

# HW1: ASYMPTOTICS

Gabriel ROMON

## Problem 1

Let  $(T_n)_{n \geq 1}$  be a sequence of random vectors of  $\mathbb{R}^d$  ( $d \geq 1$ ).  $T_n$  is said to be *bounded in probability* or *tight* (denoted  $T_n = O_{\mathbb{P}}(1)$ ) if for any  $\varepsilon > 0$ , there is some  $A > 0$  and  $n_0 \geq 1$  such that  $n \geq n_0 \implies P(\|T_n\|_2 \geq A) \leq \varepsilon$ . More generally, if  $(s_n)_{n \geq 1}$  is a sequence of real random variables we write  $T_n = O_{\mathbb{P}}(s_n)$  if for any  $\varepsilon > 0$ , there is some  $A > 0$  and  $n_0 \geq 1$  such that  $n \geq n_0 \implies P(\|T_n\|_2 \geq A s_n) \leq \varepsilon$ .

1. Show that if  $T_n = o_{\mathbb{P}}(1)$  then  $T_n = O_{\mathbb{P}}(1)$ .
2. Show that if  $T_n$  converges in probability,  $T_n$  is tight.
3. Show that if  $T_n$  converges in distribution,  $T_n$  is tight.
4. Show that if  $(\rho_n)_{n \geq 1}$  is a sequence that goes to  $\infty$  and  $\rho_n T_n$  converges in distribution, then  $T_n = o_{\mathbb{P}}(1)$ .
5. Suppose that  $T_n$  goes to 0 in probability. Let  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  be such that  $g(x) = o(\|x\|_2^p)$  as  $x \rightarrow 0$ . Show that  $g(T_n) = o_{\mathbb{P}}(\|T_n\|_2^p)$ .
6. Suppose that  $T_n$  goes to 0 in probability. Let  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  be such that  $g(x) = O(\|x\|_2^p)$  as  $x \rightarrow 0$ . Show that  $g(T_n) = O_{\mathbb{P}}(\|T_n\|_2^p)$ .
7. Let  $(X_n)_{n \geq 1}$  be a sequence of r.v.'s such that  $X_n \sim \mathcal{P}(\frac{1}{n})$ .
  - (a) Show that  $X_n = o_{\mathbb{P}}(1)$ .
  - (b) Show that for any sequence  $(u_n)_{n \geq 1}$  of positive reals,  $X_n = o_{\mathbb{P}}(u_n)$ .
  - (c) Does  $X_n$  converge almost surely ?

1. Let  $\varepsilon > 0$ . Since  $T_n = o_{\mathbb{P}}(1)$ , the sequence  $P(\|T_n\|_2 \geq \varepsilon)$  goes to 0. There exists  $n_0$  such that  $n \geq n_0 \implies P(\|T_n\|_2 \geq \varepsilon) \leq \varepsilon$  and we're done.
2. Lemma 1: If  $X_n = O_{\mathbb{P}}(1)$  and  $Y_n = O_{\mathbb{P}}(1)$  then  $X + Y = O_{\mathbb{P}}(1)$ .  
*Proof*: Let  $\varepsilon > 0$ . There are some  $A_1, A_2 > 0$  and  $n_1, n_2 \geq 1$  such that  $n \geq n_1 \implies P(\|X_n\|_2 \geq A_1) \leq \frac{\varepsilon}{2}$  and  $n \geq n_2 \implies P(\|Y_n\|_2 \geq A_2) \leq \frac{\varepsilon}{2}$ . Let  $A = \max(A_1, A_2)$ ,  $n_0 = \max(n_1, n_2)$  and note that for  $n \geq n_0$ :

$$\begin{aligned}
 P(\|X_n + Y_n\|_2 \geq 2A) &\leq P(\|X_n\|_2 + \|Y_n\|_2 \geq 2A) \\
 &\leq P(\|X_n\|_2 \geq A) + P(\|Y_n\|_2 \geq A) \\
 &\leq P(\|X_n\|_2 \geq A_1) + P(\|Y_n\|_2 \geq A_2) \\
 &\leq \varepsilon
 \end{aligned}$$

□

Let  $T$  denote the limit of  $T_n$  in probability. We have  $T_n - T = o_{\mathbb{P}}(1)$ , hence 1. yields  $T_n - T = O_{\mathbb{P}}(1)$ . Since  $T_n = T_n - T + T$  and  $T = O_{\mathbb{P}}(1)$ , the lemma yields  $T_n = O_{\mathbb{P}}(1)$ .

3. Lemma 2: Let  $(X_n^1, \dots, X_n^d)$  be a random vector.  $(X_n^1, \dots, X_n^d)$  is tight if and only if each of the  $X_n^i$  is tight.  
*Proof*:  $\Rightarrow$  Let  $\varepsilon > 0$ . There are some  $A > 0$  and  $n_0 \geq 1$  such that  $n \geq n_0 \implies P(\|X_n\|_2 \geq A) \leq \varepsilon$ . For  $n \geq n_0$ ,  $P(|X_n^i| \geq A) \leq P(\|X_n\|_{\infty} \geq A) \leq P(\|X_n\|_2 \geq A) \leq \varepsilon$  and we're

done.

