LECTURE

1

THE REAL NUMBERS: SOME BASIC CONCEPTS

The set of real numbers, denoted by $\mathbb{R}$, is a *totally ordered field*

$$(\mathbb{R}, +, \cdot, \geq)$$

meaning that

- $(\mathbb{R}, +, \cdot)$ is a field, where 0 and 1 are the neutral elements of $+$ and $\cdot$, respectively;
- $\geq$ is an order relation on $\mathbb{R}$, i.e., a binary relation, which is reflexive, transitive and antisymmetric;
- $\geq$ is total, i.e., $\forall x, y \in \mathbb{R}$ we have $x \geq y$ or $y \geq x$;
- $\geq$ is compatible with $+$, i.e., $\forall x, y, z \in \mathbb{R}$ we have $x + z \geq y + z$ whenever $x \geq y$;
- $\geq$ is compatible with $\cdot$, i.e., $\forall x, y \in \mathbb{R}$ s.t. $x \geq 0$ and $y \geq 0$, we have $xy \geq 0$.

As usual, we associate to $\geq$ the inverse order relation $\leq$ as well as the strict order relations $>$ and $<$, defined for any $x, y \in \mathbb{R}$ by

$$\begin{aligned} x \leq y &\quad\Leftrightarrow\quad y \geq x; \\ x > y &\quad\Leftrightarrow\quad x \geq y \text{ and } x \neq y; \\ x < y &\quad\Leftrightarrow\quad y > x. \end{aligned}$$

Proposition 1.1 *We have $x^2 \geq 0$ for all $x \in \mathbb{R}$. Consequently, $1 > 0$.*

Definition 1.2 *For any subset A of $\mathbb{R}$ we introduce the following (possibly empty!) sets*

$$lb(A) := \{x \in \mathbb{R} \mid x \leq a,\ \forall a \in A\};$$
$$ub(A) := \{x \in \mathbb{R} \mid x \geq a,\ \forall a \in A\}.$$

A number $x \in \mathbb{R}$ is said to be a

- lower bound *of A if $x \in lb(A)$;*
- upper bound *of A if $x \in ub(A)$;*
- least element *(or* minimum*) of A if $x \in A \cap lb(A)$;*
- greatest element *(or* maximum*) of A if $x \in A \cap ub(A)$.*

Remark 1.3 *Every set $A \subseteq \mathbb{R}$ has at most one least element and, if it exists, we denote it by* $\min A$. *Similarly, A has at most one greatest element and, if it exists, we denote it by* $\max A$.

Definition 1.4 *A subset A of $\mathbb{R}$ is said to be*

- *bounded (from) below, if A has lower bounds, i.e., $lb(A) \neq \emptyset$;*
- *bounded (from) above, if A has upper bounds, i.e., $ub(A) \neq \emptyset$;*
- *bounded, if A is both bounded above and below;*
- *unbounded, if A is not bounded.*

Remark 1.5 *The empty set is bounded. More precisely, we have*

$$lb(\emptyset) = ub(\emptyset) = \mathbb{R}.$$

Example 1.6 *(i) $A = \{a \in \mathbb{R} \mid a \geq 2\}$: unbounded (since it is not bounded above), bounded below by any $v \leq 2$, $\min A = 2$.*
(ii) $A = \{a \in \mathbb{R} \mid 0 < a < 1\}$: bounded (above by any $u \geq 1$, below by any $v \leq 0$), no minimum, no maximum.
(iii) $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\}$: bounded (above by any $u \geq 1$, below by any $v \leq 0$), $\max A = 1$, no minimum.
(iv) Every nonempty finite set has a minimum and a maximum.

