A Risk Charge Calculation Based on Conditional Probability

Topic #1: Risk Evaluation
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Abstract
In this paper, a method will be illustrated which begins at the aggregate (portfolio) level for evaluating risk, and ends by producing prices for the component individual risks, effectively allocating the total portfolio risk charge. The result is an internally consistent allocation of diversification benefits. The method effectively extends any risk-valuation theory used at the aggregate portfolio level to the individual risks comprising the portfolio. The resulting prices are additive, with each risk's price reflecting the degree to which it contributes to total portfolio risk.

Keywords: risk charge, allocation, conditional probability, additivity.

1. Background and Introduction

There are several methods for assigning risk charges to individual risks within a portfolio. Among them are utility functions, risk-adjusted probabilities, risk-adjusted weights, etc. After applying any of these methods to price individual risks, the issue of covariance and diversification must then be dealt with, because the portfolio owner's real exposure is to the aggregate portfolio result. In other words, there is no risk other than portfolio risk – risk is aggregate by its nature.

Accounting for aggregate portfolio effects in property-casualty insurance prices has historically created some difficult problems, including:

1) Additivity or sub-additivity of prices;
2) Measuring how much diversification efficiency actually exists;
3) Allocating the diversification benefits back to the individual risks; and
4) Order-dependence.

We begin with the following premise: Several separate but somewhat interdependent risk-bearing financial quantities are held as a risk portfolio over a specific time horizon. The type of value that is "at risk" can be selected in any reasonable way: liquidation value, book value, or the change over the specified time period in an alternative calculation of value. We assume that the following are given:

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- The joint distribution of outcomes for the risks at the time horizon's end; and
- The relative values to the portfolio owner of the possible aggregate outcomes (possibly reflecting risk-averse valuation).

In [1], Venter showed that covariance loadings can be used to produce additive, arbitrage-free risk charges, and also showed that a covariance loading results from a risk-adjusted distribution that is based on the conditional expectation of a "target" variable. Mango, in Appendix B of [2], demonstrated a method of allocating an overall capital cost charge to individual portfolio components using a similar concept. The ratio of price to probability (the "pricing density" function) was described and analyzed in a paper by Buhlmann [3]. Ruhm [4] analyzed arbitrage-free risk loads in terms of the price/probability ratio (the "risk discount" function).

In this paper, a method will be illustrated which synthesizes some results from each of these papers. The method begins at the aggregate level for evaluating risk, and ends by producing prices for individual risks, effectively allocating the total portfolio risk charge. The result is an internally consistent allocation of diversification benefits, avoiding the difficulties listed above. The method effectively extends any risk-valuation theory used at the aggregate portfolio level to the individual risks comprising the portfolio. The resulting prices are additive, with each risk's price reflecting the degree to which it contributes to total portfolio risk.

2. An Illustrative Example

Before providing a formal, mathematical description of the method, an example will help to illustrate the idea. (This example is summarized in Exhibit 2, which is a printout of the Microsoft Excel workbook **Bowles Ruhm-Mango Exhibit 2**, posted on the CAS website.) For clarity of presentation, the simplest possible case will be analyzed: a portfolio of only two risks, **Risk 1** and **Risk 2**, each of which has only two possible outcomes, a loss of either **100** or **200**. Net present value factors are omitted for simplicity, although in practice they would be applied to obtain a final price.

Suppose that losses for the two risks are distributed jointly as follows:

Joint Loss Distribution		**Risk 2 Loss =**		
		100	200	*Row Total*
Risk 1 Loss =	100	35%	15%	*50%*
	200	25%	25%	*50%*
Column Total		*60%*	*40%*	100%

Expected values are 150 for Risk 1 and 140 for Risk 2, with 20% correlation. The possible aggregate outcomes and their probabilities are determined by this structure:

Portfolio Outcome	Probability	Comments
200	35%	Both risks = 100
300	40%	One = 100, the other = 200
400	25%	Both risks = 200

At this point, valuation for risk comes into play. If the valuation is risk-neutral, meaning that there is no pricing adjustment for risk, then the value of the portfolio is simply its expected value:

