# Barycentric Coordinates for Star Polycons

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**Abstract**

Barycentric coordinates are needed in some applications for polygons with concave vertices. Such elements do not have rational barycentrics and mean-value coordinates are widely used. Lines meeting at concave vertices may be replaced by parabolic arcs to yield polycons for which there are rational barycentric coordinates. Theoretical foundations for this option are presented in this paper.

1. Mean-value polygons and rational polycons

Star polygons often occur in graphics application. These polygons have convex and concave vertices. Such elements do not have rational barycentric bases. Mean-value coordinates [1] are well suited for star polygons. An alternative approach enabling rational barycentric coordinates is to replace concave Vs with parabolas with the concave polygon nodes as parabola side nodes. This was suggested in [5]. Application of theory for construction of barycentric coordinates to elements with curved boundaries has been sparse. Star polycons provide impetus for new consideration.

In comparing rational and mean-value barycentrics a few properties warrant consideration. For convex polygons there are no significant differences. As an interior angle approaches $180^o$ both approaches lead to large gradients at the vertex. A simple rational remedy is to replace the offending vertex with a side node of a parabola, in which case the gradient is well behaved. In the limit the parabola reduces to a line with a side midpoint and degree-two approximation on the side. As the angle passes through $180^o$ the element becomes concave and rational barycentrics exist only for the parabolic replacement. The rationals have smaller gradients when the angle is near $180^o$. Mean-value barycentrics have not been developed for elements with curved sides. Mean-value coordinates can also treat adjacent concave vertices. Although replacement of linear sides connecting these vertices with one or more conic sides would result in elements amenable to rational barycentric coordinates, such elements are not considered here.

Rational polycon barycentrics are not positive. However, positive approximation can be ensured by increasing values at side nodes to maintain positivity on the boundary and introducing an interior hat basis function to maintain positivity within the element. There are problems where large gradients are a physical property and should be retained as, for example, in stress computations where cracks form at sharp corners. Corners may be rounded to reduce crack formation. The mean-value coordinates are preferable for an element with sharp corners while rational bases with curved sides are preferable for a rounded element. The purpose of this note is not to suggest polycons as a ubiquitous replacement for concave polygons but rather to introduce another option.

2. Star polygon vertices

An m-gon is a polygon of order m with m vertices, r of which are concave. The derived star polycon replaces the linear sides intersecting at each of the r concave vertices with a parabola. The concave vertices are now side nodes on these parabolas. The polycon has n = m - 2r linear sides and r conic sides. The order of an element is the sum of the orders of its sides. This n-sided star is an n-con of order m. The rational barycentric coordinates of a polycon share a common denominator which is known as its "adjoint". The maximum order of the adjoint curve is m - 3.

3. Parabola replacement of concave Vs

The first new challenge is construction of a parabolic replacement of a concave V. The isoparametric parabola [5] is not best. The concave vertex is in general not at the apex of the isoparametric parabola. This is easily demonstrated by considering a concave V (Fig. 1) for which MacMillan's construction [5] yields a parabola axis perpendicular to the short side. As the ratio of the lengths L1 and L2 of the arms of the V increase, the axis angle with the long side decreases. Let the V vertices be 1,2 and 3 with 2 concave. If we choose coordinates (xp,yp) with yp along the parabola axis and the origin at the concave node, the parabola $yp - c \cdot xp^2$ requires $[\frac{xp^2}{yp}]_1 = [\frac{xp^2}{yp}]_3$. If $\delta$ is the interior angle of the V and $\beta$ is the angle between the y-axis and the longer side, $L_2$, then for point 2 to be at the apex of a parabola through the three vertices:

$$V_\beta \equiv L_1[\frac{1}{\cos(\delta-\beta)} - \cos(\delta-\beta)] - L_2[\frac{1}{\cos(\beta)} - \cos(\beta)] = 0. \tag{1}$$

One may solve this transcendental equation analytically or approximate the solution by interpolation on a table of $V_\beta$ for $\beta \in [0, \delta/2]$. This parabola must then be rotated and translated to its actual (x,y) position. Suppose $\delta = 90^o + \alpha$ for some positive $\alpha < 90^o$. For equal lengths the symmetry axis is at $\beta(r=1) = 45^o + \alpha/2$. As the ratio r of $L_2/L_1$ increases, $\beta(r)$ goes from $\beta(1)$ to a minimum of $\alpha$. For example, when $\delta = 120^o$, the smallest $\beta$ is $30^o$. A value of r = 5 is satisfied with $\beta$ around $45^o$. Here, $\beta$ goes from $60^o$ to $30^o$ as r varies from unity to infinity.