$\Leftarrow$  Let  $\varepsilon > 0$ . There are some  $A_1, \dots, A_d$  and  $n_1, \dots, n_d$  linked to the  $\frac{\varepsilon}{d}$  tightness of each  $X^i$ . Let  $A = \sqrt{d} \max_{1 \leq i \leq d} A_i$  and  $n_0 = \max_{1 \leq i \leq d} n_i$ . For  $n \geq n_0$  we have

$$\begin{aligned} P(\|X_n\|_2 \geq A) &\leq P(\sqrt{d}\|X_n\|_\infty \geq A) \\ &= P\left(\bigcup_{i=1}^d |X_n^i| \geq \frac{A}{\sqrt{d}}\right) \\ &\leq \sum_{i=1}^d P(|X_n^i| \geq \frac{A}{\sqrt{d}}) \\ &\leq \varepsilon \end{aligned}$$

□

Let us prove the result for  $d = 1$ . By Lemma 1 it suffices to prove that if  $T_n \in \mathbb{R}$  converges to 0 in distribution, then  $T_n$  is tight. The cdf of 0 is continuous everywhere except at 0, so  $P(T_n \leq 1)$  goes to 1 as  $n \rightarrow \infty$ . Consequently  $P(T_n \geq 2) \rightarrow 0$ , hence  $T_n$  is tight.

Let  $d \geq 2$  and  $T_n$  be a sequence that converges in distribution. By the continuous mapping theorem, each  $T_n^i$  converges in distribution, hence each  $T_n^i$  is tight. By Lemma 2,  $T_n$  is tight.

4. Let  $\varepsilon > 0$  and  $X$  denote a random variable having the distribution of the limit of  $\rho_n T_n$ . By the continuous mapping theorem  $\|\rho_n T_n\|_2$  converges in distribution to  $\|X\|_2$ . Since  $\rho_n \rightarrow \infty$ , we may assume WLOG that  $\rho_n \geq 0$ . Let  $A > 0$  be fixed. There exists some  $n_0$  such that  $n \geq n_0 \implies \rho_n \geq A$ . For  $n \geq n_0$ ,

$$\begin{aligned} P(\|T_n\|_2 \geq \varepsilon) &= P(\rho_n \|T_n\|_2 \geq \varepsilon \rho_n) \\ &\leq P(\rho_n \|T_n\|_2 \geq \varepsilon A) \end{aligned}$$

Taking the lim sup on both side yields  $\limsup_n P(\|T_n\|_2 \geq \varepsilon) \leq \limsup_n P(\rho_n \|T_n\|_2 \geq \varepsilon A)$ . The portmanteau theorem applied to  $\rho_n \|T_n\|_2$  and the closed set  $[\varepsilon A, \infty)$  gives

$$\limsup_n P(\rho_n \|T_n\|_2 \geq \varepsilon A) \leq P(\|X\|_2 \geq \varepsilon A)$$

hence  $\limsup_n P(\|T_n\|_2 \geq \varepsilon) \leq P(\|X\|_2 \geq \varepsilon A)$ .

Letting  $A \rightarrow \infty$  yields  $\limsup_n P(\|T_n\|_2 \geq \varepsilon) = 0$ , hence  $P(\|T_n\|_2 \geq \varepsilon) \rightarrow 0$  and  $T_n = o_{\mathbb{P}}(1)$ .

5. Let  $\varepsilon > 0$ . Since  $g(x) = o(\|x\|_2^p)$  as  $x \rightarrow 0$ , there exists some  $R$  such that  $x \in B_2(0, R) \setminus \{0\} \implies \frac{|g(x)|}{\|x\|_2^p} \leq \frac{\varepsilon}{2}$ . This implies

$$P(g(T_n) \geq \varepsilon \|T_n\|_2^p) \leq P(\|T_n\|_2 > R) \xrightarrow{n \rightarrow \infty} 0$$

and we're done.

6. Let  $\varepsilon > 0$ . Since  $g(x) = O(\|x\|_2^p)$  as  $x \rightarrow 0$ , there exists some  $A > 0$  and  $R > 0$  such that  $x \in B_2(0, R) \setminus \{0\} \implies \frac{|g(x)|}{\|x\|_2^p} \leq A$ . This implies

$$P(g(T_n) \geq 2A \|T_n\|_2^p) \leq P(\|T_n\|_2 > R) \xrightarrow{n \rightarrow \infty} 0$$

Consequently there is some  $n_0$  such that  $n \geq n_0 \implies P(\|T_n\|_2 > R) \leq \varepsilon$ . Hence  $n \geq n_0 \implies P(g(T_n) \geq 2A \|T_n\|_2^p) \leq \varepsilon$  and we're done.

7. (a) Let  $\varepsilon \in (0, 1)$ . Since  $X_n$  is integer-valued, we have

$$P(X_n \geq \varepsilon) \leq P(X_n \geq 1) = 1 - P(X_n = 0) = 1 - \exp\left(-\frac{1}{n}\right) \xrightarrow{n \rightarrow \infty} 0$$

For  $\varepsilon \geq 1$ ,  $P(X_n \geq \varepsilon) \leq P(X_n \geq \frac{1}{2}) \xrightarrow{n \rightarrow \infty} 0$ , hence  $X_n$  converges in probability to 0.

(b) Let  $(u_n)_{n \geq 1}$  be a sequence of positive reals and  $\varepsilon > 0$ . Note that

$$\begin{aligned}
P(X_n \geq \varepsilon u_n) &= P(X_n \geq \varepsilon u_n) 1_{u_n > \frac{1}{\varepsilon}} + \sum_{k=0}^{\infty} P(X_n \geq \varepsilon u_n) 1_{\frac{1}{2^k \varepsilon} \geq u_n > \frac{1}{2^{k+1} \varepsilon}} \\
&\leq P(X_n \geq 1) + \sum_{k=0}^{\infty} P\left(X_n \geq \frac{1}{2^{k+1}}\right) 1_{\frac{1}{2^k \varepsilon} \geq u_n > \frac{1}{2^{k+1} \varepsilon}} \\
&\leq P(X_n \geq 1) + \sum_{k=0}^{\infty} \left(1 - \exp\left(-\frac{1}{n}\right)\right) 1_{\frac{1}{2^k \varepsilon} \geq u_n > \frac{1}{2^{k+1} \varepsilon}} \\
&\leq P(X_n \geq 1) + \sum_{k=0}^{\infty} \frac{1}{n} 1_{\frac{1}{2^k \varepsilon} \geq u_n > \frac{1}{2^{k+1} \varepsilon}} \\
&\leq P(X_n \geq 1) + \frac{1}{n} \\
&\xrightarrow{n \rightarrow \infty} 0
\end{aligned}$$

Hence  $X_n = o_{\mathbb{P}}(u_n)$ .

