LECTURE

10

PARTIALLY DIFFERENTIABLE FUNCTIONS

First order partial derivatives

Definition 10.1 *Let $A \subseteq \mathbb{R}^n$, $c = (c_1, \dots, c_n) \in intA$ and $j \in \{1, \dots, n\}$. A function $f : A \to \mathbb{R}$ is called partially differentiable w.r.t. x_j at c if the limit*

$$\lim_{x_j \to c_j} \frac{f(c_1, \dots, c_{j-1}, x_j, c_{j+1}, \dots, c_n) - f(c_1, \dots, c_n)}{x_j - c_j}$$

exists in $\mathbb{R}$. In this case, the above limit is called the partial derivative of f w.r.t. x_j at c and is denoted by $\dfrac{\partial f}{\partial x_j}(c)$ (or $f'_{x_j}(c)$, $D_j f(c)$).

Definition 10.2 *If for all $j \in \{1, \dots, n\}$, f is partially differentiable w.r.t all variables x_j at c, then f is called partially differentiable at c. In this case, the vector*

$$\left(\frac{\partial f}{\partial x_1}(c), \dots, \frac{\partial f}{\partial x_n}(c) \right) \in \mathbb{R}^n$$

is called the gradient of f at c and is denoted by $\nabla f(c)$.

Definition 10.3 *If B is an open subset of A, we say that f is partially differentiable w.r.t. x_j on B if it is partially differentiable w.r.t. x_j at every point of B. In this case, the function*

$$\frac{\partial f}{\partial x_j} : B \to \mathbb{R}, \quad x \in B \mapsto \frac{\partial f}{\partial x_j}(x) \in \mathbb{R}$$

is called the partial derivative of f w.r.t. x_j on B.

At the same tine, f is called partially differentiable on B if it is partially differentiable at every point of B. If A is open and f is partially differentiable on A, then we simply say that f is partially differentiable.

Remark 10.4 *(i) Since $c \in intA$, we can move a small distance in all directions from c while not leaving the set.*
(ii) Partial differentiation means taking the ordinary derivative w.r.t. a single variable while keeping all other variables constant. Thus, we can apply all rules of differentiation.

Example 10.5 *Consider the function* $f : \mathbb{R}^3 \to \mathbb{R}$*, defined for all* $(x, y, z) \in \mathbb{R}^3$ *by*

$$f(x, y, z) = x^3 + x\sin(yz) + y^2 e^z.$$

The partial derivatives of f *at any point* $(x, y, z) \in \mathbb{R}^3$ *are*

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y, z) &= 3x^2 + \sin(yz),\\ \frac{\partial f}{\partial y}(x, y, z) &= xz\cos(yz) + 2ye^z,\\ \frac{\partial f}{\partial z}(x, y, z) &= xy\cos(yz) + y^2 e^z.\end{aligned}$$

For instance, by considering $(x, y, z) = (1, 2, 0)$ *we get* $\frac{\partial f}{\partial x}(1, 2, 0) = 3$, $\frac{\partial f}{\partial y}(1, 2, 0) = 4$, $\frac{\partial f}{\partial z}(1, 2, 0) = 6$, *hence the gradient of* f *at* $(1, 2, 0)$ *is*

$$\nabla f(1, 2, 0) = (3, 4, 6) \in \mathbb{R}^3.$$

Remark 10.6 *Partial differentiability at a given point does not imply continuity at that point.*

Example 10.7 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x, y) \neq 0_2 \\ 0, & \text{if } (x, y) = 0_2. \end{cases}$

Function f *is partially differentiable at* 0_2*, since*

$$\frac{\partial f}{\partial x}(0_2) = \lim_{x\to 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0_2) = \lim_{y\to 0} \frac{f(0, y) - f(0, 0)}{y - 0} = 0.$$

However, as we have already seen in Example 9.13.(ii), f *is not continuous at* 0_2 *(notice that in this example* f *is partially differentiable on* $\mathbb{R}^2$*).*

Definition 10.8 *If* $A \subseteq \mathbb{R}^n$ *is open, a function* $f : A \to \mathbb{R}$ *is called continuously partially differentiable if it is partially differentiable and all partial derivatives are continuous. In this case we write* $f \in C^1(A)$.

