Geometry
Problem booklet

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Week 12

1 Week 12. Transformations

1.1 Transformations of the plane

Definition 1.1. *An affine transformation of the plane is a mapping*

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x,y) = (ax+by+c, dx+ey+f), \tag{1.1}$$

for some constant real numbers a,b,c,d,e,f.

The affine transformation (1.1) can be equally described by means of its equations

$$\begin{cases} x' = ax+by+c \\ y' = dx+ey+f. \end{cases}$$

By using the matrix language, the action of the map L can be written in the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ f \end{pmatrix},$$

or, equivalently

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \tag{1.2}$$

In order to point out the theoretical background behind representations of type (1.2), we identify the points $(x,y) \in \mathbb{R}^2$ with the points $(x,y,1) \in \mathbb{R}^3$ and even with the punctured lines of $\mathbb{R}^3$, (rx,ry,r), $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x,y) \in \mathbb{R}^2$ with the punctured lines of $\mathbb{R}^3$ represented in the form

$$\begin{pmatrix} rx \\ ry \\ r \end{pmatrix},\ r \in \mathbb{R}^*,$$

and the latter ones we shall call *homogeneous coordinates* of the point $(x,y) \in \mathbb{R}^2$. The set of homogeneous coordinates (x,y,w) will be denoted by $\mathbb{RP}^2$ and call it the real *projective plane*. The homogeneous coordinates $(x,y,w) \in \mathbb{RP}^2$, $w \neq 0$ şi $\left(\frac{x}{w}, \frac{y}{w}, 1\right)$represent the same element of $\mathbb{RP}^2$.

Observation 1.2. *The projective plane $\mathbb{RP}^2$ is actually the quotient set $(\mathbb{R}^3 \setminus \{0\})/\sim$, where $'\sim'$ is the following equivalence relation on $\mathbb{R}^3 \setminus \{0\}$:*

$$(x,y,w) \sim (\alpha,\beta,\gamma) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.î. } (x,y,w) = r(\alpha,\beta,\gamma).$$

Observe that the equivalence classes of the equivalence relation $\sim'$ are the punctured lines of $\mathbb{R}^3$ through the origin without the origin itself, i.e. the elements of the real projective plane $\mathbb{RP}^2$. Such an equivalence class of

$$(x,y,w) \in \mathbb{R}^3 \text{ wil be denoted by } [x,y,w] \text{ or by } \begin{bmatrix} x \\ y \\ w \end{bmatrix}.$$

Definition 1.3. *A* projective transformation *is a linear transformation of* $\mathbb{R}^3$*, say*

$$L:\mathbb{R}^3\longrightarrow\mathbb{R}^3,\ L\begin{pmatrix}x\\y\\w\end{pmatrix}=\begin{pmatrix}a&b&c\\d&e&f\\g&h&k\end{pmatrix}\begin{pmatrix}x\\y\\w\end{pmatrix}=\begin{pmatrix}ax+by+cw\\dx+ey+fw\\gx+hy+kw\end{pmatrix},\tag{1.3}$$

where $a,b,c,d,e,f,g,h,k\in\mathbb{R}$*, which maps the lines through the origin onto lines (obviosly through the origin).*

For example the linear invertible transformations $L:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ have such a property and they might be restericted as $L:\mathbb{R}^3\setminus\{0\}\longrightarrow\mathbb{R}^3\setminus\{0\}$. Note that such a linear map induces a map $p\circ L:\mathbb{R}^3\setminus\{0\}\longrightarrow\mathbb{RP}^2$, where $p:L:\mathbb{R}^3\setminus\{0\}\longrightarrow\mathbb{RP}^2$ stands for the canonical projection. Also $p\circ L$ maps every punctured line through the origin onto the same element of $\mathbb{RP}^2$, which shows that $p\circ L$ induces a map $\tilde{L}:\mathbb{R}^3\longrightarrow\mathbb{RP}^2$ of the projective plane $\mathbb{RP}^2$. We shall denote $\tilde{L}$ by L and call it a *projective transformation of* $\mathbb{RP}^2$. In other words the transformation

$$L:\mathbb{RP}^2\longrightarrow\mathbb{RP}^2,\ L\begin{bmatrix}x\\y\\w\end{bmatrix}=\begin{pmatrix}a&b&c\\d&e&f\\g&h&k\end{pmatrix}\begin{bmatrix}x\\y\\w\end{bmatrix}=\begin{bmatrix}ax+by+cw\\dx+ey+fw\\gx+hy+kw\end{bmatrix},\tag{1.4}$$

is well defined, as and will be denoted by L. Indeed,

$$L\begin{pmatrix}rx\\ry\\rw\end{pmatrix}=\begin{pmatrix}arx+bry+crw\\drx+ery+frw\\grx+hry+krw\end{pmatrix}=\begin{pmatrix}r(ax+by+cw)\\r(dx+ey+fw)\\r(gx+hy+kw)\end{pmatrix}.$$

