Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2016

1. (2p) Find all the solutions of each of the following equation.
(a) $x' = 2(x-1)$; (b) $x_{k+1} = 2(x_k - 1)$; (c) $x'' - 9x = 0$; (d) $x_{k+2} - 9x_k = 0$.

2. (2p) Consider the following planar system

$$\dot{x} = -y(x^2 + y^2), \quad \dot{y} = x(x^2 + y^2).$$

a) Find its equilibria.
b) Decide whether the equilibrium point $(0, 0)$ is hyperbolic or not.
c) Verify that $\varphi(t, 1, 0) = (\cos t, \sin t)$, $\varphi(t, 2, 0) = (2 \cos 4t, 2 \sin 4t)$ for all $t \in \mathbb{R}$. Find $\varphi(t, 3, 0)$.
d) Find a first integral.
e) Represent its phase portrait.
f) What remarkable property have the solutions of this system?

3. (1.5) Let $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ be such that $a_{12} \neq 0$. Show that the roots of the characteristic equation corresponding to the second-order difference equation obtained by reducing the linear system

$$x_{k+1} = a_{11}x_k + a_{12}y_k, \quad y_{k+1} = a_{21}x_k + a_{22}y_k$$

are the eigenvalues of the matrix associated to this system.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2016

1. (2p) Find all the solutions of each of the following equation.
(a) $x' = 2(x-5)$; (b) $x_{k+1} = 2(x_k-5)$; (c) $x''+9x = 0$; (d) $x_{k+2}+9x_k = 0$.

3. (2p) We consider the planar Lotka-Volterra system
$\dot{x} = x(1-y), \quad \dot{y} = y(2-x)$.
a) Find its equilibria and study their stability using the linearization method.
b) Find a first integral in $(0,\infty)\times(0,\infty)$.

3. (1.5p) Using the stair-step diagram, estimate the basin of attraction for each of the fixed points (if there is any which is an attractor) of the map

$$f : (0,\infty) \to \mathbb{R}, \quad f(x) = \frac{x^2+5}{2x} \ .$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2016

1. (1.5p) Represent the phase portrait of the scalar dynamical system
$\dot{\rho} = \rho(1 - \rho^2)$. Find $\varphi(t, 1)$ and justify. Specify the properties of $\varphi(t, 2)$ and, respectively, $\varphi(t, 0.5)$.

2. (0.25p) Find the polar coordinates of the points whose cartesian coordinates are: $(1, 0)$, $(0, 1)$, $(-2, 0)$ and $(0, -0.5)$, respectively.

3. (2.5p) We consider the planar system
$\dot{x} = -y + x(1 - x^2 - y^2),\ \dot{y} = x + y(1 - x^2 - y^2)$.
a) Study the type and stability of the equilibrium point $(0, 0)$ using the linearization method.
b) Check that $\varphi(t, 1, 0) = (\cos t, \sin t)$ for any $t \in \mathbb{R}$. Represent the corresponding orbit.
c) Transform the given system to polar coordinates.
d) Sketch the phase portrait of this planar system.

4. (1.25p) Find the solution of each of the following IVPs:
a) $x_{k+2} - 5\, x_{k+1} + 6x_k = 0,\ x_0 = 0,\ x_1 = 1;$
b) $x'' - 5x' + 6x = 0,\ x(0) = 0,\ x'(0) = 1.$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2016

1. (1.5p) Find the linear homogeneous differential equation with constant coefficients of minimal order that has as solutions:

a) e^{-3t} and $3te^{-t}$;

b) $\cos(5t)$.

Find also the general solution of each of these two equations.

2. (2p) Let $g : I \to \mathbb{R}$ be a C^1 map such that $g'(x) \neq 0$ for all x in the interval I. Assume that there exists $r \in I$ such that $g(r) = 0$. Prove that for $\eta \in I$ sufficiently close to r the unique solution $(x_k)_{k\geq 0}$ of the IVP

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad x_0 = \eta$$

satisfies

$$\lim_{k\to\infty} x_k = r.$$

3. (2p) For what values of the real parameter a the system

$$\dot{x} = ax - 5y, \ \dot{y} = x - 2y$$

has a center at the origin?

For $a = 0$ find the general solution of this system and specify its type and stability.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2016

1. (1p)

a) Write the statement of the Fundamental Theorem for linear homogeneous second order differential equations.

b) The following proposition is true or false? Justify. We remind you that $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$.

"The general solution of the differential equation $x'' - x = 0$ is
$x(t) = c_1 \cosh t + c_2 \sinh t$, where c_1, c_2 are arbitrary real constants."

2. (1p) Find a first integral in $\mathbb{R}^2$ of the pendulum equation

$$\ddot{\theta} + \omega^2 \sin\theta = 0.$$

Hint: Write first the planar system equivalent to this equation.

3. (0.75p) Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k,\ x_0 = 0,\ x_1 = 1.$$

4. (2p) We consider the following nonlinear planar systems
$\dot{x} = -x + xy, \quad \dot{y} = -2y + 3y^2.$

a) Find its equilibria and study their stability using the linearization method.

b) Find $\varphi(t, 0, 2/3)$, $\varphi(t, 4, 0)$ and $\varphi(t, 1, 2/3)$.

