COURSE 2

2.2. Lagrange interpolation

Let $[a, b] \subset \mathbb{R}$, $x_i \in [a, b]$, $i = 0, 1, ..., m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a, b] \to \mathbb{R}$.

The Lagrange interpolation problem (LIP) consists in determining the polynomial P of the smallest degree for which

$$P(x_i) = f(x_i), \;\; i = 0, 1, ..., m \tag{1}$$

i.e., the polynomial of the smallest degree which passes through the distinct points $(x_i, f(x_i))$, $i = 0, 1, ..., m$.

Since in (1) there are $m + 1$ conditions to be satisfied, we need $m + 1$ degrees of freedom. Consider the m-th degree polynomial

$$P(x) = a_0 + a_1 x + ... + a_{m-1} x^{m-1} + a_m x^m. \tag{2}$$

The $m + 1$ coefficients $\{a_i\}$ have to be determined in such way that (1) are satisfied. This leads to the linear system of equations:

$$\begin{cases} a_0 + a_1 x_0 + ... + a_{m-1} x_0^{m-1} + a_m x_0^m = f(x_0) \\ a_0 + a_1 x_1 + ... + a_{m-1} x_1^{m-1} + a_m x_1^m = f(x_1) \\ \\ a_0 + a_1 x_m + ... + a_{m-1} x_m^{m-1} + a_m x_m^m = f(x_m). \end{cases}$$

Definition 1 *A solution of (LIP) is called* **Lagrange interpolation polynomial**, *denoted by* $L_m f$.

Remark 2 *We have* $(L_m f)(x_i) = f(x_i), \; i = 0, 1, ..., m.$

$L_m f \in \mathbb{P}_m$ *($\mathbb{P}_m$ is the space of polynomials of at most m-th degree).*

The Lagrange interpolation polynomial is given by

$$(L_m f)(x) = \sum_{i=0}^{m} \ell_i(x) f(x_i), \tag{3}$$

where by $\ell_i(x)$ denote **the Lagrange fundamental interpolation polynomials.**

We have

$$u(x) = \prod_{j=0}^{m} (x - x_j),$$

$$u_i(x) = \frac{u(x)}{x - x_i} = (x - x_0)...(x - x_{i-1})(x - x_{i+1})...(x - x_m) = \prod_{\substack{j=0 \\ j \neq i}}^{m} (x - x_j)$$

and

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)} = \frac{(x - x_0)...(x - x_{i-1})(x - x_{i+1})...(x - x_m)}{(x_i - x_0)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_m)} = \prod_{\substack{j=0 \\ j \neq i}}^{m} \frac{x - x_j}{x_i - x_j},$$

(4)

for $i = 0, 1, ..., m$.

How do we know that the interpolation polynomial expanded in powers of x as in (2) and the polynomial constructed as in (3) represent the same polynomial?

Assume we have computed two interpolating polynomials $Q(x)$ and $P(x)$ each of degree m such that

$$Q(x_j) = f(x_j) = P(x_j), \quad j = 0, ..., m.$$

Then we can form the difference

$$d(x) = Q(x) - P(x),$$

that is a polynomial of degree less or equal to m.

Because of the interpolation property of P and Q, we have

$$d(x_j) = Q(x_j) - P(x_j) = 0, \quad j = 0, ..., m.$$

A non-zero polynomial of degree less than or equal to m cannot have more than m zeros. But d has $m + 1$ distint zeros, hence it must be indentically zero, so $Q(x) = P(x)$.

Proposition 3 *We also have*

$$\ell_i(x) = \frac{u(x)}{(x - x_i)u'(x_i)}, \quad i = 0, 1, ..., m. \tag{5}$$

Proof. We have $u_i(x) = \frac{u(x)}{x - x_i}$, so $u(x) = u_i(x)(x - x_i)$. We get $u'(x) = u_i(x) + (x - x_i)u_i'(x)$, whence it follows $u'(x_i) = u_i(x_i)$. So, as

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)}$$

we get

$$\ell_i(x) = \frac{u_i(x)}{u'(x_i)} = \frac{u(x)}{(x - x_i)u'(x_i)}, \quad i = 0, 1, ..., m. \tag{6}$$

∎

Theorem 4 *The operator L_m is linear.*

Proof.

