COURSE 3.

Newton interpolation polynomial

Newton interpolation polynomial is given by

$$(L_m f)(x) := (N_m f)(x) = f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})(D^i f)(x_0) \tag{1}$$

$$= f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})[x_0, ..., x_i; f],$$

where $(D^i f)(x_0)$(or denoted $[x_0, ..., x_i; f]$) is the i-th order divided difference of the function f at x_0.

Newton interpolation formula is

$$f = N_m f + R_m f,$$

where the remainder (the error) is given by

$$(R_m f)(x) = (x - x_0)...(x - x_m)[x, x_0, ..., x_m; f]. \tag{2}$$

Remark 1 *The remainder for Lagrange interpolation formula is also given by*

$$(R_m f)(x) = \frac{(x - x_0)...(x - x_m)}{(m+1)!} f^{(m+1)}(\xi),$$

with ξ *between* $x, x_0, ..., x_m$, *so, by (2), it follows that* **the divided differences are approximations of the derivatives**

$$[x, x_0, ..., x_m; f] = \frac{f^{(m+1)}(\xi)}{(m+1)!}.$$

Remark 2 *We notice that*

$$(N_i f)(x) = (N_{i-1} f)(x) + (x - x_0)...(x - x_{i-1})[x_0, ..., x_i; f]$$

so the Newton polynomials of degree $2, 3, ...,$ *can be iteratively generated, similarly to Aitken's algorithm.*

Example 3 *Find* $L_2 f$ *for* $f(x) = \sin \pi x$, *and* $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{1}{2}$, *in both forms.*

Sol. *a) We have* $u(x) = x(x-\frac{1}{6})(x-\frac{1}{2})$; $u_0(x) = (x-\frac{1}{6})(x-\frac{1}{2})$; $u_1(x) = x(x-\frac{1}{2})$; $u_2(x) = x(x-\frac{1}{6})$

$$\begin{aligned}(L_2f)(x) &= \sum_{i=0}^{2} l_i(x)f(x_i) = \sum_{i=0}^{2} \frac{u_i(x)}{u_i(x_i)} f(x_i)\\ &= \frac{(x-\frac{1}{6})(x-\frac{1}{2})}{(-\frac{1}{6})(-\frac{1}{2})}0 + \frac{x(x-\frac{1}{2})}{\frac{1}{6}(-\frac{1}{3})}\frac{1}{2} + \frac{x(x-\frac{1}{6})}{\frac{1}{2}\cdot\frac{1}{3}}1\\ &= -3x^2 + \frac{7}{2}x.\end{aligned}$$

b)

$$\begin{aligned}(N_2f)(x) &= f(0) + \sum_{i=1}^{2} (x-x_0)...(x-x_{i-1})(D^if)(x_0)\\ &= f(0) + (x-x_0)(Df)(x_0) + (x-x_0)(x-x_1)(D^2f)(x_0)\\ &= x(Df)(x_0) + x(x-\frac{1}{6})(D^2f)(x_0)\end{aligned}$$

The table of divided differences:

x	f	Df	D^2f
0	0	3	-3
$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	
$\frac{1}{2}$	1		

so

$$(N_2f)(x) = 3x - 3x(x - \frac{1}{6}) = -3x^2 + \frac{7}{2}x.$$

2.3. Hermite interpolation

Example 4 *In the following table there are some data regarding a moving car. We may estimate the position (and the speed) of the car when the time is $t = 10$ using Hermite interpolation.*

Time	0	3	5	8	13
Distance	0	225	383	623	993
Speed	75	77	80	74	72

Let $x_k \in [a, b],\ k = 0, 1, ..., m$ be such that $x_i \neq x_j$, for $i \neq j$ and let $r_k \in \mathbb{N},\ k = 0, 1, ..., m$. Consider $f : [a, b] \to \mathbb{R}$ such that there exist $f^{(j)}(x_k),\ k = 0, 1, ..., m;\ j = 0, 1, ..., r_k$ and $n = m + r_0 + ... + r_m$.

