MODELING IN THE TIME DOMAIN

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3.1 INTRODUCTION

Mathematical Models

Classical or Frequency Domain Technique

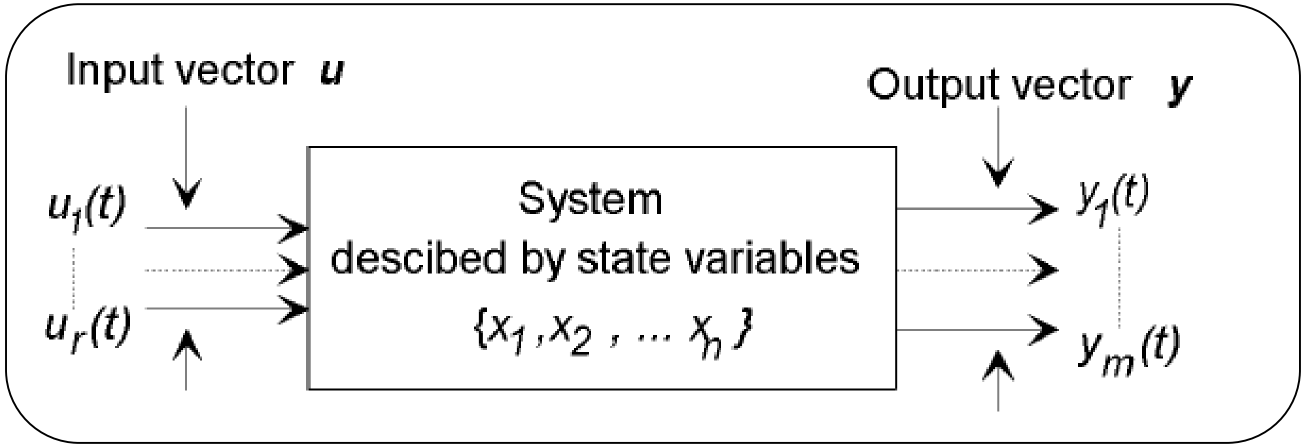
- Advantages
  - Converts system's differential equation into a transfer function, thus gives a model that *algebraically* relates output to input.
  - Rapidly provides stability and transient response information.
- Disadvantages
  - Applicable only to Linear, Time-Invariant (LTI) systems or their close approximations.

For example, LTI modeling in space applications is inadequate. Models for time-varying systems (for example, missiles with varying fuel levels) are necessary.

Modern or State-Space or Time Domain Technique

- Advantages
  - Provides a unified method for modeling, analyzing, and designing a wide range of systems using matrix algebra.
  - Useful to represent nonlinear systems that have backlash, saturation, and dead zone.
  - Nonlinear, Time-Varying, Multiple-input, multiple-output systems
- Disadvantages
  - Not as intuitive as classical method.
  - Calculations required before physical interpretation is apparent

3.3 STATE SPACE REPRESENTATION



Input vector

$$\boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

State vector

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Output vector

$$\boldsymbol{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix}$$

System matrix

Input matrix

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

Output matrix

Feedforward matrix

DEFINITIONS

System variables: Any variable that responds to an input or initial conditions.

1

State variables: In other words; **the smallest set of *linearly independent* system variables** such that *knowledge of these variables at $t=t_0$, together with knowledge of* the input, **completely determines the behavior of the system** for any time $t \geq t_0$.

Note that the concept of state is by no means limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

State Vector: A vector whose elements are the state variables.

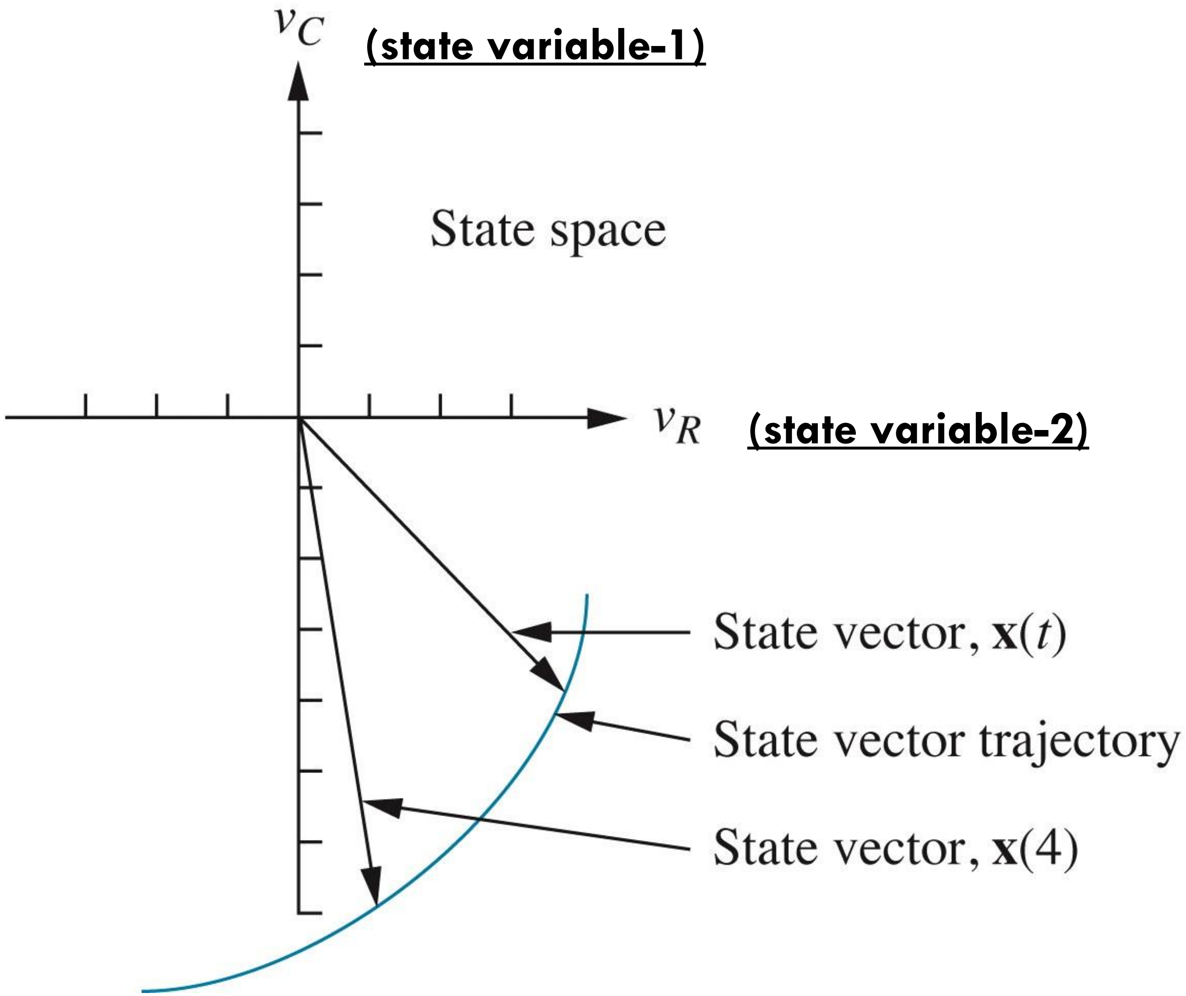
1

Linear combination: *A linear combination of n variables, x_i, for i=1 to n, is $S = K_n x_n + K_{n-1}x_{n-1} + \cdots K_1 x_1$ where eack K_i is a constant.*

Linear Independence: *A set of variables is linearly independent if none of the variables can be written as a linear combination of the others.*

State Space: The n-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, ..., x_n axis, where x_1, x_2,...... , x_n are state variables, is called a *state space*.

