Methods of Homotopy Theory in Algebraic Geometry from the Viewpoint of Cohomology Operations

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July 25, 2022

Outline

Motivation: methods of homotopy theory

$$\begin{array}{c} \text{the study of objects in geometry and topology} \\ \Big\downarrow {\scriptstyle \text{methods of homotopy theory}} \\ \text{the study of related homotopy classes} \end{array}$$

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Examples of methods of homotopy theory

Theorem (Steenrod 1951)

Let X be a paracompact space and G be a topological group, then

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Theorem (Thom 1954)

Let G be a subgroup of $\mathrm{GL}(F, k)$ for $F = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Let X be a manifold, then

$$\{\text{cobordism classes of } G\text{-submanifolds in } X\} \cong [X, MG]$$

Motivation: computational tools

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\text{the study of objects in geometry and topology} \\
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\text{the study of related homotopy classes} \\
\text{cohomology theories with their operations} \Big\downarrow \text{make them computable} \\
\text{computations in algebras}
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computations in algebras

How to process homotopy classes

(Multiplicative) cohomology theory E^*: $X \mapsto E^*(X)$ a $\mathbb{Z}$-graded module (algebra). (contravariant functors)

$$[X, Y] \xrightarrow{E^*} \underbrace{\mathbf{Maps}(E^*Y, E^*X)}_{\text{graded modules (algebras)}}$$

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$$[X, Y] \xrightarrow{E^*} \underset{\text{graded modules (algebras)}}{\mathbf{Maps}(E^*Y, E^*X)} \xrightarrow{\text{finer structure}} \underset{\text{graded } E^*E\text{-modules}}{\mathbf{Maps}(E^*Y, E^*X)}$$

compute it by homological methods!

Example: mod-2 ordinary cohomology theory

Let $H\mathbb{Z}/2$ be the mod-2 ordinary cohomology theory.

Theorem (Steenrod 1950s)

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- **Cartan's formula**: $Sq^i(uv) = \sum_{j=0}^{i} Sq^j(u) \cdot Sq^{i-j}(v)$.

The mod-2 Steenrod algebra

Let $\mathcal{A}_2^* := H\mathbb{Z}/2^* H\mathbb{Z}/2$ and it is called the **mod-2 Steenrod algebra.**

Theorem (Adem 1952)

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j, \ \text{if } 0 < a < 2b.$$

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Theorem (Serre 1953)

$\{Sq^I \mid$ all 2-admissible sequences $I\}$ is a $\mathbb{Z}/2$-basis of $\mathcal{A}_2^*$ and Adem relations determines the all the relations.

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Theorem (Steenrod 1962)

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Adem relations in mod-p ordinary cohomology theory

Let β be the mod-p Bockstein operations. If $a < pb$, then

$$P_p^a P_p^b = \sum_{j=0}^{[a/p]} \binom{(p-1)(b-j)-1}{a-pj} P_p^{a+b-j} P_p^j$$

if $a \leq b$, then

$$P_p^a \beta P_p^b = \sum_{j=0}^{[a/p]} \binom{(p-1)(b-j)-1}{a-pj} \beta P_p^{a+b-j} P_p^j + \sum_{j=0}^{[(a-1)/p]} (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} \beta P_p^{a+b-j} P_p^j$$

The mod-p Steenrod algebra

The mod-p Steenrod operation St_p^i is defiend as

$$St_p^i = \begin{cases} P_p^k, & i = 2k(p-1) \\ \beta P_p^k, & i = 2k(p-1)+1 \\ 0, & \text{otherwise.} \end{cases}$$

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Applications of the Steenrod operations

Theorem (Borel-Serre 1953)

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Theorem (Thom 1954)

Any mod-2 homology class of a finite complex K can be realized as a manifold. For any integral homology class y of K, there exists N such that Ny can be realized as an oriented manifold.

