

# Risk Sharing on Irregular Networks

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# Preferences and Dispersion

Following [Hardy et al. \(1929, 1934\)](#), and [Marshall and Olkin \(1979\)](#)

**Def** Consider two sorted vectors  $\mathbf{x}$  and  $\mathbf{y}$  ( $x_1 \geq x_2 \geq \dots \geq x_n$  and  $y_1 \geq y_2 \geq \dots \geq y_n$ )

such that  $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ , then  $\mathbf{x} \preceq_M \mathbf{y}$  (majorization order) if  $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i, \forall k$ .

$$\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}, \frac{1}{n}\right) \prec_M \left(\frac{1}{n-1}, \frac{1}{n} - 1, \dots, \frac{1}{n-1}, 0\right) \prec_M (1, 0, \dots, 0, 0).$$

**Def** Consider two vectors  $\mathbf{x}$  and  $\mathbf{y}$  such that  $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ , then  $\mathbf{x} \preceq_M \mathbf{y}$  if

$$\text{either } \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, \forall k \text{ or } \sum_{i=1}^k x_{(i)} \geq \sum_{i=1}^k y_{(i)}, \forall k$$

where  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$  (increasing) while  $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$  (decreasing).

# Preferences and Dispersion

If  $\mathbf{x} \in \mathcal{S}_n$ ,  $\frac{1}{n}\mathbf{1} \prec_M \mathbf{x} \prec_M \mathbf{e}_i$ ,  $\forall i$ , where  $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ .

**Prop**  $\mathbf{x} \preceq_M \mathbf{y}$

$$\iff \sum_{i=1}^n h(x_i) \leq \sum_{i=1}^n h(y_i) \text{ for any convex function}$$

$$\iff \sum_{i=1}^n (x_i - d)_+ \leq \sum_{i=1}^n (y_i - d)_+ \text{ for } d \in \mathbb{R}$$

$$\iff \mathbf{x} = D\mathbf{y} \text{ for some doubly stochastic matrix } D, \text{ i.e. } \sum_{k=1}^n D_{i,k} = \sum_{k=1}^n D_{k,j} = 1, \forall i, j$$

$$\iff \mathbf{x} = P_1 \cdots P_k \mathbf{y} \text{ for finitely some (Muirhead)-Pigou-Dalton transfert matrices } P_j \\ (P_j = \alpha \mathbb{I} + (1 - \alpha)T \text{ for some } \alpha \in (0, 1) \text{ and } T = 0 \text{ except } T_{i,j} = T_{j,i} = 1)$$

# Risk Aversion and Risk Sharing

**Def** Consider two random variables  $X$  and  $Y$ ,  $X \preceq_{CX} Y$  if  $\mathbb{E}[h(X)] \leq \mathbb{E}[h(Y)]$  for any convex function  $h$

$\iff Y$  is a mean-preserving spread of  $X$ , i.e.  $Y \stackrel{\mathcal{L}}{=} X + Z$ , where  $\mathbb{E}[Z|X] = 0$ .

$\iff \mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+]$  for all  $s \in \mathbb{R}$ .

$\implies \mathbb{E}[X] = \mathbb{E}[Y]$  and  $\text{Var}[X] \preceq \text{Var}[Y]$ .

**Prop** (6 in Kaas et al. (2002)) Let  $\mathbf{X} = (X_1, \dots, X_n)$  denote a collection of variables and  $\mathbf{X}^+$  a comonotonic version, then  $\mathbf{a}^\top \mathbf{X} \preceq_{CX} \mathbf{a}^\top \mathbf{X}^+$ .

**Prop** (3.4.48 in Denuit et al. (2005)) Let  $\mathbf{X} = (X_1, \dots, X_n)$  denote a collection of i.i.d. variables, if  $\mathbf{a} \prec \mathbf{b}$  for the majorization order,  $\mathbf{a}^\top \mathbf{X} \preceq_{CX} \mathbf{b}^\top \mathbf{X}$ .

As a consequence

**Prop** Let  $\mathbf{X} = (X_1, \dots, X_n)$  denote a collection of i.i.d. variables, and  $\mathbf{p}$  some  $n$ -dimensional probability vector. Then  $\mathbf{p}^\top \mathbf{X} \preceq_{CX} X_i$  for any  $i$ .

# Risk Aversion and Risk Sharing

**Def** Consider two random vectors  $\xi = (\xi_1, \dots, \xi_n)$  and  $\mathbf{X} = (X_1, \dots, X_n)$  on  $\mathbb{R}_+^n$ .  $\xi$  is a risk-sharing scheme of  $\mathbf{X}$  if  $X_1 + \dots + X_n = \xi_1 + \dots + \xi_n$  almost surely.

Following [Denuit and Dhaene \(2012\)](#) and [Carlier et al. \(2012\)](#),

**Def** Consider two random vectors  $\xi = (\xi_1, \dots, \xi_n)$  and  $\mathbf{X} = (X_1, \dots, X_n)$  on  $\mathbb{R}_+^n$ .  $\xi \preceq_{CCX} \mathbf{X}$  if  $\xi_i \preceq_{CX} X_i$ .

**Def** A risk sharing scheme  $\xi$  of  $\mathbf{X}$  is desirable if  $\xi \preceq_{CCX} \mathbf{X}$ .

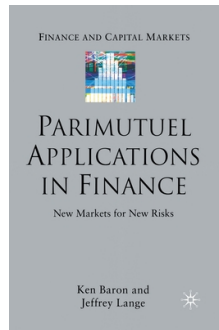
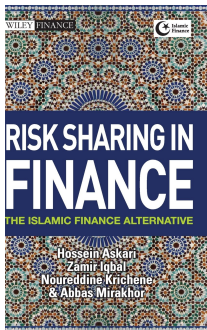
$$\text{Let } \xi_j = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i, \quad \forall j$$

- ▶ Risk sharing :  $\xi_1 + \dots + \xi_n = X_1 + \dots + X_n$
- ▶ Desirable (componentwise convex-order) :  $\xi_j \preceq_{CX} X_j, \quad \forall j$

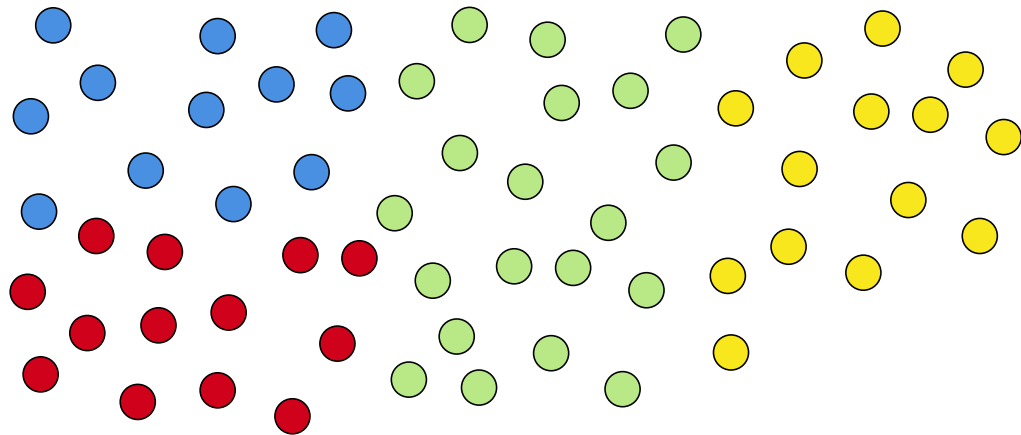
# Risk Sharing

Peer-to-peer insurance is a risk sharing network where a group of individuals pool their premiums together to insure against a risk. Peer-to-Peer Insurance mitigates the conflict that inherently arises between a traditional insurer and a policyholder when an insurer keeps the premiums that it doesn't pay out in claims

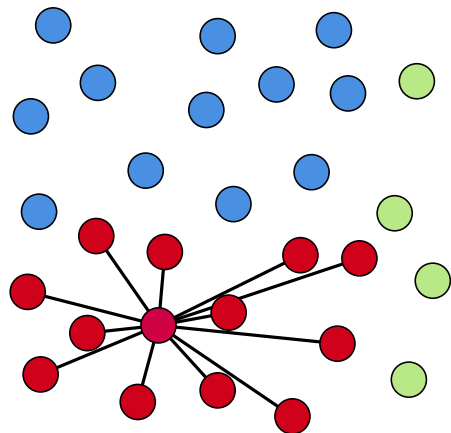
- ▶ Takaful التكافل
- ▶ Wakalah وَكَالَة
- ▶ Musharakah مُشَارَكَة
- ▶ Xiang Hu Bao 相互保
- ▶ Parimutuel



## Risk Sharing



# Risk Sharing



$$\text{Let } \xi_j = \frac{1}{n} \sum_{i=1}^n X_i, \forall j$$

► Risk sharing

$$\xi_1 + \dots + \xi_n = X_1 + \dots + X_n$$

► Componentwise convex-order

$$\xi_j \preceq_{CX} X_j, \forall j$$

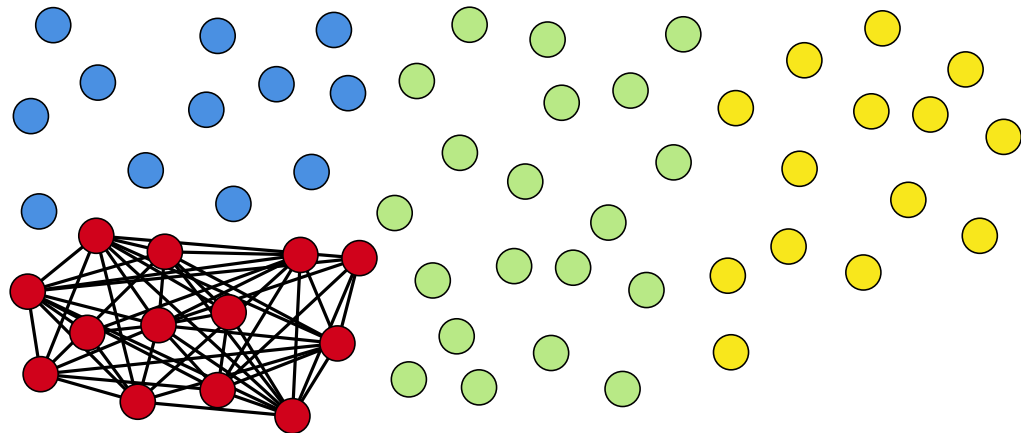
More generally, consider some linear risk sharing  $\xi = M\mathbf{X}$ , for some  $n \times n$  matrix

$$M = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{M}_k \end{bmatrix}, \quad \mathbf{M}_k = \frac{1}{n_k} \mathbf{1}_k$$