(c) Note that  $P(X_n \geq 1) = 1 - \exp(-\frac{1}{n}) \sim \frac{1}{n}$ , hence  $\sum_n P(X_n \geq 1) = \infty$ .

If the  $X_i$  are (at least pairwise) **independent**, Borel-Cantelli lemma yields  $P(\limsup_n (X_n \geq 1)) = 1$ . For almost all  $w$ , we have  $X_n(w) \geq 1$  infinitely often, hence  $X_n$  cannot converge almost surely to 0. Thus  $X_n$  does not converge almost surely.

If the  $X_i$  are not independent, a.s. convergence to 0 may occur. Let  $(\xi_n)_{n \geq 1}$  be a sequence of independent r.v.'s such that  $\xi_n \sim \mathcal{P}(\frac{1}{n} - \frac{1}{n+1})$ . Since the summands are  $\geq 0$  a.s., the series  $\sum_{n \geq 1} \xi_n$  converges a.s. in  $\overline{\mathbb{R}}$ . Since

$$E\left(\sum_{n \geq 1} \xi_n\right) = \sum_{n \geq 1} E(\xi_n) = \sum_{n \geq 1} \frac{1}{n(n+1)} < \infty$$

$\sum_{n \geq 1} \xi_n$  converges a.s. in  $\mathbb{R}$ . Thus it makes sense to define  $X_n = \sum_{i \geq n} \xi_i$ , and by what precedes  $X_n$  converges almost surely to 0.

It remains to prove that  $X_n \sim \mathcal{P}(\frac{1}{n})$ .

$$\begin{aligned}
P(X_n = p) &= E(1_{X_n=p}) = E(\lim_k 1_{\sum_{i=n}^k \xi_i = p}) \\
&= \lim_k P\left(\sum_{i=n}^k \xi_i = p\right) \\
&= \lim_k \frac{1}{p!} \left(\frac{1}{n} - \frac{1}{k+1}\right)^p \exp\left(-\left(\frac{1}{n} - \frac{1}{k+1}\right)\right) \\
&= \frac{1}{p!} \frac{1}{n^p} \exp\left(-\frac{1}{n}\right)
\end{aligned}$$

Swapping the  $\lim_k$  and the expectation is motivated by the monotone convergence theorem. For fixed  $k$ , the distribution of  $\sum_{i=n}^k \xi_i$  is that of a finite sum of independent Poisson random variables, which is well-known to be a Poisson where the parameters are summed.

## Problem 2

Let  $(T_n)_{n \geq 1}$  be a sequence of random vectors of  $\mathbb{R}^d$  ( $d \geq 1$ ) and  $T$  a random vector.

1. Show that if  $T_n$  converges almost surely to  $T$ , then  $T_n$  converges in probability to  $T$ .
2. Show that if  $T_n$  converges in probability to  $T$ , then  $T_n$  converges in distribution to  $T$ .
3. Show that if  $T$  is constant almost surely, convergence in distribution implies convergence in probability.

1. Let  $A = \{w \in \Omega, T_n(w) \xrightarrow{n \rightarrow \infty} T(w)\}$ . Note that  $A = \bigcap_{m \geq 1} \bigcup_{n \geq 1} \bigcap_{k \geq n} \|T_k - T\| < \frac{1}{m}$ .

By assumption,  $P(A^c) = 0$ , hence  $P(\bigcup_{m \geq 1} \bigcap_{n \geq 1} \bigcup_{k \geq n} \|T_k - T\| \geq \frac{1}{m}) = 0$ , which implies  $\forall m \geq 1, P(\bigcap_{n \geq 1} \bigcup_{k \geq n} \|T_k - T\| \geq \frac{1}{m}) = 0$ .

Let  $\varepsilon > 0$ . There is some  $m \geq 1$  such that  $\varepsilon \geq \frac{1}{m}$ , hence

$$P(\|T_n - T\| \geq \varepsilon) \leq P(\|T_n - T\| \geq \frac{1}{m}) \leq P(\bigcup_{k \geq n} \|T_k - T\| \geq \frac{1}{m})$$

$\bigcup_{k \geq n} \|T_k - T\| \geq \frac{1}{m}$  is a decreasing sequence of events, hence

$$P(\bigcup_{k \geq n} \|T_k - T\| \geq \frac{1}{m}) \xrightarrow{n \rightarrow \infty} P(\bigcap_{n \geq 1} \bigcup_{k \geq n} \|T_k - T\| \geq \frac{1}{m}) = 0$$

Squeezing thus yields  $P(\|T_n - T\| \geq \varepsilon) \xrightarrow{n \rightarrow \infty} 0$ , hence  $T_n$  converges to  $T$  in probability.

2. Let  $f$  be a  $K$ -Lipschitz function bounded by some  $A \geq 0$ . Note that for any  $\varepsilon > 0$ ,

$$|f(T_n) - f(T)| \leq K\varepsilon 1_{\|T_n - T\|_2 \leq \varepsilon} + 2A 1_{\|T_n - T\|_2 > \varepsilon}$$

Thus  $|E(f(T_n)) - E(f(T))| \leq E(|f(T_n) - f(T)|) \leq K\varepsilon + 2A\varepsilon P(\|T_n - T\|_2 > \varepsilon)$ .