"State space" refers to **the space whose axes are the state variables**. The state of the system can be represented as a vector within that space.



State Equations: A set of n simultaneous, first order differential equations with n state variables.

Output Equation: The algebraic equation that expresses the output variables of a system as a linear combinations of the state variables and the inputs.

State vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

State Equation $\boxed{\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}}$

Output Equation $\boxed{\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}}$

Input or control vector

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

Output vector

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

3.4 APPLYING THE STATE-SPACE REPRESENTATION

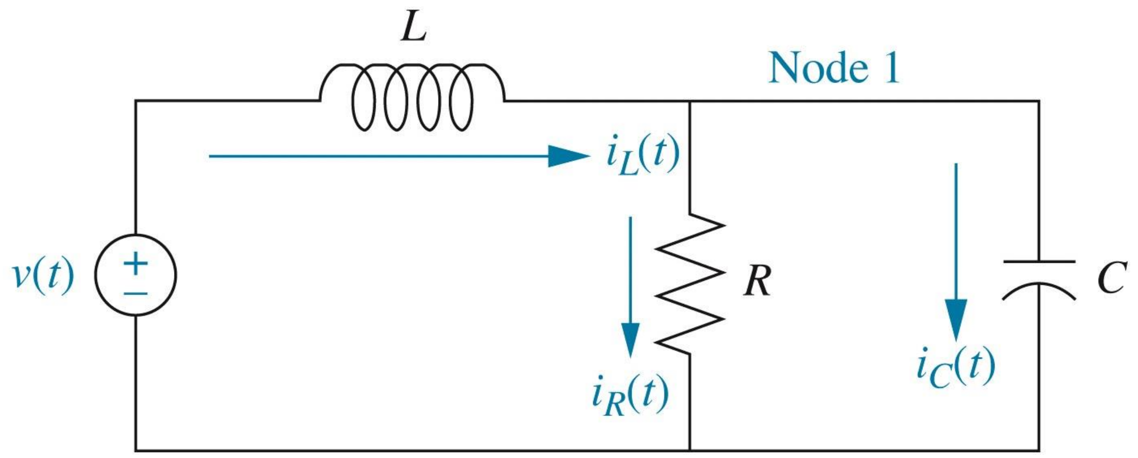
The state vector must be chosen according to the following considerations:

- A minimum number of state variables should be selected.
- The minimum number of state variables must be linearly independent.

The minimum number of state variables is equal to:

- ✓ The order of the differential equation describing the system.
- ✓ The order of the denominator of the transfer function
- ✓ The number of independent energy-storage elements

Example: Find a state-space representation of the circuit given below if the output is the current through the resistor.



1) Select the state variables by writing the derivative equation for all energy-storage elements:

$$C\frac{dv_c}{dt} = i_C \qquad L\frac{di_L}{dt} = v_L$$

Choose as the state variables

2) Obtain i_C and v_L in terms of state variables:

$$i_C = -i_R + i_L = -\frac{1}{R}v_c + i_L$$
$$v_L = -v_C + v$$

3) Substitute the equations in step 2 into the ones in step-1.(i.e. Obtain state equations)

$$C\frac{dv_c}{dt} = -\frac{1}{R}v_c + i_L \Rightarrow \frac{dv_c}{dt} = -\frac{1}{RC}v_c + \frac{1}{C}i_L$$
$$L\frac{di_L}{dt} = -v_C + v \Rightarrow \frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}v$$

4) Find the output equation:

$$i_R = \frac{1}{R}v_c$$

5) Write state-space representation in vector-matrix form:

$$\begin{bmatrix}\dot{v}_C\\ \dot{i}_L\end{bmatrix} = \begin{bmatrix}-1/(RC) & 1/C\\ -1/L & 0\end{bmatrix}\begin{bmatrix}v_C\\ i_L\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{L}\end{bmatrix}v$$

$$i_R = \begin{bmatrix}1/R & 0\end{bmatrix}\begin{bmatrix}v_C\\ i_L\end{bmatrix}$$

3.5 CONVERTING A TRANSFER FUNCTION TO STATE SPACE

- Consider the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + \frac{d\ y}{dt} + a_0 y = b_0 u$$

where y is the measure variable and u is the input.

Note: Corresponding transfer function is

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s^1 + a_0}$$

- The minimum number of state variables is *n* since the differential equation is *n*th order.
- Convenient way: Choose the output, *y(t)*, and its (*n*-1) derivatives as state variables.

$$x_1 = y$$

$$x_2 = \dot{y} \Rightarrow \dot{x}_1 = x_2$$

First row of state equations

$$\vdots$$

$$x_n = \frac{d^{n-1} y}{dt^{n-1}} \Rightarrow \dot{x}_{n-1} = x_n$$

$$\dot{x}_n = \frac{d^n y}{dt^n} = -a_0 x_1 - a_1 x_2 \cdots - a_{n-1} x_n + b_0 u$$

Last row of state equations

- Arrange in vector-matrix format.

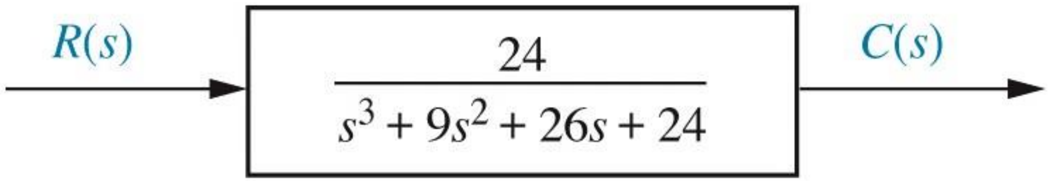
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Note the transfer function format

