Babeş-Bolyai University, Faculty of Mathematics and Computer Science

Bachelor, Computer Science, Groups 911-917, Academic Year 2016-2017

Mathematical Analysis

Seminar 5

1. Study if the following series are convergent or divergent for $\alpha \in (0, +\infty)$:

$$\text{a)} \sum_{n\geq 1} \frac{(\alpha n)^n}{n!} \qquad \text{b)} \sum_{n\geq 1} \alpha^{\ln n} \qquad \text{c)} \sum_{n\geq 1} \left[\frac{(2n-1)!!}{\sqrt{(2n)!}}\right]^{2\alpha}.$$

2. Study whether the following series are absolutely convergent, semi-convergent or divergent:

$$\text{a)} \sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n^2+1)}} \qquad \text{b)} \sum_{n\geq 1} \frac{\sqrt[3]{n}}{n+1}\cos(n\pi).$$

Prove that for any $x \in \mathbb{R}$ and $\alpha > 0$ the series $\displaystyle\sum_{n\geq 1} \frac{\sin(nx)}{n^\alpha}$ is convergent.

3. Find the set $A' := \{x_0 \in \overline{\mathbb{R}} \mid \forall V \in \mathcal{V}(x_0),\ V \cap A \setminus \{x_0\} \neq \emptyset\}$ of accumulation (i.e., cluster) points for the following sets: a) $A = [0,1) \cup \{2\}$, b) $A = \mathbb{Z}$, c) $A = \mathbb{Q}$ and d) $A = \big(0, \sqrt{2}\,\big] \cap \mathbb{Q}$.

4. Study the existence of the limit of Dirichlet's function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

at every accumulation point of its domain ($x_0 \in \overline{\mathbb{R}}$).

5. Compute the limits:

$$\text{a)} \lim_{x\to-\infty} \frac{-3x^2+x-1}{(x-1)(x-2)}, \quad \text{b)} \lim_{x\to 1} \frac{x^2-3x+2}{x^3-x}, \quad \text{c)} \lim_{x\to\infty} \sqrt{x}(\sqrt{x+1}-\sqrt{x}), \quad \text{d)} \lim_{x\to 0} \frac{\sqrt[3]{1+x}-1}{x},$$

$$\text{e)} \lim_{\substack{x\to 1\\ x>1}} \frac{x}{\sqrt[3]{x^2-4x+3}}, \quad \text{f)} \lim_{x\to-\infty} \left(\frac{x^2+1}{x^2-1}\right)^{\sqrt{-x}}, \quad \text{g)} \lim_{\substack{x\to 0\\ x<0}} \lfloor x \rfloor, \quad \text{h)} \lim_{x\to\infty} \lfloor x \rfloor, \quad \text{i)} \lim_{x\to\infty} \frac{x-\lfloor x \rfloor}{x}.$$

6. Study the continuity of the following functions and determine the type of their discontinuities:

$$\text{a)}\ f : \mathbb{R} \to \mathbb{R},\ f(x) := \lim_{n\to\infty} \frac{e^{nx}}{1+e^{nx}} \qquad \text{b)}\ g : \mathbb{R} \to \mathbb{R},\ g(x) := \begin{cases} \frac{1}{x}\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

7. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at every point in $\mathbb{R}$ and $|f|$ is continuous on $\mathbb{R}$.

8. Let $f,\ g : [0,1] \to \mathbb{R}$ be two continuous functions, such that $f(x) = g(x)$, $\forall x \in [0,1] \cap \mathbb{Q}$. Prove that $f(x) = g(x)$, $\forall x \in [0,1]$.

9. Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a,b] \to [a,b]$ be a continuous function. Prove that f has at least one fixed point $x_0 \in [a,b]$, that is, $f(x_0) = x_0$.