5. (0.75p) We consider the IVP $y' = 1 + xy^2$, $y(0) = 0$. Write the Euler numerical formula on the interval $[0, 1]$ with step-size $h = 0.02$. Specify the initial values and the number of steps necessary to find the approximate value of $\varphi(0.5)$ and, respectively, of $\varphi(1)$. Here with φ is denoted the exact solution of the given IVP.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2016

1. (1.5p) For each $k > 0$ we consider the differential equation $\dot{x} = -k\,(x - 21)$, which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time t.

(a) Find its flow.

(b) An experiment revealed the following fact. A cup of tea with initial temperature of $49°C$ has a temperature of $37°C$ after 10 minutes. Find the initial temperature of a cup of tea such that after 20 minutes the tea has $37°C$.

2. (2.5p) We consider the map

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{50} x(100 - x).$$

(a) Find its fixed points and study their stability.

(b) Using the stair-step diagram estimate the basin of attraction of the asymptotically stable fixed point.

(c) If $(x_k)_{k \geq 0}$ represent the number of fish in some lake at month k and

$$x_{k+1} = \frac{1}{50} x_k(100 - x_k), \quad x_0 = \eta$$

try to predict the fate of the fish in the case $\eta = 80$ and also in the case $\eta = 10$.

3. (1p) Find the general solution of each of the following equations.

(a) $x''' = \sin t$; (b) $x_{k+1} = 2x_k$; (c) $x_{k+1} = Ax_k$, where $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2016

1. (1.5p) We consider the differential equation

$$x'' + 4x = \cos 2t.$$

a) Find a solution of the form $x_p = t(a\cos 3t + b\sin 3t)$, with $a, b \in \mathbb{R}$.
b) Find its general solution.
c) Describe the motion of a spring-mass system governed by this equation.

2. (1p) Find the solution of

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k, \quad x_0 = 0, \quad x_1 = 0.$$

Hint: look for $a, b \in \mathbb{R}$ such that $(x_k)_p = ak + b$ is a particular solution of the difference equation.

3. (1.5p) We consider the IVP $x' = -200x, \quad x(0) = 1$.
a) Find the solution and its limit as $t \to \infty$.
b) Write the Euler's numerical formula with constant step-size h.
c) Find a range of values for the step-size h such that the solution $(x_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} x_k = 0$.

4. (1.5p) Find the fixed points and the 2–periodic points of the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$. Study their stability.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2016

1. (1p) Find the general solution of the scalar differential equation

$$x' - ax = at - 1,$$

where the unknown is the function x of variable t and $a \in \mathbb{R}^*$ is a fixed parameter.

2. (2p) We consider the scalar differential equation

$$\dot{x} = 2x(2 - x),$$

whose unknown is the function x of variable t.

a) Study the stability of its equilibria using the linearization method.

b) Depict its phase portrait.

b) Describe the properties of $\varphi(t; -1)$, $\varphi(t; 1)$, $\varphi(t; 2)$ and $\varphi(t; 5)$.

c) There exists some $\eta \in \mathbb{R}$ such that $\lim\limits_{t\to\infty} \varphi(t; \eta) = 3$?

3. (2.5p) We consider the map

$$T : \mathbb{R} \to \mathbb{R}, \quad T(x) = 1 - |2x - 1|.$$

a) Represent the graph of T.

b) Find its fixed points and the orbit of the initial state $\eta = \frac{3}{8}$.

c) Let $k \in \mathbb{N}$ be such that $k \geq 2$. Find the orbit of the initial state $\frac{3}{2^k}$.

d) Find the 2-periodic points of T.

e) Represent the graphs of T^2 and T^3. The map T has periodic orbits of period 3? Or of other periods?

Universitatea Babeş–Bolyai, Cluj-Napoca, România
Facultatea de Matematică şi Informatică

Exam on Differential Equations
June 17, 2016

1. (2p) We consider the differential equation

$$y'' + y = \cos t.$$

a) Find a solution of the form $Y(t) = t(A\cos t + B\sin t)$, where the coefficients $A, B \in \mathbb{R}$ have to be determined. [Then draw its graph. Decide whether this function $Y(t)$ is bounded or it oscillates around 0 (that is, the positive and negative values alternate).]

b) Find its general solution.

c) Find its solution that also satisfies the conditions $y(0) = 0$, $y'(0) = 0$.

2. (2p) We consider the IVP $\quad y' = -200y, \quad y(0) = 1,$
where the unknown is the function $y(t)$.

a) Find the solution and its limit as $t \to \infty$.

b) Write the Euler's numerical formula with constant step-size h.

c) For $h = 0.001$, and, respectively, $h = 0.01$ find the solution $(y_k)_{k\geq 0}$ of the difference equation found at b) and decide if it satisfies $\lim\limits_{k\to\infty} y_k = 0$.

[d) Find a range of values for the step-size h such that the solution $(y_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} y_k = 0$.]

3. (2p) Find the general solution of each of the following differential equations.

a) $ty' + 2y = 4t^2$ $\qquad$ b) $\dfrac{dy}{dx} = \dfrac{x^2}{1-y^2}$

c) $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0.$

I.

(a) Find the general solution of the differential equation $x' - 3x = t$.

(b) Find the solutions (if any) of the following BVP

$$x'' + \pi^2 x = 0, \quad x(0) = 0, \quad x(5) = \pi.$$

II.