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s^1 + a_0}$$

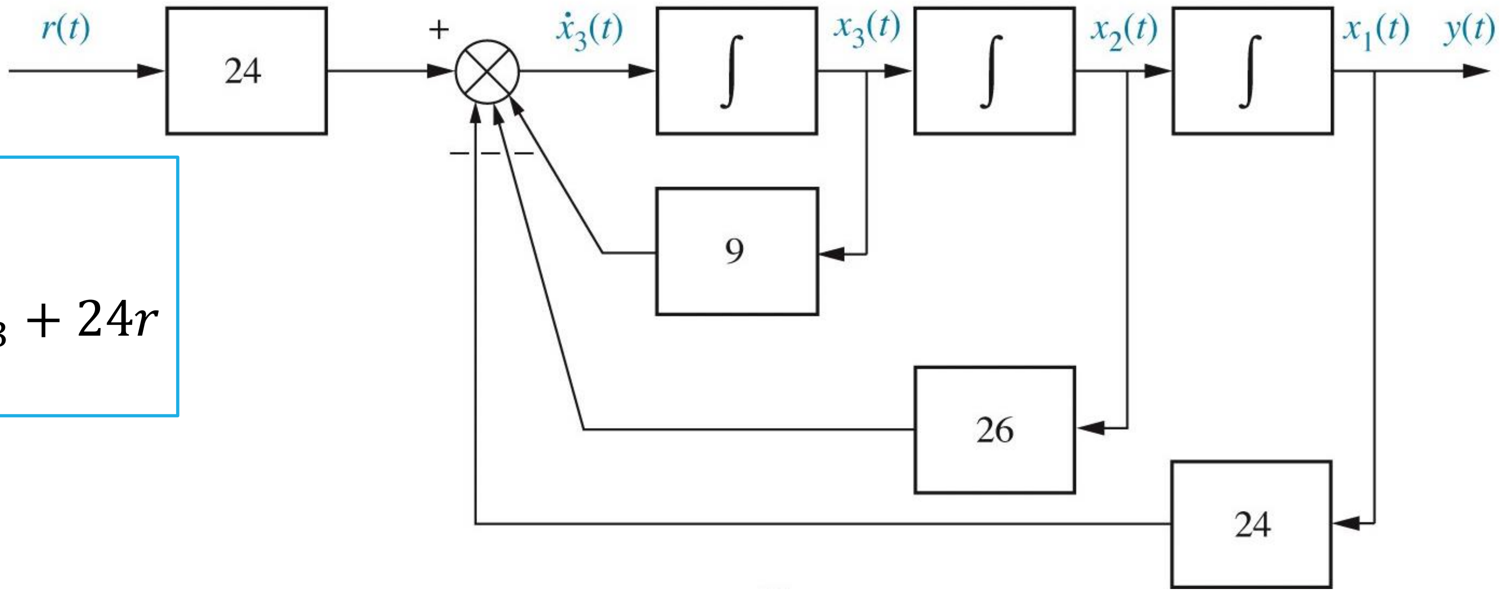
Example: Find the state-space representation for the transfer function given below.



$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)} \Rightarrow (s^3 + 9s^2 + 26s + 24)\mathrm{C(s)} = 24\mathrm{R(s)}$$

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

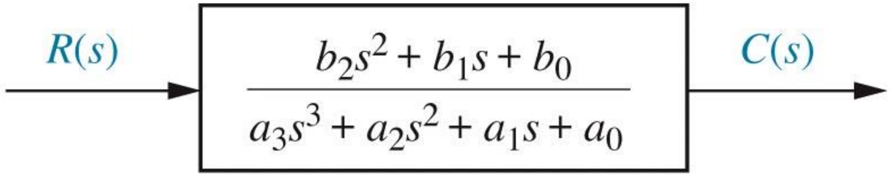
$$\left.\begin{aligned} x_1 &= c \\ x_2 &= \dot{c} \\ x_3 &= \ddot{c} \end{aligned}\right\} \quad \boxed{\begin{gathered} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r \\ y = c = x_1 \end{gathered}}$$



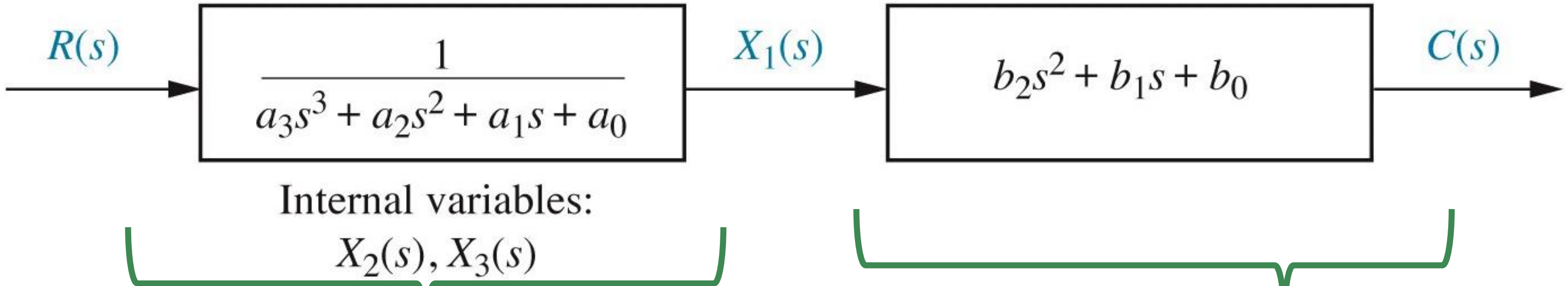
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} \mathrm{r}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

TRANSFER FUNCTION WITH NUMERATOR POLYNOMIAL



1) Seperate the transfer function into two cascaded transfer functions, as shown.



2) Obtain state equations by taking x_1 as the output.

$$\frac{X_1(s)}{R(s)} = \frac{1/a_3}{s^3 + \frac{a_2}{a_3}s^2 + \frac{a_1}{a_3}s + \frac{a_0}{a_3}}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{a_0}{a_3} & -\frac{a_1}{a_3} & -\frac{a_2}{a_3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{a_3} \end{bmatrix}u$$

3) Obtain the output equation. .

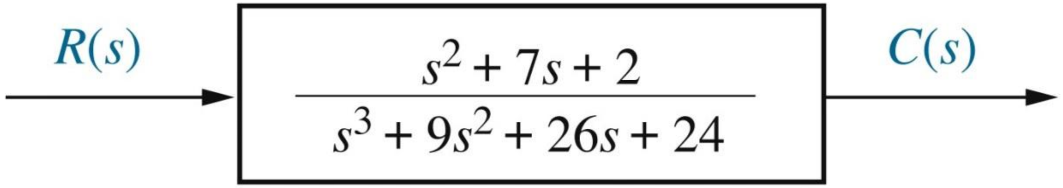
$$Y(s) = C(s) = (b_2s^2 + b_1s + b_0)X_1(s)$$

$$y(t) = b_2\frac{d^2x_1}{dt^2} + b_1\frac{dx_1}{dt} + b_0x_1$$

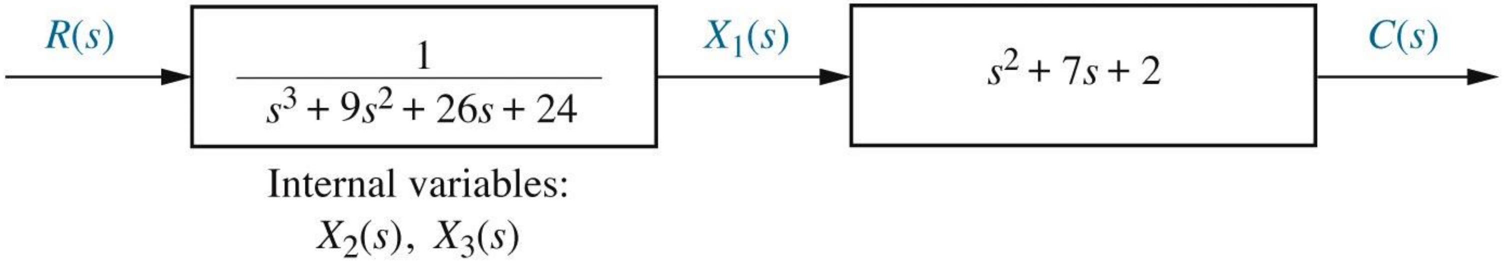
$$y(t) = b_0x_1 + b_1x_2 + b_2x_3$$

$$y = [b_0 \quad b_1 \quad b_2]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example: Find the state-space representation for the transfer function given below.