The Hermite interpolation problem (HIP) consists in determining the polynomial P of the smallest degree for which

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, ..., m;\ j = 0, ..., r_k.$$

Definition 5 *A solution of (HIP) is called* **Hermite interpolation polynomial**, *denoted by $H_n f$.*

Hermite interpolation polynomial, $H_n f$, satisfies the interpolation conditions:

$$(H_n f)^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0, ..., m; \; j = 0, ..., r_k.$$

Hermite interpolation polynomial is given by

$$(H_n f)(x) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k) \in \mathbb{P}_n, \tag{3}$$

where $h_{kj}(x)$ denote **the Hermite fundamental interpolation polynomials.** They fulfill the relations:

$$h_{kj}^{(p)}(x_\nu) = 0, \; \nu \neq k, \quad p = 0, 1, ..., r_\nu$$
$$h_{kj}^{(p)}(x_k) = \delta_{jp}, \; p = 0, 1, ..., r_k, \quad \text{for } j = 0, 1, ..., r_k \text{ and } \nu, k = 0, 1, ..., m,$$

with $\delta_{jp} = \begin{cases} 1, & j = p \\ 0, & j \neq p. \end{cases}$

We denote by

$$u(x) = \prod_{k=0}^{m} (x - x_k)^{r_k+1} \qquad \text{and } u_k(x) = \frac{u(x)}{(x - x_k)^{r_k+1}}.$$

We have

$$h_{kj}(x) = \frac{(x - x_k)^j}{j!} u_k(x) \sum_{\upsilon=0}^{r_k-j} \frac{(x - x_k)^\nu}{\nu!} \left[\frac{1}{u_k(x)}\right]^{(\nu)}_{x=x_k}. \tag{4}$$

Example 6 *Find the Hermite interpolation polynomial for a function* f *for which we know* $f(0) = 1, f'(0) = 2$ *and* $f(1) = -3$ *(equivalent with* $x_0 = 0$ *multiple node of order 2 or double node,* $x_1 = 1$ *simple node).*

Sol. We have $x_0 = 0, x_1 = 1, m = 1, r_0 = 1, r_1 = 0, n = m + r_0 + r_1 = 2$

$$\begin{aligned}(H_2 f)(x) &= \sum_{k=0}^{1} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k) \\ &= h_{00}(x) f(0) + h_{01}(x) f'(0) + h_{10}(x) f(1).\end{aligned}$$

We have h_{00}, h_{01}, h_{10}. These fulfills relations:

$h_{kj}^{(p)}(x_\nu) = 0,\ \nu \neq k, \quad p = 0, 1, ..., r_\nu$

$h_{kj}^{(p)}(x_k) = \delta_{jp},\ p = 0, 1, ..., r_k, \quad$ for $j = 0, 1, ..., r_k$ and $\nu, k = 0, 1, ..., m.$

We have $h_{00}(x) = a_1 x^2 + b_1 x + c_1 \in \mathbb{P}_2$, with $a_1, b_1, c_1 \in \mathbb{R}$, and the system

$$\begin{cases} h_{00}(x_0) = 1 \\ h'_{00}(x_0) = 0 \\ h_{00}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} h_{00}(0) = 1 \\ h'_{00}(0) = 0 \\ h_{00}(1) = 0 \end{cases}$$

that becomes

$$\begin{cases} c_1 = 1 \\ b_1 = 0 \\ a_1 + b_1 + c_1 = 0. \end{cases}$$

Solution is: $a_1 = -1, b_1 = 0, c_1 = 1$ so $h_{00}(x) = -x^2 + 1$.

We have $h_{01}(x) = a_2 x^2 + b_2 x + c_2 \in \mathbb{P}_2$, with $a_2, b_2, c_2 \in \mathbb{R}$. The system is

$$\begin{cases} h_{01}(x_0) = 0 \\ h'_{01}(x_0) = 1 \\ h_{01}(x_1) = 0 \end{cases} \Leftrightarrow \begin{cases} h_{01}(0) = 0 \\ h'_{01}(0) = 1 \\ h_{01}(1) = 0 \end{cases}$$

and we get $h_{01}(x) = -x^2 + x$.

