Algebra
Seminar 2

①. r, s, t, v – homogeneous relations defined on the set $M = \{2,3,4,5,6\}$ by

$x r y \Leftrightarrow x < y$
$x s y \Leftrightarrow x | y$
$x t y \Leftrightarrow g.c.d(x,y) = 1$
$x v y \Leftrightarrow x \equiv y \bmod 3$

Write the graphs R, S, T, V of the given relations.

a) $x r y \Leftrightarrow x < y$
$R = \{(2,3); (2,4); (2,5); (2,6); (3,4); (3,5); (3,6); (4,5); (4,6); (5,6)\}$

b) $x s y \Leftrightarrow x | y$
$S = \{(2,4); (2,6); (2,2); (3,3); (3,6); (4,4); (5,5); (6,6)\}$

c) $x t y \Leftrightarrow g.c.d(x,y) = 1$
$T = \{(2,3); (2,5); (3,4); (3,5); (4,5); (5,6); (3,2); (5,2); (4,3); (5,3); (5,4); (6,5)\}$

d) $x v y \Leftrightarrow x \equiv y \bmod 3$ (3 divides $(x-y)$)
$V = \{(2,5); (5,2); (3,6); (6,3); (2,2); (3,3); (4,4); (5,5); (6,6)\}$
(we obtain the same remainder if we divide by 3)

②. A, B – sets with n, m elements ($n, m \in \mathbb{N}^*$). Determine the number of
a) relations having the domain A and codomain B
b) homogeneous relations on A

a) $r = (A, B, R)$
$|A| = n$ $|B| = m$
$R \subseteq A \times B$
$|P(A \times B)|$ – the number of possible subsets of $A \times B$
$|P(A \times B)| = 2^{n \cdot m}$

b) $r = (A, A, R)$
$|P(A \times B)| = 2^{n^2}$

③. Give examples of (homogeneous) relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.
(r) (t) (s)

(t): $(\mathbb{N}, \mathbb{N}, <) = r$

(s): $r = (D, D, \perp)$, where D = set of lines in plane $= \{d \in P\}$

(r): $r = (M, M, R)$, $M = \{1, 2, 3\}$, $R = \{(1,1); (2,2); (3,3); (2,3); (1,2)\}$

§ Only with homogeneous relations can we talk about (r), (t) and (s).

④. Which ones of the properties (r), (t), (s) hold for the following homogeneous relations:

a) $>, <$ on $\mathbb{R}$ – (t)

b) $\vdots$ on $\mathbb{N}, \mathbb{Z}$ – (r), (t), ~~(s)~~

c) $\perp$ of lines in space – ~~(t)~~, (s)

d) $\parallel$ of lines in space – (s), ~~(t)~~, ~~(r)~~

e) $\equiv$ of triangles – (s), (t), (r) $\Rightarrow$ equivalence

f) $\sim$ of triangles – (s), (t), (r) $\Rightarrow$ equivalence

⑤. $M = \{1, 2, 3, 4\}$

r_1, r_2 – homogeneous relations on M

π_1, π_2

$R_1 = \Delta_M \cup \{(1,2); (2,1); (1,3); (3,1); (2,3); (3,2)\}$

$R_2 = \Delta_M \cup \{(1,2); (1,3)\}$

$\pi_1 = \{\{1\}, \{2\}, \{3, 4\}\}$

$\pi_2 = \{\{4\}, \{1, 2\}, \{3, 4\}\}$

a) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.

$\Delta_M = \{(1,1); (2,2); (3,3); (4,4)\}$

r_1 – equivalence

equivalence class $r\langle 1\rangle = \{x \in M \mid 1 r x\} = \{1, 2, 3\}$

$r\langle 2\rangle = \{1, 2, 3\}$

$r\langle 3\rangle = \{1, 2, 3\}$

$r\langle 4\rangle = \{4\}$

$\pi_1 = \{\{1, 2, 3\}, \{4\}\}$

r_2 - not an equivalence (not symmetric because $(1,2)$ belongs to R_2 but not $(2,1)$)

b) $\pi_1 = \{\{1\}, \{2\}, \{3,4\}\}$ - it is a partition of M

$r\langle 1\rangle = \{1\} \Rightarrow 1 r 1$

$r\langle 2\rangle = \{2\} \Rightarrow 2 r 2$

$r\langle 3\rangle = r\langle 4\rangle \Rightarrow 3r3,\ 3r4,\ 4r3,\ 4r4$

$= \{3,4\}$

$R_{\pi_1} = \Delta_M \cup \{(3,4), (4,3)\}$

$\pi_2 = \{\{1\}, \{1,2\}, \{3,4\}\}$

$\{1\} \cap \{1,2\} = \{1\} \neq \emptyset \Rightarrow \pi_2$ - not a partition of M

⑥. $z_1 r z_2 \Leftrightarrow |z_1| = |z_2|$ $\qquad z_1, z_2 \in \mathbb{C}$

$z_1 s z_2 \Leftrightarrow \arg z_1 = \arg z_2$ or $z_1 = z_2 = 0$

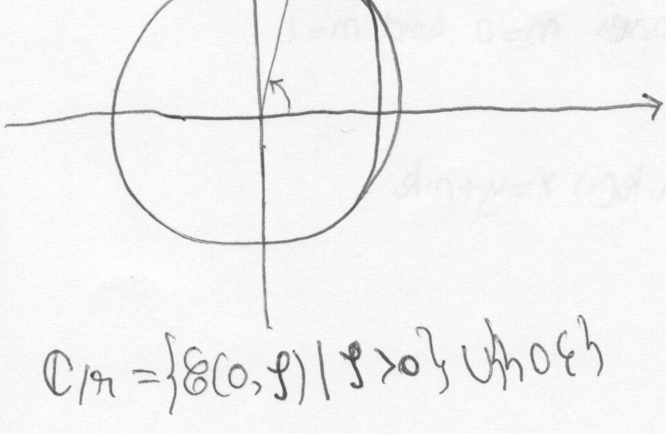
Prove that r, s - equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation)

(R): $z_1 r z_1 \Leftrightarrow |z_1| = |z_1|$ (T)

(T): $\left.\begin{array}{l} z_1 r z_2 \\ z_2 r z_3 \end{array}\right\} \Rightarrow z_1 r z_3$ (T) $\qquad \Rightarrow r$ - is equivalence

(S): $z_1 r z_2 \Leftrightarrow z_2 r z_1$ (T)

s - is equivalence (equality ~~for~~ between real numbers)



Let $z \in \mathbb{C}$, $z = \rho(\cos\varphi + i\sin\varphi)$

~~$\arg\varphi =$~~

$tg\,\varphi = \frac{y}{x}$

Let $z \in \mathbb{C}$, $|z| = \rho > 0$

$r\langle z\rangle = \{w \in \mathbb{C} \mid z r w\} =$

$= \{w \in \mathbb{C} \mid |w| = |z| = \rho\}$

$= \mathcal{C}(0, \rho)$

$\mathbb{C}/r = \{\mathcal{C}(0,\rho) \mid \rho > 0\} \cup \{\{0\}\}$

$\mathbb{C}/_{\Delta} = \{ \Delta\langle z\rangle = \{w \in \mathbb{C} \mid z \Delta w\} = \{w \in \mathbb{C} \mid \arg z = \arg w\} =$
$= \{w \in \mathbb{C} \mid \arg w = \varphi\}$

