Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems
## June 18, 2010
## I

1. (2p) Find the general solution of the following differential equations

$$x' + ax = -at + 1, \quad x'' - ax' + (a-1)x = 0,$$

where $a \in \mathbb{R} \setminus \{0, 1\}$ is a real parameter. Here the unknown is the function denoted $x$ of independent variable $t$.

2. (3p) Let $\varphi_1,\ \varphi_2 : \mathbb{R} \to \mathbb{R}$ be two distinct solutions of the differential equation $y' = \sqrt[3]{y-1}$ (the unknown is the function denoted $y$ of independent variable $x$). Decide whether or not the following situations are possible:
(a) $\varphi_1(0) = 2$ and $\varphi_1'(0) = -1$;
(b) $\varphi_1(0) = \varphi_2(0) = 2$;
(c) $\varphi_1(0) = \varphi_2(0) = 1$ and $\varphi_1'(0) \neq \varphi_2'(0)$.

3. (1.5p) Represent the phase portrait of the scalar differential equations:
(a) $\dot{x} = 2x - x^2$;
(b) $\dot{x} = 1 + x + x^2$;
(c) $\dot{x} = 1 + x + x^3$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems
June 18, 2010
II

1. (2p) Find the general solution of the following differential equations

$$x' + x = -2e^t, \quad x'' = \frac{2}{t}x'.$$

Here the unknown is the function denoted $x$ of independent variable $t$.

2. (3p) Let $\varphi_1, \ \varphi_2 : \mathbb{R} \to \mathbb{R}$ be two distinct solutions of the differential equation $y' = \sqrt[3]{y+2}$ (the unknown is the function denoted $y$ of independent variable $x$). Decide whether or not the following situations are possible:
(a) $\varphi_1(0) = 1$ and $\varphi_1'(0) = -1$;
(b) $\varphi_1(0) = \varphi_2(0) = 1$;
(c) $\varphi_1(0) = \varphi_2(0) = -2$ and $\varphi_1'(0) \neq \varphi_2'(0)$.

3. (1.5p) Decide the type and stability of the equilibrium point $(0,0)$ of the differential systems:
(a) $\dot{x} = 2x, \quad \dot{y} = -x - 3y$;
(b) $\dot{x} = 2x + y, \quad \dot{y} = -x + 3y$.
Here the unknowns are the functions denoted $x$ and, respectively, $y$, of independent variable $t$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems
June 19, 2010
I

1. (3p) Integrate the following differential equations
(a) $y' = \dfrac{-x - x^3}{y}$
(b) $y' = \dfrac{-x - y^2}{y}$
and than find a first integral of each of it.

2. (2p) Let $\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be some solution of the differential equation $y' = \dfrac{y-2}{1 - x^2 - y^2}$ (the unknown is the function denoted $y$ of independent variable $x$, while $\varepsilon > 0$ is a positive constant). Decide whether or not the following situations are possible:
(a) $\varphi(0) = -2$ and $\varphi$ is a strictly increasing function;
(b) $\varphi(0) = 2$ and $\varphi$ is a strictly increasing function;
(c) $\varphi_1(0) = -2$ and $\varphi_1'(0) = 0$.

3. (1.5p) Find the flow of the planar linear differential system:

$$\dot{x} = x + y, \quad \dot{y} = -2x + 4y.$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems
June 19, 2010
II

1. (3p) Integrate the following differential equations
(a) $y' = \dfrac{x - 2x^3}{y}$
(b) $y' = \dfrac{x - y^2}{y}$
and than find a first integral of each of it.

2. (2p) Let $\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be some solution of the differential equation $y' = \dfrac{y-1}{1+x^2+y}$ (the unknown is the function denoted $y$ of independent variable $x$, while $\varepsilon > 0$ is a positive constant). Decide whether or not the following situations are possible:
(a) $\varphi(0) = 1$ and $\varphi$ is a strictly increasing function;
(b) $\varphi(0) = 0$ and $\varphi$ is a strictly increasing function;
(c) $\varphi_1(0) = -2$ and $\varphi_1'(0) = 0$.

3. (1.5p) Find the flow of the planar linear differential system:

$$\dot{x} = x - y, \quad \dot{y} = 2x + 4y.$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems
## July 8, 2010

1. (2p) Write the general solution of $x'' - a^2 x = e^{bt}$, where $a > 0$ and $b \in \mathbb{R}$ are parameters. Here the unknown is the function denoted $x$ of independent variable $t$.

2. (1.5p) Integrate the differential equation $y' = \dfrac{-x - y^2}{y}$. Here the unknown is the function denoted $y$ of independent variable $x$.