(a) We consider the planar nonlinear system

$$\dot{x} = x(y-1), \quad \dot{y} = y(1-x).$$

(a) Find its equilibria. There exists an equilibrium point in $(0,\infty) \times (0,\infty)$?

(b) Find a first integral in $(0,\infty) \times (0,\infty)$.

(c) Prove that $f : (0,\infty) \to \mathbb{R}$, $f(x) = x - \ln x$ has a minimum at $x = 1$. Deduce that there exists a first integral $H : (0,\infty) \times (0,\infty) \to \mathbb{R}$ that has a minimum at the equilibrium point $(1,1)$. What is the shape of the orbits in $(0,\infty) \times (0,\infty)$?

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2015

1. (1p) Find the general solution of each of the following differential equations whose unknown is the function denoted $x(t)$.

(a) $x' + tx = 1$; (b) $x'' + 4x = 1$.

2. (2.5p) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$.

(a) Find the fixed points of f and study their stability using the linearization method.

(b) Represent the graph of f and find geometrically the fixed points of f.

(c) Find directly $\varphi(k, 0)$ (or, in other notation, $f^k(0)$) for any $k \geq 0$. Which is the orbit corresponding to the initial state 0? What remarkable property has this orbit? Depict this orbit using the stair-step diagram.

d) Let $\eta = 2$, and, respectively, $\eta = -1/4$. Using the stair-step diagram describe the long-term behavior of the orbit that starts at η (in other notation, of the sequence defined by $x_{k+1} = x_k^2 - 1,\ x_0 = \eta$).

3. (2p) Let $c \in [0, 1)$ be a parameter and consider the scalar dynamical system $\dot{x} = x(1 - x) - cx$.

a) Find its equilibria and study their stability using the linearization method.

b) Represent its phase portrait.

c) When $x(t) > 0$ is considered to be the number of fish in some lake, and $c \geq 0$ to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical result obtained at a) and b).

d) What will happen with the fish in the case that $c = 2$?

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2015

1. (1.5p) We consider the linear planar system $\dot{x} = -x,\ \dot{y} = -y$.

a) Find its general solution and its flow.

b) Using the definition of the orbit, find two of its orbits: the ones corresponding to the initial states $\eta = (1, 2)$, and, respectively, $\eta = (-1, -2)$.

c) Find its isocline for the slope $m = 2$. Find its isocline for the slope $m \in \mathbb{R}$. Represent few isoclines and find the shape of the orbits.

d) Represent its phase portrait.

2. (0.5p) The following proposition is true or false? Justify.

"The isoclines of a linear planar system are straight lines that pass through the origin".

3. (1.5) Find the general solution of the differential equations $x' + tx = 2t$ and $x'' + \omega^2 x = 1$ (the unknown denoted $x(t)$ and the parameter $\omega > 0$) and of the difference equation $x_{k+1} = 3\,x_k - 4$.

4. (2p) Using the stair-step diagram, estimate the basin of attraction for each of the fixed points (if there is any which is an attractor) of the map

$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x^2 + 5}{2x} .$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2015

1. (1.5p) Represent the phase portrait of the scalar dynamical system
$\dot{x} = x(1 - x^2)$. Find $\varphi(t, 1)$ and justify. Specify the monotony of $\varphi(t, 2)$ and, respectively, $\varphi(t, 0.5)$.

2. (0.25p) Find the polar coordinates of the points whose cartesian coordinates are: $(1, 0)$, $(0, 1)$, $(-2, 0)$ and $(0, -0.5)$, respectively.

3. (2.5p) We consider the planar system
$\dot{x} = -y + x(1 - x^2 - y^2),\ \dot{y} = x + y(1 - x^2 - y^2)$.
a) Study the type and stability of the equilibrium point $(0, 0)$ using the linearization method. There are other equilibria?
b) Transform the given system to polar coordinates.
c) What is the shape of the orbit corresponding to: $\varphi(t, 1, 0)$, $\varphi(t, 0, 1)$, $\varphi(t, -2, 0)$ and $\varphi(t, 0, -0.5)$, respectively? Justify.
d) What remarkable property has the function $\varphi(t, 1, 0)$?

4. (1.25p) Find all the solutions of each of the following difference equations and which also satisfies the given conditions: a) $x_{k+2} - 5\,x_{k+1} + 6x_k = 12$; b) $x_{k+1} = 1 - x_k^2,\ x_0 = 0$; c) $x_{k+2} + x_{k+1} + x_k = 0,\ x_0 = 0$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2015

1. (1.5p) Find the linear homogeneous differential equation of minimal order that has as solutions:

a) $t\,e^{2t}$ and e^{-t};

b) $\cos(\omega t)$ and $3\sin(\omega t)$ (here $\omega > 0$).

Find also the general solution of each of these two equations.

2. (2p) We consider the planar Lotka-Volterra system

$\dot{x} = x(1-y), \quad \dot{y} = y(2-x).$

a) Find its equilibria and study their stability using the linearization method.

b) Find a first integral in $(0,\infty)\times(0,\infty)$.

