Geometry
Problem booklet

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1 Week 7: Products of vectors

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Products of vectors

1.1.1 The vector product

If $[\vec{i}, \vec{j}, \vec{k}]$ is an orthonormal basis, observe that $\vec{i} \times \vec{j} \in \{-\vec{k}, \vec{k}\}$. We say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *direct* if $\vec{i} \times \vec{j} = \vec{k}$. If, on the contrary, $\vec{i} \times \vec{j} = -\vec{k}$, we say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *inverse*. Therefore, if $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis, then $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and obviously $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

Proposition 1.1. *If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and*

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}, \ \vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k},$$

then

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2) \vec{i} + (a_3b_1 - a_1b_3) \vec{j} + (a_1b_2 - a_2b_1) \vec{k}, \tag{1.1}$$

or, equivalently,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \tag{1.2}$$

One can rewrite formula (1.1) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.3}$$

the right hand side determinant being understood in the sense of its cofactor expansion along the first line.

1.1.2 Applications of the vector product

• **The area of the triangle ABC.** $S_{ABC} = \frac{1}{2} \| \overrightarrow{AB} \| \cdot \| \overrightarrow{AC} \| \sin \widehat{BAC} = \frac{1}{2} \| \overrightarrow{AB} \times \overrightarrow{AC} \|$. Since the coordinates of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are

$$(x_B - x_A, y_B - x_A, z_B - z_A) \text{ and } (x_C - x_A, y_C - x_A, z_C - z_A)$$

respectively, we deduce that

$$S_{ABC} = \frac{1}{2} \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_B - x_A & y_B - x_A & z_B - z_A \\ x_C - x_A & y_C - x_A & z_C - z_A \end{vmatrix} \right\|,$$

or, equivalently

$$4S^2_{ABC} = \begin{vmatrix} y_B-y_A & z_B-z_A \\ y_C-y_A & z_C-z_A \end{vmatrix}^2 + \begin{vmatrix} z_B-z_A & x_B-x_A \\ z_C-z_A & x_C-x_A \end{vmatrix}^2 + \begin{vmatrix} x_B-x_A & y_B-y_A \\ x_C-x_A & y_C-y_A \end{vmatrix}^2.$$

• The distance from one point to a straight line.

a) *The distance $\delta(A,BC)$ from the point $A(x_A,y_A,z_A)$ to the straight line BC*, where $B(x_B,y_B,z_B)$ şi $C(x_C,y_C,z_C)$. Since

$$S_{ABC} = \frac{\| \overrightarrow{BC} \| \cdot \delta(A,BC)}{2}$$

rezultă că

$$\delta^2(A,BC) = \frac{4S^2_{ABC}}{\| \overrightarrow{BC} \|^2}.$$

Thus, we obtain

$$\delta^2(A,BC) = \frac{\begin{vmatrix} y_B-y_A & z_B-z_A \\ y_C-y_A & z_C-z_A \end{vmatrix}^2 + \begin{vmatrix} z_B-z_A & x_B-x_A \\ z_C-z_A & x_C-x_A \end{vmatrix}^2 + \begin{vmatrix} x_B-x_A & y_B-y_A \\ x_C-x_A & y_C-y_A \end{vmatrix}^2}{(x_C-x_B)^2+(y_C-y_B)^2+(z_C-z_B)^2}.$$

(b) The distance from $\delta(A,d)$ from one point $A(A_A,y_A,z_A)$ to the straight line

$$d : \frac{x-x_0}{p} + \frac{y-y_0}{p} + \frac{z-z_0}{p}.$$

$$\delta(A,d) = \frac{\| \vec{d} \times \overrightarrow{A_0A} \|}{\vec{d}}, \quad \text{where } A_0(x_0,y_0,z_0) \in \delta.$$