$$L_m(\alpha f + \beta g)(x) = \sum_{i=0}^{m} \ell_i(x)(\alpha f + \beta g)(x_i) = \sum_{i=0}^{m} [\ell_i(x)\alpha f(x_i) + \ell_i(x)\beta g(x_i)]$$
$$= \alpha(L_m f)(x) + \beta(L_m g)(x),$$

so

$$L_m(\alpha f + \beta g) = \alpha L_m f + \beta L_m g, \qquad \forall f, g : [a, b] \to \mathbb{R} \text{ and } \alpha, \beta \in \mathbb{R}.$$

∎

Example 5 *1) Consider the nodes x_0, x_1 and a function f to be interpolated.*

We have $m = 1$,

$$u(x) = (x - x_0)(x - x_1)$$
$$u_0(x) = x - x_1$$
$$u_1(x) = x - x_0$$

$$(L_1 f)(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$
$$= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1),$$

which is the line passing through the given points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

Example 6 *Find the Lagrange polynomial that interpolates the data in the following table and find the approximative value of $f(-0.5)$.*

x	-1	0	3
$f(x)$	8	-2	4

Sol. We have $m = 2$. The Lagrange polynomial is

$$(L_2 f)(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2).$$

$u(x) = (x+1)(x-0)(x-3)$ and it follows

$$l_0(x) = \frac{(x-0)(x-3)}{(-1-0)(-1-3)} = \frac{1}{4} x(x-3)$$

$$l_1(x) = \frac{(x+1)(x-3)}{(0+1)(0-3)} = -\frac{1}{3}(x+1)(x-3)$$

$$l_2(x) = \frac{(x+1)(x-0)}{(3+1)(3-0)} = \frac{1}{12} x(x+1),$$

The polynomial is

$$(L_2 f)(x) = 2x(x-3) + \frac{2}{3}(x+1)(x-3) + \frac{1}{3} x(x+1).$$

and $(L_2 f)(-0.5) = 2.25$.

Remark 7 *Disadvantages of the form (3) of Lagrange polynomial: requires many computations and if we add or substract a point we have to start with a complete new set of computations.*

Some calculations allow us to reduce the number of operations:

$$(L_m f)(x) = \frac{(L_m f)(x)}{1} = \frac{\sum\limits_{i=0}^{m} l_i(x) f(x_i)}{\sum\limits_{i=0}^{m} l_i(x)}.$$

Dividing the numerator and the denominator by

$$u(x) = \prod_{i=1}^{m} (x - x_i)$$

and denoting

$$A_i = \frac{1}{\prod\limits_{j=0, j \neq i}^{m} (x_i - x_j)} = \frac{1}{u_i(x_i)}$$

one obtains

$$(L_m f)(x) = \frac{\sum\limits_{i=0}^{m} \frac{A_i f(x_i)}{x - x_i}}{\sum\limits_{i=0}^{m} \frac{A_i}{x - x_i}}, \qquad (7)$$

called **the barycentric form** *of Lagrange interpolation polynomial.*

Remark 8 *Formula (7) needs half of the number of arithmetic operations needed for (3) and it is easier to add or substract a point.*

The Lagrange polynomial generates **the Lagrange interpolation formula**

$$f = L_m f + R_m f,$$

where $R_m f$ denotes **the remainder** (**the error**).

Theorem 9 *Let* $\alpha = \min\{x, x_0, ..., x_m\}$ *and* $\beta = \max\{x, x_0, ..., x_m\}$. *If* $f \in C^m[\alpha, \beta]$ *and* $f^{(m)}$ *is derivable on* (α, β) *then* $\forall x \in (\alpha, \beta)$, *there exists* $\xi \in (\alpha, \beta)$ *such that*

$$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi). \tag{8}$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_m f)(z) \\ u(x) & (R_m f)(x) \end{vmatrix}.$$

From hypothesis it follows that $F \in C^m[\alpha, \beta]$ and there exists $F^{(m+1)}$ on (α, β).

We have

$$F(x) = 0, \ F(x_i) = 0, \qquad i = 0, 1, ..., m,$$

as

$$u(x_i) = \prod_{j=0}^{m} (x_i - x_j) = 0$$

and

$$(R_m f)(x_i) = f(x_i) - (L_m f)(x_i) = f(x_i) - f(x_i) = 0,$$

so F has $m+2$ distinct zeros in (α, β). Applying successively the Rolle theorem it follows that: F has $m+2$ zeros in $(\alpha, \beta) \Rightarrow F'$ has at least $m+1$ zeros in $(\alpha, \beta) \Rightarrow ... \Rightarrow F^{(m+1)}$ has at least one zero in (α, β)

So $F^{(m+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(m+1)}(\xi) = 0$.