$\varphi = (0, 2\pi]$

$\mathbb{C}/_{\Delta} = \{\{w \in \mathbb{C} \mid \arg w = \varphi\} \mid \varphi \in (0, 2\pi]\} \cup \{\{0\}\}$

(⑦ Let $m \in \mathbb{N}$. Consider the relation)

⑧ Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

$\pi_1 = \{\{1, 2, 3\}\}$

$\pi_2 = \{\{1\}, \{2, 3\}\}$

$\pi_3 = \{\{1, 2\}, \{3\}\}$

$\pi_4 = \{\{1, 3\}, \{2\}\}$

$\pi_5 = \{\{1\}, \{2\}, \{3\}\}$

$\Delta_M = \{(1,1); (2,2); (3,3)\}$

$R_{\pi_1} = \Delta_M \cup \{(1,2); (2,1); (1,3); (3,1); (2,3); (3,2)\}$ – universal relation $= M \times M$

$R_{\pi_2} = \Delta_M \cup \{(2,3); (3,2)\}$

$R_{\pi_3} = \Delta_M \cup \{(1,2); (2,1)\}$

$R_{\pi_4} = \Delta_M \cup \{(1,3); (3,1)\}$

$R_{\pi_5} = \Delta_M$ – equality relation

⑦ Let $m \in \mathbb{N}$. Consider the relation ρ_m on $\mathbb{Z}$

ρ_m – congruence modulo m, defined by: $x \rho_m y \Leftrightarrow m \mid (x - y)$

Prove that ρ_m is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_m$. Discuss the cases $m = 0$ and $m = 1$.

$\mathbb{Z}_m = \{\hat{0}, \hat{1}, \hat{2}, \ldots, \widehat{(m-1)}\}$

$x \rho_m y \Leftrightarrow m \mid (x - y) \Leftrightarrow \exists\, k \in \mathbb{N}$ s.t. $x - y = m \cdot k \Leftrightarrow x = y + mk$

$\Rightarrow m = 0$ – equality relation

$m = 1$ – universal relation

17.10.2016

Algebră

Seminar 3

①. $S_M = \{f : M \to M \mid f\text{-bijective}\}$. Show that $(S_M, \circ)$-group, called the <u>symmetric group of M</u>

$M \neq \emptyset$

1) Stable part

$(\forall) f, f' \in S_M \Rightarrow f \circ f' \in S_M$ (true)

2) ~~Associativity~~

$(\forall) f, f', f'' \in S_M \Rightarrow (f \circ f') \circ f'' = f \circ (f' \circ f'')$ (true) (for all functions)

3) Identity element

$\exists e \in S_M$ s.t. $e \circ f = f \circ e = f$, $(\forall) f \in S_M$

$e : M \to M$, $e(x) = x = 1_M$ - the identity function

4) Symmetry

$(\forall) f \in S_M$, $(\exists) f' \in S_M$ s.t. $f \circ f' = f' \circ f = e$

$f' = f^{-1}$ - the inverse of f

$\Rightarrow (S_M, \circ)$ - group

② $M \neq \emptyset$, $(R, +, \cdot)$ - ring

$R^M = \{f \mid f : M \to R\}$. On R^M we define $f+g : M \to R$, $(f+g)(x) = f(x) + g(x)$

$f \cdot g : M \to R$, $(f \cdot g)(x) = f(x) \cdot g(x)$, $(\forall) x \in M$

Show $(R^M, +, \cdot)$ - ring. If R - commutative or has identity element, is R^M the same? yes

(if „$\cdot$" is comm)

• $(R^M, +)$ - abelian group

1) ~~Associativity~~

$(\forall) f, g, h \in R^M \Rightarrow [(f+g)+h](x) = [f+(g+h)](x)$

~~$[(f+f')+f''](x) = f+f'+f''$~~ $[f(x)+g(x)]+h(x) = f(x)+[g(x)+h(x)] \Rightarrow$ True

2) <u>Commutativity</u>

$(f+g)(x) = (g+f)(x)$, $(\forall) f, g \in R^M$ and $(\forall) x \in M$

$f(x)+g(x) = g(x)+f(x) \Rightarrow$ True

3) Identity element

$\exists e \in R^M$ s.t. $(f+e)(x) = (e+f)(x) = f(x)$, $(\forall) f \in R^M$, $(\forall) x \in M$

$e : M \to R$

Let 0 be the identity element in $(R, +) \Rightarrow e(x) = 0$, $(\forall) x \in M$

4) Symmetry

$(\forall) f: M \to \mathbb{R}, (\exists) f^{-1}: M \to \mathbb{R}$ s.t. $(f + f^{-1})(x) = (f^{-1} + f)(x) = e(x), (\forall) x \in M$

Let $-f(x)$ be the symetric of $f(x)$ in $(\mathbb{R}, +) \Rightarrow f'(x) = -f(x), (\forall) x \in M$

• $(\mathbb{R}^M, \cdot)$ – semigroup

• Associativity

$(\forall) f, g, h \in \mathbb{R}^M \Rightarrow [(f \cdot g) \cdot h](x) = [f \cdot (g \cdot h)](x) \quad (\forall) x \in M$

$[f(x) \cdot g(x)] \cdot h(x) = f(x) \cdot [g(x) \cdot h(x)]$ – true, because $(\mathbb{R}, \cdot)$ – semigroup

• Distributive laws

$(\forall) f, g, h \in \mathbb{R}^M \quad [f \cdot (g + h)](x) = (f \cdot g + f \cdot h)(x), (\forall) x \in M$

$[(f + g) \cdot h](x) = (f \cdot h + g \cdot h)(x), \forall x \in M$

③. $H = \{z \in \mathbb{C} \mid |z| = 1\}$ $\quad$ H – subgroup of $(\mathbb{C}^*, \cdot)$, but not of $(\mathbb{C}, +)$

1) $H \neq \emptyset$ $(1 \in H)$

2) $(\forall) z, w \in H \Rightarrow z \cdot w^{-1} \in H$

1) $1 \in H$ because $|1| = 1$

2) $z, w \in H \Rightarrow |z| = |w| = 1$

$|z \cdot w^{-1}| = |z| \cdot |w^{-1}| = |z| \cdot \left|\frac{1}{w}\right| = 1 \cdot 1 = 1 \Rightarrow z \cdot w^{-1} \in H$

$\Rightarrow H$ – subgroup of $\mathbb{C}^*$, $(\mathbb{C}^*, \cdot)$

$e = 0 \notin H \Rightarrow H$ is not a subgroup of $(\mathbb{C}, +)$

④ $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$ $(n \in \mathbb{N}^*)$ – the set of n-th roots of unity.

