LECTURE

8

THE EUCLIDEAN SPACE $\mathbb{R}^n$. SEQUENCES OF POINTS IN $\mathbb{R}^n$

The Euclidean space $\mathbb{R}^n$

Let $n \in \mathbb{N}^*$. Consider the vector space

$$\mathbb{R}^n = \{(x_1, x_2, \dots x_n) \mid x_1, x_2, \dots, x_n \in \mathbb{R}\}$$

of all ordered n-tuples of real numbers endowed with the vector addition and the multiplication of vectors by scalars (real numbers)

$$\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \longmapsto x + y \in \mathbb{R}^n$$
$$\forall (\alpha, x) \in \mathbb{R} \times \mathbb{R}^n \longmapsto \alpha x \in \mathbb{R}^n$$

defined componentwise:

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

and

$$\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

for any $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

The zero vector (origin) of this vector space is the point

$$0_n = (0, 0, \dots, 0)$$

and the additive inverse of a point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is the point denoted by

$$-x = (-x_1, -x_2, \dots, -x_n) = (-1)x.$$

We also consider the canonical unit vectors:

$$\begin{aligned} e^1 &= (1, 0, 0, \dots, 0) \in \mathbb{R}^n \\ e^2 &= (0, 1, 0, \dots, 0) \in \mathbb{R}^n \\ &\vdots \\ e^n &= (0, 0, 0, \dots, 1) \in \mathbb{R}^n. \end{aligned}$$

The set $\{e^1, e^2, \dots e^n\}$ is a basis of the vector space $\mathbb{R}^n$ called the *standard (canonical) basis* of $\mathbb{R}^n$. If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then

$$x = x_1 e^1 + x_2 e^2 + \dots + x_n e^n.$$

Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. The real number defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

is called the *scalar product of x and y*.

The nonnegative number

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{(x_1)^2 + (x_2)^2 + \dots + (x_n)^2}$$

is called the *Euclidean norm of x*.

The *Euclidean distance between x and y* is given by

$$\mathrm{dist}(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}.$$

Remark 8.1 *For $x \in \mathbb{R}^n$, $\|x\|$ represents the Euclidean distance between x and 0_n.*

Example 8.2 *(i) When $n = 1$ every vector $x \in \mathbb{R}$ can be identified with exactly one point on the real line. If $x, y \in \mathbb{R}$, then $\langle x, y \rangle = xy$, $\|x\| = |x|$ and $\|x - y\| = |x - y|$.*

(ii) When $n = 2$ every vector $(x, y) \in \mathbb{R}^2$ can be identified with exactly one point in a plane Cartesian coordinate system Oxy. If $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2) \in \mathbb{R}^2$, then, by the Pythagoras' Theorem, the length of the segment $[P_1P_2]$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, which is exactly the Euclidean distance between the vectors (x_1, y_1) and (x_2, y_2).

(iii) When $n = 3$ every vector $(x, y, z) \in \mathbb{R}^3$ can be identified with exactly one point in a Cartesian coordinate system $Oxyz$. Let $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$. Take $P_3 = (x_2, y_2, z_1)$. Note P_2 and P_3 are on the same vertical line, so the length of the segment $[P_2P_3]$ is $|z_1 - z_2|$. Also, P_1 and P_3 are on the same horizontal plane, so the length of the segment $[P_1P_3]$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Since the points P_1, P_2 and P_3 form a right triangle with right angle at P_3, by the Pythagorean Theorem, we have that the length of the segment $[P_1P_2]$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$, which is exactly the Euclidean distance between the vectors (x_1, y_1, z_1) and (x_2, y_2, z_2).

Proposition 8.3 (Properties of the scalar product in $\mathbb{R}^n$)

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \quad \forall x, y, z \in \mathbb{R}^n$.

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle, \quad \forall \alpha \in \mathbb{R}, \forall x, y \in \mathbb{R}^n$.

(iii) $\langle x, y \rangle = \langle y, x \rangle, \quad \forall x, y \in \mathbb{R}^n$.

(iv) $\langle x, x \rangle > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0_n\}$.

