Geometry
Problem booklet

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1 Week 8: Applications of the triple scalar product. Conics

1.0.1 Applications of the triple scalar product

• The distance between two straight lines.

If $\Delta_1,\ \Delta_2$ are two straight lines, then the distance between them, denoted by $\delta(\Delta_1, \Delta_2)$, is defined as

$$\min\{\|\ \overrightarrow{M_1M_2}\ \|\ |\ M_1 \in \Delta_1,\ M_2 \in \Delta_2\}.$$

1. If Δ_1, Δ_2 are concurrent, then $\delta(\Delta_1, \Delta_2) = 0$.

2. If $\Delta_1 || \Delta_2$, then $\delta(\Delta_1, \Delta_2) = \|\ \overrightarrow{MN}\ \|$ where $\{M\} = d \cap \Delta_1,\ \ \{N\} = d \cap \Delta_2$ and d is a straight line perpendicular to the lines Δ_1 and Δ_2. Obviously $\|\ \overrightarrow{MN}\ \|$ is independent on the choice of the line d. Note that $\delta(\Delta_1, \Delta_2) = \delta(M, \Delta_2) = \delta(\Delta_1, N)$, where $M \in \Delta_1$ and $N \in \Delta_2$ are arbitrary here.

3. We now assume that the straight lines $\Delta_1,\ \Delta_2$ are noncoplanar (skew lines). In this case there exists a unique straight line d such that $d \perp \Delta_1, \Delta_2$ and $d \cap \Delta_1 = \{M_1\},\ d \cap \Delta_2 = \{M_2\}$. The straight line d is called the *common perpendicular* of the lines $\Delta_1,\ \Delta_2$ and obviously $\delta(\Delta_1, \Delta_2) = \|\ \overrightarrow{M_1M_2}\ \|$.

Assume that the straight lines $\Delta_1,\ \Delta_2$ are given by some points $A_1(x_1, y_1, z_1) \in \Delta_1, A_2(x_2, y_2, z_2) \in \Delta_2$ and some director vectors $\vec{d}_1\ (p_1, q_1, r_1) \in \vec{\Delta}_1$ and $\vec{d}_2\ (p_2, q_2, r_2) \in \vec{\Delta}_2$. In other words, their equations are

$$\Delta_1 : \frac{x - x_1}{p_1} = \frac{y - y_1}{q_1} = \frac{z - z_1}{r_1}$$

$$\Delta_2 : \frac{x - x_2}{p_2} = \frac{y - y_2}{q_2} = \frac{z - z_2}{r_2}.$$

The common perpendicular of the lines Δ_1, Δ_2 is the intersection line between the plane containing the line Δ_1 which is parallel to the vector $\vec{d}_1 \times \vec{d}_2$, and the plane containing the line Δ_2 which is parallel to $\vec{d}_1 \times \vec{d}_2$. Since

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} \vec{i} + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} \vec{j} + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \vec{k}$$

it follows that the equations of the common perpendicular are

$$\begin{cases} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ p_1 & q_1 & r_1 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0 \\ \\ \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ p_2 & q_2 & r_2 \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{vmatrix} = 0. \end{cases} \tag{1.1}$$

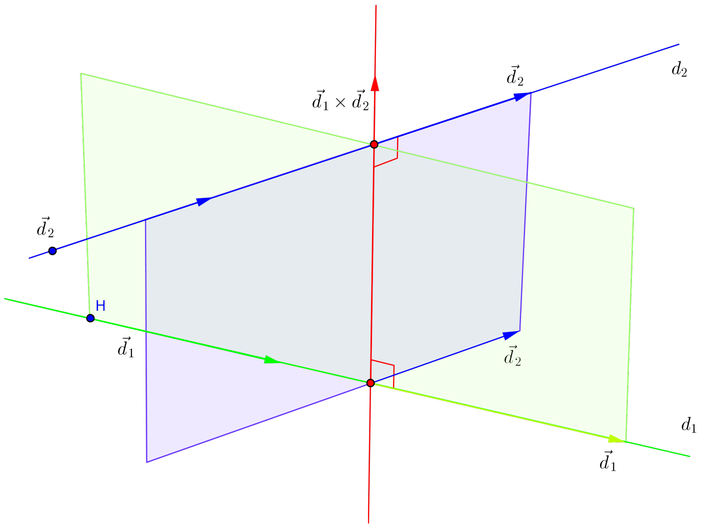


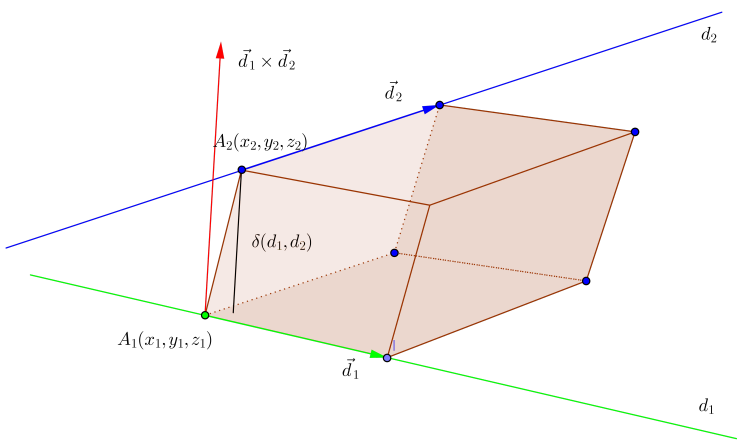
Figure 1: Prependiculara comună a dreptelor Δ_1 şi Δ_2

