As you know, during Lab 7 you will have face a Lab test which counts for 10% from the final mark. Also, please remember that two compulsory conditions to pass the final exam are:

- to have attended at least 6 Labs (Lab 7 is counted too). In this case 0.5 points will be added to your final mark. If this condition is not fulfilled you can not even participate at the final exam this year;
- to obtain at least 5 points (from the 10) at the Lab test.

During the Lab test you are allowed to consult:

- the tutorial (i.e. the file Lab-tutorial.pdf) ONLY in printed form;
- the Maple Help;
- your notebooks with the notes from the lectures, seminars and labs ONLY on paper.

During the Lab test you are NOT allowed:

- to open any webpage;
- to open any other Maple file than your worksheet;
- to open any type of file;
- to use the phones;
- to use any USB stick.

Please find in the sequel some models of Lab tests. I hope this will help you to prepare for the test.

Adriana Buică

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Secţia: Informatică engleză, Curs: Dynamical Systems

Laboratory test

1. Find the general solution to the following system (here $x = x(t)$ and $y = y(t)$):
$x' = x + y + t - 1\,, \quad y' = -2x + 4y + e^t$.

2. We consider the nonlinear planar system (here $x = x(t)$ and $y = y(t)$) $\dot{x} = y + x^2, \quad \dot{y} = -x + xy$.
a) Its equilibria are:

b) The matrix of the linearized system around the equilibrium point $(0,0)$ is $A = \begin{pmatrix} & \\ & \end{pmatrix}$ and its eigenvalues are: . The hypotheses to apply the linearization method are fulfilled in this case?

c) Remind that the cartesian differential equation of the orbits is: $\frac{dy}{dx} = \frac{-x+xy}{y+x^2}$ (here $y = y(x)$). The general solution of the cartesian differential equation of the orbits is:

d) Note that the general solution found at c) can be written in the form $H(x,y) = c$, where $H(x,y)$ has the expression:

e) Remind that H found at d) is a first integral of this planar system. In a small neighborhood of $(0,0)$ (like, for example, the box $[-0.1, 0.1] \times [-0.1, 0.1]$ the level curves of the first integral looks like:

g) It seems that the equilibrium point $(0,0)$ is a

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Laboratory test

1. a) The general solution of $x'' + 3x' + x = 1$ is (here $x = x(t)$):

b) Decide if the following statement is true or false:
All the solutions of $x'' + 3x' + x = 1$ satisfies $\lim_{t\to\infty} x(t) = 1$.

2. We consider the planar nonlinear system (here $x = x(t)$ and $y = y(t)$)

$$\dot{x} = 2x - x^2 - xy, \quad \dot{y} = -y + xy\,.$$

Find its equilibria, find the matrix of the linearized system around each equilibrium point together with its corresponding eigenvalues. Write below all these informations. Using **DEplot** represent few orbits that starts near each equilibrium point.

3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3 + 0.12\ \sin x$.

The fixed point of f is

Compute the first 30 iterates of f starting with an initial value at your choice. It seems that this orbit is a sequence with the properties:

Repeat the procedure with at least two more different initial values.

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Laboratory test

1. Consider the following differential equation ($y = y(x)$):

$$x^2 y'' - 2xy' + y = 0.$$

a) Determine the general solution of the differential equation:

....

b) Determine the solution of the equation that satisfies $y\,(1) = 2, \quad y'\,(1) = 3$:

....

c) Plot the solution obtained at item b):

....

2. Consider the nonlinear planar system

$$\begin{cases} x' = 2x - 1.2xy \\ y' = -y + 0.9xy \end{cases}$$

a) Find its equilibria and study their stability:

....

b) Use **DEplot** to represent few orbits near each equilibrium point:

....

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Laboratory test

1. Consider the following IVP ($y = y(x)$) where $\alpha \in \mathbb{R}$ is a parameter:

$$\begin{cases} y' + y = \alpha x + \alpha + 1 \\ y(1) = 2 \end{cases}$$

a) Determine the general solution of the differential equation:

....

b) Determine the solution of the IVP:

....

c) Represent simultaneously the graph of the solution of the IVP for $\alpha = -2$, $\alpha = -1$, $\alpha = 0$, $\alpha = 1$, $\alpha = 2$.

