Geometry
Problem booklet

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Contents

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1 Week 5: Two dimensional Analytic Geometry

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

1.1 Brief theoretical background. Two dimensional Analytic Geometry

1.1.1 The vector ecuation of the straight lines

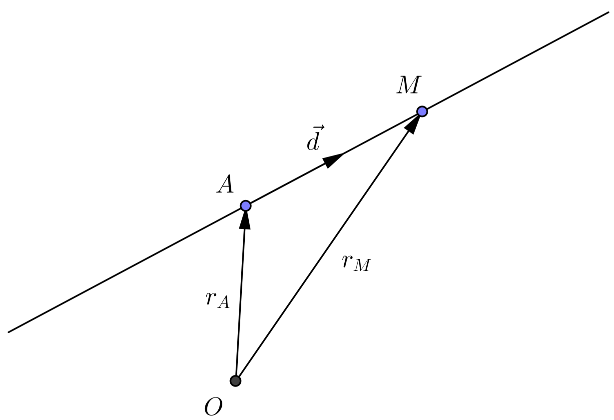
All over this section the geometric objects, such as points and lines, lie in a given plane π.

Let Δ be a straight line in π, let $A \in \Delta$ be a given point and let $\overrightarrow{d}$ be a director vector of Δ.

$$\overrightarrow{r}_M = \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{r}_A + \overrightarrow{AM}.$$

Thus

$$\begin{aligned} \{\overrightarrow{r}_M \mid M \in \Delta\} &= \{\overrightarrow{r}_A + \overrightarrow{AM} \mid M \in \Delta\} \\ &= \overrightarrow{r}_A + \{\overrightarrow{AM} \mid M \in \Delta\} \\ &= \overrightarrow{r}_A + \overrightarrow{\Delta} = \overrightarrow{r}_A + \langle \overrightarrow{d} \rangle. \\ &= \{\overrightarrow{r}_A + t\,\overrightarrow{d} : t \in \mathbb{R}\}. \end{aligned}$$



In other words, the position vectors of all points on the straight line Δ are

$$\overrightarrow{r}_M = \overrightarrow{r}_A + t\,\overrightarrow{d} : t \in \mathbb{R}. \tag{1.1}$$

This is the reason to call (1.1) the *vector equation* of the line Δ.

1.2 Cartezian equations of lines

1.2.1 Cartesian and affine reference systems

A basis of the direction $\overrightarrow{\pi}$ of the plane π is an ordered basis $[\overrightarrow{e}, \overrightarrow{f}]$ of $\overrightarrow{\pi}$.

If $b = [\overrightarrow{e}, \overrightarrow{f}]$ is a basis of $\overrightarrow{\pi}$ and $\overrightarrow{x} \in \overrightarrow{\pi}$, recall that the column vector of the coordinates of $\overrightarrow{x}$ with respect to b is being denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

whenever $\overrightarrow{x}= x_1 \overrightarrow{e} + x_2 \overrightarrow{f}$.

Definition 1.1. *A* cartesian reference system *of the plane π, is a system $R = (O, \overrightarrow{e}, \overrightarrow{f})$ where O is a point of π called the* origin *of the reference system and $b = [\overrightarrow{e}, \overrightarrow{f}]$ is a basis of the vector space $\overrightarrow{\pi}$.*

Denote by F_1, F_2 the points for which $\overrightarrow{e}=\overrightarrow{OF_1}$, $\overrightarrow{f}=\overrightarrow{OF_2}$.

Definition 1.2. *The system of points (O, F_1, F_2) is called* the affine reference system associated to the cartesian reference system $R = (O, \overrightarrow{e}, \overrightarrow{f})$.

The straight lines OF_i, $i \in \{1, 2\}$, oriented from O to F_i are called *the coordinate axes*. The coordinates x, y of the position vector $\overrightarrow{r}_M=\overrightarrow{OM}$ with respect to the basis $[\overrightarrow{e}, \overrightarrow{f}]$ are called the coordinates of the point M with respect to the cartesian system R written $M(x, y)$.

Also, for the column matrix of coordinates of the vector $\overrightarrow{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\overrightarrow{r}_M= x \overrightarrow{e} + y \overrightarrow{f}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Remark 1.3. *If $A(x_A, y_A)$, $B(x_B, y_B)$ are two points, then*

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}= x_B \overrightarrow{e} + y_B \overrightarrow{f} - (x_A \overrightarrow{e} + y_A \overrightarrow{f}) \\ &= (x_B - x_A) \overrightarrow{e} + (y_B - y_A) \overrightarrow{f}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points A and B.