Proposition 1.7 (Completeness Axiom) *The totally ordered field of real numbers $(\mathbb{R}, +, \cdot, \geq)$ is complete, meaning that every nonempty set $A \subseteq \mathbb{R}$ that is bounded above has a least upper bound, denoted by* $\sup A$ *and called the* supremum *of A. In other words, we have*

$$\sup A := \min(ub(A)).$$

Alternatively, every nonempty set $A \subseteq \mathbb{R}$ that is bounded below has a greatest lower bound, denoted by $\inf A$ *and called the* infimum *of A. In other words,*

$$\inf A := \max(lb(A)).$$

Example 1.8 *(i) $A = \{a \in \mathbb{Z} \mid 2 \leq a \leq 3\}$: $\max A = \sup A = 3$, $\min A = \inf A = 2$.*
(ii); $A = \{a \in \mathbb{R} \mid 0 < a \leq 1\}$: $\max A = \sup A = 1$, $\inf A = 0$, no minimum.

Remark 1.9 *The Completeness Axiom is also known in the literature as the* Supremum Property, *since it shows that every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$. Its counterpart shows that every nonempty subset of $\mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$. Indeed, let $A \subseteq \mathbb{R}$, $A \neq \emptyset$, bounded below. Then the set $-A = \{-a \mid a \in A\}$ is nonempty and bounded above, so, by the Supremum Property, it has a supremum in $\mathbb{R}$. Thus we have* $\inf A = -\sup(-A)$.

Remark 1.10 *Let $A \subseteq \mathbb{R}$ be a nonempty set. If A has a greatest element (resp. a least element), then $\sup A = \max A$ (resp. $\inf A = \min A$). Conversely, if A is bounded above and $\sup A \in A$ (resp. A is bounded below and $\inf A \in A$), then $\sup A = \max A$ (resp. $\inf A = \min A$).*

Definition 1.11 *We attach to $\mathbb{R}$ two elements $-\infty$ and $+\infty$ (or ∞) s.t.*

$$\forall x \in \mathbb{R}, \ -\infty < x \ \text{ and } \ x < +\infty.$$

The set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is called the extended real number system.
If a set $A \subseteq \mathbb{R}$ is not bounded above, we define $\sup A := +\infty$.
If a set $A \subseteq \mathbb{R}$ is not bounded below, we define $\inf A := -\infty$.
Also, we define $\sup \emptyset := -\infty$ and $\inf \emptyset := +\infty$ (see Remark 1.5!).

We denote by $\mathbb{N} := \{1, 2 := 1 + 1, 3 := 1 + 1 + 1, \dots\}$ the set of *natural numbers.*

Remark 1.12 *$\mathbb{N}$ is the smallest inductive subset of $\mathbb{R}$ w.r.t. inclusion (a set $A \subseteq \mathbb{R}$ is said to be inductive if $1 \in A$ and $x + 1 \in A$ whenever $x \in A$). We have $\min \mathbb{N} = 1$ and for every $n \in \mathbb{N}$, $n < n + 1$ and $\{x \in \mathbb{N} \mid n < x < n + 1\} = \emptyset$. Every nonempty subset of $\mathbb{N}$ has a least element.*

Proposition 1.13 (Principle of Mathematical Induction) *Let $n_0 \in \mathbb{N}$ and let $P(n)$ be a property defined for any number $n \in \mathbb{N}$, $n \geq n_0$. Suppose that the following two conditions hold:*

I. *$P(n_0)$ is true;*

II. *If $P(k)$ is true for some $k \in \mathbb{N}$, $k \geq n_0$, then $P(k+1)$ is also true.*

Then we have

III. *$P(n)$ is true, $\forall n \in \mathbb{N}$, $n \geq n_0$.*

The following result is a consequence of the Completeness Axiom (Supremum Property).

