Babeş-Bolyai University, Faculty of Mathematics and Computer Science

Bachelor, Computer Science, Groups 911-917, Academic Year 2017-2018

Mathematical Analysis
Solutions for Seminars 2 and 3

Seminar 2, Exercise 3. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$x_n \coloneqq \left(1+\frac{1}{n}\right)^n .$$

a) Using Bernoulli's Inequality (see Seminar 1) prove that $\dfrac{x_{n+1}}{x_n} > 1$ for all $n \in \mathbb{N}$.

b) Using Newton's Binomial Formula prove that $x_n < 3$ for all $n \in \mathbb{N}$.

Hint: notice that $\dbinom{n}{k} \leq \dfrac{n^k}{2^{k-1}}$ for all $k \in \mathbb{N}$, $k \leq n$.

c) Deduce that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and, denoting its limit by e (the Euler's number), show that $2.71 < e \leq 3$.

d) Similarly to a) prove that the sequence $(y_n)_{n\in\mathbb{N}}$, defined for all $n \in \mathbb{N}$ by

$$y_n \coloneqq \left(1+\frac{1}{n}\right) x_n,$$

is strictly decreasing. Then, observing that $x_n < y_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n$, deduce that $e < 2.72$.

Solution: d) From the definition of $(y_n)_{n\in\mathbb{N}}$ we get

$$y_n = \left(1+\frac{1}{n}\right)\cdot\left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{n}\right)^{n+1}$$

To prove that this sequence is strictly decreasing, we have to prove that $\dfrac{y_n}{y_{n+1}} > 1$.

$$\begin{aligned}
\frac{y_n}{y_{n+1}} &= \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} = \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n+2}{n+1}\right)^{n+2}} = \left[\frac{(n+1)^2}{n(n+2)}\right]^{n+1}\cdot\left(\frac{n+1}{n+2}\right) = \\
&= \left[1+\frac{1}{n(n+2)}\right]^{n+1}\cdot\left(1-\frac{1}{n+2}\right) \geq \left[1+\frac{n+1}{n(n+2)}\right]\cdot\left(1-\frac{1}{n+2}\right) = \\
&= 1-\frac{1}{n+2}+\frac{n+1}{n(n+2)}-\frac{n+1}{n(n+2)^2} = 1-\frac{n(n+2)}{n(n+2)^2}+\frac{(n+2)(n+1)}{n(n+2)^2}-\frac{n+1}{n(n+2)^2} = \\
&= 1+\frac{-n(n+2)+(n+2)(n+1)-n-1}{n(n+2)^2} = 1+\frac{-n^2-2n+n^2+3n+2-n-1}{n(n+2)^2} = \\
&= 1+\frac{1}{n(n+2)^2} > 1.
\end{aligned}$$

Note: We used Bernoulli's Inequality for $x = \dfrac{1}{n(n+2)} \in [-1,+\infty)$ to show that

$$\left(1+\frac{1}{n(n+2)}\right)^{n+1} \geq \left(1+\frac{n+1}{n(n+2)}\right)$$

To prove that $x_n < y_n$ it suffices to prove that $\frac{y_n}{x_n} > 1$. Using the definition of y_n we get

$$\frac{y_n}{x_n} = \frac{\left(1+\frac{1}{n}\right)x_n}{x_n} = \left(1+\frac{1}{n}\right) > 1$$

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To prove that $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n$ it suffices to prove that $\lim\limits_{n\to\infty} \frac{y_n}{x_n} = 1$, since they are finite.

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)x_n}{x_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$$

For $n = 800, y_n < 2.72$ and because the sequence is strictly decreasing, its limit is also smaller than 2.72, hence $e < 2.72$.

Seminar 2, Exercise 4.

