# TIME RESPONSE

# 4.1 INTRODUCTION

(1) We would like to control/evaluate

Linear, Time-invariant System

(2) Can be represented by transfer function

(3)Should be analyzed for its

- Transient response and
- Steady-state responses

To see if these characteristics yield the desired behaviour at the output.

# 4.2 POLES, ZEROS, AND SYSTEM RESPONSE

Output response= Natural response + Forced response

Homogeneous solution.
(If the system is stable, it is also called transient response.)
Depends only on the system, not the input.

Particular solution, steady-state response
Depends on the input.

Transfer Function

Numerator Polynomial Function

$$G(s) = \frac{N(s)}{D(s)}$$

Denominator Polynomial Function

The **zeros** of a transfer function are the values of s that cause the transfer function to become zero.

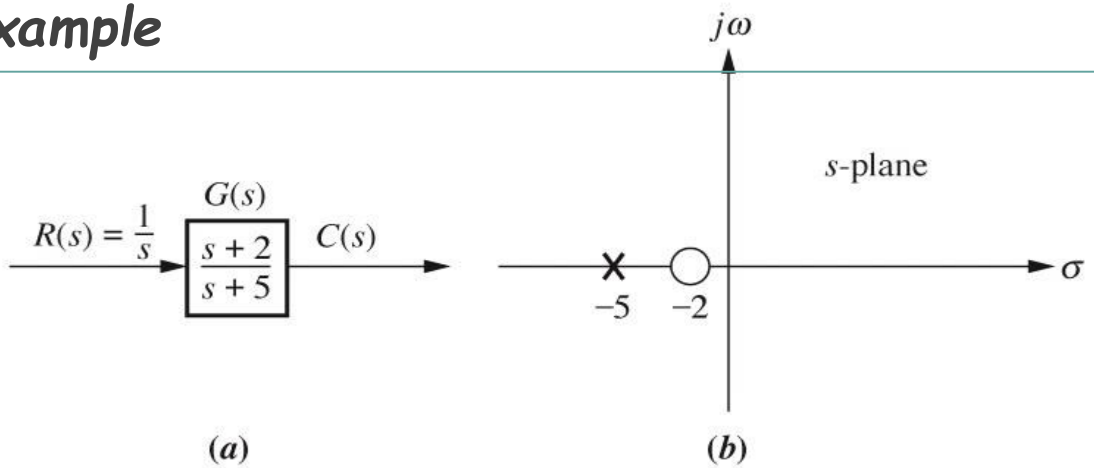
The **poles** of a transfer function are the values of s that cause the transfer function to become infinite.

Is there a relationship between the pole/zero locations and time response of a system?

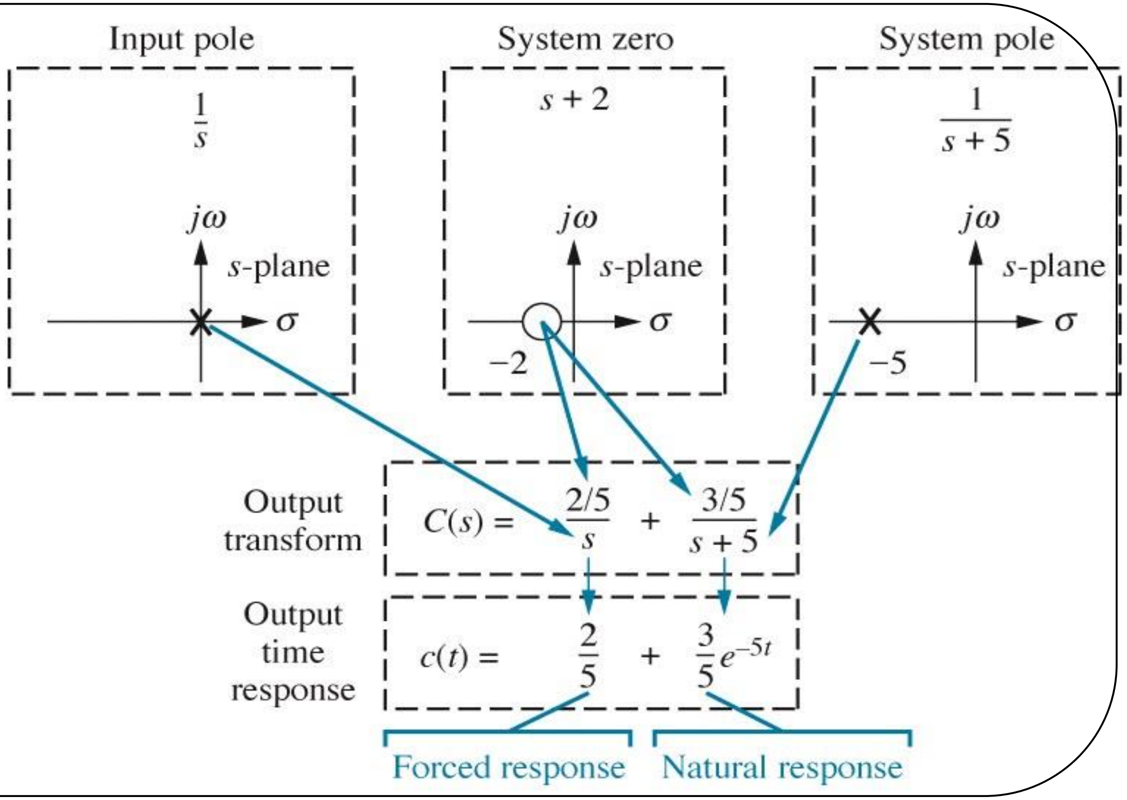
# Poles and Zeros of a first-order system: An Example

$$C(s) = \frac{s+2}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

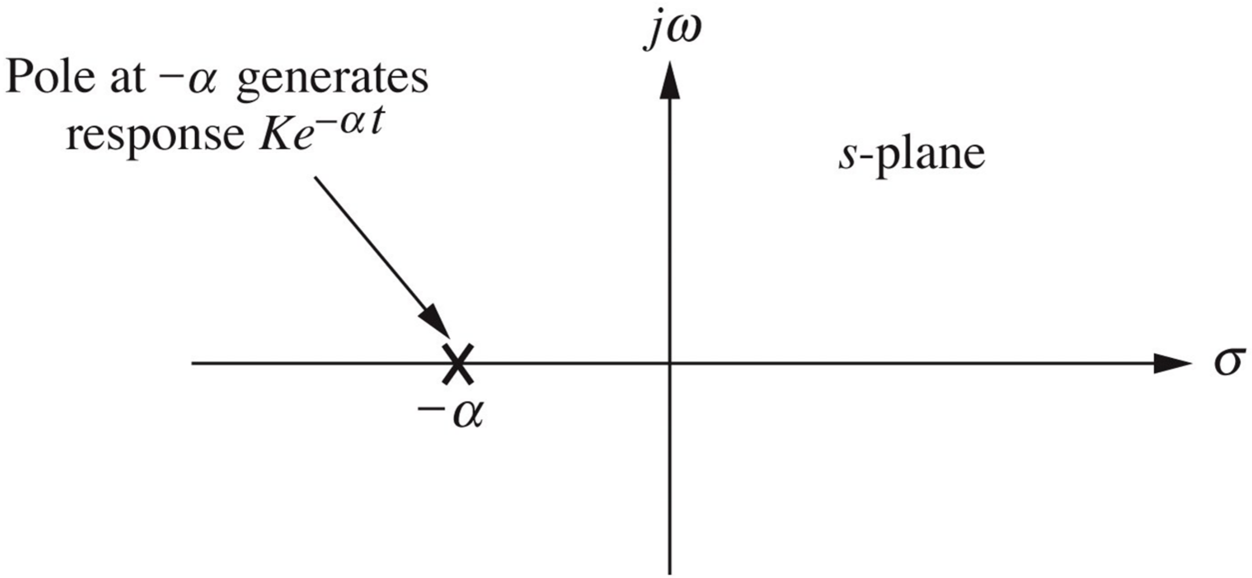


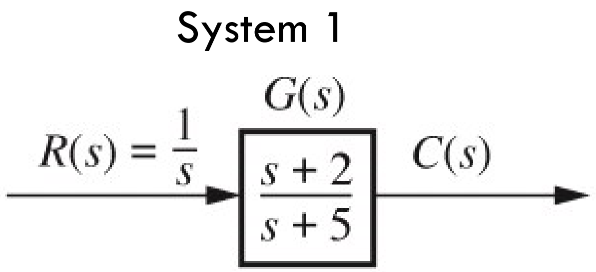

1. A pole of the input generates the form of the forced response
2. A pole of the transfer function generates the form of the natural response
3. The zeros and poles generate the amplitudes for both the forced and natural responses.

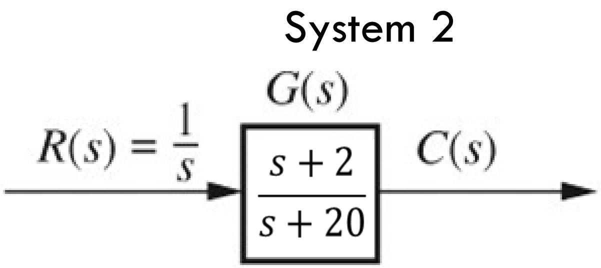


- A pole at $-\alpha$ generates an exponential response of the form $e^{-\alpha t}$.

- The farther to the left a pole is on the negative axis, the faster the exponential transent response will decay to zero.



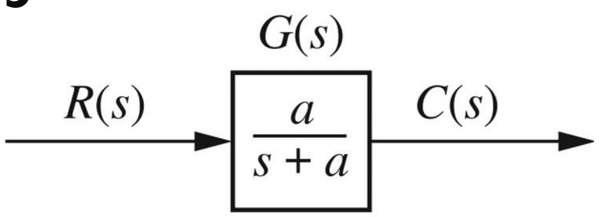




System 2 reaches to its steady-state value earlier than the System 1. (i.e., Its transient response decays faster) . System 2 is faster.

