, the most popular measure is κ, see Landis & Koch (1977). Define $\boldsymbol{N}^\perp$ from $\boldsymbol{N}$ as in the chi-square independence test. Set

$$\text{total accuracy} = \frac{\text{TP} + \text{TN}}{n}$$

$$\text{random accuracy} = \frac{\text{TP}^\perp + \text{TN}^\perp}{n} = \frac{[\text{TN}+\text{FP}] \cdot [\text{TP}+\text{FN}] + [\text{TP}+\text{FP}] \cdot [\text{TN}+\text{FN}]}{n^2}$$

and

$$\kappa = \frac{\text{total accuracy} - \text{random accuracy}}{1 - \text{random accuracy}}.$$

See Kaggle competitions.

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manière des caractéristiques $\mathbf{\Omega}$ de l'assuré, et lui réclame donc une prime pure de montant $\mathbb{E}[S]$, la même que celle qu'il réclame à tous les assurés du portefeuille. Dans ce cas, la situation est telle que présentée au Tableau 3.7.

	Assurés	Assureur
Dépense	$\mathbb{E}[S]$	$S - \mathbb{E}[S]$
Dépense moyenne	$\mathbb{E}[S]$	0
Variance	0	$\mathbb{V}[S]$

TAB. 3.7 – *Situation des assurés et de l'assureur en l'absence de segmentation.*

L'assureur prend donc l'entièreté de la variance des sinistres $\mathbb{V}[S]$ à sa charge, que celle-ci soit due à l'hétérogénéité du portefeuille, ou à la variabilité intrinsèque des montants des sinistres.

Transfert de risque en information complète

A l'autre extrême, supposons que l'assureur incorpore toute l'information $\mathbf{\Omega}$ dans la tarification. On serait alors dans la situation décrite au Tableau 3.8.

	Assurés	Assureur
Dépense	$\mathbb{E}[S\|\mathbf{\Omega}]$	$S - \mathbb{E}[S\|\mathbf{\Omega}]$
Dépense moyenne	$\mathbb{E}[S]$	0
Variance	$\mathbb{V}\Big[\mathbb{E}[S\|\mathbf{\Omega}]\Big]$	$\mathbb{V}\Big[S - \mathbb{E}[S\|\mathbf{\Omega}]\Big]$

TAB. 3.8 – *Situation des assurés et de l'assureur dans le cas où la segmentation est opérée sur base de* $\mathbf{\Omega}$.

Contrairement au cas précédent, la prime payée par un assuré prélevé au hasard dans le portefeuille est à présent une variable aléatoire: $\mathbb{E}[S|\mathbf{\Omega}]$ dépend des caractéristiques $\mathbf{\Omega}$ de cet assuré. Comme la variable aléatoire $S - \mathbb{E}[S|\mathbf{\Omega}]$ est centrée, le risque assumé par l'assureur la variance du résultat financier de l'opération d'assurance, i.e.

$$\mathbb{V}\Big[S - \mathbb{E}[S|\mathbf{\Omega}]\Big] = \mathbb{E}\Big[\Big(S - \mathbb{E}[S|\mathbf{\Omega}]\Big)^2\Big]$$

No segmentation

	Insured	Insurer
Loss	$\mathbb{E}[S]$	$S - \mathbb{E}[S]$
Average Loss	$\mathbb{E}[S]$	0
Variance	0	$\text{Var}[S]$

Perfect Information: $\mathbf{\Omega}$ observable

	Insured	Insurer
Loss	$\mathbb{E}[S\|\mathbf{\Omega}]$	$S - \mathbb{E}[S\|\mathbf{\Omega}]$
Average Loss	$\mathbb{E}[S]$	0
Variance	$\text{Var}\Big[\mathbb{E}[S\|\mathbf{\Omega}]\Big]$	$\text{Var}\Big[S - \mathbb{E}[S\|\mathbf{\Omega}]\Big]$

$$\text{Var}[S] = \underbrace{\mathbb{E}\Big[\text{Var}[S|\mathbf{\Omega}]\Big]}_{\to \text{ insurer}} + \underbrace{\text{Var}\Big[\mathbb{E}[S|\mathbf{\Omega}]\Big]}_{\to \text{ insured}}.$$

3.8. La prime pure en univers segmenté 177

On assiste dans ce cas à un partage de la variance totale de S (c'est-à-dire du risque) entre les assurés et l'assureur, matérialisé par la formule

$$\mathbb{V}[S] = \underbrace{\mathbb{E}\Big[\mathbb{V}[S|\boldsymbol{\Omega}]\Big]}_{\rightarrow \text{assureur}} + \underbrace{\mathbb{V}\Big[\mathbb{E}[S|\boldsymbol{\Omega}]\Big]}_{\rightarrow \text{assurés}}.$$

Ainsi, lorsque toutes les variables pertinentes $\boldsymbol{\Omega}$ ont été prises en compte, l'intervention de l'assureur se limite à la part des sinistres due exclusivement au hasard; en effet, $\mathbb{V}[S|\boldsymbol{\Omega}]$ représente les fluctuations de S dues au seul hasard. Dans cette situation idéale, l'assureur mutualise le risque et il n'y a donc aucune solidarité induite entre les assurés du portefeuille: chacun paie en fonction de son propre risque.

