LECTURE

6

HIGHER ORDER DERIVATIVES. TAYLOR SERIES AND POWER SERIES

Higher order derivatives

Definition 6.1 *Let $A \subseteq \mathbb{R}$, $c \in A \cap A'$ and $f : A \to \mathbb{R}$. We say that f is twice differentiable at c if $\exists V \in \mathcal{V}(c)$ such that f is differentiable on $A \cap V$ and f' is differentiable at c. If f is twice differentiable at c, then we write*

$$f^{(2)}(c) := f''(c) := (f')'(c).$$

In general, for $n \in \mathbb{N}$, ≥ 2, we say that f is n-times differentiable at c if $\exists V \in \mathcal{V}(c)$ such that f is $(n-1)$-times differentiable on $A \cap V$ and $f^{(n-1)}$ is differentiable at c. If f is n-times differentiable at c, then we write

$$f^{(n)}(c) := \left(f^{(n-1)}\right)'(c).$$

If B is a nonempty subset of A, we say that f is n-times differentiable on B if it is n-times differentiable at every point of B. In this case, the function $f^{(n)} : B \to \mathbb{R}$, $x \in B \mapsto f^{(n)}(x) \in \mathbb{R}$ is called the n^{th} derivative of f on B.

We say that f is infinitely differentiable at c if f is n-times differentiable at c for every $n \in \mathbb{N}$. Notational conventions: $f^{(0)} := f$ and $f^{(1)} := f'$.

Approximation of differentiable functions by Taylor polynomials

Let $I \subseteq \mathbb{R}$ be an interval, $x_0 \in I$, $f : I \to \mathbb{R}$ and $n \in \mathbb{N}$. Supposing that f is n-times differentiable at x_0, we want to find a polynomial function $P : \mathbb{R} \to \mathbb{R}$, of degree (at most) n, such that

$$\begin{cases} P(x_0) & = & f(x_0) \\ P'(x_0) & = & f'(x_0) \\ P''(x_0) & = & f''(x_0) \\ & \vdots & \\ P^{(n)}(x_0) & = & f^{(n)}(x_0). \end{cases} \tag{6.1}$$

We are looking for P of the form

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n.$$

By (6.1) we deduce that

$$a_0 = f(x_0), \quad a_1 = f'(x_0), \quad a_2 = \frac{f''(x_0)}{2!}, \quad \ldots, \quad a_n = \frac{f^{(n)}(x_0)}{n!}.$$

So, there is a unique polynomial P of degree (at most) n satisfying (6.1).

Definition 6.2 *Let $I \subseteq \mathbb{R}$ be an interval, $x_0 \in I$, $f : I \to \mathbb{R}$ and $n \in \mathbb{N}$. Supposing that f is n-times differentiable at x_0, the polynomial function $T_n : \mathbb{R} \to \mathbb{R}$, given by*

$$T_n(x) := f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{6.2}$$

is called the n^{th} Taylor polynomial of f (centered) at x_0.

Remark 6.3 *The n^{th} Taylor polynomial of f at x_0 is also denoted by $T_n(f; x_0)$. However, we simply write $T_n(x)$ instead of $T_n(f; x_0)(x)$ for all $x \in \mathbb{R}$.*

Remark 6.4 *Since the Taylor polynomial satisfies*

$$\begin{cases} T_n(x_0) & = & f(x_0) \\ T_n'(x_0) & = & f'(x_0) \\ T_n''(x_0) & = & f''(x_0) \\ & \vdots & \\ T_n^{(n)}(x_0) & = & f^{(n)}(x_0), \end{cases} \tag{6.3}$$

it approximates the function f on a neighborhood of x_0, i.e.,

$$f(x) \simeq f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

In particular, for $n = 1$ we obtain

$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0).$$

Definition 6.5 *Let $I \subseteq \mathbb{R}$ be an interval, $x_0 \in I$, $f : I \to \mathbb{R}$ and $n \in \mathbb{N}$. Supposing that f is n-times differentiable at x_0, the function $R_n : I \to \mathbb{R}$, defined by*

$$R_n(x) := f(x) - T_n(x), \ \forall\, x \in I, \tag{6.4}$$

is called the remainder of the approximation of f by T_n around x_0. Whenever R_n is given explicitly, we get the so-called Taylor formula:

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n}_{T_n(x)} + R_n(x), \ \forall\, x \in I.$$

Theorem 6.6 (Taylor-Lagrange) *Let $f : I \to \mathbb{R}$ be a function which is $(n+1)$-times differentiable on I for some $n \in \mathbb{N} \cup \{0\}$. Then, for any distinct points $x, x_0 \in I$ there exists a point $c \in \mathbb{R}$, $\min\{x_0, x\} < c < \max\{x_0, x\}$, such that*