The projective transformation L is completely determined by its *homogeneous transformation matrix*

$$[L]=\begin{pmatrix}a&b&c\\d&e&f\\g&h&k\end{pmatrix}.$$

If $g=h=0$ and $k=1$, then the projective transformation (1.4) is said to be *affine*. The restriction of the affine transformation (1.4), which corresponds to the situation $g=h=0$ and $k=1$, to the subspace $w=1$, has the form

$$L\begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}a&b&c\\d&e&f\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}ax+by+cw\\dx+ey+fw\\1\end{pmatrix},\tag{1.5}$$

i.e.

$$\begin{cases}x'=ax+by+c\\y'=dx+ey+f.\end{cases}\tag{1.6}$$

and it induces a transformation $L:\mathbb{RP}^2\setminus\mathbb{RP}^1_{xy}\longrightarrow\mathbb{RP}^2\setminus\mathbb{RP}^1_{xy}$

$$L\begin{bmatrix}x\\y\\1\end{bmatrix}=\begin{pmatrix}a&b&c\\d&e&f\\0&0&1\end{pmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix}=\begin{bmatrix}ax+by+cw\\dx+ey+fw\\1\end{bmatrix},\tag{1.7}$$

where $\mathbb{RP}^1_{xy}$ stands for collection of all equivalences classes $[x,y,0]$ of $\mathbb{RP}^2$. Note that the linear transformation (1.5) behaves on $\mathbb{R}^3\setminus xOy$ as a projective aplication, even when the homogeneous matrix transformation

$$\begin{pmatrix}a&b&c\\d&e&f\\0&0&1\end{pmatrix}$$

is not invertible, i.e. evey punctured line through the origin of $\mathbb{R}^3 \setminus xOy$ is mapped onto a punctured line (obviously through the origine) in $\mathbb{R}^3 \setminus xOy$.

Observation 1.4. *If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two projective applications, then their product (concatenation) transformation $L_1 \circ L_2$ is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of L_1 and L_2 .*

Indeed, if

$$L_1 \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix}$$

and

$$L_2 \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix}$$

then

$$(L_1 \circ L_2) \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \left(\begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & k_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{pmatrix} \right) \begin{pmatrix} x \\ y \\ w \end{pmatrix}$$

Observation 1.5. *If $L_1, L_2 : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ are two affine applications, then their product $L_1 \circ L_2$ is also an affine transformation.*

Proposition 1.6. *If $(aB - bA)^2 + (dB - eA)^2 > 0$, then the affine transformation (1.1) maps the line (d) $Ax + By + C = 0$ to the line*

$$(eA - dB)x + (aB - bA)y + (bf - ce)A - (af - cd)B + (ae - bd)C = 0.$$

If $aB - bA = dB - eA = 0$, then $ae - bd = 0$ and L is the constant map $\left(\frac{cB-bC}{B}, \frac{fB-eC}{B}\right)$.

Definition 1.7. *An affine transformation (1.1) is said to be* singular *if*

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0 \text{ i.e. } ae - bd = 0.$$

and non-singular otherwise.

Note that the affine transformation L is nonsingular if and only if it is invertible. In such a case the inverse L^{-1} is a non-singular affine transformation and $[L^{-1}] = [L]^{-1}$.

1.2 Examples of affine transformations

1.2.1 Translations

Definition 1.8. *The* translation *of vector $(h, k) \in \mathbb{R}^2$ is the affine transformation*

$$T(h,k) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ (T(h,k))(x,y) = (x + h, y + k).$$

Its equations are

$$\begin{cases} x' = x + h \\ y' = y + k \end{cases},$$

or, by using the matrix language and the homogeneous coordinates

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix} \text{ i.e. } \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

which shows that the homogeneous matrix transformation is

$$[T(h,k)] = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the translation $T(h,k)$ is non-singular (invertible) and $(T(h,k))^{-1} = T(-h,-k)$.

1.2.2 Scaling about the origin

Definition 1.9. *The* scaling about the origin *by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is the affine transformation*

$$S(s_x, s_y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \left(S(s_x, s_y)\right)(x,y) = (s_x \cdot x, s_y \cdot y).$$

Its equations are

$$\begin{cases} x' = s_x \cdot x \\ y' = s_y \cdot y \end{cases},$$

or, by using the matrix language and the homogeneous coordinates

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ i.e. } \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

which shows that the homogeneous matrix transformation is

$$\left[S(s_x, s_y)\right] = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the scaling about the origin by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is non-singular (invertible) and $(S(s_x, s_y))^{-1} = S(s_x^{-1}, s_y^{-1})$.

