LECTURE

9

LIMITS AND CONTINUITY OF REAL FUNCTIONS OF SEVERAL VARIABLES

Real-valued functions of several variables

Definition 9.1 *By a real-valued function of n real variables we mean any function $f : A \to \mathbb{R}$ defined on some nonempty set $A \subseteq \mathbb{R}^n$.*

Example 9.2 *(i) Let $f : A = \mathbb{R}^n \to \mathbb{R}$ be the Euclidean norm function*

$$f(x) = \sqrt{(x_1)^2 + \ldots + (x_n)^2},\ \forall\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Here $f(x)$ represents the distance between x and 0_n.
(ii) Let $f : A = (0, +\infty) \times (0, +\infty) \subseteq \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = \pi(x_1)^2 x_2,\ \forall\, x = (x_1, x_2) \in (0, +\infty) \times (0, +\infty).$$

In this case, $f(x) = f(x_1, x_2)$ represents the volume of a right circular cylinder of radius x_1 and height x_2.

Remark 9.3 *The graph of a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is a subset of $\mathbb{R}^n \times \simeq \mathbb{R}\mathbb{R}^{n+1}$, namely*

$$\begin{aligned}\operatorname{graph}(f) &:= \{(x, f(x)) \mid x \in A\}\\ &\simeq \{(x_1, \ldots, x_n, x_{n+1}) \mid x = (x_1, \ldots, x_n) \in A,\ x_{n+1} = f(x)\}.\end{aligned}$$

Limits of functions of several variables

Definition 9.4 *Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}$, $c \in A'$ and $\ell \in \overline{\mathbb{R}}$. We say that f has limit ℓ at c if*

$$\forall V \in \mathcal{V}(\ell), \exists U \in \mathcal{V}(c) \text{ such that } \forall x \in U \cap (A \setminus \{c\}) \text{ we have } f(x) \in V.$$

Remark 9.5 *Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}$, $c \in A'$.*
(i) f has at most one limit at c.
(ii) f has a limit at c if $\exists \ell \in \overline{\mathbb{R}}$, $\forall V \in \mathcal{V}(\ell)$, $\exists U \in \mathcal{V}(c)$, $\forall x \in U \cap (A \setminus \{c\})$, $f(x) \in V$. In this case ℓ is called the limit of f at c and we write

$$\lim_{x \to c} f(x) = \ell \quad \text{or} \quad \lim_{\substack{x_1 \to c_1 \\ \vdots \\ x_n \to c_n}} f(x_1, \ldots, x_n) = \ell \quad \text{or} \quad f(x) \to \ell \text{ as } x \to c.$$

We also say that $f(x)$ approaches ℓ as x approaches c.

Theorem 9.6 (ε-δ characterization of limits) *Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}$, $c \in A'$, $\ell \in \mathbb{R}$. Then*
(i) $\lim_{x \to c} f(x) = \ell \iff \forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall x \in A, 0 < \|x - c\| < \delta_\varepsilon, |f(x) - \ell| < \varepsilon.$
(ii) $\lim_{x \to c} f(x) = +\infty \iff \forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall x \in A, 0 < \|x - c\| < \delta_\varepsilon, f(x) > \varepsilon.$
(iii) $\lim_{x \to c} f(x) = -\infty \iff \forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall x \in A, 0 < \|x - c\| < \delta_\varepsilon, f(x) < -\varepsilon.$

Theorem 9.7 (Sequential characterization of limits, Heine) *Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}$, $c \in A'$, $\ell \in \overline{\mathbb{R}}$. Then the following assertions are equivalent:*
1° $\lim_{x \to c} f(x) = \ell$.
2° *For any sequence of points (x^k) in $A \setminus \{c\}$ which has the limit $\lim_{k \to \infty} x^k = c$, the sequence of real numbers $\big(f(x^k)\big)$ has the limit $\lim_{k \to \infty} f(x^k) = \ell$.*

Remark 9.8 *(i) Since, by the above result, limits of functions of several variables can be characterized using limits of sequences, limit theorems and rules for functions of several variables can be derived from corresponding ones for sequences. For instance, there exists a Sandwich Theorem for functions of several variables.*
(ii) If the limit L exists, then the same value L for the limit must be obtained along all paths to c. If there are paths to c which do not yield the same value, then the limit does not exist.