# 4.3 FIRST-ORDER SYSTEMS

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

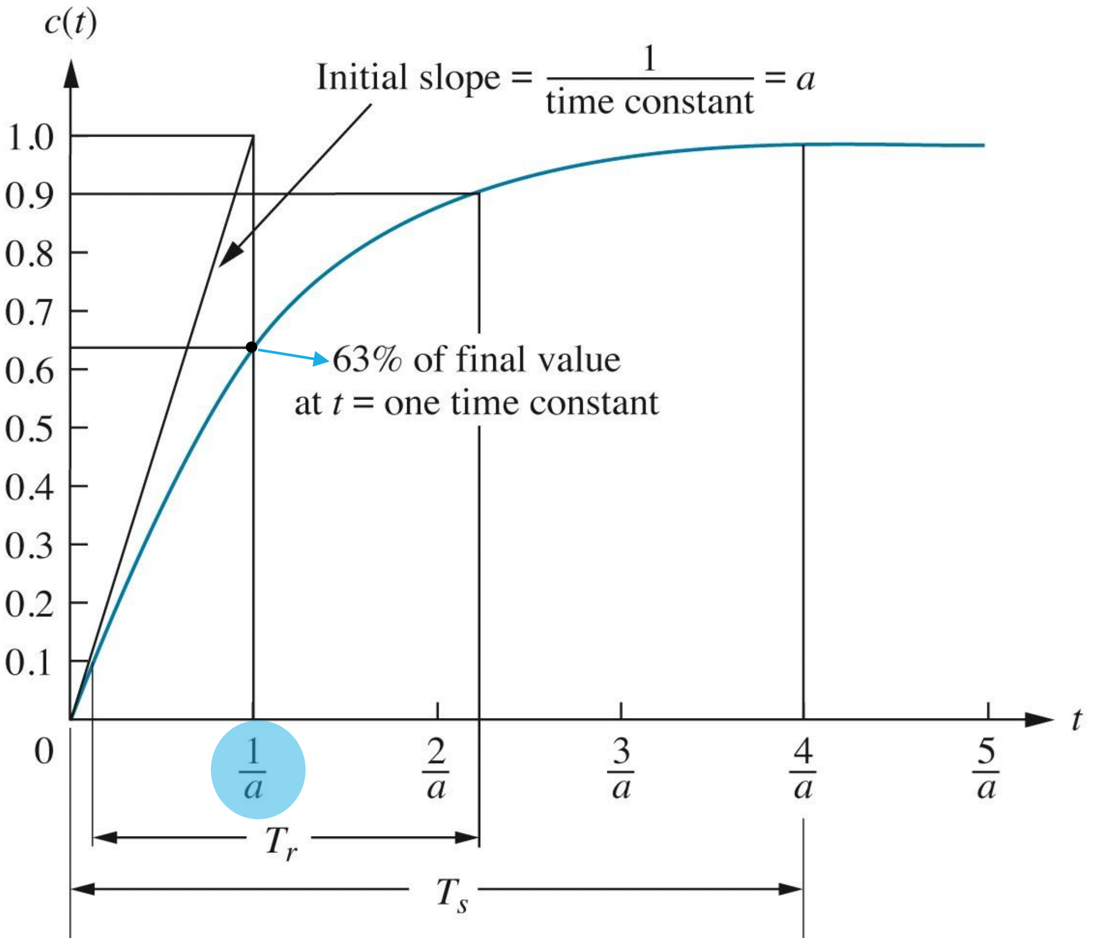
$\frac{1}{s}$




$$c(t) = c_{forced}(t) + c_{natural}(t) = 1 - e^{-at}$$

When $t = \frac{1}{a}$, $\rightarrow$ $e^{-at}|_{t=1/a} = e^{-1} = 0.37$

$$c(t)\Big|_{t=1/a} = 1 - e^{-at}\Big|_{t=\frac{1}{a}} = 1 - 0.37 = 0.63$$




**Time Constant:** Time it takes for the step response to rise to 63% of its final value.

$$\tau = \frac{1}{a}$$

**Rise time:** Time for the waveform to go from 0.1 to 0.9 of its final value.

$$T_r = \frac{2.2}{a} = 2.2\tau$$

**Settling Time:** Time for the response to reach, and stay within, 2% of its final value.

$$T_s = \frac{4}{a} = 4\tau$$

# FIRST-ORDER TRANSFER FUNCTIONS VIA TESTING

*What about if it is not possible/practical to obtain a system's transfer function analytically?*

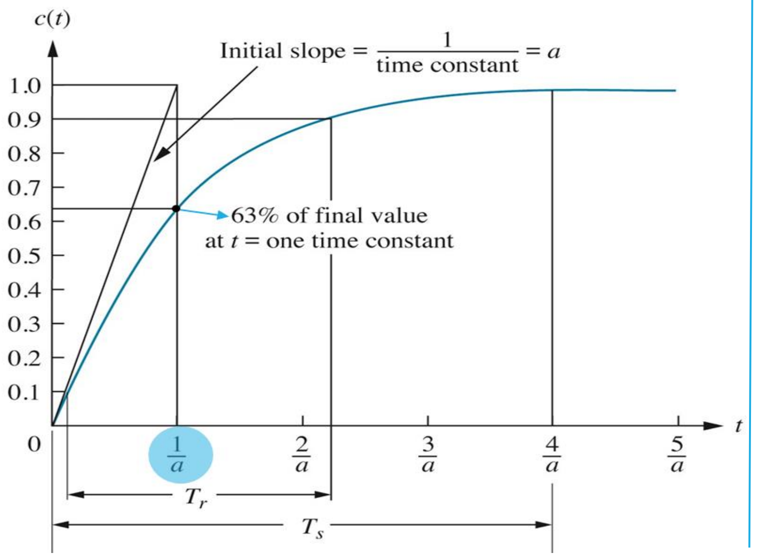
*A possible approach:*

- *Apply a step input*
- *Measure the time constant and steady-state value from the response*
- *Write the transfer function by using measured values.*

Consider a simple first-order system, $G(s) = \frac{K}{s+a}$ whose step response is

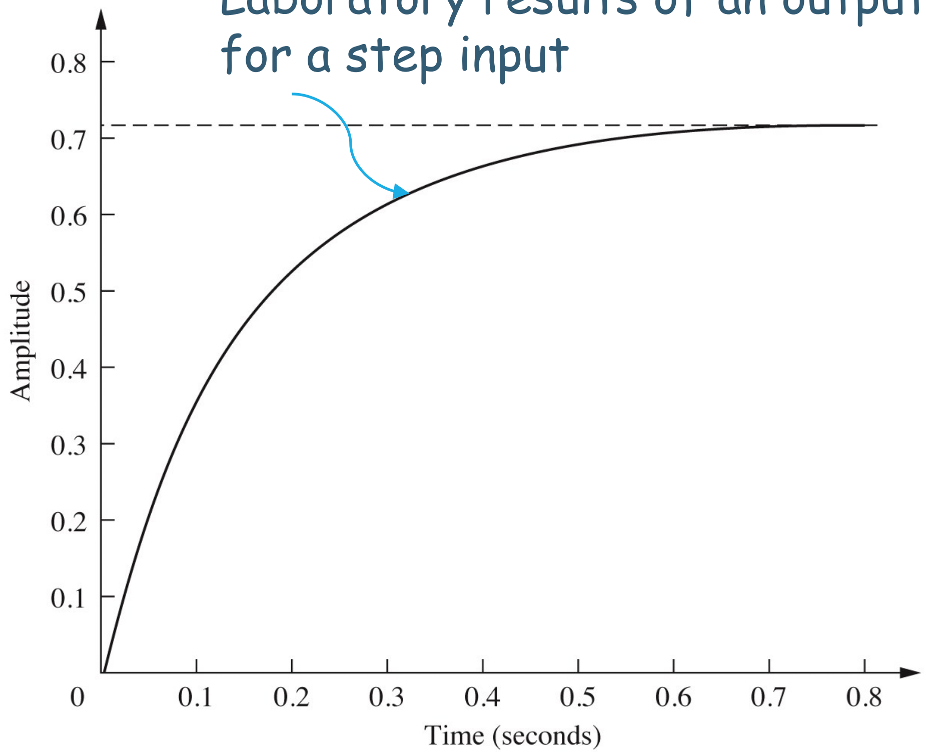
$$C(s) = R(s)G(s) = \frac{K}{s(s+a)}$$

$$= \frac{K/a}{s} - \frac{K/a}{s+a}$$



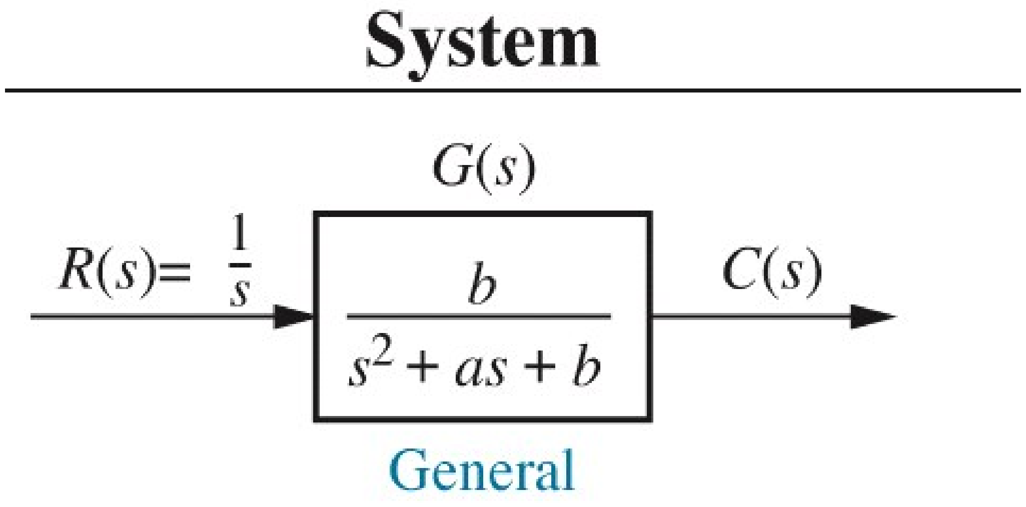

**Identify K and a from laboratory testing**

Laboratory results of an output for a step input




- ✓ Final Value: 0.72
- ✓ 0.63 x 0.72 = 0.45
- ✓ Time constant is the time where response reaches 0.45→ $\tau = 0.13\ sec$.
- ✓ Then $a = \frac{1}{0.13} = 7.7$
- ✓ Forced resonse reaches a steady-state value of $\frac{K}{a}$ =0.72. Then K=5.54
- ✓ The transfer function, $G(s) = \frac{5.54}{s+7.7}$

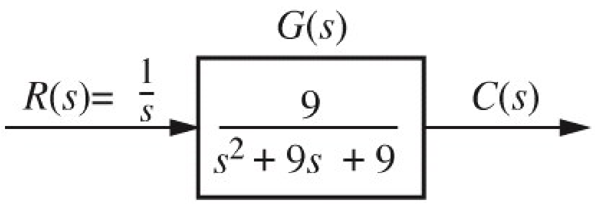
# 4.4 SECOND-ORDER SYSTEMS:INTRODUCTION



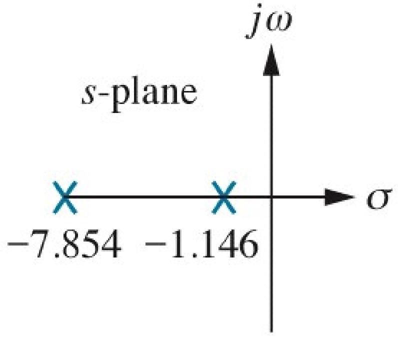

<u>**The pole locations determine the form of the response!**</u>

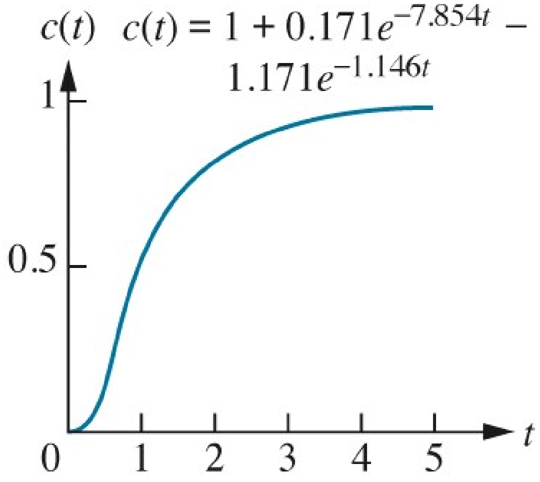
# 4.4 SECOND-ORDER SYSTEMS:INTRODUCTION

**System** **Pole-zero plot** **Response**



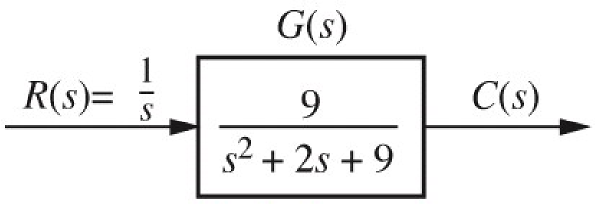

Overdamped



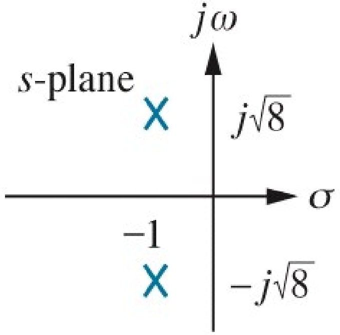



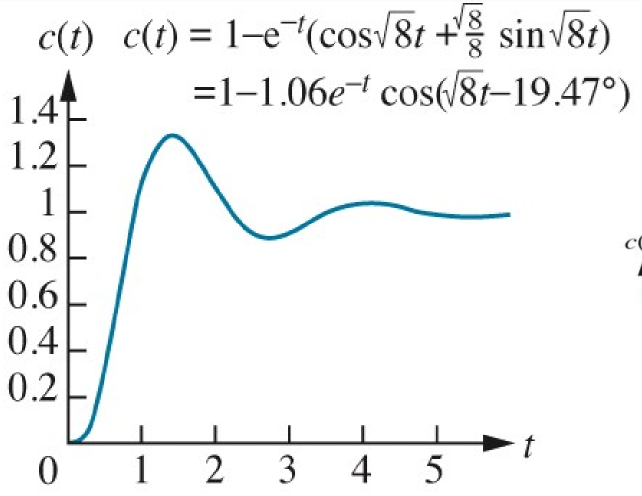

Poles: Two real at $-\sigma_1$, $-\sigma_2$
Natural Response: Two exponentials with time constants equal to the reciprocal of the pole locations