2. Consider the planar system $\dot{x} = 2 - 2x + y$, $\dot{y} = xy$.

a) Find all its equilibria:

b) For each equilibrium point, find the matrix of the linearized system around it. Calculate the eigenvalues of this matrix.

....

c) Use **DEplot** to represent a few orbits near each equilibrium point:

....

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Laboratory test

1. Let $\alpha \in \mathbb{R}$. Consider the following IVP $(y = y(x))$:

$$\begin{cases} y' - y = 2\alpha e^{-x} - 1 \\ y(0) = 2 \end{cases}$$

a) Determine the general solution of the differential equation:

....

b) Determine the solution of the IVP:

....

c) Represent simultaneously the graph of this solution for $\alpha = -2$, $\alpha = -1$, $\alpha = 0$, $\alpha = 1$, $\alpha = 2$.

2. Consider the nonlinear planar system $\dot{x} = y, \quad \dot{y} = x(1 - x^2) + y.$

a) Find all its equilibria:

b) For each equilibrium point, find the matrix of the linearized system around it. Calculate the eigenvalues of this matrix.

....

c) Use **DEplot** to represent a few orbits near each equilibrium point.

....

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Laboratory test

1. a) Find the general solution of the planar linear system $\quad x' = -x - y + 1, \quad y' = 3x - 2y + 2 \quad$.

b) Decide if the following statement is true or false:
All the solutions of $\quad x' = -x - y + 1, \quad y' = 3x - 2y + 2 \quad$ satisfy

$$\lim_{t\to\infty} (x(t), y(t)) = (1, 2).$$

2. For the IVP $\quad y' = y^2 + x^2, \quad y(0) = 0 \quad$, apply the Euler's method and the improved Euler's method in the interval $[0, 1]$ with step size $h = 0.1$. Write here the approximate values in $x = 1$ obtained in each case.

3. Find the solution to the following IVP $x'' + x = 0, \quad x(\pi/2) = 0, \quad x'(\pi/2) = -2.$
Using the expression found, represent the graph of the solution.

Laboratory test

1. Let $\alpha \in \mathbb{R}$ and $\phi(t, \alpha)$ be the solution of the IVP (here $x = x(t)$)

$$x'' - 4x = \alpha t, \quad x(0) = \alpha, \quad x'(0) = 0.$$

a) $\phi(t, 1) =$

and its graph looks like

b) $\phi(t, 0) =$

c) For each $t \in \mathbb{R}$, $\lim\limits_{\alpha \to 0} \phi(t, \alpha) =$

2. Represent the direction field of the differential equation $\quad y' = \dfrac{xy - x}{x^2 + y} \quad$ (here $y = y(x)$) in a small neighborhood of the origin.

3. We consider the nonlinear planar system $\quad \dot{x} = x^2 + y, \quad \dot{y} = xy - x \quad$ (here $x = x(t)$ and $y = y(t)$). Find its equilibria, find the matrix of the linearized system around the equilibrium point $(0, 0)$ and its eigenvalues.

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Laboratory test

1. The general solution of the differential equation

$$y' + \frac{1}{2}xy = -\frac{1}{2}$$

(here $y = y(x)$) is:

For each solution $y(x)$ of the above equation we have $\lim_{x\to\infty} y(x) =$

2. We consider the planar linear nonhomogeneous system $\dot{x} = -x - 1, \quad \dot{y} = y - 1$.

a) Its equilibrium point is:

b) Remind that the cartesian differential equation of its orbits is $\frac{dy}{dx} = \frac{y-1}{-x-1}$ (here $y = y(x)$). Represent its direction field and find its general solution.