3. (2p) Write the definition of the first integral for a differential equation in symmetrical form. Give examples.

Write the definition of the first integral for a differential equation in normal form. Give examples.

4. (1p) Find the general solution and represent the phase portrait of the planar system $\dot{x} = -y, \quad \dot{y} = x$ . Here the unknowns are the functions denoted $x$ and, respectively, $y$, of independent variable $t$.

5. (1p) (*instead of the point from the seminar partial exam*)

Find the solution of the following Initial Value Problem:

$t^2 x'' + t x' - x = 0\,, \quad x(1) = 1\,, \quad x'(1) = -1.$

Here the unknown is the function denoted $x$ of independent variable $t$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems.
# June 08, 2009

1. (1p) Find the general solution of the following differential equation

$$x' - 2x = 2t - 3.$$

2. (0.5p) Find the second order linear homogeneous differential equation with constant coefficients that has as solutions $e^{-t}$ and $5e^{-2t}$.

3. (1.5p) Find the coefficients of the power series solution (around $t = 0$) of the Initial Value Problem

$$\begin{cases} x'' + t^2 x = 0 \\ x(0) = 0 \\ x'(0) = 1. \end{cases}$$

4. (2p) Specify the type and stability of the equilibrium point $(0, 0)$ of the differential system:

$$\dot{x} = -2x, \quad \dot{y} = -y.$$

Represent the phase portrait of this system.

5. (1.5p) Find the equilibria and study their stability for the differential equation

$$\ddot{\theta} + 4\dot{\theta} + \sin\theta = 0.$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems
## June 09, 2009
## I

1. (1.5p)
(a) Find the general solution of the following differential equation

$$\varphi'' + \frac{9}{4}\varphi = 0.$$

(b) (True or False) "All the solutions of $\varphi'' + \frac{9}{4}\varphi = 0$ are periodic with a period $T = 4\pi$."

2. (1.5p) Let $I \subset \mathbb{R}$ be an open interval and $a, f : I \to \mathbb{R}$ be continuous functions. Write the general solution of the differential equations:
(a) $x' + a(t)x = 0$,
(b) $x' + a(t)x = f(t)$.

3. (0.5p) Write the Euler numerical formula for a first order differential equation.

4. (3p) We consider the differential system:

$$\dot{x} = -y + y^3, \quad \dot{y} = -x + x^3.$$

(a) Study the stability of the equilibrium point $(0, 0)$.
(b) Find a first integral.
(c) Find all the equilibria.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems
June 09, 2009
II

1. (3.5p) We consider the differential system

$$\dot{x} = -x + 2y, \quad \dot{y} = -2x - y.$$

(a) Study the type and stability of its equilibrium point $(0, 0)$.
(b) Find its general solution.
(c) Pass to polar coordinates.
(d) Represent its phase portrait.

2. (1p) Write the statements of
(a) the existence theorem (Peano)
(b) the existence and uniqueness theorem (Cauchy-Lipschitz)
for a first order differential equation.

3. (2p) (a) Find the solution of the Initial Value Problem

$$y' = \frac{2y}{x}, \quad y(1) = \pi.$$

(b) (True or False) "The solution of the previous IVP is a bounded function."

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems
## June 27, 2009
## I

1. (1.5p) Find the general solution of the differential equation:
(a) $t^2x'' - 3tx' + 3x = 0$, for $t \in (0, \infty)$;
(b) $t^2x'' + tx' + 4x = 0$, for $t \in (0, \infty)$.

2. (3p) We consider the differential equation:

$$y' = -\frac{x}{2y}.$$

(a) (True or False) "Through the point $(1, 1) \in \mathbb{R}^2$ passes one and only one integral curve of the given differential equation." Justify the answer.

(b) Find the maximal solution of the Initial Value Problem for the given differential equation with the condition $y(1) = 1$. Plot its graph.

(c) Represent the 3-level curve of the function $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = x^2 + 2y^2$. What is the relation between this curve and the one plotted at (b)? What is the relation between $H$ and the given differential equation?

3. (2p) Represent the phase portrait of:
(a) $\dot{x} = 4x - x^3$;
(b) $\dot{x} = 4x - x^3 + 1$;
(c) $\dot{x} = 4x - x^3 + 5$.

Universitatea Babeş–Bolyai
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# Exam on Dynamical Systems
June 27, 2009
II

1. (3.5p) We consider the differential system

$$\dot{x} = -x + 2y, \quad \dot{y} = -2x - y.$$

(a) Study the type and stability of its equilibrium point $(0, 0)$.
(b) Find its general solution.
(c) Pass to polar coordinates.
(d) Represent its phase portrait.