1) Seperate the transfer function into two cascaded transfer functions.



2) Obtain state equations

$$\left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= -24x_1 - 26x_2 - 9x_3 + r\end{aligned}\right\} \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\mathrm{r}$$

3) Obtain the output equation.

$$\mathrm{C(s)} = (s^2 + 7s + 2)X_1(\mathrm{s})$$

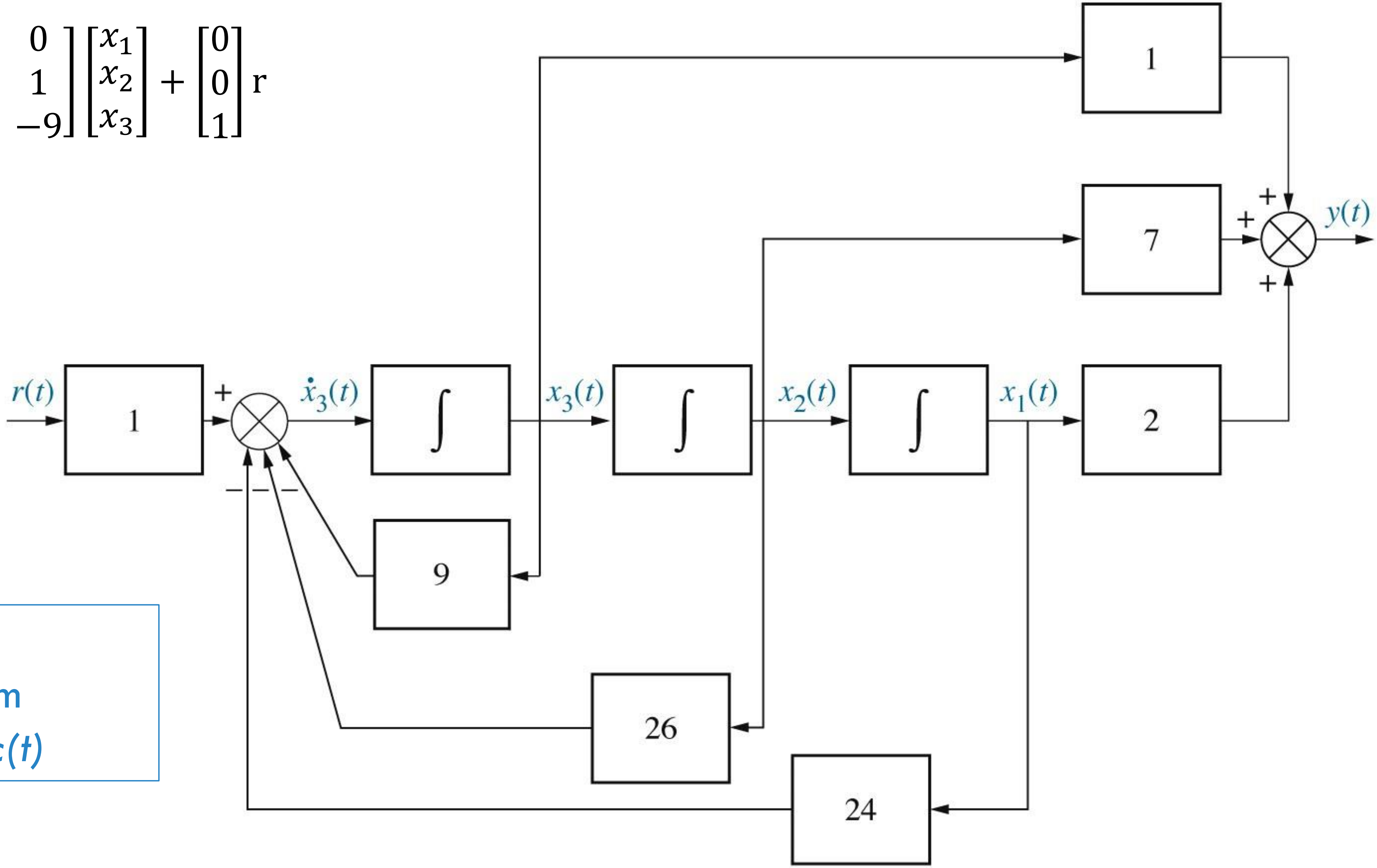
$$\ddot{x}_1 = x_3$$
$$\dot{x}_1 = x_2$$

$$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1$$

$$y(t) = 2x_1 + 7x_2 + x_3 \Rightarrow \quad y = \begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} r$$

$$y = \begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$



equivalent
block diagram
Note: $y(t) = c(t)$

3.6 CONVERTING FROM STATE SPACE TO A TRANSFER FUNCTION

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

- Take the Laplace transform assuming zero initial conditions

$$s\boldsymbol{X}(s) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s)$$
$$\boldsymbol{Y}(s) = \boldsymbol{C}\boldsymbol{X}(s) + \boldsymbol{D}\boldsymbol{U}(s)$$

Identity matrix

- Solve for $\boldsymbol{X}(s)$

$$(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{B}\boldsymbol{U}(s) \Rightarrow \boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)$$

- Use $\boldsymbol{X}(s)$ in output equation

$$\boldsymbol{Y}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s) + \boldsymbol{D}\boldsymbol{U}(s)$$
$$\boldsymbol{Y}(s) = [\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}]\boldsymbol{U}(s)$$

Transfer function matrix

- For Single-Input, Single-output (SISO) systems, the transfer function is:

$$T(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

Example: Find the transfer function for the system defined by

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}$$

$$(s\boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{adj(s\boldsymbol{I} - \boldsymbol{A})}{det(s\boldsymbol{I} - \boldsymbol{A})} = \frac{\begin{bmatrix} s^2 + 3s + 2 & s + 3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

$$\boldsymbol{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{D} = 0$$

$$T(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

Example: Find the transfer function for the multiple-input multiple-output (MIMO) system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{adj(sI - A)}{|sI - A|}$$

$$= \frac{1}{s(s+4)(s+2)+3+3s} \begin{bmatrix} s^2+6s+11 & s+2 & 3 \\ -3 & s^2+2 & 3s \\ s+4 & -s-1 & s^2+4s \end{bmatrix}$$

$$G(s) = [C(sI - A)^{-1} B + D]$$

$$= \frac{1}{s^3+6s^2+11s+3} \begin{bmatrix} s+2 & 3 \\ -(s+1) & s(s+4) \end{bmatrix} = \begin{bmatrix} \frac{Y_1}{U_1} & \frac{Y_1}{U_2} \\ \frac{Y_2}{U_1} & \frac{Y_2}{U_2} \end{bmatrix}$$

Transfer function matrix

$$\frac{Y_1}{U_1} = \frac{s+2}{s^3+6s^2+11s+3}$$

$$\frac{Y_1}{U_2} = \frac{3}{s^3+6s^2+11s+3}$$

$$\frac{Y_2}{U_1} = \frac{-(s+1)}{s^3+6s^2+11s+3}$$

$$\frac{Y_2}{U_2} = \frac{s(s+4)}{s^3+6s^2+11s+3}$$

3.7 LINEARIZATION

- A prime advantage of the state-space representation over the transfer function representation is the ability to represent systems with nonlinearities.