Transfert des risques en information partielle

Bien entendu, la situation décrite au paragraphe précédent est purement théorique puisque parmi les variables explicatives $\boldsymbol{\Omega}$ nombreuses sont celles qui ne peuvent pas être observées par l'assureur. En assurance automobile par exemple, l'assureur ne peut pas observer la vitesse à laquelle roule l'assuré, son agressivité au volant, ni le nombre de kilomètres qu'il parcourt chaque année[2]. Dès lors, l'assureur ne peut utiliser qu'un sous-ensemble $\boldsymbol{X}$ des variables explicatives contenues dans $\boldsymbol{\Omega}$, i.e. $\boldsymbol{X} \subset \boldsymbol{\Omega}$. La situation est alors semblable à celle décrite au Tableau 3.9.

	Assuré	Assureur
Dépense	$\mathbb{E}[S\|\boldsymbol{X}]$	$S - \mathbb{E}[S\|\boldsymbol{X}]$
Dépense moyenne	$\mathbb{E}[S]$	0
Variance	$\mathbb{V}\Big[\mathbb{E}[S\|\boldsymbol{X}]\Big]$	$\mathbb{E}\Big[\mathbb{V}[S\|\boldsymbol{X}]\Big]$

TAB. 3.9 – *Situation de l'assuré et de l'assureur dans le cas où la segmentation est opérée sur base de* $\boldsymbol{X} \subset \boldsymbol{\Omega}$.

Il est intéressant de constater que

$$\begin{aligned}\mathbb{E}\Big[\mathbb{V}[S|\boldsymbol{X}]\Big] &= \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{V}[S|\boldsymbol{\Omega}]\Big|\boldsymbol{X}\Big]\Big] + \mathbb{E}\Big[\mathbb{V}\Big[\mathbb{E}[S|\boldsymbol{\Omega}]\Big|\boldsymbol{X}\Big]\Big] \\ &= \underbrace{\mathbb{E}\Big[\mathbb{V}[S|\boldsymbol{\Omega}]\Big]}_{\text{mutualisation}} + \underbrace{\mathbb{E}\Big\{\mathbb{V}\Big[\mathbb{E}[S|\boldsymbol{\Omega}]\Big|\boldsymbol{X}\Big]\Big\}}_{\text{solidarité}}. \end{aligned} \quad (3.22)$$

Non-Perfect Information: $\boldsymbol{X} \subset \boldsymbol{\Omega}$ is observable

	Insured	Insurer
Loss	$\mathbb{E}[S\|\boldsymbol{X}]$	$S - \mathbb{E}[S\|\boldsymbol{X}]$
Average Loss	$\mathbb{E}[S]$	0
Variance	$\text{Var}\Big[\mathbb{E}[S\|\boldsymbol{X}]\Big]$	$\mathbb{E}\Big[\text{Var}[S\|\boldsymbol{X}]\Big]$

$$\begin{aligned}\mathbb{E}\Big[\text{Var}[S|\boldsymbol{X}]\Big] &= \mathbb{E}\Big[\mathbb{E}\Big[\text{Var}[S|\boldsymbol{\Omega}]\Big|\boldsymbol{X}\Big]\Big] \\ &+ \mathbb{E}\Big[\text{Var}\Big[\mathbb{E}[S|\boldsymbol{\Omega}]\Big|\boldsymbol{X}\Big]\Big] \\ &= \underbrace{\mathbb{E}\Big[\text{Var}[S|\boldsymbol{\Omega}]\Big]}_{\text{pooling}} \\ &+ \underbrace{\mathbb{E}\Big\{\text{Var}\Big[\mathbb{E}[S|\boldsymbol{\Omega}]\Big|\boldsymbol{X}\Big]\Big\}}_{\text{solidarity}}.\end{aligned}$$

SEGMENTATION ET MUTUALISATION
LES DEUX FACES D'UNE MÊME PIÈCE ?

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Michel Denuit
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Romuald Elie
Professeur à l'Université de Marne-la-Vallée

L'assurance repose fondamentalement sur l'idée que la mutualisation des risques entre des assurés est possible. Cette mutualisation, qui peut être vue comme une relecture actuarielle de la loi des grands nombres, n'a de sens qu'au sein d'une population de risques « homogènes » [Charpentier, 2011]. Cette condition (actuarielle) impose aux assureurs de segmenter, ce que confirment plusieurs travaux économiques (1). Avec l'explosion du nombre de données, et donc de variables tarifaires possibles, certains assureurs évoquent l'idée d'un tarif individuel, semblant remettre en cause l'idée même de mutualisation des risques. Entre cette force qui pousse à segmenter et la force de rappel qui tend (pour des raisons sociales mais aussi actuarielles, ou au moins de robustesse statistique (2)) à imposer une solidarité minimale entre les assurés, quel équilibre va en résulter dans un contexte de forte concurrence entre les sociétés d'assurance ?

Tarification sans segmentation

Sans segmentation, le « prix juste » d'un risque est l'espérance mathématique de la charge annuelle. C'est l'idée du théorème fondamental de la valorisation actuarielle : en moyenne, la somme des primes doit permettre d'indemniser l'intégralité des sinistres survenus dans l'année. Afin d'illustrer les différents aspects de la construction du tarif et ses conséquences, on va utiliser les données présentées dans le tableau 1 (voir p. xx), qui indique la fréquence annuelle de sinistres.

Les facteurs de risque sont ici le lieu d'habitation et l'âge de l'assuré, et on observe la fréquence de sinistre par classe. Le coût unitaire, supposé fixe, équivaut à 1 000 euros. La prime pure est alors E [*S*] = 1 000 x E [*N*]. Dans cet exemple, la prime pure sans segmentation sera de 82,30 euros.