(v) $\langle 0_n, x \rangle = 0, \quad \forall x \in \mathbb{R}^n$.

(vi) $\langle x, x \rangle = 0 \Leftrightarrow x = 0_n$.

Proposition 8.4 (Cauchy-Buniakowski-Schwarz inequality)

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|, \quad \forall x, y \in \mathbb{R}^n.$$

Proof. Let $x, y \in \mathbb{R}^n$.

Case 1: If $y = 0_n$, then the desired inequality holds with equality.

Case 2: If $y \neq 0_n$, then let $\alpha := -\dfrac{\langle x, y\rangle}{\langle y, y\rangle}$. Then

$$\begin{aligned}
0 &\le \langle x + \alpha y, x + \alpha y\rangle \\
&= \langle x, x + \alpha y\rangle + \langle \alpha y, x + \alpha y\rangle \\
&= \langle x + \alpha y, x\rangle + \alpha\langle x + \alpha y, y\rangle \\
&= \langle x, x\rangle + \alpha\langle y, x\rangle + \alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle \\
&= \langle x, x\rangle + 2\alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle \\
&= \langle x, x\rangle - 2\frac{(\langle x, y\rangle)^2}{\langle y, y\rangle} + \frac{(\langle x, y\rangle)^2}{\langle y, y\rangle} \\
&= \langle x, x\rangle - \frac{(\langle x, y\rangle)^2}{\langle y, y\rangle}.
\end{aligned}$$

Thus, $0 \le \langle x, x\rangle\langle y, y\rangle - (\langle x, y\rangle)^2$, so $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$. □

Proposition 8.5 (Properties of the Euclidean norm)
(i) $\|x\| = 0 \Leftrightarrow x = 0_n$.
(ii) $\|\alpha x\| = |\alpha| \cdot \|x\|, \quad \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n$.
(iii) $\|x + y\| \le \|x\| + \|y\|, \quad \forall x, y \in \mathbb{R}^n$ *(the triangle inequality).*

Proof. (i) and (ii) are immediate consequences of Proposition 8.3. Statement (iii) follows from

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \\
&\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{by the Cauchy-Buniakowski-Schwarz inequality} \\
&= (\|x\| + \|y\|)^2.
\end{aligned}$$

□

Definition 8.6 *Let $x_0 \in \mathbb{R}^n$ and $r > 0$. The set*

$$B(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$$

is called open ball of radius r centered at x_0 while the set

$$\overline{B}(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \le r\}$$

is called closed ball of radius r centered at x_0.

Example 8.7 *(i) $n = 1$: let $x_0 \in \mathbb{R}$ and $r > 0$. Then $B(x_0, r) = (x_0 - r, x_0 + r)$ and $\overline{B}(x_0, r) = [x_0 - r, x_0 + r]$.*
(ii) $n = 2$: let $(x_0, y_0) \in \mathbb{R}^2$ and $r > 0$. Then $B((x_0, y_0), r)$ is the open disc of radius r centered at (x_0, y_0) (excluding its enclosing circle) and $\overline{B}((x_0, y_0), r)$ is the closed disc of radius r centered at (x_0, y_0) (including its enclosing circle).
(iii) $n = 3$: let $(x_0, y_0, z_0) \in \mathbb{R}^3$ and $r > 0$. Then $B((x_0, y_0), r)$ consists of all points inside the sphere of radius r centered at (x_0, y_0) excluding the sphere itself and $\overline{B}((x_0, y_0), r)$ consists of all points inside the sphere of radius r centered at (x_0, y_0) including the sphere itself.

Remark 8.8 *Let $x_0 \in \mathbb{R}^n$, $r_1 > r > 0$. Then*
(i) $x_0 \in B(x_0, r)$.
(ii) $B(x_0, r) \subseteq \overline{B}(x_0, r) \subseteq B(x_0, r_1) \subseteq \overline{B}(x_0, r_1)$.
(iii) $\forall x \in B(x_0, r)$, $B(x, r - \|x_0 - x\|) \subseteq B(x_0, r)$.