The distance between the straight lines Δ_1, Δ_2 can be also regarded as the height of the parallelogram constructed on the vectors $\vec{d}_1, \vec{d}_2, \vec{d}_1 \times \vec{d}_2$. Thus

$$\delta(\Delta_1, \Delta_2) = \frac{|(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2)|}{||\vec{d}_1 \times \vec{d}_2||}. \tag{1.2}$$

Therefore we obtain

$$\delta(\Delta_1, \Delta_2) = \frac{\left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} \right|}{\sqrt{\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}^2 + \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}^2 + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}^2}} \tag{1.3}$$



• **The coplanarity condition of two straight lines.**

Using the notations of the previous section, observe that the straight lines Δ_1, Δ_2 are coplanar if and only if the vectors $\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2$ are linearly dependent (coplanar), or equivalently $(\overrightarrow{A_1A_2}, \vec{d}_1, \vec{d}_2) = 0$. Consequently the stright lines Δ_1, Δ_2 are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0 \tag{1.4}$$

1.1 Conics

1.1.1 The Ellipse

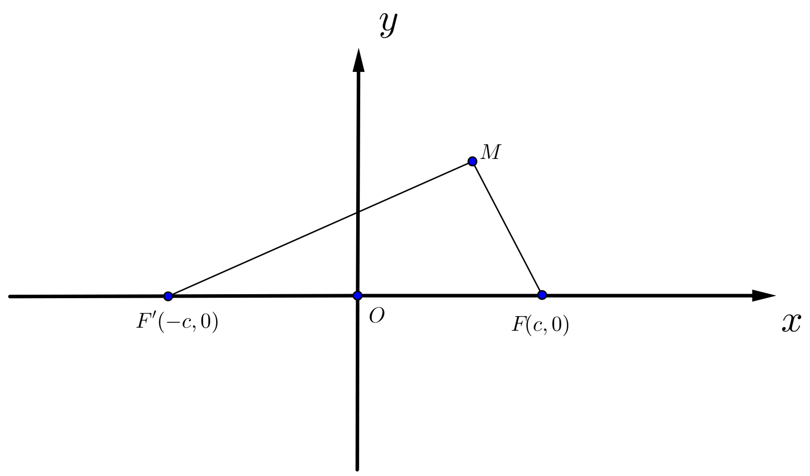
Definition 1.1. *An* ellipse *is the locus of points in a plane, the sum of whose distances from two fixed points, say F and F', called* foci *is constant.*

The distance between the two fixed points is called the *focal distance*

Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$. If M is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

We choose the Cartesian coordinate system with $F'F$ as the x-axis and the perpendicula bisector of the segment $[F'F]$ as the y axis. With such a choice we have $F(c, 0)$, $F'(-c, 0)$.



Remark 1.2. *In $\Delta MFF'$ the following inequality $|MF| + |MF'| > |FF'|$ holds. Hence $2a > 2c$. Thus, the constants a and c must verify $a > c$.*

Thus, for the generic point $M(x, y)$ of the ellipse we have that $|MF| + |MF'| = 2a$, which implies that

$$(a^2 - c^2)x^2 + a^2y^2 - a^2(a^2 - c^2) = 0,$$

as can be easily seen. Denote $a^2 - c^2$ by b^2, as $(a > c)$. Thus $b^2x^2 + a^2y^2 - a^2b^2 = 0$, i.e.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{1.5}$$

Remark 1.3. *The equation (1.5) is equivalent to*

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}; \qquad x = \pm\frac{a}{b}\sqrt{b^2 - y^2},$$

which means that the ellipse is symmetric with respect to both the x and the y axes. In fact, the line FF', determined by the foci of the ellipse, and the perpendicular line on the midpoint of the segment $[FF']$ are axes of symmetry for the ellipse. Their intersection point, which is the midpoint of $[FF']$, is the center of symmetry of the ellipse, or, simply, its center.

