Geometry
Problem booklet

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Contents

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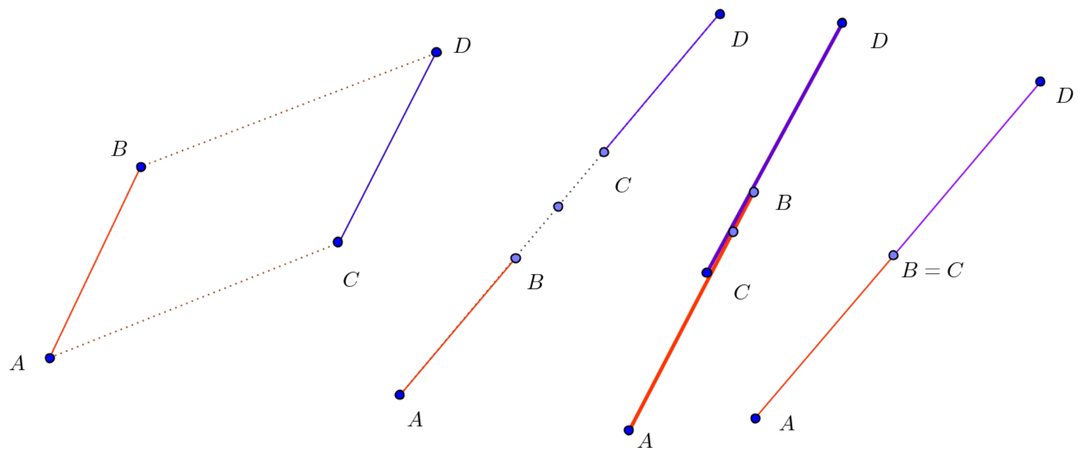
1 Week 1: Vector algebra

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Free vectors

Vectors Let $\mathcal{P}$ be the three dimensional physical space in which we can talk about points, lines, planes and various relations among them. If $(A, B) \in \mathcal{P} \times \mathcal{P}$ is an ordered pair, then A is called the *original point* or the *origin* and B is called the *terminal point* or the *extremity* of (A, B).

Definition 1.1.1. *The ordered pairs (A, B), (C, D) are said to be equipollent, written $(A, B) \sim (C, D)$, if the segments $[AD]$ and $[BC]$ have the same midpoint.*



Pairs of equipollent points $(A, B) \sim (C, D)$

Remark 1.1.2. If the points A, B, C, $D \in \mathcal{P}$ are not collinear, then $(A, B) \sim (C, D)$ if and only if $ABDC$ is a parallelogram. In fact the length of the segments $[AB]$ and $[CD]$ is the same whenever $(A, B) \sim (C, D)$.

Proposition 1.1.3. *If (A, B) is an ordered pair and $O \in \mathcal{P}$ is a given point, then there exists a unique point X such that $(A,B) \sim (O, X)$.*

Proposition 1.1.4. *The equipollence relation is an equivalence relation on $\mathcal{P} \times \mathcal{P}$.*

Definition 1.1.5. *The equivalence classes with respect to the equipollence relation are called* (free) vectors.

Denote by $\overrightarrow{AB}$ the equivalence class of the ordered pair (A, B), that is $\overrightarrow{AB} = \{(X, Y) \in \mathcal{P} \times \mathcal{P} \mid (X, Y) \sim (A, B)\}$ and let $\mathcal{V} = \mathcal{P} \times \mathcal{P} \big/_{\sim} = \{\overrightarrow{AB} \mid (A, B) \in \mathcal{P} \times \mathcal{P}\}$ be the set of (free) vectors. The *length* or the *magnitude* of the vector $\overrightarrow{AB}$, denoted by $\| \overrightarrow{AB} \|$ or by $| \overrightarrow{AB} |$, is the length of the segment $[AB]$.

Remark 1.1.6. *If two ordered pairs (A, B) and (C, D) are equippllent, i.e. the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal, then they have the same length, the same direction and the same sense. In fact a vector is determined by these three items.*

Proposition 1.1.7.