Remark 10.9 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} (x^2 + y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & \text{if } (x, y) \neq 0_2 \\ 0, & \text{if } (x, y) = 0_2. \end{cases}$

We have already seen in Example 9.13.(i) that f *is continuous. It is easy to prove that* f *is partially differentiable, but its partial derivatives are not continuous at* 0_2.

Higher order partial derivatives

Definition 10.10 *Let* $A \subseteq \mathbb{R}^n$, $c \in intA$, $i, j \in \{1, \ldots, n\}$ *and* $f : A \to \mathbb{R}$*. We say that* f *is twice partially differentiable w.r.t.* (x_i, x_j) *at* c *if* $\exists V \in \mathcal{V}(c)$, V *open,* $V \subseteq A$ *such that* f *is partially differentiable w.r.t.* x_i *on* V *and the function*

$$\frac{\partial f}{\partial x_i} : V \to \mathbb{R}, \quad x \in V \mapsto \frac{\partial f}{\partial x_i}(x) \in \mathbb{R} \tag{10.1}$$

is partially differentiable w.r.t. x_j *at* c*. The partial derivative of the function (10.1) w.r.t.* x_j *at* c *is called the second order partial derivative of* f *w.r.t.* (x_i, x_j) *at* c *and is denoted by* $\dfrac{\partial^2 f}{\partial x_j \partial x_i}(c)$ *(or* $f''_{x_i x_j}(c)$*). If* $i = j$ *we use the notation* $\dfrac{\partial^2 f}{\partial x_i^2}(c)$ *(or* $f''_{x_i^2}(c)$*). If for all* $i, j \in \{1, \ldots, n\}$, f *is twice partially differentiable w.r.t* (x_i, x_j) *at* c*, then* f *is called twice partially differentiable at* c.

Inductively, one can define partial derivatives of arbitrary order.

Remark 10.11 *(i)* $\dfrac{\partial^2 f}{\partial x_j \partial x_i}(c) = \dfrac{\partial}{\partial x_j}\left(\dfrac{\partial f}{\partial x_i}\right)(c)$, $f''_{x_i x_j}(c) = \left(f'_{x_i}\right)'_{x_j}(c)$.
Note that f has n^2 second order partial derivatives.
(ii) Higher order partial derivatives w.r.t. two or more different variables are also called mixed partial derivatives.
(iii) Partial derivatives introduced in Definition 10.1 will also be called first-order partial derivatives (in order to distinguish them from higher order partial derivatives).
(iv) As in Definition 10.1, one can introduce the notions of twice partial differentiability and second order partial derivative (as a function) on open sets. In particular, if A is open, then f is called twice partially differentiable if f is twice partially differentiable at every point of A.

Example 10.12 *Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^{xy^2}$. Let $(x,y) \in \mathbb{R}^2$. Then*

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= y^2 e^{xy^2},\\
\frac{\partial f}{\partial y}(x,y) &= 2xye^{xy^2},\\
\frac{\partial^2 f}{\partial x^2}(x,y) &= y^4 e^{xy^2},\\
\frac{\partial^2 f}{\partial y^2}(x,y) &= 2xe^{xy^2} + 4x^2y^2e^{xy^2},\\
\frac{\partial^2 f}{\partial y \partial x}(x,y) &= 2ye^{xy^2} + 2xy^3e^{xy^2} = \frac{\partial^2 f}{\partial x \partial y}(x,y).
\end{aligned}$$

Remark 10.13 *Mixed partial derivatives of a function at a point are not always equal.*

Example 10.14 *Consider $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{x^3y}{x^2+y^2}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2. \end{cases}$*
Since

$$\lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = 0 \quad \text{and} \quad \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0} = 0,$$

we have that f is partially differentiable at 0_2 and $\dfrac{\partial f}{\partial x}(0,0) = 0 = \dfrac{\partial f}{\partial y}(0,0)$. For $(x,y) \in \mathbb{R}^2 \setminus \{0_2\}$ we have that

$$\frac{\partial f}{\partial x}(x,y) = \frac{x^2y(x^2+3y^2)}{(x^2+y^2)^2} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \frac{x^3(x^2-y^2)}{(x^2+y^2)^2}.$$