1.2.2 The cartesian equations of the straight lines

Let Δ be a straight line passing through the point $A_0(x_0, y_0) \in \pi$ which is parallel to the vector $\overrightarrow{d}\ (p, q) \in\overrightarrow{\pi}$. Its vector equation is

$$\overrightarrow{r}_M=\overrightarrow{r}_{A_0} + t \overrightarrow{d},\ t \in \mathbb{R}. \tag{1.2}$$

Denoting by x, y the coordinates of the generic point M of the straight line Δ, its vector equation (1.2) is equivalent to the following system of relations

$$\begin{cases} x = x_0 + tp \\ y = y_0 + tq \end{cases},\ t \in \mathbb{R} \tag{1.3}$$

The relations (1.3) are called the *parametric equations* of the straight line Δ and they are equivalent to the following relation

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} \tag{1.4}$$

If $r = 0$, for instance, the canonical equation of the straight line Δ is $y = y_0$. If $A(x_A, y_A)$, $B(x_B, y_B)$ are different pointsof the straight line Δ, then $\overrightarrow{AB}\ (x_B - x_A, y_B - y_A)$ is a director vector of Δ, its canonical equations having, in this case, the form

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}. \tag{1.5}$$

We can put the equation (1.10) in the form

$$\begin{vmatrix} x-x_A & y-y_A \\ x_B-x_A & y_B-y_A \end{vmatrix}=0 \Longleftrightarrow \begin{vmatrix} x-x_A & y-y_A & 1 \\ x_B-x_A & y_B-y_A & 1 \\ 0 & 0 & 1 \end{vmatrix}=0 \Longleftrightarrow \begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix}=0. \qquad (1.6)$$

Given three points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$, they are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0.$$

1.3 Reduced Equations of Lines

Consider a line given by its general equation $ax + by + c = 0$, where at least one of the coefficients a and b is nonzero. One may assume that $b \neq 0$, so that the equation can be divided by b. One obtains

$$y = mx + n \qquad (1.7)$$

which is said to be the *reduced equation* of the line.

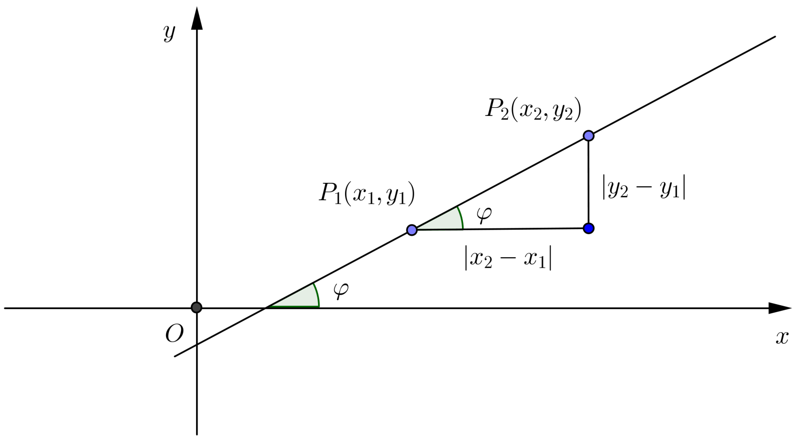
Remark: If $b = 0$, (1.10) becomes $ax + c = 0$, or $x = -\dfrac{c}{a}$, a line parallel to Oy. (In the same way, if $a = 0$, one obtains the equation of a line parallel to Ox).

Let d be a line of equation $y = mx + n$ in a Cartesian system of coordinates and suppose that the line is not parallel to Oy. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two different points on d and φ be the angle determined by d and Ox $\varphi \in [0, \pi] \setminus \{\dfrac{\pi}{2}\}$.

The points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ belong to d, hence $\begin{cases} y_1 = mx_1 + n \\ y_2 = mx_2 + n \end{cases}$, and $x_2 \neq x_1$, since d is not parallel to Oy. Then,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\varphi. \qquad (1.8)$$

The number $m = \tan\varphi$ is called the *angular coefficient* of the line d.



