Seminar 7

Dynamical Systems
May 2017

Scalar maps

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. We define the iterates of f as
$f^2=f\circ f,\ f^3=f\circ f^2, \dots, f^k=f\circ f^{k-1}, \dots \quad \forall k\geq 2$
with the convention that f^0 is the identity and $f^1=f$.
We say that η^* is a fixed point of f when $f(\eta^*)=\eta^*$.
Note that a fixed point of f is a fixed point of f^k, $\forall k\geq 2$.
We say that η^* is an attractor for f when for each η sufficiently close to η^* we have $\lim\limits_{k\to\infty} f^k(\eta)=\eta^*$.
For an attractor η^* the set $A_{\eta^*}=\{\eta\in\mathbb{R} : \lim\limits_{k\to\infty} f^k(\eta)=\eta^*\}$
is called the basin of attraction of η^*.

Theorem 1 Assume that $f\in C^1(\mathbb{R})$ and that η^* is a fixed point.
If $|f'(\eta^*)|<1$ then η^* is an attractor (asimptotically stable)
If $|f'(\eta^*)|>1$ then η^* is unstable (repellor).

We say that η^* is a p-periodic point of f when
$f^p(\eta^*)=\eta^*$ and $f^k(\eta^*)\neq\eta^*$ for any $k\in\{1,2,\dots,p-1\}$.
We say that a p-periodic point η^* of f is attractor / repellor when η^* is a fixed point attractor / repellor for f^p.
The (positive) orbit of an initial state $\eta\in\mathbb{R}$ is

$$\gamma_\eta^+=\{\eta, f(\eta), \dots, f^k(\eta), \dots\}.$$

The unique solution of the IVP $x_{k+1}=f(x_k)$, $x_0=\eta$ denoted $\varphi(k,\eta)$ can be written also as $\varphi(k,\eta)=f^k(\eta)$
$\forall k\geq 0$, $\forall \eta\in\mathbb{R}$.

Exercise 1. We consider the map $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x-\frac{1}{4}(x^2-2)$.

a) Find the fixed points of f and study their stability.

b) Write the consequences of a) for the IVP

$$x_{k+1}=x_k-\frac{1}{4}(x_k^2-2)\ ,\quad x_0=\eta.$$

c) Using the stair-step diagram study the behaviour of the sequence $(x_k)_{k\geq 0}$ given at b) for any initial value $\eta\in\mathbb{R}$.

Solution. a) The fixed points are the solutions of the equation $f(x)=x$. So we have to solve

$$x-\frac{1}{4}(x^2-2)=x \iff x^2-2=0 \iff x_1=-\sqrt{2} \text{ or } x_2=\sqrt{2}.$$

We found 2 fixed points: $\eta_1^*=-\sqrt{2}$ and $\eta_2^*=\sqrt{2}$.

In order to study the stability we compute first

$$f'(x)=1-\frac{1}{4}\cdot 2x=1-\frac{x}{2}.\quad \text{So}$$

$$|f'(-\sqrt{2})|=\left|1+\frac{\sqrt{2}}{2}\right|>1 \quad\text{and}\quad |f'(\sqrt{2})|=\left|1-\frac{\sqrt{2}}{2}\right|<1.$$

We conclude that $\eta_1^*=-\sqrt{2}$ is unstable, while $\eta_2^*=\sqrt{2}$ is an attractor.

b) Denote, as usual, the unique solution of this IVP by $\varphi(k,\eta)$. Note also that the equation is $x_{k+1}=f(x_k)$.

