# Seminar 1

**1.** Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$?

**2.** Let $A = \{a_1, a_2, a_3\}$. Determine the number of:
$(i)$ operations on $A$;
$(ii)$ commutative operations on $A$;
$(iii)$ operations on $A$ with identity element.
Generalization for a set $A$ with $n$ elements ($n \in \mathbb{N}^*$).

**3.** Decide which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups together with the usual addition or multiplication.

**4.** Let "$*$" be the operation defined on $\mathbb{R}$ by $x * y = x + y + xy$. Prove that:
$(i)$ $(\mathbb{R}, *)$ is a commutative monoid.
$(ii)$ The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

**5.** Let "$*$" be the operation defined on $\mathbb{N}$ by $x * y = \text{g.c.d.}(x, y)$.
$(i)$ Prove that $(\mathbb{N}, *)$ is a commutative monoid.
$(ii)$ Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ $(n \in \mathbb{N}^*)$ is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
$(iii)$ Fill in the table of the operation "$*$" on $D_6$.

**6.** Determine the finite stable subsets of $(\mathbb{Z}, \cdot)$.

# Seminar 2

**1.** Let $r, s, t, v$ be the homogeneous relations defined on the set $M = \{2, 3, 4, 5, 6\}$ by

$$x\, r\, y \Longleftrightarrow x < y$$

$$x\, s\, y \Longleftrightarrow x | y$$

$$x\, t\, y \Longleftrightarrow g.c.d.(x, y) = 1$$

$$x\, v\, y \Longleftrightarrow x \equiv y \pmod 3 .$$

Write the graphs $R, S, T, V$ of the given relations.

**2.** Let $A$ and $B$ be sets with $n$ and $m$ elements respectively ($m, n \in \mathbb{N}^*$). Determine the number of:
($i$) relations having the domain $A$ and the codomain $B$;
($ii$) homogeneous relations on $A$.

**3.** Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

**4.** Which ones of the properties of reflexivity, transitivity and symmetry hold for the following homogeneous relations: the strict inequality relations on $\mathbb{R}$, the divisibility relation on $\mathbb{N}$ and on $\mathbb{Z}$, the perpendicularity relation of lines in space, the parallelism relation of lines in space, the congruence of triangles in a plane, the similarity of triangles in a plane?

**5.** Let $M = \{1, 2, 3, 4\}$, let $r_1$, $r_2$ be homogeneous relations on $M$ and let $\pi_1$, $\pi_2$, where $R_1 = \Delta_M \cup \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$, $R_2 = \Delta_M \cup \{(1, 2), (1, 3)\}$, $\pi_1 = \{\{1\}, \{2\}, \{3, 4\}\}$, $\pi_2 = \{\{1\}, \{1, 2\}, \{3, 4\}\}$.
($i$) Are $r_1, r_2$ equivalences on $M$? If yes, write the corresponding partition.
($ii$) Are $\pi_1, \pi_2$ partitions on $M$? If yes, write the corresponding equivalence relation.

**6.** Define on $\mathbb{C}$ the relations $r$ and $s$ by:

$$z_1\, r\, z_2 \Longleftrightarrow |z_1| = |z_2| ; \qquad z_1\, s\, z_2 \Longleftrightarrow arg\, z_1 = arg\, z_2 \text{ or } z_1 = z_2 = 0 .$$

Prove that $r$ and $s$ are equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation).

**7.** Let $n \in \mathbb{N}$. Consider the relation $\rho_n$ on $\mathbb{Z}$, called the *congruence modulo* $n$, defined by:

$$x\, \rho_n\, y \Longleftrightarrow n | (x - y) .$$

Prove that $\rho_n$ is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_n$. Discuss the cases $n = 0$ and $n = 1$.

**8.** Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

# Seminar 3

**1.** Let $M$ be a non-empty set and let $S_M = \{f : M \to M \mid f \text{ is bijective}\}$. Show that $(S_M, \circ)$ is a group, called the *symmetric group* of $M$.