We have
$$F^{(m+1)}(z) = \begin{vmatrix} u^{(m+1)}(z) & (R_m f)^{(m+1)}(z) \\ u(x) & (R_m f)(x) \end{vmatrix},$$

with
$$u(z) = \prod_{i=0}^{m}(z - z_i) \Rightarrow u^{(m+1)}(z) = (m+1)!,$$

and
$$(R_m f)^{(m+1)}(z) = (f - (L_m f))^{(m+1)}(z)$$
$$= f^{(m+1)}(z) - (L_m f)^{(m+1)}(z) = f^{(m+1)}(z)$$

(as, $L_m f \in \mathbb{P}_m$).

We have $F^{(m+1)}(\xi) = 0$, for $\xi \in (\alpha, \beta)$, so
$$F^{(m+1)}(\xi) = \begin{vmatrix} (m+1)! & f^{(m+1)}(\xi) \\ u(x) & (R_m f)(x) \end{vmatrix} = 0,$$

i.e., $(m+1)!(R_m f)(x) = u(x) f^{(m+1)}(\xi),$

whence $(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi).$ ∎

Corolar 10 *If $f \in C^{m+1}[a,b]$ then*

$$|(R_m f)(x)| \leq \frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty, \qquad x \in [a,b]$$

where $\|\cdot\|_\infty$ denotes the uniform norm, and $\|f\|_\infty = \max\limits_{x \in [a,b]} |f(x)|$.

Example 11 *If we know that $\lg 2 = 0.301$, $\lg 3 = 0.477$, $\lg 5 = 0.699$, find $\lg 76$. Study the approximation error.*

Example 12 *Which is the limit of the error for computing $\sqrt{115}$ using Lagrange interpolation formula for the nodes $x_0 = 100$, $x_1 = 121$ and $x_2 = 144$? Find the approximative value of $\sqrt{115}$.*

The Aitken's algorithm

Let $[a, b] \subset \mathbb{R}$, $x_i \in [a, b]$, $i = 0, 1, ..., m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a, b] \to \mathbb{R}$.

Usually, for a practical approximation problem, for a given function $f : [a, b] \to \mathbb{R}$ we have to find the approximation of $f(\alpha)$, $\alpha \in [a, b]$ with an error not greater than a given $\varepsilon > 0$.

If we have enough information about f and its derivatives, we use the inequality $|(R_m f)(x)| \leq \varepsilon$ to find m such that $(L_m f)(\alpha)$ approximates $f(\alpha)$ with the given precision.

We may use the condition $\frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty \leq \varepsilon$, but it should be known $\left\| f^{(m+1)} \right\|_\infty$ or a majorant of it.

A practical method for computing the Lagrange polynomial is **the Aitken's algorithm.** This consists in generating the table:

$$
\begin{array}{c|c|c|cccc}
x_0 & f_{00} & & & & & \\
x_1 & f_{10} & f_{11} & & & & \\
x_2 & f_{20} & f_{21} & f_{22} & & & \\
x_3 & f_{30} & f_{31} & f_{32} & f_{33} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
x_m & f_{m0} & f_{m1} & f_{m2} & f_{m3} & \cdots & f_{mm}
\end{array}
$$

where

$$
f_{i0} = f(x_i), \quad i = 0, 1, ..., m,
$$

and

$$
f_{i,j+1} = \frac{1}{x_i - x_j} \begin{vmatrix} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix}, \quad i = 0, 1, ..., m; j = 0, ..., i - 1.
$$

For example,

$$f_{11} = \frac{1}{x_1 - x_0} \begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix}$$

$$= \frac{1}{x_1 - x_0}[f_{00}(x_1 - x) - f_{10}(x_0 - x)]$$

$$= \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1) = (L_1 f)(x),$$

so f_{11} is the value in x of Lagrange polynomial for the nodes x_0, x_1. We have

$$f_{ii} = (L_i f)(x),$$

$L_i f$ being Lagrange polynomial for the nodes $x_0, x_1, ..., x_i$.

