

COURSE 9

4.2. Direct methods for solving linear systems

4.2.3. Factorization methods - LU methods

The matrix A can be factored into the product of a lower triangular matrix L and an upper triangular matrix U , namely $A = LU$.

$$Ax = b \iff LUx = b,$$

where

$$L = \begin{pmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \vdots & & & \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix} \quad U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & & & \\ 0 & 0 & \dots & u_{nn} \end{pmatrix}.$$

We solve the systems in two stages:

First stage: Solve $Lz = b$,

Second stage: Solve $Ux = z$.

Methods for computing matrices L and U : **Doolittle method** where all diagonal elements of L have to be 1; **Crout method** where all diagonal elements of U have to be 1 and **Choleski method** where $l_{ii} = u_{ii}$ for $i = 1, \dots, n$.

Remark 1 *LU factorizations are modified forms of Gauss elimination method.*

Doolittle method

We consider that A is a strictly diagonally dominant matrix, so $a_{kk} \neq 0$, $k = \overline{1, n-1}$. Denote

$$l_{i,k} := \frac{a_{i,k}^{(k-1)}}{a_{k,k}^{(k-1)}}, \quad i = \overline{k+1, n}$$

$$t^{(k)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{k+1,k} \\ \dots \\ l_{n,k} \end{bmatrix},$$

having zeros for the first k -th lines, and

$$M_k = I_n - t^{(k)} e_k \in \mathcal{M}_{n \times n}(\mathbb{R}) \quad (1)$$

where $e_k = \begin{pmatrix} 0 & \dots & 1 & \dots & 0 \end{pmatrix}$ is the k -unit vector of dimension n , (has

1 on the k -th position) and $I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & 1 \end{pmatrix}$ is the identity matrix of order n .

$a_{i,k}^{(0)}$ are elements of A ; $a_{i,k}^{(1)}$ are elements of $M_1 \cdot A$; ...; $a_{i,k}^{(k-1)}$ are elements of $M_{k-1} \dots \cdot M_1 \cdot A$.

Definition 2 The matrix M_k is called **the Gauss matrix**, the components $l_{i,k}$ are called **the Gauss multiplies** and the vector $t^{(k)}$ is **the Gauss vector**.

Remark 3 If $A \in \mathcal{M}_{n \times n}(\mathbb{R})$, then the Gauss matrices M_1, \dots, M_{n-1} can be determined such that

$$U = M_{n-1} \cdot M_{n-2} \dots M_2 \cdot M_1 \cdot A$$

is an upper triangular matrix. Moreover, if we choose

$$L = M_1^{-1} \cdot M_2^{-1} \dots M_{n-1}^{-1}$$

then

$$A = L \cdot U.$$

Example 4 *Find LU factorization for the matrix*

$$A = \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}.$$

Solve the system $\begin{cases} 2x_1 + x_2 = 3 \\ 6x_1 + 8x_2 = 9 \end{cases}.$

Sol.

$$\begin{aligned} M_1 &= I_2 - t^{(1)}e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}. \end{aligned}$$

We have

$$U = M_1 A = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$$

$$L = M_1^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

So

$$A = \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} = L \cdot U = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}.$$

We have

$$L \cdot U \cdot x = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

$$Ux = z$$

and

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \Rightarrow z = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}.$$

4.3. Iterative methods for solving linear systems

Because of round-off errors, direct methods become less efficient than iterative methods for large systems ($>100\ 000$ variables).

An iterative scheme for linear systems consists of converting the system

$$Ax = b \tag{2}$$

to the form

$$x = \tilde{b} - Bx.$$

After an initial guess for $x^{(0)}$, the sequence of approximations of the solution $x^{(0)}, x^{(1)}, \dots, x^{(k)}, \dots$ is generated by computing

$$x^{(k)} = \tilde{b} - Bx^{(k-1)}, \quad \text{for } k = 1, 2, 3, \dots$$

4.3.1. Jacobi iterative method

Consider the $n \times n$ linear system,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n, \end{cases}$$

where we assume that the diagonal terms $a_{11}, a_{22}, \dots, a_{nn}$ are all nonzero.

We begin our iterative scheme by solving each equation for one of the variables:

$$\begin{cases} x_1 = u_{12}x_2 + \dots + u_{1n}x_n + c_1 \\ x_2 = u_{21}x_1 + \dots + u_{2n}x_n + c_2 \\ \dots \\ x_n = u_{n1}x_1 + \dots + u_{nn-1}x_{n-1} + c_n, \end{cases}$$

where $u_{ij} = -\frac{a_{ij}}{a_{ii}}$, $c_i = \frac{b_i}{a_{ii}}$, $i = 1, \dots, n$.

