Barycentric Coordinates for Polycons

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April, 2018

Abstract

This discussion is limited to two-space although much of the analysis has been generalized to higher diimensions. Barycentric coordinates are needed in some applications for polygons with concave vertices. Such elements do not have rational barycentrics and mean-value coordinates (Floater, 2003) are widely used. Any element wiith concave vertices may be replaced with a "polycon" which is an element with conics as well as lines as sides. There are rational barycentric coordinates for any "well-set" polycon where well-set is a generalization of convex. A concave vertex between two convex vertices may be replaced by a side node on a parabola through the three vertices. For example, star polygons have alternating convex and concave vertices. A star polygon may thus be replaced by a polycon with only parabolic sides. In general, when two or more concave vertices separate convex vertices the lines between the two convex vertices may be replaced with a least-squares fit to a conic. Theoretical foundations and MATLAB implementation for these options are presented in this paper.

1. Review of barycentric coordinates for polygons

The symbol $(j_1, j_2, ..., j_t)$ denotes a curve defined by distinct points j_s. For a line t = 2, for a conic t = 5, for a cubic t = 9, etc. The symbol $(j_1; j_2; ...; j_t)$ with semi-colons instead of commas denotes the polynomial of lowest degree in x and y that vanishes on the curve. For any u(x,y) the symbol u_s is u normalized to unity at point s and the symbol $[u]_s$ is u evaluated at point s. Thus, $u_s = \frac{u}{[u]_s}$. A polygon with n sides is an "n-gon". Polygon vertices are indexed in ccw order. All indices are mod n.

Barycentric coordinates for an element that is one of a contiguous non-overlapping set of elements must interpolate nodal values with linear precision within each element and attain global continuity. The barycentric coordinate B_j associated with node j has the following properties:

1. $B_j(x, y)$ is bounded and continuous within the element.
2. $B_j(k) = \delta(j, k)$.
3. $B_j(x, y)$ is zero on all "opposite" sides (the n - 2 sides other than the two adjacent sides);
4. $B_j(x, y)$ is linear on "adjacent" sides (j-1, j) and (j, j+1).
5. [1 x y] $= \sum_j [1\ x_j\ y_j] B_j(x, y)$.

Property 2 is a discrete orthonormality property. An element is not considered in isolation. It is often one element in a partition of a region into non-overlapping contiguous elements. Barycentric coordinates interpolate within the element nodal values j of a function f into an approximation $H(x, y) = \sum_j f(x_j, y_j) B_j(x, y)$ with H = f when f is linear (degree-one interpolation of property 5.) Global continuity requires that vertex values off a side not affect the interpolation on the side (property 3) and the interpolant be the same on elements sharing a side (property 4). A linear function on (j, j+1) is uniquely defined by its values at vertices j and j+1. These properties are necessary and sufficient for the coordinates to provide a vertex basis for continuous, piecewise degree-one, approximation over a non-overlapping polygon partition of a bounded region $\mathbf{R}(x, y)$.

That barycentric coordinates for triangle [1, 2, 3] are

$$B_1 = (2;3)_1, B_2 = (1;3)_2, \text{ and } B_3 = (1;2)_3. \tag{1}$$

has been known for many years. That these linear coordinates satisfies the five conditions is apparent. Barycentrics for parallelogram [1, 2, 3, 4] are:

$$B_1 = (2;3)_1(3;4)_1, B_2 = (3;4)_2(4;1)_2, B_3 = (4;1)_3(1;2)_3, \text{ and } B_4 = (1;2)_4(2;3)_4. \tag{2}$$

Only for parallelograms where the opposite sides are parallel do these bilinear coordinates reduce to linear on adjacent sides. For example, (3;4) is constant on side (1,2) and (2;3) is constant on side (4,1). That property 5 is satisfied requires a simple observation. The bilinear approximation of f by H is precise on the boundary (with simple components) of order four. The only polynomial of degree two that vanishes on this curve is the zero polynomial. Hence, H = f within the n-gon. These coordinates have also been known for many years. Triangles and rectangles are ubiquitous in Finite-Element computation.