Exterior intersections of adjacent sides are designated as "eip". When adjacent parabolic sides have parallel axes, two of the eip are on the absolute line (that is, at infinity). If the parabolas intersect at a point closer to the side node than the vertex, the vertex is moved to this point and the former vertex becomes an eip (Fig. 2). In rare cases the parabolas may be tangent at the vertex. This leads to an interesting study establishing stability of its barycentric coordinates. This element is said to be "essentially well-set".

4. Essentially well-set elements

Warren [6] generalized well-set convex polyhedra restricted to vertices of order three introduced in 1975 ([3], Part 1 in [5]) to allow nodes of arbitrary order. (The order at a node is equal to the number of faces sharing that node.) Although the adjoint vanishes to order p - 3 where p is the order of the node, the singularity at the node is removable. Barycentric coordinates for polyhedra with nodes of arbitrary order (based on Warren's analysis) were introduced in [4]. Application to 2D elements was not considered. However, elements with tangential intersection at vertices must be considered in construction of star basis functions. Such elements will be designated here as "essentially well-set". That rational bases may be constructed for such elements is easily demonstrated by the approach followed in the 3D development. The adjacent factor at tangential vertex j vanishes at j and thus cannot be normalized to unity there. Normalization to unity at vertex j+1 suffices. The construction for a simple element may be instructive. The vertices are: $1 = (1/\sqrt{2}, -1/\sqrt{2}); 2 = (\sqrt{2}, 0); 3 = (1/\sqrt{2}, 1/\sqrt{2})$. Sides S and adjacent factors P for the 3-con of order four are: $S_1 = \sqrt{2} + y - x, S_2 = \sqrt{2} - x - y, S_3 = x^2 + y^2 - 1, P_1 = x - (\sqrt{2}-1)y - 1$, and $P_3 = (\sqrt{2}-1)y + x - 1$ (Fig. 3).

The adjoint is $Q = \sqrt{2}x - 1$. The rational basis functions other than at 1 and 3 are standard. The basis at 3 is

$$W_3 = \frac{(\sqrt{2}+y-x)[(\sqrt{2}-1)y+x-1]]}{(2-\sqrt{2})(\sqrt{2}x-1)}. \tag{2}$$

On side $S_2$ we substitute $\sqrt{2}-x$ for y and reduce $W_3$ to $2-\sqrt{2}x$ on the side. Linearity on circular arc $S_3$ is easily demonstrated by showing the equivalence of the numerator and $QP_1$.

5.Adjacent factors at vertices

An element is well-set when all intersections of adjacent sides other than the vertex are exterior to the element If the extension of a line adjacent to a parabolic side hits the parabolic side rather than its extension, the element is not well-set. This may be remedied by introducing a concave vertex at the midpoint of the linear side and perturbing the convex vertex at which the two sides intersect in order to replace this parabola-line vertex with a parabola-parabola vertex (Fig. 4). Similarly, a parabola-parabola vertex may have to be perturbed to assure a well-set element. Programs POLY2017 and STARCON assume input leads to well-set elements without perturbation.

Adjacent factors "Padj" must be computed at all vertices as described in [3,5]. Each adjacent factor contains the vertex eip and adjacent side nodes. Padj at vertex k is normalized to unity at k. The adjacent factor at a line-line intersection is unity. The adjacent factor at a line-parabola intersection is linear (Fig. 5). The adjacent factor at a parabola-parabola intersection is quadratic (Fig. 6) except when two intersections on the absolute line reduces the quadratic to linear. When all three eip are on the absolute line one of the intersecting parabolas is just a translation of the other and the adjacent factor is the same parabola translated to contain the adjacent side nodes.