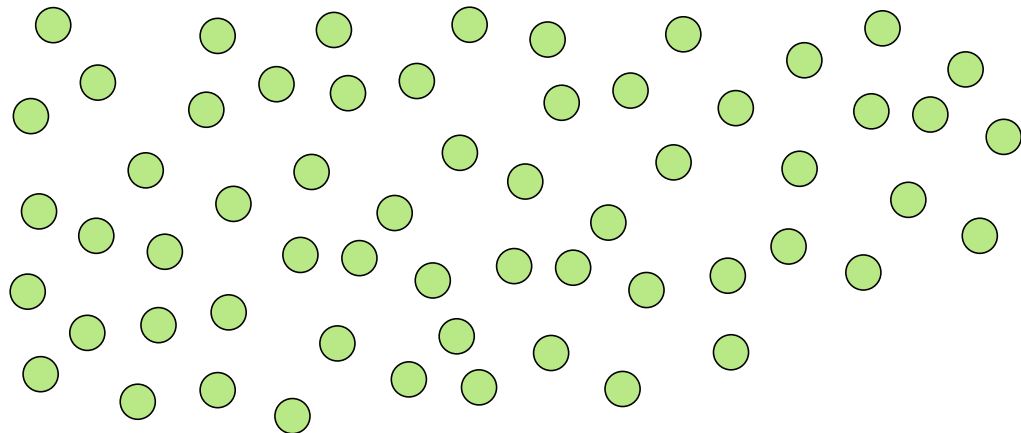
where  $\mathbf{1}_k$  is the  $n_k \times n_k$  matrix full of 1's.



