# Geometry
# Problem booklet

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## Contents



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# 1 Week 6: Products of vectors

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

## 1.1 Brief theoretical background. Products of vectors

### 1.1.1 The dot product

**Definition 1.1.** The real number

$$\vec{a} \cdot \vec{b} = \begin{cases} 0 \text{ if } \vec{a} = 0 \text{ or } \vec{b} = 0 \\ || \vec{a} || \cdot || \vec{b} || cos(\widehat{\vec{a}, \vec{b}}) \text{ if } \vec{a} \neq 0 \text{ and } \vec{b} \neq 0 \end{cases} \tag{1.1}$$

is called the *dot product* of the vectors $\vec{a}$, $\vec{b}$.

**Remark 1.2.** 1. $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$.

2. $\vec{a} \cdot \vec{a} = || \vec{a} || \cdot || \vec{a} || \cos 0 = || \vec{a} ||^2$.

**Proposition 1.3.** *The dot product has the following properties:*

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$, $\forall \vec{a}, \vec{b} \in \mathcal{V}$.
2. $\vec{a} \cdot (\lambda \vec{b}) = \lambda(\vec{a} \cdot \vec{b})$, $\forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}$.
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$, $\forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.
4. $\vec{a} \cdot \vec{a} \geq 0$, $\forall \vec{a} \in \mathcal{V}$.
5. $\vec{a} \cdot \vec{a} = 0 \Leftrightarrow \vec{a} = \vec{0}$.

**Definition 1.4.** A basis of the vector space $\mathcal{V}$ is said to be *orthonormal*, if $|| \vec{i} || = || \vec{j} || = || \vec{k} || = 1$, $\vec{i} \perp \vec{j}$, $\vec{j} \perp \vec{k}$, $\vec{k} \perp \vec{i}$ ($\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$, $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$). A cartesian reference system $R = (O, \vec{i}, \vec{j}, \vec{k})$ is said to be *orthonormal* if the basis $[\vec{i}, \vec{j}, \vec{k}]$ is orthonormal.

**Proposition 1.5.** *Let $[\vec{i}, \vec{j}, \vec{k}]$ be an orthonormal basis and $\vec{a}, \vec{b} \in \mathcal{V}$. If $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then*

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.2}$$

**Remark 1.6 1.6.** *Let $[\vec{i}, \vec{j}, \vec{k}]$ be an orthonormal basis and $\vec{a}, \vec{b} \in \mathcal{V}$. If $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then*

1. $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$ *and we conclude that* $|| \vec{a} || = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

2.
$$\cos(\widehat{\vec{a},\vec{b}}) = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||\cdot||\vec{b}||} = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}. \tag{1.3}$$

*In particular*

$$\cos(\widehat{\vec{a},\vec{i}}) = \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}};$$
$$\cos(\widehat{\vec{a},\vec{j}}) = \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}};$$
$$\cos(\widehat{\vec{a},\vec{k}}) = \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}.$$

3. $\vec{a}\perp\vec{b} \Leftrightarrow a_1b_1+a_2b_2+a_3b_3=0$

### 1.1.2 Applications of the dot product

• **The distance between two points.** Consider two points $A(x_A,y_A,z_A),\ B(x_B,y_B,z_B)\in\mathcal{P}$. The norm of the vector $\overrightarrow{AB}\ (x_B-x_A,y_B-y_A,z_B-z_A)$ is

$$||\overrightarrow{AB}|| = \sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}.$$

• **The normal vector of a plane.** Consider the plane $\pi\ :\ Ax+By+Cz+D=0$ and the point $P(x_0,y_0,z_0)\in\pi$. The equation of $\pi$ becomes

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0. \tag{1.4}$$

If $M(x,y,z)\in\pi$, the coordinates of $\overrightarrow{PM}$ are $(x-x_0,y-y_0,z-z_0)$ and the equation (1.4) tells us that $\vec{n}\cdot\overrightarrow{PM}=0$, for every $M\in\pi$, that is $\vec{n}\perp\overrightarrow{PM}=0$, for every $M\in\pi$, which is equivalent to $\vec{n}\perp\vec{\pi}$, where $\vec{n}\ (A,B,C)$. This is the reason to call $\vec{n}\ (A,B,C)$ *the normal vector* of the plane $\pi$.