Expected loss = 200*35% + 300*40% + 400*25% = **290**

Expected loss is a risk-neutral calculation; there are implicit **outcome weights** within the formula, all equal to **1.0**:

Expected loss = 200*35%***1.0** + 300*40%***1.0** + 400*25%***1.0** = **290**

One way to introduce a risk adjustment is by giving **outcome-specific** weights in the expected value calculation. To produce risk-averse valuation, the more severe (higher loss) outcomes would receive larger weights, and the less severe outcomes would receive lower weights; for example,

Portfolio Outcome	Risk-Averse Outcome Weight
200	0.500
300	1.000
400	1.250

The weights could come from a utility-based derivation, an options-formula method, or any other source (including judgment) – the technique presented here is independent of the particular portfolio risk adjustment theory, and will operate with any of them.

After normalizing these weights (scaling them so their expected value is one), the aggregate table is:

Portfolio Outcome	Outcome Probability	Normalized Weight
200	35%	0.563
300	40%	1.127
400	25%	1.408
Expected Value = **290**	Total = **100%**	Expected Value = **1.000**

The risk-adjusted price for the total portfolio can now be calculated as the expected weighted outcome:

Risk-adjusted expected loss =

200*35%***0.563** +
300*40%***1.127** +
400*25%***1.408**
= **315**

This price can also be produced by a set of risk-adjusted probabilities, which are the products of the actual probabilities and the normalized weights:

Portfolio Outcome	Actual Probability	Risk-adjusted Probability
200	35%	20%
300	40%	45%
400	25%	35%
Expected Value = **290**	Total = **100%**	Total = **100%**
Risk-adjusted Expected Value = **315**		

The risk-adjusted expected value shown is a risk-loaded price for the total portfolio. Thus, a risk charge of **25** (=315 – 290) is implied by the set of relative weights and the probability distribution of the aggregate portfolio outcomes.

The risk charge will now be allocated to the individual risks. This allocation is based on the conditional relationship between each risk's outcomes and the portfolio's possible outcomes, so that each risk receives a charge that represents **how much it contributes to undesirable portfolio outcomes.** This principle is the basis of the method. The resulting prices are additive, so that the price of any combination of risks is found by simply adding the individual prices. The major advantage of this approach, which will be explored further below, is that it can handle **any underlying dependence structure between the component risks.**

3. Application of the Conditional Structure to Calculate Individual Risk Prices

As shown above, the price of the total portfolio is found by calculating the weighted expected value of the outcomes, using a set of normalized risk-adjustment weights. The pricing calculation for individual risks proceeds in essentially the same way.

In the example, Risk 1 has two possible outcomes, 100 and 200. Each of these outcomes will be assigned a risk-adjustment weight, and the price for Risk 1 will be calculated as the weighted expected value.

If the Risk 1 = 100, there are only two possibilities for the portfolio's total outcome: 200 (if Risk 2 also = 100) or 300 (if Risk 2 = 200). Given that Risk 1 = 100, the probabilities for Risk 2 are 70% and 30%, from Bayes' Theorem:

$$P(A \mid B) = P(A \text{ and } B) / P(B)$$

$$P(\text{Risk } 2 = 100 \mid \text{Risk } 1 = 100) = 35\% / 50\% = 70\%$$
$$P(\text{Risk } 2 = 200 \mid \text{Risk } 1 = 100) = 15\% / 50\% = 30\%$$

The weight for the situation (Risk 1 = 100) is then calculated as follows:

Portfolio Outcome	Conditional Probability	Normalized Weight
200	70%	0.563
300	30%	1.127
400	0%	1.408
	Total = **100%**	Expected Value = **0.732**

By the same procedure, the weight for the (Risk 1 = 200) situation is calculated:

Portfolio Outcome	Conditional Probability	Normalized Weight
200	0%	0.563
300	50%	1.127
400	50%	1.408
	Total = **100%**	Expected Value = **1.268**

Then, the price for Risk 1 is calculated as a weighted expected value, just as in the earlier calculation of the portfolio price:

Risk 1 Outcome	Outcome Probability	Normalized Weight
100	50%	0.732
200	50%	1.268
Expected Value = **150**	Total = **100%**	Expected Value = **1.000**
Risk-adjusted Expected Value = **163**		

Note that the weights for Risk 1's outcomes have an expected value of exactly one. This means that the calculation can also be expressed in terms of risk-adjusted probabilities, which are the products of the actual probabilities and the weights:

Risk 1 Outcome	Actual Probability	Risk-adjusted Probability
100	50%	36.6%
200	50%	63.4%
Expected Value = **150**	Total = **100%**	Total = **100%**
Risk-adjusted Expected Value = **163**		

The tables for Risk 2, derived in an identical manner, are:

Risk 2 Outcome	Outcome Probability	Normalized Weight
100	60%	0.798
200	40%	1.303
Expected Value = **150**	Total = **100%**	Expected Value = **1.000**
Risk-adjusted		

Expected Value = **152**		

Risk 2 Outcome	Actual Probability	Risk-adjusted Probability
100	60%	47.9%
200	40%	52.1%
Expected Value = **150**	Total = **100%**	Total = **100%**
Risk-adjusted Expected Value = **152**		

The prices for Risk 1 and Risk 2 add to the total portfolio price, as desired.

Following Venter [1], the conditional method can be conveniently expressed as a covariance risk load formula that can be applied to any risk, including any derivative of a portfolio component (such as an excess loss layer):

$$\text{Risk Load} = \text{Cov}(Z, R),$$

where Z represents the normalized weight (as a function of the aggregate portfolio outcome) and R represents the individual risk's outcome. The reader can verify by inspection, using the definition of covariance, that all the risk loads derived in the example above are produced by this formula.

In summary, the key points just demonstrated are:

1. The total portfolio risk charge is determined by risk assessment at the aggregate level;
2. This is split to the individual risks based on the conditional relationship between the risks' outcomes and the aggregate results for the portfolio.
3. All prices are completely determined by the portfolio-level weights (which can be interpreted as risk relativities) and the probability structure, so that no other information is required.
4. Correlations between risks (and between each risk and the portfolio) are included in the prices in full detail, via the conditional probabilities.
5. Prices produced by this method are additive.
6. Being based on risk-adjusted probabilities, the prices are arbitrage-free within the context of the portfolio and its specified risk valuation structure (i.e., the specified set of weights).
7. The method can be summarized as a covariance risk load formula, where the reference variable is the set of normalized risk relativities.

4. The State-Price Structure Underlying the Example

An implicit state-price structure underlies the prices calculated by this method, where the states are defined as the possible combinations of the risks' outcomes:

State	Aggregate Outcome	Weight	Probability	State Price
(100, 100)	200	0.563	35%	0.197
(100, 200)	300	1.127	15%	0.169
(200, 100)	300	1.127	25%	0.282
(200, 200)	400	1.408	25%	0.352
	Weighted Expected = **315**	Expected = **1.000**	Total = **100%**	Total = **1.000**

Each "state price" is the product of the normalized weight and the state's probability. The state prices add to exactly one. They are the risk-adjusted probabilities underlying the risks' prices.

Any one of these state prices can be interpreted as the (undiscounted) value of an instrument (a derivative instrument of the two risks) that pays one dollar if the specified state occurs, and zero otherwise. Since exactly one of the states must occur, the states' prices should add to one, because a portfolio holding exactly one of each derivative will produce one dollar with certainty. (This is what was meant by the phrase "internally consistent in the arbitrage-free sense," as used above.) Normalizing the weights causes the state prices to add up to exactly one.

In assuming this two-risk portfolio, the portfolio-holder has effectively taken a short position in 200 of the (100,100) instruments, 300 of the (100,200) instruments, 300 of the (200,100) instruments, and 400 of the (200,200) instruments. One can multiply these amounts by their respective state prices and verify that the total price of this combination equals the total portfolio price of 315.