Corollary 1.14 (Archimedean Property) *The set of natural numbers $\mathbb{N}$ is not bounded from above. In other words, for every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ s.t. $n > x$.*

Proof. Suppose $x \geq n$, $\forall n \in \mathbb{N}$. Then $\mathbb{N}$ is nonempty and bounded above by x, so, by Theorem 1.7, it has a supremum $u \in \mathbb{R}$. Since $u - 1 < u$, $u - 1$ cannot be an upper bound of $\mathbb{N}$. This means that $\exists m \in \mathbb{N}$ s.t. $u - 1 < m$. Thus, $u < m + 1 \in \mathbb{N}$, which is a contradiction to the fact that u is an upper bound of $\mathbb{N}$. $\square$

The sets of *integer numbers* and *rational numbers* are defined as

$$\begin{aligned} \mathbb{Z} &:= \{m - n \mid m, n \in \mathbb{N}\}; \\ \mathbb{Q} &:= \{mn^{-1} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}. \end{aligned}$$

Remarks 1.15 **1.** *For every $x \in \mathbb{R}$ there is a unique $k \in \mathbb{Z}$ such that $k \leq x < k+1$; we denote this k by $[x]$ or $\lfloor x \rfloor$ and call it the* integer part *or* floor *of x.*

2. *For every $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $x \geq 0$, there exists a unique number $y \in \mathbb{R}$, $y \geq 0$ such that $x = y^n$ (when $n \geq 2$ we denote $y = \sqrt[n]{x}$).*

3. *We have $\sqrt{2} \notin \mathbb{Q}$. Therefore the set $\mathbb{R} \setminus \mathbb{Q}$ of the so-called irrational numbers is nonempty.*

As a consequence of the Archimedean Property we obtain the following result:

Corollary 1.16 (Density of $\mathbb{Q}$ in $\mathbb{R}$) *For any real numbers $a, b \in \mathbb{R}$ such that $a < b$ there exists $x \in \mathbb{Q}$ such that $a < x < b$.*

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$. By the Archimedean Property (Corollary 1.14) we can find a number $n \in \mathbb{N}$ s.t. $n > \frac{1}{b-a}$, i.e.,

$$nb - 1 > na \tag{1.1}$$

Case 1: If $nb \in \mathbb{Z}$ then (1.1) shows that $a < \frac{nb-1}{n} < b$, hence $x := \frac{nb-1}{n} \in \mathbb{Q}$ satisfies the property in demand.

Case 2: If $nb \notin \mathbb{Z}$ then we consider the integer part of nb, namely $m := [nb]$. In this case we have

$$m < nb < m + 1. \tag{1.2}$$

By (1.1) and (1.2) we deduce that $m > nb - 1 > na$ hence $na < m < nb$. Thus, in this case the number $x := \frac{m}{n} \in \mathbb{Q}$ satisfies $a < x < b$. $\square$

Remark 1.17 *$(\mathbb{Q}, +, \cdot, \geq)$ is a totally ordered field but, in contrast to $(\mathbb{R}, +, \cdot, \geq)$, it does not satisfy the Completeness Axiom. However, for every $x \in \mathbb{R}$ we have*

$$\begin{aligned} \sup\{y \in \mathbb{Q} \mid y < x\} = \quad x \quad &= \inf\{y \in \mathbb{Q} \mid y > x\}; \\ \sup\{z \in \mathbb{R} \setminus \mathbb{Q} \mid z < x\} = \quad x \quad &= \inf\{z \in \mathbb{R} \setminus \mathbb{Q} \mid z > x\}. \end{aligned}$$

Next we present some properties which are of practical interest.

Proposition 1.18 *If $A \subseteq B \subseteq \mathbb{R}$ are nonempty bounded sets, then*

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Proposition 1.19 *If A and B are nonempty subsets of $\mathbb{R}$ which are bounded above, then $A \cup B$ is bounded above and the following relations hold:*

$$\begin{aligned} \sup(A \cup B) &= \max\{\sup A, \sup B\}; \\ \inf(A \cup B) &= \min\{\inf A, \inf B\}; \end{aligned}$$

Proposition 1.20 *For any nonempty subsets A and B of $\mathbb{R}$, we have*

$$\begin{aligned} \sup(A + B) &= \sup A + \sup B, \\ \inf(A + B) &= \inf A + \inf B, \end{aligned}$$

where $A + B := \{a + b \mid a \in A, b \in B\}$.