It is immediate that the equation of the line passing through the point $P_0(x_0, y_0)$ and of the given angular coefficient m is

$$y - y_0 = m(x - x_0). \qquad (1.9)$$

1.4 General Equations of Lines

A simple computation shows that (1.4) can be written in the form

$$ax + by + c = 0, \qquad \text{with} \quad a^2 + b^2 > 0, \tag{1.10}$$

which means that every line frm π is characterized by a first degree equation. Conversely, such of an equation represents a line, since the formula (1.10) is equivalent to $\dfrac{x + \frac{c}{a}}{-\frac{b}{a}} = \dfrac{y}{1}$, when ever $a \neq 0$, and this is the equation of the line passing through $P_0\left(-\dfrac{c}{a}, 0\right)$ which is parallel to $\overline{v}\left(-\dfrac{b}{a}, 1\right)$.

The equation (1.10) is called *general equation* of the line.

2 Parallelism and Orthogonality

Remark 2.1. *The lines*

$$(d)\ ax + by + c = 0 \text{ and } (\Delta)\ \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

are parallel if and only if $ap + bq = 0$. Indeed, for the two lines we have successively:

$$d \| \Delta \iff \vec{d} = \vec{\Delta} \iff \langle \vec{u}\ (p, q) \rangle = \left\langle \vec{v}\ \left(-\frac{b}{a}, 1\right) \right\rangle \iff \exists t \in \mathbb{R} \text{ s.t. } \vec{u}\ (p, q) = t\ \vec{v}\ \left(-\frac{b}{a}, 1\right)$$

$$\iff \exists t \in \mathbb{R} \text{ s.t. } p = -t\frac{b}{a} \text{ and } q = t \iff ap + bq = 0.$$

2.1 Intersection of Two Lines

Let $d_1 : a_1x + b_1y + c_1 = 0$ and $d_2 : a_2x + b_2y + c_2 = 0$ be two lines in $\mathcal{E}_2$. The solution of the system of equation

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

will give the set of the intersection points of d_1 and d_2.

1) If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, the system has a unique solution (x_0, y_0) and the lines have a unique intersection point $P_0(x_0, y_0)$. They are *secant*.

2) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, the system is not compatible, and the lines have no points in common. They are *parallel*.

3) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the system has an infinitely many solutions, and the two lines coincide. They are *identical*.

If $d_i : a_ix + b_iy + c_i = 0$, $i = \overline{1,3}$ are three lines in $\mathcal{E}_2$, then they are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{2.1}$$

2.2 Bundle of Lines

The set of all the lines passing through a given point P_0 is said to be a *bundle* of lines. The point P_0 is called the *vertex* of the bundle.

If the point P_0 is of coordinates $P_0(x_0, y_0)$, then the equation of the bundle of vertex P_0 is

$$r(x - x_0) + s(y - y_0) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \tag{2.2}$$

Remark: One may assume that $s \neq 0$ and divide in (2.2) by s. One obtains the *reduced equation* of the bundle,

$$y - y_0 = m(x - x_0), \qquad m \in \mathbb{R}, \tag{2.3}$$

in which the line $x = x_0$ is missing. Analogously, if $r \neq 0$, one obtains the bundle, except the line $y = y_0$.

If the point P_0 is given as the intersection of two lines, then its coordinates are the solution of the system

$$\begin{cases} d_1 : a_1x + b_1y + c_1 = 0 \\ d_2 : a_2x + b_2y + c_2 = 0 \end{cases},$$

supposed to be compatible. The equation of the bundle of lines through P_0 is

$$r(a_1x + b_1y + c_1) + s(a_2x + b_2y + c_2) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \tag{2.4}$$

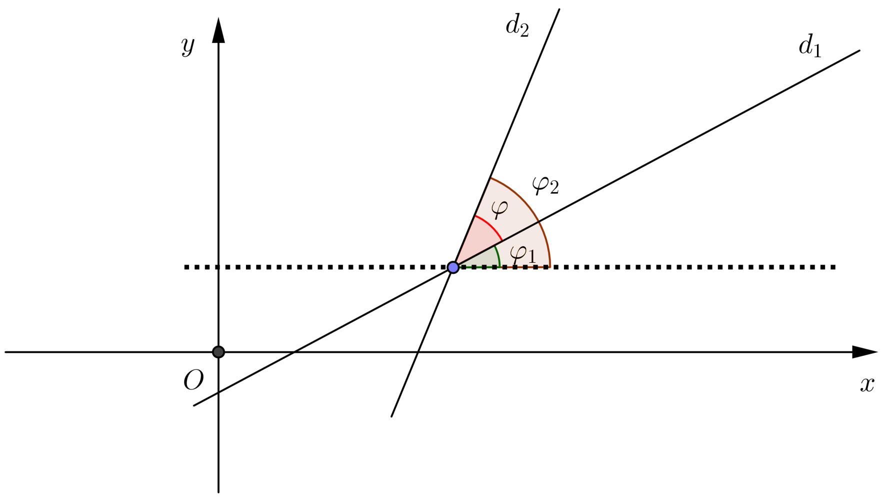
Remark: As before, if $r \neq 0$ (or $s \neq 0$), one obtains the reduced equation of the bundle, containing all the lines through P_0, except d_1 (respectively d_2).