5. A More Detailed Example

Exhibit 1 shows summarized results of applying this method to underwriting results from the Bohra/Weist paper [7] submitted to the CAS 2001 DFA Call for Papers on "DFA Insurance Company." The Microsoft Excel workbook **Bowles Ruhm-Mango Exhibit 1** demonstrating this will be posted on the CAS website.

Exhibit 1 - Conditional Risk Charge Demo using DFAIC

(1) Expected U/W Income **(96,952)**

(2) Risk Adjustment Curve Parameters

Upside Scale	**1,000,000**
Upside Shape	**200.00%**
Downside Scale	**100,000**
Downside Shape	**50.00%**

(3) Risk-Adjusted Expected U/W Income **(244,714)**

(4) Portfolio Risk Premium = (1) - (3) **147,762**

(5) LOB	(6) Expected Loss	(7) Expected U/W Income	(8) Expected Risk-adjusted U/W Income	(9) = (7) - (8) Allocated Risk Premium	(10) Risk Premium as % of E[L]
CA	115,995	(10,946)	(23,014)	12,068	10.4%
CMP	221,025	(7,910)	(23,152)	15,242	6.9%
HO	220,787	(19,460)	(67,474)	48,013	21.7%
PPA	437,352	(54,963)	(117,554)	62,591	14.3%
WC	145,131	(3,673)	(13,520)	9,847	6.8%
TOTAL	**1,140,291**	**(96,952)**	**(244,714)**	**147,762**	**13.0%**

The valuation formula used to determine the risk-averse outcome weighting is a two-sided utility transform of total underwriting income $\mathbf{UI_T}$ to risk-adjusted underwriting income $\mathbf{RUI_T}$ via the following formula:

If $\mathbf{UI_T} >= 0$ $\quad \mathbf{RUI_T} = \mathbf{UI_T} * [\ 1 + (\mathbf{UI_T} / 1M\ )^2\ ]$
Else $\quad \mathbf{RUI_T} = \mathbf{UI_T} * [\ 1 + (-\mathbf{UI_T} / 100K\ )^{0.5}\ ]$

Section (2) on Exhibit 1 shows these parameters and curve forms, which were selected to calibrate to a desired overall implied portfolio risk premium, calculated as follows:

(1) E[$\mathbf{UI_T}$] = (\$96.9M)
(3) E[$\mathbf{RUI_T}$] = (\$244.7M)

(4) Implied Portfolio Risk Premium = E[$\mathbf{UI_T}$] - E[$\mathbf{RUI_T}$] = **\$147.8M**

$\mathbf{RUI_T}$ is calculated at the scenario level. The ratio of { $\mathbf{RUI_T}$ / $\mathbf{UI_T}$ } by scenario – the "**scenario weighting**" – is then multiplied by each LOB's U/W income at the scenario level, to produce risk-adjusted U/W income by scenario, by LOB. The expected value of both the unadjusted and risk-adjusted underwriting income results for each LOB are shown in columns (7) and (8) of Exhibit 1. The Allocated Risk Premium by LOB equals the expected unadjusted U/W income minus the expected risk-adjusted U/W income—see Column (9). Column (10) displays these values as percentages of expected loss, putting them in a common format for inclusion in any premium-loading formula.

6. Derivation of Conditional Risk Charge Formulas (Discrete Case)

Assume a portfolio containing n risks with common time horizon T.

Definitions

- R_i = the outcome of the i^{th} risk at time T.
- $w = \{R_1,..,R_n\}$ = the state at time T, as defined by the portfolio.
- $N = N(w) = \Sigma R_i$ = the aggregate portfolio result.
- V(N) = the valuation function that maps the aggregate portfolio result to its value. V(N) is analogous to a utility function, but is distinct since it applies to portfolio wealth rather than total agent wealth.
- Z(N) = V(N)/N = the valuation weighting function. V(N) is scaled so that E[Z] = 1.
- p( ) denotes the probability operator, and E[] denotes the expectation operator. Unless otherwise noted, expectations are taken across state w.
- v = the risk-free present value factor corresponding to the time horizon T.
- P = vE[V] = the total value of the portfolio. The additive definition of the porfolio value is consistent with arbitrage-free valuation, and is based on the implicit assumption that V(N) completely represents the values of the possible aggregate portfolio outcomes, with no additional modification necessary.

