

Geometry

Problem booklet

Assoc. Prof. Cornel Pintea

E-mail: cpintea math.ubbcluj.ro

Contents

1	Week 4: Analytic conditions of parallelism. Projections and symmetries	1
1.1	Brief theoretical background. Analytic conditions of parallelism	1
1.2	Projections and symmetries. Pencils of planes	2
1.2.1	The intersection point of a straight line and a plane	2
1.2.2	The projection on a plane parallel to a given line	3
1.2.3	The symmetry with respect to a plane parallel to a line	4
1.2.4	The projection on a straight line parallel to a given plane	4
1.2.5	The symmetry with respect to a line parallel to a plane	5
1.3	Pencils of planes	5
1.4	Problems	6

Module leader: Assoc. Prof. Cornel Pintea

Department of Mathematics,
"Babeş-Bolyai" University
400084 M. Kogălniceanu 1,
Cluj-Napoca, Romania

1 Week 4: Analytic conditions of parallelism. Projections and symmetries

1.1 Brief theoretical background. Analytic conditions of parallelism

The equation $AX + BY + CZ = 0$ is a necessary and sufficient condition for the vector $\vec{A_0M}(X, Y, Z)$ to be contained within the direction of the plane

$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Thus the equation of the director subspace $\vec{\pi} = \{\vec{A_0M} \mid M \in \pi\}$ is $AX + BY + CZ = 0$.

Proposition 1.1. *The straight line*

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

is parallel to the plane $\pi : Ax + By + Cz + D = 0$ iff

$$Ap + Bq + Cr = 0 \quad (1.1)$$

Proposition 1.2. *Consider the planes*

$$(\pi_1) A_1x + B_1y + C_1z + D_1 = 0, (\pi_2) A_2x + B_2y + C_2z + D_2 = 0.$$

Then $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) \in \{1, 2\}$ and the following statements are equivalent

1. $\pi_1 \parallel \pi_2$.
2. $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 2$, i.e. $\vec{\pi}_1 = \vec{\pi}_2$.
3. $\text{rang} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1$.
4. The vectors $(A_1, B_1, C_1), (A_2, B_2, C_2) \in \mathbb{R}^3$ are linearly dependent.

Corollary 1.3. *Consider the planes*

$$(\pi_1) A_1x + B_1y + C_1z + D_1 = 0, (\pi_2) A_2x + B_2y + C_2z + D_2 = 0.$$

The following statements are equivalent

1. $\pi_1 \not\parallel \pi_2$.
2. $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 1$.
3. $\text{rang} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$.
4. The vectors $(A_1, B_1, C_1), (A_2, B_2, C_2) \in \mathbb{R}^3$ are linearly independent.

By using the characterization of parallelism between a line and a plane (Proposition 1.1), we shall find a necessary and sufficient condition for a vector to be contained within the direction of a straight line which is given as the intersection of two planes.

Consider the planes $(\pi_1) A_1x + B_1y + C_1z + D_1 = 0$, $(\pi_2) A_2x + B_2y + C_2z + D_2 = 0$ such that

$$\text{rang} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2,$$

alongside their intersection straight line $\Delta = \pi_1 \cap \pi_2$ of equations

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$$

Thus, $\vec{\Delta} = \vec{\pi}_1 \cap \vec{\pi}_2$ and therefore, by means of some previous Proposition, it follows that the equations of $\vec{\Delta}$ are

$$(\vec{\Delta}) \begin{cases} A_1X + B_1Y + C_1Z = 0 \\ A_2X + B_2Y + C_2Z = 0. \end{cases} \quad (1.2)$$

By solving the system (1.2) one can therefore deduce that $\vec{d}(p, q, r) \in \vec{\Delta} \Leftrightarrow \exists \lambda \in \mathbb{R}$ such that

$$(p, q, r) = \lambda \left(\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right). \quad (1.3)$$

The relation is usually (1.3) written in the form

$$\frac{p}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{q}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{r}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \quad (1.4)$$

Let us mention that the chosen values for (p, q, r) are usually precisely

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}.$$

1.2 Projections and symmetries. Pencils of planes

1.2.1 The intersection point of a straight line and a plane

Consider a straight line

$$d: \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane $\pi: Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

The parametric equations of d are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, t \in \mathbb{R}. \quad (1.5)$$