Since

$$\begin{aligned} \vec{d} \times \overrightarrow{A_0A} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ x_A-x_0 & y_A-y_0 & z_A-z_0 \end{vmatrix} \\ &= \begin{vmatrix} q & r \\ y_A-y_0 & z_A-z_0 \end{vmatrix} \vec{i} + \begin{vmatrix} r & p \\ z_A-z_0 & x_A-x_0 \end{vmatrix} \vec{j} + \begin{vmatrix} p & q \\ x_A-x_0 & y_A-y_0 \end{vmatrix} \vec{k} \end{aligned}$$

it follows that

$$\delta(A,d) = \frac{\sqrt{\begin{vmatrix} q & r \\ y_A-y_0 & z_A-z_0 \end{vmatrix}^2 + \begin{vmatrix} r & p \\ z_A-z_0 & x_A-x_0 \end{vmatrix}^2 + \begin{vmatrix} p & q \\ x_A-x_0 & y_A-y_0 \end{vmatrix}^2}}{\sqrt{p^2+q^2+r^2}}.$$

1.1.3 The double vector (cross) product

The *double vector (cross) product* of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the vector $\vec{a} \times (\vec{b} \times \vec{c})$

Proposition 1.2. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c} = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \end{vmatrix}, \quad \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}.$

Corollary 1.3. *1.* $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{a} \end{vmatrix}, \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$;

2. $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}, \ \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$ *(Jacobi's identity).*

1.1.4 The triple scalar product

The *triple scalar product* $(\vec{a}, \vec{b}, \vec{c})$ of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the real number $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

Proposition 1.4. *If* $[\vec{i}, \vec{j}, \vec{k}]$ *is a direct orthonormal basis and*

$$\begin{aligned} \vec{a} &= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \\ \vec{b} &= b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k} \\ \vec{c} &= c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}, \end{aligned}$$

then

$$(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{1.4}$$

Corollary 1.5.
1. *The free vectors* $\vec{a}, \vec{b}, \vec{c}$ *are linearly dependent (collinear) if and only if* $(\vec{a}, \vec{b}, \vec{c}) = 0$
2. *The free vectors* $\vec{a}, \vec{b}, \vec{c}$ *are linearly independent (noncollinear) if and only if* $(\vec{a}, \vec{b}, \vec{c}) \neq 0$
3. *The free vectors* $\vec{a}, \vec{b}, \vec{c}$ *form a basis of the space* $\mathcal{V}$ *if and only if* $(\vec{a}, \vec{b}, \vec{c}) \neq 0$.
4. *The correspondence* $F : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$, $F(\vec{a}, \vec{b}, \vec{c}) = (\vec{a}, \vec{b}, \vec{c})$ *is trilinear and skew-symmetric, i..e*

$$\begin{aligned} (\alpha \vec{a} + \alpha' \vec{a}\,', \vec{b}, \vec{c}) &= \alpha(\vec{a}, \vec{b}, \vec{c}) + \alpha'(\vec{a}\,', \vec{b}, \vec{c}) \\ (\vec{a}, \beta \vec{b} + \beta' \vec{b}\,', \vec{c}) &= \beta(\vec{a}, \vec{b}, \vec{c}) + \beta'(\vec{a}, \vec{b}', \vec{c}) \\ (\vec{a}, \vec{b}, \gamma \vec{c} + \gamma' \vec{c}\,') &= \gamma(\vec{a}, \vec{b}, \vec{c}) + \gamma'(\vec{a}, \vec{b}, \vec{c}\,') \end{aligned} \tag{1.5}$$

$\forall\, \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{R}$, $\forall\ \vec{a}, \vec{b}, \vec{c}, \vec{a}\,', \vec{b}\,', \vec{c}\,' \in \mathcal{V}$ *şi*

$$(\vec{a}_1, \vec{a}_2, \vec{a}_3) = sgn(\sigma)(\vec{a}_{\sigma(1)}, \vec{a}_{\sigma(2)}, \vec{a}_{\sigma(3)}),\ \forall\ \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V} \text{ şi } \forall \sigma \in S_3 \tag{1.6}$$

Remark 1.6. One can rewrite the relations (1.6) as follows:

$$\begin{aligned} (\vec{a}_1, \vec{a}_2, \vec{a}_3) &= (\vec{a}_2, \vec{a}_3, \vec{a}_1) = (\vec{a}_3, \vec{a}_1, \vec{a}_2) \\ &= -(\vec{a}_2, \vec{a}_1, \vec{a}_3) = (\vec{a}_1, \vec{a}_3, \vec{a}_2) = -(\vec{a}_3, \vec{a}_2, \vec{a}_1), \end{aligned}$$

$\forall\ \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathcal{V}$

Corollary 1.7.
1. $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})\ \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.
2. *For every* $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathcal{V}$ *the* Laplace formula

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

holds.