Conclusion 1: $P = v\Sigma_i E[ZR_i]$.

Proof: $P = vE[V] = vE[ZN] = vE[Z\Sigma_i R_i] = vE[\Sigma_i ZR_i] = v\Sigma_i E[ZR_i]$.

Additional Definitions

For fixed i, define the following variables:

- $P_i = vE[ZR_i] = v\{E[R_i] + Cov(Z , R_i)\}$. (By Lemma 1, $P = \Sigma_i P_i$.)
- X(i) = {possible values taken by R_i}
- N(r) = {possible values of $N \mid R_i = r$}

Conclusion 2: $P_i = vE[rE[Z \mid R_i = r]]$.

Proof: $P_i = vE[ZR_i] = vE[E[Zr \mid R_i = r]] = vE[rE[Z \mid R_i = r]]$.

Corollary: $P = v\Sigma_i E[rE[Z \mid R_i = r]]$.

In practice, the calculation of P_i can be performed by taking the inner expectation across values of N (since Z is determined by N), and taking the outer expectation across values of R_i:

$$P_i = vE_{r\in X(i)}[rE_{n\in N(r)}[Z(n) \mid R_i = r]].$$

This formula encapsulates the method shown above and in the exhibits.

7. A Connection to CAPM Pricing
The Capital Asset Pricing Model ("CAPM") specifies expected returns for individual securities in terms of the total market return, under certain idealized conditions [6]:

$$E[R_i] = r_f + \beta(E[R_M] - r_f)$$

By definition, expected return translates to price, provided the expected future value is known:

$$\text{Price} = E[\text{Future Value}] / (1 + E[\text{Return}])$$

The CAPM formula can therefore be viewed as a pricing formula, given the expected future value of the security. Also, the formula is similar to the conditional risk charge method, in that the portfolio-level risk premium (the spread above risk-free, $(E[R_M] - r_f)$, which corresponds to a risk charge) is taken as an input, and is used to calculate risk premia for the individual component securities which comprise the market portfolio.

If we view the market as a portfolio, we can apply the conditional risk charge method to the idealized CAPM scenario. Since the CAPM theory already generates the prices that must occur in such a market, the question that naturally occurs is, "Would the conditional risk charge method produce correct prices for the individual securities in the CAPM world?"

One would expect the answer to be "yes", since the conditional method produces a covariance risk load, and the CAPM also produces covariance risk premia. The connection is shown as follows:

Let M represent the future value of a portfolio that is comprised of all stocks in the same proportion as in the total market (the "market portfolio"), and let P represent the current price of the market portfolio. Suppose there exists a weighting function on market return, $Z(R_M)$, such that:

$$E[Z] = 1$$
$$P = E[ZM] / (1 + r_f)$$

This is the characterization of portfolio risk charge that is the basis for the conditional method. (The existence of Z will be demonstrated below by construction.) The second condition is equivalent to $E[ZR_M] = r_f$:

$$E[ZM] = P(1 + r_f)$$
$$E[ZM/P] = 1 + r_f$$

Using $M/P = (1 + R_M)$,

$$E[Z(1 + R_M)] = 1 + r_f$$
$$E[Z] + E[ZR_M)] = 1 + r_f$$

$$E[ZR_M] = r_f$$

For any stock, define ε_i by:

$$R_i = r_f + \beta(R_M - r_f) + \varepsilon_i$$

By taking expectations and covariances with respect to R_M on both sides, we obtain:

$$E[\varepsilon_i] = 0$$
$$Cov[R_M , \varepsilon_i] = 0$$

Multiplying by Z and taking expectations yields:

$$E[ZR_i] = r_f\, E[Z] + \beta(E[ZR_M] - r_f\, E[Z]) + E[\varepsilon_i]E[Z],$$

using the fact that Z is a function of R_M and the independence of R_M and ε_i. Then,

$$E[ZR_i] = r_f\, (1) + \beta(r_f - r_f\, (1)) + 0$$
$$E[ZR_i] = r_f$$
$$E[Z(1+R_i)] = 1 + r_f$$

Letting P_i and S represent the price and future value of the stock, respectively:

$$(1+R_i) = S / P_i$$
$$E[Z(S / P_i)] = 1 + r_f$$
$$P_i = E[ZS] / (1 + r_f)$$

The last equation is the conditional risk charge formula, with the present value factor made explicit. Thus the price implied by the CAPM formula is the conditional method's price.

