LECTURE

2

SEQUENCES OF REAL NUMBERS

Definition 2.1 *By a* sequence of real numbers (*or sequence in* $\mathbb{R}$) *we mean any function of type* $f : \mathbb{N} \to \mathbb{R}$. *Denoting* $x_n := f(n)$ *for all* $n \in \mathbb{N}$ *we may use one of the following notations to represent the sequence* f:

$$(x_n)_{n\in\mathbb{N}},\ (x_n)_{n\geq 1} \ \text{ or simply } \ (x_n).$$

Remark 2.2 *Any function* $g : \mathbb{N} \cap [m, \infty) \to \mathbb{R}$ *with* $m \in \mathbb{Z}$ *can be seen as a sequence, too. Indeed,* g *can be identified with* $f : \mathbb{N} \to \mathbb{R}$, *given by*

$$f(n) := g(m + n - 1),\ \forall\, n \in \mathbb{N}.$$

In this case instead of $(x_n)_{n\in\mathbb{N}}$, *with* $x_n := f(n)$ *for all* $n \in \mathbb{N}$, *we write*

$$(y_n)_{n\geq m}, \quad \text{where } \ y_n := g(n),\ \forall\, n \in \mathbb{Z},\ n \geq m.$$

Definition 2.3 *A sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$ *is said to be bounded below* (*bounded above; bounded; unbounded*) *if the set* $\{x_n \mid n \in \mathbb{N}\}$ *is bounded below* (*bounded above; bounded; unbounded*).

Remark 2.4 *For any sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *we have:*

$$\begin{aligned}
(x_n)_{n\in\mathbb{N}} \text{ is bounded below} \quad &\Longleftrightarrow \quad \exists\, a \in \mathbb{R} \text{ s.t. } x_n \geq a,\ \forall\, n \in \mathbb{N};\\
(x_n)_{n\in\mathbb{N}} \text{ is bounded above} \quad &\Longleftrightarrow \quad \exists\, a \in \mathbb{R} \text{ s.t. } x_n \leq a,\ \forall\, n \in \mathbb{N};\\
(x_n)_{n\in\mathbb{N}} \text{ is bounded} \quad &\Longleftrightarrow \quad \exists\, a \in \mathbb{R} \text{ s.t. } |x_n| \leq a,\ \forall\, n \in \mathbb{N};\\
(x_n)_{n\in\mathbb{N}} \text{ is unbounded} \quad &\Longleftrightarrow \quad \forall\, a \in \mathbb{R},\ \exists\, n \in \mathbb{N} \text{ s.t. } |x_n| > a.
\end{aligned}$$

Definition 2.5 *A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *is called*

- *increasing if* $x_{n+1} \geq x_n,\ \forall n \in \mathbb{N}$;
- *decreasing if* $x_{n+1} \leq x_n,\ \forall n \in \mathbb{N}$;
- *strictly increasing if* $x_{n+1} > x_n,\ \forall n \in \mathbb{N}$;
- *strictly decreasing if* $x_{n+1} < x_n,\ \forall n \in \mathbb{N}$;
- *monotonic* (*or monotone*) *if it is increasing or decreasing;*
- *strictly monotonic* (*or strictly monotone*) *if it is strictly increasing or strictly decreasing.*

Remark 2.6 *Every increasing sequence is bounded below. Similarly, every decreasing sequence is bounded above.*

Definition 2.7 *We say that a sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$ has a limit *(in* $\overline{\mathbb{R}}$*) if there exists* $\ell \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty; +\infty\}$ *such that*

$$\forall V \in \mathcal{V}(\ell),\ \exists n_V \in \mathbb{N} \ \ s.t. \ \ x_n \in V,\ \forall n \in \mathbb{N},\ n \geq n_V. \tag{2.1}$$

Remark 2.8 *For any* $\ell_1, \ell_2 \in \overline{\mathbb{R}}$, $\ell_1 \neq \ell_2$, *there exist* $V_1 \in \mathcal{V}(\ell_1)$ *and* $V_2 \in \mathcal{V}(\ell_2)$ *s.t.* $V_1 \cap V_2 = \emptyset$. *Hence, whenever* $(x_n)_{n\in\mathbb{N}}$ *has a limit, there is a unique* $\ell \in \overline{\mathbb{R}}$ *satisfying* (2.1).