Definition 1.8. The basis $[\vec{a}, \vec{b}, \vec{c}]$ of the space $\mathcal{V}$ is said to be *directe* if $(\vec{a}, \vec{b}, \vec{c}) > 0$. If, on the contrary, $(\vec{a}, \vec{b}, \vec{c}) < 0$, we say that the basis $[\vec{a}, \vec{b}, \vec{c}]$ is *inverse*

Definition 1.9. The *oriented volume* of the parallelipiped constructed on the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is $\varepsilon \cdot V$, where V is the volume of this parallelepiped and $\varepsilon = +1$ or -1 insomuch as the basis $[\vec{a}, \vec{b}, \vec{c}]$ is directe or inverse respectively.

Propoziţia 1.10. *The triple scalar product $(\vec{a}, \vec{b}, \vec{c})$ of the noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ is the oriented volume of the parallelepiped constructed on these vectors.*

1.2 Problems

1. Two triangles ABC şi $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices A', B', C' on the sides BC, CA, AB are concurrent. Show that, in this case, the perpendicular lines from the vertices A, B, C on the sides $B'C'$, $C'A'$, $A'B'$ are concurrent too.

2. Show the following identities:

   (a) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a}, \vec{c}, \vec{d})\, \vec{b} - (\vec{b}, \vec{c}, \vec{d})\, \vec{a} = (\vec{a}, \vec{b}, \vec{d})\, \vec{c} - (\vec{a}, \vec{b}, \vec{c})\, \vec{d}$.

   (b) $(\vec{u} \times \vec{v}, \vec{v} \times \vec{w}, \vec{w} \times \vec{u}) = (\vec{u}, \vec{v}, \vec{w})^2$.

3. The *reciprocal vectors* of the noncoplanar vectors $\vec{u}, \vec{v}, \vec{w}$ are defined by

$$\vec{u}' = \frac{\vec{v} \times \vec{w}}{(\vec{u}, \vec{v}, \vec{w})},\ \vec{v}' = \frac{\vec{w} \times \vec{u}}{(\vec{u}, \vec{v}, \vec{w})},\ \vec{w}' = \frac{\vec{u} \times \vec{v}}{(\vec{u}, \vec{v}, \vec{w})}.$$

   Show that:

   (a)
$$\begin{aligned}\vec{a} &= (\vec{a} \cdot \vec{u}')\, \vec{u} + (\vec{a} \cdot \vec{v}')\, \vec{v} + (\vec{a} \cdot \vec{w}')\, \vec{w} \\ &= \frac{(\vec{a}, \vec{v}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})}\, \vec{u} + \frac{(\vec{u}, \vec{a}, \vec{w})}{(\vec{u}, \vec{v}, \vec{w})}\, \vec{v} + \frac{(\vec{u}, \vec{v}, \vec{a})}{(\vec{u}, \vec{v}, \vec{w})}\, \vec{w}\,.\end{aligned}$$

   (b) the reciprocal vectors of $\vec{u}', \vec{v}', \vec{w}'$ are the vectors $\vec{u}, \vec{v}, \vec{w}$.

4. Let d_1, d_2, d_3, d_4 be pairwise skew straight lines. Assuming that $d_{12} \perp d_{34}$ and $d_{13} \perp d_{24}$, show that $d_{14} \perp d_{23}$, where d_{ik} is the common perpendicular of the lines d_i and d_k.

5. Find the value of the parameter α for which the pencil of planes through the straight line AB has a common plane with the pencil of planes through the straight line CD, where $A(1, 2\alpha, \alpha)$, $B(3, 2, 1)$, $C(-\alpha, 0, \alpha)$ and $D(-1, 3, -3)$.

6. Find the value of the parameter λ for which the straight lines

$$(d_1)\ \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1},\ (d_2)\ \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

   are coplanar. Find the coordinates of their intersection point in that case.

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