Since $\eta_1^*=-\sqrt{2}$ is a fixed point of f we have

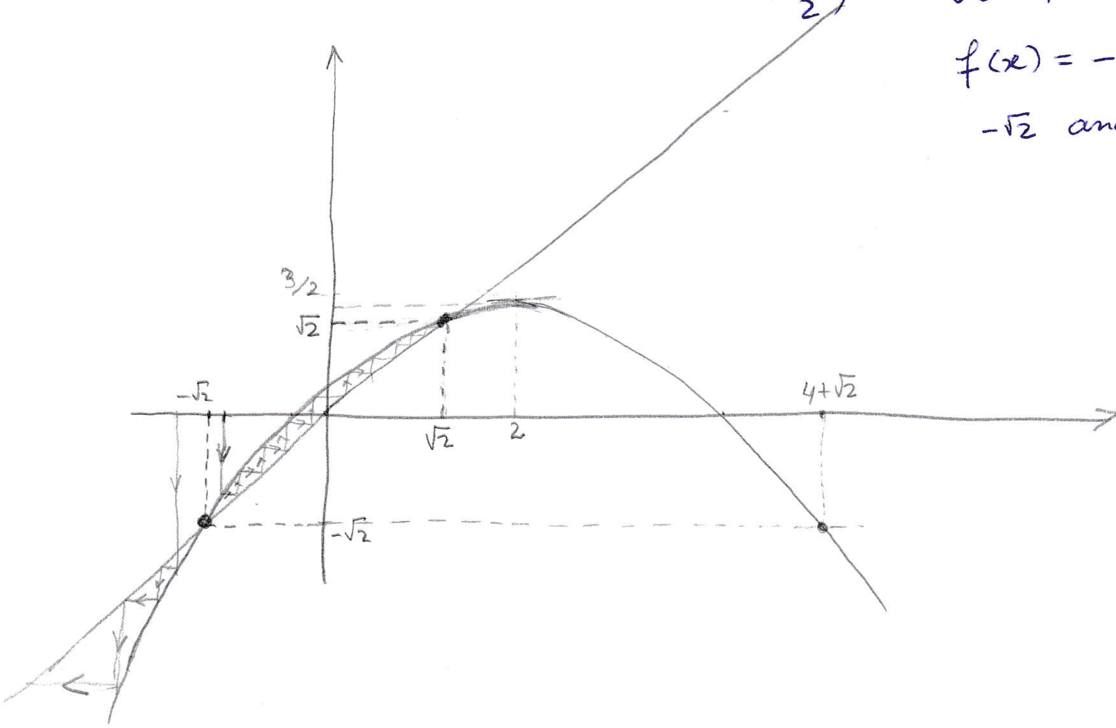
$$\varphi(k,-\sqrt{2})=-\sqrt{2}\ ,\quad \forall k\geq 0.$$

Analogously, $\varphi(k,\sqrt{2})=\sqrt{2}$, $\forall k\geq 0$.

Since $\eta_2^*=\sqrt{2}$ is an attractor for f we have that, for η sufficiently close to $\sqrt{2}$, $\lim\limits_{k\to\infty}\varphi(k,\eta)=\sqrt{2}$.

c) We need first to represent the graph of f and $y=x$.

Note that the graph of f is a parabola that intersect the line $y=x$ in the points $(-\sqrt{2},-\sqrt{2})$ and $(\sqrt{2},\sqrt{2})$; intersect the line $y=0$ at $(x_1,0)$ and $(x_2,0)$ with $x_{1,2}=2\pm\sqrt{6}$; and has a maximum at $(2,\frac{3}{2})$. Also, solving the equation $f(x)=-\sqrt{2}$ we find $-\sqrt{2}$ and $(4+\sqrt{2})$.



We notice that:

- for $x_0=\eta\in(-\infty,-\sqrt{2})$ we have $\lim\limits_{k\to\infty}\varphi(k,\eta)=-\infty$ and is strictly decreasing.
- for $x_0=\eta\in(-\sqrt{2},\sqrt{2})$ we have $\lim\limits_{k\to\infty}\varphi(k,\eta)=\sqrt{2}$ and is strictly increasing
- for $x_0=\eta\in[\sqrt{2},2]$ we have $\lim\limits_{k\to\infty}\varphi(k,\eta)=\sqrt{2}$ and is strictly decreasing
- for $x_0=\eta\in(2,4+\sqrt{2})$ we have $x_1=f(x_0)\in(-\sqrt{2},2)$ so the sequence $(\varphi(k,\eta))_{k\geq 1}$ behaves like in the previous cases.
- for $x_0=4+\sqrt{2}$ we have $\varphi(k,4+\sqrt{2})=-\sqrt{2}$, $\forall\, k\geq 1$.

We say that $4+\sqrt{2}$ is <u>eventually</u> the fixed point $-\sqrt{2}$.