The value of $t \in \mathbb{R}$ for which this line (1.5) punctures the plane π can be determined by imposing the condition on the point of coordinates

$$(x_0 + pt, y_0 + qt, z_0 + rt)$$

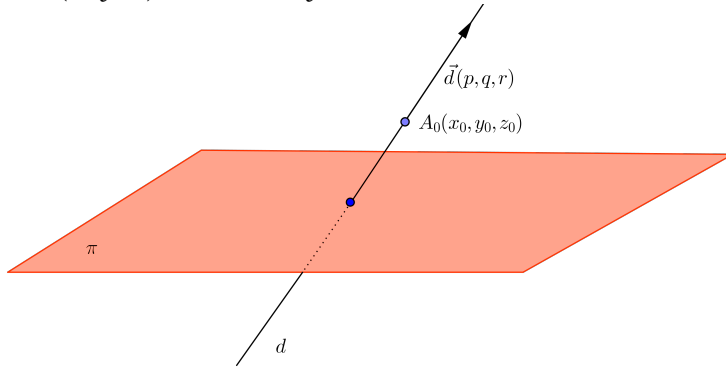
to verify the equation of the plane, namely

$$A(x_0 + pt) + B(y_0 + qt) + C(z_0 + Ct) + D = 0.$$

Thus

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} = -\frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr},$$

where $F(x, y, z) = Ax + By + Cz + D$.



The coordinates of the intersection point $d \cap \pi$ are

$$\begin{cases} x_0 - p \frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ y_0 - q \frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ z_0 - r \frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \end{cases} \quad (1.6)$$

1.2.2 The projection on a plane parallel to a given line

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

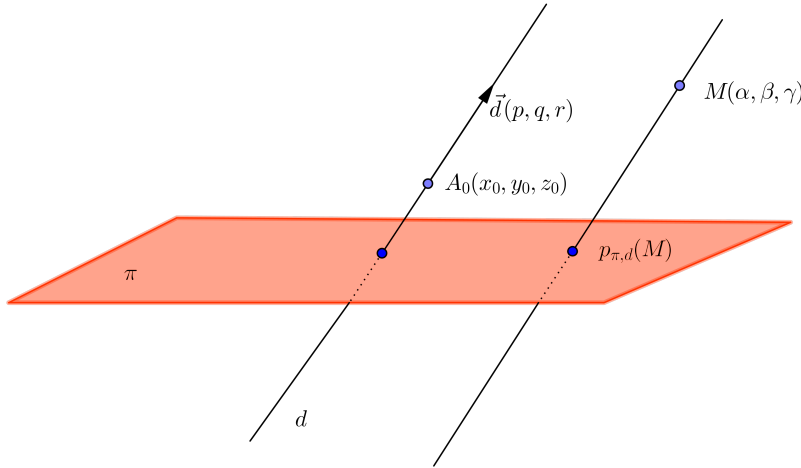
and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{\pi, d} : \mathcal{P} \rightarrow \pi$ of \mathcal{P} on π parallel to d , whose value $p_{\pi, d}(M)$ at $M \in \mathcal{P}$ is the intersection point between π and the line through M which is parallel to d . Due to relations (1.6), the coordinates of $p_{\pi, d}(M)$, in terms of the coordinates of M , are

$$\begin{cases} x_M - p \frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ y_M - q \frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ z_M - r \frac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \end{cases} \quad (1.7)$$

where $F(x, y, z) = Ax + By + Cz + D$.



Consequently, the position vector of $p_{\pi,d}(M)$ is

$$\overrightarrow{Op_{\pi,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{Ap + Bq + Cr} \vec{d}. \quad (1.8)$$

1.2.3 The symmetry with respect to a plane parallel to a line

We call the function $s_{\pi,d} : \mathcal{P} \rightarrow \mathcal{P}$, whose value $s_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{\pi,d}(M)$ the symmetry of \mathcal{P} with respect to π parallel to d . The direction of d is equally called the *direction* of the symmetry and π is called the *axis* of the symmetry. For the position vector of $s_{\pi,d}(M)$ we have

$$\overrightarrow{Op_{\pi,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\pi,d}(M)}}{2}, \text{ i.e.} \quad (1.9)$$

$$\overrightarrow{Os_{\pi,d}(M)} = 2 \overrightarrow{Op_{\pi,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2 \frac{F(M)}{Ap + Bq + Cr} \vec{d}. \quad (1.10)$$