• **The distance from a point to a plane.** Consider the plane $\pi\ :\ Ax+By+Cz+D=0$, a point $P(x_P,y_P,z_P)\in\mathcal{P}$ and $M$ the orthogonal projection of $P$ on $\pi$. The real number $\delta$ given by $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$ is called the *oriented distance* from $P$ to the plane $\pi$, where $\overrightarrow{n}_0=\frac{1}{||\vec{n}||}\overrightarrow{n}$ is the versor of the normal vector $\overrightarrow{n}\ (A,B,C)$. Since $\overrightarrow{MP}=\delta\cdot\overrightarrow{n}_0$, it follows that $\delta(P,M)=||\overrightarrow{MP}||=|\delta|$, where $\delta(P,M)$ stands for the distance from $P$ to $\pi$. We shall show that

$$\delta=\frac{Ax_P+By_P+Cz_P+D}{\sqrt{A^2+B^2+C^2}}.$$

Indeed, since $\overrightarrow{MP}= \delta\cdot \overrightarrow{n}_0$, we get successively:

$$\begin{aligned}\delta &= \overrightarrow{n}_0 \cdot \overrightarrow{MP}= \left(\frac{1}{||\overrightarrow{n}||}\overrightarrow{n}\right)\cdot \overrightarrow{MP}= \frac{\overrightarrow{n}\cdot\overrightarrow{MP}}{||\overrightarrow{n}||}\\ &= \frac{A(x_P-x_M)+B(y_P-y_M)+C(z_P-z_M)}{\sqrt{A^2+B^2+C^2}}\\ &= \frac{Ax_P+By_P+Cz_P-(Ax_M+By_M+Cz_M)}{\sqrt{A^2+B^2+C^2}}\\ &= \frac{Ax_P+By_P+Cz_P+D}{\sqrt{A^2+B^2+C^2}}.\end{aligned}$$

Consequently

$$\delta(P,M)=||\overrightarrow{MP}||=|\delta|=\frac{|Ax_P+By_P+Cz_P+D|}{\sqrt{A^2+B^2+C^2}}.$$

### 1.1.3 The vector product

**Definition 1.7.** The *vector product* or the *cross product* of the vectors $\overrightarrow{a}, \overrightarrow{b}\in \mathcal{V}$ is a vector, denoted by $\overrightarrow{a}\times\overrightarrow{b}$, which is defined to be zero if $\overrightarrow{a}, \overrightarrow{b}$ are linearly dependent (collinear), and if $\overrightarrow{a}, \overrightarrow{b}$ are linearly independent (noncollinear), then it is defined by the following data:

1. $\overrightarrow{a}\times\overrightarrow{b}$ is a vector orthogonal on the two-dimensional subspace $\langle\overrightarrow{a}, \overrightarrow{b}\rangle$ of $\mathcal{V}$;

2. if $\overrightarrow{a}=\overrightarrow{OA}$, $\overrightarrow{b}=\overrightarrow{OB}$, then the sense of $\overrightarrow{a}\times\overrightarrow{b}$ is the one in which a right-handed screw, placed along the line passing through $O$ orthogonal to the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, advances when it is being rotated simultaneously with the vector $\overrightarrow{a}$ from $\overrightarrow{a}$ towards $\overrightarrow{b}$ within the vector subspace $\langle\overrightarrow{a}, \overrightarrow{b}\rangle$ and the support half line of $\overrightarrow{a}$ sweeps the interior of the angle $\widehat{AOB}$ (Screw rule).