3. (2p) Let $f:\mathbb{R}\to\mathbb{R}$, $f(x) = 2x(1-x)$.

a) Find its fixed points and study their stability.

b) Let $I_1 = (-\infty, 0)$, $I_2 = (0,1)$ and $I_3 = (1,\infty)$. Find $f(I_1)$, $f(I_2)$ and $f(I_3)$.

c) Find the orbits corresponding to the initial states $\eta = 0$ and, respectively, $\eta = 1$.

d) Using the stair-step diagram, describe the long-term behavior of the orbits corresponding to the initial states: $\eta = 1/8$, $\eta = 7/8$, $\eta = -1/8$ and, respectively, $\eta = 9/8$.

e) Estimate the basin of attraction of the stable fixed point of f.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2015

1. (1p)

a) Write the statement of the Fundamental Theorem for linear homogeneous second order differential equations.

b) The following proposition is true or false? Justify. We remind you that $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$.

"The general solution of the differential equation $x'' - x = 0$ is
$x(t) = c_1 \cosh t + c_2 \sinh t$, where c_1, c_2 are arbitrary real constants."

2. (1p) Find a first integral in $\mathbb{R}^2$ of the pendulum equation

$$\ddot{\theta} + \omega^2 \sin \theta = 0.$$

Hint: Write first the planar system equivalent to this equation.

3. (0.75p) Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k,\ x_0 = 0,\ x_1 = 1.$$

4. (2p) We consider the following nonlinear planar systems
$\dot{x} = -x + xy, \quad \dot{y} = -2y + 3y^2.$

a) Find its equilibria and study their stability using the linearization method.

b) Find $\varphi(t, 0, 2/3)$, $\varphi(t, 4, 0)$ and $\varphi(t, 1, 2/3)$.

5. (0.75p) We consider the IVP $y' = 1 + xy^2$, $y(0) = 0$. Write the Euler numerical formula on the interval $[0, 1]$ with step-size $h = 0.02$. Specify the initial values and the number of steps necessary to find the approximate value of $\varphi(0.5)$ and, respectively, of $\varphi(1)$. Here with φ is denoted the exact solution of the given IVP.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June, 2015

1. (1.25p) For each $k > 0$ we consider the differential equation $\dot{x} = -k\,(x - 21)$, which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time t.

a) Find its flow.

b) An experiment revealed the following fact. A cup of tea with initial temperature of $49°C$ has a temperature of $37°C$ after 10 minutes. Find the initial temperature of a cup of tea such that after 20 minutes the tea has $37°C$.

2. (2.5p) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$.

(a) Find the fixed points of f and study their stability using the linearization method.

(b) Represent the graph of f and find geometrically the fixed points of f.

(c) Find the orbit corresponding to the initial state 0? What remarkable property has this orbit? Depict this orbit using the stair-step diagram.

d) Let $\eta = 2$, and, respectively, $\eta = -1/4$. Find $f(\eta)$ and $f^2(\eta)$ (here f^2 denotes the second iterate of f). Using the stair-step diagram describe the long-term behavior of the orbit corresponding to the initial state η .

3. (1.75p) For what values of the real parameter a the system

$\dot{x} = ax - 5y,\ \dot{y} = x - 2y$ has a center at the origin?

For $a = 0$ find the general solution of this system and specify its type and stability.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
July, 2015

1. (1.5p) We consider the differential equation

$$x'' + 9x = \cos 3t.$$

a) Find a solution of the form $x_p = t(a\cos 3t + b\sin 3t)$, with $a, b \in \mathbb{R}$.
b) Find its general solution.
c) Describe the motion of a spring-mass system governed by this equation.

2. (1p) Find the solution of

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k, \quad x_0 = 0, \quad x_1 = 0.$$

Hint: look for $a, b \in \mathbb{R}$ such that $(x_k)_p = ak + b$ is a particular solution of the difference equation.

3. (1.5p) We consider the IVP $x' = -200x, \quad x(0) = 1$.
a) Find the solution and its limit as $t \to \infty$.
b) Write the Euler's numerical formula with constant step-size h.
c) Find a range of values for the step-size h such that the solution $(x_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} x_k = 0$.

4. (1.5p) Find the fixed points and the 2–periodic points of the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - 2x^2$. Study the stability of the fixed points.

ST1. (1p) Find the general solution of the scalar differential equation $x' - ax = at - 1$, where the unknown is the function x of variable t and $a \in \mathbb{R}^*$ is a fixed parameter.

ST2. (1p) We consider the scalar differential equation

$$(*) \quad \dot{x} = 2x(2 - x),$$

whose unknown is the function x of variable t. We denote by $\varphi(t;\eta)$ the flow of $(*)$.

a) For $(*)$, find its equilibria, its orbits, depict its phase portrait, and study the stability of its equilibria.

b) Find $\lim\limits_{t\to\infty} \varphi(t;1)$.

c) There exists some $\eta \in \mathbb{R}$ such that $\lim\limits_{t\to\infty} \varphi(t;\eta) = 3$?

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
July 12, 2014

1. (1p) Find the linear homogeneous differential equation with constant real coefficients, of minimal order, which has as solution the function

$$\cos 3t.$$

2. (2p) We consider the scalar differential equation

$$(*)\quad \dot{x} = 2x(2-x),$$

whose unknown is the function x of variable t. We denote by $\varphi(t;\eta)$ the solution of $(*)$ satisfying $x(0)=\eta$.

a) For $(*)$, find its equilibria, its orbits, depict its phase portrait, and study the stability of its equilibria.

b) Find $\lim\limits_{t\to\infty} \varphi(t;1)$.

c) There exists some $\eta \in \mathbb{R}$ such that $\lim\limits_{t\to\infty} \varphi(t;\eta) = 3$?