Simple model $\boldsymbol{\Omega} = \{\boldsymbol{X}_1, \boldsymbol{X}_2\}$.

Four Models

$$\begin{cases} \widehat{m}_0(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathbb{E}[S] \\ \widehat{m}_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathbb{E}[S|\boldsymbol{X}_1 = \boldsymbol{x}_1] \\ \widehat{m}_2(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathbb{E}[S|\boldsymbol{X}_2 = \boldsymbol{x}_2] \\ \widehat{m}_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathbb{E}[S|\boldsymbol{X}_1 = \boldsymbol{x}_1, \boldsymbol{X}_2 = \boldsymbol{x}_2] \end{cases}$$

PolNum	CalYear	Gender	Type	Category	Occupation	Age	Group1	Bonus	Poldur	Value	Adind	SubGroup2	Group2	Density
200285786	2010	Male	E	Large	Employed	48	14	40	0	32345	1	O31	O	35.43401501
200285787	2010	Male	B	Medium	Employed	30	8	-30	9	8995	0	Q29	Q	239.4551701
200285788	2010	Female	B	Large	Housewife	47	2	-50	2	9145	1	U21	U	88.29014956
200285789	2010	Female	D	Large	Self-employed	48	13	-30	15	22075	1	R21	R	275.2822626
200285790	2010	Male	C	Medium	Housewife	57	12	-50	1	24985	1	Q5	Q	99.6400095
200285791	2010	Male	D	Medium	Self-employed	21	15	50	1	12100	1	R11	R	259.0040603
200285792	2010	Male	B	Small	Employed	44	5	-40	15	9820	1	Q10	Q	169.7885554
200285793	2010	Male	F	Small	Self-employed	37	17	-50	5	28680	1	Q5	Q	99.6400095
200285794	2010	Female	C	Large	Retired	49	3	20	4	28470	0	L94	L	84.22903844
200285795	2010	Female	A	Medium	Unemployed	35	5	20	5	8590	0	L112	L	66.06668352
200285796	2010	Male	E	Large	Self-employed	50	10	-30	3	20490	1	Q10	Q	169.7885554
200285797	2010	Female	B	Medium	Housewife	31	8	140	1	8385	1	P28	P	41.2451199
200285798	2010	Female	E	Medium	Self-employed	41	11	90	3	6410	1	L47	L	66.76541883
200285799	2010	Female	A	Medium	Housewife	44	10	-30	8	8485	0	P29	P	20.86448407
200285800	2010	Male	B	Large	Retired	69	8	-40	11	9380	1	U14	U	123.0152076
200285801	2010	Male	F	Medium	Housewife	45	11	30	0	19700	0	L40	L	76.05272599
200285802	2010	Male	E	Large	Retired	53	8	-30	6	10980	1	U19	U	61.79475865
200285803	2010	Male	C	Small	Employed	47	10	-10	9	21980	0	L96	L	45.66982293
200285804	2010	Female	D	Large	Retired	46	7	-50	1	28925	1	U12	U	54.93181221
200285805	2010	Female	C	Large	Retired	67	17	-50	9	14525	1	L52	L	73.25249905

$X_{1,i}$ $X_{k,i}$

Numtppd	Numtpbi	Indtppd	Indtpbi
0	1	0	1056.0334927
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
3	1	5800.0189068	16.507641942
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Y_i

PolNum	CalYear	Gender	Type	Category	Occupation	Age	Group1	Bonus	Poldur	Value	Adind	SubGroup2	Group2	Density
200375666	2011	Female	A	Large	Employed	46	11	50	0	42975	0	L18	L	58.91132801
200375667	2011	Male	B	Large	Unemployed	31	8	80	11	14835	0	U8	U	125.1320458
200375668	2011	Female	D	Medium	Employed	27	7	-40	13	19000	1	R30	R	296.4319078
200375670	2011	Male	B	Small	Self-employed	22	7	-10	14	33305	0	Q33	Q	129.6690079
200375672	2011	Male	B	Small	Employed	21	17	-20	14	25995	0	T25	T	28.51184808
200375674	2011	Male	C	Medium	Employed	45	19	-50	0	8320	1	N21	N	71.18027901
200375675	2011	Male	C	Medium	Housewife	51	19	30	3	8445	0	L110	L	83.90453994
200375676	2011	Male	E	Large	Self-employed	49	16	-50	3	19545	0	L58	L	64.53563007
200375677	2011	Male	C	Small	Housewife	31	11	-20	5	5030	1	Q7	Q	83.76263662
200375678	2011	Female	A	Medium	Housewife	31	9	-50	14	15480	1	P7	P	25.62227499
200375679	2011	Male	B	Large	Housewife	69	13	-50	7	29580	0	Q23	Q	205.4307964
200375682	2011	Male	A	Medium	Self-employed	43	13	140	3	3735	0	U16	U	91.54176264
200375683	2011	Female	A	Medium	Self-employed	64	18	-20	6	13670	1	O35	O	21.45273029
200375685	2011	Male	E	Large	Employed	25	8	-10	6	17315	0	O22	O	32.18545326
200375688	2011	Male	B	Small	Retired	55	7	-40	3	19410	1	R49	R	208.8164363
200375689	2011	Female	F	Medium	Self-employed	54	9	-40	14	4165	0	U12	U	54.93181221
200375690	2011	Male	D	Large	Housewife	42	9	80	0	11970	1	L125	L	44.16537902
200375692	2011	Male	E	Large	Employed	36	12	-20	7	28415	0	L48	L	71.62174491
200375693	2011	Male	F	Medium	Self-employed	26	10	-30	6	4300	0	L97	L	63.82886936
200375694	2011	Female	B	Small	Unemployed	24	6	-40	7	24005	0	M17	M	201.6569069