2.3 The Angle of Two Lines

Let d_1 and d_2 be two concurrent lines, given by their reduced equations:

$$d_1 : y = m_1x + n_1 \quad \text{and} \quad d_2 : y = m_2x + n_2.$$

The angular coefficients of d_1 and d_2 are $m_1 = \tan\varphi_1$ and $m_2 = \tan\varphi_2$. One may assume that $\varphi_1 \neq \frac{\pi}{2}$, $\varphi_2 \neq \frac{\pi}{2}$, $\varphi_2 \geq \varphi_1$, such that $\varphi = \varphi_2 - \varphi_1 \in [0, \pi] \setminus \{\frac{\pi}{2}\}$.



The angle determined by d_1 and d_2 is given by

$$\tan\varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_1 \tan\varphi_2},$$

hence

$$\tan \varphi = \frac{m_2 - m_1}{1 + m_1 m_2}. \tag{2.5}$$

1) The lines d_1 and d_2 are parallel if and only if $\tan \varphi = 0$, therefore

$$d_1 \parallel d_2 \iff m_1 = m_2. \tag{2.6}$$

2) The lines d_1 and d_2 are orthogonal if and only if they determine an angle of $\frac{\pi}{2}$, hence

$$d_1 \perp d_2 \iff m_1 m_2 + 1 = 0. \tag{2.7}$$

2.4 Projections and symmetries

2.4.1 The intersection point of two nonparallel lines

Consider two straight lines

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

and $\Delta : ax + by + c = 0$ which are not parallel to each other, i.e.

$$ap + bq \neq 0.$$

The parametric equations of d are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases}, \; t \in \mathbb{R}. \tag{2.8}$$

The value of $t \in \mathbb{R}$ for which this line (2.8) punctures the line Δ can be determined by imposing the condition on the point of coordinates

$$(x_0 + pt, y_0 + qt)$$

to verify the equation of the line Δ, namely

$$a(x_0 + pt) + b(y_0 + qt) + c = 0.$$

Thus

$$t = -\frac{ax_0 + by_0 + c}{ap + bq} = -\frac{F(x_0, y_0)}{ap + bq},$$

where $F(x, y) = ax + by + c$.

The coordinates of the intersection point $d \cap \Delta$ are

$$\begin{cases} x_0 - p\dfrac{F(x_0, y_0)}{ap + bq} \\ y_0 - q\dfrac{F(x_0, y_0)}{ap + bq}. \end{cases} \tag{2.9}$$

2.4.2 The projection on a line parallel to another given line

Consider two straight non-parallel lines

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

and $\Delta : ax + by + c = 0$ which are not parallel to each other, i.e. $ap + bq \neq 0$. For these given data we may define the projection $p_{\Delta,d} : \pi \longrightarrow \Delta$ of π on Δ parallel to d, whose value $p_{\Delta,d}(M)$ at $M \in \mathcal{P}$ is the intersection point between Δ and the line through M which is parallel to d. Due to relations (2.9), the coordinates of $p_{\Delta,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - p\dfrac{F(x_M, y_M)}{ap + bq} \\ y_M - q\dfrac{F(x_M, y_M)}{ap + bq}, \end{cases} \tag{2.10}$$

where $F(x, y) = ax + by + c$.