That polynomial barycentrics only apply to triangles and parallelograms was noted in early work. Suppose two sides of an element meet at non-vertex point p. Values at vertices are unrestricted. The linear variation on the sides will in general yield different values at p. Polynomials are single valued. Triangles and parallelograms are the only polygons that have no non-vertex intersections. Polynomial barycentrics cannot exist for any element with a non-vertex intersection of its sides.

In an attempt to broaden the class of elements for which barycentric coordinates exist, the obvious generalization was to rational bases. These were introduced in (Wachspress, 1975). [1]

The construction of rational barycentrics for convex polygons is reasonably simple. The barycentic coordinate at vertex j is $W_j = \frac{N_j}{Q}$ where Q is the same for all j. The linear forms of the sides are normalized to be positive within the n-gon. The n-gon boundary curve Γ of order n is the product of the linear forms of the n sides. Let $G \equiv \frac{\Gamma}{(j-1;j)(j;j+1)(j+1;j+2)}$. Let k_j be a normalization parameter to be determined. Then the product of the linear factors of the sides opposite j is $F_j = G(j+1;j+2)$ and $N_j = k_jF_j$ satisfies properties 1 and 3. Similarly, $F_{j+1} = G(j-1;j)$. The denominator is $Q = \sum_j N_j$. Then

$$W_j = \frac{k_jF_j}{Q} \tag{3}$$

satisfies property 2 and Q has been chosen so that the sum of the W_j is unity as required by property 5. The k_j are chosen to satisfy property 4, linearity on adjacent sides. On side (j, j+1), the only nonzero coordinates are W_j and W_{j+1}. Thus, dividing numerator and denominator by the common factor G:

$$W_j = \frac{k_jF_j}{k_jF_j + k_{j+1}F_{j+1}} \equiv \frac{k_j(j+1;j+2)}{k_j(j+1;j+2) + k_{j+1}(j-1;j)}. \tag{4}$$

The numerator is linear. Setting $k_1 = 1$, the remaining k_j may be found recursively so that the denominator is constant: $k_j[(j+1;j+2)]_j = k_{j+1}[(j-1;j)]_{j+1}$ and

$$k_{j+1} = k_j\frac{[(j+1;j+2)]_j}{[(j-1;j)]_{j+1}}. \tag{5}$$

The linear basis functions $B_{j+1} = (j-1;j)_{j+1}$ and $B_j = (j+1;j+2)_j$ sum to unity on $\mathbf{S}_j$. The numerator at j is $k_j(j+1;j+2)$ and the denominator is $k_j[(j+1;j+2)]_j$. The ratio is

[1] Graphic barycentric coordinates appear as wedges and I denoted them by W in my early work. Now they are called "Wachspress coordinates" so the W was fortuitous, although I was not unaware of this possibility at the outset. The theory has now been described in many papers and will not be repeated here. My recent book (Wachspress, 2016) reproduces the 1975 book as Part 1 and more recent work as Part 2.

$W_j = B_j$. The numerator at j+1 is $k_{j+1}(j-1;j)$ and the denominator is $k_{j+1}[(j-1;j)]_{j+1}$. The ratio is $W_{j+1} = B_{j+1}$. Linearity on the sides is established. [2]. All the factors in this analysis are positive within the n-gon so the coordinates are positive inside the boundary. This positivity is sometimes added to the properties of barycentric coordinates. It is not satisfied in the generalization to elements with curved sides. The only remaining property to be verified is 5. When f is linear, H - f = 0 on the boundary which is an irreducible curve of order n. Within the n-gon,

$$H - f = \frac{\sum_j f_j N_j(x,y) - f(x,y)Q(x,y)}{Q(x,y)}. \tag{6}$$

The numerator is a polynomial of maximal degree n-2 and the denominator is positive and bounded within the n-gon. The numerator vanishes on the boundary which is a curve of order n. Therefore, the numerator must be the zero polynomial. All five properties have now been verified and these W_j are barycentric coordinates for the n-gon.