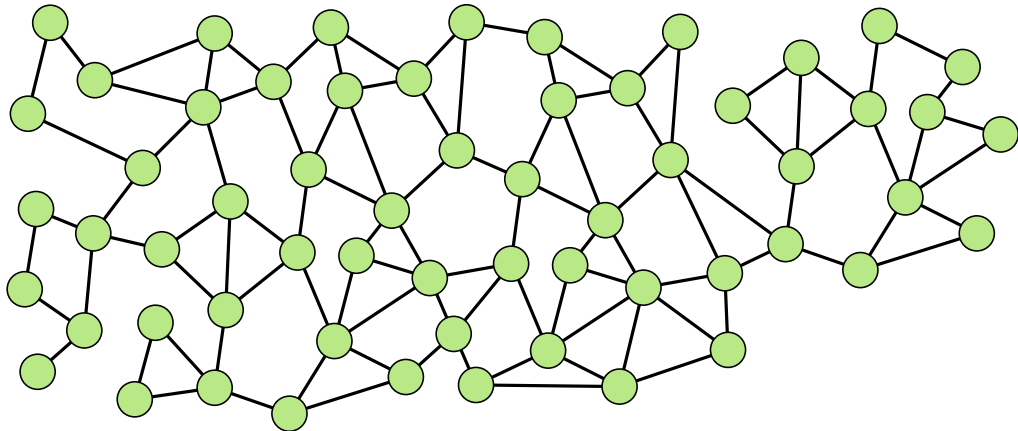
# Risk Sharing



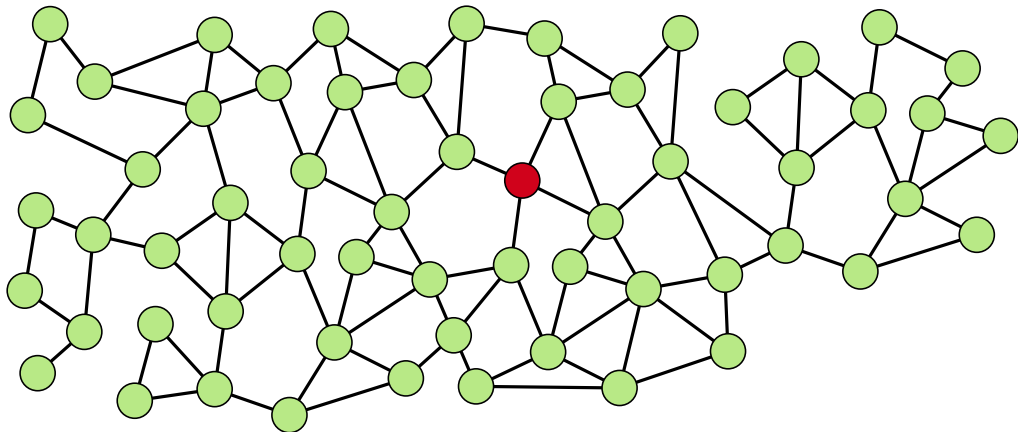
# Risk Sharing



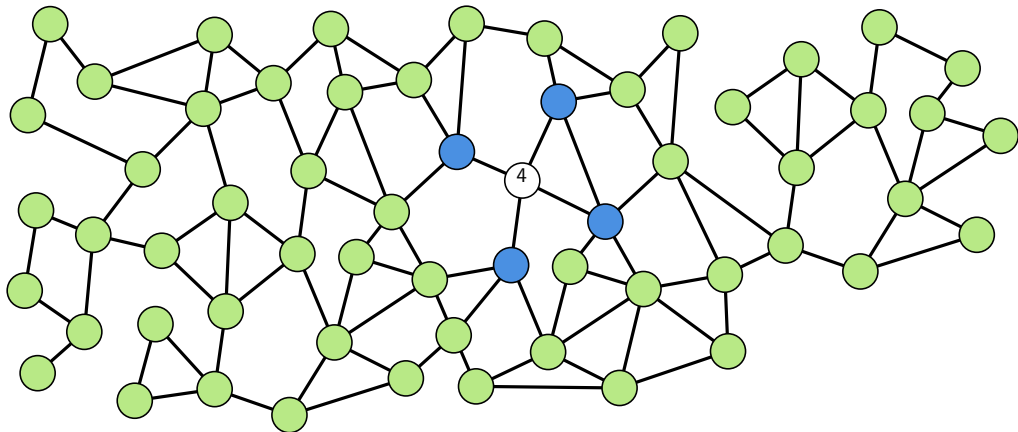
## Risk Sharing on a Network



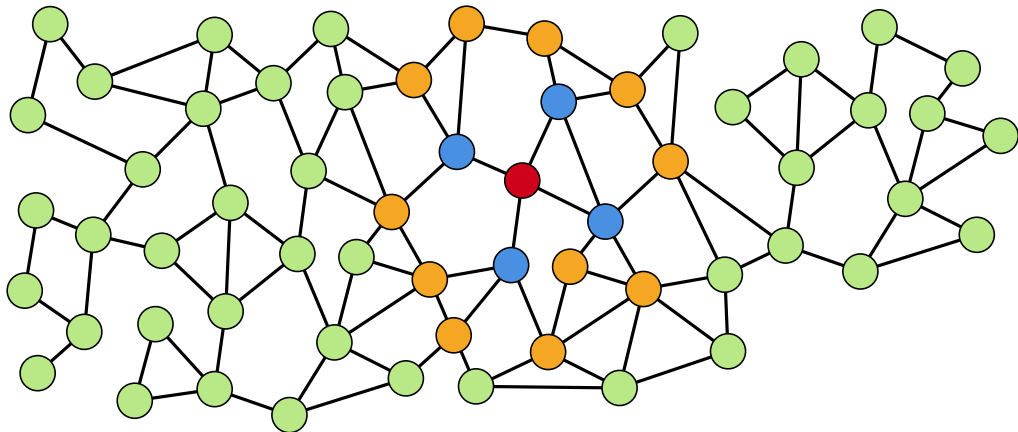
## Risk Sharing on a Network



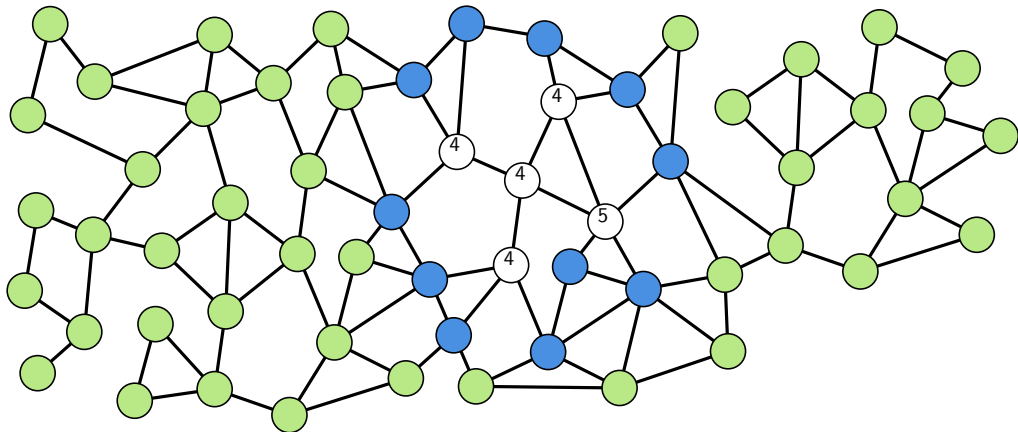
## Risk Sharing on a Network



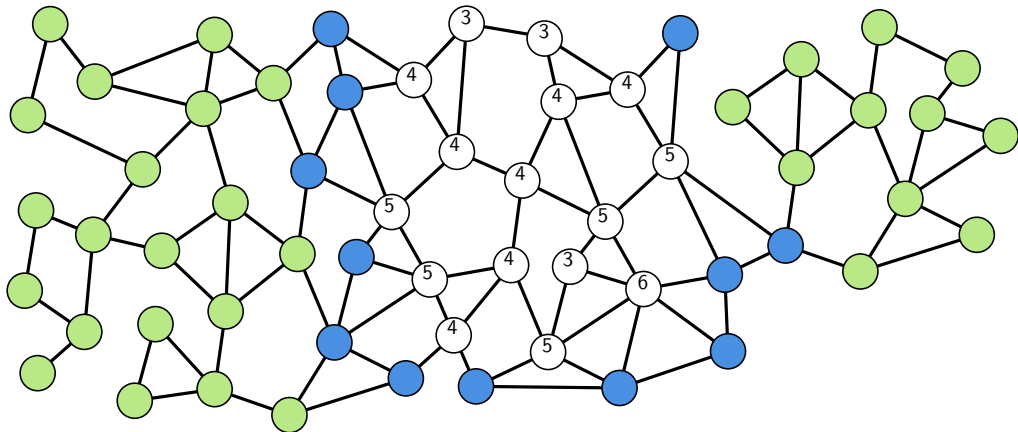
## Risk Sharing on a Network



# Risk Sharing on a Network

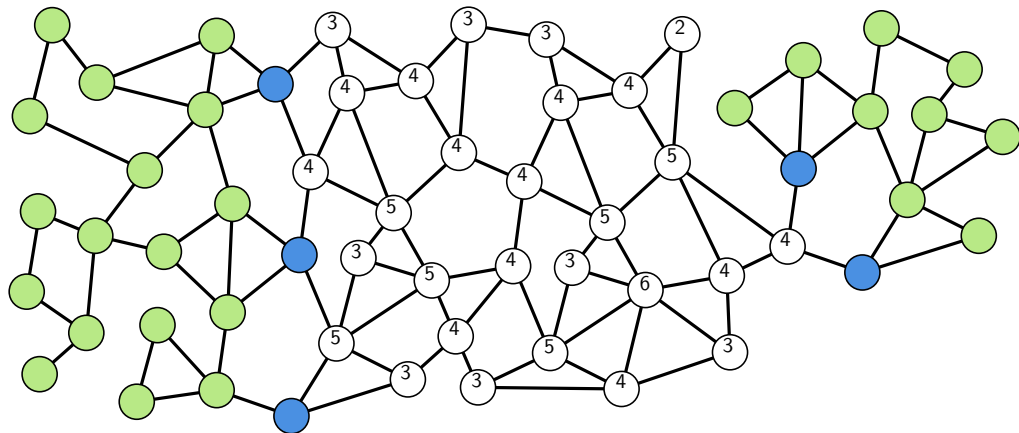


# Risk Sharing on a Network

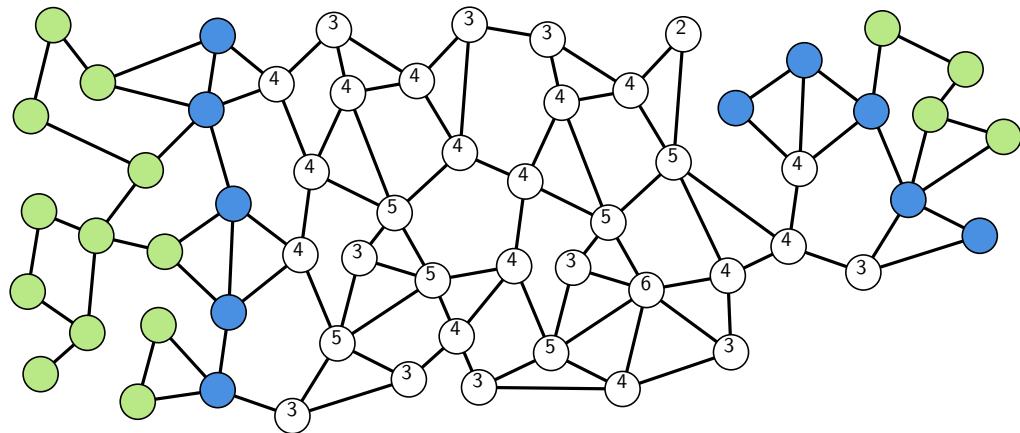




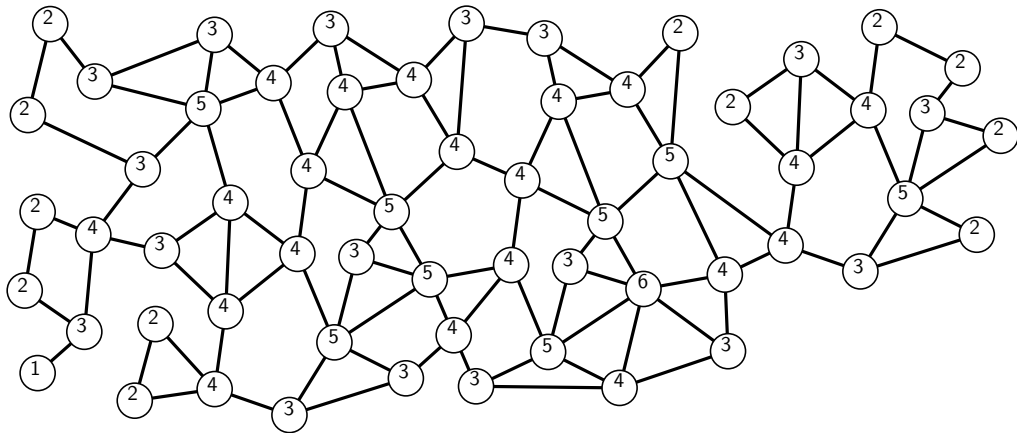
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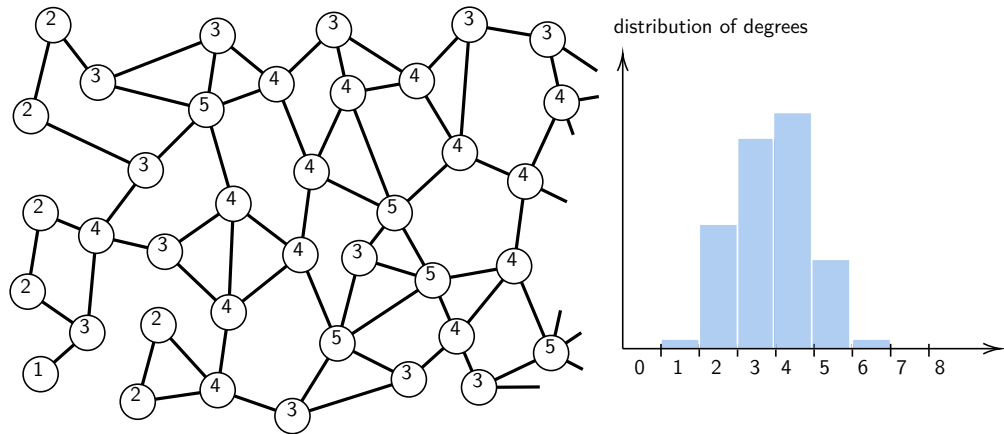
# Risk Sharing on a Network



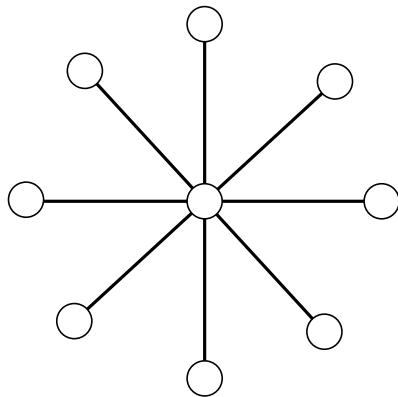
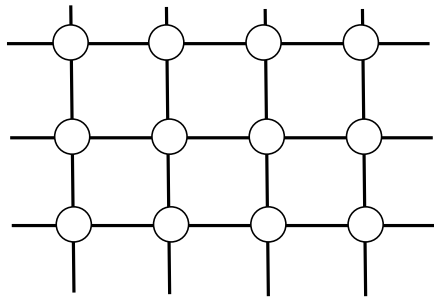
# Risk Sharing on a Network



# Risk Sharing on a Network



## Risk Sharing on a Network



Regular graph vs. star shaped graph  
(low variance vs. large variance on  $D$ )

# Linear Risk Sharings

**Def** Consider two random vectors  $\xi = (\xi_1, \dots, \xi_n)$  and  $\mathbf{X} = (X_1, \dots, X_n)$  on  $\mathbb{R}_+^n$ , such that  $\xi$  is a risk-sharing scheme of  $\mathbf{X}$ , i.e.  $\xi^\top \mathbf{1} = \mathbf{X}^\top \mathbf{1}$ . It is said to be a linear risk sharing scheme if there exists a matrix  $M$ ,  $n \times n$ , with positive entries, such that  $\xi = M\mathbf{X}$ , almost surely.

**Ex**  $\bar{M} = [\bar{M}_{i,j}]$  with  $\bar{M}_{i,j} = 1/n$ .

**Ex**  $M_\alpha = \alpha \mathbb{I} + (1 - \alpha) \bar{M}$ .

**Ex**  $M$  is a doubly stochastic matrix, (row and column conditions on  $D$  are called "zero-balance conservation" in [Feng et al. \(2021\)](#)).

**Prop** Consider two linear risk sharing schemes  $\xi_1$  and  $\xi_2$  of  $\mathbf{X}$ , such that there is a doubly stochastic matrix  $D$ ,  $n \times n$  such that  $\xi_2 = D\xi_1$ . Then  $\xi_2 \preceq_{\text{CCX}} \xi_1$ .

## Linear Risk Sharings within cliques

**Def** A network (or undirected graph) is a collection of vertices and edges,  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

**Def** A clique  $\mathcal{C}_i$  within an network  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  is a subset of vertices  $\mathcal{C}_i \subseteq \mathcal{V}$  such that every two distinct vertices are adjacent.

Thus, the induced subgraph of  $\mathcal{G}$  by  $\mathcal{C}_i$  is complete.

**Def** A clique cover is a partition of the graph  $\mathcal{G}$  into a set of cliques.

Assume that policyholders face random losses  $\mathbf{X} = (X_1, \dots, X_k, X_{k+1}, \dots, X_{k+m})$  and consider the following risk sharing

$$\xi_i = \begin{cases} \frac{1}{k} \sum_{j=1}^k X_j & \text{if } i \in \{1, 2, \dots, k\} \\ \frac{1}{m} \sum_{j=k+1}^{k+m} X_j & \text{if } i \in \{k+1, k+2, \dots, k+m\} \end{cases}$$

## Linear Risk Sharings within cliques

This is a linear risk sharing, with sharing matrix  $D_{k,m}$ , so that  $\xi = D_{k,m}\mathbf{X}$ , defined as

$$D_{k,m} = \begin{matrix} & \begin{matrix} 1 & \dots & k & k+1 & k+2 & \dots & k+m \end{matrix} \\ \begin{matrix} 1 \\ \vdots \\ k \\ k+1 \\ k+2 \\ \vdots \\ k+m \end{matrix} & \begin{pmatrix} k^{-1} & \dots & k^{-1} & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ k^{-1} & \dots & k^{-1} & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & m^{-1} & m^{-1} & \dots & m^{-1} \\ 0 & \dots & 0 & m^{-1} & m^{-1} & \dots & m^{-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & m^{-1} & m^{-1} & \dots & m^{-1} \end{pmatrix} \end{matrix} = \begin{bmatrix} \mathbf{M}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_m \end{bmatrix},$$

where  $\mathbf{M}_k = \frac{1}{k}\mathbf{1}_k$ . Since  $D_{k,m}$  is a doubly-stochastic matrix,  $\xi \prec_{CCX} \mathbf{X}$ .