Definition 8.9 *By a neighborhood of $x \in \mathbb{R}^n$ we mean any set $V \subseteq \mathbb{R}^n$ such that*

$$\exists\, r > 0 \text{ such that } B(x,r) \subseteq V.$$

We denote by $\mathcal{V}(x)$ the family of all neighborhoods of x.

Sequences in $\mathbb{R}^n$

Notation: $(x^k)_{k\geq 1}$, $(x^k)_{k\in\mathbb{N}^*}$, or (x^k) (we do not index this sequence by n since n is the dimension of $\mathbb{R}^n$; we use an upper index notation since lower indices are used for vector coordinates). Written explicitly,

$$\begin{aligned} x^1 &= (x_1^1, x_2^1, \ldots, x_n^1) \in \mathbb{R}^n \\ x^2 &= (x_1^2, x_2^2, \ldots, x_n^2) \in \mathbb{R}^n \\ &\vdots \\ x^k &= (x_1^k, x_2^k, \ldots, x_n^k) \in \mathbb{R}^n \\ &\vdots \end{aligned}$$

The sequences of real numbers

$$(x_1^k)_{k\in\mathbb{N}^*}, \quad (x_2^k)_{k\in\mathbb{N}^*}, \quad \ldots, \quad (x_n^k)_{k\in\mathbb{N}^*}$$

are called the *component sequences* of the sequence (x^k).

Definition 8.10 *We say that a sequence (x^k) in $\mathbb{R}^n$ converges to $x \in \mathbb{R}^n$ if*

$$\forall \varepsilon > 0, \exists k_\varepsilon \in \mathbb{N} \text{ such that } \forall k \in \mathbb{N}, k \geq k_\varepsilon, \text{ we have } \|x^k - x\| < \varepsilon.$$

Proposition 8.11 *A sequence in $\mathbb{R}^n$ cannot converge to two distinct vectors in $\mathbb{R}^n$.*

Proof. Let (x^k) be a sequence in $\mathbb{R}^n$ and let $x, x' \in \mathbb{R}^n$ such that (x^k) converges both to x and x'. Suppose by the contrary that $x \neq x'$. Then, choose $\varepsilon := \dfrac{\|x - x'\|}{2} > 0$. Because (x^k) converges to x, it follows that

$$\exists k_\varepsilon \in \mathbb{N} \text{ s.t. } \forall k \geq k_\varepsilon, \|x^k - x\| < \varepsilon.$$

Likewise, since (x^k) converges to x', we have that

$$\exists k'_\varepsilon \in \mathbb{N} \text{ s.t. } \forall k \geq k'_\varepsilon, \|x^k - x'\| < \varepsilon.$$

Thus, $\forall k \geq \max\{k_\varepsilon, k'_\varepsilon\}$, $\|x - x'\| \leq \|x - x^k\| + \|x^k - x'\| < \varepsilon + \varepsilon = 2\varepsilon = \|x - x'\|$, a contradiction. $\square$

Definition 8.12 *If a sequence (x^k) in $\mathbb{R}^n$ converges to some $x \in \mathbb{R}^n$, we say that (x^k) is convergent. The vector x is called the limit of (x^k) and we write*

$$\lim_{k\to\infty} x^k = x \quad \text{or} \quad x^k \to x.$$

If (x^k) does not converge to any vector in $\mathbb{R}^n$, we say that (x^k) is divergent.

Theorem 8.13 *Let $(x^k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Then*

$$\lim_{k\to\infty} x^k = x \iff \forall V \in \mathcal{V}(x), \exists k_V \in \mathbb{N} \text{ such that } \forall k \in \mathbb{N}, k \geq k_V \text{ we have } x^k \in V.$$

The following result gives a characterization of the limit of a sequence in $\mathbb{R}^n$ in terms of the limits of the component sequences.