1.2.4 The projection on a straight line parallel to a given plane

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

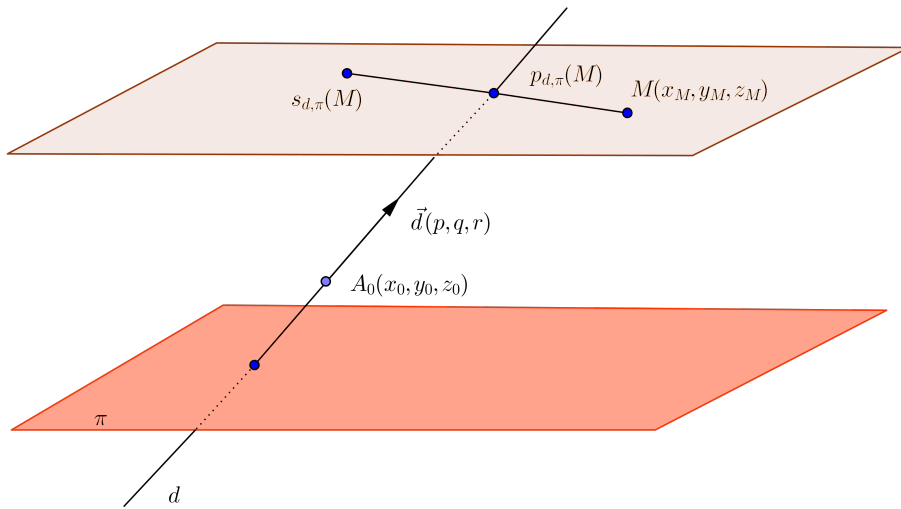
$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{d,\pi} : \mathcal{P} \rightarrow d$ of \mathcal{P} on d , whose value $p_{d,\pi}(M)$ at $M \in \mathcal{P}$ is the intersection point between d and the plane through M which is parallel to π . Due to relations (1.6), the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of M , are

$$\begin{cases} x_0 + p \frac{G(x_M, y_M, z_M)}{Ap + Bq + Cr} = x_0 + p \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \\ y_0 + q \frac{G(x_M, y_M, z_M)}{Ap + Bq + Cr} = y_0 + q \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \\ z_0 + r \frac{G(x_M, y_M, z_M)}{Ap + Bq + Cr} = z_0 + r \frac{F(M) - F(A_0)}{Ap + Bq + Cr}, \end{cases} \quad (1.11)$$

where $G(x, y, z) = A(x - x_0) + B(y - y_0) + C(z - z_0) = F(M) - F(A_0)$ and $F(x, y, z) = Ax + By + Cz + D$. The position vector of $p_{d,\pi}(M)$ is

$$\overrightarrow{Op_{d,\pi}(M)} = \overrightarrow{OA_0} + \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \vec{d}, \text{ where } A_0(x_0, y_0, z_0). \quad (1.12)$$



1.2.5 The symmetry with respect to a line parallel to a plane

We call the function $s_{d,\pi} : \mathcal{P} \rightarrow \mathcal{P}$, whose value $s_{d,\pi}(M)$ at $M \in \mathcal{P}$ is the symmetric point of M with respect to $p_{d,\pi}(M)$, the *symmetry of \mathcal{P} with respect to d parallel to π* . The direction of π is equally called the *direction* of the symmetry and d is called the *axis* of the symmetry. For the position vector of $s_{d,\pi}(M)$ we have

$$\overrightarrow{Op_{d,\pi}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{d,\pi}(M)}}{2}, \text{ i.e.} \quad (1.13)$$

$$\begin{aligned} \overrightarrow{Os_{d,\pi}(M)} &= 2 \overrightarrow{Op_{d,\pi}(M)} - \overrightarrow{OM} \\ &= 2 \overrightarrow{OA_0} - \overrightarrow{OM} + 2 \frac{F(M) - F(A_0)}{Ap + Bq + Cr} \vec{d}. \end{aligned} \quad (1.14)$$

1.3 Pencils of planes

Definition 1.4. The collection of all planes containing a given straight line

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

is called the *pencil of planes through Δ* .

Proposition 1.5. *The plane π belongs to the pencil of planes through the straight line Δ if and only if there exists $\lambda, \mu \in \mathbb{R}$, $\lambda^2 + \mu^2 > 0$ such that the equation of the plane π is*

$$\lambda(A_1x + B_1y + C_1z + D_1) + \mu(A_2x + B_2y + C_2z + D_2) = 0. \quad (1.15)$$

Remark 1.6. *The family of planes*

$$A_1x + B_1y + C_1z + D_1 + \lambda(A_2x + B_2y + C_2z + D_2) = 0,$$

where λ covers the whole real line \mathbb{R} , is the so called reduced pencil of planes through Δ and it consists in all planes through Δ except the plane of equation $A_2x + B_2y + C_2z + D_2 = 0$.

