Geometry
Problem booklet

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1 Week 2: Straight lines and planes

1.1 Brief theoretical background

1.1.1 Linear dependence and linear independence of vectors

Definition 1.1. 1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *collinear* if the points O, A, B are collinear. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are said to be *noncollinear*.

2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}, \overrightarrow{OC}$ are said to be *coplanar* if the points O, A, B, C are coplanar. Otherwise the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are *noncoplanar*.

Remark 1.2. 1. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ are linearly (in)dependent if and only if they are (non)collinear.

2. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are linearly (in)dependent if and only if they are (non)coplanar.

Proposition 1.3. *The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ form a basis of $\mathcal{V}$ if and only if they are noncoplanar.*

Corollary 1.4. *The dimension of the vector space of free vectors $\mathcal{V}$ is three.*

Proposition 1.5. *Let Δ be a straight line and let $A \in \Delta$ be a given point. The set*

$$\overrightarrow{\Delta} = \{\overrightarrow{AM} \mid M \in \Delta\}$$

is an one dimensional subspace of $\mathcal{V}$. It is independent on the choice of $A \in \Delta$ and is called the director subspace *of Δ or the* direction *of Δ.*

Remark 1.6. *The straight lines Δ, Δ' are parallel if and only if $\overrightarrow{\Delta} = \overrightarrow{\Delta}'$*

Definition 1.7. We call *director vector* of the straigh line Δ every nonzero vector *stackrel→d* $\in \overrightarrow{\Delta}$.

If $\overrightarrow{d} \in \mathcal{V}$ is a nonzero vector and $A \in \mathcal{P}$ is a given point, then there exits a unique straight line which passes through A and has the direction $\langle \overrightarrow{d} \rangle$. This stright line is

$$\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d} \rangle\}.$$

Δ is called the straight line which passes through O and is parallel to the vector $\overrightarrow{d}$.

Proposition 1.8. *Let π be a plane and let $A \in \pi$ be a given point. The set $\overrightarrow{\pi} = \{\overrightarrow{AM} \in \mathcal{V} \mid M \in \pi\}$ is a two dimensional subspace of $\mathcal{V}$. It is independent on the position of A inside π and is called the* director subspace, the director plane *or the* direction *of the plane π.*

Remark 1.9.
- *The planes π, π' are parallel if and only if $\overrightarrow{\pi} = \overrightarrow{\pi}'$.*
- *The line Δ is parallel to the plane π if and only if $\overrightarrow{\Delta} \subseteq \overrightarrow{\pi}$.*
- *If $\overrightarrow{d}_1$, $\overrightarrow{d}_2$ are two linearly independent vectors and $A \in \mathcal{P}$ is a fixed point, then there exists a unique plane through A whose direction is $\langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle$. This plane is*

$$\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle\}.$$

We call π is the plane through the point A which is parallel to the vectors $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$.

1.1.2 The vector ecuation of the straight lines and planes

Let Δ be a straight line, let $A \in \Delta$ be a given point and let $\overrightarrow{d}$ be a director vector of Δ.

$$\overrightarrow{r}_M = \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{r}_A + \overrightarrow{AM}.$$

Thus

$$\begin{aligned}\{\overrightarrow{r}_M \mid M \in \Delta\} &= \{\overrightarrow{r}_A + \overrightarrow{AM} \mid M \in \Delta\}\\ &= \overrightarrow{r}_A + \{\overrightarrow{AM} \mid M \in \Delta\}\\ &= \overrightarrow{r}_A + \overrightarrow{\Delta} = \overrightarrow{r}_A + \langle \overrightarrow{d} \rangle.\\ &= \{\overrightarrow{r}_A + t\overrightarrow{d} : t \in \mathbb{R}\}.\end{aligned}$$

In other words, the position vectors of all points on the straight line Δ are

$$\overrightarrow{r}_M = \overrightarrow{r}_A + t\overrightarrow{d} : t \in \mathbb{R}. \tag{1.1}$$

This is the reason to call (1.1) the *vector equation* of the line Δ.

