Homework 4: Hypothesis test

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For Formative assessment, submit the solutions to the Exercise 1.

Exercise 1. ($\star\star$)

1. Consider the Single vs. General alternative Bayesian hypothesis test

$$\mathrm{H}_0 : \theta = \theta_0 \qquad \text{vs} \qquad \mathrm{H}_1 : \theta \neq \theta_0.$$

or more formally

$$\mathrm{H}_0 : \begin{cases} y|\theta_0 & \sim F(y|\theta_0) \\ \theta_0 \text{ is fixed} & \end{cases} \qquad \text{vs} \qquad \mathrm{H}_1 : \begin{cases} y|\theta & \sim F(y|\theta) \\ \theta & \sim \Pi_1(\theta),\ \theta \in \Theta_1 \end{cases} \tag{1}$$

Show that the Bayes factor B_{01} of H_0 and H_1 can be computed as

$$\mathrm{B}_{01} = \frac{\pi_1\left(\theta_0|y\right)}{\pi_1\left(\theta_0\right)}$$

where $\pi_1(\cdot)$ and $\pi_1(\cdot|\cdot)$, denote the PDF of $\Pi_1(\cdot)$ and $\Pi_1(\cdot|\cdot)$.

2. In the above hypothesis test, assume that $y|\theta \sim \mathrm{Bin}(n, \theta)$ has Binomial sampling distribution with unknown parameter θ.

   (a) Find an approximation of B_{01}, when n is large.

   Hint: [From Stats 2] How the aforesaid likelihoods or the posteriors do behave as $n$becomes big?

   (b) For large n, show that $\mathrm{B}_{01} > 1$ when $z_0 = \frac{p-\theta_0}{\sqrt{u}}$, with $p = \frac{y}{n}$ and $u = \frac{p(1-p)}{n}$, satisfies $|z| < \max\left(\sqrt{k}, 0\right)$ for some k that depends of n, p, and $\pi_1\left(\theta_0\right)$.

   (c) Let the conditional prior $\Pi_1(\theta)$ be a Uniform distribution with positive mass above the interval $[0, 1]$. Show that this choice of the conditional prior $\Pi_1(\theta)$ can create a "paradox" when compared with fixed size tests.

Solution.

1. From Bayesian theorem it is

$$\pi_1(\theta|y) = \frac{f(y|\theta)\pi_1(\theta)}{f_1(y)}$$

So

$$\pi_1(\theta_0|y) = \frac{f(y|\theta_0)\pi_1(\theta_0)}{f_1(y)} = \frac{f_0(y)\pi_1(\theta_0)}{f_1(y)} = \mathrm{B}_{01}\pi_1(\theta_0)$$

because $\int_{\{\theta=\theta_0\}} f(y|\theta)1\left(\theta \in \{\theta_0\}\right)\mathrm{d}\theta = f_0(y) = \int_{\{\theta=\theta_0\}} f(y|\theta)1\left(\theta \in \{\theta_0\}\right)\mathrm{d}\theta = f(y|\theta_0)$.

Hence

$$\mathrm{B}_{01} = \frac{\pi_1\left(\theta_0|y\right)}{\pi_1\left(\theta_0\right)}$$

2.

(a) From Stats 2, we can find say about the density of the posterior of $\theta|y$ that

$$\pi_1\left(\theta|y\right) \approx \mathrm{N}\left(\theta|\hat{\theta}_{\text{MLE}}, 1/\mathscr{I}(\hat{\theta}_{\text{MLE}})\right)$$

where $\hat{\theta}_{\text{MLE}} = \frac{y}{n} = p$ and $1/\mathscr{I}(\hat{\theta}_{\text{MLE}}) = p(1-p)/n = u$.
So

$$\mathrm{B}_{01} = \frac{\pi_1\left(\theta_0|y\right)}{\pi_1\left(\theta_0\right)} \approx \frac{(2\pi u)^{-1/2}\exp\left(-\frac{1}{2}\frac{(\theta_0-p)^2}{u}\right)}{\pi_1\left(\theta_0\right)}$$

(b) For

$$\begin{aligned}
\mathrm{B}_{01} > 1 \Longleftrightarrow -\log\left(\mathrm{B}_{01}\right) < 0 \Longleftrightarrow \\
\frac{\left(\theta_0-p\right)^2}{u} <& \underbrace{-\log\left(2\pi u\right)}_{k^*} - 2\log\left(\pi_1\left(\theta_0\right)\right) \Longleftrightarrow \\
\frac{\left(\theta_0-p\right)^2}{u} <& k^* - 2\log\left(\pi_1\left(\theta_0\right)\right)
\end{aligned}$$

So re result follows for

$$k = k^* - 2\log\left(\pi_1\left(\theta_0\right)\right)$$

where

$$k^* = -\log\left(2\pi u\right) = -\log\left(2\pi\frac{p(1-p)}{n}\right)$$

(c) I have $\pi_1(\theta_0) = 1$. Then $\mathrm{B}_{01} > 1$ when $|z| < \sqrt{k^*}$.

- Assume I choose $\sqrt{k^*} = 3$, and I investigate what I get here:
- In frequentist stasts, I suppose that the classical hypothesis test would reject H_0 whenever $|z| < \sqrt{k^*} = \sqrt{3}$ (this corresponds to in sig. level 1%). So if I get $|z| = 3$, I would reject H_0.
- In Bayesian stats, $|z| = 3$ corresponds to

$$\mathrm{B}_{01} \approx \frac{(2\pi u)^{-1/2}\exp\left(-\frac{1}{2}\frac{(\theta_0-p)^2}{u}\right)}{\pi_1\left(\theta_0\right)} = \frac{\exp\left(\frac{1}{2}3\right)\exp\left(-\frac{1}{2}3\right)}{1} = 1$$

- For the record, I can get $\sqrt{k^*} = 2$ for a sample size $n = 2\pi p(1-p)\exp(9) \approx 16206\pi p(1-p)$