Mathematical Analysis
Seminar 3

1. Study whether the sequences defined by the following recurrence relations are convergent. If the sequence converges determine its limit.

a) $x_1 \in (0,1)$ and $x_{n+1} = \dfrac{2x_n + 1}{3}$ for all $n \in \mathbb{N}$;

b) $x_1 \in (0,+\infty)$ and $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$ for all $n \in \mathbb{N}$, where $a > 0$ is a priori given.

2. Consider the sequence $(\gamma_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$\gamma_n := 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n.$$

a) Using the fact that $\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$ for all $n \in \mathbb{N}$ (cf. Exercise 3 of Seminar 2), prove that $(\gamma_n)_{n\in\mathbb{N}}$ is strictly decreasing and bounded below by 0.

b) Deduce that $(\gamma_n)_{n\in\mathbb{N}}$ is convergent and, denoting its limit by γ (the Euler's constant, also known as the Euler-Mascheroni constant), show that $\gamma < 0.58$.

c) Prove that the sequence $(x_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$x_n := \gamma_n + \ln n - \ln(n+1)$$

is strictly increasing. Then, observing that $x_n < \gamma_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \gamma_n$, deduce that $\gamma > 0.57$.

3. Compute the limits:

a) $\lim\limits_{n\to\infty} \left(\dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{n+n}\right)$;

b) $\lim\limits_{n\to\infty} \left[\dfrac{1}{2\cdot 3} + \dfrac{1}{4\cdot 5} + \ldots + \dfrac{1}{2n(2n+1)}\right]$.

4. Find the sum of the following series:

a) $\displaystyle\sum_{n=1}^{\infty} (-\pi/4)^n$; b) $\displaystyle\sum_{n=1}^{\infty} 3^{1-2n}$;

c) $\displaystyle\sum_{n=1}^{\infty} \binom{n+2}{3}^{-1}$; d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1^2+2^2+\ldots+n^2}$;

e) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right)$; f) $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$;

g) $\displaystyle\sum_{n=0}^{\infty} \operatorname{arctg}\frac{1}{n^2+n+1}$; h) $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{2^n}$.