Let $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ be an initial approximation of the solution. The $k + 1$ -th approximation is:

$$\begin{cases} x_1^{(k+1)} = u_{12}x_2^{(k)} + \dots + u_{1n}x_n^{(k)} + c_1 \\ x_2^{(k+1)} = u_{21}x_1^{(k)} + u_{23}x_3^{(k)} + \dots + u_{2n}x_n^{(k)} + c_2 \\ \dots \\ x_n^{(k+1)} = u_{n1}x_1^{(k)} + \dots + u_{nn-1}x_{n-1}^{(k)} + c_n, \end{cases}$$

for $k = 0, 1, 2, \dots$

An algorithmic form:

$$x_i^{(k)} = \frac{b_i - \sum_{j=1, j \neq i}^n a_{ij}x_j^{(k-1)}}{a_{ii}}, \quad i = 1, 2, \dots, n, \quad \text{for } k \geq 1.$$

The iterative process is terminated when a convergence criterion is satisfied.

Stopping criteria: $|x^{(k)} - x^{(k-1)}| < \varepsilon$ or $\frac{|x^{(k)} - x^{(k-1)}|}{|x^{(k)}|} < \varepsilon$, with $\varepsilon > 0$ - a prescribed tolerance.

Example 5 Solve the following system using the Jacobi iterative method. Use $\varepsilon = 10^{-3}$ and $x^{(0)} = (0 \ 0 \ 0 \ 0)$ as the starting vector.

$$\begin{cases} 7x_1 - 2x_2 + x_3 &= 17 \\ x_1 - 9x_2 + 3x_3 - x_4 &= 13 \\ 2x_1 + 10x_3 + x_4 &= 15 \\ x_1 - x_2 + x_3 + 6x_4 &= 10. \end{cases}$$

These equations can be rearranged to give

$$x_1 = (17 + 2x_2 - x_3)/7$$

$$x_2 = (-13 + x_1 + 3x_3 - x_4)/9$$

$$x_3 = (15 - 2x_1 - x_4)/10$$

$$x_4 = (10 - x_1 + x_2 - x_3)/6$$

and, for example,

$$x_1^{(1)} = (17 + 2x_2^{(0)} - x_3^{(0)})/7$$

$$x_2^{(1)} = (-13 + x_1^{(0)} + 3x_3^{(0)} - x_4^{(0)})/9$$

$$x_3^{(1)} = (15 - 2x_1^{(0)} - x_4^{(0)})/10$$

$$x_4^{(1)} = (10 - x_1^{(0)} + x_2^{(0)} - x_3^{(0)})/6.$$

Substitute $x^{(0)} = (0, 0, 0, 0)$ into the right-hand side of each of these equations to get

$$x_1^{(1)} = (17 + 2 \cdot 0 - 0)/7 = 2.428\ 571\ 429$$

$$x_2^{(1)} = (-13 + 0 + 3 \cdot 0 - 0)/9 = -1.444\ 444\ 444$$

$$x_3^{(1)} = (15 - 2 \cdot 0 - 0)/10 = 1.5$$

$$x_4^{(1)} = (10 - 0 + 0 - 0)/6 = 1.666\ 666\ 667$$

and so $x^{(1)} = (2.428\ 571\ 429, -1.444\ 444\ 444, 1.5, 1.666\ 666\ 667)$.

The Jacobi iterative process:

$$x_1^{(k+1)} = \left(17 + 2x_2^{(k)} - x_3^{(k)}\right) / 7$$

$$x_2^{(k+1)} = \left(-13 + x_1^{(k)} + 3x_3^{(k)} - x_4^{(k)}\right) / 9$$

$$x_3^{(k+1)} = \left(15 - 2x_1^{(k)} - x_4^{(k)}\right) / 10$$

$$x_4^{(k+1)} = \left(10 - x_1^{(k)} + x_2^{(k)} - x_3^{(k)}\right) / 6, \quad k \geq 1.$$

We obtain a sequence that converges to

$$x^{(9)} = (2.000127203, -1.000100162, 1.000118096, 1.000162172).$$

4.3.2. Gauss-Seidel iterative method

Almost the same as Jacobi method, except that each x -value is improved using the most recent approx. of the other variables.