The classical Adams spectral sequences

Theorem (Adams 1958)

Given spaces or spectra X and Y, there exists a cohomological spectral sequence $\{E_*^{*,*}\}$ called **Adams spectral sequence** *such that*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p^*}(H\mathbb{Z}/p^* Y, H\mathbb{Z}/p^* X) \Rightarrow ([X, Y]_{t-s})_p^\wedge$$

where $([X, Y]_{t-s})_p^\wedge$ is the p-completion of the group of stable homotopy classes $\mathrm{colim}_n[\Sigma^{n+t-s} X, \Sigma^n Y]$.

If we let X, Y be points, then it converges to the p-completion of the stable homotopy group of spheres.

How Steenrod constructed mod-p power operations

$$\begin{array}{ccccc} H^n(X) & \xrightarrow{\qquad\mathcal{P}^d\qquad} & H^{nd}_{\Sigma_d}(X^d) & \xrightarrow{\Delta^*} & H^{nd}(B\Sigma_d \times X) \\ [u] & & [u^d]_{\Sigma_d} & & \\ n\text{-cocycle class} & & \Sigma_d\text{-equivariant } nd\text{-cocycle class} & & \end{array}$$

where $\mathcal{P}^d$ is called **the d-external power operation** and $\Delta^*\mathcal{P}^d$ is called **the d-total power operation**.

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Given $\alpha \in H_i(B\Sigma_d)$, then we have the cohomology operation derived from α is $[u] \mapsto \Delta^*\mathcal{P}^d([u]) \cap \alpha \in H^{nd-i}(X)$.

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If we replace Σ_d by $\mathbb{Z}/p$ and let $i = p$, then we get mod-p power operations.

Outline

Generalized cohomology theories and spectra

Definition (Spectra)

A spectrum $E = \{E_n, \varepsilon_n\}_{n\in\mathbb{Z}}$ is a sequence of pointed topological spaces E_n with basepoint-preserving maps $\varepsilon_n \colon \Sigma E_n \to E_{n+1}$. If $\varepsilon_n \colon E_n \to \Omega E_{n+1}$ is a weak homotopy equivalence, it is called an Ω-spectrum.

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Example

$H^n(X; A) = [X, K(A, n)]$. *In particular,* $(H\mathbb{Z}/p)_n := K(\mathbb{Z}/p, n)$.

Generalized cohomology theories and spectra

Definition

A morphism $f\colon E \to F$ between spectra consists of $\{f_n\colon E_n \to F_n\}$ compatible with Σ and ε_n.
Given a based space X and a spectrum E, $(E \wedge X)_n := E_n \wedge X$.
We say $f \simeq g\colon E \to F$ if there exists a map $h\colon E \wedge I_+ \to F$ such that $f = h_0$ and $g = h_1$.

the stable homotopy classes: $[E, F]^n := [E, \Sigma^n F]$.
the associated generalized cohomology: $E^*(X) := [\Sigma^\infty X, E]^*$.
the associated generalized homology: $E_*(X) := [\Sigma^\infty S^0, E \wedge X]^*$.

The stable homotopy categories

The essence is "inverting" S^1 with respect to $\wedge$ by stablizing it.

Theorem

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The construction of such categories is very complicated!
There are three popular constructions, we choose the category of $\mathbb{S}$-modules (EKMM) in this presentation.

The algebra of cohomology operations

Proposition

By Yoneda lemma and Brown's representability theorem, the algebra of cohomology operations on E is $E^*E := [E, E]^*$, the stable homotopy classes from E^* to itself.

In particular, $\mathcal{A}_p^* = H\mathbb{Z}/p^* H\mathbb{Z}/p$.

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Question

Given a ring spectrum E, how to determine power operations on E?

Extended powers and H_∞-structures

Given an $\mathbb{S}$-module E, the jth **extended power** of E is defined to be $D_j E = (E\Sigma_j)_+ \wedge E^j)/\Sigma_j$.

Definition

An H_∞-ring spectrum is a $\mathbb{S}$-module M together with $\xi_j \colon D_j M \to M$ for $j \geq 0$ satisfying some homotopy coherence conditions.

