Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Bachelor, Computer Science, Groups 911-917, Academic Year 2016-2017

Mathematical Analysis
Seminar 6

1. Show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt[3]{x}$, is not differentiable at 0 although it's derivative at 0 exists.

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0. \end{cases}$$

How many times is this function differentiable?

3. Find the n^{th} derivative ($n \in \mathbb{N}$) of the following functions:

a) $f : (-1, +\infty) \to \mathbb{R},\ f(x) = \ln(x+1)$,
b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$,
c) $f : \mathbb{R} \to \mathbb{R},\ f(x) = (\sin x - \cos x)^2 + \sin(2x)$,
d) $f : \mathbb{R} \to \mathbb{R},\ f(x) = e^{2x} x^3$,
e) $f : (-1, 1) \to \mathbb{R},\ f(x) = \ln(1 - x^2)$,
f) $f : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R},\ f(x) = \dfrac{1}{x^2 - 1}$

4. Compute the following limits:

a) $\displaystyle\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$;
b) $\displaystyle\lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^4}$;
c) $\displaystyle\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$;
d) $\displaystyle\lim_{x \to \infty} \frac{x + \ln x}{x \ln x}$;
e) $\displaystyle\lim_{\substack{x \to 0 \\ x > 0}} x \ln \sin x$;
f) $\displaystyle\lim_{\substack{x \to 0 \\ x > 0}} (\sin x)^x$.

5. Let $f : D \to \mathbb{R}$ be a function, defined on a nonempty set $D \subseteq \mathbb{R}$, and let S be a nonempty set of D. We say that f is *Lipschitzian on* S if there exists a real number $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \quad \forall x, y \in S.$$

Prove that:

1° If f is Lipschitzian on S, then f is continuous on S.

2° If f is differentiable on an interval S and $f' : S \to \mathbb{R}$ is bounded, then f is Lipschitzian on S.

3° The function $f : [0, +\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$, is not Lipschitzian on $[0, +\infty)$. However, f is Lipschitzian on $[a, +\infty)$ for any $a > 0$.

6. Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Prove that $\exists c \in (a, b)$ such that $(c - a)(c - b)f'(c) = a + b - 2c$.
Hint: Consider the function $g : [a, b] \to \mathbb{R}$, $g(x) = e^{f(x)}(x - a)(x - b)$.