Consequently, the position vector of $p_{\Delta,d}(M)$ is

$$\overrightarrow{Op_{\Delta,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{ap + bq} \overrightarrow{d}, \tag{2.11}$$

where $\overrightarrow{d} = p \overrightarrow{e} + q \overrightarrow{f}$. If we denote the coordinates of the generic point M by (x, y) with respect to the coordinate cartesian asystem R, then

$$\begin{aligned}
[p_{\Delta,d}(M)]_R &= \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} p\dfrac{F(x,y)}{ap+bq} \\ q\dfrac{F(x,y)}{ap+bq} \end{pmatrix} = \begin{pmatrix} x - p\dfrac{ax+by+c}{ap+bq} \\ y - q\dfrac{ax+by+c}{ap+bq} \end{pmatrix} \\
&= \begin{pmatrix} \left(1 - \frac{pa}{ap+bq}\right) x - \dfrac{pb}{ap+bq} y - \dfrac{pc}{ap+bq} \\ -\frac{qa}{ap+bq} x + \left(1 - \frac{qb}{ap+bq}\right) y - \dfrac{qc}{ap+bq} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{bq}{ap+bq} x - \dfrac{bp}{ap+bq} y - \dfrac{cp}{ap+bq} \\ -\frac{aq}{ap+bq} x + \dfrac{ap}{ap+bq} y - \dfrac{cq}{ap+bq} \end{pmatrix} \\
&= \frac{1}{ap+bq} \begin{pmatrix} bq & -bp \\ -aq & ap \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \frac{c}{ap+bq} \begin{pmatrix} p \\ q \end{pmatrix} \\
&= \frac{1}{ap+bq} \begin{pmatrix} bq & -bp \\ -aq & ap \end{pmatrix} [M]_R - \frac{c}{ap+bq} [\overrightarrow{d}]_b
\end{aligned}$$

2.4.3 The symmetry with respect to a line parallel to another line

We call the function $s_{\Delta,d} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{\Delta,d}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $s_{\Delta,d}(M)$ *the symmetry of π with respect to Δ parallel to d*. The direction of d is equally called the *direction* of the symmetry and π is called the *axis* of the symmetry. For the position vector of $s_{\Delta,d}(M)$ we have

$$\overrightarrow{Op_{\Delta,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\Delta,d}(M)}}{2}, \text{ i.e.} \tag{2.12}$$

$$\overrightarrow{Os_{\Delta,d}(M)} = 2\,\overrightarrow{Op_{\Delta,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{ap+bq}\,\overrightarrow{d}, \tag{2.13}$$

where $F(x,y) = ax + by + c$. Thus, the coordinates of $s_{\Delta,d}(M)$, in terms of the coordinates of M, are

$$\begin{cases} x_M - 2p\dfrac{F(x_M, y_M)}{ap+bq} \\ y_M - 2q\dfrac{F(x_M, y_M)}{ap+bq}. \end{cases} \tag{2.14}$$

If we denote by (x,y) the coordinates of the generic point M with respect to the reference cartesian system R, then

$$\begin{aligned}
[s_{\Delta,d}(M)]_R &= [\overrightarrow{Os_{\Delta,d}(M)}]_b = [\overrightarrow{OM}]_b - 2\frac{F(M)}{ap+bq}[\overrightarrow{d}]_b \\
&= \begin{pmatrix} x \\ y \end{pmatrix} - 2\begin{pmatrix} p\dfrac{ax+by+c}{ap+bq} \\ q\dfrac{ax+by+c}{ap+bq} \end{pmatrix} = \begin{pmatrix} x - 2p\dfrac{ax+by+c}{ap+bq} \\ y - 2q\dfrac{ax+by+c}{ap+bq} \end{pmatrix} \\
&= \begin{pmatrix} \left(1 - 2\dfrac{ap}{ap+bq}\right)x - 2\dfrac{pb}{ap+bq}y - 2\dfrac{pc}{ap+bq} \\ -2\dfrac{aq}{ap+bq}x + \left(1 - 2\dfrac{bq}{ap+bq}\right)y - 2\dfrac{qc}{ap+bq} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{bq-ap}{ap+bq}x - 2\dfrac{bp}{ap+bq}y - 2\dfrac{pc}{ap+bq} \\ -2\dfrac{aq}{ap+bq}x + \dfrac{ap-bq}{ap+bq}y - 2\dfrac{qc}{ap+bq} \end{pmatrix} \\
&= \frac{1}{ap+bq}\begin{pmatrix} bp-aq & -2bp \\ -2aq & ap-bq \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \frac{2c}{ap+bq}\begin{pmatrix} p \\ q \end{pmatrix} \\
&= \frac{1}{ap+bq}\begin{pmatrix} bp-aq & -2bp \\ -2aq & ap-bq \end{pmatrix}[M]_R - \frac{2c}{ap+bq}[\overrightarrow{d}]_b
\end{aligned}$$

3 Exercises [1, p. 49 & 53]

1. The sides $[BC]$, $[CA]$, $[AB]$ of the triangle ΔABC are divided by the points M, N respectively P into the same ratio k. Prove that the triangles ΔABC and ΔMNP have the same center of gravity.