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

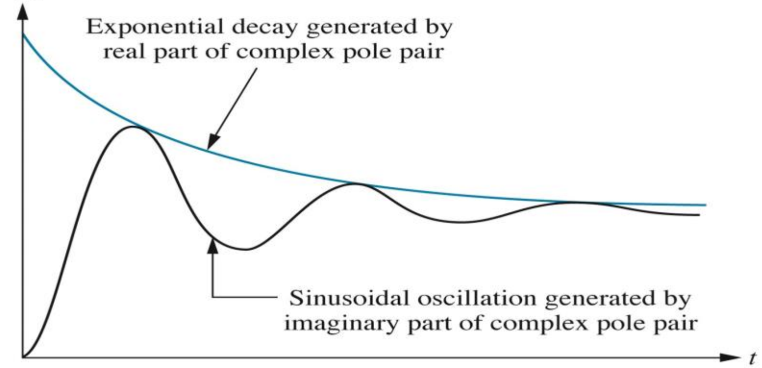



Underdamped

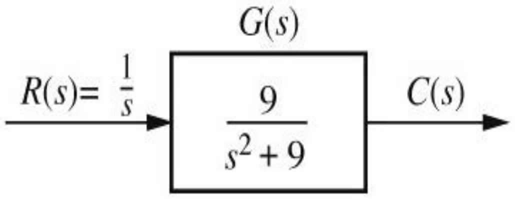


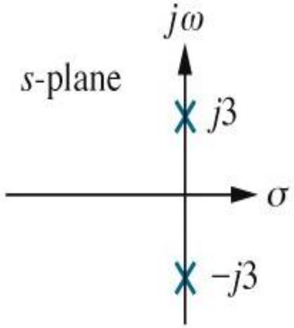

Poles: Two complex at $-\sigma_d \pm j w_d$
Natural Response: Damped sinusoid with an exponential $c(t) = A e^{-\sigma_d t} \cos(w_d t - \phi)$

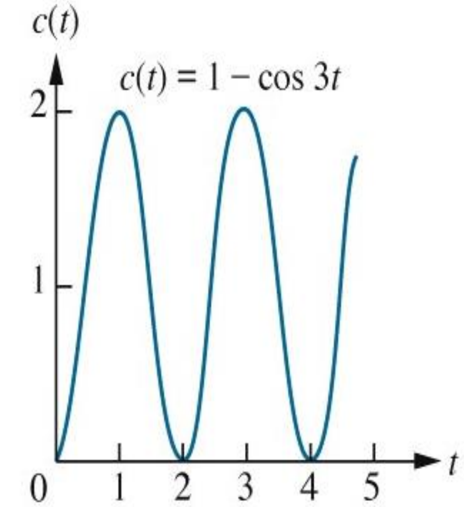


| System | Pole-zero plot | Response |
|---|---|---|



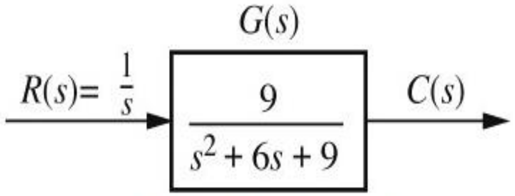

Undamped



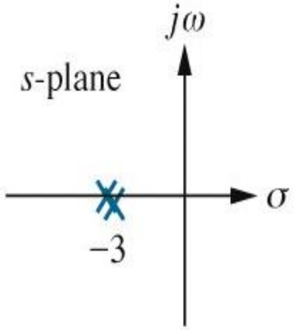



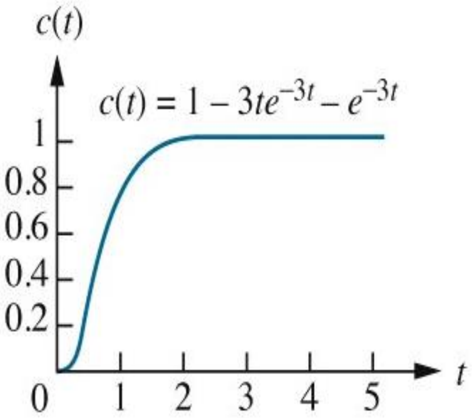

<u>Poles:</u> Two complex at $\pm j w_1$
<u>Natural Response:</u> Undamped sinuoid with radian frequency equal to the imaginary part of the poles

$$c(t) = A\cos(w_1 t - \phi)$$




Critically damped




<u>Poles:</u> Two real at $-\sigma_1$
<u>Natural Response:</u> an exponential + the product of time, t, and an exponential.

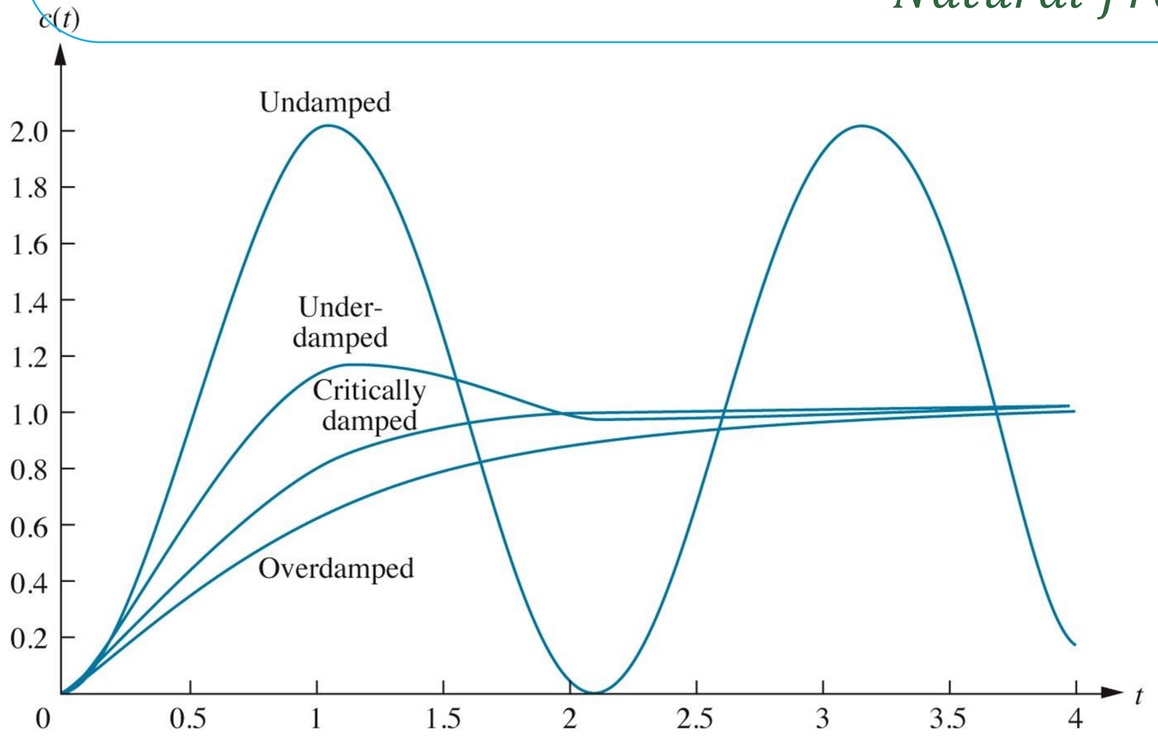
$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_2 t}$$

# 4.5 THE GENERAL SECOND-ORDER SYSTEM

**Natural Frequency, $w_n$:** The frequency of oscillation of the system without damping.

**Damping Ratio, $\zeta$ :** A quantity that compares the exponential decay frequency of the envelope to the natural frequency. It is constant regardless of the time scale of the response.

$$\zeta = \frac{Exponential\ decay\ frequency}{Natural\ frequency\ (\frac{rad}{second})} = \frac{1}{2\pi}\frac{Natural\ period\ (seconds)}{Exponential\ time\ constant}$$




$w_n$ and $\zeta$ can be used to describe the characteristics of the response.

$$G(s) = \frac{b}{s^2 + as + b}$$

$$G(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

The **poles** of a transfer function are

$$s_{1,2} = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1}$$

# SECOND-ORDER RESPONSE AS A FUNCTION OF DAMPING RATIO

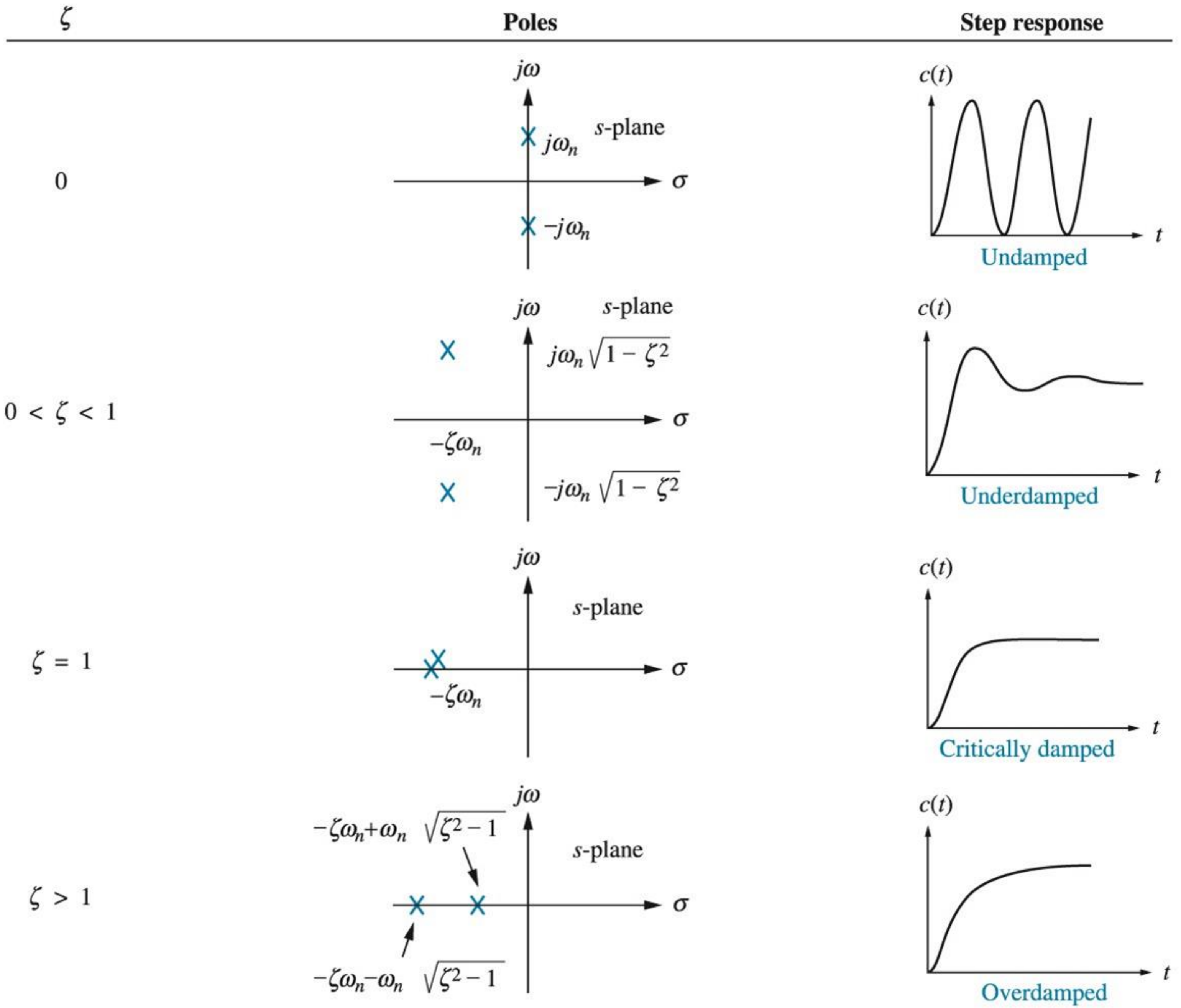
$$G(s) = \frac{b}{s^2 + as + b}$$

$$G(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

The **poles** of a transfer function are

$$s_{1,2} = -\underbrace{\zeta w_n}_{\sigma} \pm \underbrace{w_n\sqrt{\zeta^2 - 1}}_{w_d}$$