Prove that U_n – subgroup of $(\mathbb{C}^*, \cdot)$

1) $U_n \neq \emptyset$

2) $(\forall) z, w \in U_n \Rightarrow z \cdot w^{-1} \in U_n$

1) $1 \in U_n$, because $1^n = 1$

2) $z, w \in U_n \Rightarrow z^n = w^n = 1$

$(z \cdot w^{-1})^n = \left(z \cdot \frac{1}{w}\right)^n = z^n \cdot \frac{1}{w^n} = 1 \cdot 1 = 1$

$\Rightarrow U_n \leq \mathbb{C}^*, (\mathbb{C}^*, \cdot)$

⑤ $m \in \mathbb{N}, m \geq 2$. Prove that:

i) $GL_m(\mathbb{C}) = \{A \in M_m(\mathbb{C}) \mid \det(A) \neq 0\}$ - stable subset of $(M_m(\mathbb{C}), \cdot)$ - monoid

$(\forall) A, B \in GL_m(\mathbb{C}) \Rightarrow A \cdot B \in GL_m(\mathbb{C})$

$\det(A \cdot B) = \underbrace{\det A}_{\neq 0} \cdot \underbrace{\det B}_{\neq 0} \neq 0 \Rightarrow A \cdot B \in GL_m(\mathbb{C})$

ii) $(GL_m(\mathbb{C}), \cdot)$ - group (general linear group of rank m)

• Assoc. law $(\forall) A, B, C \in GL_m(\mathbb{C}) \Rightarrow (A \cdot B) \cdot C = A \cdot (B \cdot C)$ (True)

• Id. el $(\exists) E \in GL_m(\mathbb{C})$ s.t. $E \cdot A = A \cdot E = A$, $(\forall) A \in GL_m(\mathbb{C})$

$E = I_m \in GL_m(\mathbb{C})$

• Inv. law $(\forall) A \in GL_m(\mathbb{C})$ $\exists A^{-1} \in GL_m(\mathbb{C})$ s.t. $A \cdot A^{-1} = A^{-1} \cdot A = I_m$

$\det(A) \neq 0 \Rightarrow \exists A^{-1}$ ~~$\in GL_m$~~, $\det(A^{-1}) \neq 0 \Rightarrow A^{-1} \in GL_m(\mathbb{C})$

$\Rightarrow (GL_m(\mathbb{C}), \cdot)$ - group

iii) $SL_m(\mathbb{C}) = \{A \in M_m(\mathbb{C}) \mid \det A = 1\}$ - subgroup of $(GL_m(\mathbb{C}), \cdot)$ - group

$SL_m(\mathbb{C}) \neq \emptyset$ $(I_m \in SL_m(\mathbb{C}))$

$(\forall) A, B \in SL_m(\mathbb{C}) \Rightarrow AB^{-1} \in SL_m(\mathbb{C})$

$\det(A \cdot B^{-1}) = \det A \cdot \det B^{-1} = 1 \cdot 1 = 1 \Rightarrow A \cdot B^{-1} \in SL_m(\mathbb{C})$

⑥ i) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ in $(\mathbb{C}, +, \cdot)$

ii) $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$ in $(M_2(\mathbb{R}), +, \cdot)$

Show they are subrings of the given rings.

i) a) $\mathbb{Z}[i] \neq \emptyset$ $(1 + 2i \in \mathbb{Z}[i])$

b) $(\forall) x, y \in \mathbb{Z}[i] \Rightarrow x - y \in \mathbb{Z}[i]$

~~$x - y =$~~ $x = a + bi$ $\quad a, b, c, d \in \mathbb{Z}$

$y = c + di$ $(-)$

$x - y = \underset{\in \mathbb{Z}}{a - c} + i(\underset{\in \mathbb{Z}}{b - d}) \Rightarrow x - y \in \mathbb{Z}[i]$

c) $\forall x, y \in \mathbb{Z}[i] \Rightarrow x \cdot y \in \mathbb{Z}[i]$

$x \cdot y = (a + bi)(c + di) = ac + adi + bci - bd = \underset{\in \mathbb{Z}}{(ac - bd)} + i\underset{\in \mathbb{Z}}{(ad + bc)} \Rightarrow x \cdot y \in \mathbb{Z}[i]$

$\Rightarrow \mathbb{Z}[i] \subseteq \mathbb{C}$, $(\mathbb{C}, +, \cdot)$

ii) a) $M \neq \emptyset$ $\left(\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in M \right)$

b) $\forall A, B \in M \Rightarrow A - B \in M$

$$A=\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad B=\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$

$$A-B=\begin{pmatrix} a-x & b-y \\ 0 & c-z \end{pmatrix} \in M$$

iii) $\forall A,B \in M \Rightarrow A\cdot B \in M$ ...

$\Rightarrow M \leq M_2(\mathbb{R}), (M_2(\mathbb{R}),+,\cdot)$

(7). i) Let $f: \mathbb{C}^* \to \mathbb{R}$, $f(z)=|z|$. Show that f-group homomorphism between $(\mathbb{C}^*,\cdot)$ and $(\mathbb{R}^*,\cdot)$

$$\boxed{f(z_1\cdot z_2)=f(z_1)\cdot f(z_2), \ (\forall) z_1,z_2 \in \mathbb{C}^*}$$

Let $z_1,z_2 \in \mathbb{C}^*$

$f(z_1\cdot z_2)=|z_1\cdot z_2|=|z_1|\cdot|z_2|=f(z_1)\cdot f(z_2)$ (True)

ii) $g:\mathbb{C}^* \to GL_2(\mathbb{R})$, $g(a+bi)=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that g-group homomorphism between $(\mathbb{C}^*,\cdot)$ and $(GL_2(\mathbb{R}),\cdot)$

$$g[\underbrace{(a+bi)}_{z_1}\cdot\underbrace{(c+di)}_{z_2}]=g(\underbrace{a+bi}_{z_1})\cdot g(\underbrace{c+di}_{z_2}), \ (\forall) z_1,z_2 \in \mathbb{C}^*$$

$$g(z_1\cdot z_2)=g(ac-bd+bci+adi)=\begin{pmatrix} ac-bd & bc+ad \\ -(bc+ad) & ac-bd \end{pmatrix}$$

$$g(a+bi)\cdot g(c+di)=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix}=\begin{pmatrix} ac-bd & bc+ad \\ -(bc+ad) & ac-bd \end{pmatrix}$$

$\Rightarrow g(z_1\cdot z_2)=g(z_1)\cdot g(z_2)$

(8) Let $m\in\mathbb{N}, m\geq 2$. Prove that the groups $(\mathbb{Z}_m,+)$ of residue class modulo m and $(U_m,\cdot)$ of m-th roots of unity are isomorphic.

(homomorphism + bijective)

$U_m=\{z\in\mathbb{C} \mid z^m=1\}$

$1=\cos 0+i\sin 0$

$$\boxed{z^m=1 \Leftrightarrow z_k=\cos\frac{2k\pi}{m}+i\sin\frac{2k\pi}{m}}, \ k\in\{0,1,\dots,m-1\}$$

$$\boxed{(\cos x+i\sin x)^m=\cos mx+i\sin mx} \text{ - Moivre}$$

$$z_k=\underbrace{\left(\cos\frac{2\pi}{m}+i\sin\frac{2\pi}{m}\right)}_{\varepsilon}^k=\varepsilon^k$$

$\mathbb{Z}_m=\{\hat{0},\hat{1},\hat{2},\dots,\widehat{m-1}\}$

$f:\mathbb{Z}_m\to U_m$, $f(\hat{k})=\varepsilon^k$

- f - bijective (1 to 1 correspondence, $card(U_n) = card(\mathbb{Z}_n)$) !!!
- $f(\hat{k}+\hat{l}) = f(\hat{k}) \cdot f(\hat{l})$

$\Leftrightarrow f(\widehat{k+l}) = f(\hat{k}) \cdot f(\hat{l})$

$\varepsilon^x = \varepsilon^k \cdot \varepsilon^l$

$(k+l) \equiv x \pmod{n}$, $x \in \{0, \dots, n-1\}$

24.10.2016

Algebră

Seminar 4

~~K - field~~

V is a K - vector space if $(V,+)$ abelian group

(L1) $k(v_1+v_2) = kv_1 + kv_2$, $(\forall) k \in K$ $(\forall) v_1, v_2 \in V$

(L2) $(k_1+k_2) \cdot v = k_1 \cdot v + k_2 v$, $\forall k_1, k_2 \in K$ $\forall v \in V$