The denominator Q is the sum of numerators of degree m-2. Theory establishes that the curve **Q** on which Q = 0 is the curve of minimal degree that contains all the external intersection points (eip) of the boundary components. This curve is of maximal order m-3. The order is less when parallel sides (which meet at infinity) are such that the "line at infinity" (absolute line in projective coordinates) is in curve **Q**. This curve is of order m-3 in the projective plane. The linear form of the absolute line becomes unity in the affine plane and the result is that Q is of degree less than m-3. An example is the regular hexagon which has three pairs of parallel sides. In this case **Q** is a circle of order m-4 = 2 instead of m-3.

2. Polycons

A polycon may have linear and conic sides with at least one conic side. A polycon with n sides is an n-con. The order m of an n-con is the number of linear sides plus twice the number of conic sides. A conic side common to two elements is convex in one element and concave in the other. Restriction to convexity limits conic sides to convex region boundaries. Rational barycentrics exist for "well-set" elements. A well-set element need not be convex. It is well-set when all vertices are simple transverse intersections of adjacent sides and when no side contains any point off the side itself that is on the element boundary or interior to the element. A convex polygon is well-set.

A linear function has two degrees of freedom on a line. Vertex values suffice to determine linear behavior on any linear side. A linear function has three degrees of freedom on a conic side. A side node j+1/2 is chosen on each conic side (j, j+1/2, j+1). These three nodes do not determine a unique conic. The symbol just indicates that the side is a conic already defined. Similarly (j; j+1/2; j+1) is the quadratic that vanishes on the side. Two other points must be specified to define this quadratic. Barycentric coordinates for conics satisfy the five properties enumerated for convex polygons.

Property 3 requires each vertex numerator N_j to vanish on the sides opposite vertex j. When the adjacent sides are both linear F_j is of degree m-2, when one side is conic, F_j is of degree m-3, and when both sides are conic F_j is of degree m-4. Property 2 requires that N_j vanish at the adjacent side nodes. An "adjacent" factor P_j is introduced. This factor must also vanish at the exterior intersections e of the adjacent sides. P_j at a line-line vertex is unity, at a line-conic vertex (e; j-1/2) or (e; j+1/2), and at a conic-conic vertex $(e_1; e_2; e_3; j-1/2; j+1/2)$. Adjacent factor P_j is normalized to unity at j. In all cases $F_j P_j$ is of degree m-2. The adjacent factor at all side nodes is unity and the opposite factor is of degree m-2. Hence, all numerators are of degree m-2. The denominator Q is called the polycon "adjoint". It has properties in relationship to the polycon boundary curve which are similar to those of an algebraic-geometry adjoint polynomial. The sum of the degree m-2 numerators is of degree not greater than m-3 when the k_j and $k_{j+1/2}$ are determined

[2]This recursive determination of the k_j was introduced in (Dasgupta, 2003) and was subsequently generalized to the GADJ algorithm for elements with curved sides in (Dasgupta and Wachspress, 2008)

with the linearity property. Once linearity on the sides is established, all five properties are satisfied.

When the eip are distinct and there are none at infinity, computation of the adjacent factors is simple. Otherwise one must allow for a variety of possibilities. The 600 line MATLAB program poly2018 computes barycentric coordinates for polycons. The first 300 lines contain a set of test problems and generate vertex, side node and eip coordinates. The next 100 lines determine the adjacent factors. The next 100 lines determine the numerator normalization parameters k_j and $k_{j+1/2}$ with the GADJ algorithm. The final 100 lines remove spurious terms in Q due to round-off error and specify the barycentric coordinates. The major complication in passing from polygons to polycons is determination of the adjacent factors and the numerator normalization parameters.

3. Algebraic-geometry foundations

Algebraic-geometry foundations essential for rational barycentric construction were developed in (Wachspress, 1975, 2016). The set of points where algebraic curves of orders p and q intersect are in the "divisor" of these curves. When all intersections are simple these are the pq elements of the divisor. When a point is at a common tangent of the curves or at a multiple point on either curve there are fewer than pq points. Divisor theory introduces elements which expand the intersection points into a set of pq divisor elements. Conics do not have multiple points. The full theory of divisors is not needed for application to polycons. Let P be the polynomial of least degree that vanishes on algebraic curve $\mathbf{P}$. The symbol $\mathbf{P} \cdot \mathbf{Q}$ denotes the divisor of curves $\mathbf{P}$ and $\mathbf{Q}$. In this application, $\mathbf{P} \cdot \mathbf{Q}$ is just the set of intersections of curves $\mathbf{P}$ and $\mathbf{Q}$ and consists of vertices and eip (external intersection points). The assertion that $P \equiv R \bmod \mathbf{S}$ (that is, "P is congruent to R mod $\mathbf{S}$ ") means that there is a scalar b such that $P - bR = 0$ on curve $\mathbf{S}$. The crucial algebraic-geometry theorem is:

$$\text{DIVISOR THEOREM:} \qquad \text{If } \mathbf{P} \cdot \mathbf{S} = \mathbf{R} \cdot \mathbf{S}, \text{ then } P \equiv R \bmod \mathbf{S}.$$

This deceptively simple theorem provides a powerful basis for construction of rational barycentrics. It enables analysis replacing barycentric coordinate factors along a side with equivalent factors so that factors common to numerator and denominator may be cancelled to establish linear rather than rational variation on a side. This plays a crucial role in the GADJ algorithm for computing the numerator normalization parameters k_j and $k_{j+1/2}$.

3. GADJ for polycons

The seminal construction of barycentric coordinates in (Wachspress, 1975) included computation of $\mathbf{Q}_{m-3}$ directly from the m(m-3)/2 multiple points of the extended element boundary curve $\mathbf{\Gamma}$. There are m(m-3) elements in $\mathbf{Q} \cdot \mathbf{\Gamma}$. These are precisely the eip from which $\mathbf{Q}$ is generated. Thus, Q cannot vanish on the boundary of the element and may be normalized to be positive there. It has been conjectured that Q does not vanish within any well-set element. Although this has yet to be proved, no counter-example has been found. Satisfaction of Property 1 rests on this conjecture. [3] The construction of Q was simplified with GADJ. Only the eip of adjacent sides are needed for computation of the adjacent factors. This obscures the fact that all the eip are included in $\mathbf{Q}$. For each coordinate, the numerator $kN \equiv kFP$ where P = 0 at all the eip of the adjacent sides and F = 0 at all eip on the opposite sides. Since Q is the sum of these numerators, Q = 0 at all the eip. The GADJ generalization to polycons described in (Dasgupta and Wachspress, 2008) has been improved. The polycon analysis proceeds as for polygons. Let S_r denote the linear or quadratic form that vanishes on side $\mathbf{S}_r$. Then polynomial G = $\frac{\Gamma}{S_{j-1}S_jS_{j+1}}$ is common to the coordinates that do not vanish on side $\mathbf{S}_j$. When G is divided out of numerator and

[3] In seeking a well-set counter-example of lowest order I have been able to prove that it must be convex with no linear sides. Although the algorithm has succeeded for all candidates thus far considered, a general proof is forthcoming.

denominator, the variation on side $\mathbf{S}_j$ is

$$W_j = \frac{k_j P_j S_{j+1}}{k_j P_j S_{j+1} + k_{j+1/2} S_{j-1} S_{j+1} + k_{j+1} P_{j+1} S_{j-1}}, \tag{7}$$

where the second term in the denominator appears only for a conic side.

Each term contains all the eip of $\mathbf{S}_j$ with $\mathbf{S}_{j-1}$ and $\mathbf{S}_{j+1}$. A unique curve $\mathbf{R}$ of least degree contains only these eip on side $\mathbf{S}_j$. This curve may be determined from the eip. This is complicated when some of the eip fall on the absolute line due to parallel lines or asymptotes. This complication has already been addressed in determining the adjacent factors. When S_j is linear there is no side node, $R = P_j P_{j+1}$ and $\frac{P_j S_{j+1}}{P_j P_{j+1}} \equiv \frac{S_{j+1}}{P_{j+1}}$. The linear basis functions $B_{j+1} = (j-1;j)_{j+1}$ and $B_j = (j+1;j+2)_j$ sum to unity on $\mathbf{S}_j$.

