# LECTURE

# 1

## THE REAL NUMBERS: SOME BASIC CONCEPTS

The set of real numbers, denoted by $\mathbb{R}$, is a *totally ordered field*

$$(\mathbb{R}, +, \cdot, \geq)$$

meaning that

- $(\mathbb{R}, +, \cdot)$ is a field, where $0$ and $1$ are the neutral elements of $+$ and $\cdot$, respectively;
- $\geq$ is an order relation on $\mathbb{R}$, i.e., a binary relation, which is reflexive, transitive and antisymmetric;
- $\geq$ is total, i.e., $\forall x, y \in \mathbb{R}$ we have $x \geq y$ or $y \geq x$;
- $\geq$ is compatible with $+$, i.e., $\forall x, y, z \in \mathbb{R}$ we have $x + z \geq y + z$ whenever $x \geq y$;
- $\geq$ is compatible with $\cdot$, i.e., $\forall x, y \in \mathbb{R}$ s.t. $x \geq 0$ and $y \geq 0$, we have $xy \geq 0$.

As usual, we associate to $\geq$ the inverse order relation $\leq$ as well as the strict order relations $>$ and $<$, defined for any $x, y \in \mathbb{R}$ by

$$\begin{aligned}
x \leq y \quad &\Leftrightarrow \quad y \geq x; \\
x > y \quad &\Leftrightarrow \quad x \geq y \text{ and } x \neq y; \\
x < y \quad &\Leftrightarrow \quad y > x.
\end{aligned}$$

**Proposition 1.1** *We have $x^2 \geq 0$ for all $x \in \mathbb{R}$. Consequently, $1 > 0$.*

**Definition 1.2** *For any subset $A$ of $\mathbb{R}$ we introduce the following (possibly empty!) sets*

$$lb(A) := \{x \in \mathbb{R} \mid x \leq a, \ \forall a \in A\};$$
$$ub(A) := \{x \in \mathbb{R} \mid x \geq a, \ \forall a \in A\}.$$

*A number $x \in \mathbb{R}$ is said to be a*

- lower bound *of $A$ if $x \in lb(A)$;*
- upper bound *of $A$ if $x \in ub(A)$;*
- least element *(or* minimum*) of $A$ if $x \in A \cap lb(A)$;*
- greatest element *(or* maximum*) of $A$ if $x \in A \cap ub(A)$.*

**Remark 1.3** *Every set $A \subseteq \mathbb{R}$ has at most one least element and, if it exists, we denote it by* $\min A$. *Similarly, $A$ has at most one greatest element and, if it exists, we denote it by* $\max A$.

**Definition 1.4** *A subset $A$ of $\mathbb{R}$ is said to be*

- *bounded (from) below, if $A$ has lower bounds, i.e., $lb(A) \neq \emptyset$;*
- *bounded (from) above, if $A$ has upper bounds, i.e., $ub(A) \neq \emptyset$;*
- *bounded, if $A$ is both bounded above and below;*
- *unbounded, if $A$ is not bounded.*

**Remark 1.5** *The empty set is bounded. More precisely, we have*

$$lb(\emptyset) = ub(\emptyset) = \mathbb{R}.$$

**Example 1.6** (*i*) $A = \{a \in \mathbb{R} \mid a \geq 2\}$*: unbounded (since it is not bounded above), bounded below by any $v \leq 2$,* $\min A = 2$.
(*ii*) $A = \{a \in \mathbb{R} \mid 0 < a < 1\}$*: bounded (above by any $u \geq 1$, below by any $v \leq 0$), no minimum, no maximum.*
(*iii*) $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\}$*: bounded (above by any $u \geq 1$, below by any $v \leq 0$),* $\max A = 1$, *no minimum.*
(*iv*) *Every nonempty finite set has a minimum and a maximum.*

**Proposition 1.7 (Completeness Axiom)** *The totally ordered field of real numbers $(\mathbb{R}, +, \cdot, \geq)$ is complete, meaning that every nonempty set $A \subseteq \mathbb{R}$ that is bounded above has a least upper bound, denoted by* $\sup A$ *and called the* supremum *of $A$. In other words, we have*

$$\sup A := \min(ub(A)).$$

*Alternatively, every nonempty set $A \subseteq \mathbb{R}$ that is bounded below has a greatest lower bound, denoted by* $\inf A$ *and called the* infimum *of $A$. In other words,*

$$\inf A := \max(lb(A)).$$

**Example 1.8** (*i*) $A = \{a \in \mathbb{Z} \mid 2 \leq a \leq 3\}$: $\max A = \sup A = 3$, $\min A = \inf A = 2$.
(*ii*); $A = \{a \in \mathbb{R} \mid 0 < a \leq 1\}$: $\max A = \sup A = 1$, $\inf A = 0$, *no minimum.*

**Remark 1.9** *The Completeness Axiom is also known in the literature as the* Supremum Property, *since it shows that every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$. Its counterpart shows that every nonempty subset of $\mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$. Indeed, let $A \subseteq \mathbb{R}$, $A \neq \emptyset$, bounded below. Then the set $-A = \{-a \mid a \in A\}$ is nonempty and bounded above, so, by the Supremum Property, it has a supremum in $\mathbb{R}$. Thus we have* $\inf A = -\sup(-A)$.

**Remark 1.10** *Let $A \subseteq \mathbb{R}$ be a nonempty set. If $A$ has a greatest element (resp. a least element), then* $\sup A = \max A$ *(resp.* $\inf A = \min A$*). Conversely, if $A$ is bounded above and* $\sup A \in A$ *(resp. $A$ is bounded below and* $\inf A \in A$*), then* $\sup A = \max A$ *(resp.* $\inf A = \min A$*).*

**Definition 1.11** *We attach to $\mathbb{R}$ two elements $-\infty$ and $+\infty$ (or $\infty$) s.t.*

$$\forall x \in \mathbb{R},\ -\infty < x \ \text{ and } \ x < +\infty.$$

*The set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is called the extended real number system.*
*If a set $A \subseteq \mathbb{R}$ is not bounded above, we define* $\sup A := +\infty$.
*If a set $A \subseteq \mathbb{R}$ is not bounded below, we define* $\inf A := -\infty$.
*Also, we define* $\sup \emptyset := -\infty$ *and* $\inf \emptyset := +\infty$ *(see Remark 1.5!).*

We denote by $\mathbb{N} := \{1, 2 := 1+1, 3 := 1+1+1, \dots\}$ the set of *natural numbers.*

**Remark 1.12** $\mathbb{N}$ *is the smallest inductive subset of $\mathbb{R}$ w.r.t. inclusion (a set $A \subseteq \mathbb{R}$ is said to be inductive if $1 \in A$ and $x+1 \in A$ whenever $x \in A$). We have* $\min \mathbb{N} = 1$ *and for every $n \in \mathbb{N}$, $n < n+1$ and $\{x \in \mathbb{N} \mid n < x < n+1\} = \emptyset$. Every nonempty subset of $\mathbb{N}$ has a least element.*

**Proposition 1.13 (Principle of Mathematical Induction)** *Let $n_0 \in \mathbb{N}$ and let $P(n)$ be a property defined for any number $n \in \mathbb{N}$, $n \geq n_0$. Suppose that the following two conditions hold:*

**I.** *$P(n_0)$ is true;*

**II.** *If $P(k)$ is true for some $k \in \mathbb{N}$, $k \geq n_0$, then $P(k+1)$ is also true.*

*Then we have*

**III.** *$P(n)$ is true, $\forall n \in \mathbb{N}$, $n \geq n_0$.*

The following result is a consequence of the Completeness Axiom (Supremum Property).

**Corollary 1.14 (Archimedean Property)** *The set of natural numbers $\mathbb{N}$ is not bounded from above. In other words, for every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ s.t. $n > x$.*

*Proof.* Suppose $x \geq n$, $\forall n \in \mathbb{N}$. Then $\mathbb{N}$ is nonempty and bounded above by $x$, so, by Theorem 1.7, it has a supremum $u \in \mathbb{R}$. Since $u - 1 < u$, $u - 1$ cannot be an upper bound of $\mathbb{N}$. This means that $\exists m \in \mathbb{N}$ s.t. $u - 1 < m$. Thus, $u < m + 1 \in \mathbb{N}$, which is a contradiction to the fact that $u$ is an upper bound of $\mathbb{N}$. $\square$

The sets of *integer numbers* and *rational numbers* are defined as

$$\begin{aligned} \mathbb{Z} &:= \{m - n \mid m, n \in \mathbb{N}\}; \\ \mathbb{Q} &:= \{mn^{-1} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}. \end{aligned}$$

**Remarks 1.15** **1.** *For every $x \in \mathbb{R}$ there is a unique $k \in \mathbb{Z}$ such that $k \leq x < k+1$; we denote this $k$ by $[x]$ or $\lfloor x \rfloor$ and call it the* integer part *or* floor *of $x$.*

**2.** *For every $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $x \geq 0$, there exists a unique number $y \in \mathbb{R}$, $y \geq 0$ such that $x = y^n$ (when $n \geq 2$ we denote $y = \sqrt[n]{x}$).*

**3.** *We have $\sqrt{2} \notin \mathbb{Q}$. Therefore the set $\mathbb{R} \setminus \mathbb{Q}$ of the so-called irrational numbers is nonempty.*

As a consequence of the Archimedean Property we obtain the following result:

**Corollary 1.16 (Density of $\mathbb{Q}$ in $\mathbb{R}$)** *For any real numbers $a, b \in \mathbb{R}$ such that $a < b$ there exists $x \in \mathbb{Q}$ such that $a < x < b$.*

*Proof.* Let $a, b \in \mathbb{R}$ such that $a < b$. By the Archimedean Property (Corollary 1.14) we can find a number $n \in \mathbb{N}$ s.t. $n > \frac{1}{b-a}$, i.e.,

$$nb - 1 > na \tag{1.1}$$

*Case 1:* If $nb \in \mathbb{Z}$ then (1.1) shows that $a < \frac{nb-1}{n} < b$, hence $x := \frac{nb-1}{n} \in \mathbb{Q}$ satisfies the property in demand.