Note that

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y - 0} = 0,$$

while

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x \to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x-0} = \lim_{x\to 0} \frac{x^5/x^4}{x} = 1.$$

Remark 10.15 *In the previous example the mixed second order partial derivatives $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ are not continuous at 0_2.*

Definition 10.16 *If $A \subseteq \mathbb{R}^n$ is open, a function $f : A \to \mathbb{R}$ is called twice continuously partially differentiable if it is twice partially differentiable and all first and second order partial derivatives are continuous. In this case we write $f \in C^2(A)$.*

Theorem 10.17 (Schwarz) *Let $A \subseteq \mathbb{R}^n$ be open and $f \in C^2(A)$. Then for every $i, j \in \{1, \dots, n\}$,*

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Definition 10.18 *Let $A \subseteq \mathbb{R}^n$ be open, $c \in A$ and $f : A \to \mathbb{R}$. If f is twice partially differentiable at c, we can build the $n \times n$ matrix*

$$H_f(c) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(c) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(c) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(c) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(c) & \frac{\partial^2 f}{\partial x_2^2}(c) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(c) \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(c) & \frac{\partial^2 f}{\partial x_n \partial x_2}(c) & \dots & \frac{\partial^2 f}{\partial x_n^2}(c) \end{pmatrix},$$

which is called the Hessian matrix (or Hessian) of f at c, denoted also by $\nabla^2 f(c)$.

Remark 10.19 *If f is twice partially differentiable, then we can consider the Hessian matrix at all points of A. Notice that, if $f \in C^2(A)$, then $H_f(c)$ is symmetric at every $c \in A$, in view of Theorem 10.17.*

Example 10.20 *Example 10.12 revisited: let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = e^{xy^2}$. Then, for $(x, y) \in \mathbb{R}^2$,*

$$H_f(x, y) = \begin{pmatrix} y^4 e^{xy^2} & 2ye^{xy^2} + 2xy^3 e^{xy^2} \\ 2ye^{xy^2} + 2xy^3 e^{xy^2} & 2xe^{xy^2} + 4x^2y^2 e^{xy^2} \end{pmatrix}$$

and

$$H_f(1, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Vector-valued functions of several variables

Let $n, m \in \mathbb{N}^*$, $m \geq 2$. For $j \in \{1, \dots, m\}$, consider the projection mapping $pr_j : \mathbb{R}^m \to \mathbb{R}$, $pr_j(y) = y_j$, $\forall y = (y_1, \dots, y_m) \in \mathbb{R}^m$.

Definition 10.21 *Let $A \subseteq \mathbb{R}^n$. A function $f : A \to \mathbb{R}^m$ is called a vector-valued function of n variables. The components of f are the real-valued functions $f_1, \dots, f_m : A \to \mathbb{R}$ defined by $f_j = pr_j \circ f$, $\forall j \in \{1, \dots, m\}$ and we write $f = (f_1, \dots, f_m)$.*

Properties of vector-valued functions can usually be studied by considering their components one at a time.

Example 10.22 *Let $A \subset \mathbb{R}^n$, $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$.*
(i) If $c \in A'$ and $y^0 = (y_1^0, \dots, y_m^0) \in \mathbb{R}^m$, then $\lim\limits_{x \to c} f(x) = y^0 \Leftrightarrow \forall j \in \{1, \dots, m\}$, $\lim\limits_{x \to c} f_j(x) = y_j^0$.
For instance,

$$\lim_{x \to 0} \left(\frac{\sin x}{x}, (1 + x)^{\frac{1}{x}}, x \sin \frac{1}{x} \right) = \left(\lim_{x \to 0} \frac{\sin x}{x}, \lim_{x \to 0} (1 + x)^{\frac{1}{x}}, \lim_{x \to 0} x \sin \frac{1}{x} \right) = (1, e, 0).$$

(ii) If $c \in A$, then f is continuous at $c \Leftrightarrow \forall j \in \{1, \dots, m\}$, f_j is continuous.
(iii) If $c \in intA$, then f is partially differentiable at $c \Leftrightarrow \forall j \in \{1, \dots, m\}$, f_j is partially differentiable at c.