2. (1p) Write the statements of
(a) the existence theorem (Peano)
(b) the existence and uniqueness theorem (Cauchy-Lipschitz)
for a first order differential equation.

3. (2p) (a) Find the solution of the Initial Value Problem

$$y' = \frac{2y}{x}, \quad y(1) = \pi.$$

(b) (True or False) "The solution of the previous IVP is a bounded function."

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems.
## August 31, 2009
## I

1. (3p) Specify the type and stability of the equilibrium point $(0,0)$ of the differential system:

$$\dot{x} = 3x + 2y, \quad \dot{y} = 4x + y.$$

Find the general solution of the above differential system.

2. (2.5p) Represent the phase portrait of the scalar differential equation

$$\dot{x} = 2x\left(3 - \frac{x}{100}\right).$$

What remarkable property has the solution of the above equation with the initial value $x(0) = 100$ ? (*Hint: "read" its phase portrait*)
Find the general solution of the above equation.

3. (1p) Find a first integral of $(x^2 - 5xy^2)dx + (y^3 - 5x^2y + 3)dy = 0$.

4. (1p, not compulsory) Determine the equilibria and study the stability of one of them for the following planar differential system:

$$\dot{x} = -2x + y^2, \quad \dot{y} = y - 2xy.$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems.
## August 31, 2009
## II

1. (3p) Specify the type and stability of the equilibrium point $(0,0)$ of the differential system:

$$\dot{x} = 5x - 7y, \quad \dot{y} = 2x - 4y.$$

Find the general solution of the above differential system.

2. (2.5p) Represent the phase portrait of the scalar differential equation

$$\dot{x} = 3x\left(-2 + \frac{x}{100}\right).$$

What remarkable property has the solution of the above equation with the initial value $x(0) = 100$ ? (*Hint: "read" its phase portrait*)
Find the general solution of the above equation.

3. (1p) Find a first integral of $(x^3 - 3xy^2 + 2)dx + (y^2 - 3x^2y)dy = 0$.

4. (1p, not compulsory) Determine the equilibria and study the stability of one of them for the following planar differential system:

$$\dot{x} = x - 2xy, \quad \dot{y} = x^2 - 2y.$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems.
June 11, 2008

1. Find the general solutions of the following differential equations:

$$x' = -x\,, \quad x' = 3x + 2 - 3t + e^{-3t}\,, \quad x'' - x' + 2x = 0\,, \quad x''' = 0.$$

2. We consider the differential equation

$$y' = \frac{1 - \sqrt[3]{y}}{1 - xy}$$

and three Initial Value Problems for it with the conditions: $y(0) = 1$, $y(1) = 1$ and $y(0) = 0$, respectively. Here the unknown function is $y = y(x)$.

a) Are the above Initial Value Problems well–defined?

b) If they are well–defined, decide whether or not the Local Existence and Uniqueness Theorem is applicable.

c) If the Local Existence and Uniqueness Theorem is applicable, find the solution.

3. Find the differential equation of the family of planar curves described by $x^2 + 9y^2 = c$, $c \in \mathbb{R}$. Find also a planar autonomous system whose trajectories are these curves.

4. We consider the logistic map $f_\lambda : [0,1] \to [0,1]$ $f_\lambda(x) = \lambda x(1-x)$, where $\lambda \in (0,4)$ is a parameter. Find the fixed points of the logistic map and study their stability (discuss with respect to the parameter $\lambda$).

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems.
## June 12, 2008

1. We say that a differential equation exhibit resonance when all its solutions are unbounded.

For what values of the mass $m$ will $mx'' + 25x = 12\cos(36\pi t)$ exhibit resonance?

2. Find the solution of the following Initial Value Problem

$$y'' - \frac{y'}{x} = x^2 \quad y(2) = 0\,, \quad y'(2) = 4\,.$$

3. Represent the phase portrait of the following differential equation:

$$\dot{x} = 4x - x^3\,.$$

4. We consider the nonlinear autonomous planar system:

$$\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -4y + 8xy\,. \end{cases}$$

Find its equilibria and study their stability.

5. Write the definition of the first integral for a differential equation in symmetrical form.

6. Write the definition of a fixed point of some scalar map.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems.
# June 28, 2008

1. Let $\alpha \in \mathbb{R}$. We consider the differential equation $x'' + \alpha x' + 9x = 0$.

(a) Find the general solution when $\alpha = 4$ and $\alpha = 0$, respectively.

(b) Find $\alpha$ such that all the solutions are periodic. What is the period in this case? Does it depend on $\alpha$?