Market Competition

Decision Rule: the insured selects the cheapeast premium,

	A	B	C	D	E	F
	787.93	706.97	1032.62	907.64	822.58	**603.83**
	**170.04**	197.81	285.99	212.71	177.87	265.13
	473.15	447.58	**343.64**	410.76	414.23	425.23
	337.98	**336.20**	468.45	339.33	383.55	672.91

Market Competition

Decision Rule: the insured selects randomly from the three cheapeast premium

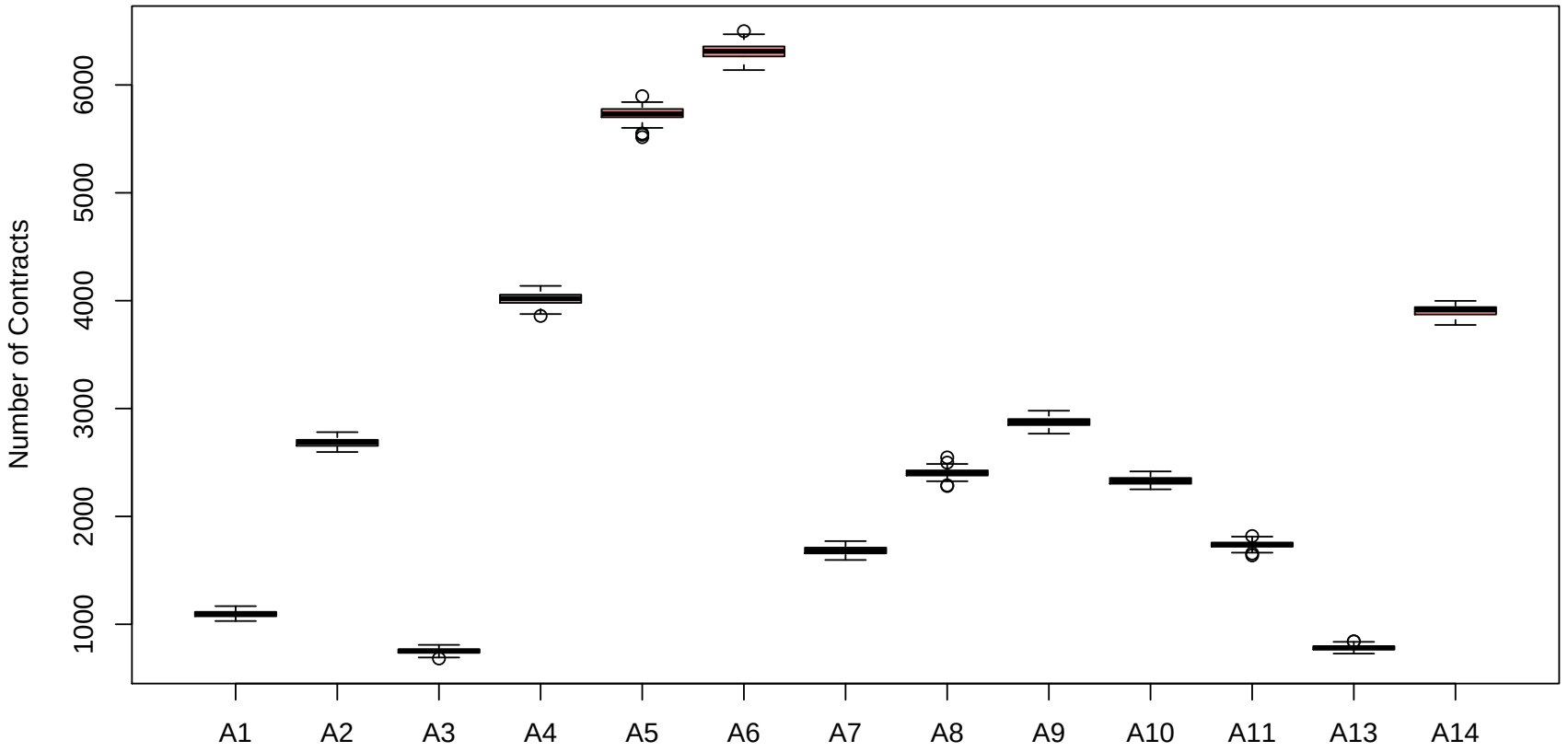
	A	B	C	D	E	F
	787.93	706.97	1032.62	907.64	822.58	603.83
	170.04	197.81	285.99	212.71	177.87	265.13
	473.15	447.58	343.64	410.76	414.23	425.23
	337.98	336.20	468.45	339.33	383.55	672.91