*Case 2:* If $nb \notin \mathbb{Z}$ then we consider the integer part of $nb$, namely $m := [nb]$. In this case we have

$$m < nb < m + 1. \tag{1.2}$$

By (1.1) and (1.2) we deduce that $m > nb - 1 > na$ hence $na < m < nb$. Thus, in this case the number $x := \frac{m}{n} \in \mathbb{Q}$ satisfies $a < x < b$. $\square$

**Remark 1.17** $(\mathbb{Q}, +, \cdot, \geq)$ *is a totally ordered field but, in contrast to $(\mathbb{R}, +, \cdot, \geq)$, it does not satisfy the Completeness Axiom. However, for every $x \in \mathbb{R}$ we have*

$$\begin{aligned} \sup\{y \in \mathbb{Q} \mid y < x\} = &\quad x \quad = \inf\{y \in \mathbb{Q} \mid y > x\}; \\ \sup\{z \in \mathbb{R} \setminus \mathbb{Q} \mid z < x\} = &\quad x \quad = \inf\{z \in \mathbb{R} \setminus \mathbb{Q} \mid z > x\}. \end{aligned}$$

Next we present some properties which are of practical interest.

**Proposition 1.18** *If $A \subseteq B \subseteq \mathbb{R}$ are nonempty bounded sets, then*

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

**Proposition 1.19** *If $A$ and $B$ are nonempty subsets of $\mathbb{R}$ which are bounded above, then $A \cup B$ is bounded above and the following relations hold:*

$$\begin{aligned} \sup(A \cup B) &= \max\{\sup A, \sup B\}; \\ \inf(A \cup B) &= \min\{\inf A, \inf B\}; \end{aligned}$$

**Proposition 1.20** *For any nonempty subsets $A$ and $B$ of $\mathbb{R}$, we have*

$$\begin{aligned} \sup(A + B) &= \sup A + \sup B, \\ \inf(A + B) &= \inf A + \inf B, \end{aligned}$$

*where $A + B := \{a + b \mid a \in A, b \in B\}$.*

If $f : D \to \mathbb{R}$ is a function, defined on a nonempty set $D$, then it will be convenient to denote

$$\inf_{x \in D} f(x) := \inf f(D) \quad \text{and} \quad \sup_{x \in D} f(x) := \sup f(D),$$

where $f(D) = \mathrm{Im}(f) := \{f(x) \mid x \in D\}$ represents the function's image.

In particular, if $D = \mathbb{N}$, a function $f : \mathbb{N} \to \mathbb{R}$ represents a sequence $(x_n)_{n \in \mathbb{N}}$. In this case we will write

$$\inf_{n \in \mathbb{N}} x_n := \inf\{x_n \mid n \in \mathbb{N}\} \quad \text{and} \quad \sup_{n \in \mathbb{N}} x_n := \sup\{x_n \mid n \in \mathbb{N}\}.$$

The following result is another important consequence of the Completeness Axiom (Supremum Property).

**Corollary 1.21 (Nested Interval Property)** *Consider a sequence of closed intervals $I_n = [a_n, b_n] \subseteq \mathbb{R}$, with $a_n < b_n$ for all $n \in \mathbb{N}$. If $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$, i.e.,*

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots \textit{ is a nested sequence of closed intervals,}$$

*then we have $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ (i.e., $\exists x \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, x \in I_n$).*

*Proof.* Let $A = \{a_k \mid k \in \mathbb{N}\}$. Then, $\forall n \in \mathbb{N}$, $b_n$ is an upper bound of $A$. Hence $A$ is nonempty and bounded above. By the Completeness Axiom (Proposition 1.7), we deduce that $A$ has a supremum in $\mathbb{R}$. Thus, $\forall n \in \mathbb{N}$, $a_n \leq \sup A \leq b_n$. This shows that $\sup A \in \bigcap_{n=1}^{\infty} I_n$. □

**Definition 1.22** *A set $V \subseteq \mathbb{R}$ is said to be*

- *a neighborhood of a number $x \in \mathbb{R}$, if there exists a real number $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq V$;*
- *a neighborhood of $-\infty$, if there exists a number $a \in \mathbb{R}$ such that $(-\infty, a) \subseteq V$;*
- *a neighborhood of $+\infty$, if there exists a number $a \in \mathbb{R}$ such that $(a, +\infty) \subseteq V$.*

**Proposition 1.23** *Let $x \in \overline{\mathbb{R}}$. Then*

(i) *if $x \in \mathbb{R}$ and $V \in \mathcal{V}(x)$, then $x \in V$.*
(ii) *if $V \in \mathcal{V}(x)$ and $U \subseteq \mathbb{R}$ s.t. $V \subseteq U$, then $U \in \mathcal{V}(x)$.*
(iii) *if $U, V \in \mathcal{V}(x)$, then $U \cap V \in \mathcal{V}(x)$.*

**Theorem 1.24** *Let $A \subseteq \mathbb{R}$ be a nonempty set, which is bounded from below by $\alpha \in \mathbb{R}$. Then the following assertions are equivalent:*

1° $\inf A = \alpha$.
2° *For every real number $\beta > \alpha$ there exists $x \in A$ such that $x < \beta$.*
3° *For every real number $\varepsilon > 0$ we have $A \cap [\alpha, \alpha + \varepsilon) \neq \emptyset$.*
4° *For every $V \in \mathcal{V}(\alpha)$ we have $V \cap A \neq \emptyset$.*

**Corollary 1.25** *Let $A \subseteq \mathbb{R}$ be a nonempty set, which is bounded from above by $\alpha \in \mathbb{R}$. Then the following assertions are equivalent:*

1° $\sup A = \alpha$.

2° *For every real number $\beta < \alpha$ there exists $x \in A$ such that $x > \beta$.*

3° *For every real number $\varepsilon > 0$ we have $A \cap (\alpha - \varepsilon, \alpha] \neq \emptyset$.*

4° *For every $V \in \mathcal{V}(\alpha)$ we have $V \cap A \neq \emptyset$.*

# LECTURE

# 2

## SEQUENCES OF REAL NUMBERS

**Definition 2.1** *By a* sequence of real numbers (*or sequence in* $\mathbb{R}$) *we mean any function of type* $f:\mathbb{N}\to\mathbb{R}$. *Denoting* $x_n := f(n)$ *for all* $n\in\mathbb{N}$ *we may use one of the following notations to represent the sequence* $f$:

$$(x_n)_{n\in\mathbb{N}},\ (x_n)_{n\geq 1}\ \text{ or simply }\ (x_n).$$

**Remark 2.2** *Any function* $g:\mathbb{N}\cap[m,\infty)\to\mathbb{R}$ *with* $m\in\mathbb{Z}$ *can be seen as a sequence, too. Indeed,* $g$ *can be identified with* $f:\mathbb{N}\to\mathbb{R}$, *given by*

$$f(n) := g(m+n-1),\ \forall\, n\in\mathbb{N}.$$

*In this case instead of* $(x_n)_{n\in\mathbb{N}}$, *with* $x_n := f(n)$ *for all* $n\in\mathbb{N}$, *we write*

$$(y_n)_{n\geq m},\quad \text{where }\ y_n := g(n),\ \forall\, n\in\mathbb{Z},\ n\geq m.$$

**Definition 2.3** *A sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$ *is said to be bounded below* (*bounded above; bounded; unbounded*) *if the set* $\{x_n \mid n\in\mathbb{N}\}$ *is bounded below* (*bounded above; bounded; unbounded*).