Attenuation    Damped frequency



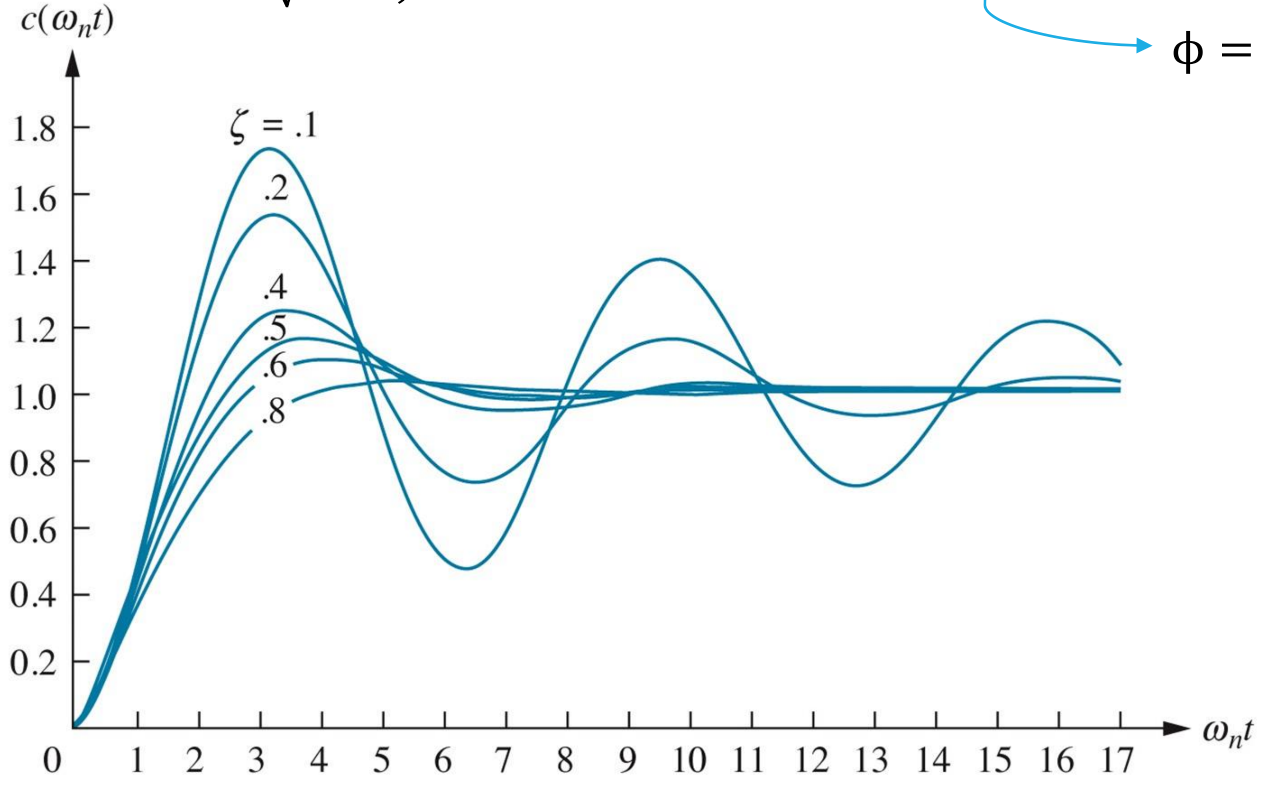
# 4.6 UNDERDAMPED SECOND-ORDER SYSTEMS

$$G(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

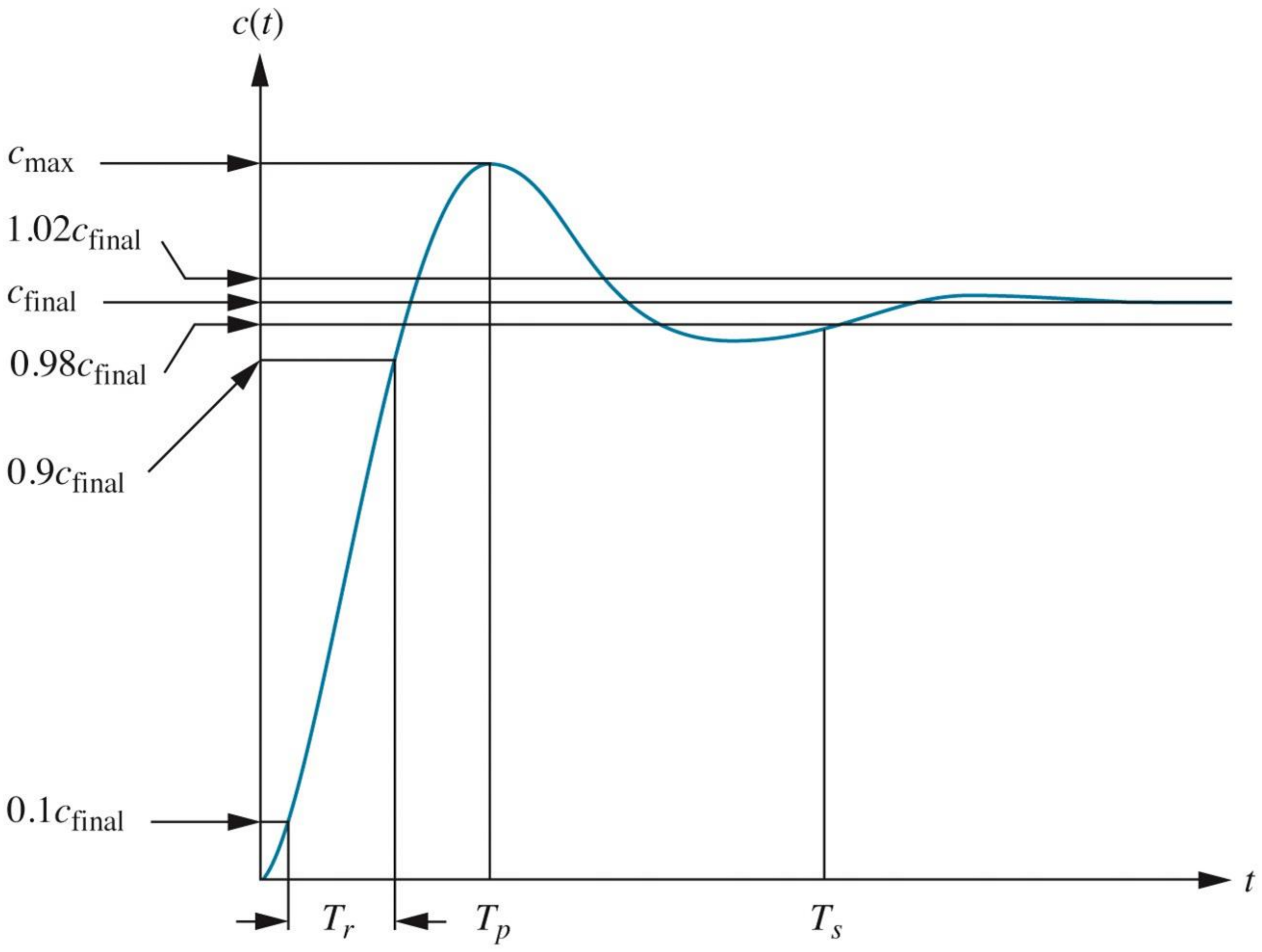
$$C(s) = \frac{w_n^2}{s(s^2 + 2\zeta w_n s + w_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta w_n s + w_n^2} = \frac{1}{s} - \frac{(s + \zeta w_n) + \frac{\zeta}{1-\zeta^2} w_n \sqrt{1-\zeta^2}}{(s + \zeta w_n)^2 + w_n^2(1-\zeta^2)}$$

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \cos(w_n \sqrt{1-\zeta^2}\,\mathrm{t} - \phi)$$

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$




The lower the value of $\zeta$ ,
the more oscillatory the response.




$$G(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} \qquad C(s) = G(s)R(s)$$

The **poles** of a transfer function are

$$s_{1,2} = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1}$$

**Rise time:** The time for the waveform to go from 0.1 to 0.9 of its final value.

$$T_r = \frac{\pi - \theta}{w_d}$$

**Peak time:** The time required to reach the first, or maximum, peak.

$$T_p = \frac{\pi}{w_n\sqrt{1-\zeta^2}}$$

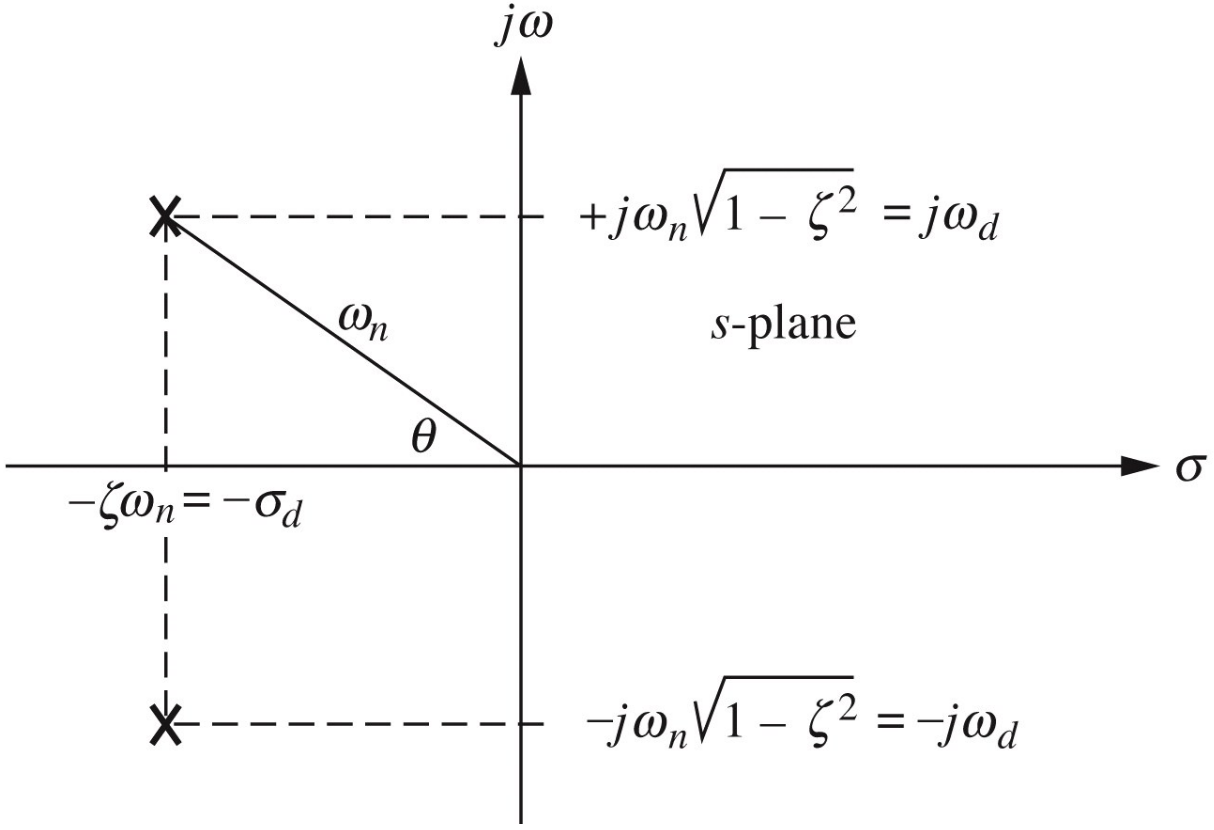
**Percent overshoot:** The amount that the waveform overshoots the final or steady-state value at the peak time, expressed as a percentage of the steady-state value.

$$\%OS = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \text{ x } 100, \qquad \%OS = \frac{c_{max} - c_{final}}{c_{max}} \text{x}100$$

**Settling time:** The time required for the transient's damped oscillations to reach and stay within ±2% of the steady-state value.

$$T_s = \frac{4}{\zeta w_n}$$

Representation of a pole in *s*-domain.