Market Competition

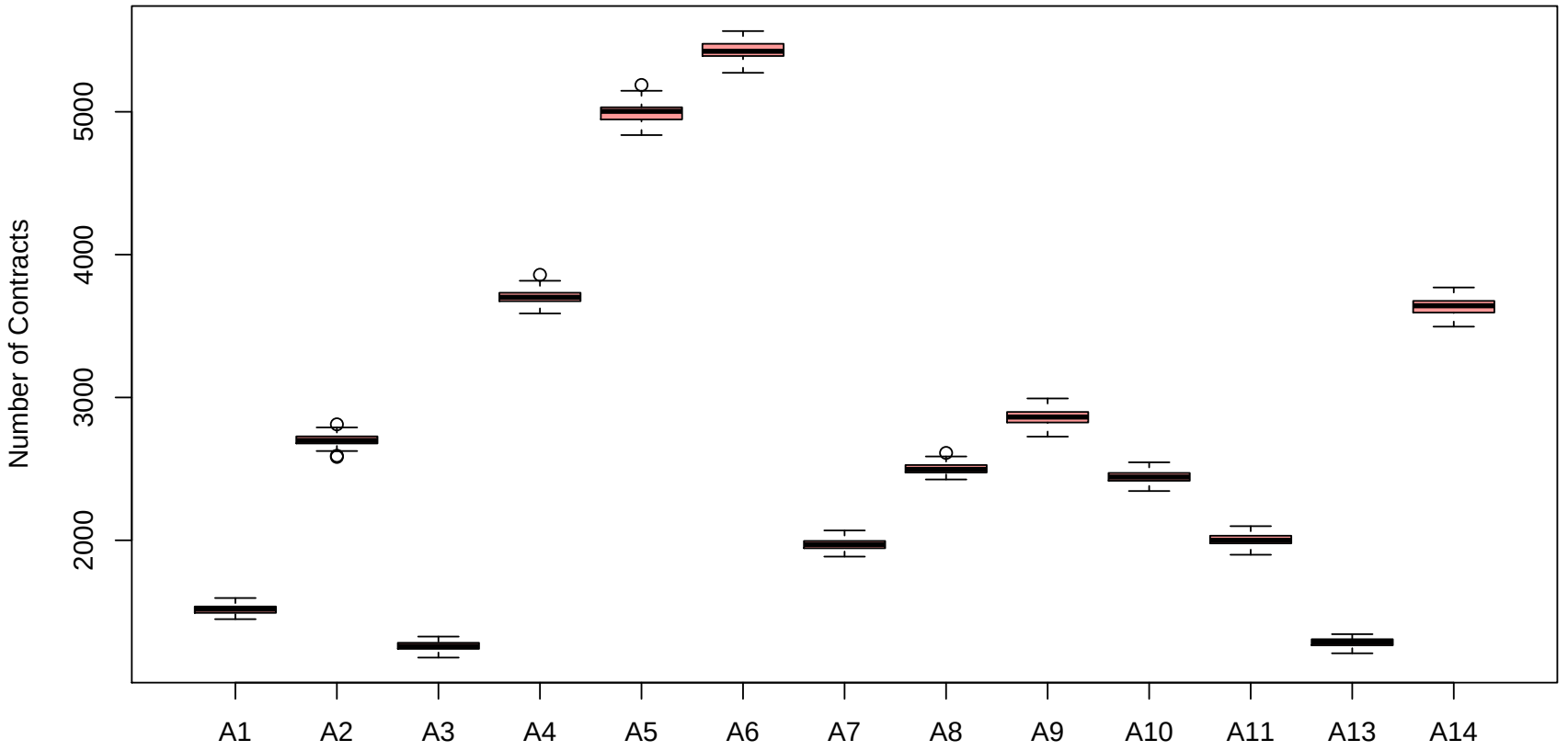
Decision Rule: the insured were assigned randomly to some insurance company for year $n-1$. For year n, they stay with their company if the premium is one of the three cheapeast premium, if not, random choice among the four

	A	B	C	D	E	F
	787.93	706.97	1032.62	907.64	822.58	603.83
	170.04	197.81	285.99	212.71	**177.87**	265.13
	473.15	447.58	343.64	410.76	414.23	425.23
	**337.98**	336.20	468.45	339.33	383.55	672.91

Market Shares (rule 2)