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Example

HR, KU and MU are H_∞-ring spectra.

H_∞-structures give rise to power operations

Let E be an H_∞-ring spectrum.

$$\begin{tikzcd}
E^*(X) \arrow[rrr, "\mathcal{P}_j"] \arrow[d, no head, "\cong"'] & & & E^*(B\Sigma_{j+} \wedge X) \arrow[d, no head, "\cong"] \\
{[\Sigma^\infty X, E]} \arrow[r, "D_j"] & {[D_j(\Sigma^\infty X), D_j E]} \arrow[r, "\circ\xi_j"] & {[\Sigma^\infty(D_j X), E]} \arrow[r, "\Delta^*"] & {[\Sigma^\infty(B\Sigma_{j+} \wedge X), E]}
\end{tikzcd}$$

Then we can derive power operations from $E_*(B\Sigma_j)$.

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$$E_2^{s,t} = \mathrm{Ext}_{E^*E}^{s,t}(E^*Y, E^*X) \Rightarrow [X, Y]_{t-s}^E$$

where $[X, Y]_{t-s}^E$ is the set of stable homotopy classes from X to Y in an E-localization shifting $t - s$.

If E is an H_∞-ring spectra, then the induced power operations appear in the E_2-page.

Outline

The construction of motivic homotopy theory

Construction (Morel-Voevodsky 1990s)

Let S be a qcqs Noetherian scheme of finite dimension and let Sm/S *be the category of smooth schemes of finite type over* S*. Let* $\Delta^{op}\mathbf{Shv}_{Nis}(\mathrm{Sm}/S)$ *be the category of* **Nisnevich sheaves** *of simplicial sets with* **projective model structure***. The unstable motivic homotopy cateogory is*

$$\mathcal{H}(S) := L_{\mathbb{A}^1}\Delta^{op}\mathbf{Shv}_{Nis}(\mathrm{Sm}/S)$$

where $L_{\mathbb{A}^1}$ *is the* **Bousfield localization with respect to the class generated by natural projections** $X \times_S \mathbb{A}^1 \to X$ **for all** $X \in \mathrm{Sm}/S$**.**

Spheres in motivic homotopy category

Definition (Spheres in motivic homotopy category)

Simplicial circle S^1_s *(or denote it* $S^{1,0}$*): the constant sheaf valued at the* $\Delta^1/\partial\Delta^1$*.*

Tate circle S^1_t *(or denote it* $S^{1,1}$*): the sheaf represented by* $\mathbb{G}_m$*.*

Given a, b *two non-negative integers with* $a \geq b$*,* **the bigraded motivic sphere** $S^{a,b} := (S^1_t)^{\wedge b} \wedge (S^1_s)^{\wedge a-b}$*.*

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Proposition

$S^{2n,n} \simeq \mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{A}^n/(\mathbb{A}^n - 0)$

The motivic stable homotopy category

Construction

Recall that we obtain classical stable homotopy category by "inverting" the circle S^1 from $\mathrm{h}(\mathcal{S}\mathrm{paces})$, we obtain motivic stable homotopy category $\mathcal{SH}(S)$ over S by "inverting" $\mathbb{P}^1 \simeq S^1_t \wedge S^1_s$ from $\mathcal{H}(S)$, whose objects are called motivc spectra.

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cohomology theory	classical spectrum	motivic spectrum
singular cohomology	$H\mathbb{Z}$	$H\mathbb{Z}_{mot}$
K-theory	KU	KGL
cobordism theory	MU	MGL

Table: Cohomology theories and spectra in classical setting and motivic setting

How motivic homotopy theory captures arithmetic data

Theorem (Morel 2004)

If k is a perfect field (with $\mathrm{char}\, k \neq 2$), then we have an isomorphism between graded rings

$$K_*^{MW}(k) \cong [S^0, S_t^1]_{\mathbb{P}^1}$$

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If k is a perfect field (with $\mathrm{char}\, k \neq 2$), then we have an isomorphism between rings

$$GW(k) \cong [S^0, S^0]_{\mathbb{P}^1}$$

The motivic Steenrod operations

Theorem (Voevodsky 2003)