3. We consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = \int_0^\infty e^{-t^2} \sin(xt)dt.$$

a) $\varphi(0) =$

b) $\varphi'(0) =$

c) $\varphi'(x) + \frac{1}{2}x\varphi(x) =$

d) Decide if the following proposition is true or false:
"φ is a solution of the differential equation $y' + \frac{1}{2}xy = -\frac{1}{2}$"

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Laboratory test

1. Consider the planar linear system $x' = 4y \quad y' = -x$.
   a) The general solution of the above system is:

....

b) The solution of the above system that satisfies the initial conditions $x(0) = 0,\ y(0) = 1$ is:

....

c) Represent the planar curve of parametric equations found at b) (or, in other words, represent the orbit of the system corresponding to the initial state $(0, 1)$.

d) Remind that the cartesian differential equation of the orbits is $\frac{dy}{dx} = \frac{-x}{4y}$ (here $y = y(x)$). Find its general solution and notice that it can be written in the form $H(x, y) = c$, $c \in \mathbb{R}$. Find the expression of $H(x, y)$ and represent the level curves of H.

....

2) Consider the differential equation $\quad x'' + 4x = 0 \quad$ (here $x = x(t)$).

a) The solution $\varphi(t)$ of the above equation that satisfies the conditions $x(0) = 0,\ x'(0) = 1$ is:

....

b) Plot the graph of the solution $\varphi(t)$ found at b):

....

c) Calculate $4\varphi(t)^2 + [\varphi'(t)]^2 =$

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Laboratory test

1. We consider the planar system $\dot{x} = y + x^2,\ \dot{y} = -x + xy$ (here $x = x(t)$ and $y = y(t)$).
   a) The equilibria are:

   b) Study the type and stability of each hyperbolic equilibrium point.

   c) Remind that the cartesian differential equation of its orbits is $\frac{dy}{dx} = \frac{-x+xy}{y+x^2}$ (here $y = y(x)$). Represent the direction field of the cartesian differential equation of its orbits. Find its general solution.

   d) Represent the level curves of the function $H(x, y) = x^2 + y^2 - x^2y$ in small boxes around each equilibrium point of the system.

   e) Knowing that the function at d) is a first integral of the system, decide the type of the equilibrium point $(0, 0)$.

   f) Putting all the things together, represent here the phase portrait of the system.

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Laboratory test

1. Using the command **dsolve**, respectively **dsolve** with the option **series**, find the solution, and, respectively, the power series solution (centered in 0) of the following IVP for the Legendre's differential equation (where $u = u(x)$ and n is a parameter)

$$(1 - x^2)u'' - 2xu' + n(n+1)u = 0, \quad u(0) = 1, \quad u'(0) = 0.$$

2. For the logistic map $f(x) = \lambda x(1 - x)$ with $\lambda = 3.4$ calculate the first 100 iterations of f starting with $x_0 = 0.5$ and represent them using **pointplot**. Interpret what you see.

3. Find the solution to the following IVP $y' = \dfrac{2y}{x}, \quad y(1) = 1$.

   Using the expression found, represent the graph of the solution.

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Laboratory test

1. Using the command **dsolve**, respectively **dsolve** with the option **series**, find the general solution, and after the power series solutions (centered in 0) of the Bessel differential equation $x^2 u'' + x u' + (x^2 - n^2)u = 0$. Here the unknown is $u(x)$ and n is a parameter.

2. For the logistic map $f(x) = \lambda x(1-x)$ with $\lambda = 3.5$ calculate the first 100 iterations starting with $x_0 = 0.5$ and represent them using **pointplot**. Interpret what you see.

3. We consider the linear planar system $\dot{x} = x + 2y,\ \dot{y} = -x - y$ (here $x = x(t)$ and $y = y(t)$).

a) Determine the type of the equilibrium point $(0, 0)$.

b) Find the solution of this system that satisfies the ICs $x(0) = 2,\ y(0) = 0$.

Using the expression found represent the corresponding orbit (that is, the planar curve with this parametric representation).