2. We consider the Initial Value Problem $x' + \alpha(t)x = f(t)$, where $\alpha, f \in C(\mathbb{R})$.

(a) Find the solution when $\alpha(t) = 2t$ and $f(t) = 3e^{-t^2}$.

(b) Find the solution (eventually only an integral representation of it) when $\alpha(t) = 2t$ and $f(t) = 1$.

(c) Write an integral representation of the solution of this IVP for arbitrary $\alpha$ and $f$.

3. Find a first integral for the differential equation

$$(5x - 2xy)dx + (3y^2 - x^2)dy = 0 .$$

4. Represent the phase portrait of the following differential equation:

$$\dot{x} = \frac{1}{2}x(1 - x) .$$

5. We consider the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}x(1 - x)$. Find its fixed points and study their stability. For what type of values of $\eta \in \mathbb{R}$ we can deduce from the above study that the sequence $(x_n)_{n \geq 0}$ given by the recurrence $x_{n+1} = \frac{1}{2}x_n(1 - x_n)$, $n \geq 0$, $\quad x_0 = \eta$ is convergent?

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

# Exam on Dynamical Systems.
# September 1, 2008

1. We consider the following differential system:

$$\begin{cases} x' = y \\ y' = -6x + 5y. \end{cases}$$

a) Find its general solution.
b) Study the type and stability of its equilibrium point.

2. Find the solution of the following Initial Value Problem:

$$y' = -\frac{2x}{y}, \quad y(0) = 3.$$

3. We consider the Initial Value Problem (IVP):

$$y' = \frac{y+x}{y-2x}, \quad y(0) = 1.$$

a) Find the domain of definition of the given differential equation and show that the point $(0, 1)$ belongs to it.

b) Find the set of all existence and uniqueness points of the given differential equation and show that the point $(0, 1)$ belongs to it.

c) Show that the solution of this IVP is strictly increasing in a small neighborhood of $x = 0$.

d) Write the Euler numerical formula for this IVP.

e) Write the Runge–Kutta numerical formula for this IVP.

f) Write the recurrence formula for the Picard sequence of successive approximations for this IVP.

4. (instead of the partial exam) Find the solution of the following Initial Value Problem:

$$y' = -\frac{y}{2x} + x, \quad y(1) = 0.$$

# Dynamical systems. Final exam 20-06-2007

1. Find the general solution of $\ddot{\theta} + \dot{\theta} + \theta = 0$. Prove that $\lim_{t \to \infty} \theta(t) = 0$ for any solution $\theta$ of this differential equation.

2. Prove that $\lim_{t \to \infty} \theta(t) = 0$ for any solution $\theta$ of the differential equation $\ddot{\theta} + \dot{\theta} + \sin \theta = 0$ with $|\theta(0)|$ sufficiently small.

3. Find the general solution of the differential equation

$$y' = \frac{3x - y}{x + 3y}.$$

(Hint: write it in symmetrical form)

4. Specify the type and study the stability of the equilibrium $(0, 0)$ of the planar system $\dot{x} = x + 3y$, $\dot{y} = 3x - y$. Find also a first integral for this system.

5. Define the notion of first integral for a planar autonomous system.

6. Write the statements of the Existence Theorem of Peano and of the Local Existence and Uniqueness Theorem for a first order scalar Initial Value Problem.

7. Prove that the Initial Value Problem

$$y' = \frac{y}{x^2 - 2x + 1}, \quad y(0) = 2$$

has a unique maximal solution and than find it.

# Dynamical systems. Final exam 06-09-2007

1. Find the solution of each of the following Initial Value Problems:
a) $x' = x\,, \quad x(0) = 1.$
b) $x' = y\,, \quad y' = x\,, \quad x(0) = 2\,, \quad y(0) = -2.$
c) $t^2x'' + tx' - x = 0\,, \quad x(1) = 1\,, \quad x'(1) = -1.$

2. Galileo's pendulum.
a) Deduce its differential equation.
b) Find the general solution of the linearized equation $\theta'' + \omega^2\theta = 0\,,$ where $\omega = g/L > 0$ ($g$ the gravitational constant and $L$ the length of the rod). Interpret the result.