3. the *norm* (*magnitude* or *length*) of $\overrightarrow{a}\times\overrightarrow{b}$ is defined by

$$||\overrightarrow{a}\times\overrightarrow{b}||=||\overrightarrow{a}||\cdot||\overrightarrow{b}||\sin(\widehat{\overrightarrow{a}, \overrightarrow{b}}).$$

**Remarks 1.8.** *1.* The *norm* (*magnitude* or *length*) of the vector $\overrightarrow{a}\times\overrightarrow{b}$ is actually the area of the parallelogram constructed on the vectors $\overrightarrow{a}, \overrightarrow{b}$.

2. The vectors $\overrightarrow{a}, \overrightarrow{b}\in\mathcal{V}$ are linearly dependent (collinear) if and only if $\overrightarrow{a}\times\overrightarrow{b}=\overrightarrow{0}$.

**Proposition 1.9.** *The vector product has the following properties:*

1. $\overrightarrow{a}\times\overrightarrow{b}=-\overrightarrow{b}\times\overrightarrow{a}, \forall\ \overrightarrow{a}, \overrightarrow{b}\in\mathcal{V}$;

2. $(\lambda\overrightarrow{a})\times\overrightarrow{b}=\overrightarrow{a}\times(\lambda\overrightarrow{b})=\lambda(\overrightarrow{a}\times\overrightarrow{b}), \forall\,\lambda\in\mathbb{R}, \overrightarrow{a}, \overrightarrow{b}\in\mathcal{V}$;

3. $\overrightarrow{a}\times(\overrightarrow{b}+\overrightarrow{c})=\overrightarrow{a}\times\overrightarrow{b}+\overrightarrow{a}\times\overrightarrow{c}, \forall\ \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\in\mathcal{V}$.

If $[\vec{i}, \vec{j}, \vec{k}]$ is an orthonormal basis, observe that $\vec{i} \times \vec{j} \in \{-\vec{k}, \vec{k}\}$. We say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *direct* if $\vec{i} \times \vec{j} = \vec{k}$. If, on the contrary, $\vec{i} \times \vec{j} = -\vec{k}$, we say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *inverse*. Therefore, if $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis, then $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and obviously $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

**Proposition 1.10.** *If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and*

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k},\ \vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k},$$

*then*

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}, \tag{1.5}$$

*or, equivalently,*

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \tag{1.6}$$

One can rewrite formula (1.5) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.7}$$

the right hand side determinant being understood in the sense of its cofactor expansion along the first line.

## 1.2 Problems

1. If two pairs of opposite edges of the tetrahedron $ABCD$ are perpendicular ($AB \perp CD$, $AD \perp BC$), show that
   (a) The third pair of opposite edges are perpendicular too ($AC \perp BD$).
   (b) $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$.
   (c) The heights of the tetrahedron are concurrent.
   (Such a tetrahedron is said to be orthocentric)
2. Two triangles $ABC$ şi $A'B'C'$ are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices $A'$, $B'$, $C'$ on the sides $BC$, $CA$, $AB$ are concurrent. Show that, in this case, the perpendicular lines from the vertices $A$, $B$, $C$ on the sides $B'C'$, $C'A'$, $A'B'$ are concurrent too.
3. Show that $\| \vec{a} \times \vec{b} \| \leq \| \vec{a} \| \cdot \| \vec{b} \|$, $\forall \vec{a}, \vec{b}, \in \mathcal{V}$.

   *Solution.* $\| \vec{a} \times \vec{b} \| = \| \vec{a} \| \cdot \| \vec{b} \| \sin(\widehat{\vec{a}, \vec{b}}) \leq \| \vec{a} \| \cdot \| \vec{b} \|$.
4. Let $\vec{a}, \vec{b}, \vec{c}$ be noncollinear vectors. Show that the necesssary and sufficient condition for the existence of a triangle $ABC$ with the properties $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{AB} = \vec{c}$ is

   $$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$$

   From the equalities of the norms deduce the low of sines.

5. Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.

6. Find the orthogonal projection

   (a) of the point $A(1,2,1)$ on the plane $\pi: \ x+y+3z+5=0$.

   (b) of the point $B(5,0,-2)$ on the straight line $(d)\ \dfrac{x-2}{3}=\dfrac{y-1}{2}=\dfrac{z-3}{4}$.

7. Compute the distance from the point $A(3,1,-1)$ to the plane $\pi: 22x+4y-20z-45=0$.

8. Find the equations of the bisector planes of the dihedral angles of the planes

$$(\pi_1)\ 2x+y-3z-5=0,\ (\pi_2)\ x+3y+2z+1=0.$$

9. Find the angle between:

   (a) the straight lines

$$(d_1)\ \begin{cases} x+2y+z-1=0 \\ x-2y+z+1=0 \end{cases} \quad (d_2)\ \begin{cases} x-y-z-1=0 \\ x-y+2z+1=0. \end{cases}$$

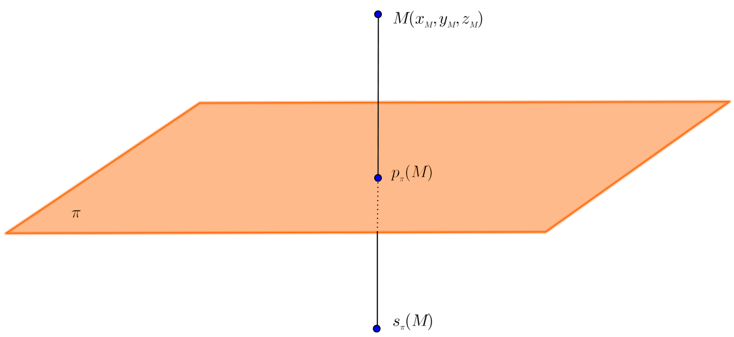
   (b) the planes

$$\pi_1: \ x+3y+2z+1=0 \text{ and } \pi_2: 3x+2y-z=6.$$

   (c) the plane $xOy$ and the straight line $M_1M_2$, where $M_1(1,2,3)$ and $M_2(-2,1,4)$.

### 1.2.1 Appendix: Orthogonal projections and orthogonal symmetries

• **The orthogonal projection on a plane $\pi$.** For a given plane $\pi: Ax+By+Cz+D=0$ and a given point $M(x_M,y_M,z_M)$, we shall determine the coordinates of its orthogonal projection on the plane $\pi$, as well as the coordinates of its (orthogonal) symmetric with respect to $\pi$. The equation of the plane and the coordinates of $M$ are considered with respect to some cartezian coordinate system $R=(O,\overrightarrow{i},\overrightarrow{j},\overrightarrow{k})$. In this respect we consider the orthogonal line on $\pi$ which passes through $M$.




Its parametric equations are

$$\begin{cases} x=x_M+At \\ y=y_M+Bt \\ z=z_M+Ct \end{cases},\ t\in\mathbb{R}. \tag{1.8}$$

The orthogonal projection $p_\pi(M)$ of $M$ on the plane $\pi$ is at its intersection point with the orthogonal line (1.8) and the value of $t \in \mathbb{R}$ for which this orthogonal line (1.8) puncture the plane $\pi$ can be determined by imposing the condition on the point of coordinates $(x_M + At, y_M + Bt, z_M + Ct)$ to verify the equation of the plane, namely $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$. Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\| \overrightarrow{n}_\pi \|^2},$$

where $F(x, y, z) = Ax + By + Cz + D$ şi $\overrightarrow{n}_\pi = A\overrightarrow{i} + B\overrightarrow{j} + C\overrightarrow{k}$ is the normal vector of the plasne $\pi$.

