Geometry
Problem booklet

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1 Week 9: Conics and Quadrics

1.1 Conics

1.1.1 The Hyperbola

Definiţia 1.1. *The* hyperbola *is defined as the geometric locus of the points in the plane, whose absolute value of the difference of their distances to two fixed points, say F and F' is constant.*

The two fixed points are called the *foci* of the hyperbola, and the distance $|FF'| = 2c$ between the foci is the *focal distance*.

Suppose that the constant in the definition is $2a$. If $M(x, y)$ is an arbitrary point of the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

Choose a Cartesian system of coordinates, having the origine at the midpoint of the segment $[FF']$ and such that $F(c, 0)$, $F'(-c, 0)$.

Remark 1.2. *In the triangle $\Delta MFF'$, $||MF| - |MF'|| < |FF'|$, so that $a < c$.*

Let us determine the equation of a hyperbola. By using the definition we get $|MF| - |MF'| = \pm 2a$, namely

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a,$$

or, equivalently

$$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}.$$

We therefore have successively

$$\begin{array}{l} x^2 - 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\ cx + a^2 = \pm a\sqrt{(x+c)^2 + y^2} \\ c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 \\ (c^2 - a^2)x^2 - a^2y^2 - a^2(c^2 - a^2) = 0. \end{array}$$

By using the notation $c^2 - a^2 = b^2$ ($c > a$) we obtain the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \tag{1.1}$$

The equation (1.1) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}; \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

Therefore, the coordinate axes are axes of symmetry of the hyperbola and the origin is a center of symmetry equally called the *center of the hyperbola*.

Remark 1.3. *To sketch the graph of the hyperbola, is it enough to represent the function*

$$f : (-\infty, -a] \cup [a, \infty) \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{x^2 - a^2},$$

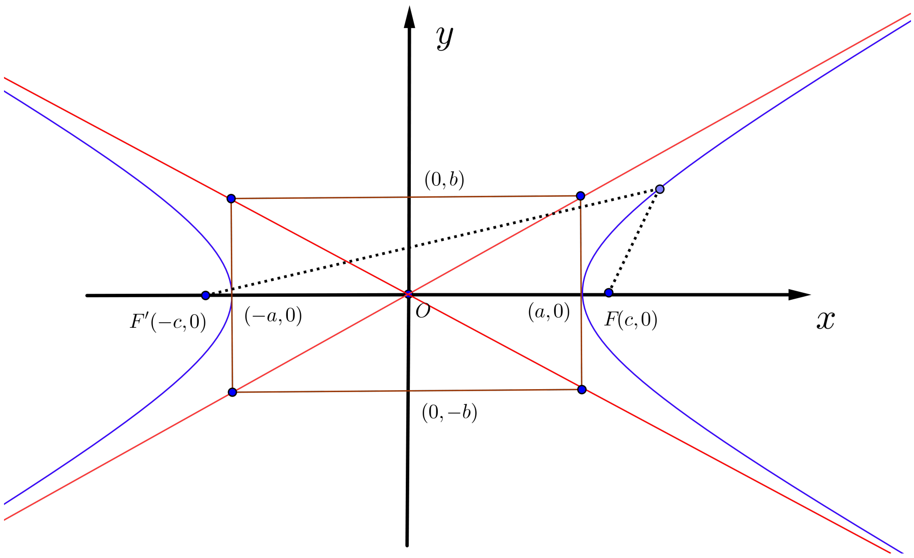
by taking into account that the hyperbola is symmetric with respect to the x-axis.

Since $\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=\dfrac{b}{a}$ and $\lim\limits_{x\to-\infty}\dfrac{f(x)}{x}=-\dfrac{b}{a}$, it follows that $y=\dfrac{b}{a}x$ and $y=-\dfrac{b}{a}x$ are asymptotes of f.

One has, also

$$f'(x)=\frac{b}{a}\frac{x}{\sqrt{x^2-a^2}},\qquad f''(x)=-\frac{ab}{(x^2-a^2)\sqrt{x^2-a^2}}.$$

x	$-\infty$		$-a$		a		∞
$f'(x)$	$-$	$-\ -\ -$	$\vert$	$/\ /\ /$	$\vert$	$+\ +\ +$	$+$
$f(x)$	∞	$\searrow$	$0\vert$	$/\ /\ /$	$\vert 0$	$\nearrow$	∞
$f''(x)$	$-$	$-\ -\ -$	$\vert$	$/\ /\ /$	$\vert$	$-\ -\ -$	$-$



1.1.2 The Parabola

Definiţia 1.4. *The* parabola *is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line d is equal to its distance to a fixed point F.*

The line d is the *director line* and the point F is the *focus*. The distance between the focus and the director line is denoted by p and represents the *parameter* of the parabola.

