Geometry
Problem booklet

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Week 11

1 Generated surfaces (Brief theoretical background)

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Conical Surfaces

Definition 1.1. *The surface generated by a variable line, called* generatrix, *which passes through a fixed point V and intersects a given curve $\mathcal{C}$, is called* conical surface. *The point V is called the* vertex *of the surface and the curve $\mathcal{C}$* director curve.

Theorem 1.2. *The conical surface, of vertex $V(x_0, y_0, z_0)$ and director curve*

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$

(V and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi\left(\frac{x-x_0}{z-z_0}, \frac{y-y_0}{z-z_0}\right) = 0. \tag{1.1}$$

Proof. The equations of an arbitrary line through $V(x_0, y_0, z_0)$ are

$$d_{\lambda\mu} : \begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \end{cases}.$$

A generatrix has to intersect the curve $\mathcal{C}$, hence the system of equations

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$

must be compatible. This happens for some values of the parameters λ and μ, which verify a *compatibility condition*

$$\varphi(\lambda, \mu),$$

obtained by eliminating x, y and z in the the previous system of equations. In these conditions, the equation of the conical surface rises from the system

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ \varphi(\lambda, \mu) = 0 \end{cases},$$

i.e.

$$\varphi\left(\frac{x-x_0}{z-z_0}, \frac{y-y_0}{z-z_0}\right) = 0.$$

□

Remark 1.3. *If φ is a polynomial function, then the equation (1.1) can be written in the form*

$$\phi(x - x_0, y - y_0, z - z_0) = 0,$$

where ϕ is homogeneous with respect to $x - x_0$, $y - y_0$ and $z - z_0$. If φ is polynomial and V is the origin of the system of coordinates, then the equation of the conical surface is $\phi(x, y, z) = 0$, with ϕ a homogeneous polynomial. Conversely, an algebraic homogeneous equation in x, y and z represents a conical surface with the vertex at the origin.

Example 1.4. *Let us determine the equation of the conical surface, having the vertex $V(1, 1, 1)$ and the director curve*

$$\mathcal{C} : \begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \end{cases}.$$

The family of lines passing through V has the equations

$$d_{\lambda\mu} : \begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases}.$$

The system of equations

$$\begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \\ x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases}$$

must be compatible. A solution is

$$\begin{cases} x = 1 - \lambda \\ y = 1 - \mu \\ z = 0 \end{cases},$$

and, replaced in the first equation of the system, gives the compatibility condition

$$[(1 - \lambda)^2 + (1 - \mu)^2]^2 - (1 - \lambda)(1 - \mu) = 0.$$

The equation of the conical surface is obtained by eliminating the parameters λ and μ in

$$\begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \\ ((1 - \lambda)^2 + (1 - \mu)^2)^2 - (1 - \lambda)(1 - \mu) = 0 \end{cases}.$$

Expressing $\lambda = \dfrac{x - 1}{z - 1}$ and $\mu = \dfrac{y - 1}{z - 1}$ and replacing in the compatibility condition, one obtains

$$\left[\left(\frac{z - x}{z - 1}\right)^2 + \left(\frac{z - y}{z - 1}\right)^2\right]^2 - \left(\frac{z - x}{z - 1}\right)\left(\frac{z - y}{z - 1}\right) = 0,$$

or

$$[(z - x)^2 + (z - y)^2]^2 - (z - x)(z - y)(z - 1)^2 = 0.$$

1.2 Conoidal Surfaces

Definition 1.5. *The surface generated by a variable line, which intersects a given line d and a given curve $\mathcal{C}$, and remains parallel to a given plane π, is called* conoidal surface. *The curve $\mathcal{C}$ is the* director curve *and the plane π is the* director plane *of the conoidal surface.*

Theorem 1.6. *The conoidal surface whose generatrix intersects the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$$

and the curve

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$

and has the director plane $\pi = 0$, (π is not parallel to d and that $\mathcal{C}$ is not contained into π), is characterized by an equation of the form

$$\varphi\left(\pi, \frac{\pi_1}{\pi_2}\right) = 0. \tag{1.2}$$

Proof. An arbitrary generatrix of the conoidal surface is contained into a plane parallel to π and, on the other hand, comes from the bundle of planes containing d. Then, the equations of a generatrix are

$$d_{\lambda\mu} : \begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \end{cases}.$$

Again, the generatrix must intersect the director curve, hence the system of equations

