Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Bachelor, Computer Science, Groups 911-917, Academic Year 2017-2018

Mathematical Analysis
Seminar 2

1. Study the boundedness, the monotony and the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ in each of the following instances:

a) $x_n = \dfrac{n!}{n^n}$;

b) $x_n = \dfrac{(-1)^n}{n}$;

c) $x_n = (-1)^n + \dfrac{n+1}{n}$;

d) $x_n = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{n(n+1)}$;

e) $x_n = \dfrac{\cos\pi}{1\cdot 2} + \dfrac{\cos 2\pi}{2\cdot 3} + \cdots + \dfrac{\cos n\pi}{n(n+1)}$;

f) $x_n = \dfrac{1}{1+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}+\sqrt{n+1}}$.

2. Compute the limit of the following sequences:

$a_n = \dfrac{\alpha n^3 + \beta n^2 + \gamma n + 1}{n^2 - n + 1}$, where $\alpha, \beta, \gamma \in \mathbb{R}$;

$b_n = \dfrac{n^\alpha}{(1+\beta)^n}$, where $\alpha \in \mathbb{N}, \beta \in (0,\infty)$;

$c_n = \dfrac{e^n - 2^n}{\pi^n - 3^n}$;

$x_n = \dfrac{\sqrt{n + \sqrt{n + \sqrt{n + \sqrt{n}}}}}{n+1}$;

$y_n = \sqrt{n}\left(\sqrt{n} - \sqrt{n+2}\right)$;

$z_n = n\left(\sqrt[3]{n+1} - \sqrt[3]{n}\right)$.

3. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$x_n \coloneqq \left(1 + \frac{1}{n}\right)^n.$$

a) Using Bernoulli's Inequality (see Seminar 1) prove that $\dfrac{x_{n+1}}{x_n} > 1$ for all $n \in \mathbb{N}$.

b) Using Newton's Binomial Formula prove that $x_n < 3$ for all $n \in \mathbb{N}$.

Hint: notice that $\dbinom{n}{k} \le \dfrac{n^k}{2^{k-1}}$ for all $k \in \mathbb{N}$, $k \le n$.

c) Deduce that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and, denoting its limit by e (the Euler's number), show that $2.71 < e \le 3$.

d) Similarly to a) prove that the sequence $(y_n)_{n\in\mathbb{N}}$, defined for all $n \in \mathbb{N}$ by

$$y_n \coloneqq \left(1 + \frac{1}{n}\right) x_n,$$

is strictly decreasing. Then, observing that $x_n < y_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n$, deduce that $e < 2.72$.

4. Compute the limits:

a) $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n^2+1}\right)^{\sqrt{n^2+1}}$;

b) $\lim\limits_{n\to\infty} \left(\dfrac{2n+1}{2n-1}\right)^n$;

c) $\lim\limits_{n\to\infty} \dfrac{(2n)^n}{(2n)!}$;

d) $\lim\limits_{n\to\infty} \dfrac{1^p + 2^p + \ldots + n^p}{n^{p+1}}$, where $p \in \mathbb{N}$;

e) $\lim\limits_{n\to\infty} \sqrt[n]{1 + 2 + \ldots + n}$;

f) $\lim\limits_{n\to\infty} \sin\left(\pi\sqrt{n^2+1}\right)$.