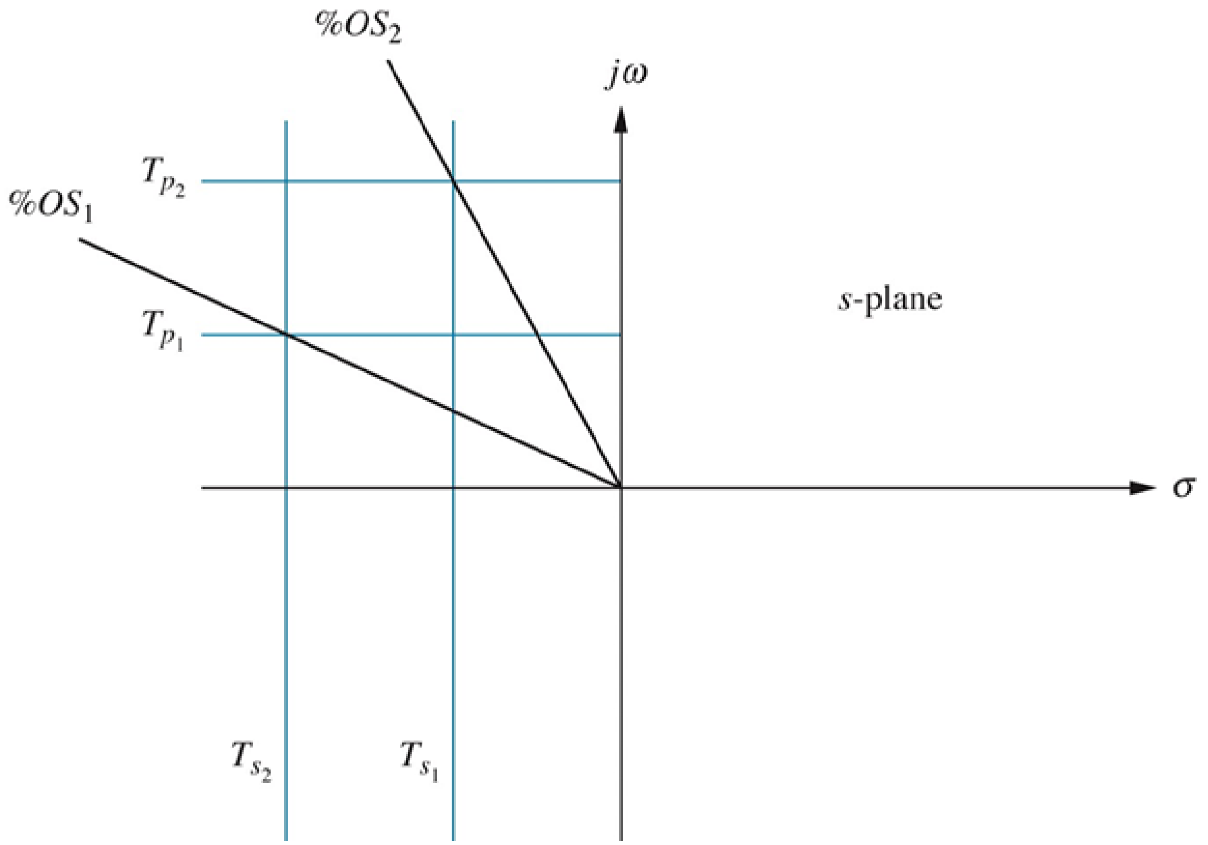



$$G(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} \qquad C(s) = G(s)R(s)$$

The **poles** of a transfer function are

$$s_{1,2} = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1}$$

Lines of constant peak time Tp, settling time, Ts, and percent overshoot %OS.



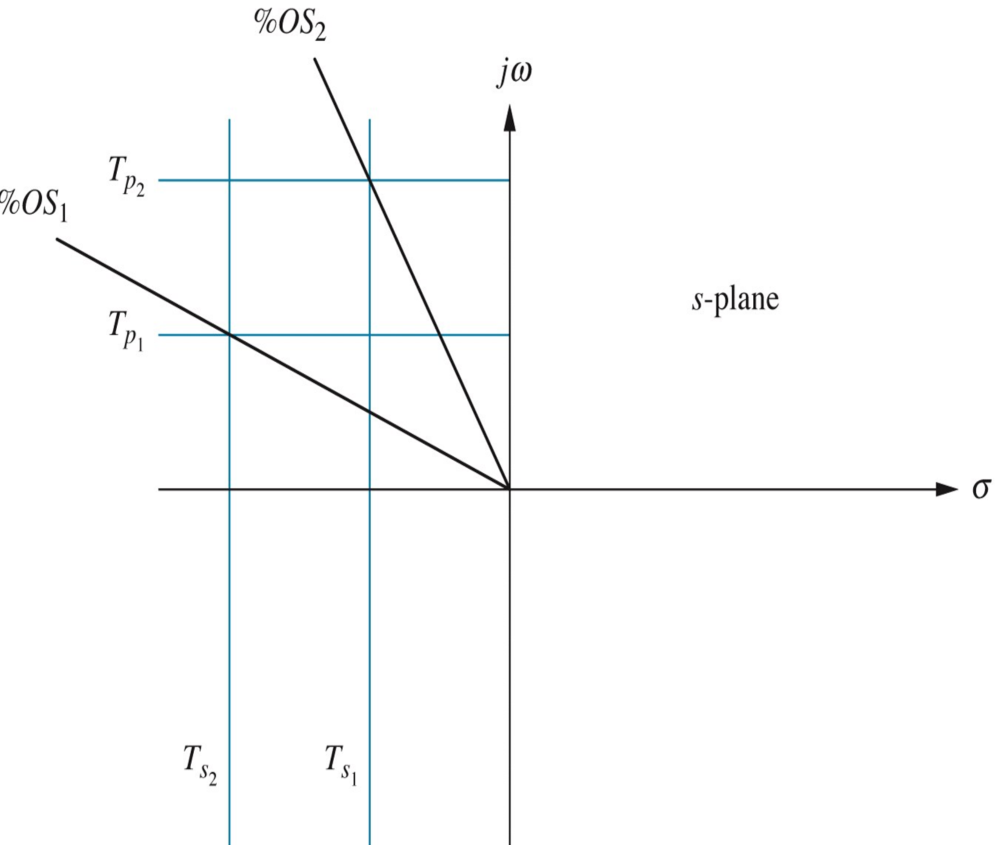

Peak time: $T_p = \frac{\pi}{w_n\sqrt{1-\zeta^2}}$

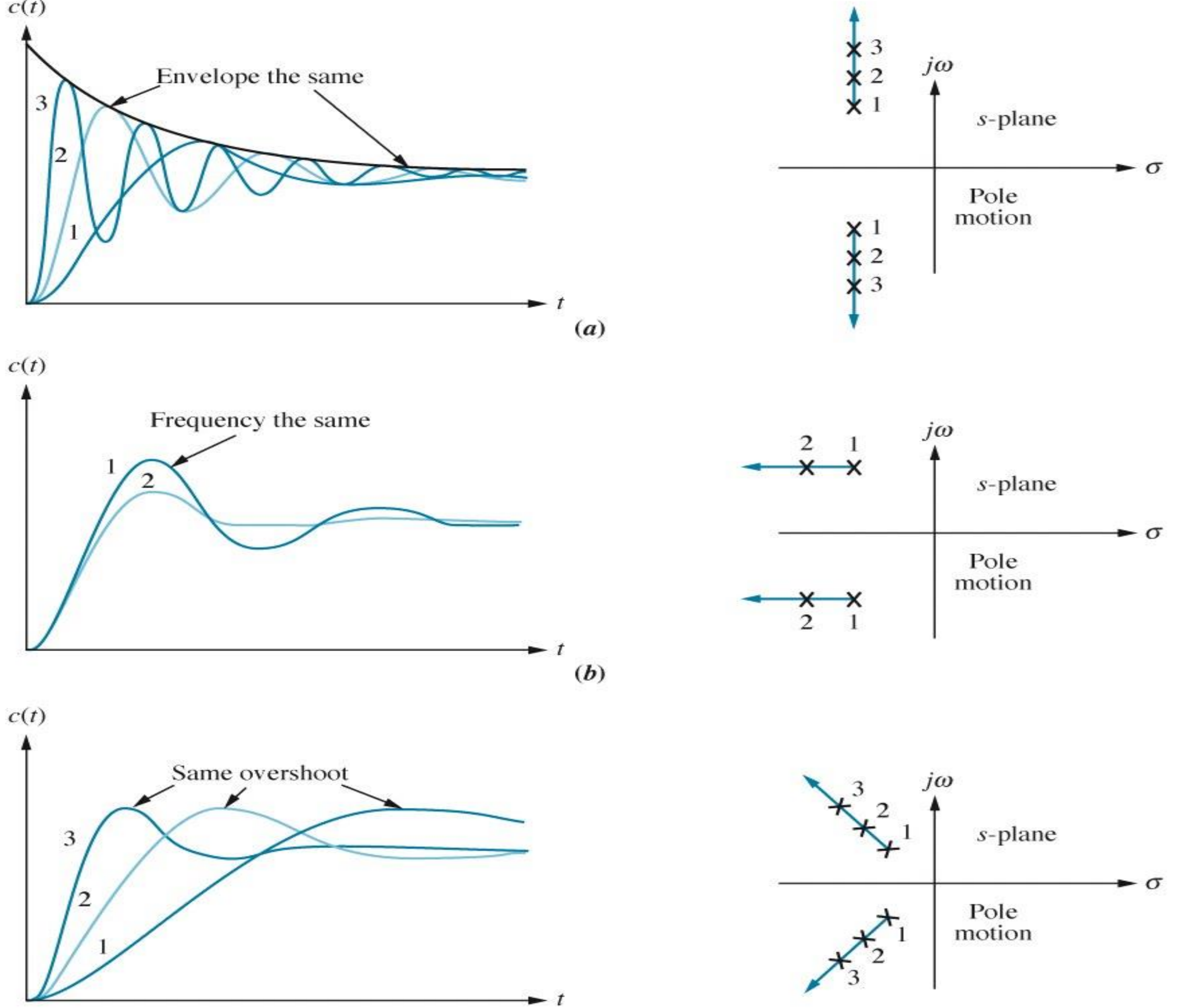
Percent overshoot: $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \text{ x } 100$

Settling time: $T_s = \frac{4}{\zeta w_n}$

$$G(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} \qquad C(s) = G(s)R(s)$$

$$s_{1,2} = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1}$$




Step responses of different 2nd-order systems having <u>same</u> (a) real parts; (b) imaginary parts; (c) damping ratios.