Proposition 1.10. *If A, B are different points of a straight line Δ, then its vector equation can be put in the form*

$$\overrightarrow{r}_M = (1-\lambda)\overrightarrow{r}_A + \lambda\overrightarrow{r}_B,\ \lambda \in \mathbb{R}. \tag{1.2}$$

Similarly, for a plane π a given point $B \in \pi$ and a basis $[\overrightarrow{d}_1, \overrightarrow{d}_2]$ of $\overrightarrow{\pi}$ we get

$$\{\overrightarrow{r}_N \mid N \in \pi\} = \overrightarrow{r}_B + \overrightarrow{\pi} = \overrightarrow{r}_B + \langle \overrightarrow{d}_1, \overrightarrow{d}_2\rangle = \{\overrightarrow{r}_B + t_1\overrightarrow{d}_1 + t_2\overrightarrow{d}_2 : t_1, t_2 \in \mathbb{R}\}.$$

In other words, the position vectors of all points on the plane π are

$$\overrightarrow{r}_N = \overrightarrow{r}_B + t_1\overrightarrow{d}_1 + t_2\overrightarrow{d}_2 : t_1, t_2 \in \mathbb{R}. \tag{1.3}$$

This is the reason to call (1.3) the *vector equation* of the plane π.

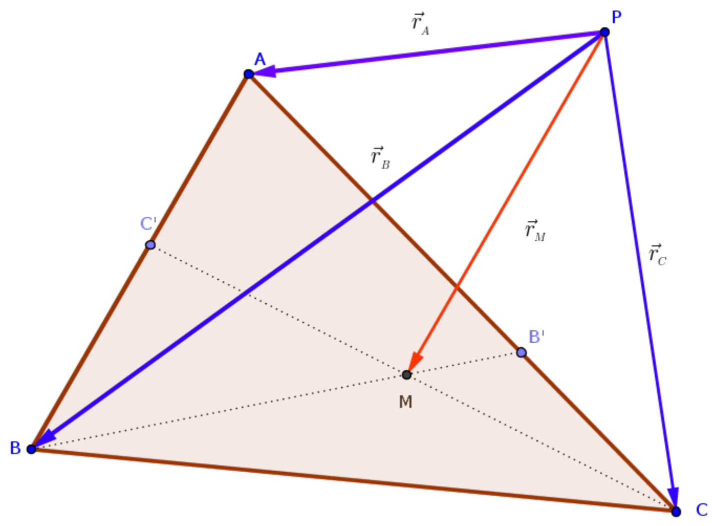
Proposition 1.11. *If A, B, C are three noncolinear points within the plane π, then the vector equation of the plane π can be put in the form*

$$\overrightarrow{r}_M = (1-\lambda_1-\lambda_2)\overrightarrow{r}_A + \lambda_1\overrightarrow{r}_B + \lambda_2\overrightarrow{r}_C,\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{1.4}$$

1.2 Problems

1. ([4, Problema 16, p. 5]) Consider the points C' and B' on the sides AB and AC of the triangle ABC such that $\overrightarrow{AC'} = \lambda\overrightarrow{BC'}$, $\overrightarrow{AB'} = \mu\overrightarrow{CB'}$. The lines BB' and CC' meet at M. If $P \in \mathcal{P}$ is a given point and $\overrightarrow{r}_A = \overrightarrow{PA}$, $\overrightarrow{r}_B = \overrightarrow{PB}$, $\overrightarrow{r}_C = \overrightarrow{PC}$ are the position vectors, with respect to P, of the vertices A, B, C respectively, show that

$$\overrightarrow{r}_M = \frac{\overrightarrow{r}_A - \lambda\overrightarrow{r}_B - \mu\overrightarrow{r}_C}{1-\lambda-\mu}. \tag{1.5}$$



