

# Homework 1: Manipulation of multivariate probability distributions, and the Posterior distribution

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For Formative assessment, submit the solutions of the parts 1 and 2 from the Exercise 1, and the solution of the Exercise 2.

**Exercise 1. (★★)**

Let  $x \sim T_d(\mu, \Sigma, \nu)$ . Recall that  $x \sim T_d(\mu, \Sigma, \nu)$  is the marginal distribution  $f_x(x) = \int f_{x|\xi}(x|\xi)f_\xi(\xi)d\xi$  of  $(x, \xi)$  where

$$x|\xi \sim N_d(\mu, \Sigma\xi v) \\ \xi \sim \text{IG}\left(\frac{v}{2}, \frac{1}{2}\right)$$

Consider partition such that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix},$$

where  $x_1 \in \mathbb{R}^{d_1}$  and  $x_2 \in \mathbb{R}^{d_2}$ .

Address the following:

1. Show that the marginal distribution of  $x_1$  is such that

$$x_1 \sim T_{d_1}(\mu_1, \Sigma_1, \nu)$$

**Hint:** Try to use the form  $f_x(x) = \int f_{x|\xi}(x|\xi)f_\xi(\xi)d\xi$ .

2. Show that

$$\xi|x_1 \sim \text{IG}\left(\frac{1}{2}(d_1 + v), \frac{1}{2}\frac{Q + v}{v}\right)$$

where  $Q = (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)$ .

**Hint:** The PDF of  $y \sim N_d(\mu, \Sigma)$  is

$$f(y) = (2\pi)^{-\frac{d}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(y - \mu)^\top \Sigma^{-1}(y - \mu)\right)$$

**Hint:** The PDF of  $y \sim \text{IG}(a, b)$  is

$$f_{\text{IG}(a,b)}(y) = \frac{b^a}{\Gamma(a)} y^{-a-1} \exp\left(-\frac{b}{y}\right) 1_{(0,+\infty)}(y)$$

3. Let  $\xi' = \xi \frac{v}{Q+v}$ , with  $Q = (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)$ , show that

$$\xi'|x_1 \sim \text{IG}\left(\frac{v + d_1}{2}, \frac{1}{2}\right)$$

4. Show that the conditional distribution of  $x_2|x_1$  is such that

$$x_2|x_1 \sim \mathbf{T}_{d_2}(\mu_{2|1}, \dot{\Sigma}_{2|1}, \nu_{2|1})$$

where

$$\begin{aligned}\mu_{2|1} &= \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1) \\ \dot{\Sigma}_{2|1} &= \frac{\nu + (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)}{\nu + d_1} \Sigma_{2|1} \\ \Sigma_{2|1} &= \Sigma_{22} - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top \\ \nu_{2|1} &= \nu + d_1\end{aligned}$$

**Hint:** You can use the Example [Marginalization & conditioning] from the Lecture Handout

**Exercise 2.** (\*\*) Let  $x$  be an observation. Consider the Bayesian model

$$\begin{cases} x|\theta & \sim \text{Pn}(\theta) \\ \theta & \sim \Pi(\theta) \end{cases}$$

where  $\text{Pn}(\theta)$  is the Poisson distribution with expected value  $\theta$ . Consider a prior  $\Pi(\theta)$  with density such as  $\pi(\theta) \propto \frac{1}{\theta}$ . Show that the posterior distribution is not always defined.

**Hint-1:** It suffices to show that the posterior is not defined in the case that you collect only one observation  $x = 0$ .

**Hint-2:** Poisson distribution:  $x \sim \text{Pn}(\theta)$  has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$