Theorem 8.14 *Let $(x^k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ with $x^k = (x_1^k, x_2^k, \ldots, x_n^k)$ for all $k \in \mathbb{N}$ and let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then*

$$\lim_{k\to\infty} x^k = x \iff \forall j \in \{1, 2, \ldots, n\}, \lim_{k\to\infty} x_j^k = x_j.$$

Proof. Suppose first that $\lim\limits_{k\to\infty} x^k = x$. Let $j \in \{1, 2, \ldots, n\}$ and $\varepsilon > 0$. Then

$$\exists k_\varepsilon \in \mathbb{N}^*, \forall k \geq k_\varepsilon, \|x^k - x\| < \varepsilon,$$

so $\forall k \geq k_\varepsilon$,

$$\left|x_j^k - x_j\right| \leq \sqrt{\left(x_1^k - x_1\right)^2 + \left(x_2^k - x_2\right)^2 + \ldots + (x_n^k - x_n)^2} = \|x^k - x\| < \varepsilon.$$

Thus, $\lim\limits_{k\to\infty} x_j^k = x_j$.

Assume now that $\forall j \in \{1, 2, \ldots, n\}$, $\lim\limits_{k\to\infty} x_j^k = x_j$. Let $\varepsilon > 0$. Then

$$\forall j \in \{1, 2, \ldots, n\}, \exists k_{\varepsilon,j} \in \mathbb{N}^*, \forall k \geq k_{\varepsilon,j}, \left|x_j^k - x_j\right| < \frac{\varepsilon}{\sqrt{n}}.$$

Take $k_\varepsilon = \max\{k_{\varepsilon,1},\ k_{\varepsilon,2}, \ldots, k_{\varepsilon,n})\}$. Then $\forall k \geq k_\varepsilon$,

$$\|x^k - x\| = \sqrt{\left(x_1^k - x_1\right)^2 + \left(x_2^k - x_2\right)^2 + \ldots + (x_n^k - x_n)^2} \leq \sqrt{\frac{\varepsilon^2}{n} + \frac{\varepsilon^2}{n} + \ldots + \frac{\varepsilon^2}{n}} = \varepsilon.$$

$\square$

Open and closed sets; interior, closure and boundary of a set

Definition 8.15 *Let $A \subseteq \mathbb{R}^n$. A point $a \in A$ is called an interior point of A if there exists $r > 0$ such that $B(a, r) \subseteq A$. The set of all interior points of A is called the interior of A and is denoted by $intA$.*

Definition 8.16 *Let $A \subseteq \mathbb{R}^n$. The set $A \subseteq \mathbb{R}^n$ is called*
- *open: if every $a \in A$ is an interior point of A.*
- *closed: if $\mathbb{R}^n \setminus A$ is open.*

Theorem 8.17 *A set $A \subseteq \mathbb{R}^n$ is closed if and only if for every sequence (x^k) in A which converges to some $c \in \mathbb{R}^n$, we have that $c \in A$.*

Remark 8.18 *(i) A set in $\mathbb{R}^n$ may be neither open, nor closed.*
(ii) $\mathbb{R}^n$ and $\emptyset$ are both open and closed (in fact these are the only sets that are both open and closed).

Definition 8.19 *Let $A \subseteq \mathbb{R}^n$. A point $c \in \mathbb{R}^n$ is called an adherent point of A if for every $r > 0$, $B(c, r) \cap A \neq \emptyset$. The set of all adherent points of A is called the closure of A and is denoted by clA.*

Remark 8.20 *For any set $A \subseteq \mathbb{R}^n$ we have*

$$clA = \left\{c \in \mathbb{R}^n \mid \exists (x^k) \text{ a sequence in } A \text{ which converges to } c\right\}.$$

Remark 8.21 (Interior vs. closure) *Let $A \subseteq \mathbb{R}^n$. The following hold:*

1° $intA \subseteq A$.
2° $intA = A$ *if and only if* A *is open.*
3° $A \subseteq clA$.
4° $A = clA$ *if and only if* A *is closed.*
5° $intA$ *is the largest open set contained in* A.
6° clA *is the smallest closed set containing* A.
7° $int(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus clA$.
8° $cl(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus intA$.