**2.** Let $M$ be a non-empty set and let $(R, +, \cdot)$ be a ring. Define on $R^M = \{f \mid f : M \to R\}$ two operations by: $\forall f, g \in R^M$,

$$f + g : M \to R\,, \quad (f+g)(x) = f(x) + g(x)\,, \quad \forall x \in M\,,$$

$$f \cdot g : M \to R\,, \quad (f \cdot g)(x) = f(x) \cdot g(x)\,, \quad \forall x \in M\,.$$

Show that $(R^M, +, \cdot)$ is a ring. If $R$ is commutative or has identity, does $R^M$ have the same property?

**3.** Prove that $H = \{z \in \mathbb{C} \mid |z| = 1\}$ is a subgroup of $(\mathbb{C}^*, \cdot)$, but not of $(\mathbb{C}, +)$.

**4.** Let $U_n = \{z \in \mathbb{C} \mid z^n = 1\} \quad (n \in \mathbb{N}^*)$ be the *set of $n$-th roots of unity*. Prove that $U_n$ is a subgroup of $(\mathbb{C}^*, \cdot)$.

**5.** Let $n \in \mathbb{N}$, $n \geq 2$. Prove that:
(*i*) $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}$ is a stable subset of the monoid $(M_n(\mathbb{C}), \cdot)$;
(*ii*) $(GL_n(\mathbb{C}), \cdot)$ is a group, called the *general linear group of rank $n$*;
(*iii*) $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) = 1\}$ is a subgroup of the group $(GL_n(\mathbb{C}), \cdot)$.

**6.** Show that the following sets are subrings of the corresponding rings:
(*i*) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ in $(\mathbb{C}, +, \cdot)$.
(*ii*) $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ in $(M_2(\mathbb{R}), +, \cdot)$.

**7.** (*i*) Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that $f$ is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(*ii*) Let $g : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $g(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that $g$ is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

**8.** Let $n \in \mathbb{N}$, $n \geq 2$. Prove that the groups $(\mathbb{Z}_n, +)$ of residue classes modulo $n$ and $(U_n, \cdot)$ of $n$-th roots of unity are isomorphic.

# Seminar 4

**1.** Let $K$ be a field. Show that $K[X]$ is a $K$-vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: $\forall\ k \in K$, $\forall f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$,

$$k \cdot f = (ka_0) + (ka_1)X + \cdots + (ka_n)X^n\,.$$

**2.** Let $K$ be a field and $m, n \in \mathbb{N}$, $m, n \geq 2$. Show that $M_{m,n}(K)$ is a $K$-vector space, with the usual addition and scalar multiplication of matrices.

**3.** Let $K$ be a field, $A \neq \emptyset$ and denote $K^A = \{f \mid f : A \to K\}$. Show that $K^A$ is a $K$-vector space, where the addition and the scalar multiplication are defined as follows: $\forall f, g \in K^A$, $\forall k \in K$, $f + g \in K^A$, $kf \in K^A$,

$$(f + g)(x) = f(x) + g(x)\,,$$

$$(k \cdot f)(x) = k \cdot f(x)\,, \forall x \in A\,.$$

In particular, $\mathbb{R}^{\mathbb{R}}$ is vector space over $\mathbb{R}$.

**4.** Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations:

$$x \perp y = xy\,,$$

$$k \top x = x^k\,,$$

$\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that $V$ is a vector space over $\mathbb{R}$.

**5.** Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
*(i)* $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
*(ii)* $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
*(iii)* $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
*(iv)* $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
*(v)* $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
*(vi)* $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$ ?

**6.** Which ones of the following sets are subspaces:
*(i)* $[-1, 1]$ of the real vector space $\mathbb{R}$;
*(ii)* $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ of the real vector space $\mathbb{R}^2$;
*(iii)* $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Q} \right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;
*(iv)* $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

**7.** Let $n \in \mathbb{N}$. Which ones of the following sets are subspaces of the $K$-vector space $K[X]$:
*(i)* $K_n[X] = \{f \in K[X] \mid \text{degree}(f) \leq n\}$;
*(ii)* $K'_n[X] = \{f \in K[X] \mid \text{degree}(f) = n\}$.

**8.** Show that the set of all solutions of the homogeneous system of equations with real coefficients

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = 0 \\ a_{21}x_1 + a_{22}x_2 = 0 \end{cases}$$

is a subspace of the real vector space $\mathbb{R}^2$.