An example of $Z(R_M)$ can be explicitly constructed. Define $Z(R_M)$ by:

$$Z(R_M) = f(R_M + E[R_M] - r_f) / f(R_M),$$

where f() is the probability density function for R_M. Then, Z satisfies the two conditions:

$$E[ZR_M] = \int R_M\, f(R_M + E[R_M] - r_f)\, dR_M$$

Substituting $u = R_M + E[R_M] - r_f$,

$$E[ZR_M] = \int (u - E[R_M] + r_f)\, f(u)\, du$$
$$E[ZR_M] = E[R_M] - E[R_M] + r_f$$
$$E[ZR_M] = r_f$$

Also,

$$E[Z] = \int f(R_M + E[R_M] - r_f)\, dR_M$$
$$E[Z] = 1$$

Under CAPM, f() is normal, and this $Z(R_M)$ function is derived by applying the Wang transform to the distribution of R_M [5].

Thus, the same mathematics can be used to derive the market price for a security in the CAPM model and an agent's price for a risk in the agent's portfolio. The only differences are the conditional probability structure and the relative risk weights specific to each situation. In this model, market pricing and agent pricing can be viewed as parallel calculations with different parameters.

8. All complete, additive pricing systems are represented by the covariance formula

To this point, we have shown that it is possible to obtain additive prices by using the conditional pricing method. Surprisingly, any set of additive prices must follow the conditional pricing formula:

$$\text{Price} = W\,(E[R] + \text{Cov}[R, Z]),$$

as long as the set of prices is "complete" (i.e., any derivative of the risks has a unique price under the pricing system). Thus, this formula characterizes all complete, additive pricing systems, and any such set of prices is fully described by its underlying Z-function and its "wealth transfer factor" W. (See Venter [1] for a related result concerning risk-adjusted probability distributions.)

This is proven as follows: For a collection of n risks with outcomes $R_1,\ldots,R_n$, let Ω represent the state-space of possible combinations of outcomes, and define the random variable $\omega\in\Omega$ as the realized outcome state (ω corresponds to the n-tuple of actual outcomes $(R_1,\ldots,R_n)$). For each $x\in\Omega$, define I_x as the indicator payoff function for the state x:

$$I_x(\omega) = 1 \text{ if } \omega=x, 0 \text{ otherwise}$$

$I_x(\omega)$ is the payoff function for the derivative that pays one dollar if state x occurs, and zero otherwise. Since the pricing system is complete, each such derivative has a price, which we will denote by $\pi(x)$. Define $Z^*(x) = \pi(x)/p(\omega=x)$, the ratio of price to probability for the state x. Then,

$$\begin{aligned} \text{Cov}[I_x(\omega), Z^*(\omega)] &= E[I_x(\omega)Z^*(\omega)] - E[I_x]E[Z^*] \\ &= \Sigma_{\omega\varepsilon\Omega}\, p(\omega=x)\, I_x(\omega)Z^*(\omega) - E[I_x]E[Z^*] \\ &= p(\omega=x)Z^*(x) - E[I_x]E[Z^*] \\ &= \pi(x) - E[I_x]E[Z^*]. \end{aligned}$$

Let $W = E[Z^*]$ and let $Z = Z^* / W$. Then $E[Z] = 1$, and:

$$\mathrm{Cov}[I_x, Z] = \mathrm{Cov}[I_x, Z^*/W] = (1/W)\mathrm{Cov}[I_x, Z^*] = (1/W)(\pi(x) - E[I_x]E[Z^*])$$
$$\mathrm{Cov}[I_x, Z] = \pi(x)/W - E[I_x]$$
$$\pi(x) = W\,(E[I_x] + \mathrm{Cov}[I_x, Z])$$

This proves the formula for the derivative corresponding to I_x. Since any combination of the risks (or their derivatives) is equivalent to a linear combination of the I_x-derivatives, the result follows from additivity of prices, expectations and covariances.