**Remark 2.4** *For any sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *we have:*

$$\begin{aligned}
(x_n)_{n\in\mathbb{N}} \text{ is bounded below} &\iff \exists\, a\in\mathbb{R} \text{ s.t. } x_n\geq a,\ \forall\, n\in\mathbb{N};\\
(x_n)_{n\in\mathbb{N}} \text{ is bounded above} &\iff \exists\, a\in\mathbb{R} \text{ s.t. } x_n\leq a,\ \forall\, n\in\mathbb{N};\\
(x_n)_{n\in\mathbb{N}} \text{ is bounded} &\iff \exists\, a\in\mathbb{R} \text{ s.t. } |x_n|\leq a,\ \forall\, n\in\mathbb{N};\\
(x_n)_{n\in\mathbb{N}} \text{ is unbounded} &\iff \forall\, a\in\mathbb{R},\ \exists\, n\in\mathbb{N} \text{ s.t. } |x_n|> a.
\end{aligned}$$

**Definition 2.5** *A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *is called*

- *increasing if* $x_{n+1}\geq x_n,\ \forall n\in\mathbb{N}$;
- *decreasing if* $x_{n+1}\leq x_n,\ \forall n\in\mathbb{N}$;
- *strictly increasing if* $x_{n+1}> x_n,\ \forall n\in\mathbb{N}$;
- *strictly decreasing if* $x_{n+1}< x_n,\ \forall n\in\mathbb{N}$;
- *monotonic* (*or monotone*) *if it is increasing or decreasing;*
- *strictly monotonic* (*or strictly monotone*) *if it is strictly increasing or strictly decreasing.*

**Remark 2.6** *Every increasing sequence is bounded below. Similarly, every decreasing sequence is bounded above.*

**Definition 2.7** *We say that a sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$ has a limit *(in* $\overline{\mathbb{R}}$*) if there exists* $\ell \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty; +\infty\}$ *such that*

$$\forall\, V \in \mathcal{V}(\ell),\ \exists\, n_V \in \mathbb{N}\ \text{ s.t. }\ x_n \in V,\ \forall\, n \in \mathbb{N},\ n \geq n_V. \tag{2.1}$$

**Remark 2.8** *For any* $\ell_1, \ell_2 \in \overline{\mathbb{R}}$, $\ell_1 \neq \ell_2$, *there exist* $V_1 \in \mathcal{V}(\ell_1)$ *and* $V_2 \in \mathcal{V}(\ell_2)$ *s.t.* $V_1 \cap V_2 = \emptyset$. *Hence, whenever* $(x_n)_{n\in\mathbb{N}}$ *has a limit, there is a unique* $\ell \in \overline{\mathbb{R}}$ *satisfying* (2.1).

**Definition 2.9** *If a sequence* $(x_n)_{n\in\mathbb{N}}$ *has a limit, then the unique* $\ell \in \overline{\mathbb{R}}$ *that satisfies* (2.1) *is called* the limit of the sequence $(x_n)_{n\in\mathbb{N}}$. *In this case we denote* $\lim\limits_{n\to\infty} x_n := \ell$ *or* $x_n \to \ell$ *and we say that* $(x_n)_{n\in\mathbb{N}}$ *tends to* $\ell$.

**Proposition 2.10** *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers that has a limit. We have*

$$\begin{aligned}
\lim_{n\to\infty} x_n = \ell \in \mathbb{R} &\iff \forall\, \varepsilon > 0,\ \exists\, n_\varepsilon \in \mathbb{N}\ \text{ s.t. }\ |x_n - \ell| < \varepsilon,\ \forall\, n \in \mathbb{N},\ n \geq n_\varepsilon;\\
\lim_{n\to\infty} x_n = -\infty &\iff \forall\, a \in \mathbb{R},\ \exists\, n_a \in \mathbb{N}\ \text{ s.t. }\ x_n < a,\ \forall\, n \in \mathbb{N},\ n \geq n_a;\\
\lim_{n\to\infty} x_n = +\infty &\iff \forall\, a \in \mathbb{R},\ \exists\, n_a \in \mathbb{N}\ \text{ s.t. }\ x_n > a,\ \forall\, n \in \mathbb{N},\ n \geq n_a.
\end{aligned}$$

**Definition 2.11** *A sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$ *is said to be*

- convergent, *if it has a finite limit, i.e.,* $\lim\limits_{n\to\infty} x_n \in \mathbb{R}$; *in this case we say that* $(x_n)_{n\in\mathbb{N}}$ *converges to the real number* $\lim\limits_{n\to\infty} x_n$;
- divergent, *if it is not convergent.*

**Proposition 2.12** *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers and let* $\ell \in \mathbb{R}$. *The following assertions are equivalent:*

1° *The sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* $\ell$, *i.e.,* $\lim\limits_{n\to\infty} x_n = \ell$.

2° *The sequence* $(|x_n - \ell|)_{n\in\mathbb{N}}$ *converges to zero, i.e.,* $\lim\limits_{n\to\infty} |x_n - \ell| = 0$.

3° *There exists a sequence* $(a_n)_{n\in\mathbb{N}}$ *of nonnegative real numbers satisfying the following two conditions:*

*(i)* $|x_n - \ell| \leq a_n$ *for all* $n \in \mathbb{N}$;

*(ii)* $(a_n)_{n\in\mathbb{N}}$ *converges to zero, i.e.,* $\lim\limits_{n\to\infty} a_n = 0$.

**Corollary 2.13** *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be two sequences of real numbers. If* $(a_n)_{n\in\mathbb{N}}$ *is bounded and* $\lim\limits_{n\to\infty} b_n = 0$, *then* $\lim\limits_{n\to\infty} (a_n \cdot b_n) = 0$.

**Proposition 2.14** *Let* $(x_n), (y_n)$ *be sequences in* $\mathbb{R}$ *such that* $\forall n \in \mathbb{N}$, $x_n \leq y_n$.

*(i) If* $(x_n)$ *and* $(y_n)$ *converge, then* $\lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} y_n$.

*(ii) If* $\lim\limits_{n\to\infty} x_n = +\infty$, *then* $\lim\limits_{n\to\infty} y_n = +\infty$.

*(iii) If* $\lim\limits_{n\to\infty} y_n = -\infty$, *then* $\lim\limits_{n\to\infty} x_n = -\infty$.

**Theorem 2.15 (Squeeze Theorem)** *Let* $(x_n), (y_n), (z_n)$ *be sequences in* $\mathbb{R}$ *such that*

$$x_n \leq y_n \leq z_n,\ \forall\, n \in \mathbb{N}.$$

*Suppose that* $(x_n)$ *and* $(z_n)$ *are convergent and* $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = \ell \in \mathbb{R}$. *Then* $(y_n)$ *is also convergent and* $\lim\limits_{n\to\infty} y_n = \ell$.

**Corollary 2.16 (Cantor's Theorem on Nested Intervals)** *Consider a nested sequence of compact intervals, i.e.,*

$$I_n := [a_n, b_n] \subseteq \mathbb{R},\quad \text{s.t. }\ a_n \leq a_{n+1} < b_{n+1} \leq b_n,\ \forall\, n \in \mathbb{N}.$$

*If* $\lim\limits_{n\to\infty} (b_n - a_n) = 0$, *then there exists* $x \in \mathbb{R}$ *such that* $\bigcap\limits_{n=1}^{\infty} I_n = \{x\}$.

**Proposition 2.17** *Every convergent sequence of real numbers is bounded.*

Notice that there are bounded sequences that are not convergent.

**Theorem 2.18 (Counterpart of the Weierstrass' Theorem)** *Let $(x_n)_{n\in\mathbb{N}}$ be a monotonic (i.e., increasing or decreasing) sequence of real numbers. The following assertions hold:*

$1^\circ$ *$(x_n)_{n\in\mathbb{N}}$ has a limit in $\overline{\mathbb{R}}$.*

$2^\circ$ *If $(x_n)_{n\in\mathbb{N}}$ is increasing, then $\lim\limits_{n\to\infty} x_n = \sup\limits_{n\in\mathbb{N}} x_n$, hence $(x_n)$ is convergent if and only if it is bounded above.*

$2^\circ$ *If $(x_n)_{n\in\mathbb{N}}$ is decreasing, then $\lim\limits_{n\to\infty} x_n = \inf\limits_{n\in\mathbb{N}} x_n$, hence $(x_n)$ is convergent if and only if it is bounded below.*

**Theorem 2.19 (Toeplitz)** *Consider an "infinite triangular matrix" of real numbers*

$$\begin{array}{ccccc}
c_{1,1} & & & & \\
c_{2,1} & c_{2,2} & & & \\
\dots & \dots & \dots & & \\
c_{n,1} & c_{n,2} & c_{n,3} & \cdots & c_{n,n} \\
\dots & \dots & \dots & \dots & \dots
\end{array}$$

*which satisfies the following three conditions:*

*(i)* $c_{n,k} \geq 0,\ \forall n \in \mathbb{N},\ \forall k \in \{1, 2, ..., n\}$;

*(ii)* $\sum\limits_{k=1}^{n} c_{n,k} = 1,\ \forall n \in \mathbb{N}$;

*(iii)* $\lim\limits_{n\to\infty} c_{n,k} \to 0,\ \forall k \in \mathbb{N}$.