# Seminar 5

**1.** Determine the following generated subspaces:
$(i)$ $< 1, X, X^2 >$ in the real vector space $\mathbb{R}[X]$.
$(ii)$ $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$.

**2.** Consider the following subspaces of the real vector space $\mathbb{R}^3$:
$(i)$ $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
$(ii)$ $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
$(iii)$ $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write $A, B, C$ as generated subspaces with a minimal number of generators.

**3.** Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that $S$ and $T$ are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

**4.** Let $S$ and $T$ be the set of all even functions and of all odd functions in $\mathbb{R}^{\mathbb{R}}$ respectively. Prove that $S$ and $T$ are subspaces of the real vector space $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} = S \oplus T$.

**5.** Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x + y, x - y),$$

$$g(x, y) = (2x - y, 4x - 2y),$$

$$h(x, y, z) = (x - y, y - z, z - x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

**6.** Which ones of the following functions are endomorphisms of the real vector space $\mathbb{R}^2$:
$(i)$ $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$;
$(ii)$ $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (a + x, b + y)$, where $a, b \in \mathbb{R}$?

**7.** Determine the kernel and the image of the endomorphisms from Exercise **5.**

**8.** Let $V$ be a vector space over $K$ and $f \in End_K(V)$. Show that the set

$$S = \{x \in V \mid f(x) = x\}$$

of fixed points of $f$ is a subspace of $V$.

## Seminar 6

**1.** Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:
($i$) $v_1$, $v_2$, $v_3$ are linearly dependent and determine a dependence relationship.
($ii$) $v_1$, $v_2$ are linearly independent.

**2.** Prove that the following vectors are linearly independent:
($i$) $v_1 = (1, 0, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, 1, 1)$ in ${}_{\mathbb{R}}\mathbb{R}^3$.
($ii$) $v_1 = (1, 2, 3, 4)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$, $v_4 = (4, 1, 2, 3)$ in ${}_{\mathbb{R}}\mathbb{R}^4$.

**3.** Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors $v_1$, $v_2$, $v_3$ are linearly independent.

**4.** Let $v_1 = (1, -2, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (0, a, 1, 2)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^4$. Determine $a \in \mathbb{R}$ such that the vectors $v_1$, $v_2$, $v_3$ are linearly dependent.

**5.** Let $v_1 = (1, 1, 0)$, $v_2 = (-1, 0, 2)$, $v_3 = (1, 1, 1)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$.
($i$) Show that the list $(v_1, v_2, v_3)$ is a basis of the real vector space $\mathbb{R}^3$.
($ii$) Express the vectors of the canonical basis $(e_1, e_2, e_3)$ of $\mathbb{R}^3$ as a linear combination of the vectors $v_1$, $v_2$ and $v_3$.
($iii$) Determine the coordinates of $u = (1, -1, 2)$ in each of the two bases.

**6.** Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists $(E_1, E_2, E_3, E_4)$ and $(A_1, A_2, A_3, A_4)$ are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in the each of the two bases.

**7.** Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial $f = a_0 + a_1 X + a_2 X^2 \in \mathbb{R}_2[X]$ in each basis.

**8.** Determine the number of bases of the vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$.

# Seminar 7

**1.** Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

**2.** Let $K$ be a field and $S = \{(x_1, \ldots, x_n) \in K^n \mid x_1 + \cdots + x_n = 0\}$.
$(i)$ Prove that $S$ is a subspace of the canonical vector space $K^n$ over $K$.
$(ii)$ Determine a basis and the dimension of $S$.

**3.** Show that $\mathbb{C}$ is a vector space over $\mathbb{R}$, and determine a basis and the dimension for it.

**4.** Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that $f$ is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

**5.** Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (-y + 5z, x, y - 5z)$. Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

**6.** Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

**7.** Let $V$ be a vector space over $K$ and let $S, T$ and $U$ be subspaces of $V$ such that $dim(S \cap U) = dim(T \cap U)$ and $dim(S + U) = dim(T + U)$. Prove that if $S \subseteq T$, then $S = T$.

**8.** Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\},$$

$$T = < (0, 1, 1), (1, 1, 0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

# Seminar 8

**1.** Let $A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Show that $A$ is invertible, determine $A^{-1}$ and solve the linear system $AX = B$.