Taking lim sup on both side yields  $\limsup_n |E(f(T_n)) - E(f(T))| \leq K\varepsilon$ . Letting  $\varepsilon \rightarrow 0$  proves that  $\lim_n E(f(T_n)) = E(f(T))$ . By the portmanteau theorem,  $T_n$  converges to  $T$  in distribution.

3. If the random vector  $(T_n^1, \dots, T_n^d)$  converges in distribution to some  $T$  with  $T = (t_1, \dots, t_d)$  a.s., then by the continuous mapping theorem each  $T_n^i$  converges in distribution to  $\delta_{t_i}$ .

We recall a useful lemma about convergence:

**Lemma 3:**  $(X_n^1, \dots, X_n^d)$  converges in probability to  $(X^1, \dots, X^d)$  if and only if each real r.v.  $X_n^i$  converges in probability to  $X^i$ .

By Lemma 3 it suffices to prove the claim in the case  $d = 1$ .

By the continuous mapping theorem applied with  $x \mapsto |x - t|$ ,  $|T_n - t|$  converges in distribution to 0. Let  $\varepsilon > 0$  and note that

$$P(|T_n - t| \geq \varepsilon) \leq P(|T_n - t| > \frac{\varepsilon}{2}) = 1 - P(|T_n - t| \leq \frac{\varepsilon}{2})$$

Since the cdf of 0 is continuous at  $\frac{\varepsilon}{2}$ , the convergence of  $|T_n - t|$  implies  $P(|T_n - t| \leq \frac{\varepsilon}{2}) \xrightarrow{n \rightarrow \infty} 1$  and we're done.

### Problem 3

Let  $\alpha \in (0, 1)$ ,  $\ell_\alpha : t \mapsto (1 - \alpha)t^+ + \alpha t^-$  and  $\phi : (x, t) \mapsto \ell_\alpha(x - t)$ . Let  $(X_n)_{n \geq 1}$  be a sequence of i.i.d. r.v.'s with positive density. For  $n \geq 1$ , let  $\hat{q}_n$  an  $M$ -estimator associated to  $\phi$ .

1. Show that  $\hat{q}_n$  is an  $\alpha$ -quantile of the sample  $X_1, \dots, X_n$ . To simplify matters,  $\hat{q}_n$  will be chosen to be maximal.
2. Find  $k$  such that  $\hat{q}_n = X_{(k)}$  where  $X_{(1)} \leq \dots \leq X_{(n)}$  are the order statistics. Show that the inequalities are strict almost surely.
3. We want to prove that  $\hat{q}_n$  is asymptotically normal.
  - (a) Show that  $X_1$  has a unique  $\alpha$ -quantile, say  $q$ .
  - (b) For  $t \in \mathbb{R}$ , show that  $P(\sqrt{n}(\hat{q}_n - q) \leq t) = P(N \geq n\alpha)$  where  $N \sim \mathcal{B}(n, F(q + t/\sqrt{n}))$  where  $F$  is the cdf of  $X_1$ .
  - (c) What is the limiting distribution of  $\frac{1}{\sqrt{n}}(N - nF(q + t/\sqrt{n}))$  as  $n \rightarrow \infty$ ?
  - (d) Use Slutsky's theorem to conclude.

1. Let  $x_1, \dots, x_n$  be fixed real numbers and  $g : t \mapsto \frac{1}{n} \sum_{i=1}^n \ell_\alpha(x_i - t)$ .  
 By definition,  $\hat{q}_n \in \arg \min_t g(t)$ . Each  $t \mapsto \ell_\alpha(x_i - t)$  is a convex function, so  $g$  is convex and  $t$  is minimal if and only if  $0 \in \partial g(t)$ .  
 Let  $p(t) = |\{i \in \llbracket 1, n \rrbracket, x_i < t\}|$  and  $q(t) = |\{i \in \llbracket 1, n \rrbracket, x_i > t\}|$ . Subgradient calculus yields

$$\begin{aligned} \partial g(t) &= -\frac{1}{n} \sum_{i=1}^n \begin{cases} \{\alpha - 1\} & \text{if } x_i < t \\ [\alpha - 1, \alpha] & \text{if } x_i = t \\ \{\alpha\} & \text{if } x_i > t \end{cases} \quad \text{where the summation is over sets} \\ &= \{p(t)(\alpha - 1) + q(t)\alpha\} + [(n - p(t) - q(t))(\alpha - 1), (n - p(t) - q(t))\alpha] \end{aligned}$$

$$\begin{aligned} \text{Thus } 0 \in \partial g(t) &\iff -p(t)(\alpha - 1) - q(t)\alpha \in [(n - p(t) - q(t))(\alpha - 1), (n - p(t) - q(t))\alpha] \\ &\iff 0 \leq n\alpha - p(t) \leq n - (p(t) + q(t)) \\ &\iff \frac{n - p(t)}{n} \geq 1 - \alpha \quad \text{and} \quad \frac{n - q(t)}{n} \geq \alpha \end{aligned}$$

Given the definition of  $p(t)$  and  $q(t)$ , this can be rephrased as:  $t$  is minimal if and only if it is an  $\alpha$ -quantile of  $x_1, \dots, x_n$ .