- We can linearize the state eq. about the equilibrium point and can use only for a limited range of operation.
- Let us represent **a nonlinear system** by the following vector-matrix formed state equations:

Note that it cannot be written in the form $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} f_1(x_1, x_2, \dots x_n, u_1, u_2, \dots u_r) \\ f_2(x_1, x_2, \dots x_n, u_1, u_2, \dots u_r) \\ \vdots \\ f_n(x_1, x_2, \dots x_n, u_1, u_2, \dots u_r) \end{bmatrix}$$

- For small-signal linearization, first determine equilibrium points $\boldsymbol{x}_0, \boldsymbol{u}_0$ from

$$\dot{\boldsymbol{x}}_0 = \boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0)$$

- Let $\boldsymbol{x}$ and $\boldsymbol{u}$ be perturbed about the equilibrium point and use taylor series to expand

$$\boldsymbol{x} = \boldsymbol{x}_0 + \delta\boldsymbol{x}$$
$$\boldsymbol{u} = \boldsymbol{u}_0 + \delta\boldsymbol{u}$$

$$\dot{\boldsymbol{x}}_0 + \delta\dot{\boldsymbol{x}} \approx f(\boldsymbol{x}_0, \boldsymbol{u}_0) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{(\boldsymbol{x}_0, \boldsymbol{u}_0)} \delta\boldsymbol{x} + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{(\boldsymbol{x}_0, \boldsymbol{u}_0)} \delta\boldsymbol{u}$$

$$\cancelto{0}{\dot{\boldsymbol{x}}_0} + \delta\dot{\boldsymbol{x}} \approx \cancelto{0}{f(\boldsymbol{x}_0,\boldsymbol{u}_0)} + \boxed{\left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)}} \delta\boldsymbol{x} + \boxed{\left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)}} \delta\boldsymbol{u}$$

$$\begin{bmatrix} \delta\dot{x}_1 \\ \vdots \\ \delta\dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_n}{\partial x_1}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} & \cdots & \left.\frac{\partial f_n}{\partial x_n}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{bmatrix} + \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial u_1}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} & \cdots & \left.\frac{\partial f_1}{\partial u_r}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_n}{\partial u_1}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} & \cdots & \left.\frac{\partial f_n}{\partial u_r}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} \end{bmatrix}}_{\boldsymbol{B}} \begin{bmatrix} \delta u_1 \\ \vdots \\ \delta u_r \end{bmatrix}$$

- Then drop $'\delta'$ and abusing notation, write the linearized differential equation

$$\boxed{\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}}$$

Example: Find the linearized state equation for the following differential equation of nonlinear spring.

$$\ddot{y} = k_1 y + k_2 y^3$$

$Let\ x_1 = y\ \ and\ \ x_2 = \dot{y}$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ k_1 x_1 + k_2 {x_1}^3 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

To find the equilibrium points, we must solve

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_2 \\ k_1 x_1 + k_2 {x_1}^3 \end{bmatrix} = 0$$

Then

$$f_1(x_1, x_2) = 0 \Rightarrow x_{02} = 0$$

$$f_2(x_1, x_2) = 0 \Rightarrow k_1 x_{01} + k_2 {x_{01}}^3 = 0 \quad \Rightarrow \quad x_{01} = 0 \text{ or } x_{01} = \pm\sqrt{{-k_1}/{k_2}}$$

$$\delta\dot{\boldsymbol{x}} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} \delta\boldsymbol{x} + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} \delta\boldsymbol{u}$$

For the state space model

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x_0} = \begin{bmatrix} 0 & 1 \\ k_1 + 3k_2{x_1}^2 & 0 \end{bmatrix}_{x_0} = \begin{bmatrix} 0 & 1 \\ k_1 + 3k_2{x_{01}}^2 & 0 \end{bmatrix}$$

For the equilibruim point $x_{01} = 0$ and $x_{02} = 0$

$$A = \begin{bmatrix} 0 & 1 \\ k_1 & 0 \end{bmatrix}$$

which are the standard dynamics of a system with just a linear spring of stiffness k_1

The linearized system is

$\delta\dot{\boldsymbol{x}} = \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ k_1 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

EXAMPLE

Given nonlinear system below, find linearized model

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -x_1(t)x_2(t) - x_2(t) + u(t), \quad x_0 = [x_{01} \quad x_{02}] = [1 \;\; 0], \;\; u_0 = 0$$

Solution

$$\dot{x}_1(t) = x_2(t) = f_1(x_1(t), x_2\,(t), u(t))$$

$$\dot{x}_2(t) = -x_1(t)x_2(t) - x_2(t) + u(t) = f_2(x_1(t), x_2\,(t), u(t))$$

$$\dot{x}_1(t) = x_2(t) = f_1(x_1(t), x_2(t), u(t))$$
$$\dot{x}_2(t) = -x_1(t)x_2(t) - x_2(t) + u(t) = f_2(x_1(t), x_2(t), u(t))$$

$$\delta\dot{\boldsymbol{x}} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{(\boldsymbol{x}_0, \boldsymbol{u}_0)} \delta\boldsymbol{x} + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{(\boldsymbol{x}_0, \boldsymbol{u}_0)} \delta\boldsymbol{u}$$

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x_0, u_0} = \begin{bmatrix} 0 & 1 \\ -x_{02} & -x_{01} - 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$$B = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\right|_{x_0, u_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

linearized system:

$$\delta\dot{\boldsymbol{x}} = \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \Rightarrow \quad \begin{aligned} &\delta\dot{x}_1(t) = x_2(t) \\ &\delta\,\dot{x}_2(t) = -2x_2(t) + u(t) \end{aligned}$$

EXAMPLE

Given the nonlinear system below, find the linearized model

$\dot{x}_1(t)=x_2(t)$

$\dot{x}_2(t)=-\sin x_1(t)-x_2(t) \; +u(t)$

Nominal (Equilibrium) point;

