Seminars 6 and 7: Linear difference equations. Discrete dynamical systems.

1. a) Find all the sequences $(x_k)_{k\geq 0}$ that satisfy $x_{k+3} = k3^k + 5k - 2$, $k \geq 0$.
b) Find all the sequences $(x_k)_{k\in\mathbb{Z}}$ that satisfy $x_{k+3} = k3^k + 5k - 2$, $k \in \mathbb{Z}$.

What can we say about the existence and uniqueness of the solutions (i.e. sequences of real numbers) of the IVPs:
c) $x_{k+2} = x_k, \quad x_0 = 1, \quad x_1 = -1$, when considering $k \geq 0$.
d) $x_{k+2} = x_k, \quad x_0 = 1, \quad x_1 = -1$, when considering $k \in \mathbb{Z}$.
e) $x_{k+2} = x_{k+1}, \quad x_0 = 1, \quad x_1 = -1$, when considering $k \geq 0$.
f) $x_{k+2} = x_{k+1}, \quad x_0 = 1, \quad x_1 = -1$, when considering $k \in \mathbb{Z}$.
g) $x_{k+1} = x_k^2, \quad x_0 = -1$, when considering $k \geq 0$.
h) $x_{k+1} = x_k^2, \quad x_0 = -1$, when considering $k \in \mathbb{Z}$.

2. Let $\eta \in \mathbb{R}$ be a fixed parameter. Find the solution $(x_k)_{k\geq 0}$ of the initial value problem

$$x_{k+1} = 2x_k, \quad x_0 = \eta.$$

Check the solution you obtained. What is the long term behavior of this sequence? Discuss with respect to η.

3. Let $\lambda \in \mathbb{R}^*$ and $\eta \in \mathbb{R}$ be fixed parameters. Find the solution $(x_k)_{k\geq 0}$ of the initial value problem

$$x_{k+1} = \lambda x_k, \quad x_0 = \eta.$$

Check the solution you obtained. What is the long term behavior of this sequence? Discuss with respect to λ and η.

4. Find the solution $(x_k)_{k\in\mathbb{Z}}$ of the initial value problem

$$x_{k+2} + x_{k+1} + x_k = 0, \quad x_0 = 0, \quad x_1 = 1.$$

Check the solution you obtained. What is the long term behavior of this sequence?

Find $\eta^* \in \mathbb{R}$ such that the constant sequence $x_k = \eta^*$ for any $k \geq 0$ satisfies:
5. $x_{k+1} = 2x_k + 1$, $k \geq 0$. What is the difference equation satisfied by $y_k = x_k - \eta^*$, $k \geq 0$? This new difference equation has constant solutions? Find the general solution

of the difference equation for (y_k) and, after, find the general solution of the difference equation for (x_k).

6. $x_{k+1} = \frac{1}{2}x_k + 1$, $k \geq 0$. The same requirements as at the previous exercise.

7. $x_{k+1} = \frac{1}{2}x_k - x_k^2$, $k \geq 0$. 8. $x_{k+1} = \frac{1}{2}x_k - x_k^2 - 3$, $k \geq 0$.

Find solutions of the form $x_k = r^k$, $k \geq 0$ of the difference equation (here we look for $r \in \mathbb{R}^*$)

9. $x_{k+2} = x_{k+1} + x_k$, $k \geq 0$.

10. $x_{k+2} = x_{k+1} + 6x_k$, $k \geq 0$.

11. Find solutions of the form $x_k = a\,3^k$ of the difference equation $x_{k+1} = 2x_k + 3^k$, $k \geq 0$. Here we look for $a \in \mathbb{R}$.

12. Find solutions of the form $x_k = ak + b$ of the difference equation $x_{k+1} = 2x_k - k$, $k \geq 0$. Here we look for $a, b \in \mathbb{R}$.

For each of the following difference equations find a particular solution of the indicated form. If no form is indicated (and the equation is nonhomogeneous), try a constant solution. Find its general solution using the fundamental theorems for linear difference equations and the characteristic equation method.

13. $x_{k+1} = \frac{1}{3}\,x_k$; 14. $x_{k+1} = \frac{1}{2}\,x_k + 2$; 15. $x_{k+1} = 2\,x_k + \frac{1}{2}$; 16. $x_{k+1} = -3x_k$;

17. $x_{k+1} = 4\,x_k + 3^{k+1}$, with $x_k^p = a \cdot 3^k$; 18. $x_{k+1} = 1/3\,x_k + 2^k$, with $x_k^p = a \cdot 2^k$;

19. $x_{k+2} - 6x_{k+1} + 9x_k = 0$;

20. $x_{k+2} + x_{k+1} + x_k = 0$. Explain how this result agrees with exercise 4.

