HW3: ASYMPTOTICS 3

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Problem 1

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function and $a \in \mathbb{R}^d$.

1. Suppose that $\|x-a\|_2 = 1 \implies f(x) > f(a)$. Show that f is bounded below and that its minimum is attained only at points such that $\|x-a\|_2 < 1$.

2. More generally, let K be a compact set with $a \in \operatorname{int} K$. Suppose that $x \in \partial K \implies f(x) > f(a)$. Show that f is bounded below and that its minimum is attained only in the interior of K.

1. Consider x such that $\|x-a\|_2 > 1$ and let $\phi : [0,\infty) \to \mathbb{R}^d, t \mapsto f(a + t\frac{x-a}{\|x-a\|_2})$.
By definition, ϕ is convex, $\phi(0) = f(a)$, $\phi(1) = f(a + \frac{x-a}{\|x-a\|_2}) > f(a)$ and $\phi(\|x-a\|_2) = f(x)$. By the inequality on slopes of convex real functions,
$$\frac{\phi(\|x-a\|_2) - \phi(0)}{\|x-a\|_2} \geq \frac{\phi(1) - \phi(0)}{1}$$
thus $\frac{f(x)-f(a)}{\|x-a\|_2} \geq f(a+\frac{x-a}{\|x-a\|_2})-f(a)$, hence $f(x) - f(a) \geq \|x-a\|_2 \underbrace{\left(f(a + \frac{x-a}{\|x-a\|_2}) - f(a)\right)}_{>0}$
Hence $\|x-a\|_2 > 1 \implies f(x) > f(a)$.
f is convex on the open set $B(a, \frac{3}{2})$, hence continuous over $B(a, \frac{3}{2})$ and thus continuous over $\overline{B}(a,1)$. It reaches therefore a minimum over $\overline{B}(a,1)$, which is also a global minimum since $\|x-a\|_2 > 1 \implies f(x) > f(a)$. Besides, since $\|x-a\|_2 = 1 \implies f(x) > f(a)$, the minimum is attained only at points such that $\|x-a\|_2 < 1$.

2. Consider $x \notin K$ and let $A = \{t \geq 0,\ a + t\frac{x-a}{\|x-a\|_2} \in K\}$. $0 \in A$ so A is non-empty, and since K is bounded, A is bounded above. Let $t_0 = \sup A$. Let us show that $a + t_0\frac{x-a}{\|x-a\|_2} \in \partial K$. Since K is closed we have clearly $a + t_0\frac{x-a}{\|x-a\|_2} \in K$. Let $\varepsilon > 0$. Since $t_0 + \varepsilon \notin A$ we have $a + (t_0+\varepsilon)\frac{x-a}{\|x-a\|_2} \notin K$ and since $||a + (t_0+\varepsilon)\frac{x-a}{\|x-a\|_2} - \left(a + t_0\frac{x-a}{\|x-a\|_2}\right)||_2 = \varepsilon$, we have $a + t_0\frac{x-a}{\|x-a\|_2} \notin \operatorname{int} K$, thus $a + t_0\frac{x-a}{\|x-a\|_2} \in \partial K$.

   Since $\|x-a\|_2 \notin A$, we have $\|x-a\|_2 > t_0$ and the same argument as in 1. shows that
$$f(x) - f(a) \geq \frac{\|x-a\|_2}{t_0}\left(f(a + t_0\frac{x-a}{\|x-a\|_2}) - f(a)\right) > 0$$
K being compact, it is bounded so there exists an open ball B such that $K \subset B$. f is convex over B, hence continuous over K, so f reaches a minimum over K. Since $x \notin K \implies f(x) > f(a)$, this minimum is global, and since $x \in \partial K \implies f(x) > f(a)$, it is attained only at points in the interior of K.