Note also that the equation $f(x)=\sqrt{2}$ has ~~to~~ 2 roots, $\sqrt{2}$ and $4-\sqrt{2}$. Hence $4-\sqrt{2}$ is eventually the fixed point $\sqrt{2}$ and $\varphi(k,4-\sqrt{2})=\sqrt{2}$, $\forall k\geq 1$.

- for $x_0=\eta\in(4+\sqrt{2},\infty)$ we have $x_1=f(x_0)\in(-\infty,-\sqrt{2})$ so the sequence $(\varphi(k,\eta))_{k\geq 1}$ behaves like in the first case.

<u>We conclude that</u>: $A_{\sqrt{2}}=(-\sqrt{2},4+\sqrt{2})$, that is

$\lim\limits_{k\to\infty}\varphi(k,\eta)=\sqrt{2}$, for any $\eta\in(-\sqrt{2},4+\sqrt{2})$. $\varphi(k,-\sqrt{2})=$
$=\varphi(k,4+\sqrt{2})=-\sqrt{2}$, $\forall k\geq 1$.

and $\lim\limits_{k\to\infty}\varphi(k,\eta)=-\infty$, for any $\eta\in(-\infty,-\sqrt{2})\cup(4+\sqrt{2},+\infty)$.

<u>Exercise 2</u> We consider the map $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2-1$.

a) Find the fixed points of f and study their stability.

b) Find the 2-periodic ~~points~~ orbits of f and study their stability.

c) Using the stair-step diagram depict the orbits corresponding to the initial states $\eta=2$ and, respectively, $\eta=-\frac{1}{4}$ and describe their behaviour.

<u>Solution</u> a) $f(x)=x \iff x^2-1=x \iff x^2-x-1=0 \iff x_{1,2}=\frac{1\pm\sqrt{5}}{2}$

We found 2 fixed points: $\eta_1^*=\frac{1-\sqrt{5}}{2}$ and $\eta_2^*=\frac{1+\sqrt{5}}{2}$.

$f'(x)=2x$

$\left|f'\left(\frac{1-\sqrt{5}}{2}\right)\right|=|1-\sqrt{5}|=\sqrt{5}-1>1$ and $\left|f'\left(\frac{1+\sqrt{5}}{2}\right)\right|=1+\sqrt{5}>1$.

Hence both fixed points are unstable.

b) $f^2(x)=f(f(x))=f(x)^2-1=(x^2-1)^2-1 \Rightarrow$

~~$=x^4-2x^2=x^2$~~ $f^2(x)=x^4-2x^2$ and $(f^2)'(x)=4x^3-4x$

$f^2(x) = x \Leftrightarrow x^4 - 2x^2 - x = 0 \Leftrightarrow$

We know that the two fixed points of f must be fixed points of f^2. Thus $(x^2 - x - 1)$ devide exactly the above polynomiol.

$x^2(x^2 - x - 1) + x^3 + x^2 - 2x^2 - x = 0 \Leftrightarrow (x^2 - x - 1)(x^2 + x) = 0$

$\Leftrightarrow x_{1,2} = \frac{1 \pm \sqrt{5}}{2}, \; x_3 = 0, \; x_4 = -1.$

So, f^2 has 4 fixed points. Two of them are the fixed points of f and the other two form the 2-periodic orbit: $0, -1, 0, -1, 0, -1, \ldots$

We have $(f^2)'(0) = 0$, hence the 2-periodic orbit is an attractor.

c) We left as homework.

~~Exercise 3~~ <u>Property</u> Assume that $f \in C^1(\mathbb{R})$ and that $\{\eta_1, \eta_2\}$ is a 2-periodic orbit of f.

If $|f'(\eta_1) \cdot f'(\eta_2)| < 1$ then this orbit is an attractor.

If $|f'(\eta_1) \cdot f'(\eta_2)| > 1$ then this orbit is unstable.

<u>Proof.</u> Since $f^2(x) = f(f(x))$ we have

$(f^2)'(x) = f'(f(x)) \cdot f'(x).$

Since $\{\eta_1, \eta_2\}$ is a 2-periodic orbit we have $f(\eta_2) = \eta_1$ and $f(\eta_1) = \eta_2$. Anyway we have

$(f^2)'(\eta_1) = f'(f(\eta_1)) \cdot f'(\eta_1) = f'(\eta_2) \cdot f'(\eta_1).$

The conclusion follows by Theorem 1 and the definition of attracting periodic point. $\square$.