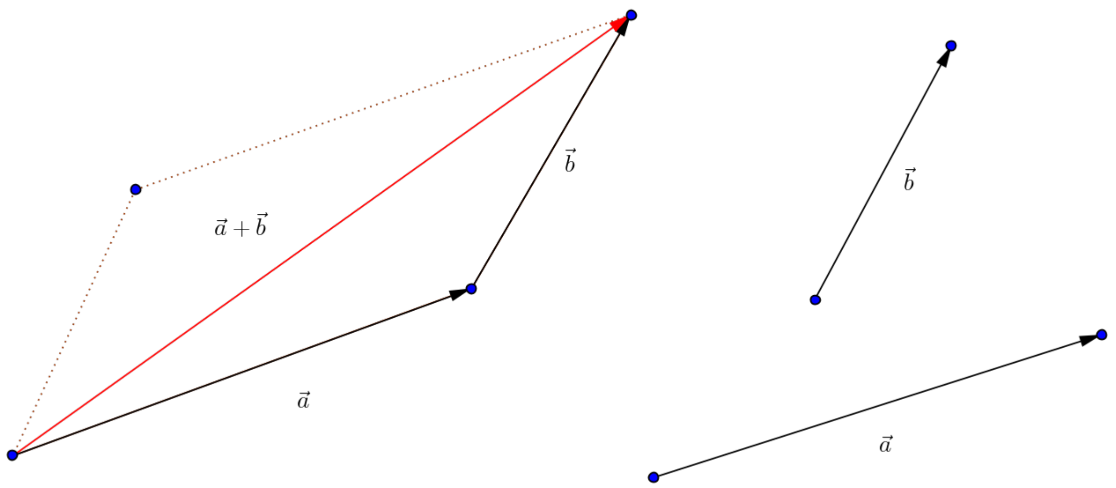
1. $\overrightarrow{AB}=\overrightarrow{CD}\Leftrightarrow\overrightarrow{AC}=\overrightarrow{BD}$.
2. $\forall A, B, O \in \mathcal{P}$, $\exists! X \in \mathcal{P}$ *such that* $\overrightarrow{AB}=\overrightarrow{OX}$.
3. $\overrightarrow{AB}=\overrightarrow{A'B'}$, $\overrightarrow{BC}=\overrightarrow{B'C'}\Rightarrow\overrightarrow{AC}=\overrightarrow{A'C'}$.

Definition 1.1.8. If O, $M \in \mathcal{P}$, the the vector $\overrightarrow{OM}$ is denoted by $\overrightarrow{r}_M$ and is called the *position vector of M with respect to O.*

Corollary 1.1.9. *The map $\varphi_O : \mathcal{P} \to \mathcal{V}$, $\varphi_O(M) = \overrightarrow{r}_M$ is one-to-one and onto, i.e bijective.*

1.1.1 Operations with vectors

• **The addition of vectors** Let $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{V}$ and $O \in \mathcal{P}$ be such that $\overrightarrow{a}=\overrightarrow{OA}$, $\overrightarrow{b}=\overrightarrow{AB}$. The vector $\overrightarrow{OB}$ is called the *sum* of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ and is written $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{a}+\overrightarrow{b}$.



Let O' be another point and A', $B' \in \mathcal{P}$ be such that $\overrightarrow{O'A'}=\overrightarrow{a}$, $\overrightarrow{A'B'}=\overrightarrow{b}$. Since $\overrightarrow{OA}=\overrightarrow{O'A'}$ and $\overrightarrow{AB}=\overrightarrow{A'B'}$ it follows, according to Proposition 1.1.4 (3), that $\overrightarrow{OB}=\overrightarrow{O'B'}$. Therefore the vector $\overrightarrow{a}+\overrightarrow{b}$ is independent on the choice of the point O.

Proposition 1.1.10. *The set $\mathcal{V}$ endowed to the binary operation $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, $(\overrightarrow{a}, \overrightarrow{b}) \mapsto \overrightarrow{a}+\overrightarrow{b}$, is an abelian group whose zero element is the vector $\overrightarrow{AA}=\overrightarrow{BB}=\overrightarrow{0}$ and the opposite of $\overrightarrow{AB}$, denoted by $-\overrightarrow{AB}$, is the vector $\overrightarrow{BA}$.*