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Laboratory test

1. Let $\alpha \in \mathbb{R}$ and $\varphi(t, \alpha)$ be the solution of the IVP

$$x'' - 4x = \alpha t, \quad x(0) = \alpha, \quad x'(0) = 0.$$

For each $t \in \mathbb{R}$, $\lim\limits_{\alpha \to 0} \varphi(t, \alpha) =$ and $\varphi(t, \alpha) =$

The graph of $\varphi(t, 1)$ looks like

The graph of $\varphi(t, 0)$ looks like

2. Represent the direction field of the differential equation $y' = \dfrac{xy - x}{x^2 + y}$ (here $y = y(x)$) in a small neighborhood of the origin.

3. Compute (better in the same **for**) the first 40 iterations of $f(x) = 4x(1 - x)$ and, respectively, $\tilde{f}(x) = 4x - 4x^2$ starting with $x_0 = 0.67$. What do you expect and what you obtain? Write here the last values.

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Laboratory test

1. Let $x'' + x = f(t), \quad x(0) = 5, \quad x'(0) = 0$, where $x = x(t)$ and $f(t) = \begin{cases} t, & t \in [0, \pi/2) \\ \pi - t, & t \in [\pi/2, \pi] \\ 0, & t \in (\pi, \infty) \end{cases}$.

The solution of this IVP is:

....

Represent simultaneously the graph of f and of the solution of this IVP:

....

2. The general solution of the differential equation $\quad y' + \dfrac{2y}{x} = \dfrac{y^3}{x^2} \quad$ (here $y = y(x)$) is:

....

The type of this differential equation is: ....

Represent the direction field of this differential equation in the box $[1, 2] \times [1, 2]$.

3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 + 0.2 \ \sin x$.

The fixed point of f is

Compute the first 20 iterates of f starting with: a) $x_0 = 0$; b) $x_0 = 7$; c) $x_0 = 70$.

What do you notice? If you do not notice anything calculate more iterates. It seems that each of these orbits is a sequence with the properties:

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Laboratory test

1. Find the solution of the IVP $x'' - 4x = t,\ x(0) = 0,\ x'(0) = 0$.
   Using the expression you found, represent the corresponding integral curve.

2. For the logistic map $f(x) = \lambda x(1-x)$ with $\lambda = 2$, $\lambda = 3.5$, respectively $\lambda = 3.8$ calculate the first 100 iterations starting with $x_0 = -0.5$, $x_0 = 0$, respectively $x_0 = 0.5$ and represent them using **pointplot**. Interpret what you see. Try with other values of λ and x_0 at your choice.

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Laboratory test

1. The general solution of the differential equation $x'' + 9x = 0$ (where $x = x(t)$) is:

2. The solution of the IVP

$$x'' + 9x = 0, \quad x(\pi/2) = 0, \quad x'(\pi/2) = -2$$

is:

and its graph looks like:

3. We consider the planar nonlinear system $\dot{x} = x^2(y - x)$, $\dot{y} = -1 + x^2 y$. Here $x = x(t)$ and $y = y(t)$.

a) Its equilibria are:

b) Specify the type and stability of each hyperbolic equilibrium point:

c) Remind that the cartesian differential equation of the orbits is $\frac{dy}{dx} = \frac{-1+x^2 y}{x^2(y-x)}$, where $y = y(x)$. The general solution of the cartesian differential equation of the orbits is:

d) From c) one can find a first integral $H(x, y) = -\frac{2-xy^2+2x^2y}{2x}$. Compute the expression

$$\frac{\partial H}{\partial x}(x, y) \cdot x^2(y - x) - \frac{\partial H}{\partial y}(x, y) \cdot (1 - x^2 y) =$$

e) the 2-level curve of H looks like:

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Laboratory test

1. The general solution of the differential equation

$$x'' + x = \sin t$$

(here $x = x(t)$) is:

2. We consider the nonlinear planar system:

$$\dot{x} = x^2 - y^2, \quad \dot{y} = 2x - 3y.$$

a) Its equilibria are:
b) The non-hyperbolic equilibria are:

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(t) = \int_0^t f(s) \sin(t - s) ds.$$

a) $\phi(0) =$

b) $\phi'(0) =$

c) $\phi''(0) + \phi(0) =$

d) Decide the validity of the following proposition:
"ϕ is a solution of the differential equation $x'' + x = f(t)$."