If $f : D \to \mathbb{R}$ is a function, defined on a nonempty set D, then it will be convenient to denote

$$\inf_{x \in D} f(x) := \inf f(D) \quad \text{and} \quad \sup_{x \in D} f(x) := \sup f(D),$$

where $f(D) = \operatorname{Im}(f) := \{f(x) \mid x \in D\}$ represents the function's image.

In particular, if $D = \mathbb{N}$, a function $f : \mathbb{N} \to \mathbb{R}$ represents a sequence $(x_n)_{n \in \mathbb{N}}$. In this case we will write

$$\inf_{n \in N} x_n := \inf\{x_n \mid n \in \mathbb{N}\} \quad \text{and} \quad \sup_{n \in N} x_n := \sup\{x_n \mid n \in \mathbb{N}\}.$$

The following result is another important consequence of the Completeness Axiom (Supremum Property).

Corollary 1.21 (Nested Interval Property) *Consider a sequence of closed intervals $I_n = [a_n, b_n] \subseteq \mathbb{R}$, with $a_n < b_n$ for all $n \in \mathbb{N}$. If $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$, i.e.,*

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots \textit{ is a nested sequence of closed intervals,}$$

then we have $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ (i.e., $\exists x \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, x \in I_n$).

Proof. Let $A = \{a_k \mid k \in \mathbb{N}\}$. Then, $\forall n \in \mathbb{N}$, b_n is an upper bound of A. Hence A is nonempty and bounded above. By the Completeness Axiom (Proposition 1.7), we deduce that A has a supremum in $\mathbb{R}$. Thus, $\forall n \in \mathbb{N}$, $a_n \leq \sup A \leq b_n$. This shows that $\sup A \in \bigcap_{n=1}^{\infty} I_n$. $\square$

Definition 1.22 *A set $V \subseteq \mathbb{R}$ is said to be*

- *a neighborhood of a number $x \in \mathbb{R}$, if there exists a real number $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq V$;*
- *a neighborhood of $-\infty$, if there exists a number $a \in \mathbb{R}$ such that $(-\infty, a) \subseteq V$;*
- *a neighborhood of $+\infty$, if there exists a number $a \in \mathbb{R}$ such that $(a, +\infty) \subseteq V$.*

Proposition 1.23 *Let $x \in \overline{\mathbb{R}}$. Then*

(*i*) *if $x \in \mathbb{R}$ and $V \in \mathcal{V}(x)$, then $x \in V$.*
(*ii*) *if $V \in \mathcal{V}(x)$ and $U \subseteq \mathbb{R}$ s.t. $V \subseteq U$, then $U \in \mathcal{V}(x)$.*
(*iii*) *if $U, V \in \mathcal{V}(x)$, then $U \cap V \in \mathcal{V}(x)$.*

Theorem 1.24 *Let $A \subseteq \mathbb{R}$ be a nonempty set, which is bounded from below by $\alpha \in \mathbb{R}$. Then the following assertions are equivalent:*

1° $\inf A = \alpha$.
2° *For every real number $\beta > \alpha$ there exists $x \in A$ such that $x < \beta$.*
3° *For every real number $\varepsilon > 0$ we have $A \cap [\alpha, \alpha + \varepsilon) \neq \emptyset$.*
4° *For every $V \in \mathcal{V}(\alpha)$ we have $V \cap A \neq \emptyset$.*

Corollary 1.25 *Let $A \subseteq \mathbb{R}$ be a nonempty set, which is bounded from above by $\alpha \in \mathbb{R}$. Then the following assertions are equivalent:*

1° $\sup A = \alpha$.

2° *For every real number $\beta < \alpha$ there exists $x \in A$ such that $x > \beta$.*

3° *For every real number $\varepsilon > 0$ we have $A \cap (\alpha - \varepsilon, \alpha] \neq \emptyset$.*

4° *For every $V \in \mathcal{V}(\alpha)$ we have $V \cap A \neq \emptyset$.*