There exists $P^i_\ell\colon H^{*,*}(X;\mathbb{Z}/\ell) \to H^{*+2i(\ell-1),*+i(\ell-1)}(X;\mathbb{Z}/\ell)$ *and* $B^i_\ell\colon H^{*,*}(X;\mathbb{Z}/\ell) \to H^{*+2i(\ell-1)+1,*+i(\ell-1)}(X;\mathbb{Z}/\ell)$ *such that*

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2. **Cartan formula**: *if* $\ell \neq 2$,

$$P^i_\ell(uv) = \sum_{j=0}^{i} P^j_p(u) P^{i-j}(v)$$

$$B^i_\ell(uv) = \sum_{j=0}^{i} B^j_\ell(u) P^{i-j}_\ell(v) + (-1)^{\deg(u)} P^j(u) B^{i-j}(v)$$

The motivic Steenrod operations

Theorem (Voevodsky 2003)

If $\ell = 2$, let $Sq^{2i} = P_2^i$, $Sq^{2i+1} = B_2^i$, τ be the generator of $H^{0,1}(K;\mathbb{Z}/2)$, and $\rho \in H^{1,1}(k;\mathbb{Z}/2)$ be the class of -1, then

$$Sq^{2i}(uv) = \sum_{j=0}^{i} Sq^{2j}(u)Sq^{2i-2j}(v) + \tau \sum_{s=0}^{i-1} Sq^{2s+1}(u)Sq^{2i-2s-1}(v)$$

$$\begin{aligned} Sq^{2i+1}(uv) = &\sum_{j=0}^{i}(Sq^{2j+1}(u)Sq^{2i-2j}(v) + Sq^{2j}(u)Sq^{2i-2j-1}(v)) \\ &+ \rho \sum_{s=0}^{i-1} Sq^{2s+1}(u)Sq^{2i-2s-1}(v) \end{aligned}$$

The Milnor conjecture and the Bloch-Kato conjecture

Voevodsky used motivic Steenrod operations to prove the following two theorems:

Theorem (Milnor conjecture, Voevodsky 2003)

Let k be a field of characteristic not equal to 2, then the norm residue homomorphisms $K_n^M(k)/2 \to H^n_{ét}(k; \mathbb{Z}/2)$ are isomorphisms for all $n \geq 0$.

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Theorem (Bloch-Kato conjecture, Voevodsky 2010)

Let k be a field of characteristic not equal to a prime ℓ, then the norm residue homomorphisms $K_n^M(k)/\ell \to H^n_{ét}(k;\mathbb{Z}/\ell)$ are isomorphisms for all $n \geq 0$.

The motivic Steenrod algebras

Theorem (Voevodsky 2003,Voevodsky 2011)

Let k be field and ℓ be a prime coprime to $\mathrm{char}(k)$, and k contains a primitive ℓth root of unity. Then the motivic cohomology

$$\mathbb{M}_\ell := H^{*,*}(k;\mathbb{Z}/\ell) \cong \frac{K_*^M(k)}{\ell}[\tau]$$

where $K_*^M(k)/\ell$ has degree (n,n) and τ is of degree $(0,1)$.

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Theorem (Voevodsky 2003)

The bigraded motivic Steenrod algebra $\mathcal{A}_\ell^{*,*}$ on mod-ℓ motivic cohomology is generated by P_ℓ^i and B_ℓ^i over $\mathbb{M}_\ell$ and is characterized by motivic Adem relations.