2. Let  $x_1, \dots, x_n$  be fixed real numbers and  $p(t), q(t)$  be defined as above. We want to find the greatest  $t$  such that  $p(t) \leq n\alpha$  and  $q(t) \leq n(1 - \alpha)$  both hold. Let us show that  $t^* = x_{\lfloor n\alpha \rfloor + 1}$  fits the bill. By definition,  $p(t^*) \leq \lfloor n\alpha \rfloor \leq n\alpha$  and

$$q(t^*) \leq n - (\lfloor n\alpha \rfloor + 2) + 1 = n(1 - \alpha) + \{n\alpha\} - 1 \leq n(1 - \alpha)$$

If  $t > x_{\lfloor n\alpha \rfloor + 1}$ , then  $p(t) \geq \lfloor n\alpha \rfloor + 1 > n\alpha$ , hence  $t^*$  is the maximal  $t$  such that  $p(t) \leq n\alpha$  and  $q(t) \leq n(1 - \alpha)$ . Thus  $\hat{q}_n = x_{\lfloor n\alpha \rfloor + 1}$ .

Remark:  $x_{\lceil n\alpha \rceil}$  is another  $\alpha$ -quantile, but it is not maximal (consider  $n = 6$  and  $\alpha = \frac{1}{2}$ ). To check that it is a quantile note that  $p(x_{\lceil n\alpha \rceil}) \leq \lceil n\alpha \rceil - 1 < n\alpha$  and

$$q(x_{\lceil n\alpha \rceil}) \leq n - (\lceil n\alpha \rceil + 1) + 1 = n - \lceil n\alpha \rceil \leq n(1 - \alpha)$$

Let  $1 \leq i \neq j \leq n$  and  $f$  denote the density of  $X_i$ .

$$\begin{aligned}
\text{Note that } P(X_i = X_j) &= E(1_{X_i=X_j}) = \int 1_{x=y} dP_{(X_i, X_j)}(x, y) \\
&= \int 1_{x=y} dP_{X_i} \otimes dP_{X_j}(x, y) \quad \text{by independence} \\
&= \int \int 1_{x=y} f(x) f(y) dx dy \\
&= \int \left( \int 1_{x=y} f(x)^2 dx \right) dy \quad \text{by Fubini}
\end{aligned}$$

For fixed  $y$ , the function  $x \mapsto 1_{x=y} f(x)^2$  is 0 almost everywhere, thus  $\int 1_{x=y} f(x)^2 dx = 0$ , hence  $P(X_i = X_j) = \int 0 dy = 0$ .

3. (a)  $q$  is an  $\alpha$ -quantile of  $X_1$  if and only if  $P(X_1 \leq q) \geq \alpha$  and  $P(X_1 \geq q) \geq 1 - \alpha$ . Since  $X_1$  has a density, its cdf  $F$  is continuous. Since the density is  $> 0$  everywhere,  $F$  is also strictly increasing, so  $F$  is a continuous increasing bijection from  $\mathbb{R}$  to  $(0, 1)$ .

Consequently there exists  $q \in \mathbb{R}$  such that  $F(q) = \alpha$ , hence  $P(X_1 \leq q) = \alpha$  and since  $X_1$  is atomless,  $P(X_1 \geq q) = P(X_1 > q) = 1 - \alpha$ . Hence  $q$  is an  $\alpha$ -quantile of  $X_1$ .

If  $q$  is an  $\alpha$ -quantile of  $X_1$ , we have both  $P(X_1 \leq q) \geq \alpha$  and  $P(X_1 < q) \leq \alpha$ . Since  $X_1$  is atomless  $P(X_1 \leq q) = P(X_1 < q) \leq \alpha$ , hence  $P(X_1 \leq q) = \alpha$  and  $q$  is unique by the injectivity of  $F$ .

- (b) **In this question it is essential that**  $\hat{q}_n = X_{[n\alpha]}$ , contrary to what's stated in Question 1.

For  $i \in \llbracket 1, n \rrbracket$ , let  $Y_i = 1_{X_i \leq \frac{t}{\sqrt{n}} + q}$  and note that

$$\begin{aligned}
P(\sqrt{n}(\hat{q}_n - q) \leq t) &= P(X_{[n\alpha]} \leq \frac{t}{\sqrt{n}} + q) \\
&= P\left(\sum_{i=1}^n Y_i \geq [n\alpha]\right) \\
&= P\left(\sum_{i=1}^n Y_i \geq n\alpha\right)
\end{aligned}$$

$\sum_{i=1}^n Y_i$  has distribution  $\mathcal{B}(n, F(t/\sqrt{n} + q))$  as a sum of  $n$  i.i.d. Bernoulli r.v.'s.

If  $\hat{q}_n = X_{[n\alpha]+1}$ , one gets  $P(\sqrt{n}(\hat{q}_n - q) \leq t) = P(\sum_{i=1}^n Y_i \geq [n\alpha] + 1)$  but the last term isn't necessarily equal to  $P(\sum_{i=1}^n Y_i \geq n\alpha)$  (if  $n\alpha \in \mathbb{N}$  and  $m \in \mathbb{N}$ ,  $m \geq n\alpha$  does not imply  $m \geq [n\alpha] + 1$ )

- (c) Note that

$$\begin{aligned}
&E \left[ \exp \left( it \frac{1}{\sqrt{n}} \left( N - nF \left( q + \frac{t}{\sqrt{n}} \right) \right) \right) \right] \\
&= E \left[ \exp \left( \frac{itN}{\sqrt{n}} \right) \right] \exp \left( -it\sqrt{n}F \left( q + \frac{t}{\sqrt{n}} \right) \right) \\
&= \left[ 1 + F \left( q + \frac{t}{\sqrt{n}} \right) \left( \exp \left( \frac{it}{\sqrt{n}} \right) - 1 \right) \right]^n \exp \left( -it\sqrt{n}F \left( q + \frac{t}{\sqrt{n}} \right) \right)
\end{aligned}$$

Since the density of  $X_1$  is continuous,  $F$  is differentiable everywhere with  $F' = f$ . This provides the following asymptotic expansion for  $F$ :

$$\begin{aligned}
F \left( q + \frac{t}{\sqrt{n}} \right) &= F(q) + \frac{t}{\sqrt{n}} f(q) + o \left( \frac{1}{\sqrt{n}} \right) \\
&= \alpha + \frac{t}{\sqrt{n}} f(q) + o \left( \frac{1}{\sqrt{n}} \right)
\end{aligned}$$