$u_0 = \frac{1}{\sqrt{2}}, \quad x_{01} = \frac{\pi}{4} \;, x_{02} = 0$

Solution

$$\dot{x}_1(t)=x_2(t) = f_1\ (x_1(t), x_2\ (t), u(t))$$
$$\dot{x}_2(t)=\text{-}\sin x_1(t) - 0.1x_2(t)\ + u(t) = f_2(x_1(t), x_2\ (t), u(t))$$

$$\delta\dot{\boldsymbol{x}} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} \delta\boldsymbol{x} + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{(\boldsymbol{x}_0,\boldsymbol{u}_0)} \delta\boldsymbol{u}$$

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x_0,u_0} = \begin{bmatrix} 0 & 1 \\ -cos x_{01} & -0.1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.707 & -0.1 \end{bmatrix}$$

$$B = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\right|_{x_0,u_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\delta\dot{\boldsymbol{x}} = \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.707 & -0.1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \Rightarrow \begin{array}{l} \delta\dot{x}_1(t)=x_2(t) \\ \delta\dot{x}_1(t)=-0.707x_1(t) - 0.1x_2(t) + u(t) \end{array}$$

4.10 LAPLACE TRANSFORM SOLUTION OF STATE EQUATIONS

- Consider the state and output equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$$

- Take the Laplace transform and solve for $\boldsymbol{X}(s)$:

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{AX}(s) + \boldsymbol{BU}(s)$$
$$\boldsymbol{Y}(s) = \boldsymbol{CX}(s) + \boldsymbol{DU}(s)$$

$$(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{x}(\mathbf{0}) + \boldsymbol{BU}(s) \quad \Rightarrow \quad \boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{BU}(s)$$

$$\boldsymbol{X}(s) = \frac{adj(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})}[\boldsymbol{x}(0) + \boldsymbol{BU}(s)]$$

- By the definition of a transfer, let $\boldsymbol{x}(0)$=0. Then use $\boldsymbol{X}(s)$ in output equation

$$\boldsymbol{Y}(s) = \boldsymbol{C}\frac{adj(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})}\boldsymbol{BU}(s) + \boldsymbol{DU}(s)$$

- The transfer function is:

$$T(s) = \frac{Y(s)}{U(s)} = \frac{\boldsymbol{C}\, adj(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} + \mathbf{D}\det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})}$$

4.10 LAPLACE TRANSFORM SOLUTION OF STATE EQUATIONS

- The transfer function is:

$$T(s) = \frac{Y(s)}{U(s)} = \frac{\boldsymbol{C}\, adj(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} + \mathbf{D}\det(s\boldsymbol{I} - \boldsymbol{A})}{\det(s\boldsymbol{I} - \boldsymbol{A})}$$

The roots of the denominator are the poles of the system. Find poles from

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = 0$$

The system poles equal the eigenvalues of the matrix **A**!!!

Note that **the poles** of the transfer function **determine the nature of the transient response** of the system.

Example: a)Solve the state equation and the output.

b)Find the eigenvalues and the system poles

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

$$y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s)$$

$$\boldsymbol{Y}(s) = \boldsymbol{C}\boldsymbol{X}(s) + \boldsymbol{D}\boldsymbol{U}(s)$$

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)$$

$$(sI - A)^{-1} = \frac{adj(sI - A)}{|sI - A|}$$

$$= \frac{1}{s^3 + 9s^2 + 26s + 24} \begin{bmatrix} s^2 + 9s + 26 & s + 9 & 1 \\ -24 & s^2 + 9s & s \\ -24s & -(26s + 24) & s^2 \end{bmatrix}$$

$$\boldsymbol{X}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s^3 + 10s^2 + 37s + 29}{(s+1)(s+2)(s+3)(s+4)} \\ \dfrac{2s^2 - 21s - 24}{(s+1)(s+2)(s+3)(s+4)} \\ \dfrac{s(2s^2 - 21s - 24)}{(s+1)(s+2)(s+3)(s+4)} \end{bmatrix}$$

$$\boldsymbol{Y}(s) = \boldsymbol{CX}(s) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = X_1(s) + X_2(s) = \frac{s^3+12s^2+16s+5}{(s+1)(s+2)(s+3)(s+4)}$$

$$\boldsymbol{Y}(s) = \frac{-6.5}{s+2} + \frac{19}{s+3} - \frac{11.5}{s+4} + \frac{0}{s+1}$$

The pole at -1
canceled a zero at -1

$$y(t) = -6.5e^{-2t} + 19e^{-3t} - 11.5e^{-4t}$$

b) The system poles equal the eigenvalues of the matrix **A**.

Therefore, both the poles of the system and eigenvalues are -2, -3, and -4.

4.11 TIME DOMAIN SOLUTION OF STATE EQUATIONS

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

The solution in the time domain is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$= \underbrace{\boldsymbol{\Phi}(t)\boldsymbol{x}(0)}_{\text{Zero-input response}} + \underbrace{\int_0^t \boldsymbol{\Phi}(t-\tau)\boldsymbol{B}\boldsymbol{u}(\tau)d\tau}_{\text{Zero-state response}}$$

Where $\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t}$ by definition, and which is called the state-transition matrix.

4.11 TIME DOMAIN SOLUTION OF STATE EQUATIONS

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

$$\boxed{\boldsymbol{x}(\mathrm{t}) = \boldsymbol{\Phi}(t)\boldsymbol{x}(0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\boldsymbol{B}\boldsymbol{u}(\tau)d\tau}$$

For the unforced system (i.e. Zero input)

$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(0)$

$$\mathcal{L}[x(t)] = \mathcal{L}[\boldsymbol{\Phi}(t)\boldsymbol{x}(0)] = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) \quad \Longrightarrow \mathcal{L}^{-1}[(s\boldsymbol{I} - \boldsymbol{A})^{-1}] = \mathcal{L}^{-1}\left[\frac{adj(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}\right] = \boldsymbol{\Phi}(t)$$

Exercise: If u(t) is a unit step find the state-transition matrix and then solve for x(t)