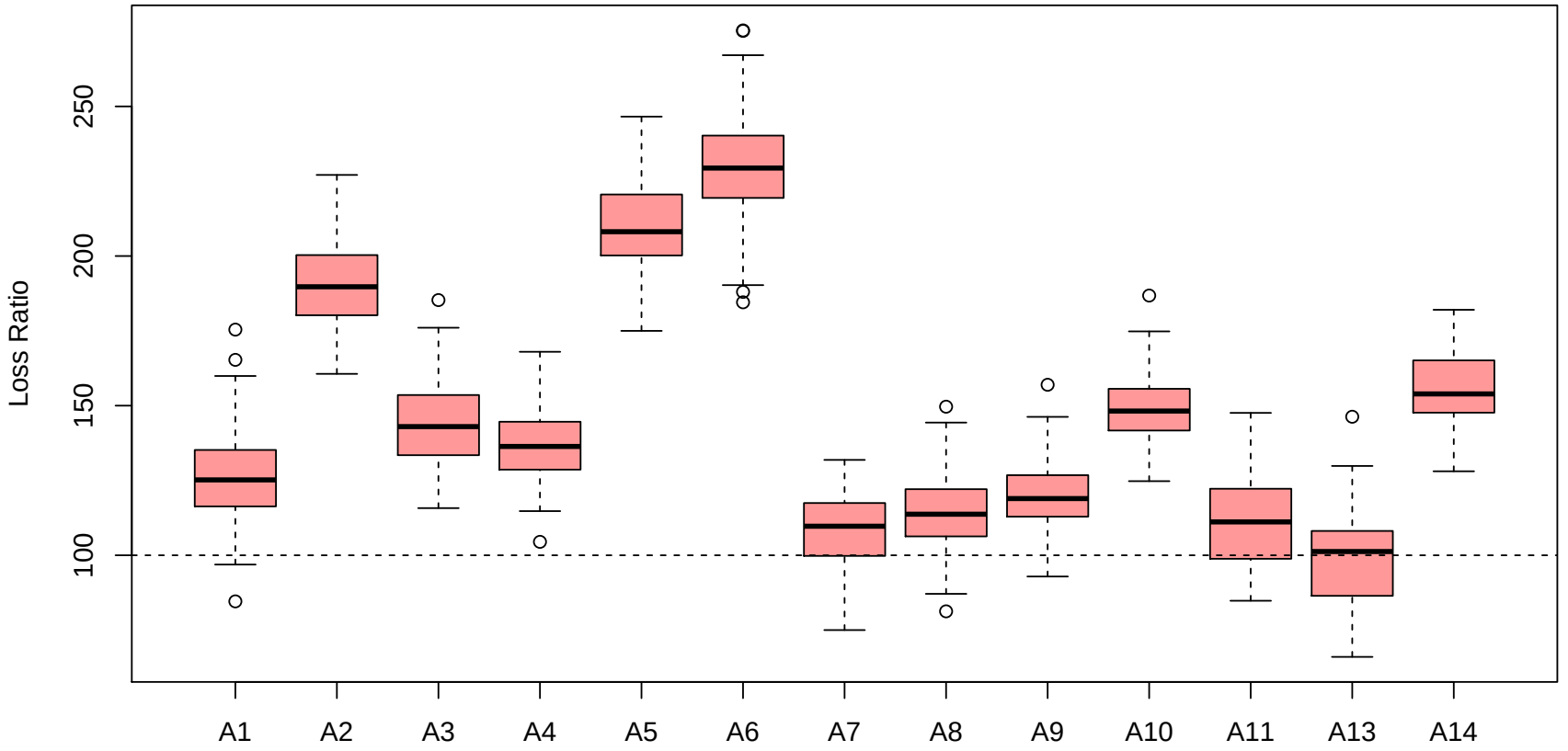


Market Shares (rule 3)



Loss Ratio, Loss / Premium (rule 2)

Market Loss Ratio $\sim 154\%$.

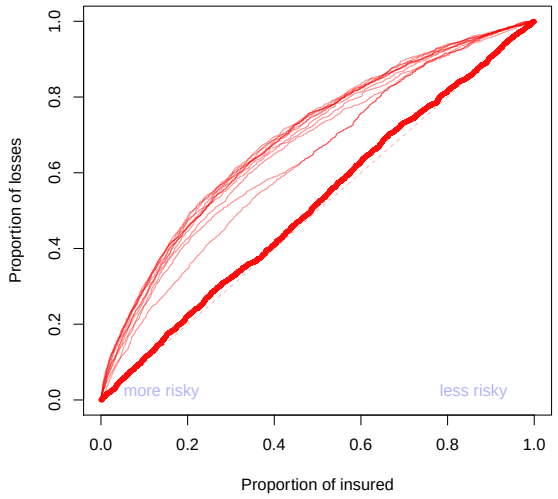


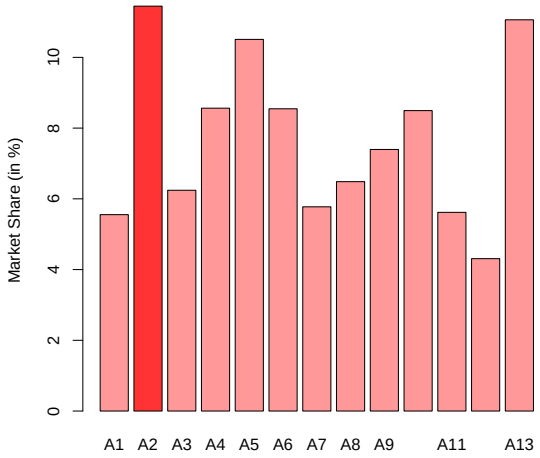
Insurer A2

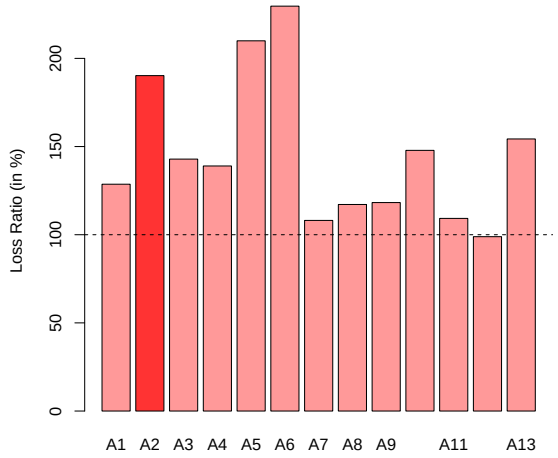
No segmentation, unique premium

Remark on normalized premiums,

$$\pi_2 = m_2(\boldsymbol{x}_i) = \frac{1}{n}\sum_{i=1}^{n} m_j(\boldsymbol{x}_i) \ \forall j$$