(L3) $(k_1 \cdot k_2) \cdot v = k_1 (k_2 \cdot v)$ $\forall k_1, k_2 \in K$ $\forall v \in V$

(L4) $1 \cdot v = v$ $\forall v \in V$

① $(K[X], +)$ is an abelian group

(L1): $\forall k \in K$, $\forall f, g \in K[X]$

$f = a_0 + a_1 X + \dots + a_n X^n$

$g = b_0 + b_1 X + \dots + b_m X^m$

$m \geq n \Rightarrow f+g = (a_0+b_0) + X(a_1+b_1) + \dots + (a_n+b_n)X^n + \dots + b_m X^m$

$k(f+g) = k(a_0+b_0) + k(a_1+b_1)X + \dots + k(a_n+b_n)X^n + \dots + k \cdot b_m X^m$

$kf + kg = ka_0 + ka_1 X + \dots + ka_n X^n + kb_0 + kb_1 X + \dots + kb_m X^m$

$= k(a_0+b_0) + \dots + k(a_n+b_n)X^n + \dots + kb_m X^m$

$\Rightarrow k(f+g) = kf + kg$

(L2): $(k_1+k_2)f = (k_1+k_2)a_0 + (k_1+k_2)a_1 X + \dots + (k_1+k_2) a_n X^n$

$k_1 f + k_2 f = k_1 a_0 + k_1 a_1 X + \dots + k_1 a_n X^n + k_2 a_0 + k_2 a_1 X + \dots + k_2 a_n X^n$

$= (k_1+k_2)a_0 + (k_1+k_2)a_1 X + \dots + (k_1+k_2) a_n X^n$

$\Rightarrow (k_1+k_2)f = k_1 f + k_2 f$

(L3): $k_1(k_2 f) = k_1(k_2 a_0 + k_2 a_1 X + \dots + k_2 a_n X^n)$

$= k_1 k_2 a_0 + k_1 k_2 a_1 X + \dots + k_1 k_2 a_n X^n$

$(k_1 \cdot k_2) \cdot f = k_1 k_2 a_0 + k_1 k_2 a_1 X + \dots + k_1 k_2 a_n X^n$

$\Rightarrow (k_1 \cdot k_2) f = k_1 (k_2 f)$

(L4): $1 \cdot f = f$ (True) $\forall f \in K[X]$

$\Rightarrow (K[X], +, \cdot)$ is a K - vector space

② $m, n \in \mathbb{N}$, $m, n \geq 2$

Show that $(\mathcal{M}_{m,n}(K), +, \cdot)$ is a K-vector space

$(\mathcal{M}_{m,n}(K), +)$ - abelian group

$(L1)$: $k(A+B) = kA + kB$, $\forall k \in K$ $\forall A, B \in \mathcal{M}(m,n)$

~~$A = a_{11}\ a_{21} \ldots a_{mn}$~~

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

$$B = (b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

$$k(A+B) = k \cdot \left((a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} + (b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} \right) = k \cdot (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} + k \cdot (b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} =$$

$$= kA + kB$$

$(L2)$: $(k_1 + k_2) \cdot A = k_1 \cdot A + k_2 \cdot A$ $\forall \ldots$

$(L3)$: $k_1 \cdot (k_2 \cdot A) = (k_1 \cdot k_2) A$ $\forall \ldots$ (with full demonstration)

$(L4)$: $1 \cdot A = A$ $\forall \ldots$

③ We already solved it in a previous seminar, but with rings.

④ $V = \{x \in \mathbb{R} \mid x > 0\}$

$x \perp y = xy$

$x \top y = x^k$

Prove V - is a vector space over $\mathbb{R}$

$(L1)$ $k_1 \top (v_1 \perp v_2) = (k_1 \top v_1) \perp (k_1 \top v_2)$

$(L2)$ $(k_1 + k_2) \top v = (k_1 \top v) \perp (k_2 \top v)$

$(L3)$ $(k_1 \top k_2) \top v = k_1 \top (k_2 \top v)$

$(L4)$ $1 \top v = v$

$((0,\infty), \perp)$ - abelian group $\quad (1 \in (0,\infty))$

$\forall x, y \in (0,\infty) \Rightarrow x * y^{-1} \in (0,\infty)$ $\quad \frac{1}{y} = y^{-1} \in (0,\infty) \Rightarrow x * \frac{1}{y} \in (0,\infty)$ $\Big\} \Rightarrow$

~~Id. element $1 \in \mathbb{R}_+^*$~~

~~$y^{-1} \in (0,\infty)$~~

$\Rightarrow ((0,\infty),+)$ - subgroup of $(\mathbb{R}^*,\cdot) \Rightarrow$ abelian group

~~(L1) $kT(x_1 x_2) = x_1^k \perp x_2^k$~~

(L1) $kT(x_1 \perp x_2) = (kTx_1) \perp (kTx_2) \quad \forall k \in \mathbb{R},\ \forall x_1, x_2 \in (0,\infty)$

$\Leftrightarrow kT(x_1 x_2) = (x_1^k) \perp (x_2^k)$

$\Leftrightarrow (x_1 x_2)^k = x_1^k \cdot x_2^k$ (true)

(L2) $(k_1 + k_2)TX = (k_1 TX) \perp (k_2 TX) \quad \forall \ldots$

$\Leftrightarrow X^{k_1 + k_2} = X^{k_1} \cdot X^{k_2}$

$\Leftrightarrow X^{k+k_2} = X^{k_1} \cdot X^{k_2}$

pt numerele din $(\mathbb{R}, +, \cdot)$

(L3) $k_1 T(k_2 TX) = (k_1 \cdot k_2) TX$

$\Leftrightarrow (X^{k_2})^{k_1} = X^{k_1 \cdot k_2}$ (true) $\forall$

(L4) $1TX = X$

$X^1 = X, \quad \forall X \in (0,\infty)$ (true)

K-field, $(V,+,\cdot)$ K-vector space, $S \leq V$

$S \leq_K V \Leftrightarrow \begin{cases} 1. & S \neq \emptyset \\ 2. & \forall k_1, k_2 \in K,\ \forall v_1, v_2 \in S \Rightarrow k_1 v_1 + k_2 v_2 \in S \end{cases}$

⑤ (i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\}$

$0 \in A \Rightarrow A \neq \emptyset$

$\forall k_1, k_2 \in \mathbb{R},\ \forall (x_1,y_1,z_1), (x_2,y_2,z_2) \in A \Rightarrow x_1 = 0,\ x_2 = 0$

$k_1(x_1,y_1,z_1) + k_2(x_2,y_2,z_2) = (\underbrace{k_1 x_1 + k_2 x_2}_{=0}, k_1 y_1 + k_2 y_2, k_1 z_1 + k_2 z_2) \in A$

$\Rightarrow A$ - subvector space

$A \leq_{\mathbb{R}} \mathbb{R}^3$

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$

$0 \in B \Rightarrow B \neq \emptyset$

Let $k_1 = k_2 = 1 \in \mathbb{R}$

$(0,1,1)$ and $(1,1,0) \in B$

$1 \cdot (0,1,1) + 1(1,1,0) = (1,2,1) \notin B$

$\Rightarrow B \not\leq_{\mathbb{R}} \mathbb{R}^3$

(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$

$0 \in C \Rightarrow C \neq \emptyset$

let $k_1 = \frac{1}{2}, k_2 = 0$

$(3,1,2)$ and $(2,0,0) \in B$

$\frac{1}{2}(3,1,2) + 0 \cdot (2,0,0) = (\underbrace{\tfrac{3}{2}}_{\notin \mathbb{Z}}, \frac{1}{2}, 1) \notin C \Rightarrow C \not\leq_{\mathbb{R}} \mathbb{R}^3$