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{6.5}$$

In other words, we have $f(x) = T_n(x) + R_n(x)$, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{6.6}$$

Proof. Consider any distinct points $x, x_0 \in I$. Without loss of generality we can assume that $x_0 < x$. By (6.3) and (6.4) we have

$$R_n^{(k)}(x_0) = 0,\ \forall\, k \in \{0, 1, \ldots, n\}.$$

By Cauchy's Generalized Mean Value Theorem 5.48, applied to the functions

$$x \longmapsto R_n(x) \ \text{ and } \ x \longmapsto (x - x_0)^{n+1}$$

on the interval $[x_0, x]$, there exists $c_1 \in (x_0, x)$ such that

$$\frac{R_n(x)}{(x - x_0)^{n+1}} = \frac{R_n(x) - R_n(x_0)}{(x - x_0)^{n+1} - (x_0 - x_0)^{n+1}} = \frac{R_n'(c_1)}{(n+1)(c_1 - x_0)^n}.$$

Applying now Cauchy's Generalized Mean Value Theorem to the functions

$$x \longmapsto R_n'(x) \ \text{ and } \ x \longmapsto (n+1)(x - x_0)^n$$

on the interval $[x_0, c_1]$, we deduce that there is $\exists c_2 \in (x_0, c_1)$ such that

$$\frac{R_n(x)}{(x - x_0)^{n+1}} = \frac{R_n'(c_1)}{(n+1)(c_1 - x_0)^n} = \frac{R_n'(c_1) - R_n'(x_0)}{(n+1)(c_1 - x_0)^n - (n+1)(x_0 - x_0)^n} = \frac{R_n''(c_2)}{(n+1)n(c_2 - x_0)^{n-1}}.$$

Continuing in this way we find $c_{n+1} \in (x_0, c_n) \subseteq (x_0, c_{n-1}) \subseteq \cdots \subseteq (x_0, x)$ such that

$$\frac{R_n(x)}{(x - x_0)^{n+1}} = \frac{R_n^{(n+1)}(c_{n+1})}{(n+1)!}. \tag{6.7}$$

On the other hand, recalling that T_n is a polynomial of degree at most n, we deduce by (6.4) that $R_n^{(n+1)}(c_{n+1}) = f^{n+1}(c_{n+1}) - T_n^{(n+1)}(c_{n+1}) = f^{n+1}(c_{n+1})$. Hence, by choosing $c := c_{n+1}$, we infer from (6.7) that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \qquad \square$$

Remark 6.7 (6.5) *is called the* Taylor's formula with the remainder in Lagrange's form (6.6)*.*

Remark 6.8 *Assume that, for some $a, b \in I$ with $a < x_0 < b$, there exists $M \in \mathbb{R}$ such that $|f^{(n+1)}(x)| \leq M$ for all $x \in [a, b]$. Then, the error of approximation of $f(x)$ by $T_n(x)$ can be estimated by*

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}(x - x_0)^{n+1},\ \forall\, x \in [a, b].$$

Corollary 6.9 (Local optimality conditions) *Let $f : I \to \mathbb{R}$ be a function, defined on an interval $I \subseteq \mathbb{R}$. If f is n-times differentiable ($n \in \mathbb{N}$, $n \geq 2$) at $x^0 \in \operatorname{int} I$ and*

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0 \neq f^{(n)}(x_0),$$

then the following assertions hold true:
1° *If n is even and $f^{(n)}(x_0) > 0$, then x_0 is a local minimum point of f.*
2° *If n is even and $f^{(n)}(x_0) < 0$, then x_0 is a local maximum point of f.*
3° *If n is odd, then x_0 is not a local extremum point of f.*

Example 6.10 *Let $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 24\cos x + 12x^2 - x^4$ for all $x \in \mathbb{R}$. It is easy to check that*

$$f'(0) = f''(0) = f^{(3)}(0) = f^{(4)}(0) = f^{(5)}(0) = 0 \neq -24 = f^{(6)}(0),$$

hence $x_0 = 0$ is local maximum point of f.

Taylor series

Definition 6.11 *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to R$ be infinitely differentiable. For $x_0 \in I$ and $x \in \mathbb{R}$, the series*

$$\sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \tag{6.8}$$

is called the Taylor series of f around x_0.

If $J \subseteq I$ is a nonempty set such that for every $x \in J$ the series (6.8) converges and its sum is $f(x)$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n, \tag{6.9}$$

then we say that f can be expanded as a Taylor series around x_0 on J. In this case, (6.9) is called the Taylor expansion of $f(x)$ around x_0 on J.

