Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Bachelor, Computer Science, Groups 911-917, Academic Year 2016-2017

Mathematical Analysis
Seminar 1

1. Prove that for any real numbers $a_1, a_2, \ldots, a_n > 0$ ($n \in \mathbb{N}$, $n \geq 2$) satisfying $a_1 \cdot a_2 \cdot \ldots \cdot a_n = 1$ we have $a_1 + a_2 + \ldots + a_n \geq n$.

2. For any real numbers $x_1, x_2, \ldots, x_n > 0$ ($n \in \mathbb{N}$, $n \geq 2$) denote

$$H(x_1, \ldots, x_n) := \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} \quad \text{(the harmonic mean);}$$

$$G(x_1, \ldots, x_n) := \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} \quad \text{(the geometric mean);}$$

$$A(x_1, \ldots, x_n) := \frac{x_1 + x_2 + \ldots + x_n}{n} \quad \text{(the arithmetic mean).}$$

Show that the following inequalities hold:

$$\min\{x_1, \ldots, x_n\} \leq H(x_1, \ldots, x_n) \leq G(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}.$$

3. Prove that for any $x \in [-1, +\infty)$ and $n \in \mathbb{N}$ we have

$$(1+x)^n \geq 1 + nx \quad \text{(Bernoulli's Inequality).}$$

Deduce that, whenever $m \in \mathbb{N}$ is even, the following inequality holds for all $y \in \mathbb{R}$:

$$(1+y)^m \geq 1 + my.$$

4. Study the boundedness of each set A_i in the list below, by finding $\inf A_i$ (or $\min A_i$ if it exists) and $\sup A_i$ (or $\max A_i$ if it exists).

$$\begin{aligned} A_1 &= \{x^2 \mid x \in \mathbb{Z}\}; & A_5 &= \{x^2 + x \mid x \in \mathbb{R}\}; \\ A_2 &= \{x \in \mathbb{Q} \mid x^2 \leq 2\}; & A_6 &= \{x^2 + y \mid x, y \in \mathbb{R}\}; \\ A_3 &= A_1 \cap A_2; & A_7 &= \{n/(n+1) \mid n \in \mathbb{N}\}; \\ A_4 &= A_1 \cup A_2; & A_8 &= \{n/(n+k) \mid n, k \in \mathbb{N}\}. \end{aligned}$$

5. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions defined on a nonempty set D. Prove that

$$\inf_{x \in D} [f(x) + g(x)] \geq \inf_{x \in D} f(x) + \inf_{x \in D} g(x) \quad \text{and} \quad \sup_{x \in D} [f(x) + g(x)] \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x).$$

Deduce that for any sequences of real numbers, $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$, we have

$$\inf_{n \in \mathbb{N}} (a_n + b_n) \geq \inf_{n \in \mathbb{N}} a_n + \inf_{n \in \mathbb{N}} b_n \quad \text{and} \quad \sup_{n \in \mathbb{N}} (a_n + b_n) \leq \sup_{n \in \mathbb{N}} a_n + \sup_{n \in \mathbb{N}} b_n.$$

Give examples where the above inequalities are strict.

6. Decide which of the following sets are neighborhoods of 0. Justify.

$$A = [-1, 1] \cup \{2\}; \quad B = (-1, 1) \cap \mathbb{Q}; \quad C = \bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right].$$

7. Let $A \subseteq \mathbb{R}$ be a nonempty set, which is bounded from below (respectively from above) by $\alpha \in \mathbb{R}$. Prove that $\inf A = \alpha$ (respectively $\sup A = \alpha$) if and only if $V \cap A \neq \emptyset$ for every $V \in \mathcal{V}(\alpha)$.