6. Algebraic-geometry review

Algebraic-geometry foundations for rational barycentric construction were developed in [3,5]. These are essential for the GADJ determination of numerator normalization. The set of points where algebraic curves of orders p and q intersect are in the "divisor" of these curves. When all intersections are simple these are the pq elements of the divisor. When a point is at a common tangent of the curves or at a multiple point on either curve there are fewer than pq points. Divisor theory introduces elements which expand the intersection points into a set of pq elements. Parabolic curves do not have multiple points. The full theory of divisors is not needed for star polycons. The symbol $P' \cdot Q'$ denotes the divisor of curves P' and Q'. In this application, $P' \cdot Q'$ is just the set of intersections of curves P' and Q' and consists of vertices and eip. Let P and R be the polynomials of least degree that vanish on algebraic curves P' and R'. The assertion that $P \equiv RmodS'$ (that is, "P is congruent to R mod S' ") means that there is a scalar b such that $P - bR = 0$ on curve S'. The crucial algebraic-geometry theorem is:

Theorem: If $P' \cdot S' = R' \cdot S', then P \equiv Rmod(S')$.

This deceptively simple theorem provides a powerful basis for construction of rational barycentrics. It enables analysis with barycentric coordinate factors along a side replaced by equivalent factors so that factors common to numerator and denominator may be cancelled to establish linear variation on the side. This motivates the GADJ algorithm.

7. GADJ normalization: rek and reno

GADJ normalization parameters rek and reno are computed to yield linearity of the barycentric coordinates on the element boundary. After rek and reno have been computed the adjoint Qadj is found as a sum over the nodes. Since normaliztion of Qadj is arbitrary $rek_1$ is chosen as unity and the remaining $rek_j$ and $reno_j$ are determined recursively. Let S' be the boundary curve which is just the product of the sides. Let jm1 = j-1 when j is not equal to 1 and jm1 = n when j = 1. Let jp1 = j+1 when j is not equal to n and equal 1 when j = n. Let jp2 be the value of jp1 at jp1 . The node numerators are summed to yield the denominator Qadj. S is the product of the n polycon sides. Let $sp_j = s_{jm1}s_js_{jp1}$ The "opposite" factor common to the nodes on side j is the polynomial $F_j = S/sp_j$ . The numerator for vertex j is the polynomial $N_j$ of degree m-2, where

$$N_j = rek_j F_j s_{jp1} Padj_j = rek_j S Padj_j/[s_{jm1} s_j]. \tag{3}$$

The term for the node j+1/2 on side j is the polynomial $N_{j+1/2}$ of degree not greater than m-2, where

$$N_{j+1/2} = reno_j F_j s_{jm1} s_{jp1} = reno_j S/s_j. \tag{4}$$

S is a common factor that may be dropped from numerator and denominator of $W_{jorj+1/2} = N_{jorj+1/2}/\sum N_{allnodes}$. In the mean-value construction for polygons (with no parabolas and therefore no side nodes and no Padj factors), multiplier $rek_j$ is computed from properties of triangles adjacent to vertex j and $N_j = rek_j/(s_{jm1}s_j)$. It is thus seen that for convex polygons the rational construction with opposite factors in $N_j$ is equivalent to the mean-value construction with local factors in the $N_j$.