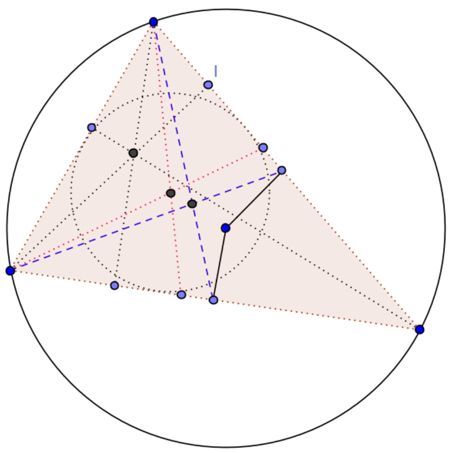
2. ([4, Problema 17, p. 5]) Consider the triangle ABC, its centroid G, its orthocenter H, its incenter I and its circumcenter O. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that:

(a) $\vec{r}_G := \overrightarrow{PG} = \dfrac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}$.

(b) $\vec{r}_I := \overrightarrow{PI} = \dfrac{a\,\vec{r}_A + b\,\vec{r}_B + c\,\vec{r}_C}{a+b+c}$.

(c) $\vec{r}_H := \overrightarrow{PH} = \dfrac{(\tan A)\,\vec{r}_A + (\tan B)\,\vec{r}_B + (\tan C)\,\vec{r}_C}{\tan A + \tan B + \tan C}$.

(d) $\vec{r}_O := \overrightarrow{PO} = \dfrac{(\sin 2A)\,\vec{r}_A + (\sin 2B)\,\vec{r}_B + (\sin 2C)\,\vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}$.



3. Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\vec{r}_M = m\frac{1-n}{1-mn}\,\vec{u} + n\frac{1-m}{1-mn}\,\vec{v} \tag{1.6}$$

and

$$\vec{r}_N = m\frac{n-1}{n-m}\,\vec{u} + n\frac{m-1}{m-n}\,\vec{v}, \tag{1.7}$$

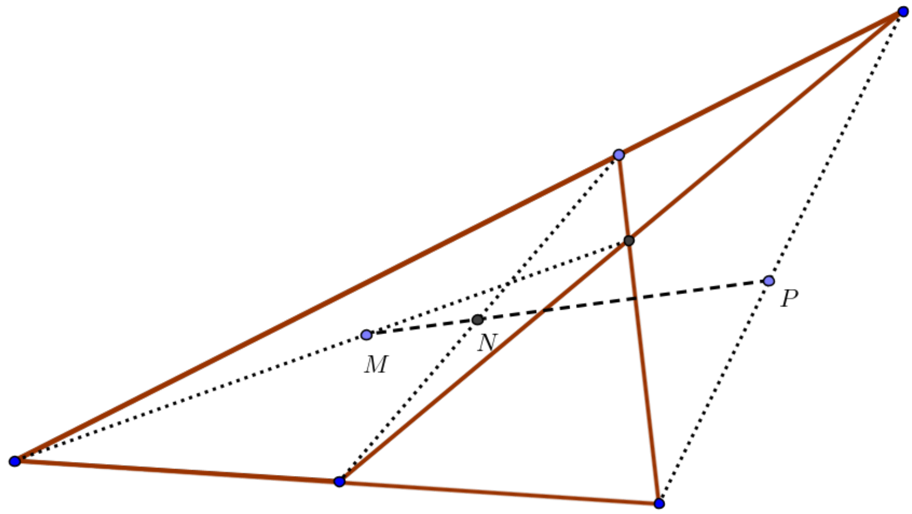
where $\{M\} = AB' \cap A'B$, $\{N\} = AA' \cap BB'$, $\vec{u} = \overrightarrow{OA}$, $\vec{v} = \overrightarrow{OA'}$, $\overrightarrow{OB} = m\,\overrightarrow{OA}$ and $\overrightarrow{OB'} =$

$n\,\overrightarrow{OA'}$. In other words

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\,\overrightarrow{OA} \quad + \quad n\frac{1-m}{1-mn}\,\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\,\overrightarrow{OA} \quad + \quad n\frac{m-1}{m-n}\,\overrightarrow{OA'}\,.$$

4. Show that the midpoints of the diagonals of a complet quadrilateral are collinear (Newton's theorem).



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