In particular the addition operation is associative and the vector

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$

is usually denoted by $\vec{a}+\vec{b}+\vec{c}$. Moreover the expression

$$((\cdots(\vec{a}_1+\vec{a}_2)+\vec{a}_3+\cdots+\vec{a}_n)\cdots), \tag{1.1}$$

is independent of the distribution of paranthesis and it is usually denoted by

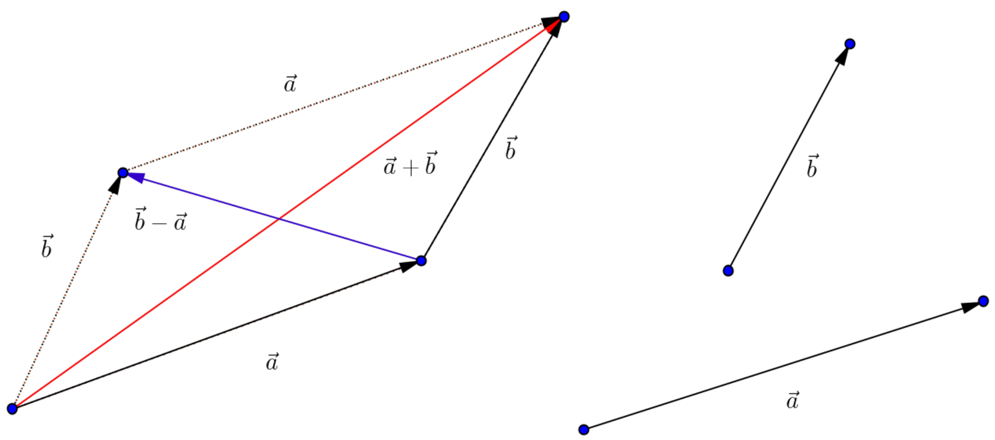
$$\vec{a}_1+\vec{a}_2+\cdots+\vec{a}_n.$$

Example 1.1.11. *If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{P}$ are some given points, then*

$$\overrightarrow{A_1A_2}+\overrightarrow{A_2A_3}+\cdots+\overrightarrow{A_{n-1}A_n}=\overrightarrow{A_1A_n}.$$

This shows that $\overrightarrow{A_1A_2}+\overrightarrow{A_2A_3}+\cdots+\overrightarrow{A_{n-1}A_n}+\overrightarrow{A_nA_1}=\vec{0}$, namely the sum of vectors constructed on the edges of a closed broken line is zero.

Corolarul 1.1.12. *If $\vec{a}=\overrightarrow{OA}$, $\vec{b}=\overrightarrow{OB}$ are given vectors, there exists a unique vector $\vec{x}\in\mathcal{V}$ such that $\vec{a}+\vec{x}=\vec{b}$. In fact $\vec{x}=\vec{b}+(-\vec{a})=\overrightarrow{AB}$ and is denoted by $\vec{b}-\vec{a}$.*



- **The multiplication of vectors with scalars**

  Let $\alpha \in \mathbb{R}$ be a scalar and $\vec{a}=\overrightarrow{OA}\in\mathcal{V}$ be a vector. We define the vector $\alpha\cdot\vec{a}$ as follows:

  - $\alpha\cdot\vec{a}=\vec{0}$ if $\alpha=0$ or $\vec{a}=\vec{0}$;
  - if $\vec{a}\neq\vec{0}$ and $\alpha>0$, there exists a unique point on the half line $]OA$ such that $||OB||=\alpha\cdot||OA||$ and define $\alpha\cdot\vec{a}=\overrightarrow{OB}$;
  - if $\alpha<0$ we define $\alpha\cdot\vec{a}=-(|\alpha|\cdot\vec{a})$.

The external binary operation

$$\mathbb{R} \times \mathcal{V} \to \mathcal{V},\ (\alpha, \vec{a}) \mapsto \alpha \cdot \vec{a}$$

is called the *multiplication of vectors with scalars*.

Proposition 1.1.13. *The following properties hold:*

$(v1)$ $(\alpha+\beta)\cdot \vec{a} = \alpha\cdot \vec{a} + \beta\cdot \vec{a},\ \forall\, \alpha,\beta \in \mathbb{R},\ \vec{a} \in \mathcal{V}.$

$(v2)$ $\alpha \cdot (\vec{a} + \vec{b}) = \alpha\cdot \vec{a} + \alpha\cdot \vec{b},\ \forall\, \alpha \in \mathbb{R},\ \vec{a}, \vec{b} \in \mathcal{V}.$

$(v3)$ $\alpha \cdot (\beta\cdot \vec{a}) = (\alpha\beta)\cdot \vec{a}, \forall\, \alpha,\beta \in \mathbb{R}.$

