Mathematical Analysis
Seminar 11

1. In each of the following cases study whether the function $f\colon \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$ has a limit at 0_2:

$$\text{a) } f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}, \qquad \text{b) } f(x,y) = \frac{x^3+y^3}{x^2+y^2}, \qquad \text{c) } f(x,y) = \frac{xy}{x^2+y^2}.$$

2. Study the continuity and the partial differentiability at 0_2 for $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$\text{a) } f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2. \end{cases} \qquad \text{b) } f(x,y) = \begin{cases} \dfrac{x^4-y^4}{2(x^4+y^4)}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2. \end{cases}$$

3. Find the second order partial derivatives of the following functions:

a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \cos x \cos y - \sin x \sin y$.

b) $f : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$, $f(x,y) = x^y$.

c) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = (x+y+z)[(2^x)^y]^z$.

d) $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$, $f(x,y,z) = xe^y/z$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x,y) = f(x^2+y^2),\ \forall\, (x,y) \in \mathbb{R}^2.$$

Prove that for any $(x,y) \in \mathbb{R}^2$ we have

$$y\frac{\partial g}{\partial x}(x,y) - x\frac{\partial g}{\partial y}(x,y) = 0.$$

5. Find the gradient $\nabla f(c)$ and the hessian matrix $\nabla^2 f(c)$ in the following cases:

a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2y^3$ and $c = (1,1)$.

b) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = e^{xyz}$ and $c = 0_3$.

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be partially differentiable functions. Prove that

$$\nabla(fg)(c) = f(c)\nabla g(c) + g(c)\nabla f(c),\ \forall\, c \in \mathbb{R}^n.$$