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

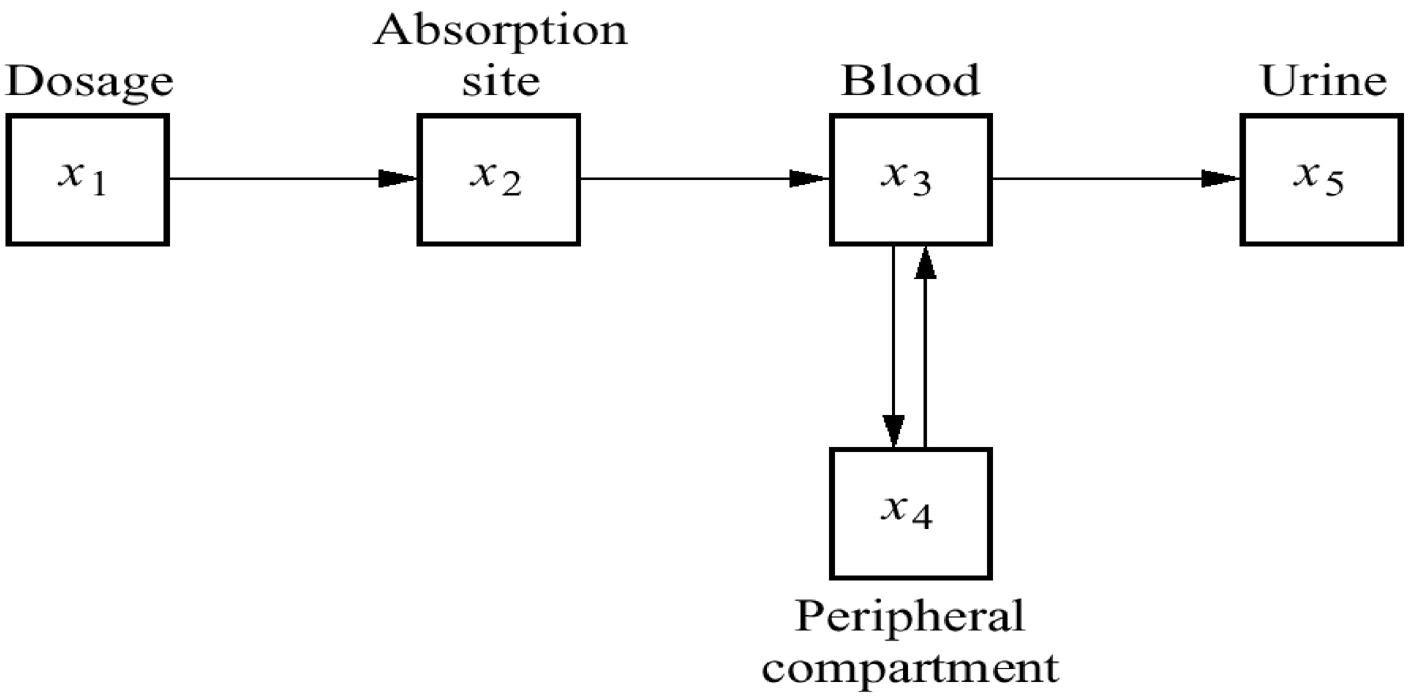
$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

You can find the solution in your textbook (see example 4.12 and example 4.13).

CASE STUDY: PHARMACEUTICAL DRUG ABSORPTION

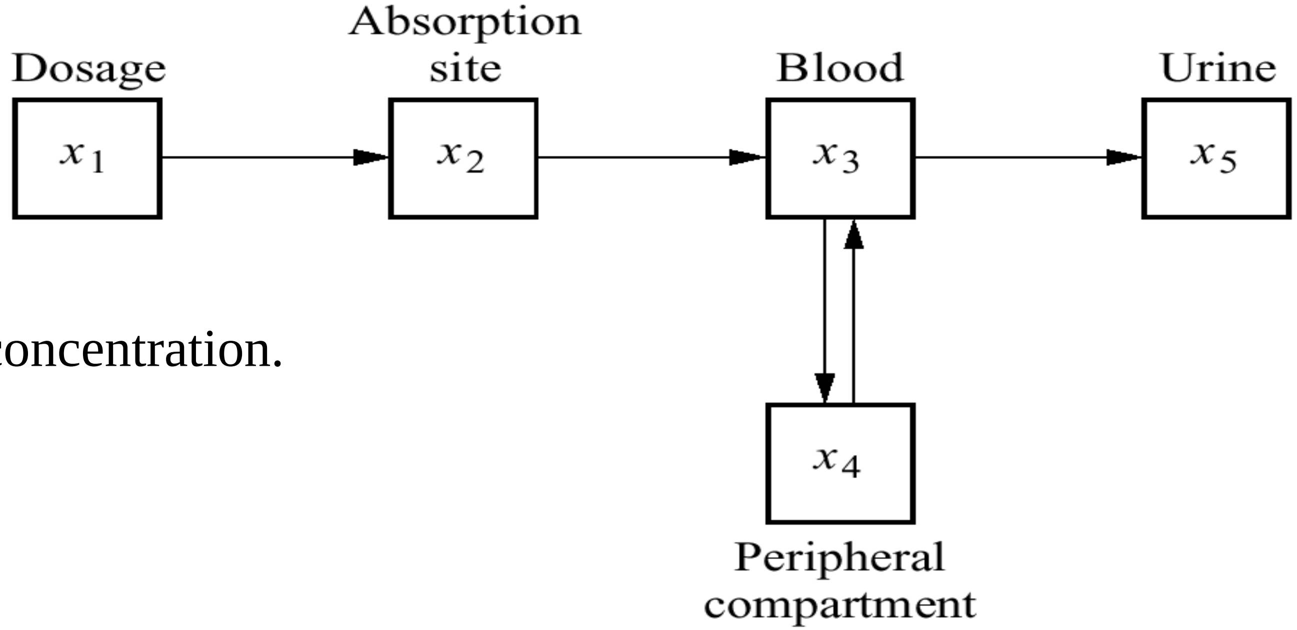
PHARMACEUTICAL DRUG ABSORPTION PROBLEM

We want to describe the distribution of a drug in the body by dividing the process into compartments: **dosage**, **absorption site**,**blood**, **peripheral compartment**, and **urine**.



- Each x_i is the amount of drug in that particular compartment.
- The **rate of change** of amount of a drug is equal to the **input flow rate diminished by the output flow rate**.

Represent the system in state space, where the outputs are the amounts of drug in each compartment.



1. Assume dosage is released at a rate proportional to concentration.

$$\frac{d}{dt}x_1 = -K_1 x_1$$

2. Assume input(/output) flow rate into(/out) any given compartment is proportional to the concentration of the drug in the previous(/its own) compartment. Then,

$$\frac{d}{dt}x_2 = K_1 x_1 - K_2 x_2$$

$$\frac{d}{dt}x_3 = K_2 x_2 - K_3 x_3 + K_4 x_4 - K_5 x_3$$

$$\frac{d}{dt}x_4 = K_5 x_3 - K_4 x_4$$

$$\frac{d}{dt}x_5 = K_3 x_3$$

3. Define the state vector as the dosage amount in each compartment. Then,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -K_1 & 0 & 0 & 0 & 0 \\ K_1 & -K_2 & 0 & 0 & 0 \\ 0 & K_2 & -(K_3+K_5) & K_4 & 0 \\ 0 & 0 & K_5 & -K_4 & 0 \\ 0 & 0 & K_3 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\boldsymbol{x}$$

CHALLANGE: STORAGE OF WATER IN AQUIFERS (UNDERGROUND WATER SUPPLIES)

An aquifer system consists of a number of interconnected natural storage tanks. Natural water flows through the sand and sandstone, changing the water levels in the tanks on its way to the sea. A water conservation policy can be established whereby water is pumped between tanks to prevent its loss to the sea.

