

## Homework 2: Conjugate priors and Jeffreys priors

Lecturer: Georgios Karagiannis

georgios.karagiannis@durham.ac.uk

For Formative assessment, submit the solutions of the parts 1, 2, and 3 from the Exercise 1, and the solution of the Exercise 2.

**Exercise 1.** (★★) Let  $x = (x_1, \dots, x_n)$  be observables. Consider a Bayesian model such as

$$\begin{cases} x_i | \lambda & \stackrel{\text{iid}}{\sim} \text{Pn}(\lambda), \quad \forall i = 1, \dots, n \\ \lambda & \sim \Pi(\lambda) \end{cases}$$

**Hint-1** Poisson distribution  $x \sim \text{Pn}(\lambda)$  has PMF:  $\text{Pn}(x|\lambda) = \frac{1}{x!} \lambda^x \exp(-\lambda) 1_{\mathbb{N}}(x)$ , where  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\lambda > 0$ .

**Hint-2** Gamma distribution  $x \sim \text{Ga}(a, b)$  has PDF:  $\text{Ga}(x|a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) 1_{(0, \infty)}(x)$ , with  $a > 0$  and  $b > 0$ .

**Hint-2** Negative Binomial distribution  $x \sim \text{Nb}(r, \theta)$  has PMF:  $\text{Nb}(x|r, \theta) = \binom{r+x-1}{r-1} \theta^r (1-\theta)^x 1_{\mathbb{N}}(x)$  with  $\theta \in (0, 1)$ ,  $r \in \mathbb{N} - \{0\}$ , and  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

1. Compute the likelihood in the aforesaid Bayesian model.
2. Show that the sampling distribution is a member of the exponential family.
3. Specify the PDF of the conjugate prior distribution  $\Pi(\lambda)$  of  $\lambda$ , and identify the parametric family of distributions as  $\lambda \sim \text{Ga}(a, b)$ , with  $a > 0$ , and  $b > 0$ . While you are deriving the conjugate prior distribution of  $\lambda$ , discuss which of the prior hyper-parameters can be considered as the ‘strength of the prior information and which can be considered as summarizing the prior information.
4. Compute the PDF of the posterior distribution of  $\lambda$ , identify the posterior distribution as a Gamma distribution  $\text{Ga}(\tilde{a}, \tilde{b})$ , and compute the posterior hyper-parameters  $\tilde{a}$ , and  $\tilde{b}$ .
5. Compute the PMF of the predictive distribution of a future outcome  $y = x_{n+1}$ , identify the name of the resulting predictive distribution, and compute its parameters.

**Exercise 2.** (★★) Assume observation  $x$  sampled from a Maxwell distribution with density

$$f(x|\theta) = \sqrt{\frac{2}{\pi}} \theta^{3/2} x^2 \exp\left(-\frac{1}{2} \theta x^2\right).$$

Find the Jeffreys prior density for the parameter  $\theta$ .