Exercise 3 We consider the map $T: [0,1] \to \mathbb{R}$, $T(x) = 1 - |2x-1|$.

a) Represent the graph of T. Is $\eta_1^* = 0$ a fixed point?
Find all the fixed points of T.

b) Compute the orbits corresponding to the initial states
$\eta = \frac{3}{4}$, $\eta = \frac{3}{8}$, $\eta = \frac{3}{2^n}$, $n \geq 2$.

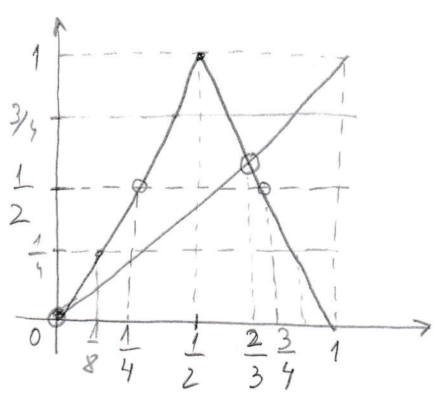
c) How many solutions have the equations:
$T(x) = 0$; $T(x) = 1$; $T(x) = \frac{1}{2}$; $T^2(x) = 0$; $T^2(x) = 1$;
$T^2(x) = \frac{1}{2}$?

d) Compute $T^2(x)$.

e) Represent the graphs of T^2 and T^3. How many fixed points they have?

f) T has a 2-periodic orbit? Or a 3-periodic orbit?
T has a 1492-periodic orbit?

Solution
$$T(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 2(1-x), & x \in (\frac{1}{2}, 1] \end{cases}$$



Since $T(0) = 0$, of course $\eta_1^* = 0$ is a fixed point of T.
By solving $2(1-x) = x$ we find that $\eta_2^* = \frac{2}{3}$ is another f.p.
Thus, T has 2 fixed points.

b) $\eta = \frac{3}{4}$, $T(\eta) = T(\frac{3}{4}) = 1 - |2 \cdot \frac{3}{4} - 1| = 1 - \frac{1}{2} = \frac{1}{2}$,
$T(\frac{1}{2}) = 1$, $T(1) = 0$.
So, the orbit is $\frac{3}{4}, \frac{1}{2}, 1, 0, 0, 0, \ldots$

$\frac{3}{8}$, $T\left(\frac{3}{8}\right) = 1 - \left|2\cdot\frac{3}{8} - 1\right| = 1 - \left|-\frac{1}{4}\right| = 1 - \frac{1}{4} = \frac{3}{4}$

So, the orbit is $\frac{3}{8}, \frac{3}{4}, \frac{1}{2}, 1, 0, 0, 0, \dots$

Let $n \geq 3$ then $T\left(\frac{3}{2^n}\right) = 2\cdot\frac{3}{2^n} = \frac{3}{2^{n-1}}$

$\frac{3}{2^n} < \frac{1}{2}$

So, the orbit is $\frac{3}{2^n}, \frac{3}{2^{n-1}}, \dots, \frac{3}{2^3}, \frac{3}{2^2}, \frac{1}{2}, 1, 0, 0, \dots$

c) $T(x) = 0$ has two solutions : 0 and 1

$T(x) = 1$ has one sol. : $\frac{1}{2}$

$T(x) = \frac{1}{2}$ has two sol : $\frac{1}{4}$ and $\frac{3}{4}$

We deduced all these from the graph.