3. (2.5p) Let $f : (0,\infty) \to \mathbb{R}$,

$$f(x) = \frac{1}{2}\left(x + \frac{3}{x}\right).$$

Fix an arbitrary $x_0 \in (0,\infty)$ and consider the sequence $(x_k)_{k\geq 0}$ satisfying the recurrence

$$x_{k+1} = f(x_k), \text{ for any } k \geq 0.$$

a) Prove that $f(x) \in [\sqrt{3},\infty)$ for any $x \in (0,\infty)$ and that $x_k \geq \sqrt{3}$ for any $k \geq 1$.

b) Prove that $x_{k+1} - x_k \leq 0$ for any $k \geq 1$.

b) Find the fixed points of f and study their stability.

c) Prove that the sequence $(x_k)_{k\geq 0}$ is convergent and $\lim\limits_{k\to\infty} x_k = \sqrt{3}$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 2014 - II

1. (1p) Find the general solution of the differential equation

$$x^2u'' - 6xu' + 10u = 0,$$

whose unknown is the function u of variable x. Hint: look for solutions of the form $u = x^r$, with $r \in \mathbb{R}$.

2. (2p) a) Write the general form of a second order linear differential equation. Formulate the Initial Value Problem for these type of equations, and write the statement of the Existence and Uniqueness Theorem for it.

b) How many solutions have each of the following problems:

$$\begin{array}{ll} \text{(i)} & x'' + t^2x = 0, \quad x(0) = 0; \\ \text{(ii)} & x'' + t^2x = 0, \quad x(0) = 0, \quad x'(0) = 0; \\ \text{(iii)} & x'' + t^2x = 0, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 1? \end{array}$$

3. (2.5p) Find the solution of each of the following difference equations and describe its long term behavior:

$$\begin{array}{ll} \text{(i)} & x_{k+1} = \dfrac{1}{5}x_k, \quad x_0 = 2; \\ \text{(ii)} & x_{k+1} = \dfrac{1}{5}x_k + 1, \quad x_0 = \dfrac{5}{4}; \\ \text{(iii)} & x_{k+1} = \dfrac{1}{5}x_k + 1, \quad x_0 = 2; \\ \text{(iv)} & x_{k+2} = x_{k+1} + x_k, \quad x_0 = 0, \quad x_1 = 1. \end{array}$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 2014 - III

1. (1p) Find the linear homogeneous difference equation with constant coefficients, of minimal order, which has as solutions the two sequences

$$1,\ \frac{1}{2}\ ,\ \frac{1}{2^2}\ ,\ \frac{1}{2^3}\ ,\ \frac{1}{2^4}\ ,\ \frac{1}{2^5}\ ,\ \dots$$

and

$$1,\ -\frac{1}{2}\ ,\ \frac{1}{2^2}\ ,\ -\frac{1}{2^3}\ ,\ \frac{1}{2^4}\ ,\ -\frac{1}{2^5}\ ,\ \dots\ .$$

2. (1.5p) We consider the scalar difference equation

$$x_{k+1} = x_k + \lambda x_k(2 - x_k),$$

whose unknown is the sequence $(x_k)_{k\geq 0}$, and where $\lambda \in (0,1)$ is a parameter. Find its fixed points and study their stability. Discuss with respect to the parameter λ.

3. (3p) We consider the planar systems

$$(*)\left\{\begin{array}{l} x' = -2x \\ y' = x - \sqrt{5}y \end{array}\right. \qquad \text{and} \qquad (**)\left\{\begin{array}{l} x' = -2x \\ y' = x + 3x^2 - \sqrt{5}(y + y^3) \end{array}\right. .$$

a) Find the general solution of (*). For any $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, find the solution, denoted $\varphi(t;\eta)$, of (*) satisfying $x(0) = \eta_1$, $y(0) = \eta_2$.

b) For any $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, find $\lim\limits_{t\to\infty} \varphi(t;\eta)$.

c) For system (**), find its equilibria and study their stability.

d) For any $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, denote by $\psi(t;\eta)$ the solution of (**) satisfying $x(0) = \eta_1$, $y(0) = \eta_2$. What can be deduced from c) about $\lim\limits_{t\to\infty} \psi(t;\eta)$?

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Exam on Dynamical Systems
June 2014 - IV

1. (1.5p) Find the general solution of the difference equation

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k,$$

Hint: look for a particular solution of the form $(x_k)_p = ak + b$, with $a, b \in \mathbb{R}$.

2. (1p) How many solutions have each of the following problems:

$$\begin{array}{rl} \text{(i)} & x_{k+2} + k^2 x_k = 0, \quad x_0 = 0; \\ \text{(ii)} & x_{k+2} + k^2 x_k = 0, \quad x_0 = 0, \quad x_1 = 0; \\ \text{(iii)} & x_{k+2} + k^2 x_k = 0, \quad x_0 = 0, \quad x_1 = 0, \quad x_2 = 1? \end{array}$$

3. (1.5p) Find the general solution of each of the following differential equations and describe the long term behavior of such a solution:

$$\begin{array}{rl} \text{(i)} & x' = -5x; \\ \text{(ii)} & x' = -5x + 1; \\ \text{(iii)} & x'' + x' + x = 0. \end{array}$$

4. (1.5p) Find the fixed points and the 2–periodic points of the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - 2x^2$.