When side j is parabolic only bases at vertices j, j+1/2 and jp1 affect the variation on side j. Thus, on side j the barycentric coordinate $W_j$ varies as

$$W_j = \frac{rek_j s_{jp1} Padj(j)}{rek_j s_{jp1} Padj(j) + reno_j s_{jm1} s_{jp1} + rek_{jp1} s_{jm1} Padj_{jp1}}. \tag{5}$$

When the three sides in this equation are all linear there is no side node or Padj factor and the numerator is linear. The denominator must be constant with the same value at j and jp1. It follows that $rek_j s_{jp1}(j) = rek_{jp1} s_{jm1}(jp1)$ so that

$$rek_{jp1} = rek_j \frac{s_{jp1}(j)}{s_{jm1}(jp1)}. \tag{6}$$

This is true for all vertices of a convex polygon. When $s_j$ is linear there is no side node. If just one of the other sides is a parabola the algebraic geometry congruence theorem may be applied. Let $s_{jp1}$ be the parabolic side. Similar analysis applies when $s_{jm1}$ is the parabolic side. Numerator and denominator may be divided by $Padj_j$ so that the numerator is then linear. It is a constant times the barycentric coordinate $B_j$ at j for the line $s_j$. For $W_j$ to be linear on side $s_j$ the denominator must be a constant. We note that (parabola jp1) $\cdot$ (side j) $= (jp1, eip_{jp1})$ so $s_{jp1} \equiv (jp1; jp2)(jp1 + 1/2; eip_{jp1}) \equiv (jp1; jp2) Padj_{jp1}$ mod side j. Thus, since $Padj_{jp1}$ is normalized to one at vertex jp1 and $Padj_j = 1$,

$$rek_{j+1} = rek_j \frac{s_{jp1} Padj_j(jp1)}{s_{jm1}(jp1)}. \tag{7}$$

When both $s_{jm1}$ and $s_{jp1}$ are quadratic, $Padj_j = (sino_{jm1}; eip_j)$ and $Padj_{jp1} = (sino_{jp1}; eip_{jp1})$ yield

$$Padj_j Padj_{jp1} \equiv (eip_j; eip_{jp1}) mod(sidej). \tag{8}$$

It follows that

$$s_{jp1} Padj_j \equiv (jp1; sino_{jp1})(eip_j; eip_{jp1}) \equiv (jp1; sino_{jp1}) Padj_j Padj_{jp1} mod(sidej) \tag{9}$$

and

$$s_{jm1} Padj_{jp1} \equiv (jm1; sino_{jm1})(eip_j; eip_{jp1}) \equiv (jm1; sino_{jm1}) Padj_j Padj_{jp1} mod(sidej). \tag{10}$$

All terms on side j share a common factor $Q_j = Padj_j Padj_{jp1}$. The normalization $Padj_j(j) = 1$ and $Padj_{jp1}(jp1) = 1$ results in:

$$rek_{jp1} = rek_j \frac{s_{jp1}(j) Padj_j(jp1)}{s_{jm1}(jp1) Padj_{jp1}(j)}. \tag{11}$$

The analysis is more delicate when side (j , j + 1/2 , jp1) is parabolic. Both $reno_j$ and $rek_{jp1}$ may be computed from $rek_j$. The congruence theorem reveals that the common factor R at the three nodes vanishes at all the eip. When the numerator is divided by R it is a constant times the barycentric coordinate $B_j$ of triangle (j, j+1/2, jp1). The terms in the denominator are just the same constant times the barycentric coordinates for this triangle. The sum of these coordinates is unity. When the adjacent sides are both linear R is linear. When one adjacent side is parabolic R is quadratic. When both adjacent sides are parabolic R is cubic. The normalization of R is arbitrary since only ratios at the nodes appear in the GADJ algorithm. Now consider the product of the adjacent factors, $Padj_j Padj_{jp1}$. When both adjacent sides are linear this product is quadratic, when one is parabolic the product is cubic, and when both adjacent sides are parabolic the product is quartic. It is always one degree higher than R. This is due to the fact that on side j this product has a double point at node j+1/2 in addition to the eip. There is only one line with a double point with side j at the side node, and that is the tangent to the side at j+1/2. The Padj product divided by the linear form of the tangent is a suitable $R_j$. Evaluation of this $R_j$ at the vertices to compute $rek_{jp1}$ is a simple task. However, the evaluation for reno is more difficult since $R_j$ is a double point of both numerator and denominator at side node j+1/2. The limit as a point moves along the side is the value of $R_j(j+1/2)$. One may either apply L'Hospital's rule or evaluate $R_j$ at points on the parabola close to the side node and compute the limit as these points approach j+1/2.