Consider a Cartesian system of coordinates xOy, in which $F\left(\dfrac{p}{2},0\right)$ and $d: x=-\dfrac{p}{2}$. If $M(x,y)$ is an arbitrary point of the parabola, then it verifies

$$|MN|=|MF|,$$

where N is the orthogonal projection of M on Oy.

Thus, the coordinates of a point of the parabola verify

$$\sqrt{\left(x+\frac{p}{2}\right)^2+0}=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}$$

$$\left(x+\frac{p}{2}\right)^2=\left(x-\frac{p}{2}\right)^2=y^2$$

$$x^2 + px + \frac{p^2}{4} = x^2 - px + \frac{p^2}{4} + y^2,$$

and the equation of the parabola is

$$y^2 = 2px. \tag{1.2}$$

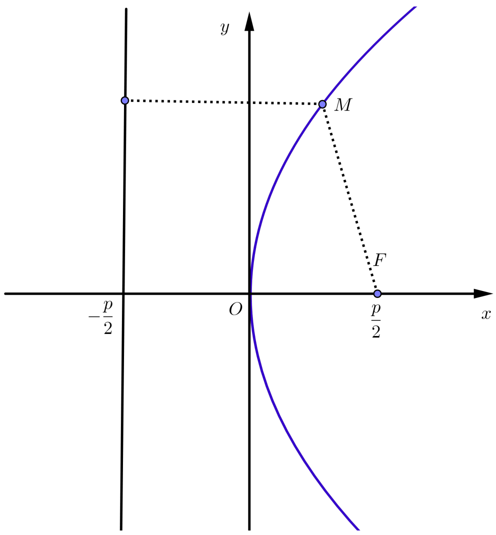
Remark 1.5. *The equation (1.2) is equivalent to $y = \pm\sqrt{2px}$, so that the parabola is symmetric with respect to the x-axis.*

Representing the graph of the function $f : [0, \infty) \to [0, \infty)$ and using the symmetry of the curve with respect to he x-axis, one obtains the graph of the parabola.

One has

$$f'(x) = \frac{p}{\sqrt{2px_0}}; \qquad f''(x) = -\frac{p}{2x\sqrt{2x}}.$$

x	0		∞
$f'(x)$	$\vert$	$+ + +$	$+$
$f(x)$	0	$\nearrow$	∞
$f''(x)$	$-$	$- - -$	$-$



Theorem 1.6. *(The preimage theorem) If $U \subseteq \mathbb{R}^2$ is an open set, $f : U \longrightarrow R$ is a C^1-smooth function and $a \in Imf$ is a regular value[1] of f, then te inverse image of a through f,*

$$f^{-1}(a) = \{(x, y) \in U | f(x, y) = a\}$$

is a planar regular curve called the regular curve of implicit cartezian equation $f(x, y) = a$.

Proposition 1.7. *The equation of the tangent line $T_{(x_0,y_0)}(C)$ of the planar egular curve C of implicit cartezian equation $f(x, y) = a$ at the point $p = (x_0, y_0) \in C$, is*

$$T_{(x_0,y_0)}(C) : f_x(p)(x - x_0) + f_y(p)(y - y_0) = 0, \tag{1.3}$$

and the equation of the normal line $N_{(x_0,y_0)}(C)$ of C at p is

$$N_{(x_0,y_0)}(C) : \frac{x - x_0}{f_x(p)} = \frac{y - y_0}{f_y(p)}.$$

[1]The value $a \in \mathrm{Im}(f)$ of the function f is said to be *regular* if $(\nabla f)(x, y) \neq 0$, $\forall (x, y) \in f^{-1}(a)$

By using the general equation (1.3) of the tangent line to an implicite curve from Proposition (1.7), one can easily show that:

1. The equation of the tangent line to the ellipse $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ at some point $M_0(x_0, y_0) \in \mathcal{E}$ is
$$T_{M_0}(\mathcal{E}) : \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1. \tag{1.4}$$

2. The equation of the tangent line to the hyperbola $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ at some point $M_0(x_0, y_0) \in \mathcal{H}$ is
$$T_{M_0}(\mathcal{H}) : \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1. \tag{1.5}$$

3. The equation of the tangent line to the parabola $\mathcal{P} : y^2 = 2px$ at some point $M_0(x_0, y_0) \in \mathcal{P}$ is
$$T_{M_0}(\mathcal{P}) : y_0 y = p(x + x_0). \tag{1.6}$$

2 Quadrics

2.1 The ellipsoid

The *ellipsoid* is the quadric surface given by the equation

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{2.1}$$

- The coordinate planes are all planes of symmetry of $\mathcal{E}$ since, for an arbitrary point $M(x, y, z) \in \mathcal{E}$, its symmetric points with respect to these planes, $M_1(-x, y, z)$, $M_2(x, -y, z)$ and $M_3(x, y, -z)$ belong to $\mathcal{E}$; therefore, the coordinate axes are axes of symmetry for $\mathcal{E}$ and the origin O is the center of symmetry of the ellipsoid (2.1);