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Exam on Dynamical Systems
June 2014 - V

1. (2.5p) We consider the planar differential system

$$x' = -4y, \quad y' = x \ .$$

a) Find its general solution.
b) Specify the type and stability of this linear system.
c) Represent its phase portrait. What type of curves are the orbits?

2. (1.5p) We consider the initial value problem

$$y' = 1 - xy^2, \quad y(0) = 0.$$

Write the Euler numerical formula for this IVP, in the interval $[0, 1]$, with constant stepsize $h = 0.02$. Compute the approximate values in $x = 0.02$ and, respectively $x = 0.04$. How many steps there are needed in order to calculate the approximate value in $x = 1$?

3. (1.5p) We consider the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - \frac{1}{4}(x^2 - 2)$ and, given $x_0 \in \mathbb{R}$, consider the sequence $(x_k)_{k\geq 0}$ satisfying the recurrence

$$x_{k+1} = f(x_k) \ .$$

a) Find the fixed points of f, and study their stability.
b) Find $(x_k)_{k\geq 0}$ when $x_0 = \sqrt{2}$.
c) There exists some $x_0 \in \mathbb{R} \setminus \{\sqrt{2}\}$ such that $\lim\limits_{k\to\infty} x_k = \sqrt{2}$?
d) There exists some $x_0 \in \mathbb{R}$ such that $\lim\limits_{k\to\infty} x_k = 2$?

Universitatea Babeş–Bolyai
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Exam on Dynamical Systems
June 2014 - VI

1. (2p) We consider the differential equation

$$x'' + 9x = \cos 3t.$$

a) Find a solution of the form $x_p = t(a\cos 3t + b\sin 3t)$, with $a, b \in \mathbb{R}$.
b) Find its general solution.
c) Describe the motion of a spring-mass system governed by this equation.

2. (2p) We consider the differential system

$$x' = x - y, \quad y' = x + y.$$

a) Find the type and stability of this linear system.
b) Pass to polar coordinates, i.e. find the differential system in the unknowns $(\rho(t), \theta(t))$ when

$$x(t) = \rho(t)\cos\theta(t), \quad y(t) = \rho(t)\sin\theta(t).$$

c) What type of curves are the orbits?

3. (0.75p) We consider the difference equation $x_{k+1} = -2x_k + 3^k$.
a) Find a solution of the form $x_k = a3^k$, with $a \in \mathbb{R}$.
b) Find its general solution.
c) Find the solution with $x_0 = 0$.

4. (0.75p) Write the definition of a fixed point and, respectively, of a 2-periodic point for some map $f : \mathbb{R} \to \mathbb{R}$.

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Exam on Dynamical Systems
June 2014 - VII

1. (2.5p) We consider the difference equation

$$x_{k+2} + x_k = \cos\frac{k\pi}{2}\ .$$

a) Find a solution of the form $(x_k)_p = ak\cos\frac{k\pi}{2}$, with $a \in \mathbb{R}$. (Hint: we remind that $\cos(x+\pi) = -\cos x$ for any $x \in \mathbb{R}$)
b) Find its general solution.
c) Find the solution with $x_0 = x_1 = 0$ and describe its long-time behavior.

2. (2p) We consider the difference system

$$x_{k+1} = \frac{3}{5}x_k + \frac{1}{5}y_k, \quad y_{k+1} = \frac{1}{5}x_k + \frac{3}{5}y_k.$$

a) Study the stability of this linear system.
b) Find the general solution.

3. (1p) We consider the differential equation $x' = -2x + e^{3t}$.
a) Find a solution of the form $x_p = ae^{3t}$, with $a \in \mathbb{R}$.
b) Find its general solution.
c) Find the solution with $x(0) = 0$.

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Exam on Dynamical Systems
June 2014 - VIII

1. (1.5p) Find the general solution of
(i) $x' + 4x = 0$; (ii) $x' + tx = 0$; (iii) $x'' + 2x' + x = 0$.
Here the unknown is the function x of variable t.

2. (1p) We consider the differential equation

$$y' = 1 - \frac{x}{y^2} \; .$$

Compute the slope of its direction field in the points $(0, 1)$ and, respectively, $(1, 1)$. What type of curve is the 1-isocline, respectively, the 0-isocline?

3. (1p) Find (directly) the solution of

$$x_{k+2} + x_{k+1} + x_k = 0, \quad x_0 = 0, \quad x_1 = 1 \; .$$

Describe its long-term behavior.

4. (2p) We consider the IVP $x' = -200x, \quad x(0) = 1$.
a) Find the solution and its limit as $t \to \infty$.
b) Write the Euler's numerical formula with constant step-size h.
c) Find a range of values for the step-size h such that the solution $(x_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} x_k = 0$.