So $f_{11}, f_{22}, ..., f_{ii}, ..., f_{mm}$ is a sequence of approximations of $f(x)$.

If the interpolation procedure is convergent then the sequence is also convergent, i.e., $\lim_{m \to \infty} f_{mm} = f(x)$. By Cauchy convergence criterion it follows

$$\lim_{i \to \infty} |f_{ii} - f_{i-1,i-1}| = 0.$$

This could be used as a stopping criterion, i.e.,

$$\left| f_{ii} - f_{i-1,i-1} \right| \leq \varepsilon, \quad \text{for a given precision } \varepsilon > 0.$$

Recommendation is to sort the nodes $x_0, x_1, ..., x_m$ with respect to the distance to x, such that

$$\left| x_i - x \right| \leq \left| x_j - x \right| \text{ if } i < j, \quad i, \quad j = 1, ..., m.$$

Example 13 *Approximate $\sqrt{115}$ with precision $\varepsilon = 10^{-3}$, using Aitken's algorithm.*

Newton interpolation polynomial

A useful representation for Lagrange interpolation polynomial is

$$(L_m f)(x) := (N_m f)(x) = f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})(D^i f)(x_0)$$

$$(9)$$

$$= f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})[x_0, ..., x_i; f],$$

which is called **Newton interpolation polynomial;** where $(D^i f)(x_0)$ (or denoted $[x_0, ..., x_i; f]$) is the i-th order divided difference of the function f at x_0, given by the table

	f	$\mathcal{D}f$	$\mathcal{D}^2 f$	...	$\mathcal{D}^{\mathbf{m-1}} f$	$\mathcal{D}^m f$
x_0	f_0	$\mathcal{D}f_0$	$\mathcal{D}^2 f_0$	...	$\mathcal{D}^{m-1} f_0$	$\mathcal{D}^m f_0$
x_1	f_1	$\mathcal{D}f_1$	$\mathcal{D}^2 f_1$		$\mathcal{D}^{m-1} f_1$	
x_2	f_2	$\mathcal{D}f_2$	$\mathcal{D}^2 f_2$			
...	...	...				
x_{m-2}	f_{m-2}	$\mathcal{D}f_{m-2}$	$\mathcal{D}^2 f_{m-2}$			
x_{m-1}	f_{m-1}	$\mathcal{D}f_{m-1}$				
x_m	f_m					

Newton interpolation formula is

$$f = N_m f + R_m f,$$

where $R_m f$ denotes the remainder.

Assume that we add the point $(x, f(x))$ at the top of the table of divided differences:

	f	Df	$\ldots$	$D^{m+1}f$
x	$f(x)$	$(Df)(x) = [x, x_0; f]$		$[x, x_0, \ldots, x_m; f]$
x_0	$f(x_0)$	$(Df)(x_0) = [x_0, x_1; f]$	$\ldots$	
x_1	$f(x_1)$	$(Df)(x_1) = [x_1, x_2; f]$		
$\ldots$	$\ldots$	$\ldots$		
x_{m-1}	$f(x_{m-1})$	$(Df)(x_{m-1}) = [x_{m-1}, x_m; f]$		
x_m	$f(x_m)$			

For obtaining the interpolation polynomial we consider

$$[x, x_0; f] = \frac{f(x_0) - f(x)}{x_0 - x} \implies f(x) = f(x_0) + (x - x_0)[x, x_0; f] \quad (10)$$

$$[x, x_0, x_1; f] = \frac{[x_0, x_1; f] - [x, x_0; f]}{x_1 - x} \quad (11)$$

$$\implies [x, x_0; f] = [x_0, x_1; f] + (x - x_1)[x, x_0, x_1; f].$$

Inserting (11) in (10) we get

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1; f] + (x - x_0)(x - x_1)[x, x_0, x_1; f].$$

If we continue eliminating the divided differences involving x in the same way, we get

$$f(x) = (N_m f)(x) + (R_m f)(x)$$

with

$$(N_m f)(x) = f(x_0) + \sum_{i=1}^{m} (x - x_0)...(x - x_{i-1})[x_0, ..., x_i; f]$$

and the remainder (the error) given by

$$(R_m f)(x) = (x - x_0)...(x - x_m)[x, x_0, ..., x_m; f]. \qquad (12)$$