Solution: f)

$$\begin{aligned}
\lim_{n\to\infty} \sin\left(\pi\sqrt{n^2+1}\right) &= \lim_{n\to\infty} \sin\left[\pi\left(\sqrt{n^2+1}-n\right)+\pi n\right] = \\
&= \lim_{n\to\infty} \left[\sin\left(\pi\left(\sqrt{n^2+1}-n\right)\right)\cdot\cos(\pi n) + \cos\left(\pi\left(\sqrt{n^2+1}-n\right)\right)\cdot\underbrace{\sin(\pi n)}_{0}\right] = \\
&= \lim_{n\to\infty} \sin\left(\pi\left(\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}\right)\right)\cdot\cos(\pi n) = \\
&= \lim_{n\to\infty} \underbrace{\sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right)}_{\to 0}\cdot\underbrace{\cos(\pi n)}_{\in[-1,1]} = 0
\end{aligned}$$

Seminar 3, Exercise 2. Consider the sequence $(\gamma_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$\gamma_n := 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n.$$

a) Using the fact that $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$ for all $n \in \mathbb{N}$ (cf. Exercise 3 of Seminar 2), prove that $(\gamma_n)_{n\in\mathbb{N}}$ is strictly decreasing and bounded below by 0.

b) Deduce that $(\gamma_n)_{n\in\mathbb{N}}$ is convergent and, denoting its limit by γ (the Euler's constant, also known as the Euler-Mascheroni constant), show that $\gamma < 0.58$.

c) Prove that the sequence $(x_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$x_n := \gamma_n + \ln n - \ln(n+1)$$

is strictly increasing. Then, observing that $x_n < \gamma_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \gamma_n$, deduce that $\gamma > 0.57$.

Solution: c) Using the definition of $(x_n)_{n\in\mathbb{N}}$ from c) we get

$$x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n+1)$$

To prove that this sequence is strictly increasing, we have to prove that $x_{n+1} - x_n > 0$ for all $n \in \mathbb{N}$.

$$\begin{aligned}
x_{n+1} - x_n &= 1 + \frac{1}{2} + \ldots + \frac{1}{n} + \frac{1}{n+1} - \ln(n+2) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n+1)\right) = \\
&= \frac{1}{n+1} - \ln(n+2) + \ln(n+1) = \frac{1}{n+1} - \ln\left(\frac{n+2}{n+1}\right) = \frac{1}{n+1} - \ln\left(1+\frac{1}{n+1}\right) \; (*)
\end{aligned}$$

Taking the logarithm of both sides of the first inequality from a) and using the fact that the function ln is strictly increasing, we get the following

$$n \ln\left(1 + \frac{1}{n}\right) < 1, \text{for all } n \in \mathbb{N}$$

$$(n+1)\ln\left(1 + \frac{1}{n+1}\right) < 1, \text{for all } n \in \mathbb{N}$$

$$\ln\left(1 + \frac{1}{n+1}\right) < \frac{1}{n+1}$$

$$\frac{1}{n+1} - \ln\left(1 + \frac{1}{n+1}\right) > 0$$

From this inequality and from $(*)$ it follows that $x_{n+1} - x_n > 0$, hence $(x_n)_{n\in\mathbb{N}}$ is strictly increasing.

Using the definition of $(x_n)_{n\in\mathbb{N}}$ we know that $x_n = \gamma_n + \ln n - \ln(n+1)$, but ln is a strictly increasing function so $\ln n - \ln(n+1) < 0$, hence $x_n < \gamma_n$.

To prove that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \gamma_n$, it suffices to prove that $\lim\limits_{n\to\infty} (x_n - \gamma_n) = 0$, since both of the limits are finite.

$$\lim_{n\to\infty}(x_n - \gamma_n) = \lim_{n\to\infty}(\gamma_n + \ln n - \ln(n+1) - \gamma_n) = \gamma - \gamma + \lim_{n\to\infty}(\ln n - \ln(n+1)) =$$
$$= \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = \ln 1 = 0$$

For $n = 80, \gamma_n > 0.57$ and since γ_n is an increasing sequence, it follows that its limit is also greater than 0.57, hence $\gamma > 0.57$.