- The traces in the coordinates planes are ellipses of equations

$$\begin{cases} \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \end{cases}, \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ z = 0. \end{cases}$$

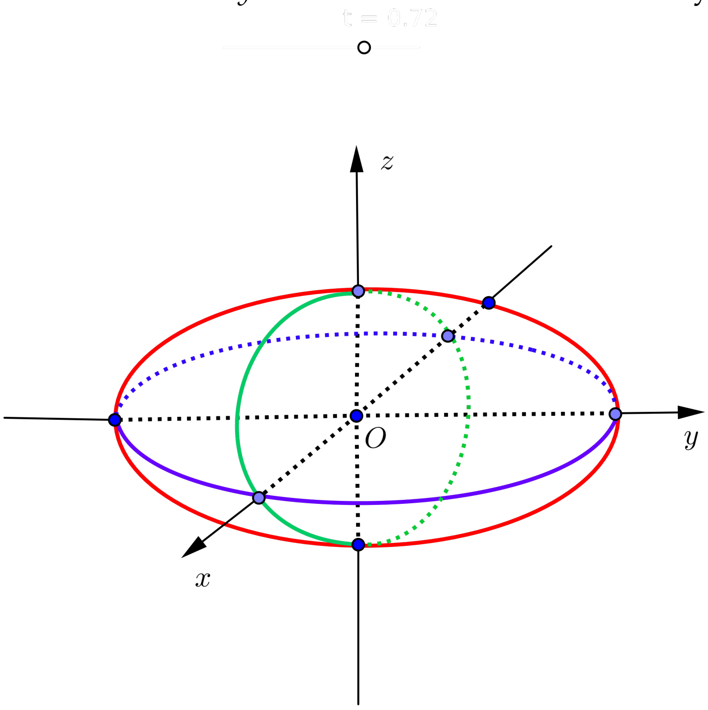
- The sections with planes parallel to xOy are given by setting $z = \lambda$ in (2.1). Then, a section is of equations $\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{\lambda^2}{c^2} \\ z = \lambda \end{cases}$.

- If $|\lambda| < c$, the section is an ellipse

$$\begin{cases} \frac{x^2}{\left(a\sqrt{1 - \frac{\lambda^2}{c^2}}\right)^2} + \frac{y^2}{\left(b\sqrt{1 - \frac{\lambda^2}{c^2}}\right)^2} = 1 \\ z = \lambda \end{cases};$$

- If $|\lambda| = c$, the intersection is reduced to one (tangency) point $(0, 0, \lambda)$;
- If $|\lambda| > c$, the plane $z = \lambda$ does not intersect the ellipsoid $\mathcal{E}$.

The sections with planes parallel to xOz or yOz are obtained in a similar way.



2.2 Hyperboloids of One Sheet

The surface of equation

$$\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{2.2}$$

is called *hyperboloid of one sheet*.

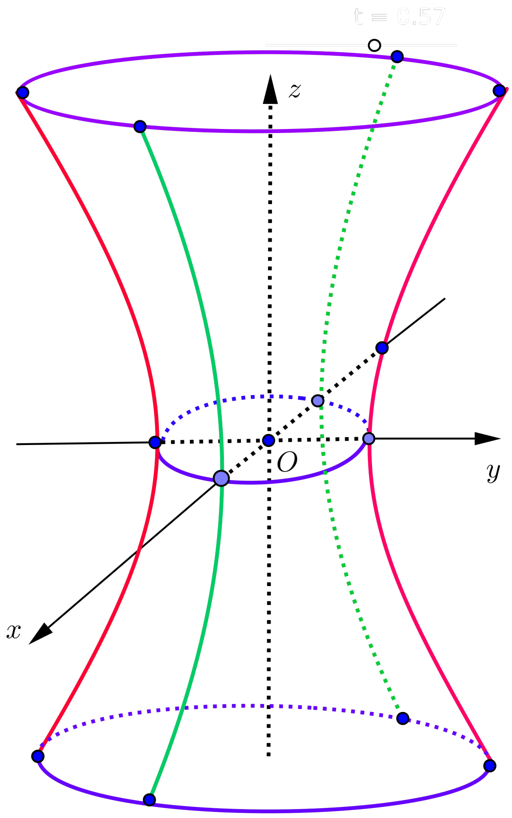
- The coordinate planes are planes of symmetry for $\mathcal{H}_1$; hence, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively, of equations

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ x = 0 \\ \text{a hyperbola} \end{cases}; \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{cases}; \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0 \\ z = 0 \\ \text{an ellipse} \end{cases};$$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases}; \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{cases};$$

