

Homework 2: Conjugate priors and Jeffreys priors

Lecturer: Georgios Karagiannis

georgios.karagiannis@durham.ac.uk

For Formative assessment, submit the solutions of the parts 1, 2, and 3 from the Exercise 1, and the solution of the Exercise 2.

Exercise 1. (★★) Let $x = (x_1, \dots, x_n)$ be observables. Consider a Bayesian model such as

$$\begin{cases} x_i | \lambda & \stackrel{\text{iid}}{\sim} \text{Pn}(\lambda), \forall i = 1, \dots, n \\ \lambda & \sim \Pi(\lambda) \end{cases}$$

Hint-1 Poisson distribution $x \sim \text{Pn}(\lambda)$ has PMF: $\text{Pn}(x|\lambda) = \frac{1}{x!} \lambda^x \exp(-\lambda) 1_{\mathbb{N}}(x)$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\lambda > 0$.

Hint-2 Gamma distribution $x \sim \text{Ga}(a, b)$ has PDF: $\text{Ga}(x|a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) 1_{(0, \infty)}(x)$, with $a > 0$ and $b > 0$.

Hint-2 Negative Binomial distribution $x \sim \text{Nb}(r, \theta)$ has PMF: $\text{Nb}(x|r, \theta) = \binom{r+x-1}{r-1} \theta^r (1-\theta)^x 1_{\mathbb{N}}(x)$ with $\theta \in (0, 1)$, $r \in \mathbb{N} - \{0\}$, and $\mathbb{N} = \{0, 1, 2, \dots\}$.

1. Compute the likelihood in the aforesaid Bayesian model.
2. Show that the sampling distribution is a member of the exponential family.
3. Specify the PDF of the conjugate prior distribution $\Pi(\lambda)$ of λ , and identify the parametric family of distributions as $\lambda \sim \text{Ga}(a, b)$, with $a > 0$, and $b > 0$. While you are deriving the conjugate prior distribution of λ , discuss which of the prior hyper-parameters can be considered as the ‘strength of the prior information and which can be considered as summarizing the prior information.
4. Compute the PDF of the posterior distribution of λ , identify the posterior distribution as a Gamma distribution $\text{Ga}(\tilde{a}, \tilde{b})$, and compute the posterior hyper-parameters \tilde{a} , and \tilde{b} .
5. Compute the PMF of the predictive distribution of a future outcome $y = x_{n+1}$, identify the name of the resulting predictive distribution, and compute its parameters.

Solution.

Exercise 2. (★★) Assume observation x sampled from a Maxwell distribution with density

$$f(x|\theta) = \sqrt{\frac{2}{\pi}} \theta^{3/2} x^2 \exp\left(-\frac{1}{2} \theta x^2\right).$$

Find the Jeffreys prior density for the parameter θ .

Solution.