Paired factor analysis for tree reconstruction

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The model

Let D_{nj} be the data corresponding to n-th sample and j-th feature. We assume for now that the data is Gaussian in its distribution. We assume there are K factors or nodes of the tree. We assume the model

$$E\left[D_{nj}|Z_n = (k_1, k_2), \lambda_n = q, F\right] = qF_{k1,j} + (1-q)F_{k2,j} \tag{1}$$

We assume a prior on λ,

$$Pr\left[\lambda_n = q\right] = \pi_q$$

Then we can write

$$Pr\left[D_n|Z_n = (k_1, k_2), F, s^2_{j=1,2,\cdots,J}\right] = \sum_q \pi_q Pr\left[D_n|Z_n = (k_1, k_2), \lambda_n = q, F, s^2_{j=1,2,\cdots,J}\right]$$

where s^2_j is the variance of the jth feature.

We also assume the prior

$$Pr\left[Z_n = (k_1, k_2)\right] = \pi_{k_1,k_2} \qquad k_1 < k_2$$

Then we can write

$$Pr\left[D_n|\pi, F\right] = \sum_{k_1 < k_2} \pi_{k_1,k_2} Pr\left[D_n|Z_n = (k_1, k_2), F, s^2_{j=1,2,\cdots,J}\right] \tag{2}$$

We define the joint prior over the edges and the fraction of the edge represented as

$$\pi_{k_1,k_2,q} = \pi_{k_1,k_2}\pi_q \qquad k_1 < k_2$$

The overall likelihood

$$L(\pi, F) = \prod_{n=1}^{N} Pr\left[D_n|\pi, F, s^2_{j=1,2,\cdots,J}\right]$$

or we can write it as

$$L(\pi, F) = \prod_{n=1}^{N} \sum_{k_1 < k_2} \sum_q \left[\pi_{k_1,k_2,q} \times \prod_{j=1}^{G} N\left(D_{nj}; qF_{k_1,g} + (1-q)F_{k_2,g}, s^2_j\right)\right]$$

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And the log likelihood

$$logL(\pi, F) = \sum_{n=1}^{N} log \left(\sum_{k_1 < k_2} \sum_{q} \left[\pi_{k_1, k_2, q} \times \prod_{j=1}^{G} N\left(D_{nj}; qF_{k_1,g} + (1-q)F_{k_2,g}, s_j^2 \right) \right] \right) \tag{3}$$

This is the quantity we want to maximize.

EM updates

We assume that q can take a finite set of values between 0 and 1, say $1/100, 2/100, \cdots, 90/100, 1$.

Suppose we have run upto m iterations. For the $(m+1)$th iteration, we have

$$
\begin{aligned}
\delta_{n,k_1,k_2,q}^{(m+1)} &= Pr\left[Z_n = (k_1, k_2), \lambda_n = q | \pi^{(m)}, F^{(m)}, s_{j=1,2,\cdots,J}^{(m)}, D_n \right] & (4)\\
&\propto Pr\left[Z_n = (k_1, k_2) \right] Pr\left[\lambda_n = q \right] Pr\left[D_n | \pi^{(m)}, F^{(m)}, s_{j=1,2,\cdots,J}^{(m)}, Z_n = (k_1, k_2), \lambda_n = q \right] & (5)\\
&\propto \pi_{k_1,k_2,q}^{(m)} \prod_j N\left(D_{nj} | qF_{k_1,j}^{(m)} + (1-q)F_{k_2,j}^{(m)}, s_j^{(m)^2} \right) & (6)
\end{aligned}
$$

where $s_j^{(m)^2}$ is the residual variance of feature j.

We normalize δ so that

$$\sum_{k_1 < k_2} \sum_{q} \delta_{n,k_1,k_2,q}^{(m+1)} = 1 \qquad \forall n$$

We define

$$\pi_{k_1,k_2,q}^{(m+1)} = \frac{1}{N} \sum_{n=1}^{N} \delta_{n,k_1,k_2,q}^{(m+1)}$$

We have therefore updated $\pi_{k_1,k_2,q}^{(m)}$ to $\pi_{k_1,k_2,q}^{(m+1)}$.