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{\lambda^2}{c^2} \\ z = \lambda \\ \text{ellipses} \end{cases};$$



Remark: The surface $\mathcal{H}_1$ contains two families of lines, as

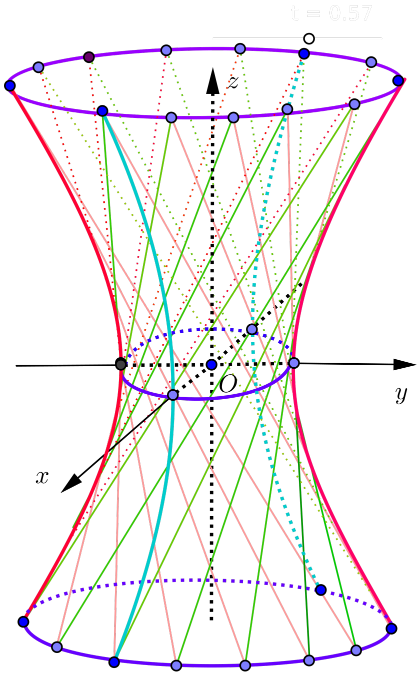
$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} \Leftrightarrow \left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right).$$

The equations of the two families of lines are:

$$d_\lambda : \begin{cases} \lambda\left(\dfrac{x}{a} + \dfrac{z}{c}\right) = 1 + \dfrac{y}{b} \\ \dfrac{x}{a} - \dfrac{z}{c} = \lambda\left(1 - \dfrac{y}{b}\right) \end{cases}, \lambda \in \mathbb{R},$$

$$d'_\mu : \begin{cases} \mu\left(\dfrac{x}{a} + \dfrac{z}{c}\right) = 1 - \dfrac{y}{b} \\ \dfrac{x}{a} - \dfrac{z}{c} = \mu\left(1 + \dfrac{y}{b}\right) \end{cases}, \mu \in \mathbb{R}.$$

Through any point on $\mathcal{H}_1$ pass two lines, one line from each family.



3 Problems

1. Find the equations of the tangent lines to the ellipse $\mathcal{E}: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ having a given angular coefficient $m \in \mathbb{R}$. (see [1, p. 110]).

2. Find the equations of the tangent lines to the ellipse $\mathcal{E}: \ x^2 + 4y^2 - 20 = 0$ which are orthogonal to the line $d: \ 2x - 2y - 13 = 0$.

3. Find the equations of the tangent lines to the ellipse $\mathcal{E}: \dfrac{x^2}{25} + \dfrac{y^2}{16} - 1 = 0$, passing through $P_0(10, -8)$.

4. If $M(x, y)$ is a point of the tangent line $T_{M_0}(E)$ of the ellipse $\mathcal{E}: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at one of its points $M_0(x_0, y_0) \in \mathcal{E}$, show that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \geq 1$.

5. Find the intersection points between the line $(d)\ 2x - y - 10 = 0$ and the hyperbola $\mathcal{H}: \dfrac{x^2}{20} - \dfrac{y^2}{5} - 1 = 0$.

6. Find the equations of the tangent lines to the hyperbola $\mathcal{H}: \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$ having a given angular coefficient $m \in \mathbb{R}$. (see [1, p. 115]).

7. Find the equations of the tangent lines to the hyperbola $\mathcal{H}: \dfrac{x^2}{20} - \dfrac{y^2}{5} - 1 = 0$ which are orthogonal to the line $d: \ 4x + 3y - 7 = 0$.

8. Find the equation of the parabola having the focus $F(-7, 0)$ and the director line $d: x - 7 = 0$.

9. Find the equations of the tangent lines to the parabola $\mathcal{P}: y^2 = 2px$ having a given angular coefficient $m \in \mathbb{R}$. (see [1, p. 119]).

10. Find the equation of the tangent line to the parabola $\mathcal{P}: y^2 - 8x = 0$, parallel to $d: 2x + 2y - 3 = 0$.

11. Find the equation of the tangent line to the parabola $\mathcal{P}: y^2 - 36x = 0$, passing through $P(2, 9)$.

12. Show that the sum of the distances from any point inside the ellipse to its foci is less than the length of the major axis.

13. Find the locus of the orthogonal projections of the center $O(0, 0)$ of the ellipse $\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on its tangents.

14. Find the locus of the orthogonal projections of the center $O(0, 0)$ of the hyperbola $\mathcal{H}: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ on its tangents.

15. Show that a ray of light through a focus of an ellipse reflects to a ray that passes through the other focus (optical property of the ellipse).

16. Show that a ray of light through a focus of a hyperbola reflects to a ray that passes through the other focus (optical property of the hyperbola).

17. Show that a ray of light through a focus of a parabola reflects to a ray parallel to the axis of the parabola (optical property of the parabola).

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