The *scaling $S_P(s_x, s_y)$ by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ about an arbitrary point $P(x_0, y_0)$* acts in a similar way as $S(s_x, s_y)$, but the role of the origin is played bt P. Thus

$$S_P(s_x, s_y) = T(x_0, y_0) \circ S(s_x, s_y) \circ T(-x_0, -y_0),$$

i.e. its homogeneous transformation matrix is

$$\left[S_P(s_x, s_y)\right] = \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & (1-s_x)x_0 \\ 0 & s_y & (1-s_y)y_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.2.3 Reflections

Definition 1.10. *The* reflections about the x-axis *and the y-axis respectively are the affine transformation*

$$r_x, r_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ r_x(x,y) = (x,-y),\ r_y = (-x,y).$$

Their equations are

$$r_x : \begin{cases} x' &= x \\ y' &= -y \end{cases} \quad \text{and } r_y : \begin{cases} x' &= -x \\ y' &= y \end{cases}$$

or, by using the matrix language and the homogeneous coordinates

$$r_x : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } r_y : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ i.e.}$$

$$r_x : \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \text{ and } r_y : \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

which shows that the homogeneous matrices transformations are

$$[r_x] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } (r_y) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $r_x = S(1,-1)$ and $r_y = S(-1,1)$. Thus the two reflections are non-singular (invertible) and $r_x^{-1} = r_x$, $r_y^{-1} = r_y$.

Definition 1.11. *The* reflection $r_l : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the line l *maps a given point M to the point M' defined by the property that l is the perpendicular bisector of the segment MM'. One can show that the action of the reflection about the line $l : ax + by + c = 0$ is*

$$r_l(x,y)=\left(\frac{b^2-a^2}{a^2+b^2}x-\frac{2ab}{a^2+b^2}y-\frac{2ac}{a^2+b^2}, -\frac{2ab}{a^2+b^2}x+\frac{a^2-b^2}{a^2+b^2}y-\frac{2bc}{a^2+b^2}\right).$$

Its equations are

$$\begin{cases} x' &= \dfrac{b^2-a^2}{a^2+b^2}x-\dfrac{2ab}{a^2+b^2}y-\dfrac{2ac}{a^2+b^2} \\ y' &= -\dfrac{2ab}{a^2+b^2}x+\dfrac{a^2-b^2}{a^2+b^2}y-\dfrac{2bc}{a^2+b^2} \end{cases},$$

or, by using the matrix language and the homogeneous coordinates

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \dfrac{b^2-a^2}{a^2+b^2} & -\dfrac{2ab}{a^2+b^2} \\ -\dfrac{2ab}{a^2+b^2} & \dfrac{a^2-b^2}{a^2+b^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \dfrac{2ac}{a^2+b^2} \\ \dfrac{2bc}{a^2+b^2} \end{pmatrix} \text{ i.e.}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{b^2-a^2}{a^2+b^2} & -\dfrac{2ab}{a^2+b^2} & -\dfrac{2ac}{a^2+b^2} \\ -\dfrac{2ab}{a^2+b^2} & \dfrac{a^2-b^2}{a^2+b^2} & -\dfrac{2bc}{a^2+b^2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

which shows that the homogeneous matrix transformation is

$$[r_l] = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc}{a^2+b^2} \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{a^2+b^2}\begin{pmatrix} b^2-a^2 & -2ab & -2ac \\ -2ab & a^2-b^2 & -2bc \\ 0 & 0 & a^2+b^2 \end{pmatrix}.$$

Note that the reflection r_l is non-singular (invertible) and $r_l^{-1} = r_l$.

1.2.4 Rotations

Definition 1.12. *The* rotation $\mathrm{rot}_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *about the origin through an angle θ maps a point $M(x,y)$ into a point $M'(x',y')$ with the properties that the segments $[OM]$ and $[OM']$ are congruent and the $m(\widehat{MOM'}) = \theta$. If $\theta > 0$ the rotation is supposed to be* anticlockwise *and for $\theta < 0$ the rotation is* clockwise. *If $(x,y) = (r\cos\varphi, r\sin\varphi)$, then the coordinates of the rotated point are $\big(r\cos(\theta+\varphi), r\sin(\theta+\varphi)\big) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, i.e.*

$$\mathrm{rot}_\theta = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Its equations are

$$\begin{cases} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{cases},$$

or, by using the matrix language and the homogeneous coordinates

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \text{ i.e. } \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

which shows that the homogeneous matrix transformation is

$$[\mathrm{rot}_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the rotation rot_θ *is non-singular (invertible) and* $\mathrm{rot}_\theta^{-1} = \mathrm{rot}_{-\theta}$.