(iv) $D = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$

$0 \in D \Rightarrow D \neq \emptyset$

$\forall k_1,k_2 \in \mathbb{R} \quad \forall (x_1,y_1,z_1),(x_2,y_2,z_2) \in D \Rightarrow S = k_1(x_1,y_1,z_1) + k_2(x_2,y_2,z_2) \in D$

$x_1+y_1+z_1 = 0 \quad | \cdot k_1$

$x_2+y_2+z_2 = 0 \quad | \cdot k_2 \qquad (+)$

$\cancel{k_1(x_1,y_1,z_1) + k_2(x_2,y_2,z_2)} \Rightarrow k_1x_1 + k_2x_2 + k_1y_1 + k_2y_2 + k_1z_1 + k_2z_2 = 0$

$S \in D \Rightarrow D \leq_{\mathbb{R}} \mathbb{R}^3$

(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=1\}$

$0 \notin E \Rightarrow (E,+)$ - is not an Abelian group

$\Rightarrow E \not\leq_{\mathbb{R}} \mathbb{R}^3$

(vi) $F = \{(x,y,z) \in \mathbb{R}^3 \mid x=y=z\}$

$0 \in F \Rightarrow F \neq \emptyset$

$\forall k_1,k_2 \in \mathbb{R} \quad \forall (x_1,y_1,z_1),(x_2,y_2,z_2) \in F \Rightarrow S = k_1(x_1,y_1,z_1) + k_2(x_2,y_2,z_2) \in F$

$x_1 = y_1 = z_1 = x$

$x_2 = y_2 = z_2 = y$

$S = (k_1x + k_2y, k_1x + k_2y, k_1x + k_2y) \in F$

$\Rightarrow F \leq_{\mathbb{R}} \mathbb{R}^3$

6. (i) $10 \cdot 1 = 10 \notin [-1,1]$

(ii) $(100, 200) \Rightarrow 100^2 + 200^2 \leq 1$ (F)

(iii) yes and no

(iv) yes

⑦ $m \in \mathbb{N}$

(i) $K_m[X] = \{ f \in K[X] \mid \text{degree}(f) \leq m \}$

$K_m[X] \neq \emptyset \quad (f = 0 \in K_m[X])$
$(\text{degree}(f) = -\infty)$

$\forall k_1, k_2 \in K \quad \forall f, g \in K_m[X] \Rightarrow k_1 f + k_2 g \in K_m[X]$

$\text{degree}(f) \leq m$
$\text{degree}(g) \leq m$
$\text{degree}(k_1 f) \leq m$
$\text{degree}(k_2 f) \leq m$

$\Leftrightarrow \text{degree}(f+g) \leq m \Rightarrow \text{degree}(k_1 f + k_2 g) \leq m$

$\Rightarrow K_m[X] \leq_K K[X]$

(ii) $K'_m[X] = \{ f \in K[X] \mid \text{degree}(f) = m \}$

Let $f = a_0 + a_1 X + \dots + a_{m-1} X^{m-1} + 1 \cdot X^m$

$g = b_0 + b_1 X + \dots + b_{m-1} X^{m-1} + (-1) X^m$

$k_1 = k_2 = 1$

$k_1 \cdot f + k_2 \cdot g = (a_0 + b_0) + \dots + (a_{m-1} + b_{m-1}) X^{m-1} = h$

$\Rightarrow \text{degree}(h) \neq m$

$\Rightarrow K'_m[X] \not\leq_K K[X]$

⑧ $S = \{ (x, y) \in \mathbb{R}^2 \mid a_{11} x + a_{12} y = 0 \text{ and } a_{21} x + a_{22} y = 0 \}$

$0 \in S \Rightarrow S \neq \emptyset$

$\forall k_1, k_2 \in \mathbb{R} \quad \forall (x_1, y_1), (x_2, y_2) \in S \Rightarrow k_1 (x_1, y_1) + k_2 (x_2, y_2) \in S$

31.10.2016

Algebră

Seminar 5

①. V-vector space over K, $x_1, \dots, x_m \in V$

$\langle x_1, x_2, \dots, x_m \rangle = \{k_1x_1 + k_2x_2 + \dots + k_mx_m \mid k_i \in K, i = \overline{1,m}\}$

(i) $\langle 1, X, X^2 \rangle$, $\mathbb{R}[X]$

$\langle 1, X, X^2 \rangle = \{k_1 + k_2X + k_3X^2 \mid k_1, k_2, k_3 \in \mathbb{R}\}$

$= \{f \in \mathbb{R}[X] \mid \text{degree}(f) \leq 2\}$

$= \mathbb{R}_2[X]$

(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = S$, $\mathcal{M}_2(\mathbb{R})$

$S = \left\{ a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$

$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$

$= \mathcal{M}_2(\mathbb{R})$

②. (i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$

$\cancel{= \{(0, y, z) \in \mathbb{R}^3}$

$= \{(0, y, z) \mid y, z \in \mathbb{R}\}$

$= \{(0, y, 0) + (0, 0, z) \mid y, z \in \mathbb{R}\}$

$= \{y(0, 1, 0) + z(0, 0, 1) \mid y, z \in \mathbb{R}\}$

$= \langle (0, 1, 0), (0, 0, 1) \rangle$

(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

$= \{(-y-z, y, z) \mid y, z \in \mathbb{R}\}$

$= \{(-y, y, 0) + (-z, 0, z) \mid y, z \in \mathbb{R}\}$

$= \{y(-1, 1, 0) + z(-1, 0, 1) \mid y, z \in \mathbb{R}\}$

$= \langle (-1, 1, 0), (-1, 0, 1) \rangle$

(iii) $C=\{(x,y,z)\in\mathbb{R}^3 \mid x=y=z\}$

$=\{(x,x,x) \mid x\in\mathbb{R}\}$

$=\{x(1,1,1) \mid x\in\mathbb{R}\}$

$=\langle(1,1,1)\rangle$

3. $S=\{(x,y,z)\in\mathbb{R}^3 \mid x+y+z=0\}$

$T=\{(x,y,z)\in\mathbb{R}^3 \mid x=y=z\}$

Prove that S,T - subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3=S\oplus T$

$$\boxed{\begin{array}{l}\underline{\text{Th}}\ V\text{- vector space over } K,\ S,T\leq_K V \\ \Rightarrow V=S\oplus T \Leftrightarrow \forall v\in V,\ \exists!\ s\in S,\ t\in T \text{ s.t. } v=s+t\end{array}}$$

$\forall v\in\mathbb{R}^3,\ v=(x,y,z)$ $\quad s=(a,b,-a-b)\in S$

$t=(c,c,c)\in T$

$?\ s\in S,\ t\in T \Rightarrow v=s+t$

$\Leftrightarrow (x,y,z)=(a+c,\ b+c,\ -a-b+c)$

$$\begin{cases} a+c=x \\ b+c=y \\ -a-b+c=z \end{cases} \Rightarrow \cancel{\begin{cases} a=x-c \\ b=y-c \end{cases}}$$