Universitatea Babeş–Bolyai
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Exam on Dynamical Systems
June 19, 2013

1. Find the general solution of each of the following equations.
a) $x' - tx = t$, b) $t^2 x'' - 3tx' + 3x = 0$, $t \in (0, \infty)$.
(Hint: At b) look for solutions of the form $x = t^r$ with $r \in \mathbb{R}$).

2. Describe the motion of a spring-mass system whose equation is
$x'' + k/m\ x = 0$, where $k, m > 0$.

3. We consider the IVP $\dot{x} = -200\ x$, $x(0) = 1$.
a) Find the solution and its limit as $t \to \infty$.
b) Write the Euler's numerical formula with constant step-size h.
c) Find a range of values for the step-size h such that the solution $(x_k)_{k \geq 0}$ of the difference equation found at b) satisfies $\lim_{k\to\infty} x_k = 0$.

Universitatea Babeş–Bolyai
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Exam on Dynamical Systems
June 12, 2013

1. Find the solution of the IVP

$$x'' + 4x' + 5x = 0, \quad x(0) = 1, \quad x'(0) = -2.$$

Represent the corresponding integral curve and describe its long-term behavior.

2. Find (directly) the solution of

$$x_{k+2} + x_{k+1} + x_k = 0, \quad x_0 = 0, \quad x_1 = 1.$$

3. Study the stability of the linear difference system

$$x_{k+1} = \frac{1}{3}x_k - \frac{1}{3}y_k, \quad y_{k+1} = \frac{1}{3}x_k + \frac{1}{3}y_k.$$

4. Let $k, t_0, x_0 \in \mathbb{R}$ be fixed parameters. Find the solution of the IVP

$$x' = k(21 - x), \quad x(t_0) = x_0.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Define the notions of fixed point and p-periodic point of the map f. Which is the corresponding orbit in each case?

Find the fixed points of $f(x) = 3x(1 - x)$.

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Exam on Dynamical Systems
June 14, 2013

1. Find the solution of each of the following equations and its limit as $k \to \infty$.

a) $x_{k+2} + x_{k+1} - 2x_k = 0, \quad x_0 = 1, \quad x_1 = 1.$
b) $x_{k+2} - 6x_{k+1} + 9x_k = 0, \quad x_0 = 0, \quad x_1 = 1.$
c) $4x_{k+2} - 2\sqrt{2}x_{k+1} + x_k = 0.$

2. We consider the linear differential system $\dot{x} = -2y, \quad \dot{y} = x.$

a) Find its general solution.
b) Represent its phase portrait. Find a first integral.
c) What is the stability character of this system?

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Exam on Dynamical Systems
June 22, 2013

1. We consider the equation $x'' - x = te^{-2t}$.

a) Find a particular solution of the form $x_p(t) = (at + b)e^{-2t}$, where $a, b \in \mathbb{R}$.

b) Find the general solution.

c) Find the solution that satisfies the initial conditions $x(0) = 0$, $x'(0) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x - (x^2 - 2)/4$.

a) Find the fixed points of f and study their stability.

b) What can we say about the solution of $x_{k+1} = f(x_k)$ with $x_0 = \sqrt{2}$ and, respectively, with $x_0 \in \mathbb{R}$ such that $|x_0 - \sqrt{2}|$ is sufficiently small?

3. Find the general solution of $dy/dx = -2y/x$.

Universitatea Babeş–Bolyai
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Exam on Dynamical Systems.
June 11, 2012

1. Represent the phase portrait of $\dot{x} = (N - x)x - c$ where $N > 0$ is a fixed constant related with the birth rate of some population of fishes in a lake. Discuss with respect to the parameter $c \geq 0$ that represents the fishing rate. Interpret the results.

2. We consider the linear differential system $\dot{x} = y, \quad \dot{y} = -4x$.
a) Show that all its solutions are periodic with the same principal period.
b) Represent its phase portrait.
c) Find a real function $H(x, y)$ that takes constant value on each orbit.

3. Study the long term behavior of the solution of the IVP $x' = -120x, \quad x(0) = x_0$, (where $x_0 > 0$) and of the solution of the corresponding difference equation obtained by Euler's numerical formula. What is the largest safe stepsize in this numerical integration?

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Exam on Dynamical Systems.
June 12, 2012

1. Find the general solution of the following equations.
a) $x' - 3t^2 x = t^3$; b) $x' - 3t^2 x = f(t)$ where $f \in C(\mathbb{R})$;
b) $x_{k+1} + 3x_k = 0$; c) $x_{k+1} + 3x_k = b$ where $b \in \mathbb{R}$.

2. a) Study the stability of the equilibria of the differential equation $\dot{x} = \frac{1}{2}(x^2 - a)$ where $a > 0$. Represent the phase portrait.
b) Study the stability of the fixed points of the difference equation $x_{k+1} = \frac{1}{2}(x_k + \frac{a}{x_k})$ where $a > 0$. Represent the stair-step diagram.

3. We consider the IVP $y' = 2xy^2 + x^3, x \in [0, 1]$, $y(0) = 0$.
a) Describe the corresponding Euler's numerical algorithm.
b) Write the recurrence formula for the Picard sequence of successive approximations $(\varphi_n)_{n \geq 0}$. Starting with $\varphi_0(x) \equiv 0$ calculate $\varphi_1(x)$.