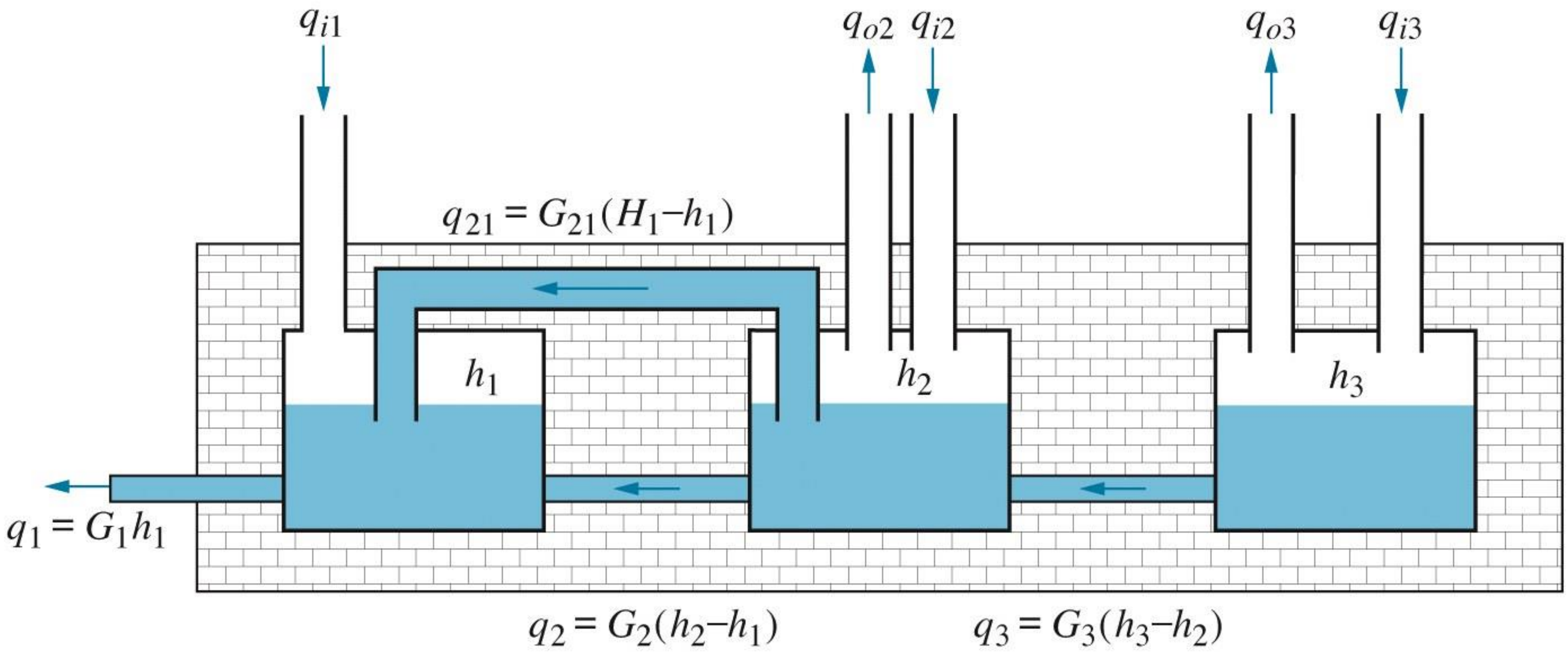
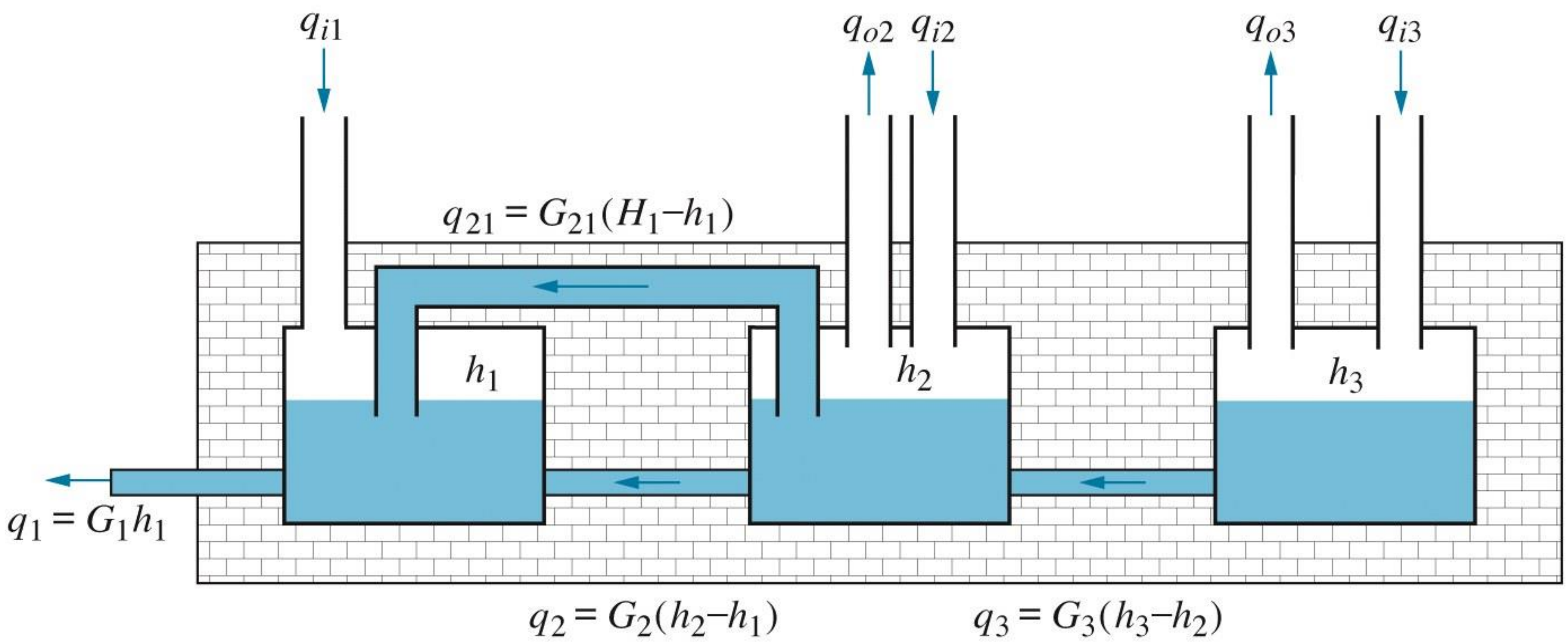


Figure 3.17



Represent the aquifer system in state space, where the state variables and the outputs are the heads of each tank.

- Each q_n is the natural water flow to the sea and $q_n = G_n(h_n - h_{n-1})$
- q_{02} , q_{03} are flows from the tanks for irrigation, industry, and homes

- q_{21} is created by the water conservation policy to prevent loss to the sea
  If $h_1<H_1$ then water will be pumped from Tank2 to Tank1
  If $h_1>H_1$ then water will be pumped back to Tank2 to prevent loss to the sea.
  Flow is proportional to the difference between H_1 *and* h_1 : $q_{21} = G_{21}(H_1 - h_1)$

- The net flow into a tank is proportional to the rate of change of head in each tank. Thus,

$$C_n \frac{dh_n}{dt} = q_{in} - q_{0n} + q_{n+1} - q_n + q_{(n+1)n} - q_{n(n-1)}$$