Geometry
Problem booklet

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Contents

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Week 10

1 Quadrics. Brief theoretical background

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 The hyperboloid of two sheets

The *hyperboloid of two sheets* is the surface of equation

$$\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{1.1}$$

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively,

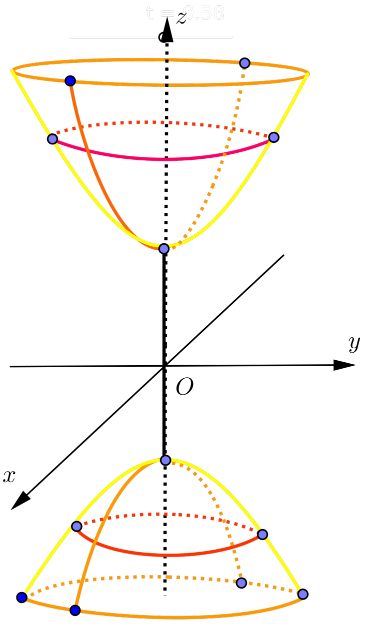
$$\begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0 \\ x = 0 \\ \text{a hyperbola;} \end{cases}, \begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} + 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0 \\ z = 0 \\ \text{the empty set} \end{cases};$$

- The intersections with planes parallel to the coordinate planes are

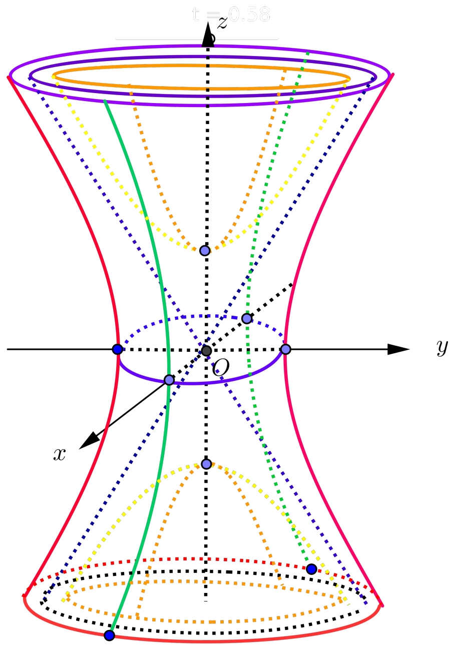
$$\begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 - \frac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases}, \quad \begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} = -1 - \frac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{cases}$$

and $\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 + \frac{\lambda^2}{c^2} \\ z = \lambda \end{cases}.$

  - If $|\lambda| > c$, the section is an ellipse;
  - If $|\lambda| = c$, the intersection reduces to a point $(0, 0, \lambda)$;
  - If $|\lambda| < c$, one obtains the empty set.



The hyperboloid of two sheets



The hyperboloids of one and two sheets and their common asymptotic cone

1.2 Elliptic Cones

The surface of equation

$$\mathcal{C}: \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{1.2}$$

is called *elliptic cone*.

- The coordinate planes are planes of symmetry for $\mathcal{C}$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{C}$;

- The intersections with the coordinates planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0 \\ x = 0 \\ \text{two lines} \end{cases}, \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ y = 0 \\ \text{two lines} \end{cases},$$

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \\ z = 0 \\ \text{the origin } O(0,0,0). \end{cases}$$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -\dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases}; \quad \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = -\dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas.} \end{cases}$$

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{\lambda^2}{c^2} \\ z = \lambda \\ \text{ellipses} \end{cases}$$

1.3 Elliptic Paraboloids

The surface of equation

$$\mathcal{P}_e : \frac{x^2}{p} + \frac{y^2}{q} = 2z, \qquad p, q \in \mathbb{R}_+^*, \tag{1.3}$$

is called *elliptic paraboloid*.

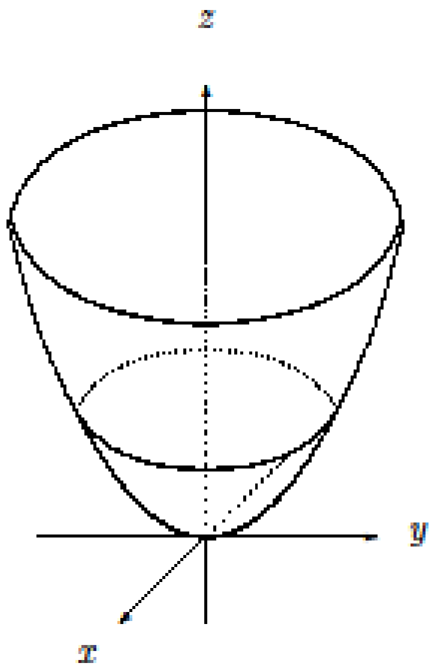
- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are

$$\begin{cases} \dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{cases}, \begin{cases} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{cases}, \begin{cases} \dfrac{x^2}{p} + \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{the origin } O(0,0,0). \end{cases}$$