## Linear Risk Sharings within cliques

Pick someone at random (let  $I$  denote a uniform variable over  $\{1, 2, \dots, n\}$ , and  $\xi' = \xi_I$ ), with i.i.d. losses  $X_i$

$$\mathbb{E}[\xi'] = \mathbb{E}[\mathbb{E}[\xi_I|I]] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \mathbb{E}[X]$$

$$\begin{aligned} \text{Var}[\xi'] &= \mathbb{E}[\text{Var}[\xi_I|I]] = \frac{1}{n} \sum_{i=1}^n \text{Var}[\xi_i] = \frac{1}{n} \left( k \frac{\text{Var}[X]}{k^2} + m \frac{\text{Var}[X]}{m^2} \right) \\ &= \frac{\text{Var}[X]}{n} \left( \frac{1}{k} + \frac{1}{m} \right) = \frac{\text{Var}[X]}{k(n-k)} < \text{Var}[X] \end{aligned}$$

$k \mapsto k(n-k)$  is maximal when  $k = \lfloor n/2 \rfloor$ , which means that risk sharing benefit is maximal (socially maximal, for a randomly chosen representative policyholder) when the two cliques have the same size.

## A Weaker Linear Risk Sharing

Following [Martínez Pería et al. \(2005\)](#), a weaker ordering can be considered with row-stochastic matrices, instead of doubly stochastic matrices. A row-stochastic matrix (or left-stochastic matrix)  $C$  satisfies

$$R = [R_{i,j}] \text{ where } R_{i,j} \geq 0, \text{ and } \sum_{i=1}^n R_{i,j} = 1 \ \forall j.$$

**Prop** Let  $R$  be some  $n \times n$  row-stochastic matrix, and given  $\mathbf{X}$ , a positive vector in  $\mathbb{R}^+$ , define  $\xi = R\mathbf{X}$ . Then  $\xi$  is a linear risk sharing of  $\mathbf{X}$ .

Following [Martínez Pería et al. \(2005\)](#), we can define an ordering based those row-stochastic matrices

**Def** Consider two linear risk sharing schemes  $\xi_1$  and  $\xi_2$  of  $\mathbf{X}$ .  $\xi_1$  weakly dominates  $\xi_2$ , denoted  $\xi_2 \preceq_{wCX} \xi_1$  if and only if there is a column-stochastic matrix  $R$ ,  $n \times n$  such that  $\xi_2 = R\xi_1$ .

## Linear Risk Sharings with Friends

**Def** Given  $\mathbf{b} \in \mathbb{R}^m$  and  $A \in \mathbb{R}^{m \times n}$ ,

$$\mathbf{b} \otimes A = \begin{bmatrix} b_1 \mathbf{a}_{1\cdot}^\top \\ b_2 \mathbf{a}_{2\cdot}^\top \\ \vdots \\ b_m \mathbf{a}_{m\cdot}^\top \end{bmatrix} = \begin{bmatrix} b_1 a_{11} & b_1 a_{12} & \cdots & b_1 a_{1n} \\ b_2 a_{21} & b_2 a_{22} & \cdots & b_2 a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_m a_{m1} & b_m a_{m2} & \cdots & b_m a_{mn} \end{bmatrix}$$

With cliques, we had

$$M = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{M}_k \end{bmatrix}, \quad \mathbf{M}_k = \frac{1}{n_k} \mathbf{1}_k \text{ or } M = \begin{bmatrix} \mathbf{n}_1^{-1} \\ \mathbf{n}_2^{-1} \\ \vdots \\ \mathbf{n}_k^{-1} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{1}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_k \end{bmatrix},$$

where  $\mathbf{1}_k$  is the  $n_k \times n_k$  matrix full of 1's,  $\mathbf{n}_k = (n_k, \cdots, n_k)$ .

## Linear Risk Sharings with Friends

But we can consider risk sharings based on more general networks,

**Lemma** Given an adjacency matrix  $A$  of an undirected graph, if  $\mathbf{d}$  denotes the vector of degrees  $\mathbf{d} = A\mathbf{1}$ , then  $M = \mathbf{d}^{-1} \otimes A$  is a row-stochastic matrix.

$$\text{Ex } A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} . \text{ and } M = \mathbf{d}^{-1} \otimes A = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} .$$

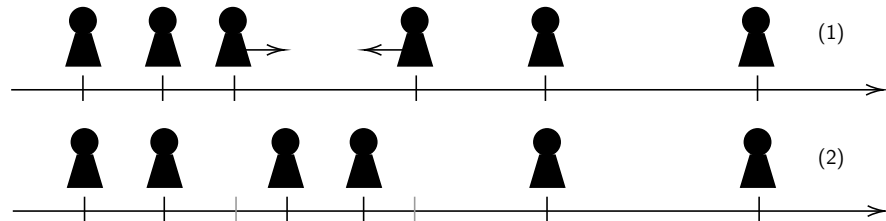
**Def** Define  $\mathcal{I}_n = \{1, 2, \dots, n\}$ , let  $\mathcal{G}_A$  denote the network  $\mathcal{G}_A = (\mathcal{I}_n, \mathcal{E}_A)$ , associated with adjacency matrix  $A$ .

**Def** Given a network  $\mathcal{G}_A = (\mathcal{I}_n, \mathcal{E}_A)$ , the linear risk sharing  $\xi$  of  $\mathbf{x} \in \mathbb{R}_+^n$ , through  $\mathcal{G}_A$  is  $\xi_A = \mathbf{d}^{-1} \otimes A\mathbf{x}$ ,

$$\xi_A = \mathbf{d}^{-1} \otimes A\mathbf{x} = \text{i.e. } \xi_j = \frac{1}{d_j} A_j^\top \mathbf{x} = \frac{1}{d_j} \sum_{i \in \mathcal{V}_j} x_i .$$

# Linear Risk Sharings with Friends

# Stochastic Dominance and Networks



(Muirhead)-Pigou-Dalton transferts ([Muirhead \(1903\)](#)[Dalton \(1920\)](#))  
see also [Atkinson \(2015\)](#),

$$\mathbf{y}^{(2)} \preceq_M \mathbf{y}^{(1)} \leftarrow \begin{cases} y_i^{(2)} = y_i^{(1)}, \forall i \neq j, k \\ y_j^{(2)} = y_j^{(1)} + h, \\ y_k^{(2)} = y_k^{(1)} - h, y_j^{(2)} > y_j^{(1)} \end{cases}$$

see martingale property of mean-preserving spread,  $Y^{(1)} \stackrel{\mathcal{L}}{=} Y^{(2)} + Z$ , where  $\mathbb{E}[Z|Y^{(2)}] = 0$  (convex order is a dispersion order)

## From Marshall and Olkin (1979),

**B.2. Theorem** (Hardy, Littlewood, and Pólya, 1929). A necessary and sufficient condition that  $x \prec y$  is that there exist a doubly stochastic matrix  $P$  such that  $x = yP$ .

The lemma involves a special kind of linear transformation called a *T-transformation*, or more briefly a *T-transform*. The matrix of a *T-transform* has the form

$$T = \lambda I + (1 - \lambda)Q,$$

where  $0 \leq \lambda \leq 1$  and  $Q$  is a permutation matrix that just interchanges two coordinates. Thus  $xT$  has the form

$$xT = (x_1, \dots, x_{j-1}, \lambda x_j + (1 - \lambda)x_k, x_{j+1}, \dots, x_{k-1}, \\ \lambda x_k + (1 - \lambda)x_j, x_{k+1}, \dots, x_n).$$

**B.1. Lemma** (Muirhead, 1903; Hardy, Littlewood, and Pólya, 1934, 1952, p. 47). If  $x \prec y$ , then  $x$  can be derived from  $y$  by successive applications of a finite number of *T-transforms*.

## D Majorization in Integers

Consider the basic Lemma 2.B.1, which states that if  $x \prec y$ , then  $x$  can be derived from  $y$  by successive applications of a finite number of “*T-transforms*.” Recall that a *T-transform* leaves all but two components of a vector unchanged, and replaces these two components by averages. If  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  are integers and  $a \prec b$ , can  $a$  be derived from  $b$  by successive applications of a finite number of *T-transforms* in such a way that after the application of each *T-transform* a vector with integer components is obtained? An affirmative answer was given by Muirhead (1903) and by Folkman and Fulkerson (1969). Using the same term as Dalton (1920), Folkman and Fulkerson (1969) define an operation called a *transfer*. If  $b_1 \geq \dots \geq b_n$  are integers and  $b_i > b_j$ , then the transformation

$$\begin{aligned} b'_i &= b_i - 1, \\ b'_j &= b_j + 1, \\ b'_k &= b_k, \quad k \neq i, j, \end{aligned}$$

is called a *transfer from  $i$  to  $j$* . This transfer is a *T-transform*, because

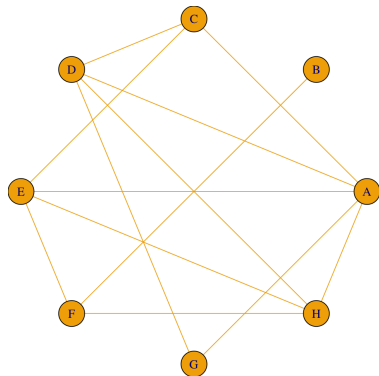
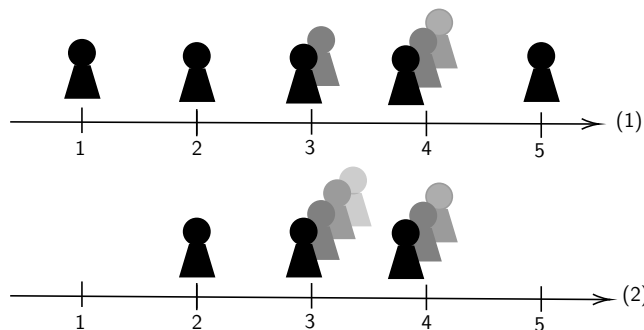
$$b'_i = \alpha b_i + (1 - \alpha)b_j, \quad b'_j = (1 - \alpha)b_i + \alpha b_j,$$

where  $\alpha = (b_i - b_j - 1)/(b_i - b_j)$ .