Definition 2.9 *If a sequence* $(x_n)_{n\in\mathbb{N}}$ *has a limit, then the unique* $\ell \in \overline{\mathbb{R}}$ *that satisfies* (2.1) *is called* the limit of the sequence $(x_n)_{n\in\mathbb{N}}$. *In this case we denote* $\lim\limits_{n\to\infty} x_n := \ell$ *or* $x_n \to \ell$ *and we say that* $(x_n)_{n\in\mathbb{N}}$ *tends to* ℓ.

Proposition 2.10 *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers that has a limit. We have*

$$\begin{aligned}
\lim_{n\to\infty} x_n = \ell \in \mathbb{R} &\iff \forall\, \varepsilon > 0,\ \exists\, n_\varepsilon \in \mathbb{N} \ \ s.t. \ \ |x_n - \ell| < \varepsilon,\ \forall\, n \in \mathbb{N},\ n \geq n_\varepsilon;\\
\lim_{n\to\infty} x_n = -\infty &\iff \forall\, a \in \mathbb{R},\ \exists\, n_a \in \mathbb{N} \ \ s.t. \ \ x_n < a,\ \forall\, n \in \mathbb{N},\ n \geq n_a;\\
\lim_{n\to\infty} x_n = +\infty &\iff \forall\, a \in \mathbb{R},\ \exists\, n_a \in \mathbb{N} \ \ s.t. \ \ x_n > a,\ \forall\, n \in \mathbb{N},\ n \geq n_a.
\end{aligned}$$

Definition 2.11 *A sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$ *is said to be*

- convergent, *if it has a finite limit, i.e.,* $\lim\limits_{n\to\infty} x_n \in \mathbb{R}$*; in this case we say that* $(x_n)_{n\in\mathbb{N}}$ *converges to the real number* $\lim\limits_{n\to\infty} x_n$*;*
- divergent, *if it is not convergent.*

Proposition 2.12 *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers and let* $\ell \in \mathbb{R}$. *The following assertions are equivalent:*

1° *The sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* ℓ, *i.e.,* $\lim\limits_{n\to\infty} x_n = \ell$.

2° *The sequence* $(|x_n - \ell|)_{n\in\mathbb{N}}$ *converges to zero, i.e.,* $\lim\limits_{n\to\infty} |x_n - \ell| = 0$.

3° *There exists a sequence* $(a_n)_{n\in\mathbb{N}}$ *of nonnegative real numbers satisfying the following two conditions:*

(*i*) $|x_n - \ell| \leq a_n$ *for all* $n \in \mathbb{N}$;

(*ii*) $(a_n)_{n\in\mathbb{N}}$ *converges to zero, i.e.,* $\lim\limits_{n\to\infty} a_n = 0$.

Corollary 2.13 *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be two sequences of real numbers. If* $(a_n)_{n\in\mathbb{N}}$ *is bounded and* $\lim\limits_{n\to\infty} b_n = 0$, *then* $\lim\limits_{n\to\infty} (a_n \cdot b_n) = 0$.

Proposition 2.14 *Let* $(x_n), (y_n)$ *be sequences in* $\mathbb{R}$ *such that* $\forall n \in \mathbb{N}$, $x_n \leq y_n$.

(*i*) *If* (x_n) *and* (y_n) *converge, then* $\lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} y_n$.

(*ii*) *If* $\lim\limits_{n\to\infty} x_n = +\infty$, *then* $\lim\limits_{n\to\infty} y_n = +\infty$.

(*iii*) *If* $\lim\limits_{n\to\infty} y_n = -\infty$, *then* $\lim\limits_{n\to\infty} x_n = -\infty$.

