Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză, Curs: Dynamical Systems, Primăvara 2017

Laboratories 2 and 3. Solving Linear Differential Equations

1. Decide whether the functions $\sin t,\ \cos t,\ \sinh t,\ \cosh t$ are solutions to $x^{(4)} - x = 0$.

Find the general solution to each of the following differential equations. Write each equation and its corresponding general solution in your notebooks.

2. $x' + tx = 0$; 3. $x'' + x = 0$; 4. $4x'' + 8x' + 5x = 0$; 5. $x'' - 3x' + 2x = 0$.

Find the solution to each of the following IVPs and then represent its graph on $-\infty, \infty$. Write everything in your notebooks. If Maple does not return you directly an accurate graph, then use Maple, for example, to find the zeros of the function, or its limit at infinity, or portions of graphs on small intervals.

Describe the long-term behavior of the solution using some of the words: exponential increasing or decreasing, periodic with minimal period ..., bounded between ..., oscillatory around zero, with constant amplitude, with amplitude decreasing to zero as $t \to \infty$, with amplitude increasing to infinity as $t \to -\infty$... . Write everything in your notebooks.

6. $x'' + x = 0,\quad x(\pi/2) = 1,\quad x'(\pi/2) = -2$
7. $4x'' + 8x' + 5x = 0,\quad x(0) = 0,\quad x'(0) = 0.5$
8. $x'' - 3x' + 2x = 0,\quad x(0) = 2,\quad x'(0) = 3$

9. Find the general solution to the second order linear homogeneous differential equation $x'' + a(t)x = 0$ taking first $a(t) = 5$, then $a(t) = t$ and finally $a(t) = t^5$. In order to find details on how Maple "thinks" when trying to solve it, set infolevel[dsolve] to 3. Ask Wikipedia what are the Bessel functions. Use Maple to plot on different intervals at your choice (preferably large intervals of positive real numbers) the Bessel function BesselJ(1,t). Describe its long-term behavior.

If you want, set back infolevel[dsolve] to 1.

Now let $a \in C(\mathbb{R})$ be an arbitrary function and consider the following IVP:

$x'' + a(t)x = 0,\quad x(0) = 0,\quad x'(0) = 0.$

Notice that the Existence and Uniqueness Theorem applies in this case and yields that this IVP has a unique solution. We also notice that the null solution, $x = 0$, verifies all the conditions, hence it is the unique solution of this IVP.

Now find with Maple the solution of this IVP taking first $a(t) = 5$, then $a(t) = t$ and finally $a(t) = t^5$. What is your opinion about what Maple returns?

Find all solutions to each of the following BVPs (boundary value problems). Notice that, indeed, these are not IVPs. Why?

10. $x'' + x = 0, \quad x(0) = x(\pi) = 0$ 11. $x'' + x = 0, \quad x(0) = x(1) = 0.$

12. Find the solution of $x'' + x = 0, \quad x(0) = 0.$ Than find $a > 0$ such that the BVP $x'' + x = 0, \quad x(0) = x(a) = 0$ has non-null solutions.

Find the general solution to each of the following linear nonhomogeneous differential equations. Notice that, taking $c_1 = 0$, we obtain a particular solution of the equation which has the same form as the nonhomogeneous term. Try to formalize the concept *of the same form* and write everything in your notebooks.

13. $x' + x = 15$ 14. $x' + x = 2e^t - 7e^{-3t}$ 15. $x' + x = -t^2 + 3t - 7$

16. $x' + x = \sin t + 3\cos t$ 17. $x' + x = \sin t$ 18. $x' + x = 3\cos t$

19. Find the general solution to $x' + x = \frac{2}{\sqrt{\pi}} e^{-t^2 - t}$. Compute the primitive of e^{t^2} and then of $\frac{2}{\sqrt{\pi}} e^{-t^2}$. Ask Maple what is erf.

Decide whether the following statements are true or false.

20. All the solutions of $x'' + 3x' + x = 1$ satisfy $\lim\limits_{t\to\infty} x(t) = 1$.

21. The solution of the IVP $x'' + 4x = 1, \quad x(0) = 5/4, \quad x'(0) = 0$ satisfies $x(\pi) = 5/4$.

22. The equation $x' = 3x + t^3$ admits a polynomial solution.

First introduce in Maple the following function f, then represent its graph using the option discont=true. Is this function continuous in any point? Is this function differentiable in any point? Decide this looking at the graph. In your notebooks draw the graph and write the explanations.