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$

has to be compatible. This leads to a compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

and the equation of the conoidal surface is obtained from

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ \varphi(\lambda,\mu) = 0 \end{cases}.$$

By expressing λ and μ, one obtains (1.2). □

Example 1.7. *Let us find the equation of the conoidal surface, whose generatrices are parallel to xOy and intersect Oz and the curve*

$$\begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}.$$

The equations of xOy and Oz are, respectively,

$$xOy : z = 0, \quad \text{and} \quad Oz : \begin{cases} x = 0 \\ z = 0 \end{cases},$$

so that the equations of the generatrix are

$$d_{\lambda,\mu} : \begin{cases} x = \lambda y \\ z = \mu \end{cases}.$$

From the compatibility of the system of equations

$$\begin{cases} x = \lambda y \\ z = \mu \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases},$$

one obtains the compatibility condition

$$2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0,$$

and, replacing $\lambda = \dfrac{y}{x}$ and $\mu = z$, the equation of the conoidal surface is

$$2x^2z - 2y^2z - 2x^2 + y^2 = 0.$$

1.3 Revolution Surfaces

Definition 1.8. *The surface generated after the rotation of a given curve $\mathcal{C}$ around a given line d is said to be a* revolution surface.

Theorem 1.9. *The equation of the revolution surface generated by the curve*

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$

in its rotation around the line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r},$$

is of the form

$$\varphi((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, px + qy + rz) = 0. \tag{1.3}$$

Proof. An arbitrary point on the curve $\mathcal{C}$ will describe, in its rotation around d, a circle situated into a plane orthogonal on d and having the center on the line d. This circle can be seen as the intersection between a sphere, having the center on d and of variable radius, and a plane, orthogonal on d, so that its equations are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}.$$

The circle has to intersect the curve $\mathcal{C}$, therefore the system

$$\begin{cases} F_1(x,y,z)=0 \\ F_2(x,y,z)=0 \\ (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=\lambda \\ px+qy+rz=\mu \end{cases}$$

must be compatible. One obtains the compatibility condition

$$\varphi(\lambda,\mu)=0,$$

which, after replacing the parameters, gives the equation of the surface (1.3). □

2 Problems

1. Show that a ray of light through a focus of an ellipse reflects to a ray that passes through the other focus (optical property of the ellipse).

   Solution. Let $F_1(-c,0)$, $F_2(c,0)$ be the foci of the ellipse $\mathcal{E}:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Recall that the gradient $\mathrm{grad}(f)(x_0,y_0)=(f_x(x_0,y_0),f_y(x_0,y_0))$ is a normal vector of the ellipse $\mathcal{E}$ to its point $M_0(x_0,y_0)$, where

   $$f(x,y)=\delta(F_1,M)+\delta(F_2,M)=\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}$$

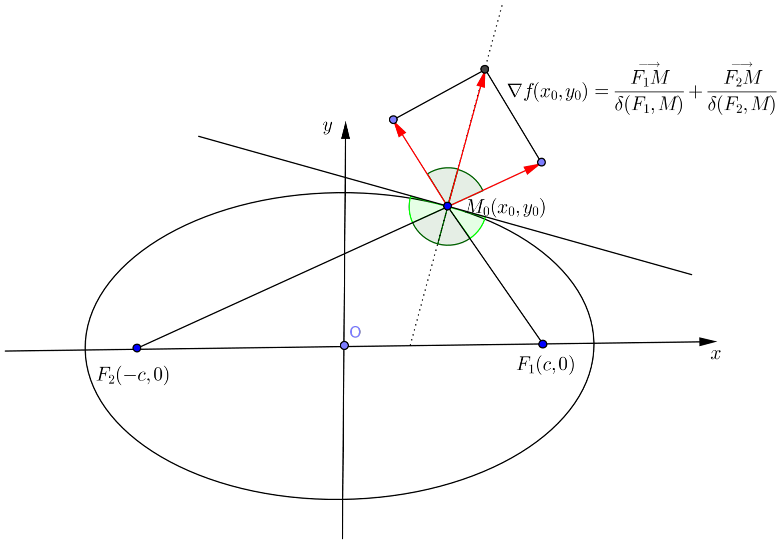
   and $M(x,y)$, as the ellipse is a level set of f. Note that

   $$f_x(x_0,y_0)=\frac{x_0+c}{\delta(F_1,M_0)}+\frac{x_0-c}{\delta(F_2,M_0)} \text{ and } f_y(x_0,y_0)=\frac{y}{\delta(F_1,M_0)}+\frac{y}{\delta(F_2,M_0)},$$