Theorem 2.15 (Squeeze Theorem) *Let* $(x_n), (y_n), (z_n)$ *be sequences in* $\mathbb{R}$ *such that*

$$x_n \leq y_n \leq z_n,\ \forall\, n \in \mathbb{N}.$$

Suppose that (x_n) *and* (z_n) *are convergent and* $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = \ell \in \mathbb{R}$. *Then* (y_n) *is also convergent and* $\lim\limits_{n\to\infty} y_n = \ell$.

Corollary 2.16 (Cantor's Theorem on Nested Intervals) *Consider a nested sequence of compact intervals, i.e.,*

$$I_n := [a_n, b_n] \subseteq \mathbb{R}, \ \ s.t. \ \ a_n \leq a_{n+1} < b_{n+1} \leq b_n,\ \forall\, n \in \mathbb{N}.$$

If $\lim\limits_{n\to\infty} (b_n - a_n) = 0$, *then there exists* $x \in \mathbb{R}$ *such that* $\bigcap\limits_{n=1}^{\infty} I_n = \{x\}$.

Proposition 2.17 *Every convergent sequence of real numbers is bounded.*

Notice that there are bounded sequences that are not convergent.

Theorem 2.18 (Counterpart of the Weierstrass' Theorem) *Let $(x_n)_{n\in\mathbb{N}}$ be a monotonic (i.e., increasing or decreasing) sequence of real numbers. The following assertions hold:*

1° *$(x_n)_{n\in\mathbb{N}}$ has a limit in $\overline{\mathbb{R}}$.*

2° *If $(x_n)_{n\in\mathbb{N}}$ is increasing, then $\lim\limits_{n\to\infty} x_n = \sup\limits_{n\in\mathbb{N}} x_n$, hence (x_n) is convergent if and only if it is bounded above.*

2° *If $(x_n)_{n\in\mathbb{N}}$ is decreasing, then $\lim\limits_{n\to\infty} x_n = \inf\limits_{n\in\mathbb{N}} x_n$, hence (x_n) is convergent if and only if it is bounded below.*

Theorem 2.19 (Toeplitz) *Consider an "infinite triangular matrix" of real numbers*

$$\begin{array}{lllll} c_{1,1} & & & & \\ c_{2,1} & c_{2,2} & & & \\ \cdots & \cdots & \cdots & & \\ c_{n,1} & c_{n,2} & c_{n,3} & \cdots & c_{n,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

which satisfies the following three conditions:

(i) $c_{n,k} \geq 0,\ \forall\, n \in \mathbb{N},\ \forall\, k \in \{1, 2, ..., n\}$;

(ii) $\sum\limits_{k=1}^{n} c_{n,k} = 1,\ \forall n \in \mathbb{N}$;

(iii) $\lim\limits_{n\to\infty} c_{n,k} \to 0,\ \forall\, k \in \mathbb{N}$.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers that has a limit $\ell \in \overline{\mathbb{R}}$ then the sequence $(y_n)_{n\in\mathbb{N}}$, given by

$$y_n = c_{n,1}x_1 + c_{n,2}x_2 + ... + c_{n,n}x_n,\ \forall\, n \in \mathbb{N}$$

has the same limit ℓ.

Theorem 2.20 (Stolz-Cesàro) *Let $(a_n), (b_n)$ be sequences in $\mathbb{R}$ such that*

(i) (b_n) *is strictly increasing and* $\lim\limits_{n\to\infty} b_n = +\infty$,

(ii) $\lim\limits_{n\to\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = L \in \overline{\mathbb{R}}$.

Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$.

Proof. We can apply Toeplitz's Theorem, letting $a_0 = b_0 := 0$,

$$x_n := \frac{a_n - a_{n-1}}{b_n - b_{n-1}},\ \forall\, n \in \mathbb{N} \quad \text{and} \quad c_{n,k} := \frac{b_k - b_{k-1}}{b_n},\ \forall\, k \in \{1, 2, ..., n\}.$$

Since $y_n = \frac{a_n}{b_n}$, the conclusion follows. □

Corollary 2.21 *Let (x_n) be a sequence in $\mathbb{R}$. The following assertions hold:*