- The intersection with the planes parallel to the coordinate planes are $\begin{cases} \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \end{cases}$,
  - If $\lambda > 0$, the section is an ellipse;
  - If $\lambda = 0$, the intersection reduces to the origin;
  - If $\lambda < 0$, one has the empty set;

  and

$$\begin{cases} \dfrac{y^2}{q} = 2z - \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{cases}; \quad \begin{cases} \dfrac{x^2}{p} = 2z - \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas} \end{cases};$$



1.4 Hyperbolic Paraboloids

The *hyperbolic paraboloid* is the surface given by the equation

$$\mathcal{P}_h : \frac{x^2}{p} - \frac{y^2}{q} = 2z, \qquad p, q > 0. \tag{1.4}$$

- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are, respectively,

$$\left\{\begin{array}{c} -\dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{array}\right. ; \quad \left\{\begin{array}{c} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{array}\right. ; \quad \left\{\begin{array}{c} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{two lines.} \end{array}\right. ;$$

- The intersection with the planes parallel to the coordinate planes are

$$\left\{\begin{array}{c} \dfrac{y^2}{q} = -2z + \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{array}\right. ; \quad \left\{\begin{array}{c} \dfrac{x^2}{p} = 2z + \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas.} \end{array}\right.$$

$$\left\{\begin{array}{c} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \\ \text{hyperbolas} \end{array}\right.$$

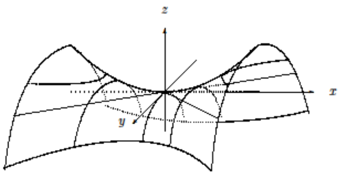
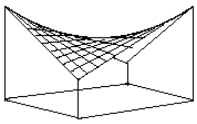
Remark: The hyperbolic paraboloid contains two families of lines. Since

$$\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right)\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) = 2z,$$

then the two families are, respectively, of equations

$$d_\lambda : \begin{cases} \dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} = \lambda \\ \lambda \left(\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} \right) = 2z \end{cases}, \lambda \in \mathbb{R} \text{ and}$$

$$d'_\mu : \begin{cases} \dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = \mu \\ \mu \left(\dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} \right) = 2z \end{cases}, \mu \in \mathbb{R}.$$

1.5 Singular Quadrics

1.5.1 Elliptic Cylinder, Hyperbolic Cylinder, Parabolic Cylinder

- The *elliptic cylinder* is the surface of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0. \tag{1.5}$$

or

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0, \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

- The *hyperbolic cylinder* is the surface of equation

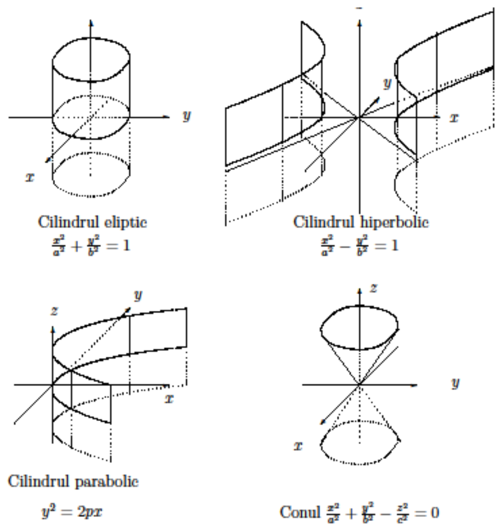
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0, \tag{1.6}$$

or

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0, \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0.$$

- The *parabolic cylinder* is the surface of equation

$$y^2 = 2px, \quad p > 0, \quad (\text{or an alternative equation}). \tag{1.7}$$



Theorem 1.1. *(The preimage theorem) If $U \subseteq \mathbb{R}^3$ is an open set, $f : U \longrightarrow R$ is a C^1-smooth function and $a \in Imf$ is a regular value[1] of f, then te inverse image of a through f,*

$$f^{-1}(a) = \{(x, y, z) \in U | f(x, y, z) = a\}$$

is a regular surface called the regular surface of implicit cartezian equation $f(x, y, z) = a$.