We have $h_{10}(x) = a_3 x^2 + b_3 x + c_3 \in \mathbb{P}_2$, with $a_3, b_3, c_3 \in \mathbb{R}$. The system is

$$\begin{cases} h_{10}(x_0) = 0 \\ h'_{10}(x_0) = 0 \\ h_{10}(x_1) = 1 \end{cases} \Leftrightarrow \begin{cases} h_{10}(0) = 0 \\ h'_{10}(0) = 0 \\ h_{10}(1) = 1 \end{cases}$$

and we get $h_{10}(x) = x^2$.

The Hermite polynomial is

$$(H_2 f)(x) = -x^2 + 1 - 2x^2 + 2x - 3x^2 = -6x^2 + 2x + 1.$$

The Hermite interpolation formula is

$$f = H_n f + R_n f,$$

where $R_n f$ denotes the remainder term (the error).

Theorem 7 *If* $f \in C^n[\alpha, \beta]$ *and* $f^{(n)}$ *is derivable on* (α, β)*, with* $\alpha = \min\{x, x_0, ..., x_m\}$ *and* $\beta = \max\{x, x_0, ..., x_m\}$*, then there exists* $\xi \in (\alpha, \beta)$ *such that*

$$(R_n f)(x) = \frac{u(x)}{(n+1)!} f^{(n+1)}(\xi). \tag{5}$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_n f)(z) \\ u(x) & (R_n f)(x) \end{vmatrix}.$$

$F \in C^n[\alpha, \beta]$ and there exists $F^{(n+1)}$ on (α, β).

We have

$$F(x) = 0, \quad F^{(j)}(x_k) = 0, \qquad k = 0, ..., m; \;\; j = 0, ..., r_k;$$

because

$$u(x) = \prod_{k=0}^{m} (x - x_k)^{r_k+1} \Rightarrow u^{(j)}(x_k) = 0,\ \ j = 0, ..., r_k$$

and

$$(R_m f)^{(j)}(x_k) = f^{(j)}(x_k) - (H_n f)^{(j)}(x_k) = f^{(j)}(x_k) - f^{(j)}(x_k) = 0.$$

So, F and its derivatives have $n+2$ distinct zeros in (α, β). Applying successively Rolle's theorem it follows that F' has at least $n+1$ zeros in $(\alpha, \beta) \Rightarrow ... \Rightarrow F^{(n+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(n+1)}(\xi) = 0$.

We have

$$F^{(n+1)}(z) = \begin{vmatrix} u^{(n+1)}(z) & (R_n f)^{(n+1)}(z) \\ u(x) & (R_n f)(x) \end{vmatrix},$$

with $u(z) = \prod\limits_{k=0}^{m} (z - z_k)^{r_k+1} \in \mathbb{P}_{n+1} \Rightarrow u^{(n+1)}(z) = (n+1)!$, and $(R_n f)^{(n+1)}(z) = f^{(n+1)}(z) - (H_n f)^{(n+1)}(z) = f^{(n+1)}(z)$ (as, $H_n f \in$

$\mathbb{P}_n$). We get

$$F^{(n+1)}(\xi) = \begin{vmatrix} (n+1)! & f^{(n+1)}(\xi) \\ u(x) & (R_n f)(x) \end{vmatrix} = 0,$$

whence it follows (5). ■

Corolar 8 *If $f \in C^{n+1}[a,b]$ then*

$$|(R_n f)(x)| \leq \frac{|u(x)|}{(n+1)!} \left\| f^{(n+1)} \right\|_\infty, \qquad x \in [a,b]$$

where $\|\cdot\|_\infty$ denotes the uniform norm ($\|f\|_\infty = \max\limits_{x\in[a,b]} |f(x)|$).

Remark 9 *In case of $m = 0$, i.e., $n = r_0$, (HIP) becomes* **Taylor interpolation problem**. *Taylor interpolation polynomial is*

$$(T_n f)(x) = \sum_{j=0}^{n} \frac{(x-x_0)^j}{j!} f^{(j)}(x_0).$$