A portfolio containing exactly one I_x-derivative for each $x \in \Omega$ would pay \$1 with certainty. This means that $\Sigma_{x\varepsilon\Omega}\, \pi(x)$ represents the price for \$1 certain under the pricing system, which is what the factor "W" represents:

$$W = E[Z^*] = \Sigma_{x\varepsilon\Omega}\, p(\omega=x)Z^*(x) = \Sigma_{x\varepsilon\Omega}\, \pi(x)$$

If W differs from the risk-free discounted value of \$1, the pricing system implicitly includes a wealth transfer factor:

$$\text{Wealth Transfer Factor} = W(1+r)$$

In the case of a market, such as the insurance market, a conservative pricing system might rely on the availability of implicit wealth transfer from the market, which could be expected to disappear if and when market efficiency increases.

In summary, one can construct a complete, additive pricing structure by defining what constitutes risk (e.g., portfolio aggregate loss), assigning relative risk-weights, normalizing them, and selecting a wealth transfer factor. The main covariance pricing formula would then be applied to price any risk or derivative (e.g., risk layer or aggregate layer). Any additive, complete set of prices has an underlying set of normalized risk relativities (the Z function), and a wealth transfer scalar (W), and can be written as:

$$\text{Price} = W\,(E[R] + \mathrm{Cov}[R, Z])$$

9. Conclusion

The conditional risk charge method described in this paper can be used to extend a portfolio risk measure down to the level of individual risks and their derivatives, such as excess loss layers. The risk load can be expressed conveniently as covariance with portfolio risk relativity. The resulting prices are additive, and reflect complex dependence relationships between the risks. In this way, the price for a risk is representative of the extent to which it contributes to each potential aggregate outcome and the relative values those outcomes have to the portfolio holder.

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Conditional Risk Load: 2x2 Example

Exhibit 2

Shading indicates inputs

Non-Conditional Probabilities for Each Risk

Risk 1			Total / E[L]
Loss:	100	200	150
P[Loss]:	50%	50%	100%

Risk 2			Total / E[L]
Loss:	100	200	140
P[Loss]:	60%	40%	100%

Correlation Matrix of Risks

		Risk 2			
		100	200		
Risk 1	100	35%	15%	50%	
	200	25%	25%	50%	
		60%	40%	Correlation:	20%

Values of Possible Portfolio States

	Outcome State w	p[w]	Outcome's Relative Weighting	Normalized Weight Z[w]	Risk-Adjusted Probabilities	Weighted "Utility" Value
	200	35%	0.500	0.563	20%	113
	300	40%	1.000	1.127	45%	338
	400	25%	1.250	1.408	35%	563
Total / Exp'd	**290.00**	100%	0.888	1.000	100%	**315.49**
					Risk Load:	**25.49**

Decomposition of Z[w] ---> Z[Risk] by Conditional Analysis

		Individual Risk Events Braiding into States			
w	Z[w]	P[w|R1=100]	P[w|R1=200]	P[w|R2=100]	P[w|R2=200]
200	0.563	70.00%	0.00%	58.33%	0.00%
300	1.127	30.00%	50.00%	41.67%	37.50%
400	1.408	0.00%	50.00%	0.00%	62.50%
Total		100.00%	100.00%	100.00%	100.00%
E[Z | Rx=y]		**0.732**	**1.268**	**0.798**	**1.303**

Risk Loaded Pricing for Each Risk

Risk 1			Total / Exp'd
Loss:	100	200	150.00
P[Loss]:	50%	50%	100%
Z[Loss]:	0.732	1.268	**163.38**
Risk Load:			**13.38**

Risk 2			Total / Exp'd
Loss:	100	200	140
P[Loss]:	60%	40%	100%
Z[Loss]:	0.798	1.303	**152.11**
Risk Load:			**12.11**