A domain decomposition method for 3-D Controlled Source Electromagnetics

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SUMMARY

Three dimensional controlled source electromagnetics (CSEM) forward modelling in the frequency domain requires the solution of a large scale, complex and linear system.
Such a system when derived from finite elements and finite differences formulations is sparse and ill-conditioned. The use of direct solvers is prohibitively expensive for realistic-sized problems since several million degrees of freedom are usually involved. To avoid inherent hardware limitations iterative solvers are an option, however their potential efficiency relies on the use of efficient pre-conditioning. We present an alternative pre-conditioner for the CSEM forward modelling problem, based on the Schur complement method, and will discuss some limitations and possible solutions.

INTRODUCTION

During the past few years the CSEM technique has achieved recognition in the oil industry as an important and valuable tool to reduce exploration risk. The use of this technique often requires the use of sophisticated mathematical and computational methods and efficient forward modelling algorithms which are unavoidable for parameter estimation from the collected data sets. The solution of the CSEM forward modelling problem in the frequency domain requires the solution of large scale complex symmetric linear systems of equations for several sources, and the inverse problem requires multiple forward modelling solutions for sources, receivers, frequencies whether sensitivities are calculated using a back-propagation algorithm or McGillivray's method (McGillivray and Oldenburg, 1990). A significant effort has been devoted to the development of different strategies to achieve an accurate and stable solution. Direct methods based on matrix factorization are suitable for multi-source problems in the sense that a single decomposition is sufficient to solve as many right-hand sides as necessary, nonetheless memory requirements make such methods prohibitive when 3-D geometries are considered. On the other hand iterative methods are less demanding in terms of hardware requirements but are inefficient unless efficient preconditioning is used due to the high condition numbers that are usually associated with the linear systems obtained from the numerical discretization wether finite differences or finite elements are used.
We are going to present a method that results from combining both strategies (direct and iterative), consisting in the reordering of the linear system and the use of the Schur complement to achieve a divide and conquer approach. The resulting submatrices are distributed across nodes and factorized locally on each processor. The linear system resulting from the Schur complement for the interface nodes is solved using an iterative solver and pre-conditioned with the ILU decomposition.

THE ELECTRIC FIELD IN THE FREQUENCY DOMAIN

For the electric field calculation we consider its decomposition in primary and secondary components. For an isotropic conductive medium the secondary electric field in the frequency domain is given by:

$$\nabla \times \nabla \times \mathbf{E}_s - i\omega\mu\sigma\mathbf{E}_s = i\omega\mu\Delta\sigma\mathbf{E}_p \tag{1}$$

where $\mathbf{E}_p$ and $\mathbf{E}_s$ are the primary and secondary field respectively and $\Delta\sigma$ represents the conductivity anomalies. All the other symbols have their usual meanings. The primary field can be calculated using a homogeneous or layered background medium. To ensure solution uniqueness the perfect electrically conductive boundary condition is considered:

$$\mathbf{n} \times \mathbf{E}\,|_{\partial\Omega} = \mathbf{0} \tag{2}$$

where $\partial\Omega$ is the boundary of the modelling domain.
The corresponding system of finite element equations for the secondary field equation is derived using the weighted residual method or Galerkin's method (Zienkiewicz and Taylor, 2005), enforcing the condition:

$$R_k = \sum_{e=1}^{N_e} \int_{\Omega^e} \mathbf{N}^e_{i(k)} \cdot \mathbf{r}\, dV = 0 \tag{3}$$

where $\mathbf{r} = \nabla \times \nabla \times \mathbf{E}_s - i\omega\mu\sigma\mathbf{E}_s - i\omega\mu\Delta\sigma\mathbf{E}_p$ is the vector residual and R_k is the weighted residual for the k-th edge.
Assembling the equations leads to a large, sparse, complex symmetric and linear system of equations yields:

$$\mathbf{AE} = \mathbf{s} \tag{4}$$

were $\mathbf{A}$ is the matrix of complex coefficients resulting from the assembly, $\mathbf{E}$ is the electric field discretized over the grid and $\mathbf{s}$ is the source vector.