$T(T(x)) = 0 \iff T(x) = 0$ or $T(x) = 1 \iff x \in \{0, \frac{1}{2}, 1\}$

$T(T(x)) = 1 \iff T(x) = \frac{1}{2} \iff x \in \{\frac{1}{4}, \frac{3}{4}\}$

$T(T(x)) = \frac{1}{2} \iff T(x) = \frac{1}{4}$ or $T(x) = \frac{3}{4} \iff x \in \{\frac{1}{8}, \frac{7}{8}, \frac{3}{8}, \frac{5}{8}\}$

d) $$T^2(x) = T(T(x)) = \begin{cases} 2T(x), & T(x) \in [0, \frac{1}{2}] \\ 2 - 2T(x), & T(x) \in (\frac{1}{2}, 1] \end{cases} =$$

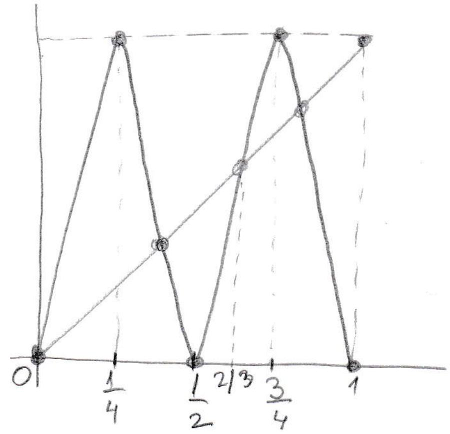
$$= \begin{cases} 4x, & x \in [0, \frac{1}{4}] \\ 2 - 4x, & x \in (\frac{1}{4}, \frac{1}{2}] \\ 2 - 4(1-x), & x \in (\frac{1}{2}, \frac{3}{4}] \\ 4(1-x), & x \in (\frac{3}{4}, 1] \end{cases}$$

e) From c) we know that $T^2(0) = T^2(\frac{1}{2}) = T^2(1) = 0$

and $T^2(\frac{1}{4}) = T^2(\frac{3}{4}) = 1$.

From d) we know that T is linear on each of the intervals $(0, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{4})$ and $(\frac{3}{4}, 1)$.

Hence:



We notice that T^2 has 4 fixed points.

Two of them are, of course, the fixed points of T, 0 and $\frac{2}{3}$.

In order to represent the graph of T^3 we solve the eq.:

$T^3(x) = 0 \iff T^2(T(x)) = 0 \iff T(x) \in \{0, \frac{1}{2}, 1\} \iff$

$\iff x \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$

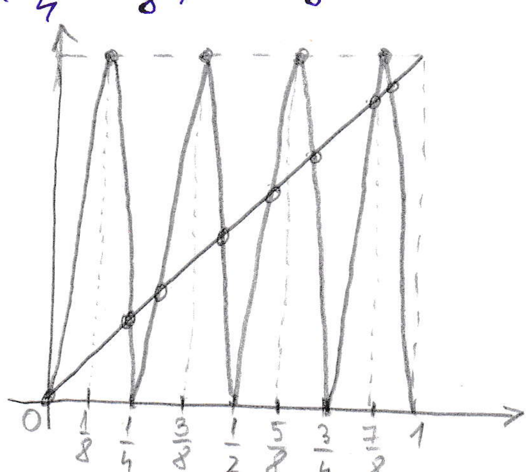
and also the eq

$T^3(x) = 1 \iff T(T^2(x)) = 1 \iff T^2(x) = \frac{1}{2} \iff x \in \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$

We realize that T^3 is linear on each of the intervals

$(0, \frac{1}{8})$, $(\frac{1}{8}, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{8})$, $(\frac{3}{8}, \frac{1}{2})$, $(\frac{1}{2}, \frac{5}{8})$, $(\frac{5}{8}, \frac{3}{4})$, &

$(\frac{3}{4}, \frac{7}{8})$, $(\frac{7}{8}, 1)$. Hence



We notice that T^3 has 8 fixed points.

Two of them are, of course, the fixed points of T, 0 and $\frac{2}{3}$.

f) Since T^2 has two fixed points which are not fixed points of T, ~~th~~ we deduce that they form a 2-periodic orbit for T. Actually, they can be found exactly by solving $2-4x = x \Leftrightarrow x = \frac{2}{5}$ and ~~computing~~ ~~th~~ $4-4x = x \Leftrightarrow x = \frac{4}{5}$.

So, T has a unique 2-periodic orbit: $\{\frac{2}{5}, \frac{4}{5}\}$.

Since T^3 has 6 fixed points which are not fixed points of T, we deduce that they form <u>two</u> 3-periodic orbits.

Since T has at least a 3-periodic orbit, we deduce using the Sharkowski Theorem that T has periodic orbits of any period.