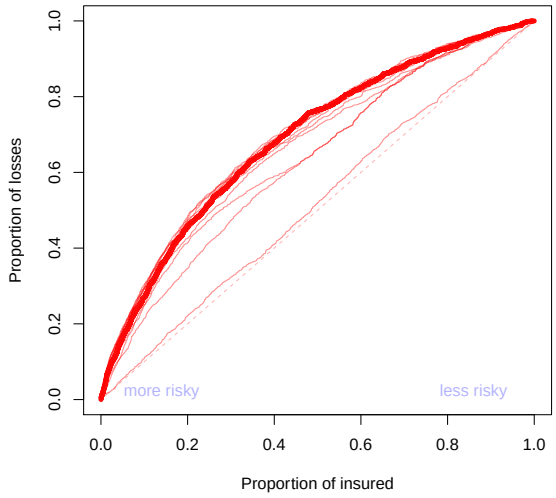
Insured A1

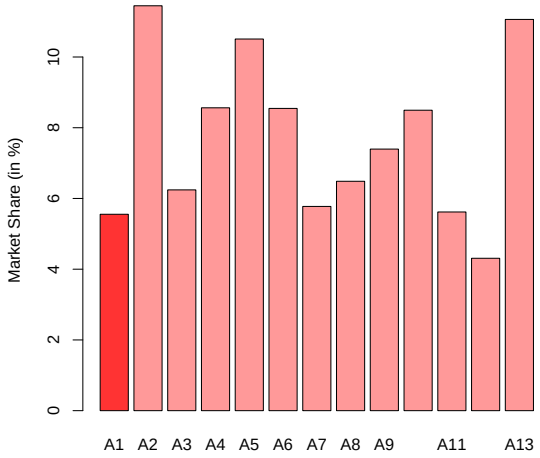
GLM, frequency material / bodily injury, individual losses material

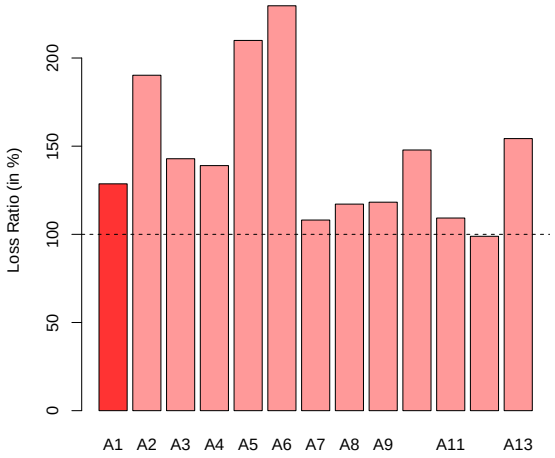
Ages in classes [18-30], [30-45], [45-60] and [60+], crossed with occupation

Manual smoothing, SAS and Excel

Actuaries in a Mutual Fund (in France)







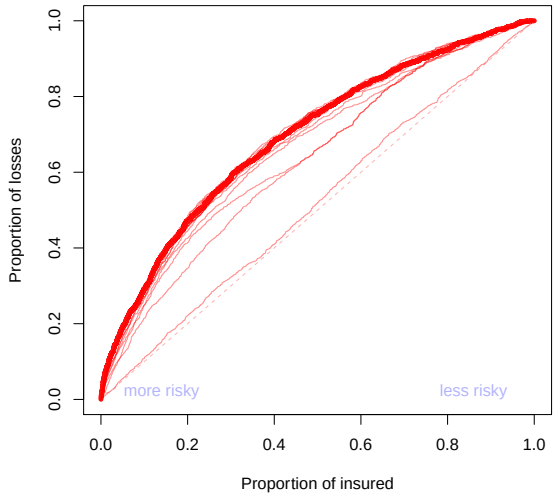
Insurer A8/A9

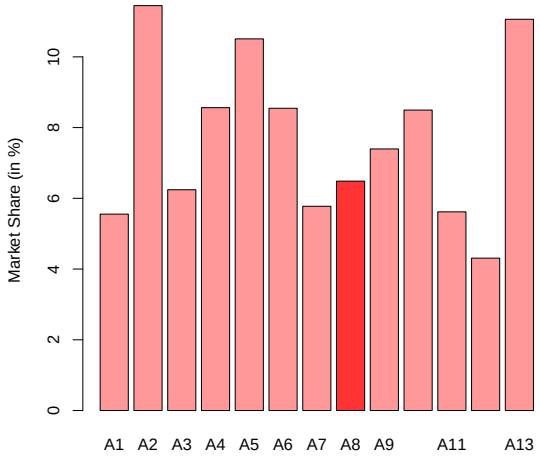
GLM, frequency and losses, without major losses (>15k)

