Babeş-Bolyai University, Faculty of Mathematics and Computer Science

Bachelor, Computer Science, Groups 911-917, Academic Year 2016-2017

Mathematical Analysis
Seminar 7

1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$. Find the second Taylor polynomial $T_2(x)$ of f at 0 and the remainder term $R_2(x)$ of the corresponding Taylor's formula in Lagrange's form. Deduce that $1 - \frac{x^2}{2} \leq \cos x$ for all $x \in \mathbb{R}$.

2. For each function $f : \mathbb{R} \to \mathbb{R}$ given below check that $f'(0) = 0$ and find the smallest number $n \in \mathbb{N}$ such that $f^{(n)}(0) \neq 0$. Then, deduce whether 0 is a local extremum point of f or not; in the affirmative, specify if 0 is a global extremum point or just a local one.

a) $f(x) = e^x + e^{-x} - x^2$; b) $f(x) = \cos(x^2)$; c) $f(x) = 6\sin x - 6x + x^3$.

3. Let $f : (-1, \infty) \to \mathbb{R}$, $f(x) = \ln(x + 1)$. Show that f can be expanded as a Taylor series around 0 on $[0, 1]$ and find the corresponding Taylor series expansion.

4. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

is infinitely differentiable, but f is not expandable as a Taylor series around 0 on any neighborhood of 0. Find all global extremum points of f.

5. Find the radius of convergence and the convergence set for each of the following power series:

a) $\displaystyle\sum_{n\geq 1}(x - e)^n$; b) $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}(x + 1)^n$; c) $\displaystyle\sum_{n\geq 1} \frac{1}{(2n)!!}x^n$; d) $\displaystyle\sum_{n\geq 1}(2n + 1)!!\, x^n$.

6. Compute the following limits as Riemann integrals:

a) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)$; b) $\displaystyle\lim_{n\to\infty} \frac{\sqrt[n]{e} + 2\sqrt[n]{e^2} + \cdots + n\sqrt[n]{e^n}}{n^2}$.

7. Compute the improper integrals:

a) $\displaystyle\int_0^1 \frac{1}{x}dx$; b) $\displaystyle\int_1^2 \frac{1}{x(x-2)}$; c) $\displaystyle\int_{-\infty}^0 xe^{-x^2}dx$; d) $\displaystyle\int_0^{+\infty} e^{-x}\sin x dx$.