$(+)$

$3c=x+y+z \Rightarrow c=\dfrac{x+y+z}{3}$ $\quad \forall x,y,z\in\mathbb{R}$

$a=x-\dfrac{x+y+z}{3}=\dfrac{2x\cancel{+y+z}}{3}$

$b=\dfrac{2y\cancel{+x+z}}{3}$

$$(x,y,z)=\underbrace{\left(\frac{2x-y-z}{3},\ \frac{2y-x-z}{3},\ \frac{-x-y+2z}{3}\right)}_{s}+\underbrace{\left(\frac{x+y+z}{3},\ \frac{x+y+z}{3},\ \frac{x+y+z}{3}\right)}_{t}$$

$\Rightarrow \mathbb{R}^3=S\oplus T$

<u>Def</u>: ~~$V=S\oplus T$~~

$V=S+T=\{s+t \mid s\in S,\ t\in T\}$

$\oplus$ if $S\cap T=\{0\}$

④ $S = \{ f \in \mathbb{R}^{\mathbb{R}} \mid f(-x) = f(x),\ \forall x \in \mathbb{R} \}$

$T = \{ g \in \mathbb{R}^{\mathbb{R}} \mid g(-x) = -g(x),\ \forall x \in \mathbb{R} \}$

I $f : \mathbb{R} \to \mathbb{R},\ f(x) = 0 \Rightarrow f \in S \Rightarrow S \neq \emptyset$

$g : \mathbb{R} \to \mathbb{R},\ g(x) = 0 \Rightarrow g \in T \Rightarrow T \neq \emptyset$

II $\forall f, g \in S,\ \forall k_1, k_2 \in \mathbb{R} \Rightarrow k_1 f + k_2 g \in S$

$(k_1 f + k_2 g)(x) = k_1 f(x) + k_2 g(x) = k_1 \cdot f(-x) + k_2 g(-x) = (k_1 f + k_2 g)(-x) \Rightarrow$

$\Rightarrow k_1 f + k_2 g \in S \quad \forall f, g \in S$
$\forall k_1, k_2 \in \mathbb{R}$

Similar for odd functions

I, II $\Rightarrow S, T$ - subspaces

$S \leq_{\mathbb{R}} \mathbb{R}^{\mathbb{R}},\ T \leq_{\mathbb{R}} \mathbb{R}^{\mathbb{R}}$

Prove $\mathbb{R}^{\mathbb{R}} = S \oplus T$

$\mathbb{R}^{\mathbb{R}} = S + T$

$S + T = \{ f + g \mid f \in S, g \in T \}$

~~$\forall h \in \mathbb{R}^{\mathbb{R}} \quad h = f + g$~~

$\forall h \in \mathbb{R}^{\mathbb{R}}$ Find $f \in S, g \in T$ s.t $h = f + g$

$h(x) = f(x) + g(x) \qquad \forall x \in \mathbb{R}$

$h(-x) = f(x) - g(x) \qquad (+), (-)$

$f(x) = \dfrac{h(x) + h(-x)}{2} \qquad \forall x \in \mathbb{R}$

$g(x) = \dfrac{h(x) - h(-x)}{2}$

Prove $S \cap T = \{0\}$

Let $f \in S \cap T \Rightarrow \left.\begin{array}{l} f(x) = f(-x) \\ f(x) = -f(x) \end{array}\right\} \Rightarrow f(x) = -f(x)$

$\Rightarrow f(x) = 0$

$\Rightarrow S \cap T = \{0\}$

<u>Remark</u>: $ch\, x = \dfrac{e^x + e^{-x}}{2}$ - even

$sh\, x = \dfrac{e^x - e^{-x}}{2}$ - odd

- hyperbolic functions

⑤ **Def** V, V' over K, $f: V \to V'$

f - K-linear map $\overset{def}{\Leftrightarrow}$ $\forall k \in K$ $\forall v_1, v_2 \in V$

$$f(v_1+v_2) = f(v_1) + f(v_2)$$
$$f(kv_1) = k f(v_1)$$

<u>Theorem</u> f K-linear map $\Leftrightarrow$ $\forall k_1, k_2 \in K$ $\forall v_1, v_2 \in V$

$$f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)$$

Let $f, g: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x,y) = (x+y, x-y)$

$$g(x,y) = (2x-y, 4x-2y)$$

$h: \mathbb{R}^3 \to \mathbb{R}^3$, $h(\cancel{x}, y, z) = (x-y, y-z, z-x)$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$

↑ endomorphism (K-linear map)

Applying the def

$\forall k \in \mathbb{R}$ $\forall \underbrace{(x,y)}_{v_1}, \underbrace{(a,b)}_{v_2} \in \mathbb{R}^2$ ~~$f(x_1$~~

$\Rightarrow f(v_1+v_2) = f(x+a, y+b) = (x+y+a+b, x+a-y-b)$

$f(v_1) + f(v_2) = \cancel{(x,y)+(a,b)}\ (x+y, x-y) + (a+b, a-b) = (x+y+a+b, x+a-y-b)$

$\Rightarrow f(v_1+v_2) = f(v_1) + f(v_2)$ ⓣ

$f(kv_1) = (kx+ky, kx-ky)$

$kf(v_1) = k(x+y, x-y) = (kx+ky, kx-ky)$

$\Rightarrow f(kv_1) = k(f(v_1))$ ⓣ

$\Rightarrow f \in End_{\mathbb{R}}(\mathbb{R}^2)$

Applying the th.

$\forall k_1, k_2 \in \mathbb{R}$ $\quad \forall v_1 v_2 \in \mathbb{R}^2$ $\quad g(k_1 v_1 + k_2 v_2) = k_1 g(v_1) + k_2 g(v_2)$

$v_1 = (x,y)$

$v_2 = (a,b)$

$g(k_1 v_1 + k_2 v_2) = (2k_1 x + 2k_2 a - k_1 y - k_2 b, 4(k_1 x + k_2 a) - 2(k_1 y + k_2 b))$

$k_1 v_1 + k_2 v_2 = (k_1 x, k_1 y) + (k_2 a, k_2 b) = (k_1 x + k_2 a, k_1 y + k_2 b)$

$k_1 g(v_1) + k_2 g(v_2) = k_1(2x-y, 4x-2y) + k_2(2a-b, 4a-2b) = (2k_1 x - k_1 y + 2k_2 a - k_2 b, 4k_1 x - 2k_1 y + 4k_2 a - 2k_2 b)$

$\Rightarrow g(k_1 v_1 + k_2 v_2) = k_1 g(v_1) + k_2 g(v_2)$ ⓣ $\Rightarrow$ $g \in End_{\mathbb{R}}(\mathbb{R}^2)$

(7). Determine the kernel and the image of the endomorphisms from ex 5.