*If $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers that has a limit $\ell \in \overline{\mathbb{R}}$ then the sequence $(y_n)_{n\in\mathbb{N}}$, given by*

$$y_n = c_{n,1}x_1 + c_{n,2}x_2 + ... + c_{n,n}x_n,\ \forall n \in \mathbb{N}$$

*has the same limit $\ell$.*

**Theorem 2.20 (Stolz-Cesàro)** *Let $(a_n), (b_n)$ be sequences in $\mathbb{R}$ such that*

*(i)* $(b_n)$ *is strictly increasing and* $\lim\limits_{n\to\infty} b_n = +\infty$,

*(ii)* $\lim\limits_{n\to\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = L \in \overline{\mathbb{R}}$.

*Then* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$.

*Proof.* We can apply Toeplitz's Theorem, letting $a_0 = b_0 := 0$,

$$x_n := \frac{a_n - a_{n-1}}{b_n - b_{n-1}},\ \forall n \in \mathbb{N} \quad \text{and} \quad c_{n,k} := \frac{b_k - b_{k-1}}{b_n},\ \forall k \in \{1, 2, ..., n\}.$$

Since $y_n = \frac{a_n}{b_n}$, the conclusion follows. □

**Corollary 2.21** *Let $(x_n)$ be a sequence in $\mathbb{R}$. The following assertions hold:*

$1^\circ$ *If* $\lim\limits_{n\to\infty} x_n = \ell \in \overline{\mathbb{R}}$, *then* $\lim\limits_{n\to\infty} \dfrac{x_1 + x_2 + \ldots + x_n}{n} = \ell$.

$2^\circ$ *If $x_n > 0$ for all $n \in \mathbb{N}$ and* $\lim\limits_{n\to\infty} x_n = \ell \in \overline{\mathbb{R}}$, *then* $\lim\limits_{n\to\infty} \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} = \ell$.

$3^\circ$ *If $x_n > 0$ for all $n \in \mathbb{N}$ and* $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = L \in \overline{\mathbb{R}}$, *then* $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = L$.

**Definition 2.22** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequences of real numbers. A sequence $(y_k)_{k\in\mathbb{N}}$ is said to be a* subsequence *of $(x_n)_{n\in\mathbb{N}}$ if there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers (i.e., $n_k \in \mathbb{N}$ and $n_k < n_{k+1}$ for all $k \in \mathbb{N}$) such that*

$$y_k = x_{n_k},\ \forall\, k \in \mathbb{N}.$$

**Proposition 2.23** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ that has a limit $x = \lim\limits_{n\to\infty} x_n \in \overline{\mathbb{R}}$. Then any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ has the same limit, i.e., $\lim\limits_{k\to\infty} x_{n_k} = x$.*

**Theorem 2.24 (Bolzano-Weierstrass)** *Every bounded sequence of real numbers has a convergent subsequence.*

**Theorem 2.25 (Cauchy's criterion for convergence of sequences)** *For any sequence $(x_n)_{n\in\mathbb{N}}$ the following assertions are equivalent:*

$1^\circ$ *$(x_n)_{n\in\mathbb{N}}$ is convergent.*
$2^\circ$ *For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, with $m \geq n_\varepsilon$ and $n \geq n_\varepsilon$, we have $|x_m - x_n| < \varepsilon$.*
$3^\circ$ *For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq n_\varepsilon$ and any $p \in \mathbb{N}$ we have $|x_{n+p} - x_n| < \varepsilon$.*

**Corollary 2.26 (Sufficient condition for convergence of sequences)** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Assume that there is a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative real numbers satisfying the following two conditions:*

$1^\circ$ *$|x_{n+p} - x_n| \leq a_n$ for all $n, p \in \mathbb{N}$;*
$2^\circ$ *$(a_n)_{n\in\mathbb{N}}$ converges to zero, i.e., $\lim_{n\to\infty} a_n = 0$.*

*Then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent.*

**Limit Laws**

$x + \infty = \infty + x = \infty,\ \forall x \in \mathbb{R},$

$x + (-\infty) = (-\infty) + x = -\infty,\ \forall x \in \mathbb{R},$

$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty,$

$$x \cdot \infty = \infty \cdot x = \begin{cases} \infty, \text{ if } x \in (0, \infty) \\ -\infty, \text{ if } x \in (-\infty, 0), \end{cases}$$

$$x \cdot (-\infty) = (-\infty) \cdot x = \begin{cases} -\infty, \text{ if } x \in (0, \infty) \\ \infty, \text{ if } x \in (-\infty, 0), \end{cases}$$

$\infty \cdot \infty = \infty, \quad (-\infty) \cdot (-\infty) = \infty, \quad \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty,$

$\frac{x}{\infty} = \frac{x}{-\infty} = 0,\ \forall x \in \mathbb{R},$

$\frac{1}{0+} = \infty, \quad \frac{1}{0-} = -\infty,$

$$x^{\infty} = \begin{cases} \infty, \text{ if } x \in (1, \infty) \\ 0, \text{ if } x \in [0, 1), \end{cases}$$

$$x^{-\infty} = \begin{cases} 0, \text{ if } x \in (1, \infty) \\ \infty, \text{ if } x \in (0, 1), \end{cases}$$

$$(\infty)^{x} = \begin{cases} \infty, \text{ if } x \in (0, \infty) \\ 0, \text{ if } x \in (-\infty, 0), \end{cases}$$

$\infty^{\infty} = \infty, \quad \infty^{-\infty} = 0.$

**The following limits are undefined**

$\infty + (-\infty), \quad (-\infty) + \infty,$

$0 \cdot \infty, \quad \infty \cdot 0, \quad 0 \cdot (-\infty), \quad (-\infty) \cdot 0,$

$\frac{\infty}{\infty}, \quad \frac{-\infty}{-\infty}, \quad \frac{\infty}{-\infty}, \quad \frac{-\infty}{\infty},$

$1^{\infty}, \quad 0^{0}, \quad \infty^{0}, \quad 1^{-\infty}.$

# LECTURE

# 3

# SERIES OF REAL NUMBERS. SERIES WITH NONNEGATIVE TERMS (I)

**Definition 3.1** *To any given sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers we attach another sequence, $(s_n)_{n\in\mathbb{N}}$, defined for all $n \in \mathbb{N}$ by*

$$s_n := x_1 + x_2 + \ldots + x_n = \sum_{k=1}^{n} x_k \, .$$

*The couple $\big((x_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}\big)$ is called* series *and it is denoted by*

$$\sum_{n\geq 1} x_n \, .$$

*For any $n \in \mathbb{N}$, the number $s_n$ is called the* partial sum of the series up to rank $n$*. If the sequence $(s_n)_{n\in\mathbb{N}}$ of partial sums converges (resp. diverges), we say that* the series $\sum_{n\geq 1} x_n$ is convergent (resp. divergent)*. If the sequence $(s_n)_{n\in\mathbb{N}}$ of partial sums has a limit we say that* the series has a sum*; in this case, the sum of the series is denoted by*

$$\sum_{n=1}^{\infty} x_n := \lim_{n\to\infty} s_n \, .$$

**Remark 3.2** *If $(x_n)_{n\geq m}$ is a sequence of real numbers (where $m \in \mathbb{Z}$), then we consider a series of type*

$$\sum_{n\geq m} x_n.$$

*It is easy to check that, for any $p \in \mathbb{N}$, the series $\sum\limits_{n\geq m} x_n$ has a sum (in $\overline{\mathbb{R}}$) if and only if the series $\sum\limits_{n\geq m+p} x_n$ has a sum (in $\overline{\mathbb{R}}$) and, in this case, we have*

$$\sum_{n=m}^{\infty} x_n = x_m + x_{m+1} + \ldots x_{m+p-1} + \sum_{n=m+p}^{\infty} x_n.$$

**Example 3.3 (The geometric series)** *For any number $q \in \mathbb{R}$, consider the so-called geometric series*

$$\sum_{n\geq 0} q^n$$

*where, by convention, $q^0 = 1$ even if $q = 0$. We distinguish three cases:*

- *If $q \in (-\infty, -1]$, then the geometric series has no sum, hence it is divergent;*
- *If $q \in (-1, 1)$, i.e., $|q| < 1$, then the geometric series is convergent and has the sum*

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q};$$

- *If $q \in [1, \infty)$, then the geometric series has the sum $\sum_{n=0}^{\infty} q^n = +\infty$, hence it is divergent.*

Indeed, the sequence of partial sums of the geometric series is given by

$$s_n := 1 + q + \ldots + q^n = \begin{cases} \frac{1-q^{n+1}}{1-q}, & \text{if } q \neq 1, \\ n+1, & \text{if } q = 1. \end{cases}$$

Therefore, if $|q| < 1$, then $\lim_{n\to\infty} s_n = \dfrac{1}{1-q}$. If $q \leq -1$, the sequence $(s_n)$ has no limit, hence it diverges. Finally, when $q \geq 1$, the sequence $(s_n)$ diverges while $\lim_{n\to\infty} s_n = +\infty$.