The alternative is to determine $R_j$ directly from the eip. When both jm1 and jp1 are linear, $R_j$ is just the line (e;f) through the exterior points e and f. When jm1 is linear and jp1 is parabolic, $R_j$ is just the quadratic factor of the curve through exterior point e on side jm1 and the three eip, say $f_1, f_2, f_3$, of sides jp1 and j. Similarly, when jp1 is linear and jm1 is parabolic, $R_j$ is the quadratic factor through eip f on side jp1 and the eip $e_1, e_2, e_3$ on side jm1. When both jm1 and jp1 are parabolic, $R_j$ is the degree-three factor of a cubic curve through the six eip of sides jm1 and jp1 with side j. The eip of two parabolas may always be computed so that the third point is real. Thus, when jm1 is linear $R_j$ may be chosen as $(e, f_3)(f_1, f_2)$. Similar analysis applies when jm1 is parabolic and jp1 is linear. When both of these sides are parabolic, $R_j$ may be chosen as the product of three lines: $(e_3, f_3)(e_1, e_2)(f_1, f_2)$. Eip on the absolute line replace absent affine eip points with slope conditions on the lines in $R_j$.

These alternatives were applied in MATLAB programs "starcon" and "stargeip". The GADJ algorithm with the Padj scheme in starcon required about 100 lines while direct determination from the eip in stargeip required about 250 lines. The larger value of the latter was a consequence of need to address all possible combinations of points on the absolute line. These points are accounted for in the prior computation of the adjacent Padj factors. In starcon, $R_j(j+1/2)$ was computed as a limit of values computed at four points on the side spaced on the order of $10^{-4}$ times the distance between vertices j and jp1. This was found to be a balance between loss in accuracy due to points too far from j+1/2 and loss due to computation of a ratio of values too close to zero. Results with these programs agreed to within five significant digits for several problems which were chosen to test eip alternatives. Rounding error in computation of the adjacent factor at a conic-conic vertex must be treated carefully. The numerator and denominator of the ratio at j+1/2 are close to zero and a linear system of order five is solved to obtain the Padj values for the numerator. Rounding error in the significant components of Padj do not affect accuracy. There may be terms in Padj due only to rounding error. These terms of $O(10^{-16})$ affect accuracy. They are eliminated in the starcon Padj computation.

The GADJ algorithm may be summarized as follows: First, all terms in Eq. 5 on side j are divided by a common factor of $R_j = Padj_j Padj_{jp1}/t$ where t is unity on a linear side and t = $t_{j+1/2}$ is the tangent to curved side j at side node j+1/2. These ratios are evaluated at the nodes on side j:

$$b_j = [s_{jp1}/R_j]_j, b_{j+1/2} = [s_{jp1}s_{jm1}/R_j]_{j+1/2}, b_{jp1} = [s_{jm1}/R_j]_{jp1}. \tag{12}$$

The linear numerator in Eq. 5 is then $Num_j = b_j B_j$, where $B_j$ is the barycentric coordinate of triangle (j , j+1/2 , jp1). The denominator is:

$$Den_j = b_j B_j + b_{j+1/2} B_{j+1/2} + b_{jp1} B_{jp1}. \tag{13}$$

Next, $rek_{jp1}$ and $reno_j$ are computed so that $Den_j = b_j(B_j + B_{j+1/2} + B_{jp1})$:

$$rek_{jp1} = b_j rek_j / b_{jp1}, reno_j = b_j rek_j / b_{j+1/2}. \tag{14}$$

Since the sum of the triangle barycentric coordinates is unity, $W_j = B_j$ on side j.

Computation of the barycentric coordinates once rek and reno have been found is straightforward. One sums the normalized numerators to obtain the adjoint, Qadj. This adjoint remains for higher degree coordinates with un-normalized numerators $N_s$ at nodes s. The higher degree basis function at node s is $W_s = k_s N_s / Qadj$, where $k_s$ is chosen to normalize $W_s$ to unity at s.

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