Remark 1.7. *If*

$$(\Delta_1) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ Ax + By + Cz + D = 0 \end{cases} \quad (\Delta_2) \begin{cases} A_2x + B_2y + C_2z + D_2 = 0 \\ Ax + By + Cz + D = 0 \end{cases}$$

are two concurrent lines at some point P , then the family of lines

$$(\Delta_{\lambda,\mu}) \begin{cases} A_1x + B_1y + C_1z + D_1 + \lambda(Ax + By + Cz + D) = 0 \\ A_2x + B_2y + C_2z + D_2 + \mu(Ax + By + Cz + D) = 0 \end{cases}$$

consists in all lines through P except those lines which lie in the plane

$$\pi : Ax + By + Cz + D = 0.$$

1.4 Problems

1. Write the equation of the line which passes through the point $M(1,0,7)$, is parallel to the plane $(\pi) 3x - y + 2z - 15 = 0$ and intersects the line

$$(d) \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

2. Write the equations of the projection of the line

$$(d) \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

on the plane $\pi : x + 2y - z = 0$ parallel to the direction $\vec{u} (1, 1, -2)$. Write the equations of the symmetry of the line d with respect to the plane π parallel to the direction $\vec{u} (1, 1, -2)$.

3. Write the equation of the plane determined by the line

$$(d) \begin{cases} x - 2y + 3z = 0 \\ 2x + z - 3 = 0 \end{cases}$$

and the point $A(-1, 2, 6)$.

4. Find the equation of the plane containing the origin and the line

$$(d) \frac{x-2}{3} = \frac{y-4}{1} = \frac{z-1}{1}.$$

5. Show that two different parallel lines are either projected onto parallel lines or on two points by a projection $p_{\pi,d}$, where

$$(\pi) Ax + By + Cz + D = 0, \quad (d) \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and $\pi \nparallel d$.

6. Show that two parallel lines are mapped onto parallel lines by a symmetry $s_{\pi,d}$, where

$$(\pi) Ax + By + Cz + D = 0, \quad (d) \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

and $\pi \nparallel d$.

7. Consider a plane $(\pi) Ax + By + Cz + D = 0$ and a line

$$(d) \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$$

with respect a coordinate Cartesian reference system $R = (O, b)$, where $b = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$ is a basis of \mathcal{V} . If $\pi \nparallel d$, show that:

(a) $\overrightarrow{p_{\pi,d}(M)p_{\pi,d}(N)} = \overrightarrow{p(MN)}$, $\forall M, N \in \mathcal{P}$, where $p : \mathcal{V} \rightarrow \mathcal{V}$ is the linear transformation with the matrix representation

$$[p]_b = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix}.$$

(b) $\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)} = \overrightarrow{s(MN)}$, $\forall M, N \in \mathcal{P}$, where $s : \mathcal{V} \rightarrow \mathcal{V}$ is the linear transformation with the matrix representation

$$[s]_b = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} -Ap + Bq + Cr & -2Bp & -2Cp \\ -2Aq & Ap - Bq + Cr & -2Cq \\ -2Ar & -2Br & Ap + Bq - Cr \end{pmatrix}.$$

8. Consider a plane $(\pi) Ax + By + Cz + D = 0$ and a line

$$(d) \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}.$$

If $\pi \nparallel d$, show that:

$$(a) p_{\pi,d} \circ p_{\pi,d} = p_{\pi,d};$$

$$(b) s_{\pi,d} \circ s_{\pi,d} = id_{\mathcal{P}}.$$

References

- [1] Andrica, D., Țopan, L., Analytic geometry, Cluj University Press, 2004.
- [2] Galbură Gh., Radó, F., Geometrie, Editura didactică și pedagogică-București, 1979.
- [3] Pinte, C. Geometrie. Elemente de geometrie analitică. Elemente de geometrie diferențială a curbelor și suprafețelor, Presa Universitară Clujeană, 2001.
- [4] Radó, F., Orban, B., Groze, V., Vasile, A., Culegere de Probleme de Geometrie, Lit. Univ. "Babeş-Bolyai", Cluj-Napoca, 1979.