(D) $f : V \to V'$, f K-linear map

$\operatorname{Ker} f = \{ x \in V \mid f(x) = 0' \} \leq V$

$\operatorname{Im} f = \{ f(x) \mid x \in V \} \leq V'$

(T) $X \subseteq V$, $f(\langle X \rangle) = \langle f(X) \rangle$

$f(x,y) = (x+y, x-y)$

$f(x,y) = (0,0) \Rightarrow \begin{cases} x+y=0 \\ x-y=0 \end{cases} \Rightarrow x=y=0 \Rightarrow \operatorname{Ker} f = \{(0,0)\}$

$\operatorname{Im} f = \{(x+y, x-y) \mid x,y \in \mathbb{R}\} = \{x(1,1)+y(1,-1) \mid x,y \in \mathbb{R}\} = \langle (1,1),(1,-1) \rangle =$
$= \mathbb{R}^2$

$g(x,y) = (2x-y, 4x-2y)$

$g(x,y) = (0,0) \Rightarrow \begin{cases} 2x-y=0 \\ 4x-2y=0 \end{cases} \Rightarrow y = 2x$

$\operatorname{Ker} g = \{(x,2x) \mid x \in \mathbb{R}\} = \langle (1,2) \rangle$

$\operatorname{Im} g = \{ \underbrace{(2x-y)}_{\in \mathbb{R}} \cdot (1,2) \mid x,y \in \mathbb{R} = \langle (1,2) \rangle$

$h(x,y,z) = (x-y, y-z, z-x)$

$h(x,y,z) = (0,0,0) \Rightarrow \begin{cases} x-y=0 \\ y-z=0 \\ z-x=0 \end{cases} \Rightarrow x=y=z$

$\operatorname{Ker} h = \{ x\cdot(1,1,1) \mid x \in \mathbb{R}\} = \langle (1,1,1) \rangle$

~~$\operatorname{Im} h = \{ x(1,1,1) \mid x \in \mathbb{R}\}$~~

$\operatorname{Im} h = \{(x-y, y-z, z-x) \mid x,y,z \in \mathbb{R}\}$
$= \{ x\cdot(1,0,-1) + y(-1,1,0) + z(0,-1,1) \mid x,y,z \in \mathbb{R}\}$
$= \langle (1,0,-1), (-1,1,0), (0,-1,1) \rangle$

⑧. V - vector space over K , $f \in End_K(V)$

Show that the set $S = \{x \in V \mid f(x) = x\}$ of fixed points of f is a subspace of V

I $S \neq \emptyset$

$f \in End_K(V) \Rightarrow \forall k \in K, \forall v \in V$

$f(kv) = k f(v)$

$k = 0 \Rightarrow f(0) = 0 \Rightarrow 0 \in S \Rightarrow S \neq \emptyset$

II $\forall k_1, k_2 \in K \ \forall x_1, x_2 \in S$

$\Rightarrow k_1 x_1 + k_2 x_2 \in S$

$x_1 \in S \Rightarrow f(x_1) = x_1$

$x_2 \in S \Rightarrow f(x_2) = x_2$

$f(k_1 x_1 + k_2 x_2) = k_1 f(x_1) + k_2 f(x_2) = k_1 x_1 + k_2 x_2$

$\uparrow$ $f \in End_K(V)$ $\uparrow$ $x_1 \in S$, $x_2 \in S$

Algebră

Seminar 6

7.11.2016

① $v_1=(1,-1,0)$

$v_2=(2,1,1) \quad \in \mathbb{R}^3$

$v_3=(1,5,2)$

Prove that: i) v_1, v_2, v_3 - are linearly dependent + ~~det.~~ a dependence relationship

ii) v_1, v_2 - linearly independent

i) v_1, v_2, v_3 - linearly dependent $\Leftrightarrow v_3 = k_1 v_1 + k_2 v_2$, $k_1, k_2 \in \mathbb{R}$

$(1,5,2) = k_1(1,-1,0) + k_2(2,1,1)$

$(1,5,2) = (k_1, -k_1, 0) + (2k_2, k_2, k_2)$

$\begin{cases} k_1+2k_2=1 \\ k_2-k_1=5 \\ k_2=2 \end{cases} \Rightarrow \begin{cases} k_1=-3 \\ k_2=2 \end{cases} \Rightarrow$ we found $k_1=-3$ and $k_2=2 \Rightarrow$

$\Rightarrow v_1, v_2, v_3$ are linearly dependent

ii) $k_1 v_1 + k_2 v_2 = 0 \overset{?}{\Rightarrow} k_1 = k_2 = 0$

~~We close~~

$k_1(1,-1,0)+k_2(2,1,1)=(0,0,0)$

$(k_1+2k_2, k_2-k_1, k_2) = (0,0,0)$

$\begin{cases} k_1+2k_2=0 \\ k_2-k_1=0 \\ k_2=0 \end{cases} \Rightarrow k_1=k_2=0 \Rightarrow v_1, v_2$ - linearly independent

② Prove that the following vectors are linearly independent:

i) $v_1=(1,0,2)$, $v_2=(-1,2,1)$, $v_3=(3,1,1) \quad \in_{\mathbb{R}} \mathbb{R}^3$

ii) $v_1=(1,2,3,4)$, $v_2=(2,3,4,1)$, $v_3=(3,4,1,2)$, $v_4=(4,1,2,3) \quad \in_{\mathbb{R}} \mathbb{R}^4$

i) v_1, v_2, v_3 - li. ind. $\overset{\text{Th } n \text{ vectors in } K^n}{\Longleftrightarrow} \begin{vmatrix} 1 & -1 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} \neq 0$

$\Leftrightarrow -13 \neq 0$ (True)

ii) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = 5 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & -2 & -2 \\ 0 & -1 & -1 & -1 \end{vmatrix} = \cancel{5 \cdot \begin{vmatrix} 1 & 1 & -3 \\ 2 & & \end{vmatrix}}$

$= 5 \cdot (-2) \cdot (-1) \cdot \begin{vmatrix} 1 & 1 & -3 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 10 \cdot 8 = 80 \neq 0$

$\Rightarrow v_1, v_2, v_3, v_4$ – lin. indep.

4) $v_1 = (1, -2, 0, -1)$

$v_2 = (2, 1, 1, 0) \quad \in_{\mathbb{R}} \mathbb{R}^4$

$v_3 = (0, a, 1, 2)$

$a \in \mathbb{R}$, $a = ?$ s.t. v_1, v_2, v_3 – lin. dep.

v_1, v_2, v_3 – lin dep $\Leftrightarrow v_3 = k_1 v_1 + k_2 v_2$, $k_1, k_2 \in \mathbb{R}$

$(0, a, 1, 2) = k_1(1, -2, 0, -1) + k_2(2, 1, 1, 0)$

$(0, a, 1, 2) = (k_1, -2k_1, 0, -k_1) + (2k_2, k_2, k_2, 0)$

$\begin{cases} k_1 + 2k_2 = 0 \\ k_2 - 2k_1 = a \\ k_2 = 1 \\ -k_1 = 2 \end{cases} \Rightarrow \begin{cases} k_1 = -2 \\ k_2 = 1 \end{cases}$

$\Rightarrow a = -2 - 2 = -4$

3) cu det $\neq 0$

⑤ $v_1 = (1,1,0)$

$v_2 = (-1,0,2)$ $\quad \in_{\mathbb{R}} \mathbb{R}^3$

$v_3 = (1,1,1)$

i) Show that the list (v_1, v_2, v_3) is a basis ~~for~~ of ~~the~~ $\mathbb{R}^3$

v_1, v_2, v_3 - lin. indep. $\Leftrightarrow D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} \neq 0$

(v_1, v_2, v_3) - basis $\overset{Th}{\Leftrightarrow}$ v_1, v_2, v_3 - lin. indep.