The *rotation* $\mathrm{rot}_\theta(P)$ *about an arbitrary point* $P(x_0, y_0)$ acts in a similar way as rot_θ, but the role of the origin is played bt P. Thus $\mathrm{rot}_\theta(P) = T(x_0,y_0) \circ \mathrm{rot}_\theta \circ T(-x_0,-y_0)$, i.e. its homogeneous transformation matrix is

$$\begin{aligned} [\mathrm{rot}_\theta(P)] &= \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta & x_0(1-\cos\theta) + y_0\sin\theta \\ \sin\theta & \cos\theta & -x_0\sin\theta + y_0(1-\cos\theta) \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

1.2.5 Shears

Definition 1.13. *Given a fixed direction in the plane specified by a unit vector $v = (v_1, v_2)$, consider the lines d with direction v and the oriented distance d from the origin. The* shear *about the origin of factor r in the direction v is defined to be the transformation which maps a point $M(x, y)$ on d to the point $M' = M + rdv$. The equation of the line through M of direction v is*

$$v_2 X - v_1 Y + (v_1 y - v_2 x) = 0.$$

The oriented distacnce from the origin to this line is $v_1 y - v_2 x$. Thus the action of the shear $Sh(v, r) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ about the origin of factor r in the direction v is

$$\begin{aligned} Sh(v,r)(x,y) &= (x,y) + rd(v_1, v_2) \\ &= (x,y) + (r(v_1 y - v_2 x)v_1, r(v_1 y - v_2 x)v_2) \\ &= (x,y) + \left(-rv_1 v_2 x + rv_1^2 y, -rv_2^2 x + rv_1 v_2 y\right) \\ &= \left((1 - rv_1 v_2)x + rv_1^2 y, -rv_2^2 x + (1 + rv_1 v_2)y\right) \end{aligned}$$

Its equations are

$$\begin{cases} x' &= (1 - rv_1 v_2)x + rv_1^2 y \\ y' &= -rv_2^2 x + (1 + rv_1 v_2)y \end{cases},$$

or, by using the matrix language and the homogeneous coordinates

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 - rv_1 v_2 & rv_1^2 \\ -rv_2^2 & 1 + rv_1 v_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ i.e. } \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - rv_1 v_2 & rv_1^2 & 0 \\ -rv_2^2 & 1 + rv_1 v_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

which shows that the homogeneous matrix transformation is

$$[Sh(v,r)(x,y)] = \begin{pmatrix} 1 - rv_1 v_2 & rv_1^2 & 0 \\ -rv_2^2 & 1 + rv_1 v_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The *shear $Sh_P(v, r)$ about an arbitrary point $P(x_0, y_0)$* of factor r in the direction v acts in a similar way as $Sh(v, r)$, but the role of the origin is played bt P. Thus $Sh_P(v, r) = T(x_0, y_0) \circ Sh(v, r) \circ T(-x_0, -y_0)$, i.e. its homogeneous transformation matrix is

$$\begin{aligned} [Sh_P(v,r)] &= \begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - rv_1 v_2 & rv_1^2 & 0 \\ -rv_2^2 & 1 + rv_1 v_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - rv_1 v_2 & rv_1^2 & rv_1(x_0 v_2 - y_0 v_1) \\ -rv_2^2 & 1 + rv_1 v_2 & rv_2(x_0 v_2 - y_0 v_1) \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

1.3 Problems

1. Consider a quadrilateral with vertices $A(1,1)$, $B(3,1)$, $C(2,2)$, and $D(1.5,3)$. Find the image quadrilaterals through the translation $T(1,2)$, the scaling $S(2,2.5)$, the reflections about the x and y-axes, the clockwise and anticlockwise rotations through the angle $\pi/2$ and the shear $Sh\left(\left(2/\sqrt{5}, 1/\sqrt{5}\right), 1.5\right)$.

2. Find the concatenation (product) of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$ followed by a scaling by a factor of 3 units in the x-direction and 2 units in the y-direction. (Hint: $S(3,2)R_{3\pi/2}$)

3. Find the homogeneous matrix of the product (concatenation) $S(3,2) \circ R_{\frac{3\pi}{2}}$.

4. Find the equations of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ.

5. Show that the concatenation (product) of two rotations, the first through an angle θ about a point $P(x_0, y_0)$ and the second about a point $Q(x_1, y_1)$ (distinct from P) through an angle $-\theta$ is a translation.

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