To apply the Divisor Theorem and find a recursion formula for k, we note that

$$\mathbf{P_j S_{j+1}} \cdot \mathbf{S_j} \text{ and } \mathbf{(j+1;j+2)R} \cdot \mathbf{S_j} = e_j, e_{j+1}, j+1.$$

Therefore, on side $\mathbf{S}_j$, $P_j S_{j+1} = b(j+1;j+2)R$ and if we define $b_j \equiv \left[\frac{S_{j+1}}{P_{j+1}}\right]_j$, the Divisor Theorem yields $b = b_j[(j+1;j+2)]_j$ and $\frac{S_{j+1}}{P_{j+1}} = b_j B_j$. Similarly, $\frac{S_{j-1}}{P_j} = b_{j+1} B_{j+1}$, where $b_{j+1} \equiv \left[\frac{S_{j-1}}{P_j}\right]_{j+1}$. The denominator $k_j b_j B_j + k_{j+1} b_{j+1} B_{j+1} = k_j b_j$ when

$$k_{j+1} = k_j \frac{b_j}{b_{j+1}}. \tag{8}$$

The situation is more complicated when $\mathbf{S}_j$ is conic. The product $P_j P_{j+1}$ now has an added double point at j+1/2. It is actually one degree higher than R. The only linear form that the product can be divided by to remove this double point is the tangent to $\mathbf{S}_j$ at j+1/2. Let $\mathbf{T}$ be this tangent. Then

$$R = \frac{P_j P_{j+1}}{T}. \tag{9}$$

The barycentric coordinates for triangle [j, j+1/2, j+1] are $B_j = (j+1/2;j+1)_j$ $B_{j+1/2} = (j;j+1)_{j+1/2}$ $B_{j+1} = (j;j+1/2)_{j+1}$. These coordinates sum to unity. The divisors for the conic side are now

$$\mathbf{P_j S_{j+1}} \cdot \mathbf{S_j} \text{ and } \mathbf{(j+1/2;j+1)R} \cdot \mathbf{S_j} = e_j, e_{j+1}, j+1/2, j+1,$$

and

$$\mathbf{P_{j+1} S_{j-1}} \cdot \mathbf{S_j} \text{ and } \mathbf{(j;j+1/2)R} \cdot \mathbf{S_j} = e_j, e_{j+1}, j, j+1/2,$$

and

$$\mathbf{S_{j+1} S_{j-1}} \cdot \mathbf{S_j} \text{ and } \mathbf{(j;j+1)R} \cdot \mathbf{S_j} = e_j, e_{j+1}, j, j+1.$$

The Divisor Theorem then yields $P_j S_{j+1} = b(j+1/2;j+1)R$ and when $b_j \equiv \left[\frac{P_j S_{j+1}}{R}\right]_j$,

$$\frac{P_j S_{j+1}}{R} = b_j B_j.$$

Similarly,

$$\frac{P_{j+1} S_{j-1}}{R} = b_{j+1} B_{j+1}.$$

For the side node, $S_{j-1} S_{j+1} = b(j-1;j+1)R$ and with $b_{j+1/2} \equiv \left[\frac{S_{j-1} S_{j+1}}{R}\right]_{j+1/2}$

$$\frac{S_{j+1} S_{j-1}}{R} = b_{j+1/2} B_{j+1/2}.$$

Dividing numerator and denominator in W_j by R, we find that the denominator is constant when $k_j b_j = k_{j+1/2} b_{j+1/2} = k_{j+1} b_{j+1}$ which gives the recursion:

$$k_{j+1} = k_j \frac{b_j}{b_{j+1}} \text{ and } k_{j+1/2} = k_j \frac{b_j}{b_{j+1/2}}. \tag{10}$$

Evaluating R at the vertices is straightforward. However, at the side node j+1/2 the numerator $P_j P_{j+1}$ and the denominator T each have a double point. The value of R at j+1/2 is computed with L'Hopital's rule. First $S_j = 0$ yields y = f(x) or x = f(y) on this side. The choice is made according to the slope of T. Let the derivative of any function V with respect to the retained variable be denoted by V'. Both adjacent factors vanish at j+1/2. Thus, the second derivative of $P_j P_{j+1}$ is $U \equiv 2P'_j P'_{j+1}$. The second derivative of T which is no longer linear with respect to the retained variable is $T\,''$. $[R]_{j+1/2} = [\frac{U}{T\,''}]_{j+1/2}$. On side $\mathbf{S}_j$,

$$W_j = \frac{k_j b_j B_j}{k_j b_j} = B_j.$$

Similarly, $W_{j+1/2} = B_{j+1/2}$ and $W_{j+1} = B_{j+1}$.