Let  $\text{Log}$  denote the principal branch of the logarithm. For  $|z| < 1$ ,

$$\text{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

thus  $\frac{|\text{Log}(1+z) - z + \frac{z^2}{2}|}{|z^2|} = |z| \left| \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n} z^{n-3} \right|$ .  $z \mapsto \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n} z^{n-3}$  is a power series with radius  $\geq 1$ , it is therefore bounded over  $\overline{B}(0, \frac{1}{2})$ . As a result

$$\lim_{z \rightarrow 0} \frac{\text{Log}(1+z) - z + \frac{z^2}{2}}{z^2} = 0$$

and  $\text{Log}(1+z) = z - \frac{z^2}{2} + o(z^2)$ . A bit of algebra shows that

$$\begin{aligned} \text{Log} \left[ 1 + F \left( q + \frac{t}{\sqrt{n}} \right) \left( \exp \left( \frac{it}{\sqrt{n}} \right) - 1 \right) \right] &= \text{Log} \left[ 1 + \frac{i\alpha t}{\sqrt{n}} + \frac{t^2}{n} \left( if(q) - \frac{\alpha}{2} \right) + o \left( \frac{1}{n} \right) \right] \\ &= \frac{i\alpha t}{\sqrt{n}} + \frac{t^2}{n} \left( if(q) - \frac{\alpha}{2} \right) + \frac{\alpha^2 t^2}{2n} + o \left( \frac{1}{n} \right) \\ &= \frac{i\alpha t}{\sqrt{n}} + \frac{t^2}{n} \left( if(q) - \frac{\alpha}{2} + \frac{\alpha^2}{2} \right) + o \left( \frac{1}{n} \right) \end{aligned}$$

The original expectation turns into

$$\begin{aligned} E \left[ \exp \left( it \frac{1}{\sqrt{n}} \left( N - nF \left( q + \frac{t}{\sqrt{n}} \right) \right) \right) \right] &= \exp \left[ n \left( \frac{i\alpha t}{\sqrt{n}} + \frac{t^2}{n} \left( if(q) - \frac{\alpha}{2} + \frac{\alpha^2}{2} \right) + o \left( \frac{1}{n} \right) \right) \right] \\ &\quad \cdot \exp \left( -it\alpha\sqrt{n} - it^2 f(q) + o(1) \right) \\ &= \exp \left[ -\frac{\alpha(1-\alpha)}{2} t^2 + o(1) \right] \\ &\xrightarrow{n \rightarrow \infty} \exp \left[ -\frac{\alpha(1-\alpha)}{2} t^2 \right] \end{aligned}$$

The characteristic function of  $\frac{1}{\sqrt{n}} \left( N - nF \left( q + \frac{t}{\sqrt{n}} \right) \right)$  converges pointwise to that of a  $\mathcal{N}(0, \alpha(1-\alpha))$ , hence  $\frac{1}{\sqrt{n}} \left( N - nF \left( q + \frac{t}{\sqrt{n}} \right) \right)$  converges in distribution to  $\mathcal{N}(0, \alpha(1-\alpha))$ .

(d) Let  $Z_n = \frac{1}{\sqrt{n}} \left( N - nF \left( q + \frac{t}{\sqrt{n}} \right) \right)$ . Note that

$$\begin{aligned} P(N \geq n\alpha) &= P \left( Z_n \geq \sqrt{n} \left( \alpha - F \left( q + \frac{t}{\sqrt{n}} \right) \right) \right) = P \left( -Z_n \leq \sqrt{n} \left( F \left( q + \frac{t}{\sqrt{n}} \right) - F(q) \right) \right) \\ \sqrt{n} \left( F \left( q + \frac{t}{\sqrt{n}} \right) - F(q) \right) &\text{ is a deterministic sequence that converges (everywhere,} \\ &\text{hence almost surely, thus in probability) to } tf(q). \text{ We have} \end{aligned}$$

$$P(N \geq n\alpha) = P \left( -Z_n \underbrace{\frac{tf(q)}{\sqrt{n} \left( F \left( q + \frac{t}{\sqrt{n}} \right) - F(q) \right)} \frac{1}{f(q)}}_{\text{converges in probability to } \frac{1}{f(q)}} \leq t \right)$$

By Slutsky's theorem, the random variable on the left of the  $\leq$  sign converges in distribution to  $-\frac{1}{f(q)} \mathcal{N}(0, \alpha(1-\alpha)) = \mathcal{N}(0, \frac{\alpha(1-\alpha)}{f(q)^2})$ .

Hence  $P(\sqrt{n}(\hat{q}_n - q) \leq t) = P(N \geq n\alpha)$  converges to the cdf of a  $\mathcal{N}(0, \frac{\alpha(1-\alpha)}{f(q)^2})$  evaluated at  $t$  (and this cdf is continuous).

This proves that  $\sqrt{n}(\hat{q}_n - q)$  converges in distribution to a  $\mathcal{N}(0, \frac{\alpha(1-\alpha)}{f(q)^2})$ .

## Problem 4

Let  $\theta > 0$  and  $(X_n)_{n \geq 1}$  be i.i.d. random variables following  $\mathcal{U}([0, \theta])$ . Show that the MLE  $\hat{\theta}_n$  of  $\theta$  is asymptotically exponential with convergence rate  $\frac{1}{n}$ .