We define the parameter

$$\theta := \left(\pi_{k_1,k_2,q}, F, s_{j=1,2,\cdots,J} \right)$$

We define the complete loglikelihood

$$logL_c\left(\theta; D, Z, \lambda \right) = log\pi_{k_1,k_2,q} + logL(D|Z, \lambda, q, F)$$

We take the expectation of this quantity with respect to $\left[Z, \lambda | D, \theta^{(m)} \right]$.

$$Q(\theta | \theta^{(m)}) \propto - \sum_{n=1}^{N} \sum_{k_1 < k_2} \sum_{q} \delta_{n,k_1,k_2,q}^{(m+1)} \sum_{j} \left[logs_j^{(m+1)} + \frac{(D_{nj} - qF_{k_1,j} - (1-q)F_{k_2,j})^2}{2s_j^{(m+1)^2}} \right] \tag{7}$$

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We try to maximize this quantity with respect to F, So, we can take derivative with respect to F and try to solve the resulting normal equation.

This equation, conditional on $\left[Z, \lambda | D, \theta^{(m)}\right]$, can be written as

$$D_{N \times J} = L_{N \times K} F_{K \times J} + E_{N \times J} \qquad (8)$$

where

$$e_{nj} \sim N(0, s_j^2)$$

We define

$$D_{nj}' := \frac{D_{nj}}{s_j}$$

If we consider finding the factors on a feature-by-feature basis, we do not need to worry about s_j.

$$L_{nk} = \begin{cases} q \text{ or } (1-q) & \lambda_n = q \\ 0 & \text{o.w.} \end{cases}$$

We have

$$E_{Z,\lambda|D,\theta^{(m)}}\left[L_{nk}\right] = \sum_q \sum_{k_2 > k} q \delta_{n,k,k_2,q}^{(m+1)} + \sum_q \sum_{k_1 < k} (1-q) \delta_{n,k_1,k,q}^{(m+1)}$$

$$E_{Z,\lambda|D,\theta^{(m)}}\left[L_{nk}^2\right] = \sum_q \sum_{k_2 > k} q^2 \delta_{n,k,k_2,q}^{(m+1)} + \sum_q \sum_{k_1 < k} (1-q)^2 \delta_{n,k_1,k,q}^{(m+1)}$$

Also for any $k \neq l$,

$$E_{Z,\lambda|D,\theta^{(m)}}\left[L_{nk} L_{nl}\right] = \sum_q q(1-q) \delta_{n,k,l,q}^{(m+1)}$$

We use these to solve for the equation

$$\left[E_{Z,\lambda|D,\theta^{(m)}}\left(L^T L\right)\right] F \approx \left[E_{Z,\lambda|D,\theta^{(m)}}(L)\right]^T D$$

The solution therefore is

$$F \approx \left[E_{Z,\lambda|D,\theta^{(m)}}\left(L^T L\right)\right]^{-1} \left[E_{Z,\lambda|D,\theta^{(m)}}(L)\right]^T D$$

For $W = L^T L$

$$W_{kl} = \sum_n L_{kn} L_{nl}$$

$$E_{Z,\lambda|D,\theta^{(m)}}\left(W_{kl}\right) = \sum_n E_{Z,\lambda|D,\theta^{(m)}}\left(L_{nk} L_{nl}\right)$$

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We use the definition of $E_{Z,\lambda|D,\theta^{(m)}}\left[L_{nk}^2\right]$ and $E_{Z,\lambda|D,\theta^{(m)}}\left[L_{nk}L_{nl}\right]$ from above to solve F.

In the same way as we computed F by solving for the normal equation obtained from taking derivative of the function $Q(\theta|\theta^{(m)})$, we take derivative of the latter with respect to s_j^2 to obtain EM updates of the residual variance terms. Taking the derivative, we obtain the estimate as

$$\widehat{s_j^{(m+1)}}^2 = \frac{1}{N}\sum_{n=1}^{N}\sum_{k_1<k_2}\sum_q \delta_{n,k_1,k_2,q}^{(m+1)}(D_{nj}-qF_{k_1,j}-(1-q)F_{k_2,j})^2 \tag{9}$$

where the F are the estimated values of the factors from the previous step.

We then continue this procedure described above for multiple iterations.