• **The orthogonal projection on the plane $\pi$.**

The coordinates of the orthogonal projection $p_\pi(M)$ of $M$ on th eplane $\pi$ are

$$\begin{cases} x_M - A\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ y_M - B\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ z_M - C\dfrac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2}. \end{cases}$$

Therefore, the position vector of the orthogonal projection $p_\pi(M)$ is

$$\overrightarrow{Op_\pi(M)} = \overrightarrow{OM} - \frac{F(M)}{\| \overrightarrow{n}_\pi \|^2}\overrightarrow{n}_\pi . \tag{1.9}$$

• **The orthogonal symmetry with respect to the plane $\pi$.** In order to find the position vector of the orthogonally symmetric point $s_\pi(M)$ of $M$ w.r.t. $\pi$, we use the relation

$$\overrightarrow{Op_\pi(M)} = \frac{1}{2}\left(\overrightarrow{OM} + \overrightarrow{Os_\pi(M)}\right),$$

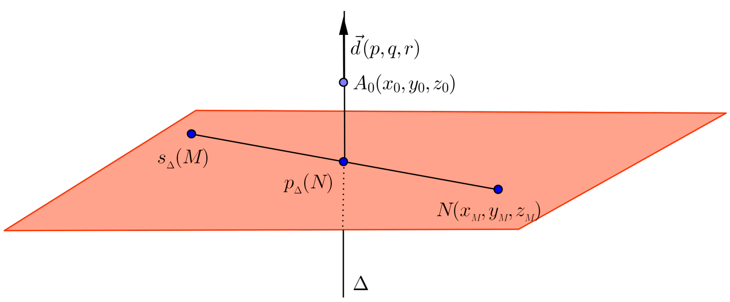
namely

$$\overrightarrow{Os_\pi(M)} = 2\,\overrightarrow{Op_\pi(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{\| \overrightarrow{n}_\pi \|^2}\overrightarrow{n}_\pi .$$

• **The orthogonal projection on a line $\Delta$.** For a given line

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a point $N(x_N, y_N, z_N)$, we shall find the coordinates of its orthogonal projection on the line $\Delta$, as well as the coordinates of the orthogonally symmetric point $M$ with respect to $\Delta$. The equations of the line and the coordinates of the point $N$ are considered with respect to an orthonormal coordinate system $R = (O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$. In this respect we consider the plane $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$ orthogonal on the line $\Delta$ which passes through the point $N$.




The parametric equations

of the line $\Delta$ are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \; t \in \mathbb{R}. \tag{1.10}$$

The orthogonal projection of $N$ on the line $\Delta$ is at its intersection point and the plane $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$, and the value of $t \in \mathbb{R}$ for which the line $\Delta$ puncture the orthogonal plane $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$ can be found by imposing the condition on the point of coordinate $(x_0 + pt, y_0 + qt, z_0 + rt)$ to verify the equation of the plane, namely $p(x_0 + pt - x_N) + q(y_0 + qt - y_N) + r(z_0 + rt - z_N) = 0$. Thus

$$t = -\frac{p(x_0 - x_N) + q(y_0 - y_N) + r(z_0 - z_N)}{p^2 + q^2 + r^2} = -\frac{G(x_0, y_0, z_0)}{\| \overrightarrow{d}_\Delta \|^2},$$

where $G(x, y, z) = p(x - x_N) + q(y - y_N) + r(z - z_N)$ and $\overrightarrow{d}_\pi = p \overrightarrow{i} + q \overrightarrow{j} + r \overrightarrow{k}$ is the director vectoir of the line $\Delta$. Ths coordinates of the orthogonal projection $p_\Delta(N)$ of $N$ on the line $\Delta$ are therefore

$$\begin{cases} x_0 - p\dfrac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ y_0 - q\dfrac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ z_0 - r\dfrac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2}. \end{cases}$$

Thus, the position vector of the orthogonal projection $p_\Delta(N)$ is

$$\overrightarrow{Op_\Delta(N)} = \overrightarrow{OA}_0 - \frac{G(A_0)}{\| \overrightarrow{d}_\Delta \|^2} \overrightarrow{d}_\Delta, \tag{1.11}$$

where $A_0(x_0, y_0, z_0) \in \Delta$.