<u>Exercise 4</u> We consider the map $G : [0,1] \to \mathbb{R}$, $G(x) = 4x(1-x)$.

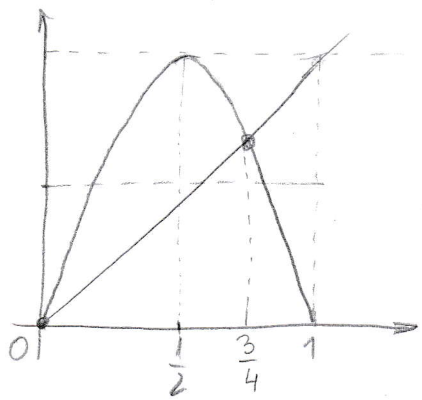
a) Represent the graph of G. Is $x_1^* = 0$ a fixed point?
Find all the fixed points of G.

b) How many solutions have the equations:
$G(x) = 0$; $G(x) = 1$; $G(x) = \frac{1}{2}$; $G^2(x) = 0$, $G^2(x) = 1$; $G^2(x) = \frac{1}{2}$;
$G^3(x) = 0$; $G^3(x) = 1$?

c) Represent the graphs of G^2 and G^3. How many fixed points they have?

d) G has a 2-periodic orbit? Or a 3-periodic ~~point~~ orbit?
G has a 1492-periodic orbit?

__Solution__ We have $G(0) = G(1) = 0$ and the graph of G is a parabola with a maximum at $\left(\frac{1}{2}, 1\right)$.



Since $G(0)=0$, of course $\eta_1^* = 0$ is a fixed point of G.

We note that G has another f.p which can be found ~~by~~ solving

$4x(1-x) = x \quad , \quad x \neq 0$

$4 - 4x = 1 \qquad x = \frac{3}{4} \qquad \eta_2^* = \frac{3}{4}$.

Thus, G has two fixed points 0 and $\frac{3}{4}$.

b) From the graph we see that

$G(x) = 0$ has two solutions: 0 and 1

$G(x) = 1$ has one solution: $\frac{1}{2}$

$G(x) = \frac{1}{2}$ has two solutions: $a_1 \in (0, \frac{1}{2})$ and $a_2 \in (\frac{1}{2}, 1)$

$G^2(x) = 0 \Leftrightarrow G(G(x)) = 0 \Leftrightarrow G(x) \in \{0, 1\} \Leftrightarrow x \in \{0, \frac{1}{2}, 1\}$

$G^2(x) = 1 \Leftrightarrow G(G(x)) = 1 \Leftrightarrow G(x) = \frac{1}{2} \Leftrightarrow x \in \{a_1, a_2\}$

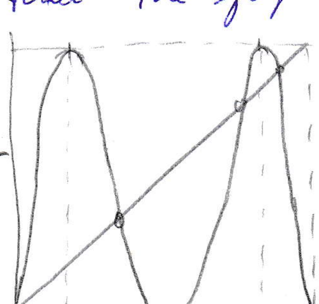
$G^2(x) = \frac{1}{2} \Leftrightarrow G(G(x)) = \frac{1}{2} \Leftrightarrow G(x) \in \{a_1, a_2\}$ and this equation has 4 solutions.

$G^3(x) = 0 \Leftrightarrow G(G^2(x)) = 0 \Leftrightarrow G^2(x) \in \{0, 1\} \Leftrightarrow x \in \{0, a_1, \frac{1}{2}, a_2, 1\}$

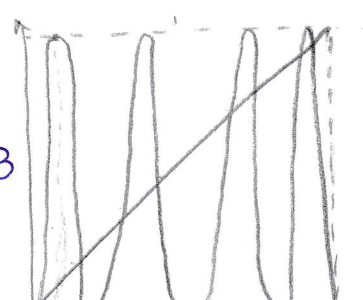
$G^3(x) = 1 \Leftrightarrow G(G^2(x)) = 1 \Leftrightarrow G^2(x) = \frac{1}{2}$ has 4 solutions.

c) The graph of G^2 looks like the graph of T^2, only that the graph of G^2 is smooth.

G^2



G^3



The discussions are like for T^2 and T^3, respectively.