SOLUTION OF THE SYSTEM OF EQUATIONS

Before reordering, the linear system is scaled symmetrically:

$$\left[\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}\right]\mathbf{D}^{1/2}\mathbf{E} = \mathbf{D}^{-1/2}\mathbf{s} \tag{5}$$

where $\mathbf{D}$ represents the diagonal of $\mathbf{A}$. The matrix $\mathbf{A}$ in equation 5 is then reordered to a diagonal and bordered block form:

$$\begin{pmatrix} A^1_{ii} & & & & A^1_{ib} \\ & A^2_{ii} & & & A^2_{im} \\ & & & & \vdots \\ & & & & \vdots \\ & & & A^n_{ii} & A^n_{im} \\ A^1_{im} & A^2_{im} & \dots & A^n_{im} & \widehat{A}_{mm} \end{pmatrix} \begin{pmatrix} E^1_i \\ \vdots \\ \vdots \\ \vdots \\ E^n_i \\ \widehat{E}_m \end{pmatrix} = \begin{pmatrix} S^1_i \\ \vdots \\ \vdots \\ \vdots \\ S^n_i \\ \widehat{S}_m \end{pmatrix} \tag{6}$$

the resultant block matrices A_{ii}^n in the main diagonal represent the impedance matrices on each sub-domain, whereas the border block matrices A_{im}^n establish the relation between the interface edges and the inner edges for the sub-domain n.
The elimination of E_i^n followed by the application of the Schur complement yields:

$$\left(\widehat{A}_{mm} - A_{mi}A_{ii}^{-1}A_{im}\right)\widehat{E}_m = s_b - A_{mi}A_{ii}^{-1}S_i \qquad (7)$$

also known as the reduced system.
To achieve maximum efficiency the main diagonal block matrices are distributed across nodes on a parallel computing system. After calculating locally on each corresponding cpu the factorization of each block matrix, the right hand side for the reduced system can be constructed since $A_{ii}^{-1} = U_{ii}^{-1}L_{ii}^{-1}$. The parallel factorization was computed using the open source package MUMPS (Multifrontal Massively Parallel Solver) (Amestoy et. al., 2000). The solution for equation 7 can therefore be achieved using a preconditioned Krylov subspace method (van der Vorst, 2003). The preconditioner was based in the ILU(0) decomposition, obtained by discarding the small elements in the reduced linear system using a threshold value. From our experience the threshold is highly dependent on a particular heuristic associated to each particular linear system. Usually values below 10^{-3} allow a good convergence. If convergence is not reached, lower threshold numbers usually improve the convergence rate at the expense of increasing the run time. After solving equation 7 the solution for inner nodes can be computed concurrently using the sub-linear system for each sub-domain $A_{ii}^j E_i^j = S_j - A_{ib}\widehat{E}_m$.

MODEL 1: RESISTIVE LAYER IN A HALF SPACE

To test the accuracy and performance of the proposed domain decomposition method we considered firstly a layered model. The resistivity model was discretized using a $135 \times 135 \times 37$ (1.99 million equations in double complex arithmetic) irregular grid representing a computational domain of $18.4 \times 18.4 \times 3$ km. The grid utilized was finer near the source positions ranging between 10 and 150 m. The sea-water layer is 1 km thick. Figure 1 shows the resistivity model:

$$\rho(z) = \begin{cases} 0.33\ \Omega m, & \text{if } z > 0\ m \\ 1.0\ \Omega m, & \text{if } -200 < z < 0\ m \\ 10.0\ \Omega m, & \text{if } -400 < z < -200\ m \\ 1.0\ \Omega m, & \text{if } z < -400\ m \end{cases} \qquad (8)$$

The source position was $(0,0,30)$ m with frequency 1 Hz. We performed several tests with different numbers of subdomains. If the number of processors is kept constant, increasing the number of subdomains will require more memory to compute the factorizations of each block matrix. On the other hand if not divided in enough subdomains, the factorization of the block matrices will take too much cpu time and consequently the performance of the solver degrades. The best trade-off between the number of MPI processes and number of sub-domains was obtained using 16 MPI processes and 10000 subdomains. Convergence was observed for a threshold of 10^{-3} for the same number of sub-domains, nevertheless the

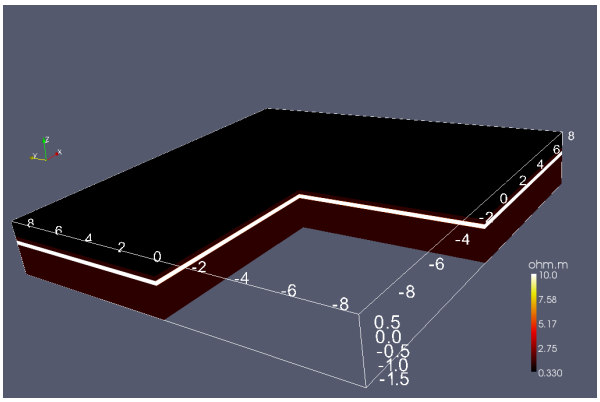

Figure 1: Display of the resistivity model used to test the accuracy of the domain decomposition solver. The distance units are in km.

number of iterations reached more than 1000, and therefore the threshold value was decreased to 10^{-4}. The GMRES solver converged in 479 iterations. Table 1 summarizes the computational performance of the domain decomposition solver for model 1.