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Exam on Dynamical Systems.
June 13, 2012

1. Find the general solution of the following equations.
a) $x' = 2x$; b) $x_{k+1} = 2x_k$; c) $x'' - x' - 6x = 3t - \sin 2t$.

2. a) For what values of the real parameter a the system
$\dot{x} = ax - 5y, \quad \dot{y} = x - 2y$ has a center at the origin?
b) Find the equilibria and study their stability for
$\dot{x} = 1 - xy, \quad \dot{y} = x - y^2$.

3. Find the fixed points and the periodic points of minimal period 2 for the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - 2x^2$. Study their stability.

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Exam on Dynamical Systems.
July 2, 2012

1. Find the general solution of the following equations.
a) $x' = \lambda x$, where $\lambda \in \mathbb{R}$ is a fixed parameter; b) $x' = tx$;
c) $x''' - x'' = 0$; d) $t^2 x'' - 3tx' + 3x = 0$.

2. Find the equilibria and study their stability for the pendulum equation $\ddot{\theta} + \frac{\nu}{m}\dot{\theta} + \frac{g}{l}\sin\theta = 0$ where all the parameters are positive real numbers.

3. a) Prove that the fixed point $(0,0)$ of the difference system
$x_{k+1} = \frac{3}{5}x_k + \frac{1}{5}y_k, \quad y_{k+1} = \frac{1}{5}x_k + \frac{3}{5}y_k$ is asymptotically stable.
b) We consider the Fibonacci sequence $(x_k)_{k\geq 0}$ satisfying
$x_{k+2} = x_{k+1} + x_k$ for any $k \geq 0$.
Find its general solution and study the stability of its fixed point 0.

4. (instead of the test seminar) Represent the phase portrait of the scalar differential equation $\dot{x} = 2x - x^2$ and of the planar differential system
$\dot{x} = -x, \quad \dot{y} = y.$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems.
June 08, 2011

1. (0.5p) Find the solution of the initial value problem

$$y' = 2xy, \quad y(0) = -2.$$

2. (2.5p) We consider the differential equation $x'' + 2ax' + 4x = 0$, where $a > 0$ is a real parameter. Write the general solution and describe the long-term behavior of the solutions (for $t \in (0, \infty)$). Discuss with respect to the parameter a.

3. (1p) Write the statement of The Superposition Principle. Give an example.

4. We consider the scalar differential equation $\dot{x} = -x^2 + x + 2$.
a) (1p) Represent its phase portrait.
b) (0.5p) Denote by $\varphi(t)$ its solution with $x(0) = 3$. Find $\lim\limits_{t\to\infty} \varphi(t)$.
c) (0.5p) Write the Euler numerical formula with constant step size h for this differential equation.

Universitatea Babeş–Bolyai
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Exam on Dynamical Systems.
June 10, 2011

1. Find the general solution of the following differential equations
a) (1p) $x' - 3x = 5t$.

b) (1p) $x' = y, \quad y' = -x - 2y$.

c) (0.5p) $y' = 2y/x$.

2. a) (0.5p) Write the statement of the existence and uniqueness theorem for first order nonlinear differential equations.

b) (0.5p) Prove that the Initial Value Problem $y' = 1 - 4y^2, y(0) = 1/2$ has a unique solution and than name it.

c) (0.5p) Give an example of an Initial Value Problem for which existence and uniqueness theorem is not applicable. Justify.

3. (2p) We consider the nonlinear planar autonomous system

$$\dot{x} = -x + xy, \quad \dot{y} = -4y + 8xy.$$

Find its equilibria and study their stability.

Universitatea Babeş–Bolyai
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Exam on Dynamical Systems.
June 24, 2011

1. (2.5p) We say that a differential equation exhibit resonance when all its solutions are unbounded.

For what values of the mass m will $mx'' + 25x = 12\cos(36\pi t)$ exhibit resonance?

2. (0.75p) Find the general solution of the following differential equation

$$t^2x'' - 3tx' + 3x = 0.$$

3. We consider the differential system $\quad x' = -x, \quad y' = -3y.$

a) (0.5p) Find its general solution.

b) (0.5p) What is the type of its equilibrium point $(0, 0)$?

c) (0.5p) Find a first integral.

d) (0.5p) Represent its phase portrait.

4. (0.75p) Write the statement of the Stability Theorem in First Order Approximation for an equilibrium point of a nonlinear planar system.

Universitatea Babeş–Bolyai
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Exam on Dynamical Systems.
July 09, 2011

1. (1.5p) Find the general solution of the following differential equation $x'' - x = e^{at}$. Discuss with respect to the real parameter a.

2. (0.5p) The classification of the singular point of a linear planar autonomous system.

3. We consider the differential equation $y' = -2x/y$.

a) (1p) Represent its direction field (hint: represent the 1, -1, 0, 2, -2 isoclines together with the corresponding directions).

b) (0.5p) Find a first integral.

c) (0.5p) Write the Euler numerical formula with constant step size h for this differential equation.

4. (2p) Study the stability of the equilibria at the positions $\theta = 0$, and $\theta = \pi$, respectively, of the differential equation $\ddot{\theta} + 4\dot{\theta} + \sin\theta = 0$.

5. (1p) (not compulsory) Find the general solution of the differential equation $t^2x'' - 3tx' + 4x = 0$.