**2.** Using the Kronecker-Capelli theorem, decide if the following linear systems are compatible and then solve the compatible ones:

$$(i) \begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases} \qquad (ii) \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$(iii) \begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

**3.** Using the Rouché theorem, decide if the systems from **2.** are compatible and then solve the compatible ones.

**4.** Decide when the following linear system is compatible determinate and in that case solve it by using Cramer's method:

$$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases} \quad (a, b, c \in \mathbb{R}).$$

Solve the following linear systems by the Gauss and Gauss-Jordan methods:

**5.** $$(i) \begin{cases} 2x + 2y + 3z = 3 \\ x - y \quad = 1 \\ -x + 2y + z = 2 \end{cases} \qquad (ii) \begin{cases} 2x + 5y + z = 7 \\ x + 2y - z = 3 \\ x + y - 4z = 2 \end{cases} \qquad (iii) \begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

**6.** $$\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases} \qquad (\lambda \in \mathbb{R})$$

**7.** $$\begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases} \qquad (a \in \mathbb{R})$$

**8.** Determine the positive solutions of the following non-linear system:

$$\begin{cases} xyz = 1 \\ x^3y^2z^2 = 27 \\ \frac{z}{xy} = 81 \end{cases}$$

# Seminar 9

Compute by applying elementary operations the ranks of the matrices:

**1.** $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$; $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **2.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Compute by applying elementary operations the inverses of the matrices:

**3.** $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **4.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

*For the following exercises, for a list $X$ of vectors in a canonical vector space $\mathbb{R}^n$, use that dim $< X >$ is equal to the rank of an echelon form $C$ of the matrix consisting of the components of the vectors of $X$, and a basis of $< X >$ is given by the non-zero rows of $C$.*

**5.** In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine *dim* $< X >$ and a basis of $< X >$.

**6.** In the real vector space $\mathbb{R}^4$ consider the list $X = (v_1, v_2, v_3)$, where $v_1 = (1, 0, 4, 3)$, $v_2 = (0, 2, 3, 1)$ and $v_3 = (0, 4, 6, 2)$. Determine *dim* $< X >$ and a basis of $< X >$.

**7.** Determine the dimension of the subspaces $S$, $T$, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S =< (1, 0, 4), (2, 1, 0), (1, 1, -4) > ,$$

$$T =< (-3, -2, 4), (5, 2, 4), (-2, 0, -8) > \ .$$

**8.** Determine the dimension of the subspaces $S$, $T$, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^4$ and a basis for the first three of them, where

$$S =< (1, 2, -1, -2), (3, 1, 1, 1), (-1, 0, 1, -1) > ,$$

$$T =< (2, 5, -6, -5), (-1, 2, -7, -3) > \ .$$

# Seminar 10

**1.** Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by

$$f(x, y, z) = (x + y, y - z, 2x + y + z).$$

Determine the matrix $[f]_E$, where $E = (e_1, e_2, e_3)$ is the canonical basis for $\mathbb{R}^3$.

**2.** Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1, 1, 0), (0, 1, 1), (1, 0, 1))$ of $\mathbb{R}^3$, $B' = (v'_1, v'_2) = ((1, 1), (1, -2))$ of $\mathbb{R}^2$ and let $E' = (e'_1, e'_2)$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

**3.** Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis $E$ of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

$(i)$ Show that $v = (1, 4, 1, -1) \in Ker\, f$ and $v' = (2, -2, 4, 2) \in Im\, f$.
$(ii)$ Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
$(iii)$ Define $f$.

**4.** In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1, 2), (1, 3))$ and $B' = (v'_1, v'_2) = ((1, 0), (2, 1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f + g]_B$ and $[f \circ g]_{B'}$.

**5.** Consider the endomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$f(x, y) = (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha) \quad (\alpha \in \mathbb{R}).$$

Write its matrix in the canonical basis of $\mathbb{R}^2$ and show that $f$ is an automorphism.

**6.** Let $V$ be a vector space of dimension 2 over the field $K = \mathbb{Z}_2$. Determine $|V|$, $|End_K(V)|$ and $|Aut_K(V)|$.