Remark 1.4. *In order to sketch the graph of the ellipse, observe that it is enough to represent the function*

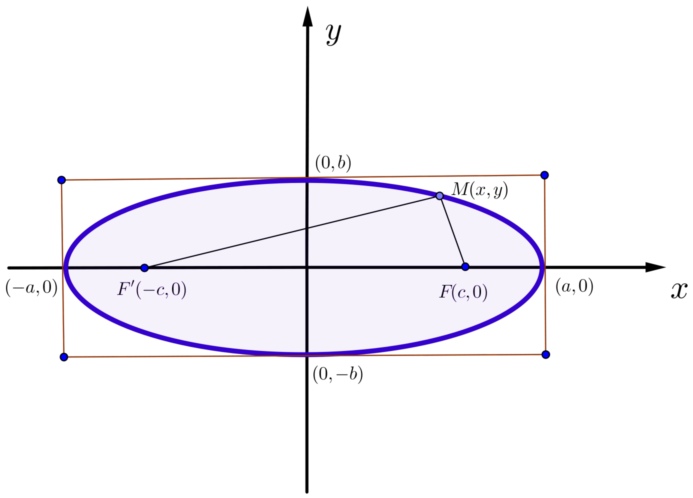
$$f : [-a, a] \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{a^2 - x^2},$$

and to complete the ellipse by symmetry with respect to the x-axis.

One has

$$f'(x) = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}, \qquad f''(x) = -\frac{ab}{(a^2 - x^2)\sqrt{a^2 - x^2}}.$$

x	$-a$		0		a
$f'(x)$	$\vert$	$+\ +\ +$	0	$-\ -\ -$	$\vert$
$f(x)$	0	$\nearrow$	b	$\searrow$	0
$f''(x)$	$\vert$	$-\ -\ -$	$-$	$-\ -\ -$	$\vert$



1.1.2 The Hyperbola

Definition 1.5. *The* hyperbola *is defined as the geometric locus of the points in the plane, whose absolute value of the difference of their distances to two fixed points, say F and F', called foci, is constant.*

The two fixed points are called the *foci* of the hyperbola, and the distance $|FF'| = 2c$ between the foci is the *focal distance*.

Assume that the constant in the definition is $2a$. If $M(x, y)$ is an arbitrary point of the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

We choose the Cartesian coordinate system with $F'F$ as the x-axis and the perpendicula bisector of the segment $[F'F]$ as the y axis. With such a choice we have $F(c, 0)$, $F'(-c, 0)$.

Remark 1.6. *In the triangle $\Delta MFF'$, $||MF| - |MF'|| < |FF'|$, so that $a < c$.*

Let us determine the equation of a hyperbola. By using the definition we get for a point $M(x, y)$ on the hyperbola that $|MF| - |MF'| = \pm 2a$, which implies that

$$(c^2 - a^2)x^2 - a^2y^2 - a^2(c^2 - a^2) = 0,$$

as can be easily seen. By using the notation $c^2 - a^2 = b^2$ ($c > a$) we obtain the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \tag{1.6}$$

The equation (1.6) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}; \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

Therefore, the coordinate axes are axes of symmetry of the hyperbola and the origin is a center of symmetry equally called the *center of the hyperbola*.

1.2 Problems

1. Find the distance from the point $P(1, 2, -1)$ to the straight line (d) $x = y = z$.

2. Find the distance between the straight lines

$$(\Delta_1)\ \frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1}, \ (\Delta_2)\ \frac{x+1}{3} = \frac{y}{4} = \frac{z-1}{3}$$

   as well as the equations of the common perpendicular.

3. Find the distance between the straight lines M_1M_2 and d, where $M_1(-1, 0, 1)$, $M_2(-2, 1, 0)$ and

$$(d)\ \begin{cases} x + y + z = 1 \\ 2x - y - 5z = 0. \end{cases}$$

   as well as the equations of the common perpendicular.

4. The points $A(1, 2\alpha, \alpha)$, $B(3, 2, 1)$, $C(-\alpha, 0, \alpha)$ and $D(-1, 3, -3)$ are being considered with respect to some orthonormal cartezian system. Find the value of the parameter α for which the pencil of planes through the straight line AB has a common plane with the pencil of planes through the straight line CD.

5. Find the value of the parameter λ for which the straight lines

$$(\Delta_1)\ \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{1}, \ (\Delta_2)\ \frac{x+1}{4} = \frac{y-3}{1} = \frac{z}{\lambda}$$

   are coplanar. Find the coordinates of their intersection point in that case.

6. Determine the coordinates of the foci of the ellipse (E) $9x^2 + 25y^2 - 225 = 0$.

7. Sketch the graph of $y = -\frac{3}{4}\sqrt{16 - x^2}$.

8. Find the intersection points between the line (d) $x + 2y - 7 = 0$ and the ellipse (E) $x^2 + 3y^2 - 25 = 0$.

9. Determine the coordinates of the foci of the hyperbola $\mathcal{H} : \dfrac{x^2}{92} - \dfrac{y^2}{4} - 1 = 0$.

10. Find the intersection points between the line $(d)\ 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \dfrac{x^2}{20} - \dfrac{y^2}{5} - 1 = 0$.

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