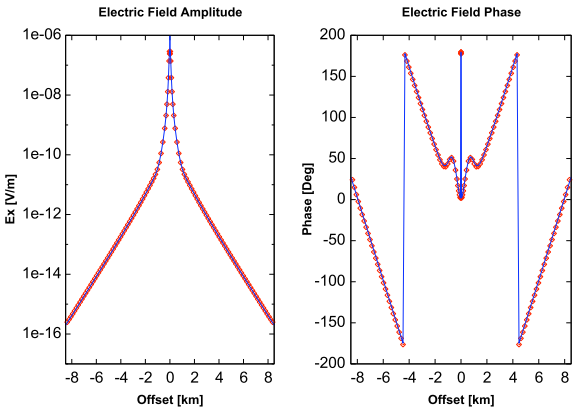

Figure 2: Comparison of the x component of the electric field computed using the FEM code (circles) and the 1D code (solid line) in the inline direction at y = 0 m for a signal of 1 Hz.

Figures 2 and 3 show a good agreement between the solutions obtained using the finite element code with the presented solver and the ones obtained using Chave and Cox's (1982) solution for 1-D resistivity models. In Figures 4 and 5 it is possible to observe the relative error distribution on the sea floor for amplitude and phase of the x-component of the electric field. The electric field computed using the finite element code was compared with the one computed using the 1-D solution. With a few exceptions it is possible to observe an error below 5 per cent.

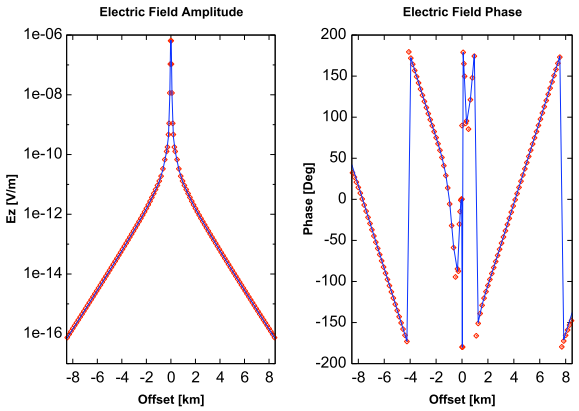

Figure 3: Comparison of the z component of the electric field computed using the FEM code (circles) and the 1D code (solid line) in the inline direction at y = 0 m for a signal of 1 Hz.

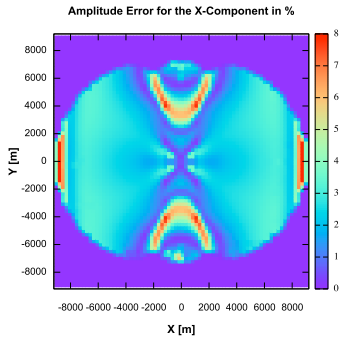

Figure 4: Amplitude error in percentage for the x-component.

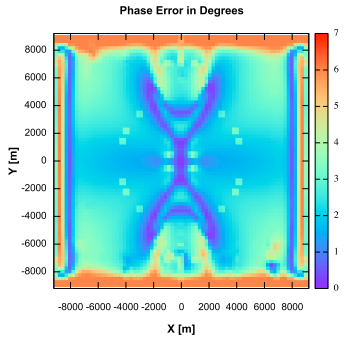

Figure 5: Phase error in degrees for the x-component.

MODEL 2: 3-D MODEL

We tested the domain decomposition solver on a deep sea-water 3-D model (Figure 6) and compared the solution with the one obtained using a Dyadic Green's function (IE) (Zhdanov and Keller, 1994). This model was discretized using a

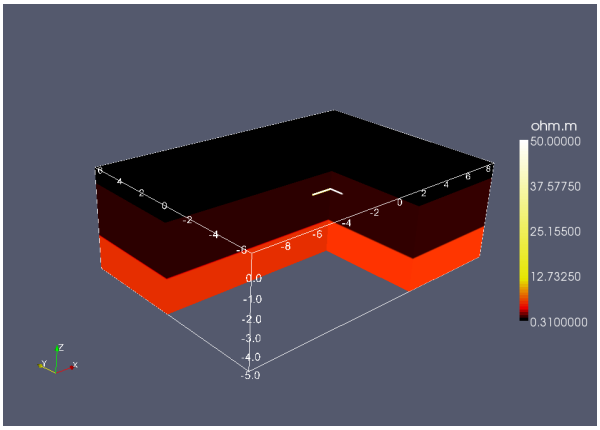

Figure 6: Display of the 3-D resistivity model. The distance units are in km.