1° *If $\lim\limits_{n\to\infty} x_n = \ell \in \overline{\mathbb{R}}$, then $\lim\limits_{n\to\infty} \dfrac{x_1 + x_2 + \ldots + x_n}{n} = \ell$.*

2° *If $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = \ell \in \overline{\mathbb{R}}$, then $\lim\limits_{n\to\infty} \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} = \ell$.*

3° *If $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = L \in \overline{\mathbb{R}}$, then $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = L$.*

Definition 2.22 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequences of real numbers. A sequence $(y_k)_{k\in\mathbb{N}}$ is said to be a* subsequence *of $(x_n)_{n\in\mathbb{N}}$ if there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers (i.e., $n_k \in \mathbb{N}$ and $n_k < n_{k+1}$ for all $k \in \mathbb{N}$) such that*

$$y_k = x_{n_k}, \ \forall\, k \in \mathbb{N}.$$

Proposition 2.23 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ that has a limit $x = \lim\limits_{n\to\infty} x_n \in \overline{\mathbb{R}}$. Then any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ has the same limit, i.e., $\lim\limits_{k\to\infty} x_{n_k} = x$.*

Theorem 2.24 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

Theorem 2.25 (Cauchy's criterion for convergence of sequences) *For any sequence $(x_n)_{n\in\mathbb{N}}$ the following assertions are equivalent:*

1° *$(x_n)_{n\in\mathbb{N}}$ is convergent.*
2° *For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, with $m \geq n_\varepsilon$ and $n \geq n_\varepsilon$, we have $|x_m - x_n| < \varepsilon$.*
3° *For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq n_\varepsilon$ and any $p \in \mathbb{N}$ we have $|x_{n+p} - x_n| < \varepsilon$.*

Corollary 2.26 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Assume that there is a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative real numbers satisfying the following two conditions:*

1° *$|x_{n+p} - x_n| \leq a_n$ for all $n, p \in \mathbb{N}$;*
2° *$(a_n)_{n\in\mathbb{N}}$ converges to zero, i.e., $\lim_{n\to\infty} a_n = 0$.*

Then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent.

Limit Laws

$x + \infty = \infty + x = \infty, \ \forall x \in \mathbb{R},$

$x + (-\infty) = (-\infty) + x = -\infty, \ \forall x \in \mathbb{R},$

$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty,$

$$x \cdot \infty = \infty \cdot x = \begin{cases} \infty, \text{ if } x \in (0,\infty) \\ -\infty, \text{ if } x \in (-\infty,0), \end{cases}$$

$$x \cdot (-\infty) = (-\infty) \cdot x = \begin{cases} -\infty, \text{ if } x \in (0,\infty) \\ \infty, \text{ if } x \in (-\infty,0), \end{cases}$$

$\infty \cdot \infty = \infty, \quad (-\infty)\cdot(-\infty) = \infty, \quad \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty,$

$\frac{x}{\infty} = \frac{x}{-\infty} = 0, \ \forall x \in \mathbb{R},$

$\frac{1}{0+} = \infty, \quad \frac{1}{0-} = -\infty,$

$$x^\infty = \begin{cases} \infty, \text{ if } x \in (1,\infty) \\ 0, \text{ if } x \in [0,1), \end{cases}$$

$$x^{-\infty} = \begin{cases} 0, \text{ if } x \in (1,\infty) \\ \infty, \text{ if } x \in (0,1), \end{cases}$$

$$(\infty)^x = \begin{cases} \infty, \text{ if } x \in (0,\infty) \\ 0, \text{ if } x \in (-\infty,0), \end{cases}$$

$\infty^\infty = \infty, \quad \infty^{-\infty} = 0.$

The following limits are undefined

$\infty + (-\infty), \quad (-\infty) + \infty,$

$0 \cdot \infty, \quad \infty \cdot 0, \quad 0 \cdot (-\infty), \quad (-\infty) \cdot 0,$

$\frac{\infty}{\infty}, \quad \frac{-\infty}{-\infty}, \quad \frac{\infty}{-\infty}, \quad \frac{-\infty}{\infty},$

$1^\infty, \quad 0^0, \quad \infty^0, \quad 1^{-\infty}.$