• **The orthogonal symmetry with respect to a line** $\Delta$**.** In order to find the position vector of the orthogonally symmetric point $s_\Delta(N)$ of $N$ with respect to the line $\Delta$ we use the relation

$$\overrightarrow{Op_\Delta(N)} = \frac{1}{2}\left(\overrightarrow{ON} + \overrightarrow{Os_\Delta(N)}\right)$$

i.e.

$$\overrightarrow{Os_\Delta(N)} = 2\, \overrightarrow{Op_\Delta(N)} - \overrightarrow{ON} = 2\, \overrightarrow{OA}_0 - 2\frac{G(A_0)}{\| \overrightarrow{d}_\Delta \|^2} \overrightarrow{d}_\Delta - \overrightarrow{ON}.$$

**Remark 1.11.** *1. The distance from the point $M(x_M, y_M, z_M)$ to the plane $\pi : Ax + By + Cz + D = 0$ can be equally computed by means of* (1.9). *Indeed,*

$$\begin{aligned} \delta(M, \pi) &= \| \overrightarrow{Mp_\pi(M)} \| = \| \overrightarrow{Op_\pi(M)} - \overrightarrow{OM} \| \\ &= \left| -\tfrac{F(M)}{\| \overrightarrow{n}_\pi \|^2} \right| \cdot \| \overrightarrow{n}_\pi \| = \frac{|F(M)|}{\| \overrightarrow{n}_\pi \|}. \end{aligned}$$

*2. The distance from the point $N(x_N, y_N, z_N)$ to the straight line* $\Delta : \dfrac{x - x_0}{p} = \dfrac{y - y_0}{q} = \dfrac{z - z_0}{r}$

*can be computed by means of* (1.11). *Indeed,*

$$\delta(M,\Delta) = \| \overrightarrow{Np_{\Delta}(N)} \| = \| \overrightarrow{NO} + \overrightarrow{Op_{\Delta}(N)} \| \tag{1.12}$$
$$= \left\| \overrightarrow{NA_0} - \frac{G(A_0)}{\| \overrightarrow{d}_{\Delta} \|^2} \overrightarrow{d}_{\Delta} \right\| = \left\| \overrightarrow{NA_0} - \frac{\overrightarrow{d}_{\Delta} \cdot \overrightarrow{NA_0}}{\| \overrightarrow{d}_{\Delta} \|^2} \overrightarrow{d}_{\Delta} \right\|.$$

**Proposition 1.12.** *Taking into account the formula* (1.12) *for the distance* $\delta(M,\Delta)$ *from the point* $N(x_N, y_N, z_N)$ *to the straight line* $\Delta : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$ *as well as Proposition* **??** *we deduce that*

$$\delta(M,\Delta) = \left\| \overrightarrow{NA_0} - \frac{\overrightarrow{d}_{\Delta} \cdot \overrightarrow{NA_0}}{\| \overrightarrow{d}_{\Delta} \|^2} \overrightarrow{d}_{\Delta} \right\| = \frac{\| \left(\overrightarrow{d}_{\Delta} \cdot \overrightarrow{d}_{\Delta}\right) \overrightarrow{NA_0} - \left(\overrightarrow{d}_{\Delta} \cdot \overrightarrow{NA_0}\right) \overrightarrow{d}_{\Delta} \|}{\| \overrightarrow{d}_{\Delta} \|^2}$$
$$= \frac{\| \overrightarrow{d}_{\Delta} \times (\overrightarrow{NA_0} \times \overrightarrow{d}_{\Delta}) \|}{\| \overrightarrow{d}_{\Delta} \|^2} = \frac{\| \overrightarrow{NA_0} \times \overrightarrow{d}_{\Delta} \|}{\| \overrightarrow{d}_{\Delta} \|}.$$

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