Example 8.22 *(i)* $n = 1$*:*

- *Let* $A = [0, 1)$*. Then* $intA = (0, 1)$*,* $clA = [0, 1]$*,* A *is neither closed nor open.*
- *Let* $A = \mathbb{R}^*$*. Then* $intA = A$*,* A *is open,* $clA = \mathbb{R}$.
- *Let* $A = \mathbb{N}$*. Then* $intA = \emptyset$*,* $clA = A$*,* A *is closed.*

(ii) $n = 2$*:*

- *Let* $A = [0, 1] \times [0, 2] \setminus \{0_2\}$*. Then* $intA = (0, 1) \times (0, 2)$*,* $clA = [0, 1] \times [0, 2]$*,* A *is neither closed nor open.*
- *Let* $A = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y \neq 0\}$*. Then* $intA = A$*,* A *is open, and* $clA = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$.

(iii) Any open ball in $\mathbb{R}^n$ *is an open set.*
(iv) Any closed ball in $\mathbb{R}^n$ *is a closed set. Indeed, let* $x \in \mathbb{R}^n$*,* $r > 0$*. We show that* $\mathbb{R}^n \setminus \overline{B}(x, r)$ *is open. Let* $y \in \mathbb{R}^n \setminus \overline{B}(x, r)$*. Then* $r_y = \|x - y\| - r > 0$*. For any* $z \in B(y, r_y)$*,* $\|z - y\| < r_y = \|x - y\| - r \leq \|x - z\| + \|z - y\| - r$*. Thus,* $\|x - z\| > r$*, so* $z \in \mathbb{R}^n \setminus \overline{B}(x, r)$*. Hence,* $B(y, r_y) \subseteq \mathbb{R}^n \setminus \overline{B}(x, r)$*. It follows that* $\mathbb{R}^n \setminus \overline{B}(x, r)$ *is open, which means that* $\overline{B}(x, r)$ *is closed.*

Definition 8.23 *Let* $A \subseteq \mathbb{R}^n$*. A point* $c \in \mathbb{R}^n$ *is called a boundary point of* A *if for every* $r > 0$*,* $B(c, r) \cap A \neq \emptyset$ *and* $B(c, r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$*. The set of all boundary points of* A *is called the boundary of* A *and is denoted by* bdA *(sometimes we write* $bd(A)$*).*

Remark 8.24 *Let* $A \subseteq \mathbb{R}^n$.
(i) $bdA = clA \setminus intA = clA \cap cl(\mathbb{R}^n \setminus A)$.
(ii) $bdA = bd(\mathbb{R}^n \setminus A)$.
(iii) bdA *is closed.*

Definition 8.25 *Let* $A \subseteq \mathbb{R}^n$*. A point an* $c \in \mathbb{R}^n$ *is called an accumulation point of* A *if for every* $r > 0$*,* $B(c, r) \cap (A \setminus \{c\}) \neq \emptyset$*. The set of all accumulation points of* A *is called the derived set of* A *and is denoted by* A' *(sometimes we write* $(A)'$*).*

Remark 8.26 *For any set* $A \subseteq \mathbb{R}^n$ *we have*

$$A' = \left\{c \in \mathbb{R}^n \mid \exists (x^k) \text{ a sequence in } A \setminus \{c\} \text{ which converges to } c\right\}.$$

Remark 8.27 *Let* $A \subseteq \mathbb{R}^n$.
(i) $intA \subseteq A' \subseteq clA$.
(ii) $clA = A' \cup A$.
(iii) A *is closed if and only if* $A' \subseteq A$.
(iv) A' *is closed.*

Example 8.28 *(i) For* $n = 1$ *let* $A = \{0\} \cup [1, 2] \cup (3, 4)$*. Then* $intA = (1, 2) \cup (3, 4)$*,* $clA = \{0\} \cup [1, 2] \cup [3, 4]$*,* A *is neither closed, nor open,* $bdA = \{0, 1, 2, 3, 4\}$*,* $A' = [1, 2] \cup [3, 4]$.
(ii) For $n = 2$ *let* $A = \{(0, 2)\} \cup (\{1\} \times [0, 2])$*. Then* $intA = \emptyset$*,* $clA = A$*,* A *is closed,* $bdA = A$*,* $A' = \{1\} \times [0, 2]$.