For a $n \times n$ system, the $k + 1$ -th approximation is:

$$\begin{cases} x_1^{(k+1)} = u_{12}x_2^{(k)} + \dots + u_{1n}x_n^{(k)} + c_1 \\ x_2^{(k+1)} = u_{21}x_1^{(k+1)} + u_{23}x_3^{(k)} + \dots + u_{2n}x_n^{(k)} + c_2 \\ \dots \\ x_n^{(k+1)} = u_{n1}x_1^{(k+1)} + \dots + u_{nn-1}x_{n-1}^{(k+1)} + c_n, \end{cases}$$

with $k = 0, 1, 2, \dots$; $u_{ij} = -\frac{a_{ij}}{a_{ii}}$, $c_i = \frac{b_i}{a_{ii}}$, $i = 1, \dots, n$ (as in Jacobi method).

Algorithmic form:

$$x_i^{(k)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^n a_{ij}x_j^{(k-1)}}{a_{ii}}$$

for each $i = 1, 2, \dots, n$, and for $k \geq 1$.

Stopping criteria: $|x^{(k)} - x^{(k-1)}| < \varepsilon$, or $\frac{|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}|}{|\mathbf{x}^{(k)}|} < \varepsilon$, with ε - a prescribed tolerance, $\varepsilon > 0$.

Remark 6 *Because the new values can be immediately stored in the location that held the old values, the storage requirements for \mathbf{x} with the Gauss-Seidel method is half than that for Jacobi method and the rate of convergence is faster.*

Example 7 *Solve the following system using the Gauss-Seidel iterative method. Use $\varepsilon = 10^{-3}$ and $\mathbf{x}^{(0)} = (0 \ 0 \ 0 \ 0)$ as the starting vector.*

$$\begin{cases} 7x_1 - 2x_2 + x_3 & = 17 \\ x_1 - 9x_2 + 3x_3 - x_4 & = 13 \\ 2x_1 & + 10x_3 + x_4 = 15 \\ x_1 - x_2 + x_3 + 6x_4 & = 10 \end{cases}$$

We have

$$x_1 = (17 + 2x_2 - x_3)/7$$

$$x_2 = (-13 + x_1 + 3x_3 - x_4)/9$$

$$x_3 = (15 - 2x_1 - x_4)/10$$

$$x_4 = (10 - x_1 + x_2 - x_3)/6,$$

and, for example,

$$x_1^{(1)} = (17 + 2x_2^{(0)} - x_3^{(0)})/7$$

$$x_2^{(1)} = (-13 + x_1^{(1)} + 3x_3^{(0)} - x_4^{(0)})/9$$

$$x_3^{(1)} = (15 - 2x_1^{(1)} - x_4^{(0)})/10$$

$$x_4^{(1)} = (10 - x_1^{(1)} + x_2^{(1)} - x_3^{(1)})/6,$$

which provide the following Gauss-Seidel iterative process:

$$x_1^{(k+1)} = \left(17 + 2x_2^{(k)} - x_3^{(k)} \right) / 7$$

$$x_2^{(k+1)} = \left(-13 + x_1^{(k+1)} + 3x_3^{(k)} - x_4^{(k)} \right) / 9$$

$$x_3^{(k+1)} = \left(15 - 2x_1^{(k+1)} - x_4^{(k)} \right) / 10$$

$$x_4^{(k+1)} = \left(10 - x_1^{(k+1)} + x_2^{(k+1)} - x_3^{(k+1)} \right) / 6, \quad \text{for } k \geq 1.$$

Substitute $\mathbf{x}^{(0)} = (0, 0, 0, 0)$ into the right-hand side of each of these equations to get

$$x_1^{(1)} = (17 + 2 \cdot 0 - 0) / 7 = 2.428\ 571\ 429$$

$$x_2^{(1)} = (-13 + 2.428\ 571\ 429 + 3 \cdot 0 - 0) / 9 = -1.1746031746$$

$$x_3^{(1)} = (15 - 2 \cdot 2.428\ 571\ 429 - 0) / 10 = 1.0142857143$$

$$\begin{aligned} x_4^{(1)} &= (10 - 2.428\ 571\ 429 - 1.1746031746 - 1.0142857143) / 6 \\ &= 0.8970899472 \end{aligned}$$

and so

$$\mathbf{x}^{(1)} = (2.428571429 - 1.1746031746, 1.0142857143, 0.8970899472).$$

Similar procedure generates a sequence that converges to

$$\mathbf{x}^{(5)} = (2.000025, -1.000130, 1.000020, 0.999971).$$

4.3.3. Relaxation method

In case of convergence, the Gauss-Seidel method is faster than Jacobi method. The convergence can be more improved using **relaxation method (SOR method)** (SOR=Successive Over Relaxation)

Algorithmic form of the method:

$$x_i^{(k)} = \frac{\omega}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^n a_{ij}x_j^{(k-1)} \right) + (1 - \omega)x_i^{(k-1)}$$

for each $i = 1, 2, \dots, n$, and for $k \geq 1$.