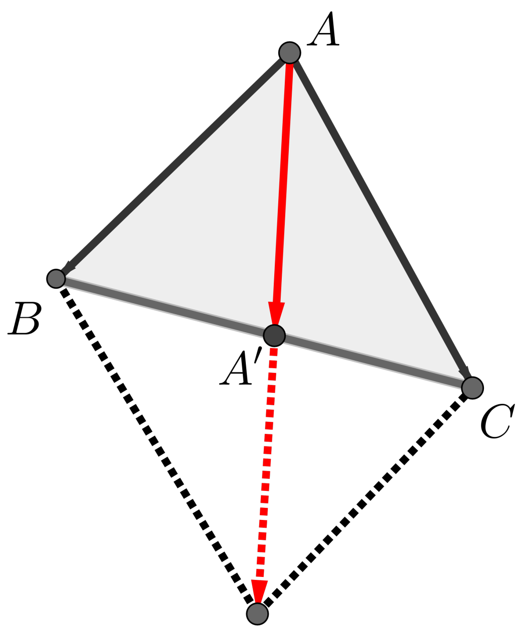
$(v4)$ $1\cdot \vec{a} = \vec{a},\ \forall\ \vec{a} \in \mathcal{V}.$

1.1.2 The vector structure on the set of vectors

Theorem 1.1.14. *The set of (free) vectors endowed with the addition binary operation of vectors and the external binary operation of multiplication of vectors with scalars is a real vector space.*

Example 1.1.15. *If A' is the midpoint of the egde $[BC]$ of the triangle ABC, then*

$$\overrightarrow{AA'} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}).$$



1.2 Problems

1. ([4, Problema 3, p. 1]) Let $OABCDE$ be a regular hexagon in which $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OE} = \vec{b}$. Express the vectors $\overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}$ in terms of the vectors $\vec{a}$ and $\vec{b}$. Show that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE} = 3\,\overrightarrow{OC}$.

2. Consider a tetrahedron $ABCD$. Find the the following sums of vectors:

   (a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$.

   (b) $\overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{DC}$.

   (c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{CD}$.

3. Consider a pyramid with the vertex at S and the basis a parallelogram $ABCD$ whose diagonals are concurrent at O. Show the equality $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\,\overrightarrow{SO}$.

4. Let E and F be the midpoints of the diagonals of a quadrilateral $ABCD$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right) = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{CB}\right).$$

5. In a triangle ABC we consider the height AD from the vertex A ($D \in BC$). Find the decomposition of the vector AD in terms of the vectors $\vec{c} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AC}$.

6. ([4, Problema 12, p. 3]) Let M, N be the midpoints of two opposite edges of a given quadrilateral $ABCD$ and P be the midpoint of $[MN]$. Show that

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 0$$

7. ([4, Problema 12, p. 7]) Consider two perpendicular chords AB and CD of a given circle and $\{M\} = AB \cap CD$. Show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2\,\overrightarrow{OM}.$$

8. ([4, Problema 13, p. 3]) If G is the centroid of a tringle ABC and O is a given point, show that

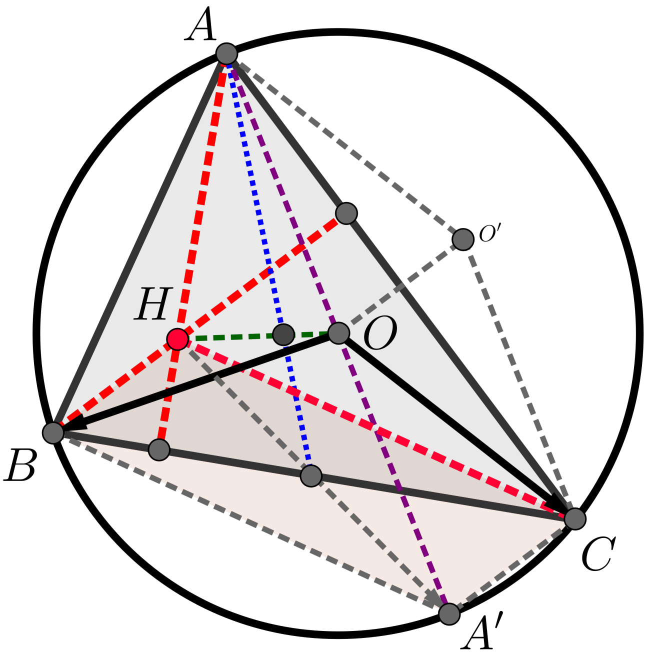
$$\overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}.$$

9. ([4, Problema 14, p. 4]) Consider the triangle ABC alongside its orthocenter H, its circumcenter O and the diametrically opposed point A' of A on the latter circle. Show that:

   (a) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$.