Proposition 1.2. *The equation of the tangent plane $T_{(x_0,y_0,z_0)}(S)$ of the regular surface S of implicit cartezian equation $f(x, y, z) = a$ at the point $p = (x_0, y_0, z_0) \in S$, is*

$$T_{(x_0,y_0,z_0)}(S) : f_x(p)(x - x_0) + f_y(p)(y - y_0 + f_z(p)(z - z_0)) = 0, \tag{1.8}$$

and the equation of the normal line $N_{(x_0,y_0,z_0)}(S)$ of S at p is

$$N_{(x_0,y_0,z_0)}(S) : \frac{x - x_0}{f_x(p)} = \frac{y - y_0}{f_y(p)} = \frac{z - z_0}{f_z(p)}.$$

By using the general equation (1.8) of the tangent line to an implicite curve from Proposition (1.2), one can easily show that:

1. The equation of the tangent plane to the ellipsoid $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ at some point $M_0(x_0, y_0, z_0) \in \mathcal{E}$ is

$$T_{M_0}(\mathcal{E}) : \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1. \tag{1.9}$$

2. The equation of the tangent line to the hyperboloid of one sheet $\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ at some point $M_0(x_0, y_0, z_0) \in \mathcal{H}_1$ is

$$T_{M_0}(\mathcal{H}_1) : \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = 1. \tag{1.10}$$

[1]The value $a \in \text{Im}(f)$ of the function f is said to be *regular* if $(\nabla f)(x, y, z) \neq 0$, $\forall (x, y, z) \in f^{-1}(a)$

3. The equation of the tangent line to the hyperboloid of one sheet $\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ at some point $M_0(x_0, y_0, z_0) \in \mathcal{H}_2$ is

$$T_{M_0}(\mathcal{H}_2) : \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = -1. \tag{1.11}$$

4. The equation of the tangent plane to the elliptic paraboloid $\mathcal{P}_e : \frac{x^2}{p} + \frac{y^2}{q} = 2z, \quad p, q \in \mathbb{R}_+^*$, at some point $M_0(x_0, y_0, z_0) \in \mathcal{P}_e$ is

$$T_{M_0}(\mathcal{P}_e) : \frac{x_0 x}{p} + \frac{y_0 y}{q} = z + z_0. \tag{1.12}$$

5. The equation of the tangent plane to the hyperbolic paraboloid $\mathcal{P}_h : \frac{x^2}{p} - \frac{y^2}{q} = 2z, \quad p, q \in \mathbb{R}_+^*$, at some point $M_0(x_0, y_0, z_0) \in \mathcal{P}_h$ is

$$T_{M_0}(\mathcal{P}_h) : \frac{x_0 x}{p} - \frac{y_0 y}{q} = z + z_0. \tag{1.13}$$

2 Generated Surfaces

Consider the 3-dimensional Euclidean space $\mathcal{E}_3$, together with a Cartesian system of coordinates $Oxyz$. Generally, the set

$$S = \{M(x, y, z) : F(x, y, z) = 0\},$$

where $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a real function and D is a domain, is called *surface* of implicit equation $F(x, y, z) = 0$. For example the quadric surfaces, defined in the previous chapter for F a polynomial of degree two, are such of surfaces. On the other hand, the set

$$S_1 = \{M(x, y, z) : x = x(u, v), y = y(u, v), z = z(u, v)\},$$

where $x, y, z : D_1 \subseteq \mathbb{R}^2 \to \mathbb{R}$, is a *parameterized surface*, of parametric equations

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \quad (u, v) \in D_1.$$

The intersection between two surfaces is a *curve* in 3-space (remember, for instance, that the intersection between a quadric surface and a plane is a conic section, hence the conics are plane curves). Then, the set

$$C = \{M(x, y, z) : F(x, y, z) = 0, G(x, y, z) = 0\},$$

where $F, G : D \subseteq \mathbb{R}^3 \to \mathbb{R}$, is the curve of *implicit* equations

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}.$$

As before, one can parameterize the curve. The set

$$C_1 = \{M(x,y,z) : x = x(t), y = y(t), z = z(t)\},$$

where $x, y, z : I \subseteq \mathbb{R} \to \mathbb{R}$ and I is open, is called *parameterized curve* of parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in I.$$

Let be given a family of curves, depending on one single parameter λ,

$$\mathcal{C}_\lambda : \begin{cases} F_1(x,y,z;\lambda) = 0 \\ F_2(x,y,z;\lambda) = 0 \end{cases}.$$

In general, the family $\mathcal{C}_\lambda$ does not cover the entire space. By eliminating the parameter λ between the two equations of the family, one obtains the equation of the surface *generated* by the family of curves.

Suppose now that the family of curves depends on two parameters λ, μ,

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} F_1(x,y,z;\lambda,\mu) = 0 \\ F_2(x,y,z;\lambda,\mu) = 0 \end{cases},$$

and that the parameters are related through $\varphi(\lambda,\mu) = 0$ If it can be obtained an equation which does not depend on the parameters (by eliminating the parameters between the three equations), then the set of all the points which verify it is called surface *generated* by the family (or the sub-family) of curves.