4.Parabola replacement of concave vertices

Star polygons often occur in graphics application. These polygons have alternating convex and concave vertices. Such elements do not have rational barycentric bases. Mean-value coordinates are well suited for star polygons. An alternative approach enabling rational barycentric coordinates is to replace concave Vs with parabolas with the concave polygon vertices as parabola side nodes. This was suggested in (Wachspress, 2016). These star polycons have only parabolic sides. Application of theory for construction of barycentric coordinates to elements with curved boundaries has been sparse. Star polycons provide impetus for new consideration. In general there are polygons and polycons with concave vertices that may be replaced with side nodes of parabolas. Contiguous elements must be constructed with the same curved sides (even though convex vertices may be replaced by side nodes.) In comparing rational and mean-value barycentrics a few properties warrant consideration. For convex polygons there are no significant differences. As an interior angle approaches 180^o both approaches lead to large gradients at the vertex. When the offending vertex is replaced with a parabola side node the gradient is well behaved. In the limit the parabola reduces to a line with a side midpoint and degree-two approximation on the side. As the angle passes through 180^o the element becomes concave and rational barycentrics exist only for the parabolic replacement. The rationals have smaller gradients when the angle is near 180^o. Mean-value barycentrics have not been developed for elements with curved sides. Mean-value coordinates can also treat adjacent concave vertices. Replacement of linear sides connecting these vertices with one or more conic sides yields elements amenable to rational barycentric coordinates.

Rational polycon barycentrics are not positive. However, positive approximation can be ensured by increasing values at side nodes to maintain positivity on the boundary and introducing an interior hat basis function to maintain positivity within the element. There are problems where large gradients are a physical property and should be retained as, for example, in stress computations where cracks form at sharp corners. Corners may be rounded to reduce crack formation. The mean-value coordinates are preferable for an element with sharp corners while rational bases with curved sides are preferable for a rounded element. The purpose of this note is not to suggest side nodes of parabolas as a ubiquitous replacement for concave vertices but rather to introduce this option.

The first new challenge is construction of a parabolic replacement of a concave V. The isoparametric parabola suggested in (Wachspress, 2016) is not best. The concave vertex is in general not at the vertex of the isoparametric parabola, but there is a unique parabola whose vertex is the replaced concave vertex. As the ratio of the lengths L1 and L2 of the arms of the V increase, the axis angle with the long side decreases. Let the V vertices be 1, 2 and 3 with 2 concave. If coordinates (xp,yp) are chosen with yp along the parabola axis and (0,0) as the parabola vertex, the parabola $yp = c \cdot xp^2$ requires $[\frac{xp^2}{yp}]_1 = [\frac{xp^2}{yp}]_3$. If δ is the interior angle of the V and β is the angle between the y-axis and the longer side, L_2, then for point 2 to be at the apex of a parabola through the three vertices:

$$V_\beta \equiv L_1[\frac{1}{\cos(\delta-\beta)} - \cos(\delta-\beta)] - L_2[\frac{1}{\cos(\beta)} - \cos(\beta)] = 0. \tag{11}$$

One may solve this transcendental equation analytically or approximate the solution by interpolation on a table of V_β for $\beta \in [0, \delta/2]$. This parabola must then be rotated and translated to its actual (x,y) position. Suppose $\delta = 90^o + \alpha$ for some positive $\alpha < 90^o$. For equal lengths the symmetry axis is at $\beta(r=1) = 45^o + \alpha/2$. As the ratio r of L_2/L_1 increases, $\beta(r)$ goes from $\beta(1)$ to a minimum of α. For example, when $\delta = 120^o$, the smallest β is 30^o. A value of r $= 5$ is satisfied with β around 45^o. Here, β goes from 60^o to 30^o as r varies from unity to infinity. When adjacent parabolic sides have parallel axes, two of the eip are on the absolute line (that is, at infinity). If the parabolas intersect at a point closer to the side node than the vertex, the vertex is moved to this point and the former vertex becomes an eip . In rare cases the parabolas may be tangent at the vertex. This leads to an interesting study establishing stability of its barycentric coordinates. This element is said to be "essentially well-set".