23. $f(t) = \begin{cases} t, & t \le 2 \\ 3 - t, & t > 2 \end{cases}$; 24. $f(t) = \begin{cases} t, & 0 \le t \le \pi \\ \pi e^{\pi - t}, & t > \pi \end{cases}$.

25. We consider the IVP $x'' + x = f(t), \quad x(0) = 0, \quad x'(0) = 1,$ where f is given at 24. Find its solution and represent the corresponding integral curve.

26. Let $\omega > 0$ and $\varphi(t, \omega)$ be the solution of the IVP

$$x'' + x = \cos(\omega t), \quad x(0) = x'(0) = 0.$$

Find $\varphi(t, \omega)$ and note that it is not defined in $\omega = 1$. Find separately the solution when $\omega = 1$. Prove that $\lim_{\omega\to 1} \varphi(t, \omega) = \varphi(t, 1)$ for each $t \in \mathbb{R}$.

Notice that $\varphi(t, \omega)$ is bounded for each $\omega \neq 1$ fixed. Find a bound for $|\varphi(t, \omega)|$. Plot $\varphi(t, 1)$ and note that it is unbounded. Write everything in your notebooks.

27. Let $\alpha > 0$ and $\varphi(t, \alpha)$ be the solution of the IVP

$$x'' - 4x = e^{\alpha t}, \quad x(0) = x'(0) = 0.$$

Find $\varphi(t, \alpha)$ and note that it is not defined in $\alpha = 2$. Find separately the solution when $\alpha = 2$. Compute $\lim_{\alpha \to 2} \varphi(t, \alpha)$ for each $t \in \mathbb{R}$.

28. Find the solution of the IVP $x' = x, \quad x(0) = 1$ using **dsolve** with the option **series**. You can increase the order of truncation by setting, for example, **Order:=15**. In this way polynomial approximations of the solution of the IVP are obtained. In your notebooks write the exact solution and few polynomial approximations of it.

29. Find the solution of the IVP $x^2 u''(x) + x u'(x) + x^2 u(x) = 0, \quad u(0) = 1, \quad u'(0) = 0$ using **dsolve**. Than apply again **dsolve** with the option **series**. In your notebooks write the exact solution and a polynomial approximation of degree 10.

30. Find the general solution of $y''(x) - 2xy'(x) + 4y(x) = 0$ using **dsolve** with the option **series**. From the formula you obtained notice that, taking $y'(0) = 0$, a second degree polynomial solution is obtained. In your notebooks write this exact solution and check it.

31. The motion of a spring-mass system is given by the IVP

$$x'' + x = f(t), \quad x(0) = 5, \quad x'(0) = 0,$$

where $f(t) = \begin{cases} t, & t \in [0, \pi/2) \\ \pi - t, & t \in [\pi/2, \pi] \\ 0, & t \in (\pi, \infty) \end{cases}$. Find the solution of this IVP and draw its graph.

32. Check how the following works:

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>solve(r^2-k=0,r); solve(r^2+k=0,r);
>assume(k<0); from now on, the variable k is considered <0
>solve(r^2-k=0,r); solve(r^2+k=0,r);
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33. Find the general solution to

$$x'' + kx = 0$$

where $k \in \mathbb{R}$ is a parameter (consider all the possibilities).

34. We consider the IVP

$$x'' + x = f(t), \quad x(0) = 0, \quad x'(0) = 0$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Find its solution using **dsolve** and show that this is given by the formula

$$\varphi(t) = \int_0^t f(s) \sin(t - s) ds.$$

We consider the following linear planar system where the unknown functions are x and y of independent variable t. Denote by $\varphi(t, \eta_1, \eta_2)$ its solution satisfying $x(0) = \eta_1$, $y(0) = \eta_2$. For different fixed values of $(\eta_1, \eta_2) \in \{(1,1), (-2,1), (1,0), (-2,0), (-1,-2), ...\}$ find $\varphi(t, \eta_1, \eta_2)$ and plot simultaneously the planar curves with the parametric equations $(x, y) = \varphi(t, \eta_1, \eta_2)$, $t \in \mathbb{R}$. Write the system and the orbits in your notebooks.

35. $x' = -2x, \quad y' = -3y,$ 36. $x' = -2x, \quad y' = 3y,$ 37. $x' = -y, \quad y' = 4x,$
38. $x' = -x + 3y, \quad y' = -3x - y,$ 39. $x' = -x + 3y, \quad y' = 3x - y.$