**D.1. Lemma** (Muirhead, 1903). If  $a_1, \dots, a_n, b_1, \dots, b_n$  are integers and  $a \prec b$ , then  $a$  can be derived from  $b$  by successive applications of a finite number of transfers.

# Stochastic Dominance and Networks

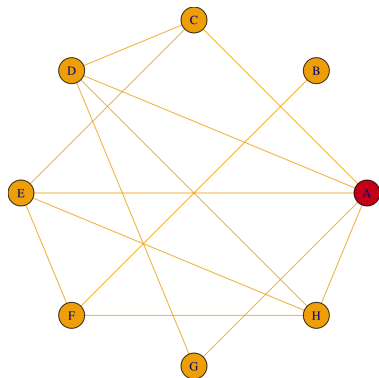
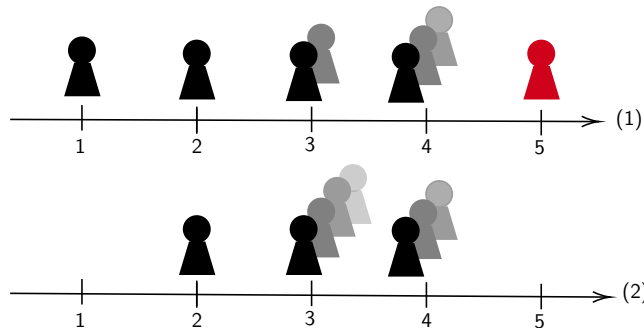
$$\mathbf{y}^{(1)} = (5, 4, 4, 4, 3, 3, 2, 1) \rightarrow (4, 4, 4, 3, 3, 3, 3, 2) = \mathbf{y}^{(2)}$$





# Stochastic Dominance and Networks

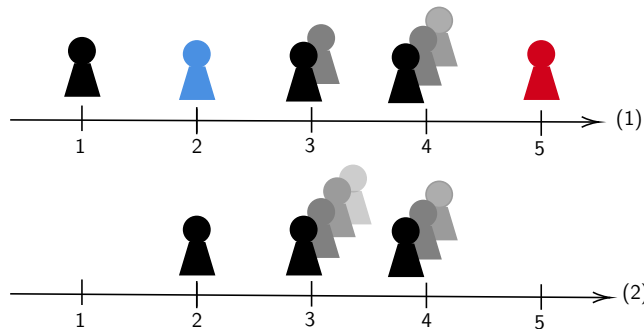
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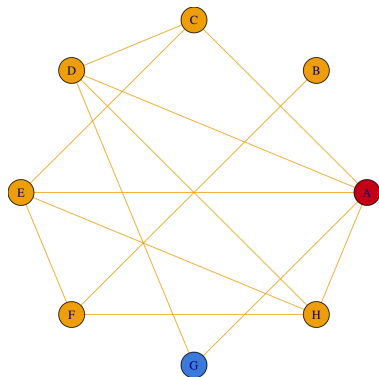
Find largest  $j$  such that  $\sum_{i=1}^j y_i^{(1)} < \sum_{i=1}^j y_i^{(2)}$

# Stochastic Dominance and Networks

$$\mathbf{y}^{(1)} = (5, 4, 4, 4, 3, 3, 2, 1) \rightarrow (4, 4, 4, 3, 3, 3, 3, 2) = \mathbf{y}^{(2)}$$

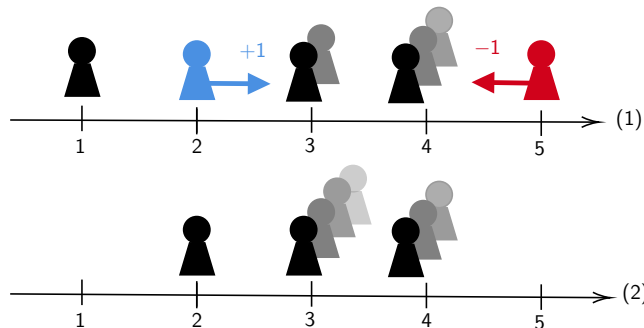


Find largest  $k < j$  such that  $y_k^{(2)} > y_k^{(1)} > y_{j+1}^{(2)} > y_{j+1}^{(1)}$

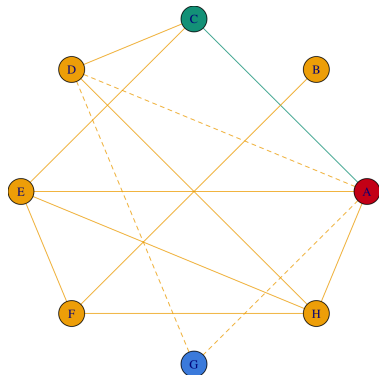


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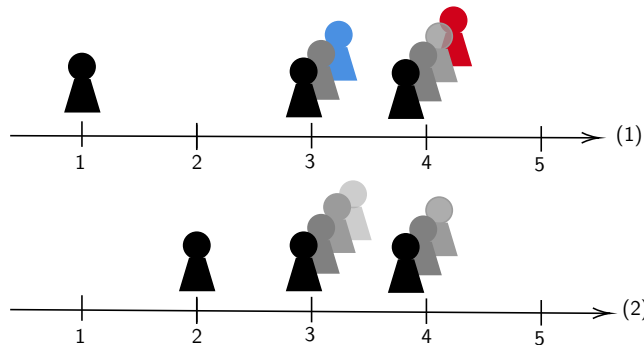


To get a transfert, we need to select  $\ell$

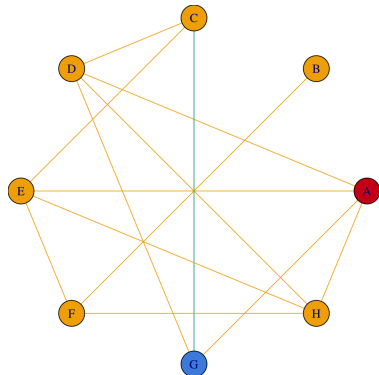


# Stochastic Dominance and Networks

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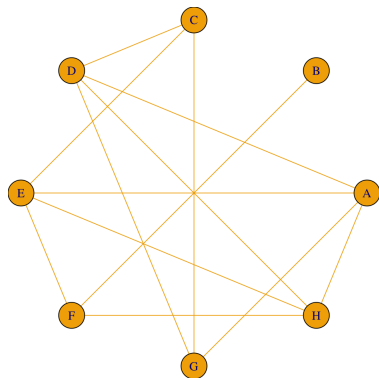
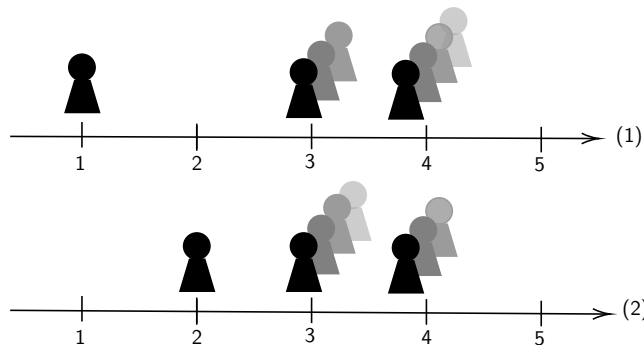


Remove edge  $(\ell, j)$  and create  $(\ell, k)$



# Stochastic Dominance and Networks

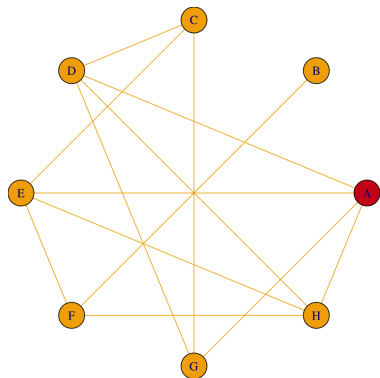
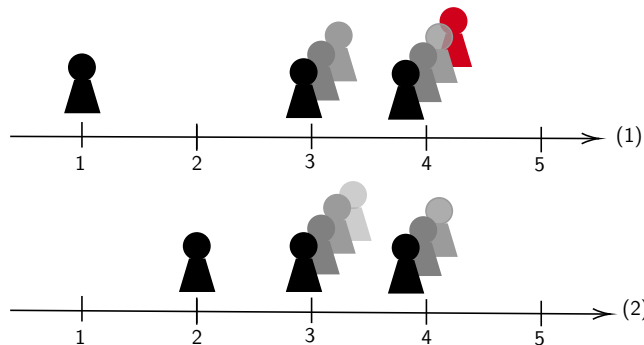
$$\mathbf{y}^{(1)} = (5, 4, 4, 4, 3, 3, 2, 1) \rightarrow (4, 4, 4, 3, 3, 3, 3, 2) = \mathbf{y}^{(2)}$$



... we're back at stage 1...

# Stochastic Dominance and Networks

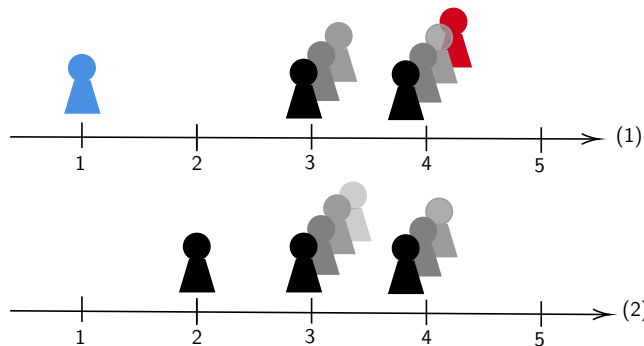
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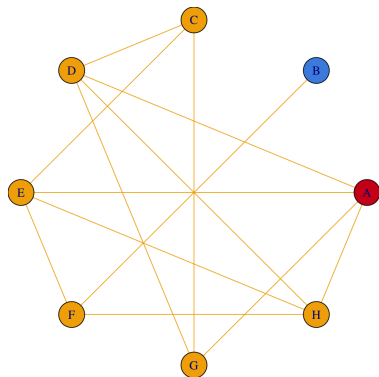
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# Stochastic Dominance and Networks

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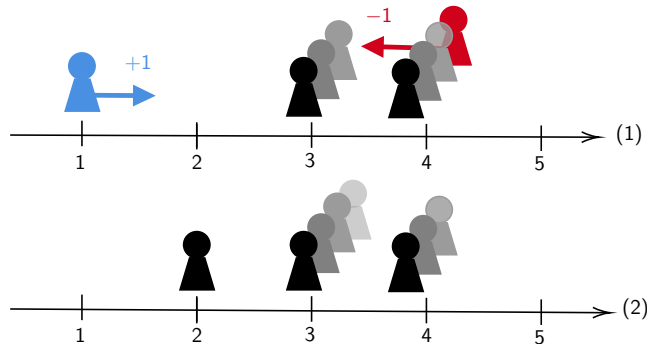