5. Essentially well-set elements

In three space dimensions, convex polyhedra restricted to vertices of order three were introduced in (Wachspress, 1975). Although the adjoint vanishes to order p - 3 where p is the order of the vertex, (Warren, 1996) realized that the singularity at the vertex is removable and constructed barycentric coordinates for all convex polyhedra. An alternative to Warren's construction was described in (Wachspress, 2010). Application to 2D elements was not considered. However, polycons with tangential intersection at parabola vertices may be considered. Such elements are designated here as "essentially well-set". That rational bases may be constructed for such elements is easily demonstrated by the approach followed in the 3D development. The adjacent factor at tangential vertex j vanishes at j and thus cannot be normalized to unity there. Normalization to unity at vertex j+1 suffices. The construction for a simple element may be instructive. The vertices are: $1 = (1/\sqrt{2}, -1/\sqrt{2}); 2 = (\sqrt{2}, 0); 3 = (1/\sqrt{2}, 1/\sqrt{2})$. Sides S and adjacent factors P for the 3-con of order four are: $S_1 = \sqrt{2}+y-x, S_2 = \sqrt{2}-x-y, S_3 = x^2+y^2-1, P_1 = x-(\sqrt{2}-1)y-1$, and $P_3 = (\sqrt{2}-1)y+x-1$. The adjoint is $Q = \sqrt{2}x - 1$. The rational basis functions other than at 1 and 3 are standard. The basis at 3 is

$$W_3 = \frac{(\sqrt{2}+y-x)[(\sqrt{2}-1)y+x-1]]}{(2-\sqrt{2})(\sqrt{2}x-1)}. \tag{12}$$

On side S_2 we substitute $\sqrt{2} - x$ for y and reduce W_3 to $2 - \sqrt{2}x$ on the side. Linearity on circular arc S_3 is easily demonstrated by showing the equivalence of the numerator and QP_1.

If the extension of a line adjacent to a parabolic side hits the parabolic side rather than its extension, the element is not well-set. This may be remedied by introducing a concave vertex at the midpoint of the linear side and perturbing the convex vertex at which the two sides intersect in order to replace this parabola-line vertex with a parabola-parabola vertex. Similarly, a parabola-parabola vertex may have to be perturbed to assure a well-set element. Programs poly2018 and starcon assume input leads to well-set elements without perturbation.

6. Adjacent concave vertices

When two or more concave vertices separate a pair of convex vertices a least squares conic fit to them may be constructed. Points on the midpoints of the sides between the convex vertices are added to the fitted points in the MATLAB conpq program. This yields seven points when there are only two concave points. Let t be the number of points to be fit to a conic. In conpq the points are weighted. The convex vertices are weighted with

$q_1 = q_t = 10$, the concave vertices with $q_p = 2$, and the midpoints with $q_p = 1$. This leads to relatively small movement of the convex vertices and reduces the influence of the side midpoints. This choice was arbitrary and a user is welcome to modify it. The curve is normalized to -1 at the midpoint of the line connecting the two convex vertices. The equation to be solved with a generalized inverse is $M\mathbf{z} = \mathbf{r}$ with M of order (t+1,6) and $\mathbf{r}$ a vector of length t+1. The first t elements in $\mathbf{r}$ are zero and the last element is -1. Row p of M is $q_p[1 \;\; x_p \;\; y_p \;\; {x_p}^2 \;\; x_p y_p \;\; {y_p}^2]$. Let $G \equiv (M^\top M)^{-1}$. Then $\mathbf{z} = GM^\top \mathbf{r}$. The conic is $[1 \;\; x \;\; y \;\; x^2 \;\; xy \;\; y^2]\mathbf{z}$. A hyperbola has two branches. To avoid a second branch cutting through the element a hyperbola is replaced by a parabola. The vertex of the parabola is chosen as the point on the hyperbolic curve where the tangent is parallel to the line connecting the convex vertices. A conic is hyperbolic when its discriminant $D = z_5 - 4z_4z_6$ is positive.

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