**Example 3.4 (The harmonic series)** *The so-called* harmonic series

$$\sum_{n\geq 1} \frac{1}{n}$$

*is divergent and has the sum*

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Indeed, denoting the partial sums by $s_n := 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} s_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \ldots + \left(\frac{1}{2^n} + \ldots + \frac{1}{2^n}\right) = 1 + \frac{n}{2} \end{aligned}$$

hence $\sup_{n\in\mathbb{N}} s_n \geq \sup_{n\in\mathbb{N}} s_{2^n} = +\infty$. On the other hand, we have $s_n < s_{n+1}$ for all $n \in \mathbb{N}$. By Theorem 2.18 (Weierstrass), we infer that $\lim_{n\to\infty} s_n = +\infty$.

**Example 3.5 (Euler's number as a sum of a series)** *The series*

$$\sum_{n\geq 0} \frac{1}{n!}$$

*is convergent and its sum is the Euler's number, i.e.,*

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Indeed, let $s_n := 1 + 1 + \dfrac{1}{2!} + \ldots + \dfrac{1}{n!}$, $n \in \mathbb{N}$. Recall that $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$ (see Exercise 3 of Seminar 2). By Newton's Binomial Formula,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \ldots + \frac{n(n-1)\cdot \ldots \cdot 1}{n!} \cdot \frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot \ldots \cdot \left(1 - \frac{n-1}{n}\right) \le s_n.\end{aligned}$$

Now, consider an arbitrary given $n \in \mathbb{N}^*$. Then, for any $m \ge n$, we have

$$\begin{aligned}\left(1+\frac{1}{m}\right)^m &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{m}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{m}\right) + \\ &\quad + \ldots + \frac{1}{m!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdot\ldots\cdot\left(1-\frac{m-1}{m}\right) \\ &\ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{m}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{m}\right).\end{aligned}$$

Letting $m \to \infty$, we have that $e \ge s_n$. Thus, $\forall n \in \mathbb{N}^*$, $\left(1+\dfrac{1}{n}\right)^n \le s_n \le e$. Letting $n \to \infty$, we obtain that $\lim\limits_{n\to\infty} s_n = e$, so $\sum\limits_{n\ge 1} \dfrac{1}{n!}$ is convergent and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n!} = e$.

**Example 3.6 (Telescoping series)** *Given a sequence* $(x_n)_{n\in\mathbb{N}}$ *of real numbers, we say that*

$$\sum_{n\ge 1}(x_n - x_{n+1})$$

*is a* telescoping series. *This series is convergent if and only if the sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent. More precisely, we have*

$$\sum_{n=1}^{\infty}(x_n - x_{n+1}) = x_1 - \lim_{n\to\infty} x_n.$$

For instance, consider the series

$$\sum_{n\ge 1} \frac{1}{n(n+1)}.$$

It is easily seen that

$$\frac{1}{n(n+1)} = \frac{n+1-1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},\ \forall n \in \mathbb{N},$$

hence we have a telescopic series. Denoting its partial sums by

$$s_n := \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n\cdot(n+1)},\ n \in \mathbb{N},$$

it follows that $s_n = 1 - \dfrac{1}{2} + \dfrac{1}{2} - \dfrac{1}{3} + \ldots + \dfrac{1}{n} - \dfrac{1}{n+1} = 1 - \dfrac{1}{n+1}$.

Thus, $\lim\limits_{n\to\infty} s_n = 1$, so $\sum\limits_{n\ge 1} \dfrac{1}{n(n+1)}$ is convergent and its sum is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

**Proposition 3.7** *Let $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ be convergent series and let $c \in \mathbb{R}$. Then, the following assertions hold:*

*a) The series $\sum_{n\geq 1}(x_n + y_n)$ is convergent and*

$$\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n.$$

*b) The series $\sum_{n\geq 1}(c\, x_n)$ is convergent and*

$$\sum_{n=1}^{\infty}(c\, x_n) = c\sum_{n=1}^{\infty} x_n.$$

**Proposition 3.8 (The $n^{th}$ Term Test – necessary condition for convergence)** *If a series of real numbers $\sum_{n\geq 1} x_n$ converges, then its general term converges to zero, i.e., $\lim_{n\to\infty} x_n = 0$.*

**Remark 3.9** *The condition $\lim\limits_{n\to\infty} x_n = 0$ is not sufficient for the convergence of a series $\sum_{n\geq 1} x_n$. For instance, the harmonic series is divergent while its general term converges to zero (see Example 3.4).*

**Corollary 3.10 (Sufficient conditions for divergence of series)** *A series $\sum_{n\geq 1} x_n$ is divergent whenever*

*(i) the sequence $(x_n)$ is divergent*

*or*

*(ii) the sequence $(x_n)$ converges and $\lim_{n\to\infty} x_n \neq 0$.*

**Theorem 3.11 (Cauchy's Criterion for convergence of series)** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. The following assertions are equivalent:*

*1° The series $\sum_{n\geq 1} x_n$ is convergent.*

*2° For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_{n+1} + x_{n+2} + \cdots + x_{n+p}| < \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$ and $p \in \mathbb{N}$.*

**Corollary 3.12 (Sufficient condition for convergence of series)** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Assume that there is a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative real numbers satisfying the following two conditions:*

*1° $|x_{n+1} + x_{n+2} + \cdots + x_{n+p}| \leq a_n$ for all $n, p \in \mathbb{N}$;*

*2° $(a_n)_{n\in\mathbb{N}}$ converges to zero, i.e., $\lim_{n\to\infty} a_n = 0$.*

*Then the series $\sum_{n\geq 1} x_n$ is convergent.*

## Series with nonnegative terms (I)

**Lemma 3.13 (Convergence of series vs boundedness of their partial sums)** *Let $\sum_{n\geq 1} x_n$ be a series with nonnegative terms (i.e., $x_n \geq 0$ for all $n \in \mathbb{N}$) and let $(s_n)_{n\in\mathbb{N}}$ be the sequence of its partial sums. Then the series $\sum_{n\geq 1} x_n$ has a sum in $\mathbb{R} \cup \{+\infty\}$, namely*

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} s_n.$$

*Moreover, the following assertions are equivalent:*

*1° The series $\sum_{n\geq 1} x_n$ converges.*

*2° The sequence $(s_n)_{n\in\mathbb{N}}$ is bounded.*

*Proof.* For any $n \in \mathbb{N}$ we have $x_{n+1} \geq 0$, hence $s_{n+1} = s_n + x_{n+1} \geq s_n$. Therefore the sequence $(s_n)$ is increasing. By Theorem 2.18 (Weierstrass) it follows that $(s_n)$ has a limit in $\overline{\mathbb{R}}$. More precisely, $(s_n)$ is convergent if and only if it is bounded. $\square$

**Remark 3.14** *If a series $\sum_{n\geq 1} x_n$ is convergent, then (in view of Propositions 2.17 and 3.8) the sequence $(s_n)$ must be bounded, but this condition is not equivalent to the convergence of $\sum_{n\geq 1} x_n$. For instance, consider the series*

$$\sum_{n\geq 1} (-1)^n.$$

*The sequence of partial sums of this series is given by*

$$s_n = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

*Obviously, the sequence $(s_n)$ is bounded, but does not converge (because it possesses two subsequences converging to different limits). Therefore the series $\sum_{n\geq 1}(-1)^n$ is divergent.*

**Theorem 3.15 (Cauchy's Condensation Criterion)** *Let $\sum\limits_{n\geq 1} x_n$ be a series with nonnegative terms. If the sequence $(x_n)_{n\in\mathbb{N}}$ is decreasing, then the following assertions are equivalent:*

1° *The given series, $\sum\limits_{n\geq 1} x_n$, converges.*

2° *The series $\sum\limits_{n\geq 0} 2^n \cdot x_{2^n}$ converges.*

**Example 3.16 (The generalized harmonic series)** *For every number $p \in \mathbb{R}$ consider the so-called* generalized harmonic series

$$\sum_{n\geq 1} \frac{1}{n^p}\,.$$

*This series is convergent if and only if $p > 1$.*

Indeed, denote $x_n := \frac{1}{n^p}$ for all $n \in \mathbb{N}$. If $p \leq 0$, then we clearly have $\lim_{n\to\infty} x_n \neq 0$, hence the series $\sum_{n\geq 1} x_n$ diverges according to Corollary 3.10. Assume now that $p > 0$. Then the sequence $(x_n)$ is decreasing and has positive terms. In this case, according to Cauchy's condensation criterion, the series $\sum_{n\geq 1} x_n$ converges if and only if the series $\sum\limits_{n\geq 0} 2^n \cdot x_{2^n}$ converges. The latter series actually is a geometric series, since for every $n \in \mathbb{N} \cup \{0\}$ we have $2^n \cdot x_{2^n} = 2^n \cdot \frac{1}{2^{np}} = (2^{1-p})^n$. In view of Example 3.3 we deduce that the series $\sum_{n\geq 1} x_n$ converges if and only if $2^{1-p} < 1$, i.e., $p > 1$.