Find largest  $k < j$  such that  $y_k^{(2)} > y_k^{(1)} > y_{j+1}^{(2)} > y_{j+1}^{(1)}$

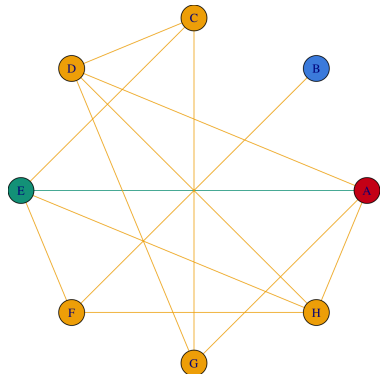


# Stochastic Dominance and Networks

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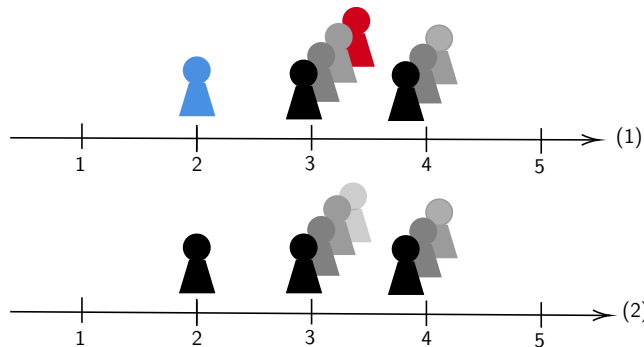
To get a transfert, we need to select  $\ell$



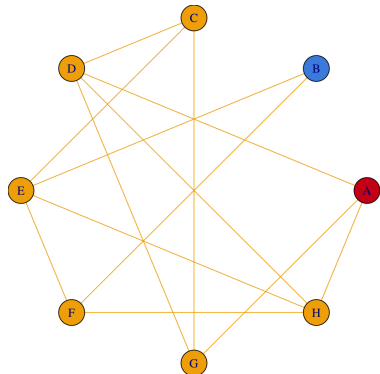


# Stochastic Dominance and Networks

$$\mathbf{y}^{(1)} = (5, 4, 4, 4, 3, 3, 2, 1) \rightarrow (4, 4, 4, 3, 3, 3, 3, 2) = \mathbf{y}^{(2)}$$

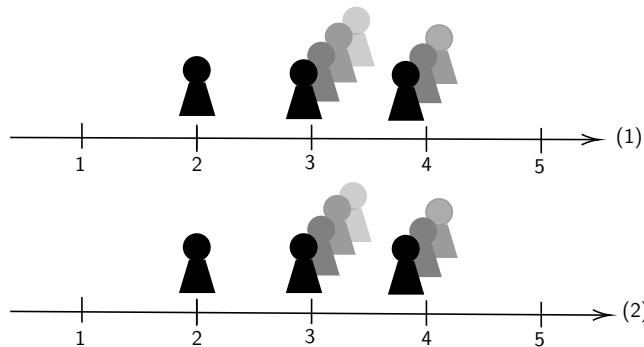


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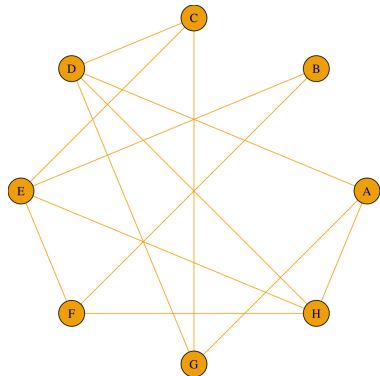


# Stochastic Dominance and Networks

$$\mathbf{y}^{(1)} = (5, 4, 4, 4, 3, 3, 2, 1) \rightarrow (4, 4, 4, 3, 3, 3, 3, 2) = \mathbf{y}^{(2)}$$



Remove edge  $(\ell, j)$  and create  $(\ell, k)$



# Stochastic Dominance and Networks

$$A_1 = \begin{bmatrix} . & . & 1 & 1 & 1 & . & 1 & 1 \\ . & . & . & . & . & 1 & . & . \\ 1 & . & . & 1 & 1 & . & . & . \\ 1 & . & 1 & . & . & 1 & . & 1 \\ . & 1 & . & . & 1 & . & . & 1 \\ 1 & . & . & 1 & . & . & . & . \\ 1 & . & . & 1 & 1 & 1 & . & . \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \\ 4 \\ 4 \\ 3 \\ 2 \\ 4 \end{bmatrix}, A_2 = \begin{bmatrix} . & . & . & 1 & . & . & 1 & 1 \\ . & . & . & . & 1 & 1 & . & . \\ . & . & . & 1 & 1 & . & 1 & . \\ 1 & . & 1 & . & . & . & 1 & 1 \\ . & 1 & 1 & . & . & 1 & . & 1 \\ . & 1 & . & . & 1 & . & . & 1 \\ 1 & . & 1 & 1 & . & . & . & . \\ 1 & . & . & 1 & 1 & 1 & . & . \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \\ 4 \\ 4 \\ 3 \\ 3 \\ 4 \end{bmatrix}$$

Note that  $\mathbf{d}_2 = \mathbf{y}^{(2)} \preceq \mathbf{y}^{(1)} = \mathbf{d}_1$  and  $A_2 \preceq A_1$ .

# Linear Sharing Risks with Friends

Define  $\mathcal{I}_n = \{1, 2, \dots, n\}$ , consider two networks

$$\begin{cases} \mathcal{G}_1 = (\mathcal{I}_n, \mathcal{E}_1), \text{ with adjacency matrix } A_1, \text{ with degree vectors } \mathbf{d}_1 \\ \mathcal{G}_2 = (\mathcal{I}_n, \mathcal{E}_2), \text{ with adjacency matrix } A_2, \text{ with degree vectors } \mathbf{d}_2 \end{cases}$$

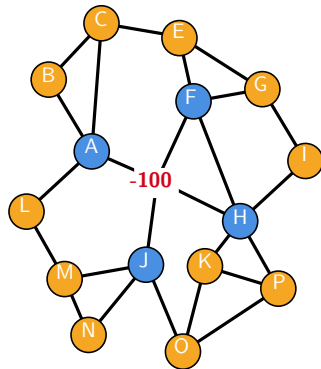
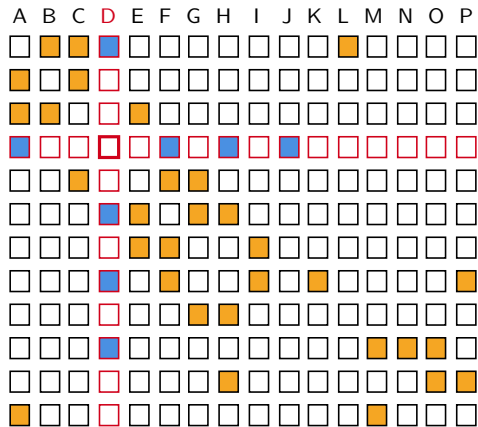
**Def** Consider two networks  $\mathcal{G}_1 = (\mathcal{I}_n, \mathcal{E}_1)$  and  $\mathcal{G}_2 = (\mathcal{I}_n, \mathcal{E}_2)$ , write  $\mathcal{G}_1 \preceq_D \mathcal{G}_2$  if  $\mathbf{d}_1 \preceq \mathbf{d}_2$

**Def** Consider two networks  $\mathcal{G}_1 = (\mathcal{I}_n, \mathcal{E}_1)$  and  $\mathcal{G}_2 = (\mathcal{I}_n, \mathcal{E}_2)$ , write  $\mathcal{G}_1 \preceq_A \mathcal{G}_2$  if  $A_1 \preceq A_2$

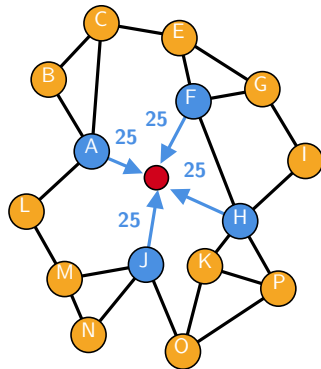
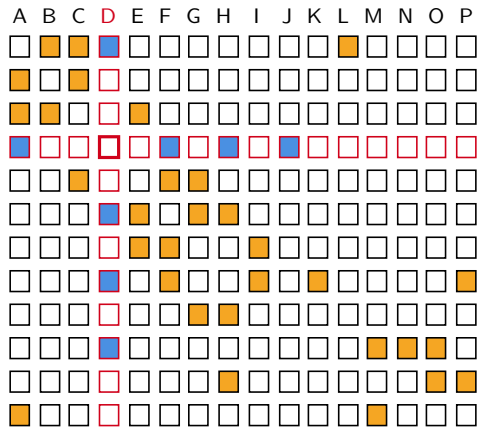
**Prop** If  $\mathcal{G}_1 \preceq_A \mathcal{G}_2$ , then  $\xi_{A_1} \preceq \xi_{A_2} \preceq \mathbf{x}$



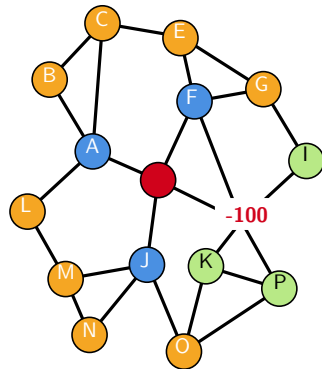
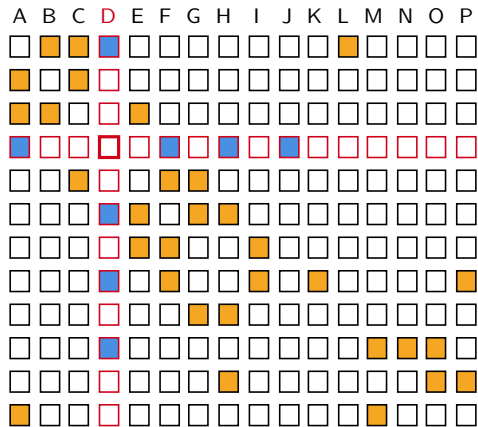
# Sharing Risks with Friends