   (b) $\overrightarrow{HB} + \overrightarrow{HC} = \overrightarrow{HA'}$.

   (c) $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\,\overrightarrow{HO}$.



10. ([4, Problema 15, p. 4]) Consider the triangle ABC alongside its centroid G, its orthocenter H and its circumcenter O. Show that O, G, H are collinear and $3\ \overrightarrow{HG} = 2\ \overrightarrow{HO}$.

11. ([4, Problema 11, p. 3]) Consider two parallelograms, $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ in $\mathcal{P}$, and M_1, M_2, M_3, M_4 the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$, $[A_4B_4]$ respectively. Show that:

    - $2\ \overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}$ and $2\ \overrightarrow{M_3M_4} = \overrightarrow{A_3A_4} + \overrightarrow{B_3B_4}$.
    - M_1, M_2, M_3, M_4 are the vertices of a parallelogram.

12. ([4, Problema 27, p. 13]) Consider a tetrahedron $A_1A_2A_3A_4$ and the midpoints A_{ij} of the edges A_iA_j, $i \neq j$. Show that:

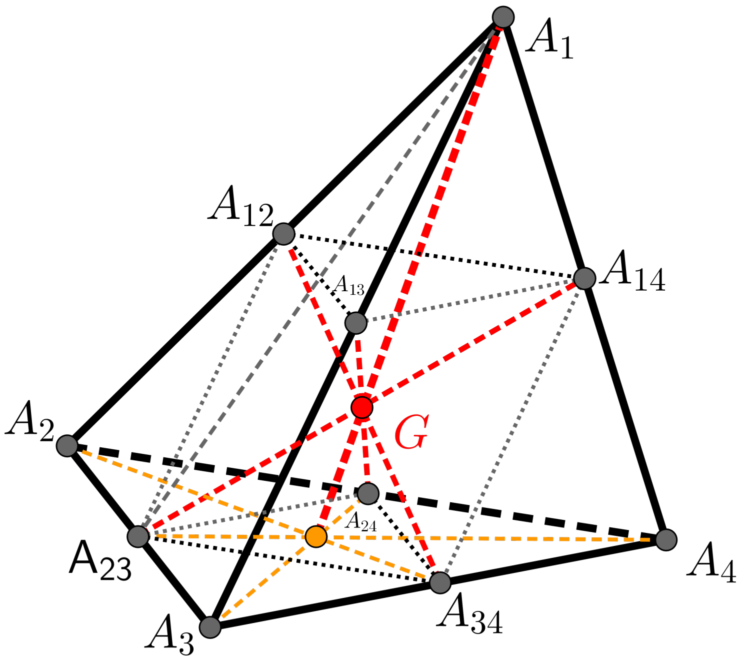
    (a) The lines $A_{12}A_{34}$, $A_{13}A_{24}$ and $A_{14}A_{23}$ are concurrent in a point G.

    (b) The medians of the tetrahedron (the lines passing through the vertices and the centroids of the opposite faces) are also concurrent at G.

    (c) Determine the ratio in which the point G divides each median.

    (d) Show that $\overrightarrow{GA_1} + \overrightarrow{GA_2} + \overrightarrow{GA_3} + \overrightarrow{GA_4} = \overrightarrow{0}$.

    (e) If M is an arbitrary point, show that $\overrightarrow{MA_1} + \overrightarrow{MA_2} + \overrightarrow{MA_3} + \overrightarrow{MA_4} = 4\ \overrightarrow{MG}$.



13. In a triangle ABC consider the points M, L on the side AB and N, T on the side AC such that $3\overrightarrow{AL} = 2\overrightarrow{AM} = \overrightarrow{AB}$ and $3\overrightarrow{AT} = 2\overrightarrow{AN} = \overrightarrow{AC}$. Show that $\overrightarrow{AB} + \overrightarrow{AC} = 5\overrightarrow{AS}$, where $\{S\} = MT \cap LN$.

14. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$, not necesarily in the same plane, alongside their centroids G_1, G_2. Show that $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = 3\overrightarrow{G_1G_2}$.

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