2. Sketch the graph of $x^2 - 4xy + 3y^2 = 0$.

3. Find the equation of the line passing through the intersection point of the lines

$$d_1 : 2x - 5y - 1 = 0, \quad d_2 : x + 4y - 7 = 0$$

and through a point M which divides the segment $[AB]$, $A(4,-3)$, $B(-1,2)$, into the ratio $k = \dfrac{2}{3}$.

4. Let A be a mobile point on the Ox axis and B a mobile point on Oy, so that
$$\frac{1}{OA}+\frac{1}{OB}=k\ (constant).$$
Prove that the lines AB pass through a fixed point.

5. Find the equation of the line passing through the intersection point of
$$d_1 : 3x-2y+5=0, \quad d_2 : 4x+3y-1=0$$
and crossing the Oy axis at the point A with $OA=3$.

6. Find the parametric equations of the line through P_1 and P_2, when
   a) $P_1(3,-2)$, $P_2(5,1)$;
   b) $P_1(4,1)$, $P_2(4,3)$.

7. Find the parametric equations of the line through $P(-5,2)$ and parallel to $\overline{v}(2,3)$.

8. Show that the equations
$$x=3-t, y=1+2t \qquad \text{and} \qquad x=-1+3t, y=9-6t$$
represent the same line.

9. Find the vector equation of the line passing through P_1 and P_2, when
   a) $P_1(2,-1)$, $P_2(-5,3)$;
   b) $P_1(0,3)$, $P_2(4,3)$.

10. Given the line $d : 2x+3y+4=0$, find the equation of a line d_1 passing through the point $M_0(2,1)$, in the following situations:
    a) d_1 is parallel with d;
    b) d_1 is orthogonal on d;
    c) the angle determined by d and d_1 is $\varphi=\frac{\pi}{4}$.

11. The vertices of the triangle ΔABC are the intersection points of the lines
$$d_1 : 4x+3y-5=0, \quad d_2 : x-3y+10=0, \quad d_3 : x-2=0.$$
    (a) Find the coordinates of A, B, C.
    (b) Find the equations of the median lines of the triangle.
    (c) Find the equations of the heights of the triangle.

12. Find the coordinates of the point P on the line $d : 2x-y-5=0$ for which the sum $AP+PB$ is minimum, when $A(-7,1)$ and $B(-5,5)$.

13. Find the coordinates of the circumcenter (the center of the circumscribed circle) of the triangle determined by the lines $4x-y+2=0$, $x-4y-8=0$ and $x+4y-8=0$.

14. Prove that, in any triangle ΔABC, the orthocenter H, the center of gravity G and the circumcenter O are collinear.

15. Given the bundle of lines of equations $(1-t)x+(2-t)y+t-3=0$, $t \in \mathbb{R}$ and $x+y-1=0$, find:

    a) the coordinates of the vertex of the bundle;

    b) the equation of the line in the bundle which cuts Ox and Oy in M respectively N, such that $OM^2 \cdot ON^2 = 4(OM^2 + ON^2)$.

16. Let $\mathcal{B}$ be the bundle of vertex $M_0(5,0)$. An arbitrary line from $\mathcal{B}$ intersects the lines $d_1 : y-2=0$ and $d_2 : y-3=0$ in M_1 respectively M_2. Prove that the line passing through M_1 and parallel to OM_2 passes through a fixed point.

17. The vertices of the quadrilateral $ABCD$ are $A(4,3)$, $B(5,-4)$, $C(-1,-3)$ and $D((-3,-1)$.

    a) Find the coordinates of the points
    $$\{E\} = AB \cap CD \;\&\; \{F\} = BC \cap AD;$$

    b) Prove that the midpoints of the segments $[AC]$, $[BD]$ and $[EF]$ are collinear.

18. Let M be a point whose coordinates satisfy
    $$\frac{4x+2y+8}{3x-y+1} = \frac{5}{2}.$$

    a) Prove that M belongs to a fixed line;

    b) Find the minimum of x^2+y^2, when $M \in d \setminus \{M_0(-1,-2)\}$.

19. Find the geometric locus of the points whose distances to two orthogonal lines have a constant ratio.

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