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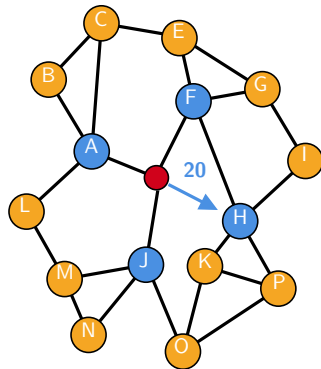
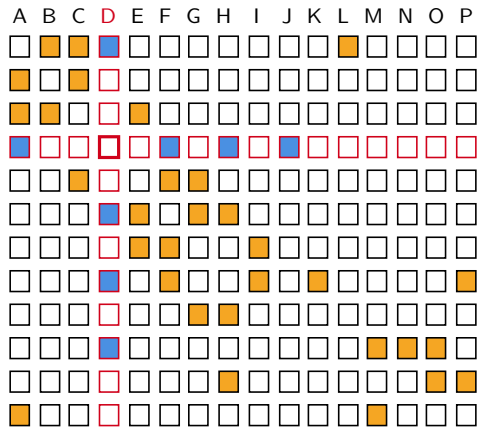


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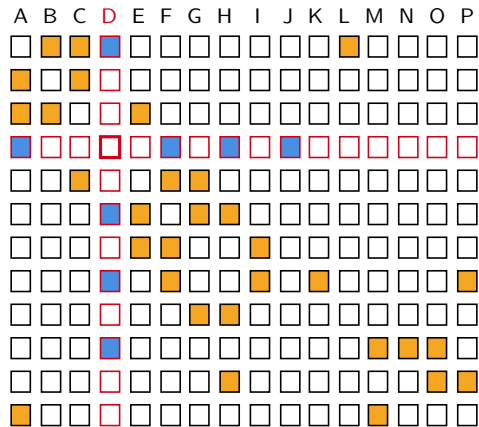




# Sharing Risks with Friends



# Sharing Risks with Friends



Looks like a linear risk sharing mechanism,

$$\xi = B\mathbf{X} \text{ a.s., where } B_{i,j} = A_{i,j}/d_i,$$

$A$  being the adjacency matrix of the network  
Here,  $B$  is a row-stochastic matrix.

But it suffers some drawbacks...

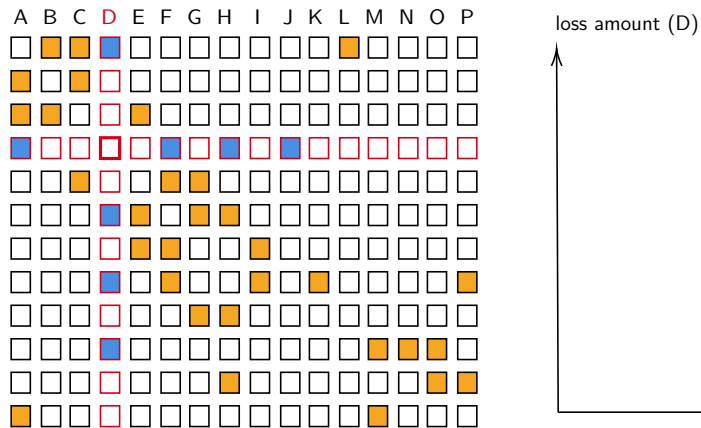
- ▶ need an upper bound
- ▶ unfairness ( $B_{i,i} = 0, \forall i$ )

(no longer "linear" risk sharing mechanism)

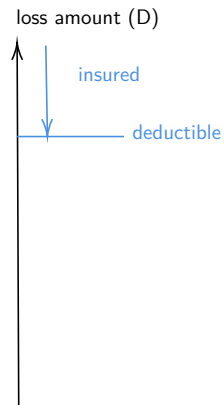
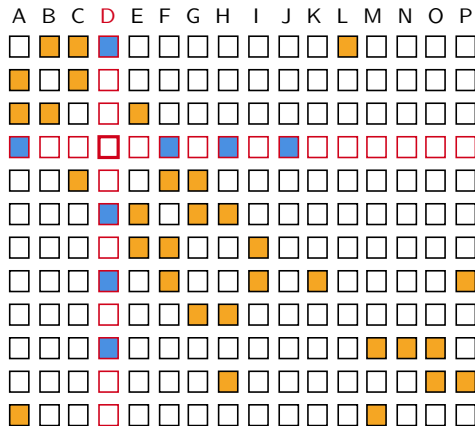




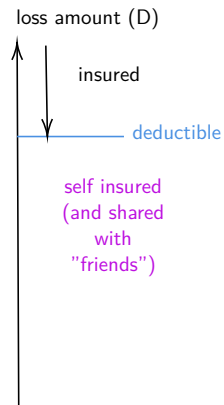
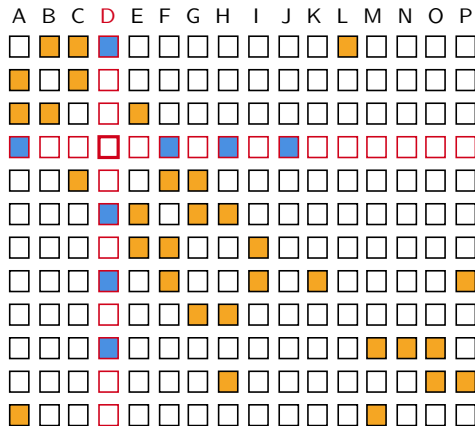
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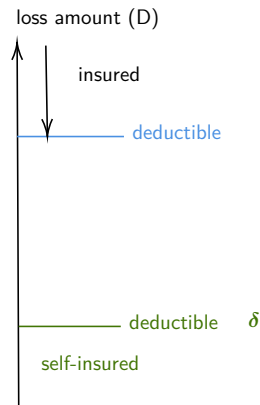
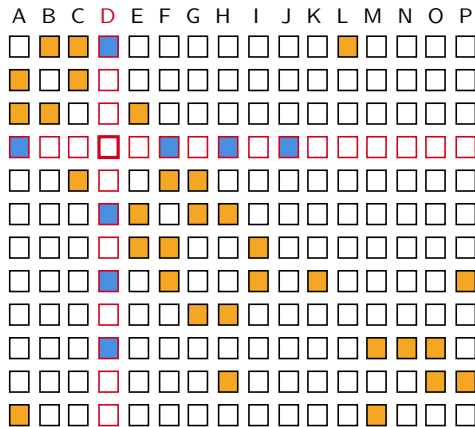
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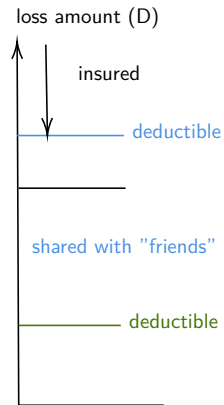
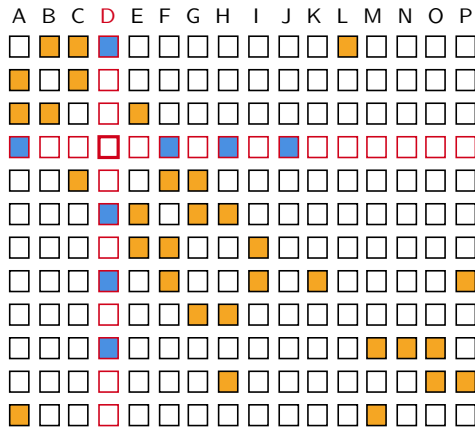


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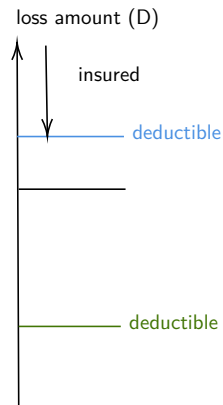
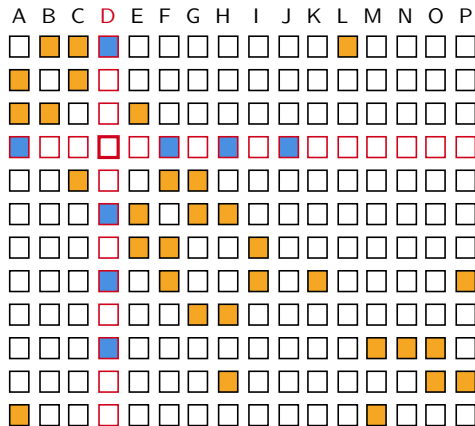