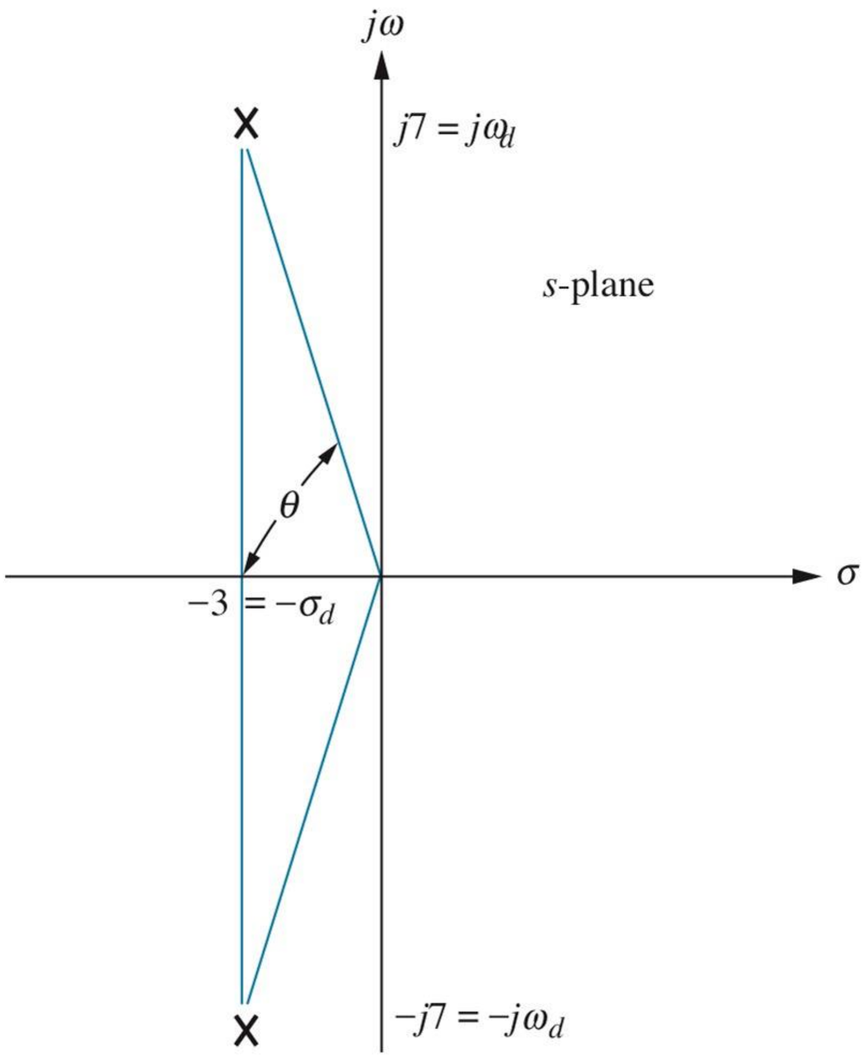
Except for certain applications where oscillations cannot be tolerated, it is desirable that the ***transient response*** *be* ***sufficiently fast*** *&* ***sufficiently damped***.

Thus, for a desirable transient response of a *second-order system*, the **damping ratio must be between 0.4 and 0.8.**

( Small values of $\zeta$ ($\zeta<0.4$) yield excessive overshoot.
Large values of $\zeta$ ($\zeta>0.8$) responds sluggishly. )

Note that, fastest response without oscillation is the critically damped response.

**Problem:** *Given the pole plot shown in figure below, find* $\zeta, w_n, T_p, \%OS, and T_s$.




$$s_{1,2} = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1} = -3 \pm j7$$

$$\zeta = \cos\theta = \cos[\tan^{-1}(\frac{7}{3})] = 0.394$$

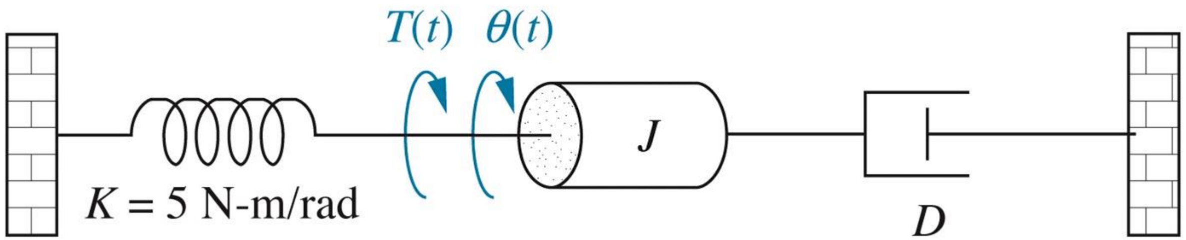
$$w_n = \sqrt{7^2 + 3^2 = 7.616}$$

$$T_p = \frac{\pi}{w_n\sqrt{1-\zeta^2}} = \frac{\pi}{7} = 0.449$$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \text{ x } 100 = 26\%$$

$$T_s = \frac{4}{\zeta w_n} = \frac{4}{3} = 1.33 \ seconds$$

**Problem:** *Given the system shown in figure below, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input torque T(t).*




$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

From the transfer function G(s): $w_n = \sqrt{\frac{K}{J}}$ *and*

$$2\zeta w_n = \frac{D}{J}$$

From the problem statement : $T_s = \frac{4}{\zeta w_n} = 2$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \text{ x } 100 = 20\% \Rightarrow \zeta = 0.456$$

$$2\zeta w_n = 4 = \frac{D}{J}$$

$$\zeta = \frac{4}{2w_n} = 2\sqrt{\frac{J}{K}}$$

$$\zeta = 2\sqrt{\frac{J}{K}} = 0.456$$

$$\frac{J}{K} = 0.052 \Rightarrow$$

$$D = 1.04$$

$$J = 0.26$$

# 4.7 SYSTEM RESPONSE WITH ADDITIONAL POLES

- Up to now, we analyzed systems with one or two poles.
- Note that the formulae describing OS%, Ts and Tp are derived only for a system with two complex poles and no zeros!
- What about if a system has more than two poles or has zeros ?

One of the Approaches:

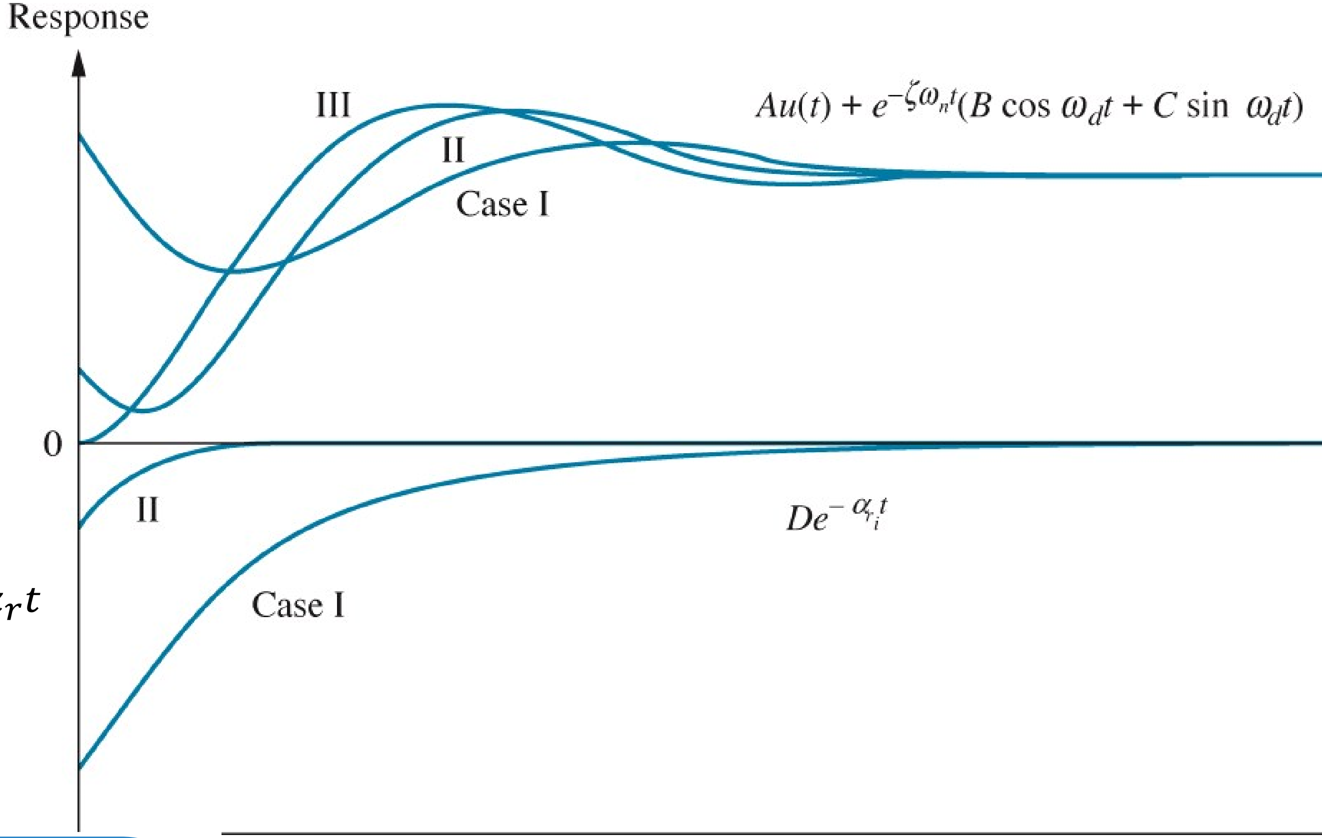
- Approximate it as a second-order system that has just two complex **dominant** poles.

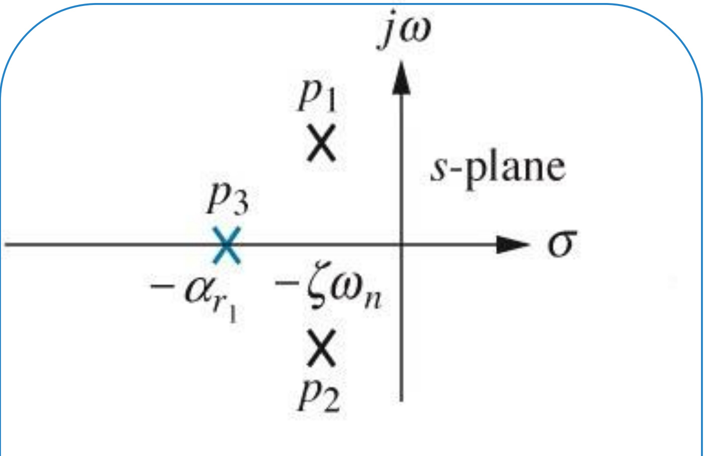
Those closed-loop poles that have dominant effects on the transient-response behavior are called dominant closed-loop poles.

<u>Consider a three-pole system</u> with complex poles and a third pole on the real axis. Step response of a such system is

$$C(s) = \frac{A}{s} + \frac{B(s + \zeta w_n) + C w_d}{(s + \zeta w_n)^2 + {w_d}^2} + \frac{D}{s + \alpha_r}$$

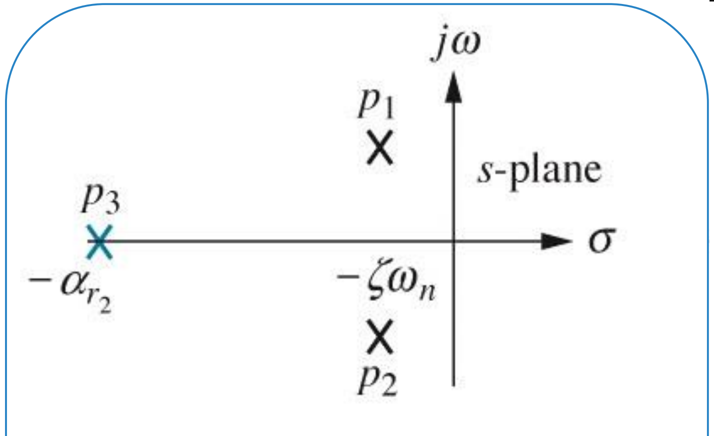
$$c(t) = Au(t) + e^{-\zeta w_n t}(B \cos w_d t + C \sin w_d t) + De^{-\alpha_r t}$$