**Remark 3.17** *For $p = 1$ we recover the classical harmonic series (see Example 3.4), which is divergent and has the sum*

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

*Actually, the generalized harmonic series has a sum in $\overline{\mathbb{R}}$ for every $p \in \mathbb{R}$. More precisely, we have*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = +\infty \text{ if } p \in (-\infty, 1] \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} =: \zeta(p) \in (1, +\infty) \text{ if } p \in (1, +\infty)$$

*where $\zeta : (1, \infty) \to (1, +\infty)$ represents the Riemann zeta function. Notice that $\zeta$ is strictly decreasing. In particular, for $p \in \{2, 3, 4\}$ we have*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645,\ \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202 \ (\textit{Apéry's constant}),\ \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.082\,.$$

**Theorem 3.18 (Comparison Test)** *Let* $\sum\limits_{n\geq 1} x_n$ *and* $\sum\limits_{n\geq 1} y_n$ *be series with nonnegative terms. If there is* $n_0 \in \mathbb{N}$ *s.t.*

$$x_n \leq y_n \text{ for all } n \geq n_0,$$

*then the following assertions hold:*

*(i) If* $\sum\limits_{n\geq 1} y_n$ *is convergent, then* $\sum\limits_{n\geq 1} x_n$ *is convergent.*

*(ii) If* $\sum\limits_{n\geq 1} x_n$ *is divergent, then* $\sum\limits_{n\geq 1} y_n$ *is divergent.*

*Proof.* (i) Without loss of generality assume that $n_0 = 1$. Consider the partial sums

$$s_n := x_1 + x_2 + \ldots + x_n \quad \text{and} \quad \tilde{s}_n := y_1 + y_2 + \ldots + y_n, \ \forall n \in \mathbb{N}.$$

Since $\sum_{n\geq 1} y_n$ is convergent, it follows that $(\tilde{s}_n)$ is bounded (by Lemma 3.13), hence $\exists M > 0$ such that $\tilde{s}_n \leq M$, $\forall n \in \mathbb{N}$. Then $s_n \leq \tilde{s}_n \leq M$, $\forall n \in \mathbb{N}$. Thus, $(s_n)$ is bounded and therefore $\sum_{n\geq 1} x_n$ is convergent (by Lemma 3.13).

Assertion (ii) is an equivalent counterpart of (i). □

**Corollary 3.19 (Comparison Test in practical form)** *Let* $\sum\limits_{n\geq 1} x_n$ *be a series with nonnegative terms and let* $\sum\limits_{n\geq 1} y_n$ *be a series with positive terms, such that the following limit exists:*

$$\ell := \lim_{n\to\infty} \frac{x_n}{y_n} \in [0, +\infty) \cup \{+\infty\}.$$

*The following assertions hold:*

1° *If* $\ell \in (0, +\infty)$, *then the series* $\sum\limits_{n\geq 1} x_n$ *and* $\sum\limits_{n\geq 1} y_n$ *have the same nature, i.e., they are both convergent or both divergent.*

2° *If* $\ell = 0$, *then*

*a) If* $\sum\limits_{n\geq 1} y_n$ *converges, then* $\sum\limits_{n\geq 1} x_n$ *converges.*

*b) If* $\sum\limits_{n\geq 1} x_n$ *diverges, then* $\sum\limits_{n\geq 1} y_n$ *diverges.*

3° *If* $\ell = +\infty$, *then*

*a) If* $\sum\limits_{n\geq 1} x_n$ *converges, then* $\sum\limits_{n\geq 1} y_n$ *converges.*

*b) If* $\sum\limits_{n\geq 1} y_n$ *diverges, then* $\sum\limits_{n\geq 1} x_n$ *diverges.*

**Example 3.20** *Let* $\sum_{n\geq 1} x_n$ *be a series with positive terms and let* $p \in \mathbb{R}$. *Assume that the following limit exists*

$$\ell := \lim_{n\to\infty} (n^p \cdot x_n) \in [0, \infty) \cup \{+\infty\}.$$

*Applying the Comparison Test in practical form (Corollary 3.19) for the given series and the generalized harmonic series* $\sum_{n\geq 1} y_n := \sum_{n\geq 1} \frac{1}{n^p}$, *we deduce that (see Exercise 3.16):*

1° *If* $0 \leq \ell < \infty$ *and* $p > 1$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

2° *If* $0 < \ell \leq \infty$ *and* $p \leq 1$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

**Corollary 3.21** *Let* $\sum\limits_{n\geq 1} x_n$ *and* $\sum\limits_{n\geq 1} y_n$ *be series with positive terms. If there is* $n_0 \in \mathbb{N}$ *s.t.*

$$\frac{x_{n+1}}{x_n} \leq \frac{y_{n+1}}{y_n} \text{ for all } n \geq n_0,$$

*then the following assertions hold:*

*a) If* $\sum_{n\geq 1} y_n$ *converges, then* $\sum_{n\geq 1} x_n$ *converges.*

*b) If* $\sum_{n\geq 1} x_n$ *diverges, then* $\sum_{n\geq 1} y_n$ *diverges.*

**Example 3.22** *The following series is divergent:*

$$\sum_{n\geq 1}(2-\sqrt{e})\cdot(2-\sqrt[3]{e})\cdot\ldots\cdot(2-\sqrt[n]{e}).$$

*Indeed, letting* $y_n := (2-\sqrt{e})\cdot(2-\sqrt[3]{e})\cdot\ldots\cdot(2-\sqrt[n]{e})$ *and taking into account that* $e < \left(1+\frac{1}{n}\right)^{n+1}$ *for all* $n \in \mathbb{N}$ *(see Exercise 3 of Seminar 2), we infer*

$$\frac{y_{n+1}}{y_n} = 2 - \sqrt[n+1]{e} > 1 - \frac{1}{n} = \frac{n-1}{n} = \frac{y_{n+1}}{y_n}$$

*where* $y_n := \frac{1}{n-1}$ *for all* $n \geq 2$. *Since the harmonic series* $\sum_{n\geq 2} y_n$ *diverges, we deduce by Corollary 3.21 that the given series diverges, too.*

**Theorem 3.23 (d'Alembert's Ratio Test)** *Let* $\sum_{n\geq 1} x_n$ *be a series with positive terms. The following assertions hold:*

1° *If* $\exists q \in (0,1), \exists n_0 \in \mathbb{N}$ *s.t.* $\frac{x_{n+1}}{x_n} \leq q,\ \forall\, n \geq n_0$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

2° *If* $\exists n_0 \in \mathbb{N}$ *s.t.* $\frac{x_{n+1}}{x_n} \geq 1,\ \forall\, n \geq n_0,$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

3° *If the following limit exists*

$$D := \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \in [0,+\infty) \cup \{+\infty\},$$

*then we have*

*a) If* $D < 1$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

*b) If* $D > 1$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

**Example 3.24** *The series* $\sum_{n\geq 1} \frac{(n!)^2}{(2n)!}$ *is convergent. Indeed, since*

$$D := \lim_{n\to\infty} \frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{n+1}{2(2n+1)} = \frac{1}{4} < 1,$$

*it follows by de Ratio Test (Theorem 3.23) that the given series is convergent.*

**Theorem 3.25 (Cauchy's Root Test)** *Let* $\sum_{n\geq 1} x_n$ *be a series with nonnegative terms.*

1° *If* $\exists q \in [0,1), \exists n_0 \in \mathbb{N}$ *s.t.* $\sqrt[n]{x_n} \leq q,\ \forall\, n \geq n_0$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

2° *If* $\exists n_0 \in \mathbb{N}$ *s.t.* $\sqrt[n]{x_n} \geq 1,\ \forall\, n \geq n_0$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

3° *If the following limit exists*

$$C = \lim_{n\to\infty} \sqrt[n]{x_n} \in [0,+\infty) \cup \{+\infty\},$$

*then we have*

*a) If* $C < 1$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

*b) If* $C > 1$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

**Example 3.26** *The series* $\sum_{n\geq 1} \frac{n^p}{2^n}$ *is convergent for every* $p \in \mathbb{R}$. *Indeed, since*

$$C := \lim_{n\to\infty} \sqrt[n]{\frac{n^p}{2^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^p}{2} = \frac{1}{2} < 1,$$