Let  $x_1, \dots, x_n$  be an  $n$ -sample. The likelihood of the model writes as

$$L(\theta) = \prod_{i=1}^n \frac{1}{\theta} 1_{[0, \theta]}(x_i) = \frac{1}{\theta^n} 1_{\min x_i \geq 0} 1_{\max x_i \leq \theta}$$

If  $\min x_i < 0$ ,  $L = 0$  and the MLE is not defined, so we may assume WLOG that  $\min x_i \geq 0$ .  $L$  is 0 when  $\theta < \max x_i$  and positive decreasing for  $\theta \geq \max x_i$ . Thus  $L$  has a unique maximum at  $\theta = \max x_i$ , hence  $\hat{\theta}_n = \max x_i$ .

Let us compute the cdf of  $\hat{\theta}_n$ . Let  $F$  denote the cdf of  $X_1$ .

$$\begin{aligned} P(\max X_i \leq t) &= P\left(\bigcap_{i=1}^n X_i \leq t\right) = F(t)^n \\ &= \begin{cases} 0 & \text{if } t < 0 \\ \frac{t^n}{\theta^n} & \text{if } t \in [0, \theta] \\ 1 & \text{if } t \geq \theta \end{cases} \end{aligned}$$

The cdf is continuous so the distribution of  $\max X_i$  is atomless.

Let  $t \geq 0$ . Since  $\hat{\theta}_n$  is atomless,

$$\begin{aligned} P(n(\theta - \hat{\theta}_n) \leq t) &= P(\hat{\theta}_n \geq \theta - \frac{t}{n}) = 1 - P(\hat{\theta}_n \leq \theta - \frac{t}{n}) \\ &= 1 - \left(\theta - \frac{t}{n}\right)^n \frac{1}{\theta^n} = 1 - \left(1 - \frac{t}{\theta n}\right)^n \\ &\xrightarrow{n \rightarrow \infty} 1 - \exp\left(-\frac{t}{\theta}\right) \end{aligned}$$

If  $t < 0$  similar computations show that  $P(n(\theta - \hat{\theta}_n) \leq t) \xrightarrow{n \rightarrow \infty} 0$

The limiting cdf is that of a  $\mathcal{E}(\frac{1}{\theta})$  (and it is continuous), so  $n(\theta - \hat{\theta}_n)$  converges in distribution to  $\mathcal{E}(\frac{1}{\theta})$ .



## Problem 5

Let  $a \in \mathbb{R}$ ,  $\lambda > 0$  and  $f : x \mapsto \lambda e^{-\lambda(x-a)} 1_{x \geq a}$ . Let  $(X_n)_{n \geq 1}$  be an i.i.d sequence of r.v.'s with density  $f$ . For  $n \geq 1$ , let  $(\hat{a}_n, \hat{\lambda}_n)$  the MLE of  $(a, \lambda)$ . Show that  $\hat{a}_n$  is asymptotically exponential with convergence rate  $\frac{1}{n}$  and  $\hat{\lambda}_n$  is asymptotically normal.

Let  $x_1, \dots, x_n$  be an  $n$ -sample. The likelihood of the model writes as

$$L(a, \lambda) = \prod_{i=1}^n \lambda e^{-\lambda(x_i-a)} 1_{x_i \geq a} = \lambda^n 1_{\min x_i \geq a} e^{-\lambda \sum_{i=1}^n (x_i-a)}$$

When  $a > \min x_i$ ,  $L(a, \lambda) = 0$  and the likelihood is minimized. We may therefore assume that  $a \leq \min x_i$ . If  $a = \frac{1}{n} \sum_{i=1}^n x_i$ , then  $a = x_1 = \dots = x_n$  and  $L(a, \lambda) = \lambda^n \xrightarrow{\lambda \rightarrow \infty} \infty$ , so the MLE does not exist. We may thus assume additionally that the equalities  $x_1 = \dots = x_n$  do not hold, so  $a < \frac{1}{n} \sum_{i=1}^n x_i$ .

We have  $\log L(a, \lambda) = n \log \lambda - \lambda \sum_{i=1}^n (x_i - a)$ . Studying the derivative w.r.t  $\lambda$  shows that  $\lambda \mapsto \log L(a, \lambda)$  reaches a unique maximum at  $\lambda^*(a) = \frac{n}{\sum_{i=1}^n (x_i - a)}$  (which is well-defined given the previous assumption). Since  $\log$  is strictly monotonic,  $\lambda \mapsto L(a, \lambda)$  also has its unique maximum at  $\lambda^*(a)$ .

Consider  $(-\infty, \min x_i] \rightarrow \mathbb{R}, a \mapsto L(a, \lambda(a^*)) = \frac{n^n}{[\sum_{i=1}^n (x_i - a)]^n}$ . This function is increasing in  $a$ , so it reaches its maximum at  $a = \min x_i$ .

Thus  $\hat{a}_n = \min x_i$  and  $\hat{\lambda}_n = \lambda(\hat{a}_n) = \frac{n}{\sum_{i=1}^n (x_i - \min x_i)}$ .

The cdf of  $X_1$  is given by  $P(X_1 \leq t) = \begin{cases} 0 & \text{if } t < a \\ 1 - e^{-\lambda(t-a)} & \text{if } t \geq a \end{cases}$  and the cdf of  $\min X_i$  by  $P(\min X_i \leq t) = 1 - (1 - P(X_1 \leq t))^n$ .

Let  $t \geq 0$ . We have  $P(n(\min X_i - a) \leq t) = 1 - (1 - (1 - e^{-\lambda \frac{t}{n}}))^n = 1 - e^{-\lambda t}$ . For  $t < 0$  we get  $P(n(\min X_i - a) \leq t) = 0$  in a similar fashion. The cdf of  $n(\min X_i - a)$  is that of a  $\mathcal{E}(\lambda)$ , hence  $n(\min X_i - a) \sim \mathcal{E}(\lambda)$  (and remarkably this holds for finite  $n$ ).