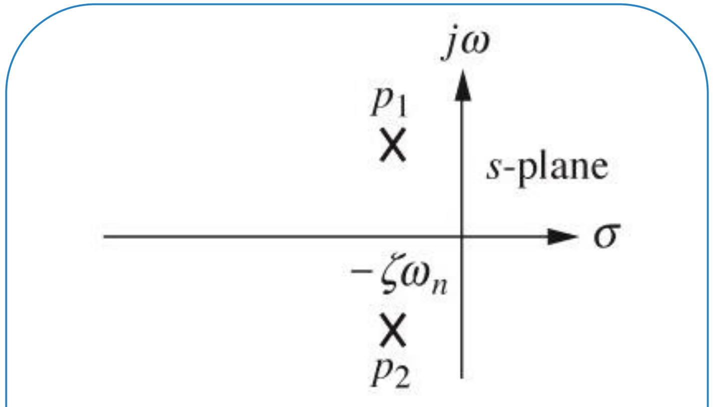

Case I

$\alpha_r = \alpha_{r_1}$ and $\alpha_r \cong \zeta w_n$




Case II

$\alpha_r = \alpha_{r_2}$ and $\alpha_r \gg \zeta w_n$



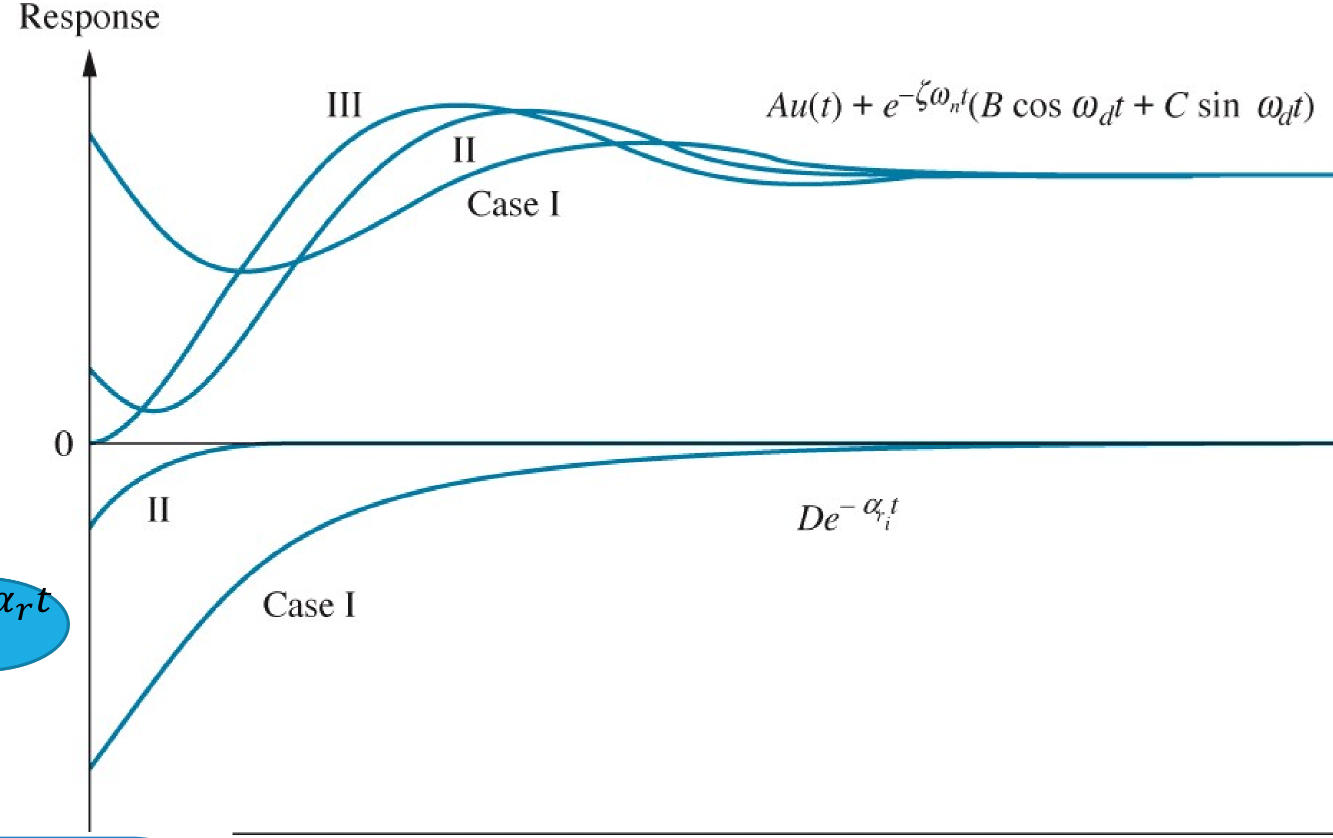

Case III

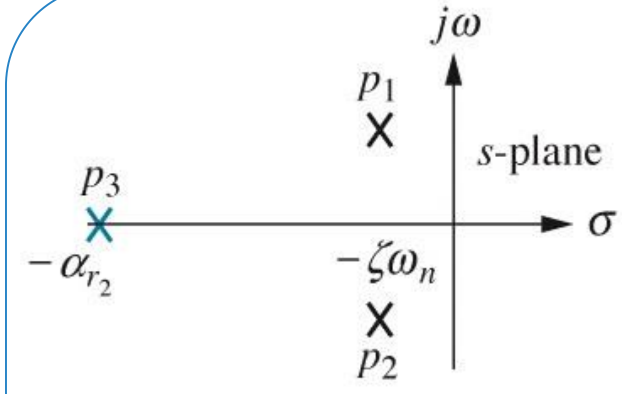
$\alpha_r = \infty$

Consider a three-pole system with complex poles and a third pole on the real axis. Step response of a such system is

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta w_n) + C w_d}{(s+\zeta w_n)^2 + {w_d}^2} + \frac{D}{s+\alpha_r}$$

$$c(t) = Au(t) + e^{-\zeta w_n t}(B\cos w_d t + C\sin w_d t) + De^{-\alpha_r t}$$




Case II

$\alpha_r = \alpha_{r_2}$ and $\alpha_r \gg \zeta w_n$

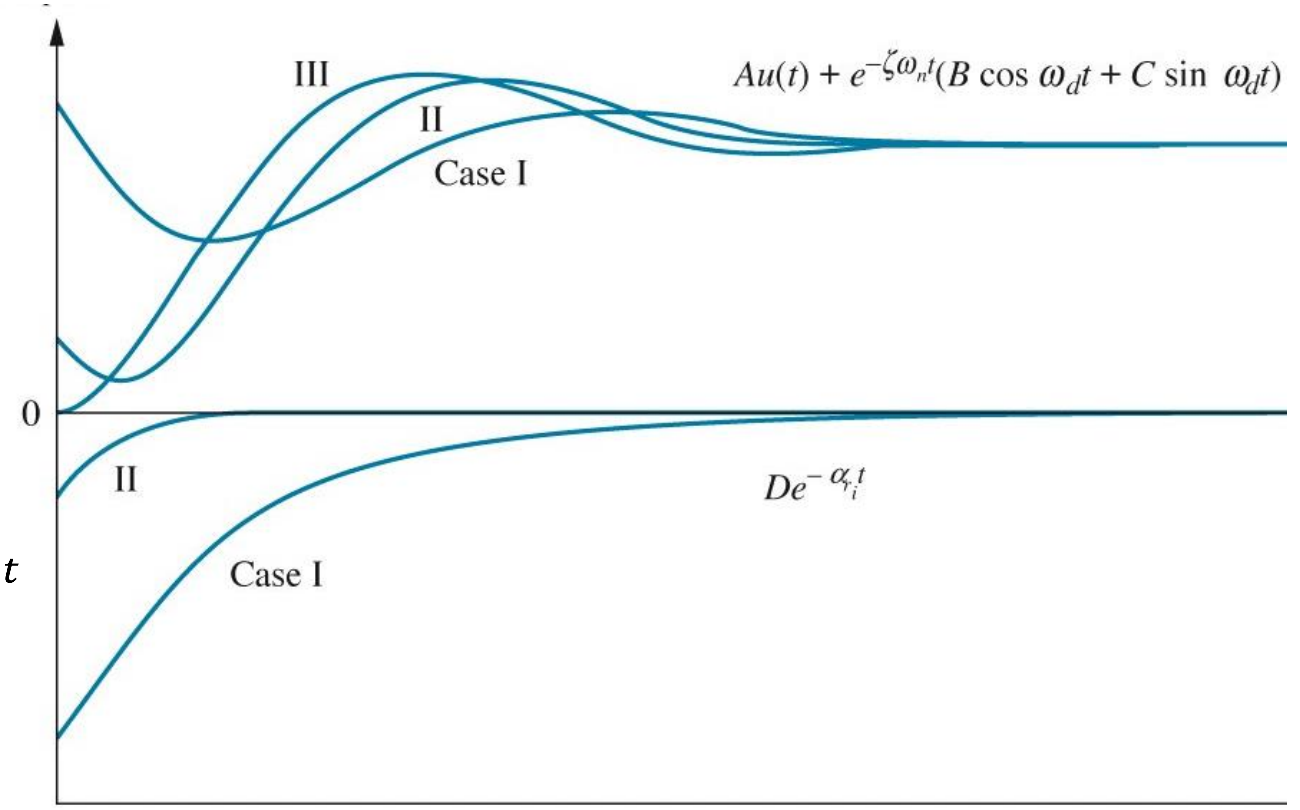
The pure exponential will die out much more rapidly than the second-order underdamped step response.

OS%,Ts,Tp can be calculated from the second-order underdamped step respose component.

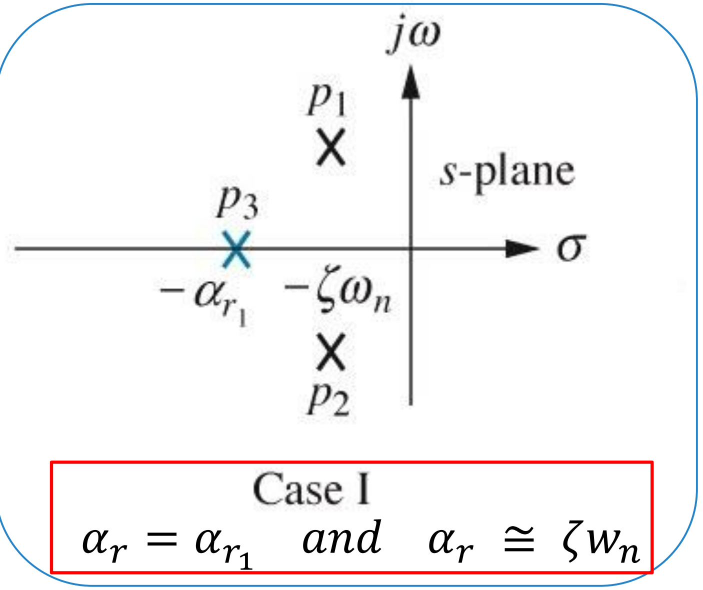
Consider a three-pole system with complex poles and a third pole on the real axis. Step response of a such system is

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta w_n) + C w_d}{(s+\zeta w_n)^2 + {w_d}^2} + \frac{D}{s+\alpha_r}$$

$$c(t) = Au(t) + e^{-\zeta w_n t}(B\cos w_d t + C \sin w_d t) + De^{-\alpha_r t}$$



(b)




The real pole's transient respose will not decay to insignificance at the settling time generated by the second-order pair.

In this case, the exponential decay is significant, and the system cannot be represented as a second order system.

