Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Bachelor, Computer Science, Groups 911-917, Academic Year 2016-2017

Mathematical Analysis
Seminar 2

1. Study the boundedness, the monotony and the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ in each of the following instances:

a) $x_n = \dfrac{n!}{n^n}$;

b) $x_n = \dfrac{(-1)^n}{n}$;

c) $x_n = (-1)^n + \dfrac{n+1}{n}$;

d) $x_n = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{n(n+1)}$;

e) $x_n = \dfrac{\cos\pi}{1\cdot 2} + \dfrac{\cos 2\pi}{2\cdot 3} + \cdots + \dfrac{\cos n\pi}{n(n+1)}$;

f) $x_n = \dfrac{1}{1+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}+\sqrt{n+1}}$.

2. Compute the limit of the following sequences:

$a_n = \dfrac{\alpha n^3 + \beta n^2 + \gamma n + 1}{n^2 - n + 1}$, where $\alpha, \beta, \gamma \in \mathbb{R}$;

$b_n = \dfrac{n^\alpha}{(1+\beta)^n}$, where $\alpha \in \mathbb{N}, \beta \in (0,\infty)$;

$c_n = \dfrac{\pi^n - 3^n}{e^n - 2^n}$;

$x_n = \dfrac{\sqrt{n + \sqrt{n + \sqrt{n + \sqrt{n}}}}}{n+1}$;

$y_n = \sqrt{n}\left(\sqrt{n} - \sqrt{n+3}\right)$;

$z_n = n\left(\sqrt[3]{n+1} - \sqrt[3]{n}\right)$.

3. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$x_n := \left(1 + \frac{1}{n}\right)^n.$$

a) Using Bernoulli's Inequality (see Seminar 1) prove that $\dfrac{x_{n+1}}{x_n} > 1$ for all $n \in \mathbb{N}$.

b) Using Newton's Binomial Formula prove that $x_n < 3$ for all $n \in \mathbb{N}$.

Hint: notice that $\dbinom{n}{k} \leq \dfrac{n^k}{2^{k-1}}$ for all $k \in \mathbb{N}$, $k \leq n$.

c) Deduce that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and, denoting its limit by e (the Euler's number), show that $2.71 < e \leq 3$.

d) Similarly to a) prove that the sequence $(y_n)_{n\in\mathbb{N}}$, defined for all $n \in \mathbb{N}$ by

$$y_n := \left(1 + \frac{1}{n}\right) x_n,$$

is strictly decreasing. Then, observing that $x_n < y_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n$, deduce that $e < 2.72$.

4. Compute the limits:

a) $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n^2+1}\right)^{\sqrt{n^2+1}}$;

b) $\lim\limits_{n\to\infty} \left(\dfrac{2n+1}{2n-1}\right)^n$;

c) $\lim\limits_{n\to\infty} \dfrac{(2n)^n}{(2n)!}$;

d) $\lim\limits_{n\to\infty} \dfrac{1^p + 2^p + \ldots + n^p}{n^{p+1}}$, where $p \in \mathbb{N}$;

e) $\lim\limits_{n\to\infty} \sqrt[n]{1 + 2 + \ldots + n}$;

f) $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n}$.

Mathematical Analysis
Seminar 3

1. Study whether the sequences defined by the following recurrence relations are convergent. If the sequence converges determine its limit.

a) $x_1 \in (0,1)$ and $x_{n+1} = \dfrac{2x_n + 1}{3}$ for all $n \in \mathbb{N}$;

b) $x_1 \in (0,+\infty)$ and $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$ for all $n \in \mathbb{N}$, where $a > 0$ is a priori given.

2. Consider the sequence $(\gamma_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$\gamma_n := 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n.$$

a) Using the fact that $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$ for all $n \in \mathbb{N}$ (cf. Exercise 3 of Seminar 2), prove that $(\gamma_n)_{n\in\mathbb{N}}$ is strictly decreasing and bounded below by 0.
b) Deduce that $(\gamma_n)_{n\in\mathbb{N}}$ is convergent and, denoting its limit by γ (the Euler's constant, also known as the Euler-Mascheroni constant), show that $\gamma < 0.58$.
c) Prove that the sequence $(x_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$x_n := \gamma_n + \ln n - \ln(n+1)$$

is strictly increasing. Then, observing that $x_n < \gamma_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \gamma_n$, deduce that $\gamma > 0.57$.

3. Compute the limits:
a) $\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{n+n}\right)$;
b) $\lim\limits_{n\to\infty}\left[\dfrac{1}{2\cdot 3} + \dfrac{1}{4\cdot 5} + \ldots + \dfrac{1}{2n(2n+1)}\right]$.

4. Find the sum of the following series:

a) $\displaystyle\sum_{n=1}^{\infty} (-\pi/4)^n$; b) $\displaystyle\sum_{n=1}^{\infty} 3^{1-2n}$;

c) $\displaystyle\sum_{n=1}^{\infty} \binom{n+2}{3}^{-1}$; d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1^2+2^2+\ldots+n^2}$;

e) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right)$; f) $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$;

g) $\displaystyle\sum_{n=0}^{\infty} \operatorname{arctg}\frac{1}{n^2+n+1}$; h) $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{2^n}$.

Mathematical Analysis
Seminar 4

1. Consider the series
$$\sum_{n\geq 1} \frac{2n-1}{2^n}.$$
Compute its sum and deduce whether the series converges or not.

2. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Using the Cauchy's general criterion of convergence for series, prove that the following series are convergent:

a) $\displaystyle\sum_{n\geq 1} \frac{\sin(\alpha_n)}{n(n+1)}$;
b) $\displaystyle\sum_{n\geq 1} \frac{\alpha_n}{2^n(1+|\alpha_n|)}$.

3. Prove that the following series are divergent:

a) $\displaystyle\sum_{n\geq 1} \operatorname{arctg} n$;
b) $\displaystyle\sum_{n\geq 1} \cos(n\pi/6)$;
c) $\displaystyle\sum_{n\geq 1} \sin n$.

4. Study if the following series are convergent or divergent:

a) $\displaystyle\sum_{n\geq 1} \frac{e^n}{n+3^n}$; b) $\displaystyle\sum_{n\geq 1} \frac{1}{n^2-\ln n+\sin n}$;

c) $\displaystyle\sum_{n\geq 1} \frac{\sqrt{n+1}}{1+2+\cdots+n}$; d) $\displaystyle\sum_{n\geq 1} \frac{2^n\cdot n!}{n^n}$;

e) $\displaystyle\sum_{n\geq 1} \frac{5^{n/2}}{n2^n}$; f) $\displaystyle\sum_{n\geq 1} (\arctan n)^n$;

g) $\displaystyle\sum_{n\geq 1} \frac{n^2}{2^{n^2}}$; h) $\displaystyle\sum_{n\geq 1} \frac{(n+1)^n}{n^{n+2}}$.

5. Let $\displaystyle\sum_{n\geq 1} x_n$ be a convergent series with nonnegative terms. Study which of the following series are convergent:

a) $\displaystyle\sum_{n\geq 1} \frac{x_n}{1+x_n}$, b) $\displaystyle\sum_{n\geq 1} x_n^2$, c) $\displaystyle\sum_{n\geq 1} \sqrt{x_n}$, d) $\displaystyle\sum_{n\geq 1} \frac{\sqrt{x_n}}{n}$.

Mathematical Analysis

Seminar 5

1. Study if the following series are convergent or divergent for $\alpha \in (0, +\infty)$:

$$\text{a)} \sum_{n\geq 1} \frac{(\alpha n)^n}{n!} \qquad \text{b)} \sum_{n\geq 1} \alpha^{\ln n} \qquad \text{c)} \sum_{n\geq 1} \left[\frac{(2n-1)!!}{\sqrt{(2n)!}}\right]^{2\alpha}.$$

2. Study whether the following series are absolutely convergent, semi-convergent or divergent:

$$\text{a)} \sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n^2+1)}} \qquad \text{b)} \sum_{n\geq 1} \frac{\sqrt[3]{n}}{n+1}\cos(n\pi).$$

Prove that for any $x \in \mathbb{R}$ and $\alpha > 0$ the series $\sum_{n\geq 1} \frac{\sin(nx)}{n^\alpha}$ is convergent.

3. Find the set $A' := \{x_0 \in \overline{\mathbb{R}} \mid \forall V \in \mathcal{V}(x_0),\ V \cap A \setminus \{x_0\} \neq \emptyset\}$ of accumulation (i.e., cluster) points for the following sets: a) $A = [0,1) \cup \{2\}$, b) $A = \mathbb{Z}$, c) $A = \mathbb{Q}$ and d) $A = \left(0, \sqrt{2}\,\right] \cap \mathbb{Q}$.

4. Study the existence of the limit of Dirichlet's function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

at every accumulation point of its domain ($x_0 \in \overline{\mathbb{R}}$).

5. Compute the limits:

$$\text{a)} \lim_{x\to-\infty} \frac{-3x^2+x-1}{(x-1)(x-2)}, \quad \text{b)} \lim_{x\to 1} \frac{x^2-3x+2}{x^3-x}, \quad \text{c)} \lim_{x\to\infty} \sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right), \quad \text{d)} \lim_{x\to 0} \frac{\sqrt[3]{1+x}-1}{x},$$

$$\text{e)} \lim_{\substack{x\to 1\\ x>1}} \frac{x}{\sqrt[3]{x^2-4x+3}}, \quad \text{f)} \lim_{x\to-\infty} \left(\frac{x^2+1}{x^2-1}\right)^{\sqrt{-x}}, \quad \text{g)} \lim_{\substack{x\to 0\\ x<0}} \lfloor x \rfloor, \quad \text{h)} \lim_{x\to\infty} \lfloor x \rfloor, \quad \text{i)} \lim_{x\to\infty} \frac{x-\lfloor x \rfloor}{x}.$$

6. Study the continuity of the following functions and determine the type of their discontinuities:

$$\text{a)}\ f : \mathbb{R} \to \mathbb{R},\ f(x) := \lim_{n\to\infty} \frac{e^{nx}}{1+e^{nx}} \qquad \text{b)}\ g : \mathbb{R} \to \mathbb{R},\ g(x) := \begin{cases} \frac{1}{x}\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

7. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at every point in $\mathbb{R}$ and $|f|$ is continuous on $\mathbb{R}$.

8. Let $f,\ g : [0,1] \to \mathbb{R}$ be two continuous functions, such that $f(x) = g(x)$, $\forall x \in [0,1] \cap \mathbb{Q}$. Prove that $f(x) = g(x)$, $\forall x \in [0,1]$.

9. Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a,b] \to [a,b]$ be a continuous function. Prove that f has at least one fixed point $x_0 \in [a,b]$, that is, $f(x_0) = x_0$.