$X_1$  is square-integrable with  $E(X_1) = \frac{1}{\lambda} + a$  and  $V(X_1) = \frac{1}{\lambda^2}$ . Note that

$$\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n X_i - \min X_i - \frac{1}{\lambda} \right) = \sqrt{n} \cdot \frac{1}{n} \sum_{i=1}^n (X_i - E(X_i)) + \sqrt{n} (a - \min X_i)$$

Since  $\sqrt{n} \cdot \sqrt{n}(\min X_i - a)$  converges in distribution, Question 4 from Problem 1 implies that  $\sqrt{n}(\min X_i - a) = o_{\mathbb{P}}(1)$ . The CLT yields the convergence in distribution of  $\sqrt{n} \cdot \frac{1}{n} \sum_{i=1}^n (X_i - E(X_i))$  to  $\mathcal{N}(0, \frac{1}{\lambda^2})$ . By Slutsky's theorem  $\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n X_i - \min X_i - \frac{1}{\lambda} \right)$  converges in distribution to  $\mathcal{N}(0, \frac{1}{\lambda^2})$ .

The Delta Method applied with  $x \mapsto \frac{1}{x}$  yields the convergence in distribution of

$$\sqrt{n} \left( \frac{1}{\frac{1}{n} \sum_{i=1}^n X_i - \min X_i} - \lambda \right)$$

to  $\mathcal{N}(0, \frac{1}{\lambda^2} \cdot \lambda^4) = \mathcal{N}(0, \lambda^2)$ .

## Problem 6

Let  $\theta \in \mathbb{R}$  and  $(X_n)_{n \geq 1}$  a sequence of i.i.d. r.v.'s following  $\mathcal{N}(\theta^3, 1)$ .

1. For  $n \geq 1$  compute  $\hat{\theta}_n$  the MLE of  $\theta$ .
2. Show that  $\hat{\theta}_n$  is consistent.
3. For what values of  $\theta$  is  $\hat{\theta}_n$  asymptotically normal ?
4. Depending on  $\theta$  find  $\alpha > 0$  such that  $|\hat{\theta}_n - \theta| = O_{\mathbb{P}}\left(\frac{1}{n^\alpha}\right)$

1. Let  $x_1, \dots, x_n$  be an  $n$ -sample. The likelihood of the model writes as

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_i - \theta^3)^2}{2}\right) \\ &= \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^n (x_i - \theta^3)^2\right) \end{aligned}$$

Thus  $\log L(\theta) = -\frac{n}{2} \log(2\pi) - \frac{1}{2} \sum_{i=1}^n (\theta^3 - x_i)^2$  which is a degree 6 polynomial in  $\theta$  with leading coefficient  $-\frac{n}{2}$ . It is therefore coercive and reaches a global maximum at a critical point. We have

$$(\log L)'(\theta) = 0 \iff 6\theta^2 \sum_{i=1}^n (\theta^3 - x_i) = 0 \iff \theta = 0 \text{ or } \theta = \left(\frac{1}{n} \sum_{i=1}^n x_i\right)^{1/3} = \bar{x}^{1/3}$$

Up to a constant we have  $(\log L)(0) = -\frac{1}{2} \sum_{i=1}^n x_i^2$  and

$$(\log L)(\bar{x}^{1/3}) = -\frac{1}{2} \sum_{i=1}^n (x_i - \bar{x})^2 = -\frac{1}{2} \left[ \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2 \right] \geq (\log L)(0)$$

The MLE is thus  $\hat{\theta}_n = \bar{x}^{1/3}$ .

2. By the weak Law of Large Numbers  $\bar{X}$  converges in probability to  $\theta^3$ . The continuous mapping theorem applied with  $x \mapsto x^{1/3}$  yields convergence in probability of  $\bar{X}^{1/3}$  to  $\theta$ , thus  $\hat{\theta}_n$  is consistent.
3. By the CLT  $\sqrt{n}(\bar{X} - \theta^3)$  converges in distribution to  $\mathcal{N}(0, 1)$ . If  $\theta \neq 0$  the function  $x \mapsto x^{1/3}$  is differentiable at  $\theta$  and the Delta Method yields convergence in distribution of  $\sqrt{n}(\bar{X}^{1/3} - \theta)$  to  $\mathcal{N}(0, \frac{1}{9\theta^{4/3}})$ .  
Let  $Y$  be a r.v. with distribution  $\mathcal{N}(0, 1)$ . When  $\theta = 0$ , combining the CLT with the continuous mapping theorem gives convergence in distribution of  $n^{1/6} \bar{X}^{1/3}$  to  $Y^{1/3}$ , which rewrites as  $\left[n^{1/2} \bar{X}^{1/3}\right] \frac{1}{n^{1/3}} \rightarrow Y^{1/3}$ . If  $n^{1/2} \bar{X}^{1/3}$  converged in distribution, Slutsky's theorem would imply that  $\left[n^{1/2} \bar{X}^{1/3}\right] \frac{1}{n^{1/3}} \rightarrow 0$  in distribution, a contradiction. Consequently, when  $\theta = 0$ ,  $\hat{\theta}_n$  is not asymptotically normal.
4. For  $\theta \neq 0$  we proved that  $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \frac{1}{9\theta^{4/3}})$ . Question 3 of Problem 1 implies that  $\sqrt{n}(\hat{\theta}_n - \theta)$  is tight, hence  $\hat{\theta}_n - \theta = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$ .

For  $\theta = 0$ ,  $n^{1/6} \hat{\theta}_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}} Y^{1/3}$ , and by the same argument  $\hat{\theta}_n - \theta = O_{\mathbb{P}}\left(\frac{1}{n^{1/6}}\right)$ .