*it follows by de Root Test (Theorem 3.25) that the given series is convergent.*

# LECTURE

## 4

## SERIES WITH NONNEGATIVE TERMS (II). SERIES WITH ARBITRARY TERMS

### Series with nonnegative terms (II)

**Theorem 4.1 (Kummer's Test)** *Let* $\sum_{n\geq 1} x_n$ *be a series with positive terms.*

1° *If* $\exists\,(c_n)_{n\in\mathbb{N}}$ *in* $(0,+\infty)$, $\exists\, r>0$ *and* $\exists\, n_0\in\mathbb{N}$, *such that*

$$c_n\frac{x_n}{x_{n+1}}-c_{n+1}\geq r,\ \forall\, n\in\mathbb{N},\ n\geq n_0,$$

*then the series* $\sum\limits_{n\geq 1} x_n$ *is divergent.*

2° *If* $\exists\,(c_n)_{n\in\mathbb{N}}$ *in* $(0,+\infty)$ *and* $\exists\, n_0\in\mathbb{N}$, *such that*

$$\sum_{n=1}^{n}\frac{1}{c_n}=+\infty \qquad \text{and} \qquad c_n\frac{x_n}{x_{n+1}}-c_{n+1}\leq 0,\ \forall\, n\in\mathbb{N},\ n\geq n_0,$$

*then the series* $\sum\limits_{n\geq 1} x_n$ *is divergent.*

*Proof.* 1° Since $c_nx_n-c_{n+1}x_{n+1}\geq rx_{n+1}$, $\forall n\geq n_0$, it follows that for any $n\geq n_0+1$,

$$\sum_{k=n_0}^{n-1}(c_kx_k-c_{k+1}x_{k+1})\geq r\sum_{k=n_0}^{n-1}x_{k+1}.$$

Denoting $s_n:=x_1+\ldots+x_n$, we deduce that $c_{n_0}x_{n_0}-c_nx_n\geq r\,(s_n-s_{n_0})$ and therefore

$$s_n\leq s_{n_0}+\frac{1}{r}\,(c_{n_0}x_{n_0}-c_nx_n)\leq s_{n_0}+\frac{c_{n_0}x_{n_0}}{r}.$$

Hence, the sequence of partial sums $(s_n)$ is bounded, which means that the series $\sum\limits_{n\geq 1} x_n$ is convergent (by Lemma 3.13)

2° Since $c_n x_n \le c_{n+1} x_{n+1}$, $\forall n \ge n_0$, we have $c_{n_0} x_{n_0} \le c_n x_n$, $\forall n \ge n_0$. This yields

$$\frac{1}{c_n} \le \frac{1}{c_{n_0} x_{n_0}} x_n, \ \forall\, n \ge n_0.$$

Since the series $\sum_{n\ge1} \frac{1}{c_n}$ is divergent, we conclude that the series $\sum_{n\ge1} x_n$ is divergent as well, according to the Comparison Test (Theorem 3.18) □

**Theorem 4.2 (Raabe-Duhamel's Test)** *Let $\sum_{n\ge1} x_n$ be a series with positive terms.*

1° *If $\exists\, q > 1$, $\exists\, n_0 \in \mathbb{N}$ such that $n\left(\frac{x_n}{x_{n+1}} - 1\right) \ge q$, $\forall\, n \ge n_0$, then $\sum_{n\ge1} x_n$ is convergent.*

2° *If $\exists n_0 \in \mathbb{N}$ such that $n\left(\frac{x_n}{x_{n+1}} - 1\right) \le 1$, $\forall\, n \ge n_0$, then $\sum_{n\ge1} x_n$ is divergent.*

3° *If the following limit exists*

$$R := \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) \in \overline{\mathbb{R}},$$

*then we have*

*a) If $R > 1$, $\sum_{n\ge1} x_n$ is convergent.*

*b) If $R < 1$, $\sum_{n\ge1} x_n$ is divergent.*

*Proof.* Follows from Kummer's Test (Theorem 4.1)for $c_n := n$ for all $n \in \mathbb{N}$. □

**Example 4.3** *For any $a > 0$ consider the series*

$$\sum_{n\ge1} \frac{n!}{a(a+1)\cdot\ldots\cdot(a+n)}.$$

*This series is convergent for $a > 1$ and divergent for $a \in (0,1]$.*

*Indeed, denoting $x_n := \frac{n!}{a(a+1)\cdot\ldots\cdot(a+n)}$, we have*

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)!}{a(a+1)\cdot\ldots\cdot(a+n+1)} \cdot \frac{a(a+1)\cdot\ldots\cdot(a+n)}{n!} = \frac{n+1}{a+n+1}.$$

*Note that $D := \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$, hence the Ratio Test is inconclusive. However,*

$$R := \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{a+n+1}{n+1} - 1\right) = \lim_{n\to\infty} n\frac{a}{n+1} = a,$$

*which allows us to conclude, by Raabe-Duhamel's Test, that the given series is convergent if $a > 1$ and divergent if $a \in (0,1)$.*

*Finally, for $a = 1$ the given series becomes $\sum_{n\ge1} \frac{1}{n+1}$, which is divergent.*

**Theorem 4.4 (Bertrand's Test)** *Let $\sum_{n\geq 1} x_n$ be a series with positive terms. If the following limits exists*

$$B := \lim_{n\to\infty} (\ln n) \left[ n \left( \frac{x_n}{x_{n+1}} - 1 \right) - 1 \right] \in \overline{\mathbb{R}},$$

*then we have*

*a) If $B > 1$, then $\sum\limits_{n\geq 1} x_n$ is convergent.*

*b) If $B < 1$, then $\sum\limits_{n\geq 1} x_n$ is divergent.*

*Proof.* Follows from Kummer's Test (Theorem 4.1)for $c_n := n \cdot \ln n$, $n \in \mathbb{N}$, $n \geq 2$. □

**Example 4.5** *The series $\sum\limits_{n\geq 1} \left[ \frac{(2n-1)!!}{(2n)!!} \right]^2$ is divergent.*

*Indeed, denoting $x_n := \left[ \frac{(2n-1)!!}{(2n)!!} \right]^2 = \left[ \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot \ldots \cdot (2n)} \right]^2$ we have $\frac{x_{n+1}}{x_n} = \left( \frac{2n+1}{2n+2} \right)^2$ for all $n \in \mathbb{N}$. It is a simple exercise to check that*

$$\begin{aligned}
D &:= \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1; \\
R &:= \lim_{n\to\infty} n \left( \frac{x_n}{x_{n+1}} - 1 \right) = \lim_{n\to\infty} n \left[ \left( \frac{2n+2}{2n+1} \right)^2 - 1 \right] = \lim_{n\to\infty} \frac{4n^2+3n}{4n^2+4n+1} = 1,
\end{aligned}$$

*hence both the Ratio Test and the Raabe-Duhamel's Test are inconclusive.*

*On the other hand, we have*

$$B := \lim_{n\to\infty} (\ln n) \left[ n \left( \frac{x_n}{x_{n+1}} - 1 \right) - 1 \right] = \lim_{n\to\infty} (\ln n) \left( \frac{4n^2+3n}{4n^2+4n+1} - 1 \right) = 0 < 1.$$

*We conclude by Bertrand's Test that the given series is divergent.*

## Series with arbitrary terms

**Theorem 4.6 (Abel-Dirichlet's Test)** *Let $\sum\limits_{n\geq 1} x_n$ be a series of real numbers. Assume that there exist two sequences of real numbers, $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, satisfying the following three conditions:*

*(i) $x_n = a_n \cdot b_n$, $\forall\, n \in \mathbb{N}$.*

*(ii) $\exists\, M > 0$ s.t. $-M \leq A_n := a_1 + \cdots + a_n \leq M$, $\forall\, n \in \mathbb{N}$, i.e., the sequence $(A_n)_{n\in\mathbb{N}}$ is bounded.*

*(iii) The sequence $(b_n)_{n\in\mathbb{N}}$ is monotone and convergent to 0.*

*Then the series $\sum\limits_{n\geq 1} x_n$ is convergent.*

*Proof.* Without loss of generality we can assume in (iii) that $(b_n)$ is decreasing. We will prove that $\sum\limits_{n\geq 1} x_n$ converges by using Cauchy's Criterion (Theorem 3.11)To this aim, consider an arbitrary $\varepsilon > 0$.