Example 9.9 *Let $A = (0, +\infty) \times \mathbb{R} \setminus \{(1, 0)\}$, $f : A \to \mathbb{R}$, $f(x, y) = \dfrac{(x-1)^2 \ln x}{(x-1)^2 + y^2}$. Note that $(1, 0) \in A'$. Then $\lim_{(x,y) \to (1,0)} f(x, y) = 0$.*

Notice that for any $(x, y) \in A$,

$$0 \leq \left| \frac{(x-1)^2 \ln x}{(x-1)^2 + y^2} \right| \leq |\ln x|$$

and $\lim_{(x,y) \to (1,0)} |\ln x| = 0$. Apply the Sandwich Theorem.

Example 9.10 *Let $f : \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$, $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Notice that $0_2 \in (\mathbb{R}^2 \setminus \{0_2\})'$. We claim that f has no limit at 0_2.*
Indeed, consider the sequences (a^k) and (b^k) defined for $k \in \mathbb{N}^*$ by $a^k = (1/k, 1/k)$ and $b^k = (1/k, 0)$. Then $\lim_{k \to \infty} a^k = 0_2 = \lim_{k \to \infty} b^k$, but $\lim_{k \to \infty} f(a^k) = 0$ and $\lim_{k \to \infty} f(b^k) = 1$. By Theorem 9.7, f does not have a limit at 0_2.

Definition 9.11 *Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}$, $c \in A$. We say that f is continuous at c if*

$$\forall V \in \mathcal{V}(f(c)), \exists U \in \mathcal{V}(c) \text{ such that } \forall x \in U \cap A \text{ we have } f(x) \in V.$$

If B is a subset of A, we say that f is continuous on B if it is continuous at every point of B. If f is continuous on A, then f is simply called continuous.

Theorem 9.12 (Characterizations of continuity) *Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}$, $c \in A$. Then f is continuous at c if and only if one of the following conditions are met:*
(i) $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$, $\forall x \in A$, $\|x - c\| < \delta$, $|f(x) - f(c)| < \varepsilon$.
(ii) $\forall$ sequence (x^k) in A with $\lim\limits_{k\to\infty} x^k = c$ we have that $\lim\limits_{k\to\infty} f(x^k) = f(c)$.

Example 9.13 *(i) Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2. \end{cases}$*
We show that f is continuous at 0_2.
Indeed, let $\varepsilon > 0$. Take $\delta = \sqrt{\varepsilon}$. Then, $\|(x,y) - 0_2\| = \sqrt{x^2+y^2} < \delta$ yields

$$|f(x,y) - f(0_2)| = \left|(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}\right| \leq x^2+y^2 < \delta^2 = \varepsilon.$$

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2. \end{cases}$
We show that f is not continuous at 0_2.
Indeed, consider the sequence (a^k) defined for $k \in \mathbb{N}^*$ by $a^k = (1/k, 1/k)$. Then $\lim\limits_{k\to\infty} a^k = 0_2$ and $\lim\limits_{k\to\infty} f(a^k) = 1/2$. Since $f(0_2) = 0$, it follows that f is not continuous at 0_2.

Theorem 9.14 *Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}$, $a \in A$, $f : A \to B$ and $g : B \to \mathbb{R}$. If f is continuous at $a \in A$ an g is continuous at $f(a)$, then $g \circ f : A \to \mathbb{R}$ is continuous at a.*

Remark 9.15 *(i) Polynomials in n several variables are continuous on $\mathbb{R}^n$.*
$n = 2$: $P(x,y) = xy^2 + 7x^3y + y - 3$.
$n = 3$: $P(x,y,z) = 4x^2y^3 + 3x^2y^2z^2 - 5x + 4z + 1$.
(ii) Rational functions (a quotient of two polynomials) are continuous on their maximal domain of definition.
$n = 2$: $f : \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 \mid x + y = 0\} \to \mathbb{R}$, $f(x,y) = \dfrac{x^2+5y}{x+y}$.
$n = 3$: $f : \mathbb{R}^3 \setminus \{0_3\} \to \mathbb{R}$, $f(x,y,z) = \dfrac{x^3+x-y}{x^2+y^2+z^2}$.
(iii) Sums, products and quotients (when defined) of continuous real-valued functions of several variables are continuous.
(iv) One can construct continuous functions of several variables by taking an elementary function g in Theorem 9.14. For instance, let $f : \mathbb{R}^n \to \mathbb{R}$, $f(x_1,\ldots,x_n) = (x_1)^2 + \ldots + (x_n)^2$, $g : [0,\infty) \to \mathbb{R}$, $g(u) = \sqrt{u}$. Then $g \circ f : \mathbb{R}^n \to \mathbb{R}$, $(g \circ f)(x_1,\ldots,x_n) = \sqrt{(x_1)^2 + \ldots + (x_n)^2} = \|(x_1,\ldots,x_n)\|$ is continuous on $\mathbb{R}^n$.