$D = 0 + 2 + 0 - 0 - 2 + 1 = 1 \neq 0 \Rightarrow v_1, v_2, v_3$ - lin. indep. $\Rightarrow (v_1, v_2, v_3)$ - basis

ii) Express the vectors of the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$ as a linear combination of v_1, v_2, v_3

$e_1 = (1,0,0)$

$e_2 = (0,1,0)$

$e_3 = (0,0,1)$

$$\begin{cases} v_1 = e_1 + e_2 \\ v_2 = -e_1 + 2e_3 \\ v_3 = e_1 + e_2 + e_3 \end{cases}$$

$v_3 - v_1 = e_3$

$v_2 = -e_1 + 2v_3 - 2v_1 \Rightarrow e_1 = 2v_3 - 2v_1 - v_2$

$v_1 = e_1 + e_2 \Rightarrow e_2 = v_1 - 2v_3 + 2v_1 + v_2 = 3v_1 + v_2 - 2v_3$

$$\begin{cases} e_1 = 2v_3 - 2v_1 - v_2 \\ e_2 = 3v_1 + v_2 - 2v_3 \\ e_3 = v_3 - v_1 \end{cases} \Leftrightarrow \begin{cases} e_1 = -2v_1 - v_2 + 2v_3 \\ e_2 = 3v_1 + v_2 - 2v_3 \\ e_3 = -v_1 + v_3 \end{cases}$$

iii) Det. the coordinates of $u=(1,-1,2)$ in each of the two bases

$u = e_1 - e_2 + 2e_3 = (1,-1,2)$

$u = -2v_1 - v_2 + 2v_3 - 3v_1 - v_2 + 2v_3 + -2v_1 + 3v_3$

$= -7v_1 - 2v_2 + 7v_3$

$= (-7,-2,7)$

⑥ $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Prove that (E_1, E_2, E_3, E_4), (A_1, A_2, A_3, A_4) - bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in the each of the two bases.

(E_1, E_2, E_3, E_4) - basis $\Leftrightarrow \begin{cases} E_1, E_2, E_3, E_4 \text{ - lin. ind.} \\ \langle E_1, E_2, E_3, E_4 \rangle = M_2(\mathbb{R}) \text{ - see Seminar 5} \end{cases}$ (Proved)

$k_1E_1 + k_2E_2 + k_3E_3 + k_4E_4 = 0 \overset{?}{\Leftrightarrow} k_1 = k_2 = k_3 = k_4 = 0$ (Obvious)

(A_1, A_2, A_3, A_4) - basis $\Leftrightarrow \begin{cases} \langle A_1, A_2, A_3, A_4 \rangle = M_2(\mathbb{R}) & 1. \\ A_1, A_2, A_3, A_4 \text{ - lin. ind.} & 2. \end{cases}$

1. ~~...~~

2. A_1, A_2, A_3, A_4 - lin. ind ~~...~~

$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = 0 \overset{?}{\Rightarrow} k_1 = k_2 = k_3 = k_4 = 0$

$$\begin{pmatrix} k_1+k_2+k_3+k_4 & k_2+k_3+k_4 \\ k_3+k_4 & k_1+k_4 \end{pmatrix} = 0$$

$$\begin{cases} k_1+k_2+k_3+k_4 = 0 & (1) \\ k_2+k_3+k_4 = 0 & (2) \\ k_3+k_4 = 0 & (3) \\ k_1+k_4 = 0 & (4) \end{cases}$$

$(1)-(2) \Rightarrow k_1 = 0 \Rightarrow k_4 = 0 \Rightarrow k_3 = 0 \Rightarrow k_2 = 0$

$\Rightarrow k_1 = k_2 = k_3 = k_4 = 0$

$\Rightarrow A_1, A_2, A_3, A_4$ - lin ind $\overset{Th}{\Rightarrow}$ (A_1, A_2, A_3, A_4) - basis

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$B = 2E_1 + E_2 + E_3 = (2, 1, 1, 0)$ with resp. to (E_1, E_2, E_3, E_4)

$B = k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1+k_2+k_3+k_4 & k_2+k_3+k_4 \\ k_3+k_4 & k_1+k_4 \end{pmatrix}$$

$$\begin{cases} k_1+k_2+k_3+k_4 = 2 & (1) \\ k_2+k_3+k_4 = 1 & (2) \\ k_3+k_4 = 1 & (3) \\ k_1+k_4 = 0 & (4) \end{cases}$$

$(1)-(2) \Rightarrow k_1 = 1$

$(4) \Rightarrow k_4 = -1$

$(3) \Rightarrow k_3 = 2$

$(2) \Rightarrow k_2 = 0$

$\Rightarrow B = A_1 + 2A_3 - A_4 = (1, 0, 2, -1)$ with resp. to (A_1, A_2, A_3, A_4)

7. Let $\mathbb{R}_2[X] = \{ f \in \mathbb{R}[X] \mid \text{degree}(f) \le 2 \}$

Show that the lists $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)$, $a \in \mathbb{R}$ are bases of $\mathbb{R}_2[X]$

and det. the coordinates of a polynomial $f = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$ in each basis.

$\mathbb{R}_2[X] = \langle 1, X, X^2 \rangle \Rightarrow E$ is a system of generators

$k_1 \cdot 1 + k_2 \cdot X + k_3 \cdot X^2 = 0 \Leftrightarrow k_1 = k_2 = k_3 = 0 \Rightarrow E$ - linear ind

$\Rightarrow E$ - basis

For B:

~~$k_1 \cdot 1 + k_2 \cdot X + k_3$~~ $k_0 \cdot 1 + k_1(X-a) + k_2(X-a)^2 = 0 \overset{?}{\Leftrightarrow} k_0 = k_1 = k_2 = 0$

$k_0 + k_1X - k_1a + k_2X^2 - 2k_2aX + k_2a^2 = 0$

$k_2X^2+(k_1-2k_2a)X+(k_0-k_1a+k_2a^2)=0$

$$\begin{cases} k_2=0 \\ k_1-2k_2a=0 \\ k_0-k_1a+k_2a^2=0 \end{cases} \Rightarrow k_0=k_1=k_2=0 \Rightarrow B\text{-lin. ind.}$$

B-has the same number of el. as E => also generator

=> B-basis

$f=a_0+a_1X+a_2X^2=(a_0,a_1,a_2)$ with resp to $(1,X,X^2)$

$f=a_0+a_1X+a_2X^2=k_0\cdot 1+k_1(X-a)+k_2(X-a)^2=k_0+k_1X-k_1a+k_2X^2-2aXk_2-k_2a^2$

$=(k_0-k_2a^2+k_1a)+X(k_1-2ak_2)+k_2\cdot X^2$

$$\begin{cases} k_0-k_1a-k_2a^2=a_0 \\ k_1-2ak_2=a_1 \\ k_2=a_2 \end{cases} \Rightarrow \begin{cases} k_0=a_0+a_1\cdot a+2a_2\cdot a^2+a_2\cdot a^2 \\ k_1=a_1+2a_2\cdot a \\ k_2=a_2 \end{cases}$$

(8) No. of bases of the vector $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$

$\mathbb{Z}_2^3=\{(a,b,c) \mid a,b,c\in\{\hat{0},\hat{1}\}\}$

in total we have 8 triples

$((\hat{1},\hat{0},\hat{0}),(\hat{0},\hat{1},\hat{0}),(\hat{0},\hat{0},\hat{1}))$ => bases contain 3 vectors

(v_1,v_2,v_3) $\qquad 0=(\hat{0},\hat{0},\hat{0})$

$v_1\in\mathbb{Z}_2^3-\{0\}$ – 7 choises

$v_2\in\mathbb{Z}_2^3-\{0,v_1\}$ – 6 choises (because they can't be lin. dependent)

$v_3\in\mathbb{Z}_2^3-\{0,v_1,v_2,v_1+v_2\}$ – 4 choises

$Nr=4\cdot7\cdot6=168$