Assume that the exponential decay is negligible after five time constants.
Thus, if $|5\alpha_r| > |\zeta w_n|$ then represent the system by its dominant second-order pair of poles

Consider a three-pole system with complex poles and a third pole on the real axis. Step response of a such system is

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta w_n) + C w_d}{(s+\zeta w_n)^2 + {w_d}^2} + \frac{D}{s+\alpha_r}$$

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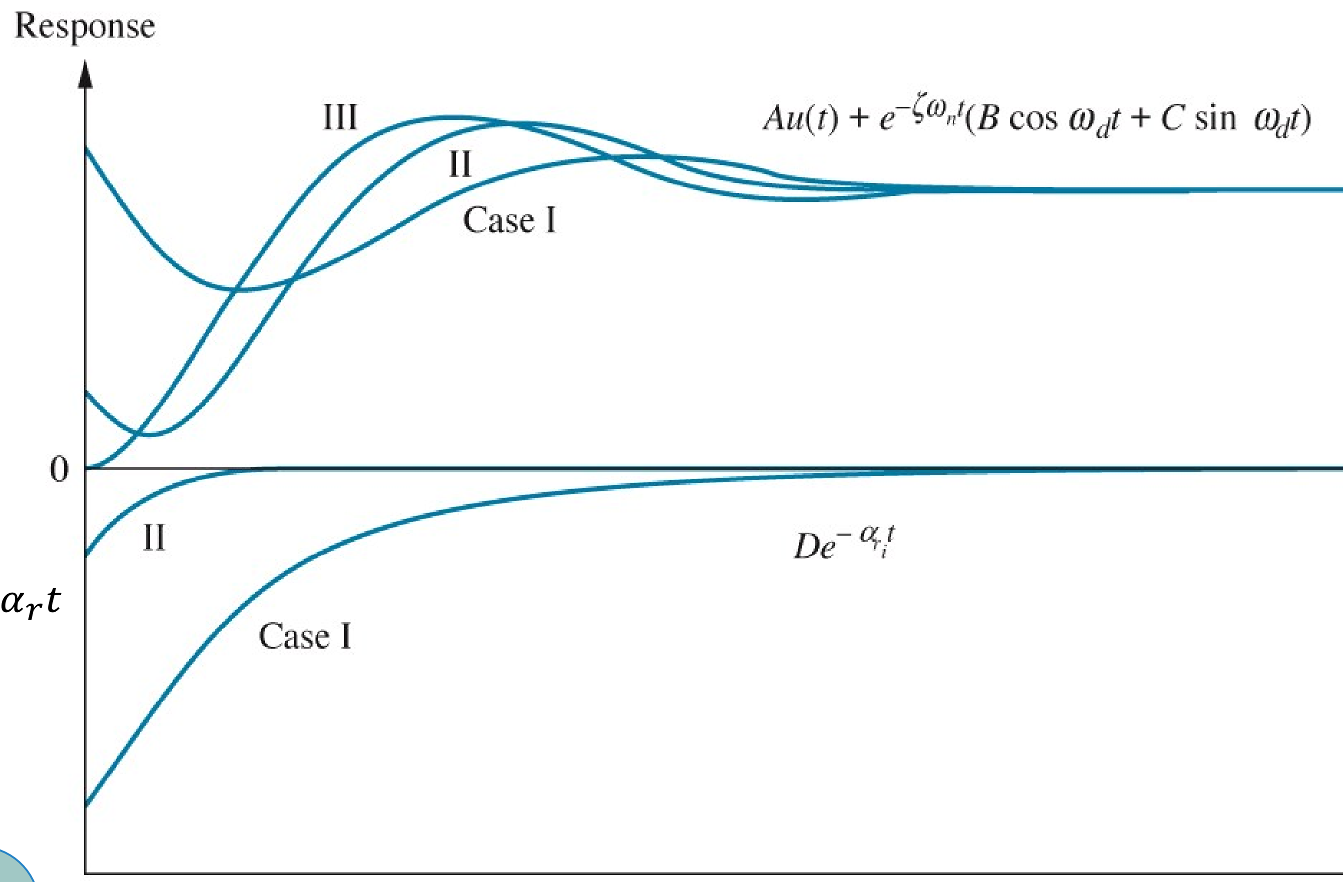
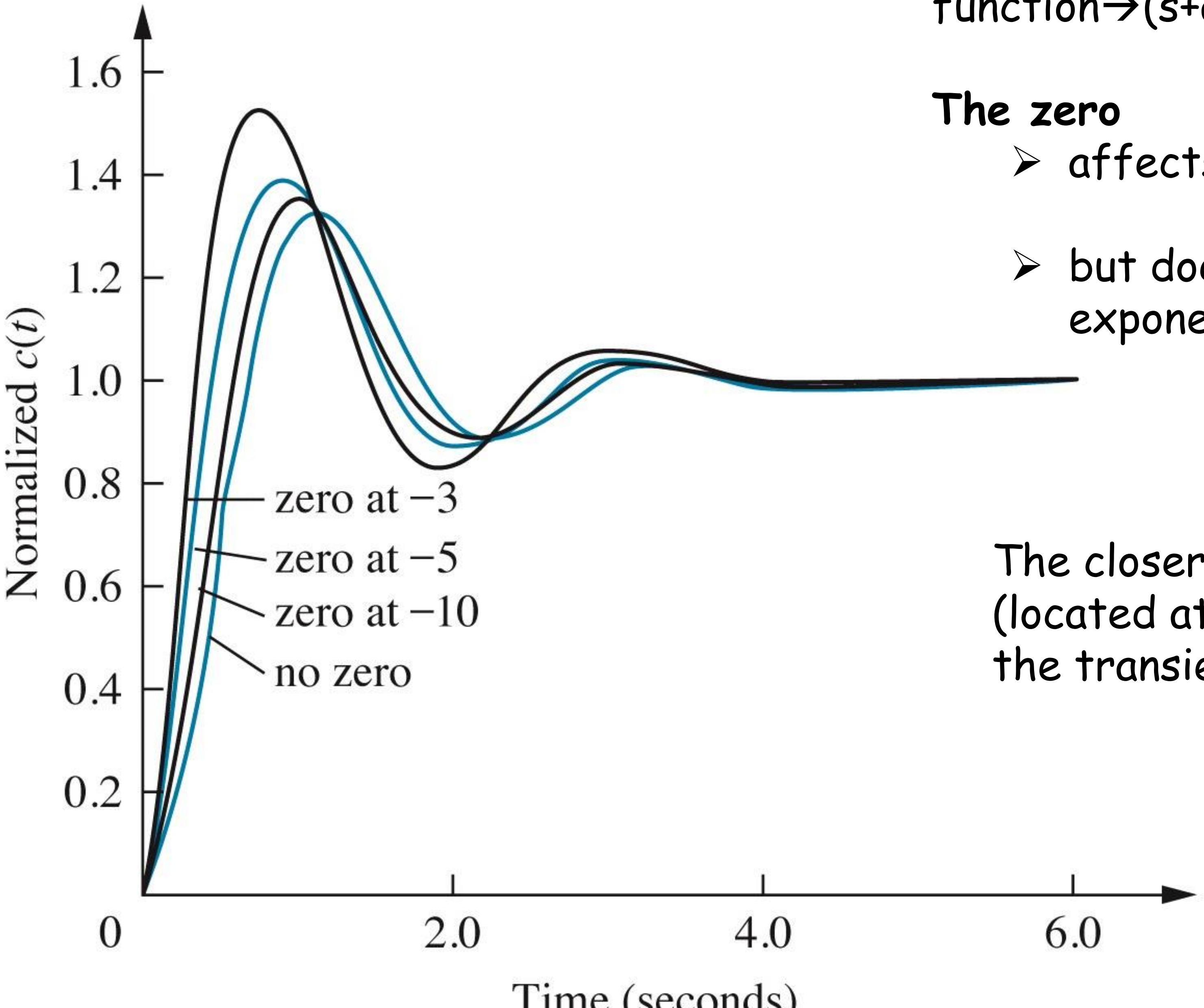



- What about the magnitude D of the pure exponential decay, $e^{-\alpha_r t}$?
- Can it be so large that its contribution is not negligible?

No. Because

✓ In a three-pole system with dominant second-order poles and no zeros, the residue of the third pole,D, will actually decrease in magnitude as the third pole is moved farther into the left half plane. Therefore, D cannot be too large.

# 4.8 SYSTEM RESPONSE WITH ZEROS



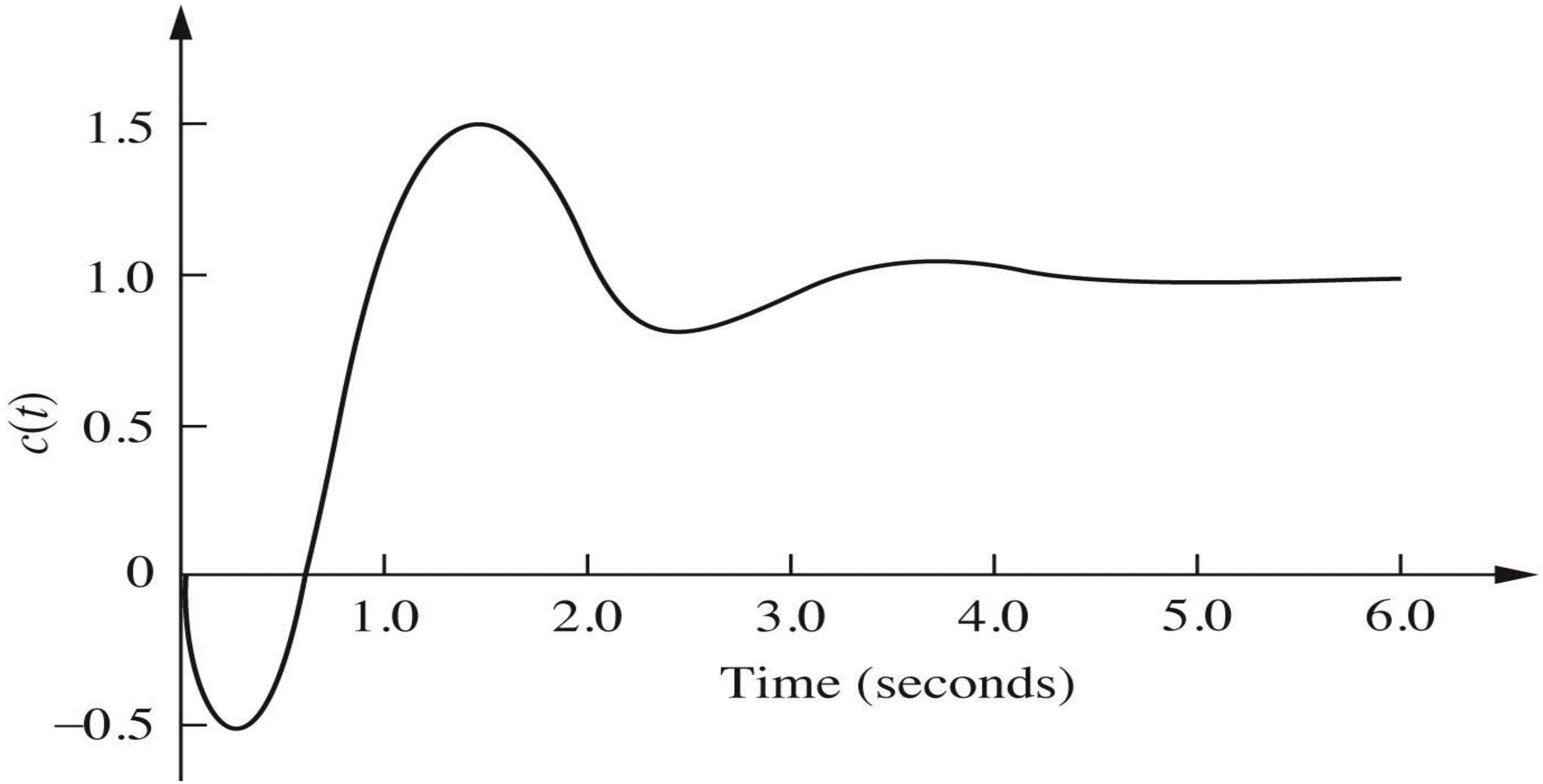

If we add a zero at –a to the transfer function→(s+a)T(s)

**The zero**

- affects the amplitude of a response
- but does not affect the nature of the response-exponential, damped sinusoid, and so on.

The closer the zero is to the dominant poles (located at $-1 \pm j2.8$), the greater its effect on the transient response.

An interesting phenomenon occurs if the zero is in the right half-plane (i.e a is negative.)




- Initially, the response may turn toward the negative direction even though the final value is positive.

- A system that exhibits this phenomenon is known as a nonminimum-phase system.

- If a motorcycle or airplane was a nonmimimum-phase system, it would initially veer left when commanded to steer right.