3. Represent the phase portrait and find a first integral for the planar system:

$$\dot{x} = -y\,, \quad \dot{y} = x\,.$$

4. The statement of the Local Existence and Uniqueness Theorem for a first order scalar Initial Value Problem.

5. Determine the equilibria and study their stability for the planar system:

$$\dot{x} = x - 2xy\,, \quad \dot{y} = x^2 - 2y\,.$$

6. (*instead of the point from the seminar partial exam*) Find the value of the real parameter $b$ for which the given equation is exact and than find a first integral using the value of $b$:

$$(2xy^2 + bx^2y)dx + x^2(x + 2y)dy = 0\,.$$

# Dynamical systems. Final exam 25-01-2005

1. Find a first integral of $(2x+1)dx + 2ydy = 0$.

2. Represent the phase portrait of $x' = \lambda - x^2$ , where $\lambda \in \mathbb{R}$ is a parameter. Study the stability of the equilibrium points.

3. Find the maximal solution of the Initial Value Problem:
$x^2y'' - 2xy' + 2y = x^3\,, \quad y(1) = 1\,, \quad y'(1) = 1\,.$

4. We consider the system $\dot{x} = 3x + 2y\,, \quad \dot{y} = -x + y\,.$
a) Write its general solution.
b) Specify the type and study the stability of the equilibrium point.

5. Write the Euler's numerical formula to find the approximate solution of the Initial Value Problem: $y' = y + \sin y\,, \quad y(0) = 2$ on the interval $[0,\, 1.5]\,.$

6. We consider the Initial Value Problem: $y' = f(x,y)\,, \quad y(x_0) = y_0$ , where $f : [x_0 - a,\, x_0 + a] \times \mathbb{R} \to \mathbb{R}$ and $a > 0,$ $y_0 \in \mathbb{R}.$
a) Write the definition and state sufficient conditions for the function $f$ to be Lipschitz with respect to $y$.
b) Write the statement of the Global Existence and Uniqueness Theorem.
c) Prove the convergence of the sequence of functions $\varphi_n \in C[-1,1]$, for all $n \geq 0$ given by the recurrence:

$$\varphi_{n+1}(x) = 1 + 2\int_0^x s\varphi_n(s)ds\,,\ n \geq 0\,,\ \varphi_0(x) = 1 \text{ for all } x \in [-1,1]\,.$$

(Hint: use b) and the formula $\left(\int_0^x u(s)ds\right)' = u(x)$ .)

# Dynamical systems. Final exam 26-01-2005

1. We consider the system $\dot{x} = x\,, \quad \dot{y} = 1 + y\,.$ Write its general solution and represent its phase portrait.

2. Represent the phase portrait of $\dot{x} = 2x - \sin x\,.$

3. Write the general solution of $y'' - a^2 y = e^{bx}$, where $a > 0$ and $b \in \mathbb{R}$ are parameters.

4. a) Verify that $y_1 = x$ and $y_2 = e^{-2x}$ are solutions of
$(2x+1)y'' + 4xy' - 4y = 0\,.$
b) Find the maximal solution of the Initial Value Problem:
$(2x+1)y'' + 4xy' - 4y = (2x+1)^2\,, \quad y(0) = 1\,, \quad y'(0) = 0.$

5. Write the definitions for a fixed point of a scalar map for an asymptotically stable fixed point.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map and $\eta\,, \eta^* \in \mathbb{R}$ be such that $f^n(\eta) \to \eta^*$ as $n \to \infty$. Prove that $\eta^*$ is a fixed point of $f$.

7. Let $\eta \in \mathbb{R}$ be such that $|\eta|$ is sufficiently small. Study the convergence of the sequence given by the recurrence

$$x_{n+1} = \frac{1}{2}x_n - 3x_n^3 \quad , n \geq 0,\ x_0 = \eta.$$

# Dynamical systems. Final exam 13-02-2005

1. Find the general solution of $y' = 3y + x^2$.

2. We consider the Initial Value Problem:

$$y' = \frac{1}{y - x^2} + 2x\,, \quad y(0) = -1.$$

a) Write the domain of the differential equation, denoted $D_f$, as $D_f = U_1 \cup U_2$, where $U_1$ and $U_2$ are open and connected.

b) Do the change of the variable $u = y - x^2$ , where $u = u(x)$.

c) Find the maximal solution of this IVP.

3. We consider the differential equation $y' = \lambda + 2y - y^2$ where $\lambda \in \mathbb{R}$ is a parameter.

a) Find the equilibrium points and study their stability.

b) Write the Euler's numerical formula to find the approximate solution of this differential equation on the interval $[0, 1]$ that satisfies $y(0) = 0.5$.

4. We consider the scalar map $f : \mathbb{R} \to \mathbb{R}$ $f(x) = (1 + x)/2$ . Find the fixed points of $f$. Draw the stair–step diagram starting with $x_0 = 3$.

5. Write the definition of the Wronski–an of two $C^1$ functions.

6. Linear homogeneous second order differential equations with constant coefficients.