2.1 Cylindrical Surfaces

Definition 2.1. *The surface generated by a variable line, called* generatrix, *which remains parallel to a fixed line d and intersects a given curve $\mathcal{C}$, is called* cylindrical surface. *The curve $\mathcal{C}$ is called the* director curve *of the cylindrical surface.*

Theorem 2.1. *The cylindrical surface, with the generatrix parallel to the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases},$$

which has the director curve

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$

(d and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi(\pi_1, \pi_2) = 0. \tag{2.1}$$

Proof. The equations of an arbitrary line, which is parallel to

$$d : \begin{cases} \pi_1(x,y,z) = 0 \\ \pi_2(x,y,z) = 0 \end{cases}, \text{ are } d_{\lambda,\mu} : \begin{cases} \pi_1(x,y,z) = \lambda \\ \pi_2(x,y,z) = \mu \end{cases}.$$

Not every line from the family $d_{\lambda,\mu}$ intersects the curve $\mathcal{C}$. This happens only when the system of equations

$$\begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \\ \pi_1(x,y,z) = \lambda \\ \pi_2(x,y,z) = \mu \end{cases}$$

is compatible. By eliminating λ and μ between four equations of the system, one obtains a *necessary condition* $\varphi(\lambda, \mu) = 0$ for the parameters λ and μ in order to nonempty intersection between the line $d_{\lambda,\mu}$. The equation of the surface can be determined now from the system

$$\begin{cases} \pi_1(x,y,z) = \lambda \\ \pi_2(x,y,z) = \mu \\ \varphi(\lambda,\mu) = 0 \end{cases},$$

and it is immediate that $\varphi(\pi_1, \pi_2) = 0$. $\square$

Remark 2.2. *Any equation of the form (2.1), where π_1 and π_2 are linear function of x, y and z, represents a cylindrical surface, having the generatrices parallel to* $d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$.

Example 2.3. *Let us find the equation of the cylindrical surface having the generatrices parallel to*

$$d : \begin{cases} x + y = 0 \\ z = 0 \end{cases}$$

and the director curve given by

$$\mathcal{C} : \begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \end{cases}.$$

The equations of the generatrices d are

$$d_{\lambda,\mu} : \begin{cases} x + y = \lambda \\ z = \mu \end{cases}.$$

They must intersect the curve $\mathcal{C}$, i.e. the system

$$\begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \\ x + y = \lambda \\ z = \mu \end{cases}$$

has to be compatible. A solution of the system can be obtained using the three last equations

$$\begin{cases} x = 1 \\ y = \lambda - 1 \\ z = \mu \end{cases}$$

and, replacing in the first one, one obtains the compatibility condition

$$2(\lambda - 1)^2 + \mu - 1 = 0.$$

Thus, the equation of the required cylindrical surface is

$$2(x + y - 1)^2 + x - 1 = 0.$$

3 Problems

1. Find the intersection points of the ellipsoid

$$(\mathcal{E})\ \frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1 \text{ with the line } (d)\ \frac{x-4}{2}=\frac{y+6}{-3}=\frac{z+2}{-2}.$$

   Write the equations of the tangent plane and the normal line to the ellipsoid $(\mathcal{E})$ at those intersection points $\mathcal{E}\cap d$.

2. Find the rectilinear generatrices of the hyperboloid of one sheet

$$(\mathcal{H}_1)\ \frac{x^2}{36}+\frac{y^2}{9}-\frac{z^2}{4}=1$$

   which are parallel to the plane $(\pi)\ x+y+z=0$.

3. Find the locus of points on the hyperbolic paraboloid $(\mathcal{P}_h)\ y^2-z^2=2x$ through wich the rectilinear generatrices are perpendicular.

4. Find the locus $L_{\mathcal{E}}$ of the orthogonal projections of the center $O(0,0,0)$ of the ellipsoid $\mathcal{E}:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{yz^2}{c^2}=1$ on its tangent planes.

5. Find the locus $L_{\mathcal{H}_1}$ of the orthogonal projections of the center $O(0,0,0)$ of the hyperboloid of one sheet $(H_1)\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ on its tangent planes.

6. Find the locus of points in the space equidistant to two given straight lines.

7. Find the equation of the cylindrical surface whose director curve is the planar curve

$$(C)\ \begin{cases} y^2+z^2=x \\ x=2z \end{cases}$$

   and the generatrix is perpendicular to the plane of the director curve.

8. Find the equation of the cylindrical surface generated by a variable straight line parallel to the line

$$(\Delta)\ \begin{cases} y-3z=0 \\ y+2z=0 \end{cases} \text{ which is tangent to the surface } (\mathcal{E})\ 4x^2+3y^2+2z^2=1.$$

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