On the one hand, by (i), (ii) and the assumption that $(b_n)$ is decreasing, we have

$$\begin{aligned}
& |x_{n+1} + x_{n+2} + \cdots + x_{n+p}| \\
= & |a_{n+1}b_{n+1} + a_{n+2}b_{n+2} + \ldots + a_{n+p}b_{n+p}| \\
= & |(A_{n+1} - A_n)\, b_{n+1} + (A_{n+2} - A_{n+1})\, b_{n+2} + \cdots + (A_{n+p} - A_{n+p-1})\, b_{n+p}| \\
= & |-A_n b_{n+1} + A_{n+1}\,(b_{n+1} - b_{n+2}) + \cdots + A_{n+p-1}\,(b_{n+p-1} - b_{n+p}) + A_{n+p}b_{n+p}| \\
\leq & |-A_n| \cdot |b_{n+1}| + |A_{n+1}| \cdot |b_{n+1} - b_{n+2}| + \cdots + |A_{n+p-1}| \cdot |b_{n+p-1} - b_{n+p}| + |A_{n+p}| \cdot |b_{n+p}| \\
= & |A_n| \cdot b_{n+1} + |A_{n+1}| \cdot (b_{n+1} - b_{n+2}) + \cdots + |A_{n+p-1}| \cdot (b_{n+p-1} - b_{n+p}) + |A_{n+p}| \cdot b_{n+p} \\
\leq & M\, [b_{n+1} + (b_{n+1} - b_{n+2}) + (b_{n+2} - b_{n+3}) + \ldots + (b_{n+p-1} - b_{n+p}) + b_{n+p}] \\
= & 2M b_{n+1},\ \forall\, n, p \in \mathbb{N}.
\end{aligned}$$

On the other hand, since $\lim_{n\to\infty} b_n = 0$ by (iii), there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|b_n| < \frac{\varepsilon}{2M}, \ \forall\, n \in \mathbb{N},\ n \geq n_\varepsilon.$$

We conclude that $|x_{n+1} + x_{n+2} + \cdots + x_{n+p}| < \varepsilon,\ \forall\, n \in \mathbb{N},\ n \geq n_\varepsilon,\ \forall\, p \in \mathbb{N}$. □

**Definition 4.7** *A series* $\sum\limits_{n\geq 1} x_n$ *is called* alternating *if either*

$$x_1 \geq 0,\ x_2 \leq 0, x_3 \geq 0, \ldots (i.e.,\ x_n = (-1)^{n+1}|x_n| \text{ for all } n \in \mathbb{N})$$

*or*

$$x_1 \leq 0,\ x_2 \geq 0, x_3 \leq 0, \ldots (i.e.,\ x_n = (-1)^n\,|x_n| \text{ for all } n \in \mathbb{N}).$$

**Theorem 4.8 (Leibniz's Criterion for Alternating Series)** *Consider an alternating series* $\sum\limits_{n\geq 1} x_n$. *If the sequence* $(|x_n|)_{n\in\mathbb{N}}$ *is decreasing, then the following assertions are equivalent:*

1° *The series* $\sum\limits_{n\geq 1} x_n$ *is convergent.*

2° *The sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* $0$.

*Proof.* Assume that $x_n = (-1)^{n+1}|x_n|$ for all $n \in \mathbb{N}$. Then the conclusion follows by Abel-Dirichlet's Test for $a_n := (-1)^{n+1}$ and $b_n := |x_n|$. □

**Definition 4.9** *A series of real numbers* $\sum\limits_{n\geq 1} x_n$ *is called absolutely convergent if the series* $\sum\limits_{n\geq 1} |x_n|$ *is convergent.*

**Theorem 4.10** *If a series of real numbers* $\sum\limits_{n\geq 1} x_n$ *is absolutely convergent, then it is also convergent.*

*Proof.* Let $\varepsilon > 0$. Since $\sum\limits_{n\geq 1} |x_n|$ is convergent, there exists in view of the Cauchy's Criterion (Theorem 3.11) a number $n_\varepsilon \in \mathbb{N}$ such that

$$\big||x_{n+1}| + \cdots + |x_{n+p}|\big| < \varepsilon,\ \forall\, n \in \mathbb{N},\ n \geq n_\varepsilon,\ \forall\, p \in \mathbb{N}.$$

Noting that $|x_{n+1} + \cdots + x_{n+p}| \leq |x_{n+1}| + \cdots + |x_{n+p}| = \big||x_{n+1}| + \cdots + |x_{n+p}|\big|$, we infer

$$|x_{n+1} + \cdots + x_{n+p}| < \varepsilon,\ \forall\, n \in \mathbb{N},\ n \geq n_\varepsilon,\ \forall\, p \in \mathbb{N}.$$

By Cauchy's Criterion (Theorem 3.11)we conclude that $\sum\limits_{n\geq 1} x_n$ is convergent. □

**Definition 4.11** *A series of real numbers* $\sum\limits_{n\geq 1} x_n$ *is called* semi-convergent *(or* conditionally convergent*) if it is convergent but not absolutely convergent.*

**Remark 4.12** *A series* $\sum\limits_{n\geq 1} x_n$ *with nonnegative terms is absolutely convergent if and only if it is convergent.*

**Example 4.13 (The alternating generalized harmonic series)** *Let $p \in \mathbb{R}$. The so-called* alternating generalized harmonic series

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^p}$$

*is divergent for $p \in (-\infty, 0]$, semi-convergent for $p \in (0, 1]$ and absolutely convergent for $p \in (1, \infty)$.*

*In particular, for $p = 1$ we get the* alternating harmonic series, *whose sum is*

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = \ln 2.$$

**Example 4.14** *The series $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{n\sqrt{n}}$ is absolutely convergent.*

**Example 4.15** *The series $\displaystyle\sum_{n\geq 1} (-1)^{n+1} \sin\frac{1}{n}$ is semi-convergent.*

**Example 4.16** *The series $\displaystyle\sum_{n\geq 1} (-1)^{n+1} \frac{n}{n+1}$ is divergent.*

**Example 4.17** *The series $\displaystyle\sum_{n\geq 1} \cos(n\pi)$ is divergent.*

**Theorem 4.18 (Cauchy)** *If a series $\displaystyle\sum_{n\geq 1} x_n$ is absolutely convergent, then for any bijection (permutation) $\sigma : \mathbb{N} \to \mathbb{N}$ the series $\displaystyle\sum_{n\geq 1} x_{\sigma(n)}$ is absolutely convergent and its sum coincides with the sum of the initial series, i.e., $\displaystyle\sum_{n=1}^{\infty} x_{\sigma(n)} = \sum_{n=1}^{\infty} x_n$.*

**Theorem 4.19 (Riemann)** *If a series $\displaystyle\sum_{n\geq 1} x_n$ is semi-convergent, then for every $s \in \overline{\mathbb{R}}$ there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\displaystyle\sum_{n=1}^{\infty} x_{\sigma(n)} = s$.*

**Example 4.20** *Consider the alternating harmonic series (see Example 4.13), whose sum is*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots = \ln 2.$$

*If we permute its terms by alternating $p := 2$ positive terms followed by $q := 3$ negative terms we obtain*

$$1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{5} + \frac{1}{7} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \ldots = \sqrt{\frac{p}{q}} \ln 2.$$

Indeed, consider the Euler's constant $\gamma := \lim\limits_{n\to\infty} \gamma_n$ (see Exercise 2 of Seminar 3), where $\gamma_n := \dfrac{1}{n} + \ldots + \dfrac{1}{n} - \ln n$ for all $n \in \mathbb{N}$.

Denote by $(s_n)_{n\in\mathbb{N}}$ the sequence of partial sums of the permuted series. Then, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} s_{5k} &= \left(1+\frac{1}{3}-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{8}-\frac{1}{10}-\frac{1}{12}\right)+\ldots+ \\ &\quad +\left(\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{6k-4}-\frac{1}{6k-2}-\frac{1}{6k}\right) \\ &= \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{4k}-\ln 4k\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2k}-\ln 2k\right)- \\ &\quad -\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{3k}-\ln 3k\right)+\ln 4k-\frac{1}{2}\ln 2k-\frac{1}{2}\ln 3k \\ &= \gamma_{4k}-\frac{1}{2}\gamma_{2k}-\frac{1}{2}\gamma_{3k}+\ln\frac{4k}{\sqrt{6}k}, \end{aligned}$$

hence

$$\lim_{k\to\infty} s_{5k} = \gamma-\frac{1}{2}\gamma-\frac{1}{2}\gamma+\ln\frac{4}{\sqrt{6}} = \sqrt{\frac{2}{3}}\ln 2.$$

On the other hand, we also have

$$\begin{aligned} s_{5k+1} &= s_{5k}+\frac{1}{4k+1}, \\ s_{5k+2} &= s_{5k}+\frac{1}{4k+1}+\frac{1}{4k+3}, \\ s_{5k+3} &= s_{5k}+\frac{1}{4k+1}+\frac{1}{4k+3}-\frac{1}{6k+2}, \\ s_{5k+4} &= s_{5k}+\frac{1}{4k+1}+\frac{1}{4k+3}-\frac{1}{6k+2}-\frac{1}{6k+4}, \end{aligned}$$

which show that $\lim\limits_{k\to\infty} s_{5k} = \lim\limits_{k\to\infty} s_{5k+1} = \lim\limits_{k\to\infty} s_{5k+2} = \lim\limits_{k\to\infty} s_{5k+3} = \lim\limits_{k\to\infty} s_{5k+4}$.

We conclude that

$$\lim_{n\to\infty} s_n = \sqrt{\frac{2}{3}}\ln 2.$$