For $0 < \omega < 1$ the procedure is called **under relaxation method**, that can be used to obtain convergence for systems which are not convergent by Gauss-Siedel method.

For $\omega > 1$ the procedure is called **over relaxation method**, that can be used to accelerate the convergence for systems which are convergent by Gauss-Siedel method.

By Kahan's Theorem follows that the method converges for $0 < \omega < 2$.

Remark 8 For $\omega = 1$, relaxation method is Gauss-Seidel method.

Example 9 Solve the following system, using relaxation iterative method. Use $\varepsilon = 10^{-3}$, $\mathbf{x}^{(0)} = (1 \ 1 \ 1)$ and $\omega = 1.25$,

$$\begin{array}{rclcl} 4x_1 & + & 3x_2 & & = & 24 \\ 3x_1 & + & 4x_2 & - & x_3 & = & 30 \\ & & - & x_2 & + & 4x_3 & = & -24 \end{array}$$

We have

$$\begin{aligned} x_1^{(k)} &= 7.5 - 0.937x_2^{(k-1)} - 0.25x_1^{(k-1)} \\ x_2^{(k)} &= 9.375 - 9.375x_1^{(k)} + 0.3125x_3^{(k-1)} - 0.25x_2^{(k-1)} \\ x_3^{(k)} &= -7.5 + 0.3125x_2^{(k)} - 0.25x_3^{(k-1)}, \quad \text{for } k \geq 1. \end{aligned}$$

The solution is $(3, 4, -5)$.

4.3.4 The matriceal formulations of the iterative methods

Split the matrix A into the sum

$$A = D + L + U,$$

where D is the diagonal of A , L the lower triangular part of A , and U the upper triangular part of A . That is,

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} 0 & \cdots & & 0 \\ a_{21} & \ddots & & \ddots \\ \vdots & \ddots & \ddots & \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{bmatrix},$$
$$U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & \cdots & & 0 \end{bmatrix}$$

The system $Ax = b$ can be written as

$$(D + L + U)x = b.$$

The **Jacobi method** in matriceal form is given by:

$$D\mathbf{x}^{(k)} = -(L + U)\mathbf{x}^{(k-1)} + \mathbf{b}$$

the **Gauss-Seidel method** in matriceal form is given by:

$$(D + L)\mathbf{x}^{(k)} = -U\mathbf{x}^{(k-1)} + \mathbf{b}$$

and **the relaxation method** in matriceal form is given by:

$$(D + \omega L)\mathbf{x}^{(k)} = ((1 - \omega)D - \omega U)\mathbf{x}^{(k-1)} + \omega \mathbf{b}$$

Convergence of the iterative methods

Remark 10 *The convergence (or divergence) of the iterative process in the Jacobi and Gauss-Seidel methods does not depend on the initial guess, but depends only on the character of the matrices themselves. However, a good first guess in case of convergence will make for a relatively small number of iterations.*

A sufficient condition for convergence:

Theorem 11 (Convergence Theorem) *If A is strictly diagonally dominant, then the Jacobi, Gauss-Seidel and relaxation methods converge for any choice of the starting vector $\mathbf{x}^{(0)}$.*

Example 12 *Consider the system of equations*

$$\begin{bmatrix} 3 & 1 & 1 \\ -2 & 4 & 0 \\ -1 & 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

The coefficient matrix of the system is strictly diagonally dominant since

$$|a_{11}| = |3| = 3 > |1| + |1| = 2$$

$$|a_{22}| = |4| = 4 > |-2| + |0| = 2$$

$$|a_{33}| = |-6| = 6 > |-1| + |2| = 3.$$

Hence, if the Jacobi or Gauss-Seidel method are used to solve the system of equations, they will converge for any choice of the starting vector $\mathbf{x}^{(0)}$.

Example 13 *Consider the linear system*

$$\begin{aligned}4x_1 + x_2 &= 3 \\2x_1 + 5x_2 &= 1.\end{aligned}$$

Perform two iterations of Jacobi, Gauss-Seidel and relaxation methods to this system, beginning with the vector $x = [3, 11]$ and for $\omega = 1.25$.

(Solutions of the system are $7/9$ and $-1/9$).