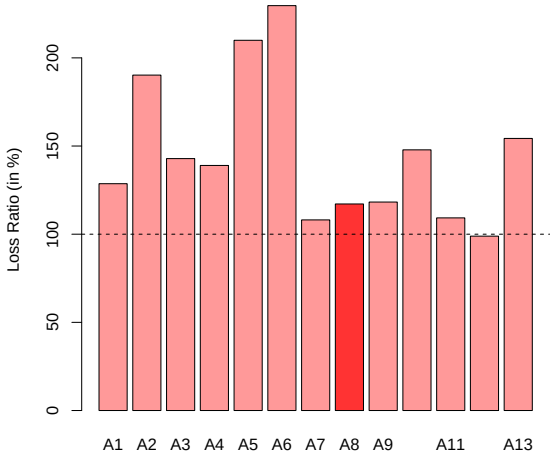
Age-gender interaction

Use of a commercial pricing software

Actuary in a French Mutual Fund





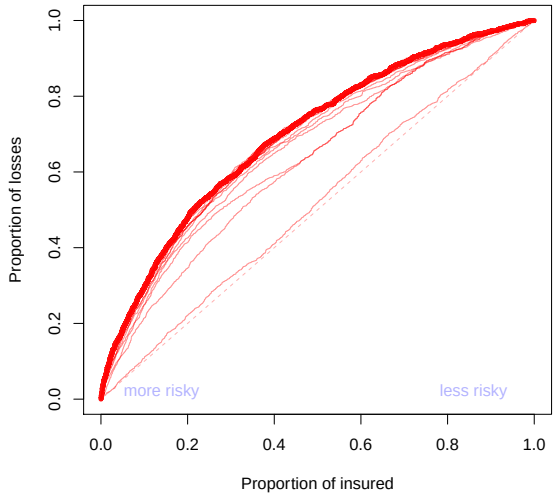


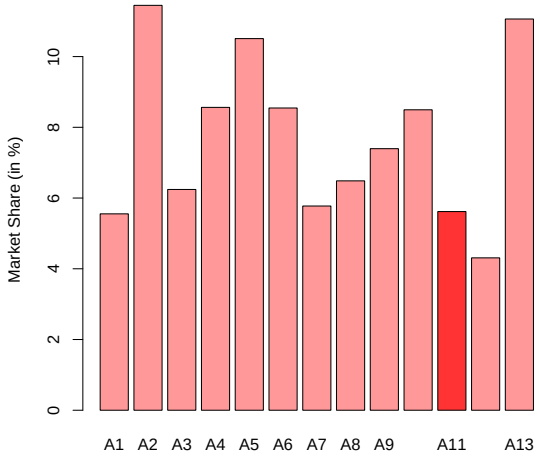
Insurer A11

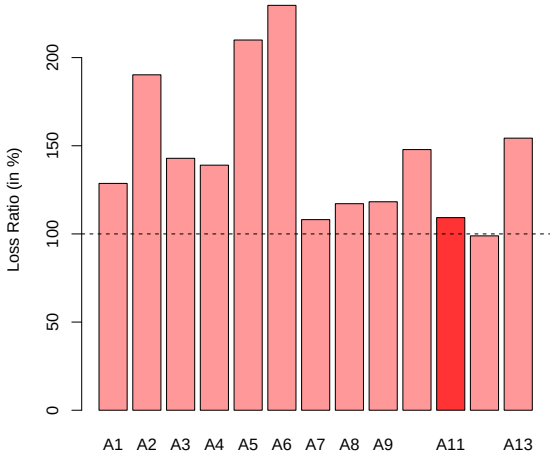
All features, but one XGBoost (gradient boosting)

Correction for negative premiums

Coded in Python actuary in an insurance company.





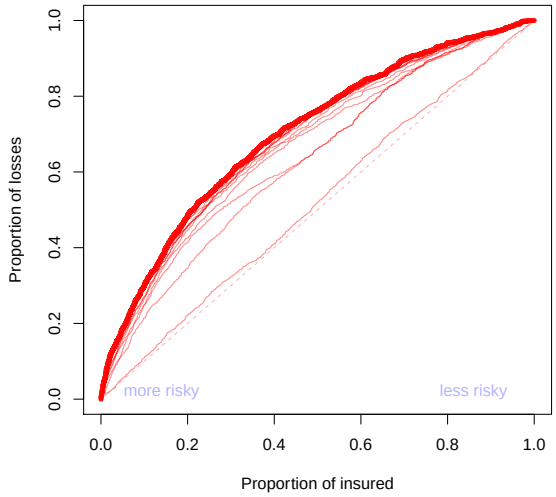


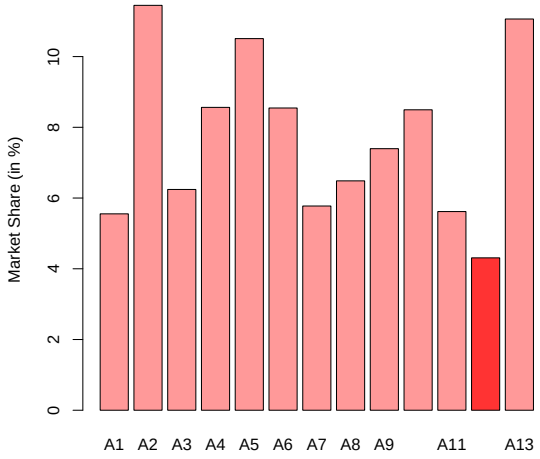
Insurer A12

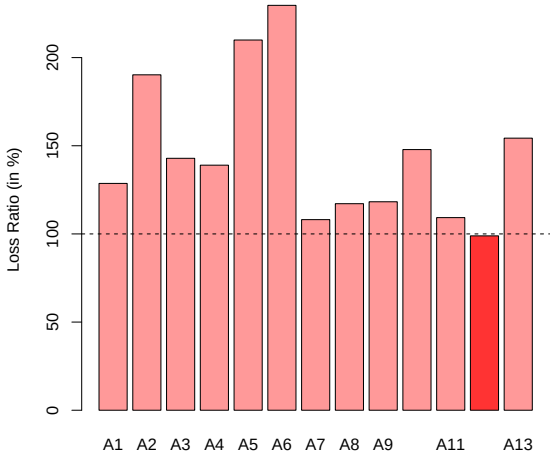
All features, use of two XGBoost (gradient boosting) models

Correction for negative premiums

Coded in R by an actuary in an Insurance company.







Back on the Pricing Game