Barycentrics for Incompatible Neighbors

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Abstract

When nodes on the common edge of adjacent elements do not coincide the elements are "incompatible". Standard barycentrics for such elements result in loss of continuity. Various approaches, including restricted variation of edge nodes or introduction of compatible transition elements, restore continuity. An alternative is to modify the element barycentrics in order to restore continuity. Concave polygons may be treated with mean-value coordinates. Alternative replacement of concave vertices with parabola side nodes allows "Wachspress" rational coordinates for all elements. Incompatible parabolic sides may be resolved but this discussion is restricted to polygons.

SOURCE OF INCOMPATIBLE ELEMENTS

Error analysis for compatible grids may indicate a need for local grid refinement. Interfaces of different compatible grids yield incompatible elements. Natural phenomena may have a cell structure with incompatible elements. One example is dried mud flats.

RESTORING CONTINUITY

Barycentric coordinates are a basis for degree-one approximation. Higher degree bases include added side nodes with associated interpolation bases. Vertex barycentrics are multiplied by "adjacent" factors that vanish on adjacent side nodes (Wachspress; 1975, 2016). The adjoint (denominator of the element barycentrics) is independent of side node placement. Continuity is restored by increasing the degree of variation on the incompatible side. Higher degree approximation over the element is not sought. Degree-one is retained. A simple incompatible side is shown in Fig. 1 where side node 3 is introduced on element A to yield quadratic variation determined by values at the three nodes on side (1,2). Vertex 2 is chosen as an exterior side-node of B and vertex 1 as an exterior side-node of C. The linear forms associated with these side nodes are of the horizontal lines through the nodes. This results in common quadratic variation on A, B and C determined by values at nodes 1, 2 and 3. This may be clarified with a specific example. Suppose a portion of a grid of unit squares is refined to a grid of squares with sides equal to 1/2 (four elements within each formerly unit square.) This leads to incompatible sides shown in Fig. 2. The barycentric coordinates for these three elements are:

For A, the vertex barycentrics are $W(1) = 2xy(y - .5)$, $W(5) = 2x(1 - y)(.5 - y)$, $W(6) = (1-x)(1-y)$, $W(7) = y(1-x)$. and the side-node barycentric is $W(2) = 4xy(1-y)$,

For B, the vertex barycentrics are $W(1) = 4y(y-.5)(1.5-x)$, $W(2) = 8y(1-y)(1.5-x)$, $W(3) = 4(1-y)(x-1)$, $W(4) = 4(y-.5)(x-1)$, and the side-node barycentric is $W(5) = 4(1-y)(.5-y)(1.5-x)$,

For C, the vertex barycentrics are $W(2) = 8y(1-y)(1.5-x)$, $W(3) = 4y(x-1)$, $W(8) = 4(.5-y)(x-1)$, $W(5) = 4(.5-y)(1-y)(1.5-x)$, and the side-node barycentric is $W(1) = 4y(.5-y)(1.5-x)$.

These barycentrics yield continuous piecewise degree-one approximation with linear variation over all sides other than on (1,2,5) and quadratic variation along (1,2,5).

For general polygons the linear forms at the side nodes must be chosen carefully. A quarilateral is adequate for demonstrating the choice (Fig. 3). The linear forms at side-node 5 must not disrupt linearity on sides (1,4) and (2,3). Thus, the adjacent factor R at

1 must be that of the line through 5 parallel to (1,4). The value of this linear is constant on (1,4) and may be normalized to unity at 1. Similarly the adjacent factor R at vertex 2 must be parallel to (2,3). The side coordinate is the product of the opposite sides divided by the adjoint. The incompatibility can be more complex. For example, consider the three to one refinement in Fig. 4 . Four values yield unique cubic variation on a line. The adjacent factors for the four elements are specified in Fig. 4.

BRICKWORK

Another example of incompatible elements is a brick wall (Fig. 5). All horizontal edges are incompatible. A midside node may be introduced on each horizontal edge. Now each A brick may be aligned with the B brick sharing the right half of its horizontal edge. Vertex 1 of A is side node 2.5 of B. Side node 1.5 of A is vertex 3 of B. Vertex 2 of A is side node 3.5 of B. Side node 2.5 of A is vertex 4 of B (Fig. 6). The barycentric coordinates for the original bricks are linear on this edge. Introduction of a linear factor for each side node increases the variation to cubic along the common edge. Values at these four nodes determine a unique cubic common to A and B on the edge.

CONCAVE SIDES TO PARABOLAS

A concave V may be replaced with a parabola whose apex is the side node of a parabolic side. The polygon becomes a well-set polycon. If the neighbor sharing the parabolic side is compatible all is well. However, when there are two neighbors as in Fig. 7 with (1,2,3) the parabolic side, the neighbors are incompatible. B and C now require side nodes 4 and 5 for barycentric coordinates. Compatibility is restored by quadratic variation determined by nodes 1-5. Side nodes 5 and vertex 2 of C are exterior side nodes of B and side node 4 and vertex 1 of B are exterior side nodes of C. Details of the construction of coordinates for polycons are beyond the scope of this note.

REFERENCES

E.L. Wachspress, A Rational Finite Element Basis (Academic Press, New York, 1975); Rational Bases and Generalized Barycentrics, (Springer, New York, 2016)

GitHub site genepol/barycentric