$140 \times 102 \times 59$ irregular mesh (nearly 2.5 million equations), representing a computational domain of $19.1 \times 13.4 \times 6.5$ km. The grid spacing varies between 10 and 370 m. The source was positioned at $(-2000, 0, 5)$ m and with frequency of 1 Hz. The sea-water layer was 1 km thick. We were only able to obtain an efficient convergence rate at the expense of decreasing the ILU decomposition threshold to 10^{-5}. Consequentely the

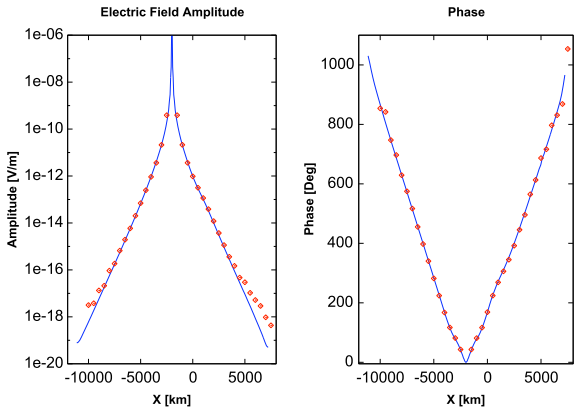

Figure 7: Comparison of the x component of the electric field computed using the FEM code (solid line) and the IE code (circles) in the inline direction at y = 0 m for a signal of 1 Hz.

computing time for the construction of the preconditioner increased substantially when compared with the one for model 1, considering that the number of equations did not increase substantially. This decrease in the threshold value can be explained by the bigger grid spacing ratios in the mesh used for the discretization of model 2, when compared with the ones

in the mesh utilized for the discretization of model 1. Higher grid spacing ratios imply an increase in the condition number of the matrix, hence it is more difficult to approximate its inverse and therefore we must increase the fill-in of the preconditioner by decreasing the ILU threshold. Nonetheless as seen on table 1, the number of itearations is quite acceptable as it is the overall computational time considering the dimensions of the linear system. The best trade-off obtained for the computational time, memory requirements, versus number of MPI processes and number of sub-domains was obtained using 10000 sub-domains. Figures 7 and 8 show there is a good agreement between the solutions obtained for the electric field using the finite element code with the proposed solver and the ones obtained using the Dyadic Green's function. If is possible to observe instabilities for amplitude values lower than the numerical precision of the cpu, nonetheless those instabilities have no practical effect since they are also below the level of noise usually observed in real data.

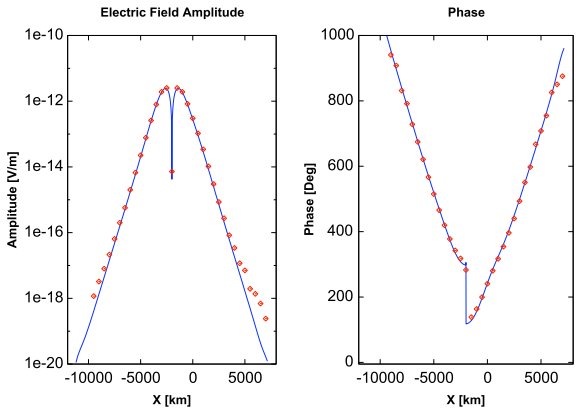

Figure 8: Comparison of the y component of the electric field computed using the FEM code (solid line) and the IE code (circles) in the inline direction at y = 1500 m for a signal of 1 Hz.

	Model 1	Model 2
DOF	1994760	2499002
t_{factor} / s	312	851
t_{iter} / s	1.8	2.4
$N_{iterations}$	479	85
tolerance	10^{-7}	10^{-7}
Mem_{node} / Gb	7 Gb	7 Gb
Mem_{total} / Gb	28 Gb	28 Gb
ILU(0) threshold	10^{-4}	10^{-5}
N MPI proc	16	16

Table 1

THE AIR-WAVE CASE

When dealing with the air wave problem we observed a lack of convergence in the proposed solver. This is due to the almost singular behaviour of the linear system, since it is dominated by the curl-curl term of equation 1. To overcome this problem a possible strategy could be truncating the modelling domain with a boundary integral equation. However doing so has the inevitable cost of increasing the code complexity and probably execution time since such integrals are represented numericaly by dense matrices. This is still an open problem and research is under way.

CONCLUSIONS

We presented a robust domain decompostion solver based on the Schur complement method for the CSEM forward problem in the frequency domain. From the examples shown we can conclude that the method is stable and converges for linear systems arising from discretizations with high grid spacing ratios. This is an important aspect since the use of irregular meshes is necessary to optimize the modelling grid. On the other hand, the solver is entirely algebraic avoiding the previous knowledge of the geometry for each particular problem one wishes to solve. The solver performance is sensitive to the threshold selected for the preconditioner and the number of subdomains. These parameters have to be adjusted for each particular problem accordingly to the size of the problem, available hardware and the condition number of the linear system. We observed very slow convergence rates when an air wave is considered due to the large null space associated with the resulting linear system requiring further research to generalize the use of the domain decomposition solver to all possible 3-D CSEM modelling cases.

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EDITED REFERENCES

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