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Laboratory test

1. The general solution of the differential equation

$$y' + \frac{1}{2}xy = \frac{1}{2}$$

(here $y = y(x)$) is:

2. a) Represent the direction fields of the differential equation $\quad \dfrac{dy}{dx} = \dfrac{3x - 2y}{-x + y} \quad$.

b) Consider the planar linear system

$$\dot{x} = -x + y, \quad \dot{y} = 3x - 2y. \tag{1}$$

Compute the eigenvalues of the matrix of this system.

Specify the type and stability of this system.

3. We consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = \int_0^\infty e^{-t^2} \cos(xt)dt.$$

a) $\phi(0) =$

b) $\phi'(0) =$

c) $\phi'(x) + \frac{1}{2}x\phi(x) =$
c) Decide the validity of the following proposition
"ϕ is a solution of the differential equation $y' + \dfrac{1}{2}xy = \dfrac{1}{2}$."

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Laboratory test

1. The general solution of the differential equation

$$x' + \frac{1}{2}tx = 0$$

(here $x = x(t)$) is:

For each solution $x(t)$ of the above equation we have $\lim_{t\to\infty} x(t) =$

2. a) Represent the direction field of the differential equation $\dfrac{dy}{dx} = -\dfrac{y-1}{x+1}$ (here $y = y(x)$).

b) Find the general solution of the equation at a).

3. Take $\eta \in \{0, 2, -1/4, (1+\sqrt{5})/2\}$. Compute the first 100 values of the sequence $(x_k)_{k\geq 0}$ that satisfies the IVP

$$x_{k+1} = x_k^2 - 1, \quad x_0 = \eta.$$

Describe its long-term behavior.

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Laboratory test

1) The general solution of the linear planar system

$$\begin{cases} x' = 9y \\ y' = -x \end{cases}$$

(here $x = x(t)$ and $y = y(t)$) is:

2) We consider the differential equation $\quad x'' + 9x = 0 \quad$ (here $x = x(t)$).

a) Its general solution is:

b) The solution $\varphi(t)$ of the above equation that satisfies the initial conditions $x(0) = 0,\ x'(0) = 1$ is:

c) Plot the graph of the solution $\varphi(t)$ found at b).

d) Calculate $9\varphi(t)^2 + [\varphi'(t)]^2 =$

3) Take $\eta \in \{\sqrt{5}, 2.1, 2, 1\}$. Compute the first 100 values of the sequence $(x_k)_{k\geq 0}$ that satisfies the IVP

$$x_{k+1} = \frac{x_k^2 + 5}{2x_k}, \quad x_0 = \eta.$$

Describe its long-term behavior.

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Laboratory test

We consider the differential equation $\dot{x} = 9 - x^2$, where $x = x(t)$.

a) e^{3t} is a solution: (True or False)

b) Find its equilibria.

c) Its solution, denoted $\varphi(t, 3)$, with initial condition $x(0) = 3$ is:

d) Its solution, denoted $\varphi(t, \eta)$, with initial condition $x(0) = \eta$ has

$$\lim_{\eta \to 3} \varphi(t, \eta) =$$

e) Represent the graphs of $\varphi(t, 0)$ and, respectively, of $\varphi(t, 3.1)$ on the interval $(0, \infty)$.

f) Represent its direction field in the box $[-4, 4] \times [-4, 4]$.

g) the Euler numerical formula is:

(remind that the Euler numerical formula with stepsize h for the equation $\dot{x} = f(x)$ is $x_{k+1} = x_k + hf(x_k)$.)

2. For the IVP $y' = 9 - y^2$, $y(0) = 0$ use Euler's method with step size h = 0.1 to find approximate values of the solution in the interval $[0, 1]$.