Remark 6.12 *For any $x \in I$, the partial sums of the Taylor series (6.8) are given by*

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = T_n(x),\ \forall\, n \in \mathbb{N} \cup \{0\}.$$

Thus, the series (6.8) converges if and only if its sum is finite, i.e.,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n := s(x) := \lim_{n\to+\infty} T_n(x) \in \mathbb{R}$$

and, according to (6.4), we have

$$\lim_{n\to+\infty} R_n(x) = f(x) - \lim_{n\to+\infty} T_n(x) = f(x) - s(x).$$

Therefore, by Definition 6.11, f can be expanded as a Taylor series around x_0 on J if and only if

$$\lim_{n\to+\infty} R_n(x) = 0,\ \forall\, x \in J.$$

Example 6.13 (Taylor expansion of the exponential function around 0) *Let $f : \mathbb{R} \to \mathbb{R}$,*

$$f(x) = e^x.$$

Note that $\forall k \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $f^{(k)}(x) = e^x$, so $\forall k \in \mathbb{N}$, $f^{(k)}(0) = 1$. Let $n \in \mathbb{N}$, $x \in \mathbb{R}$. Then there exists c between 0 and x such that

$$e^x = 1 + \frac{1}{1!}x + \ldots + \frac{1}{n!}x^n + R_n(x),$$

where $R_n(x) = \dfrac{e^c}{(n+1)!}x^{n+1}$.

Since $0 \leq |R_n(x)| \leq e^c \dfrac{|x|^{n+1}}{(n+1)!}$ and $\lim\limits_{n\to\infty} \dfrac{|x|^n}{n!} = 0$, it follows that $\lim\limits_{n\to\infty} R_n(x) = 0$.

Therefore, f can be expanded as a Taylor series around 0 on $\mathbb{R}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x \in \mathbb{R}.$$

Example 6.14 (Taylor expansion of sine function around 0) *The function $f : \mathbb{R} \to \mathbb{R}$,*

$$f(x) = \sin x,$$

can be expanded as a Taylor series around 0 on $\mathbb{R}$:

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}, \quad \forall x \in \mathbb{R}.$$

Example 6.15 (Taylor expansion of cosine function around 0) *The function $f : \mathbb{R} \to \mathbb{R}$,*

$$f(x) = \cos x,$$

can be expanded as a Taylor series around 0 on $\mathbb{R}$:

$$\cos x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!}, \quad \forall x \in \mathbb{R}.$$

Power series

Definition 6.16 *Let $(a_n)_{n \geq 0}$ be a sequence of real numbers and let $c \in \mathbb{R}$. A series of type*

$$\sum_{n \geq 0} a_n (x-c)^n, \quad \text{where } x \in \mathbb{R}, \tag{6.10}$$

is called power series centered at x with coefficients a_n. *The set*

$$C := \{x \in \mathbb{R} \mid \text{the series (6.10) converges}\}$$

is called the convergence set *of the power series.*

Theorem 6.17 (Abel) *There exists $R \in [0, +\infty) \cup \{+\infty\}$ such that the power series (6.10) converges absolutely whenever $0 \leq |x - c| < R$ and diverges whenever $|x - c| > R$.*

Definition 6.18 *The unique R satisfying the conditions of Abel's Theorem 6.17 is called the* radius of convergence *of the power series.*

Theorem 6.19 (Cauchy-Hadamard) *If the limit*

$$L := \lim_{n \to \infty} \sqrt[n]{|a_n|} \in [0, \infty) \cup \{+\infty\}$$

exists, then the power series (6.10) has the radius of convergence $R = \begin{cases} 1/L & \text{if } 0 < L < +\infty \\ 0 & \text{if } L = +\infty \\ +\infty & \text{if } L = 0. \end{cases}$

Corollary 6.20 *If the limit*

$$L := \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \in [0, \infty) \cup \{+\infty\}$$

exists, then the power series (6.10) has the radius of convergence $R = \begin{cases} 1/L & \text{if } 0 < L < +\infty \\ 0 & \text{if } L = +\infty \\ +\infty & \text{if } L = 0. \end{cases}$

Example 6.21 *1) For the geometric series* $\sum_{n\geq 1}(x-c)^n$*, centered at any number* $c\in\mathbb{R}$*, we have* $R=1$ *and* $C=(c-1,c+1)$*.*

2) For $\sum_{n\geq 1}\frac{1}{n}x^n$ *we have* $R=1$ *and* $C=[-1,1)$*.*

3) For $\sum_{n\geq 1}\frac{(-1)^n}{n}x^n$ *we have* $R=1$ *and* $C=(-1,1]$*.*

4) For $\sum_{n\geq 1}\frac{1}{n!}(x-c)^n$*, centered at any* $c\in\mathbb{R}$*, we have* $R=0$ *and* $C=\mathbb{R}$*.*

5) For $\sum_{n\geq 1}n!(x+1)^n$ *we have* $R=+\infty$ *and* $C=\{-1\}$*.*