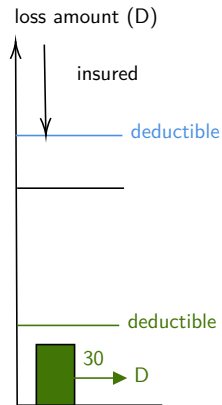
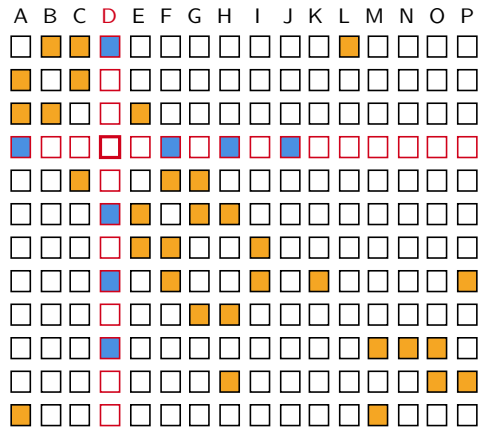
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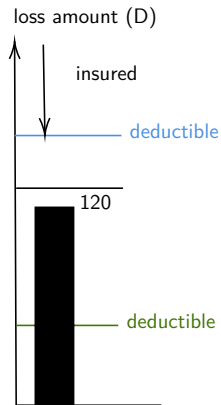
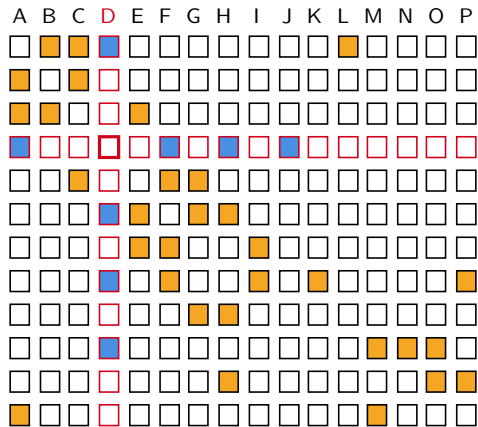
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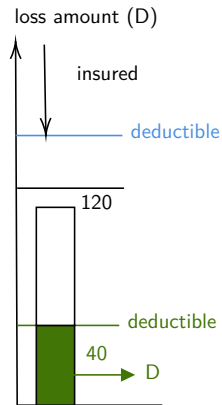
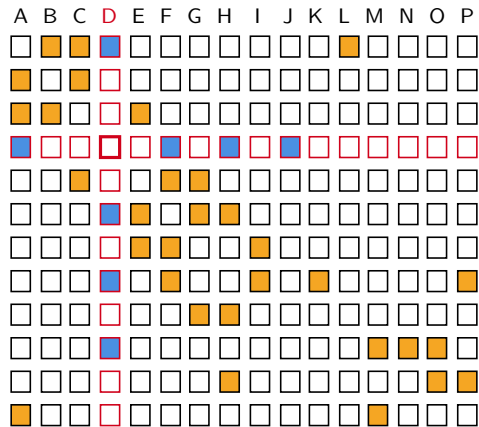
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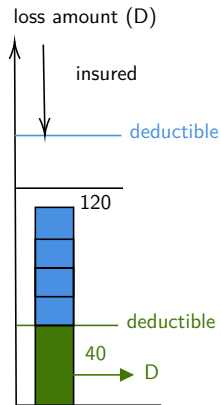
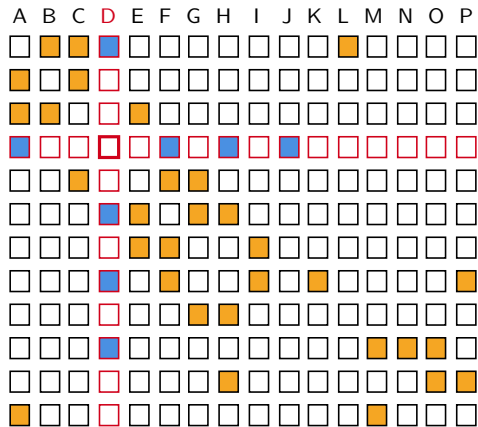
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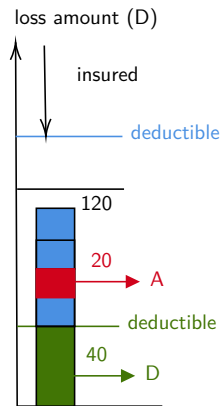
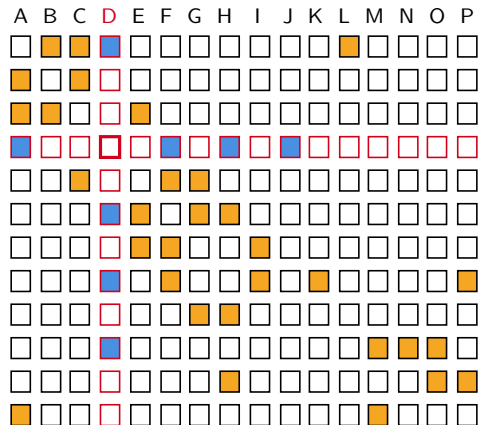
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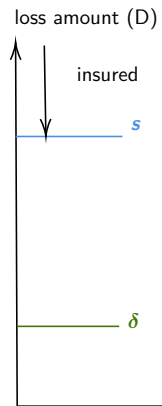
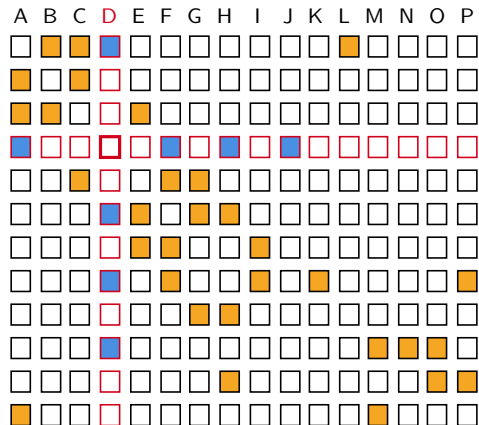
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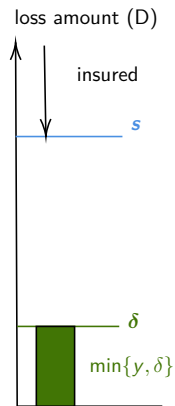
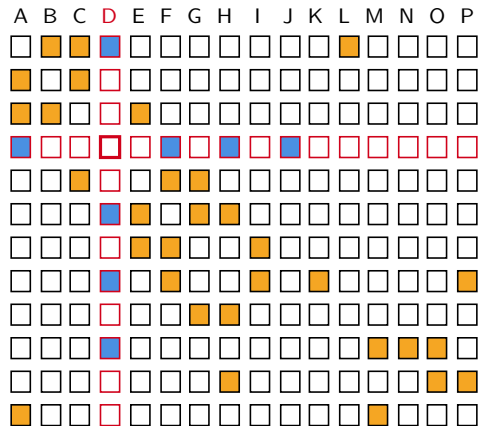


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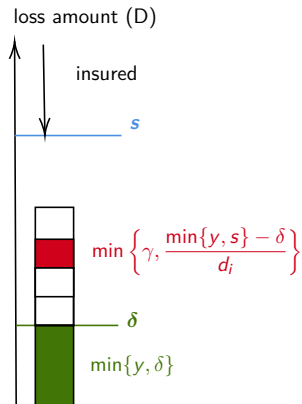
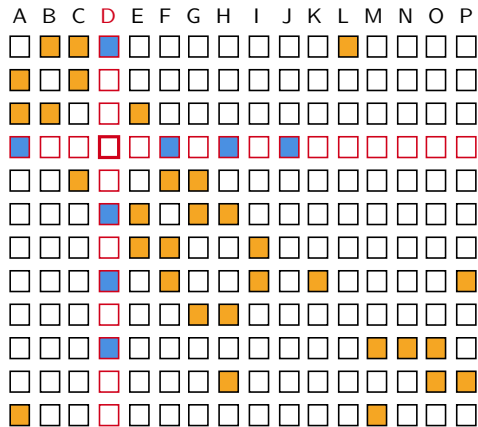




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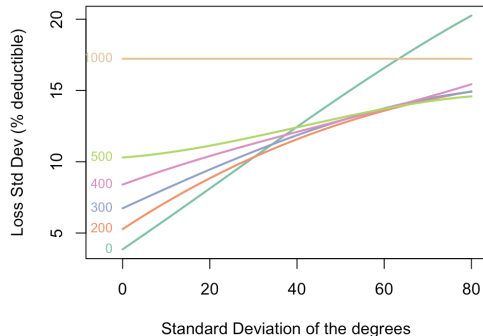
## Sharing Risks with Friends



# Sharing Risks with Friends

- ▶  $Y_i$  loss of insured  $i$ ,  $Z_i = \mathbf{1}(Y_i > 0)$
- ▶  $\mathcal{V}_i$  is the set of friends of insured  $i$ ,  $d_i = \text{Card}(\mathcal{V}_i)$
- ▶  $s$  deductible of insurance contracts
- ▶  $\gamma$  is the maximum amount shared between  $i$  and  $j$  (reciprocal contracts)

$$\begin{aligned} \xi_i = & Z_i \cdot \min\{s, Y_i\} \\ & + \sum_{j \in \mathcal{V}_i} Z_j \min \left\{ \gamma, \frac{\min\{s, Y_j\} - \delta}{d_j} \right\} \\ & - Z_i \cdot \min\{d_i \gamma, \min\{s, Y_i\} - \delta\} \end{aligned}$$



## Optimization\* of the Risk Sharing Mechanism

$$\begin{cases} \max \left\{ \sum_{(i,j) \in \mathcal{E}} \gamma_{(i,j)} \right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0, \gamma], \quad \forall (i,j) \in \mathcal{E} \\ \sum_{j \in \mathcal{V}_i} \gamma_{(i,j)} \leq s, \quad \forall i \in \mathcal{V} \end{cases}$$

Given losses  $\mathbf{X} = (X_1, \dots, X_n)$ , define contributions  $C_{i \rightarrow j}^* = \min \left\{ \frac{\gamma_{(i,j)}^*}{\sum_{i \in \mathcal{V}_j} \gamma_{(i,j)}^*} \cdot X_j, \gamma_{(i,j)}^* \right\}$ ,

and  $\xi_i^* = X_i + \sum_{j \in \mathcal{V}_i} [Z_j C_{i \rightarrow j}^* - Z_i C_{j \rightarrow i}^*]$  is a risk sharing, called optimal risk sharing.

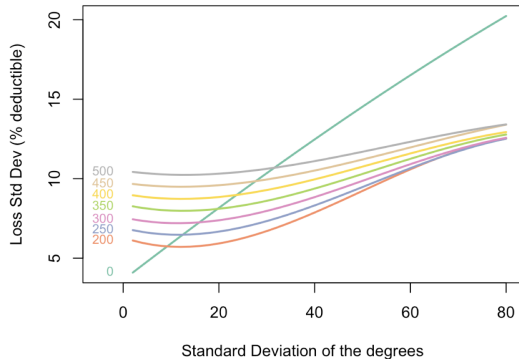
\* from a welfare (social planner) perspective

# Sharing Risks with Friends, and Friends of Friends

We can also consider friends of friends

$$\left\{ \begin{array}{l} \gamma_1^* = \operatorname{argmax} \left\{ \sum_{(i,j) \in \mathcal{E}^{(1)}} \gamma_{(i,j)} \right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0, \gamma_1], \forall (i,j) \in \mathcal{E}^{(1)} \\ \sum_{j \in \mathcal{V}_i^{(1)}} \gamma_{(i,j)} \leq s, \forall i \end{array} \right.$$
  

$$\left\{ \begin{array}{l} \gamma_2^* = \operatorname{argmax} \left\{ \sum_{(i,j) \in \mathcal{E}^{(2)}} \gamma_{(i,j)} \right\} \\ \text{s.t. } \gamma_{(i,j)} \in [0, \gamma_2], \forall (i,j) \in \mathcal{E}^{(2)} \\ \sum_{j \in \mathcal{V}_i^{(1)}} \gamma_{1:(i,j)}^* + \sum_{j \in \mathcal{V}_i^{(2)}} \gamma_{(i,j)} \leq s, \forall i \end{array} \right.$$



# Take-away

- ▶ Back to the roots of insurance with risk sharing,
- ▶ Important to better model interactions
- ▶ Nice mathematical properties of linear risk sharing (connexions with convex ordering)
- ▶ More complex to derive a more realistic insurance product (with lower and upper limits)
- ▶ ... ongoing work...



Collaborative insurance sustainability and network structure

Lariosse Kouakou, Matthias Löwe,  
Philipp Ratz & Franck Vermet



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