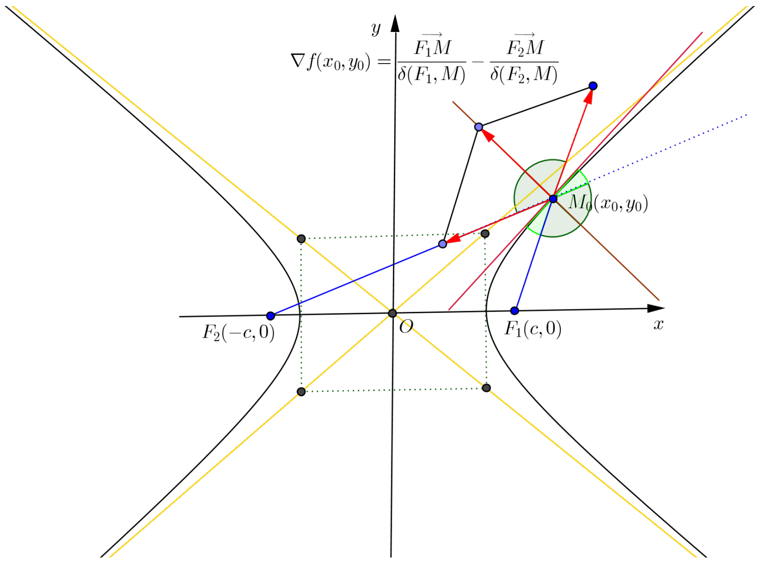
   and shows that

   $$\begin{aligned}\nabla f &= (f_x(x_0,y_0),f_y(x_0,y_0))=\left(\frac{x_0+c}{\delta(F_1,M_0)}+\frac{x_0-c}{\delta(F_2,M_0)},\frac{y_0}{\delta(F_1,M_0)}+\frac{y_0}{\delta(F_2,M_0)}\right)\\ &=\frac{(x_0+c,y)}{\delta(F_1,M_0)}+\frac{(x_0-c,y)}{\delta(F_2,M_0)}=\frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}+\frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}.\end{aligned}$$

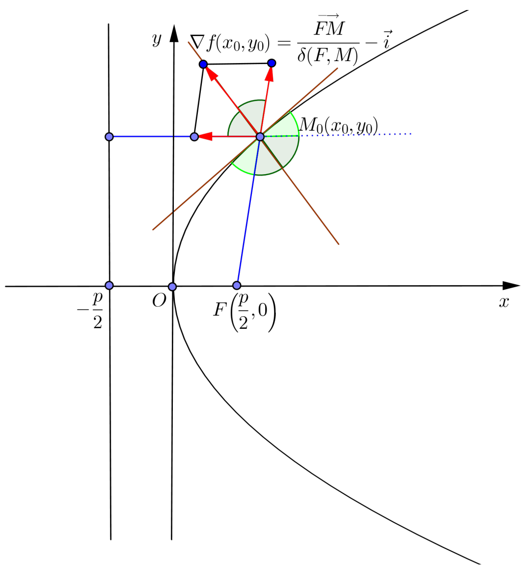
   The versors $\dfrac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}$ and $\dfrac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}$ point towards the exterior of the ellipse $\mathcal{E}$ and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{F_1M_0}$ and $\overrightarrow{F_2M_0}$ and (the sum) is also orthogonal to the tangent $T_{M_0}(\mathcal{E})$ of the ellipse at $M_0(x_0,y_0)$. This shows that the angle between the ray F_1M and the tangent $T_{M_0}(\mathcal{E})$ equals the angle between the ray F_2M and the tangent $T_{M_0}(\mathcal{E})$.



2. Show that a ray of light through a focus of a hyperbola reflects to a ray that passes through the other focus (optical property of the hyperbola).



3. Show that a ray of light through a focus of a parabola reflects to a ray parallel to the axis of the parabola (optical property of the parabola).



4. Consider a circle and a line parallel with the plane of the circle. Find the equation of the conoidal surface generated by a variable line which intersects the line (d) and the circle (C) and remains orthogonal to (d). (The Willis conoid)

5. Find the equation of the revolution surface generated by the rotation of a variable line through a fixed line.

6. The *torus* is the revolution surface obtained by the rotation of a circle C about a fixed line (d) within the plane of the circle such that $d \cap C = \emptyset$. Find the equation of the torus.

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