Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Bachelor, Computer Science, Groups 911-917, Academic Year 2016-2017

Mathematical Analysis
Seminar 10

1. Prove that the following properties hold for any $x, y \in \mathbb{R}^n$:
a) $\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$.
b) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (the generalized parallelogram identity).

2. Let $x, y \in \mathbb{R}^n$. Prove that the following statements are equivalent:
1° $\langle x, y\rangle = 0$ (i.e., x and y are orthogonal).
2° $\|x+y\| = \|x-y\|$.
3° $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

3. A set $S \subseteq \mathbb{R}^n$ is said to be convex if for any points $x, y \in S$ and any number $t \in [0,1]$ we have $(1-t)x + ty \in S$. Prove that, for all $x^0 \in \mathbb{R}^n$ and $r > 0$, the open ball $B(x^0, r)$ as well as the closed ball $\overline{B}(x^0, r)$ are convex sets.

4. Show that if $x, y \in \mathbb{R}^n$, $x \neq y$, then there exist $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$ such that $U \cap V = \emptyset$.

5. In each of the following instances, determine if the sequence $(x^k)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$ is convergent or not. If the sequence is convergent, find also its limit.

a) $n = 2,\ x^k = \left(\dfrac{1}{k}, \dfrac{k^2+4k}{2k^2+1}\right)$, b) $n = 2,\ x^k = \big((-1/2)^k, (-1)^k\big)$,

c) $n = 2,\ x^k = \left(\sin k, \dfrac{1}{k^2}\right)$, d) $n = 2,\ x^k = \left(\left(\dfrac{\sqrt{k}}{1+\sqrt{k}}\right)^k, \dfrac{1^1 + 2^2 + \ldots + k^k}{k^k}\right)$,

e) $n = 3,\ x^k = \big(e^{-k}\cos k, e^{-k}\sin k, k\big)$, f) $n = 3,\ x^k = \left(\dfrac{2^k}{k!}, \dfrac{1-4k^7}{k^7+12k}, \dfrac{\sqrt{k}}{e^{3k}}\right)$,

g) $n = 4,\ x^k = \left(\dfrac{2^{2k}}{\left(2+\frac{1}{k}\right)^{2k}}, \dfrac{1}{\sqrt[k]{k!}}, (e^k + k)^{\frac{1}{k}}, \dfrac{\alpha^k}{k}\right)$, where $\alpha \geq 0$ is fixed.

6. Find the interior, the closure and the boundary for each of the following subsets of $\mathbb{R}^2$. Specify whether the sets are open and/or closed.
a) $A = [0,1] \times [1,2]$, b) $A = [0,1) \times (1,2]$, c) $A = \{(x,0) \mid x < 0\} \cup \{(x,y) \mid y < 0\}$,
d) $A = \mathbb{Q} \times \mathbb{Q}$, e) $A = \{0_2\}$, f) $A = \mathbb{R} \times \{0\}$, g) $A = \mathbb{R}^2$, h) $A = \emptyset$.