21. Find the general solution of the linear planar system $x_{k+1} = 1/2\,x_k + y_k$, $\quad y_{k+1} = -1/5\,y_k$ in two ways. One way must be by reducing it to a second order difference equation in x_k.

22. Find the general solution of the linear planar system $x_{k+1} = -x_k - y_k$, $\quad y_{k+1} = -x_k + y_k$ by reducing it to a second order difference equation.

23. Write the Euler numerical formula with stepsize $h > 0$ for the IVP $x' = -120x$, $x(0) = 1$ and solve the difference equation you obtained in two cases: $h = 0.1$ and $h = 0.001$. Discuss on the long term behavior of the solution of the differential equation and, also, of the solution of the difference equation.

24. Write the Euler numerical formula with stepsize $h > 0$ for the IVP $x' = 2x(1-x),\ x(0) = 1$.

25. Find the fixed points of the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - \frac{1}{4}(x^2 - 2)$ and study their stability.

26*. Represent the stair-step diagram of the function from the previous exercise and try to discuss the long term behavior of the solution of $x_{k+1} = x_k - \frac{1}{4}(x_k^2 - 2)$, $k \geq 0$ with respect to an arbitrary $x_0 \in \mathbb{R}$.

27. Prove that the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = m + \varepsilon \sin x$, where $m > 0$ and $0 < \varepsilon < 1$ has a unique fixed point which is asymptotically stable. The equation $x = m + \varepsilon \sin x$ is known as *Kepler equation* and arises in the study of planetary motion.

28. Study the stability of the fixed point $(0, 0)$ of the following planar linear difference system:

a) $x_{k+1} = 3/5\, x_k + 1/5\, y_k, \quad y_{k+1} = 1/5\, x_k + 3/5\, y_k;$

b) $x_{k+1} = 1/2\, x_k + y_k, \quad y_{k+1} = -1/5\, y_k;$

c) $x_{k+1} = -x_k - y_k, \quad y_{k+1} = -x_k + y_k.$

29. Find the fixed points and the 2-periodic points of the maps $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = -x^3$ and $g(x) = x^2 - 1$.

30. In order to study the stability of the fixed point 0 of the maps $f, g, h : \mathbb{R} \to \mathbb{R}$, $f(x) = x + x^3$, $g(x) = x - x^3$, $h(x) = x + x^2$, note that the linearization method can not be applied. Study the stability of the fixed point 0 using the stair-step diagram.

31. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$.

(a) Find the fixed points of f and study their stability using the linearization method.

(b) Represent the graph of f and find geometrically the fixed points of f.

(c) Find directly $\varphi(k, 0)$ (or, in other notation, $f^k(0)$) for any $k \geq 0$. Which is the orbit corresponding to the initial state 0? What remarkable property has this orbit? Depict this orbit using the stair-step diagram.

d) Let $\eta = 2$, and, respectively, $\eta = -1/4$. Using the stair-step diagram describe the long-term behavior of the orbit that starts at η (in other notation, of the sequence defined by $x_{k+1} = x_k^2 - 1,\ x_0 = \eta$).

32. Using the stair-step diagram, estimate the basin of attraction for each of the fixed points (if there is any which is an attractor) of the map

$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x^2 + 5}{2x} .$$

33. Find all the solutions of each of the following difference equations and which also satisfy the given conditions: a) $x_{k+2} - 5\, x_{k+1} + 6x_k = 12$; b) $x_{k+1} = 1 - x_k^2,\ x_0 = 0$; c) $x_{k+2} + x_{k+1} + x_k = 0,\ x_0 = 0$.

34. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x(1 - x)$.

a) Find its fixed points and study their stability.

b) Let $I_1 = (-\infty, 0)$, $I_2 = (0, 1)$ and $I_3 = (1, \infty)$. Find $f(I_1)$, $f(I_2)$ and $f(I_3)$.

c) Find the orbits corresponding to the initial states $\eta = 0$ and, respectively, $\eta = 1$.

d) Using the stair-step diagram, describe the long-term behavior of the orbits corresponding to the initial states: $\eta = 1/8$, $\eta = 7/8$, $\eta = -1/8$ and, respectively, $\eta = 9/8$.

e) Estimate the basin of attraction of the stable fixed point of f.

35. Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k,\ x_0 = 0,\ x_1 = 1.$$

36. Find the solution of

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k, \quad x_0 = 0, \quad x_1 = 0.$$

Hint: look for $a, b \in \mathbb{R}$ such that $(x_k)_p = ak + b$ is a particular solution of the difference equation.