The motivic Adams spectral sequences

Theorem (Dugger-Isaksen 2010, Hu-Kriz-Ormsby 2011, Kylling-Wilson 2019)

Let k be a field of characteristic not equal to a prime ℓ, let $\mathbb{M}_\ell := H^{*,*}(k;\mathbb{Z}/\ell)$, there is spectral sequence called **motivic Adams spectral sequence** *such that*

$$E_2 = \mathrm{Ext}_{\mathcal{A}_\ell^{*,*}}(\mathbb{M}_\ell, \mathbb{M}_\ell) \Rightarrow [\Sigma^\infty_{s,t}\mathrm{Spec}(k), \Sigma^\infty_{s,t}\mathrm{Spec}(k)]^{\mathbb{A}^1_k}_{*,*}$$

Outline

How the Adams spectral sequences detect information

We summerize Bruner's mechanism in the following diagram.

$$\begin{tikzcd}
\text{power operations} \arrow[d, "\text{displayed}"'] & H_\infty\text{-structure} \arrow[l, "\text{induces at}"', "\text{cohomology level}"] \arrow[r, "\text{induces at}", "\text{homotopy level}"'] & \text{extended powers} \arrow[d, "\text{displayed}"] \\
\underset{E_2\text{-page}}{\mathrm{Ext}^{s,t}_{E^*E}(E^*Y, E^*X)} \arrow[rr, Rightarrow, "\text{differentials in the Adams spectral sequence}", "\text{converges along the Adams spectral sequence}"'] & & \underset{E_\infty\text{-page}}{[X, Y]^E_{t-s}}
\end{tikzcd}$$

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 & & \\
 & \boxed{\textbf{How to convert cohomology operations into homotopy operations}} \arrow[uu, Rightarrow, "\text{encoded}"'] \arrow[d, Rightarrow, "\text{presented}"] & \\
\mathrm{Ext}^{s,t}_{E^*E}(E^*Y, E^*X) \arrow[rr, Rightarrow, "\text{differentials in the Adams spectral sequence}", "\text{converges along the Adams spectral sequence}"'] & & {[X, Y]^E_{t-s}} \\
E_2\text{-page} & & E_\infty\text{-page}
\end{tikzcd}
$$

What hides behind the power operations

$$\begin{tikzcd}
\begin{array}{c} E_\infty\text{-structure} \\ \scriptstyle E_\infty\text{-operad action} \end{array} \arrow[d, "\text{pass to homotopy}"] & \text{multiplicative coherence data} \arrow[l, "\text{encoded in}"'] \\
\begin{array}{c} H_\infty\text{-structure} \\ \scriptstyle E_\infty\text{-operad homotopy action} \end{array} \arrow[d, "\text{cohomological presentation}"] & \\
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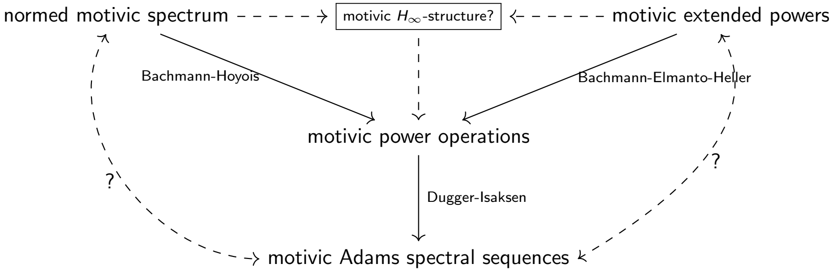
\begin{array}{c} E_\infty\text{-structure} \\ \scriptstyle E_\infty\text{-operad action} \end{array} \arrow[d, "\text{pass to homotopy}"] & \text{multiplicative coherence data} \arrow[l, "\text{encoded in}"'] \arrow[r, "\text{encoded in}"] & \text{motivic norm structure} \arrow[dd, dashed, "\text{Bachmann-Hoyois}"'] \\
\begin{array}{c} H_\infty\text{-structure} \\ \scriptstyle E_\infty\text{-operad homotopy action} \end{array} \arrow[d, "\text{cohomological presentation}"] & \boxed{\text{intrinsic symmetry}} \arrow[u, "\text{encoded in}"'] & \\
\text{power operations} & & \text{motivic power operations}
\end{tikzcd}
$$

How motivic extended powers emerge in the motivic Adams spectral sequences



Question & Answer

Thank you!