Bayesian Statistics III/IV (MATH3361/4071)

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## Homework 4: Hypothesis test

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For Formative assessment, submit the solutions to the Exercise 1.

## Exercise 1. $(\star\star)$

1. Consider the Single vs. General alternative Bayesian hypothesis test

$$H_0: \theta = \theta_0$$
 vs  $H_1: \theta \neq \theta_0$ .

or more formally

$$\mathbf{H}_0: \begin{cases} y|\theta_0 & \sim F(y|\theta_0) \\ \theta_0 \text{ is fixed} \end{cases} \quad \text{vs} \quad \mathbf{H}_1: \begin{cases} y|\theta & \sim F(y|\theta) \\ \theta & \sim \Pi_1(\theta), \ \theta \in \Theta_1 \end{cases} \tag{1}$$

Show that the Bayes factor  $B_{01}$  of  $H_0$  and  $H_1$  can be computed as

$$\mathbf{B}_{01} = \frac{\pi_1 \left(\theta_0 | y\right)}{\pi_1 \left(\theta_0\right)}$$

where  $\pi_1(\cdot)$  and  $\pi_1(\cdot|\cdot)$ , denote the PDF of  $\Pi_1(\cdot)$  and  $\Pi_1(\cdot|\cdot)$ .

- 2. In the above hypothesis test, assume that  $y|\theta \sim \text{Bin}(n,\theta)$  has Binomial sampling distribution with unknown parameter  $\theta$ .
  - (a) Find an approximation of  $B_{01}$ , when n is large.

**Hint:** [From Stats 2] How the aforesaid likelihoods or the posteriors do behave as *n*becomes big?

- (b) For large n, show that  $B_{01} > 1$  when  $z_0 = \frac{p-\theta_0}{\sqrt{u}}$ , with  $p = \frac{y}{n}$  and  $u = \frac{p(1-p)}{n}$ , satisfies  $|z| < \max\left(\sqrt{k}, 0\right)$  for some k that depends of n, p, and  $\pi_1\left(\theta_0\right)$ .
- (c) Let the conditional prior  $\Pi_1(\theta)$  be a Uniform distribution with positive mass above the interval [0,1]. Show that this choice of the conditional prior  $\Pi_1(\theta)$  can create a "paradox" when compared with fixed size tests.

## Solution.

1. From Bayesian theorem it is

$$\pi_1(\theta|y) = \frac{f(y|\theta)\pi_1(\theta)}{f_1(y)}$$

So

$$\pi_1(\theta_0|y) = \frac{f(y|\theta_0)\pi_1(\theta_0)}{f_1(y)} = \frac{f_0(y)\pi_1(\theta_0)}{f_1(y)} = \mathbf{B}_{01}\pi_1(\theta_0)$$

because  $\int_{\{\theta=\theta_0\}} f(y|\theta) \mathbf{1}\left(\theta \in \{\theta_0\}\right) \mathrm{d}\theta = f_0(y) = \int_{\{\theta=\theta_0\}} f(y|\theta) \mathbf{1}\left(\theta \in \{\theta_0\}\right) \mathrm{d}\theta = f(y|\theta_0).$ 

Hence

$$\mathbf{B}_{01} = \frac{\pi_1 \left(\theta_0 | y\right)}{\pi_1 \left(\theta_0\right)}$$

2.

(a) From Stats 2, we can find say about the density of the posterior of  $\theta|y$  that

$$\pi_1\left(\theta|y\right) \approx \mathbf{N}\left(\theta|\hat{\theta}_{\mathrm{MLE}}, 1/\mathscr{I}(\hat{\theta}_{\mathrm{MLE}})\right)$$

where  $\hat{\theta}_{\rm MLE} = \frac{y}{n} = p$  and  $1/\mathscr{I}(\hat{\theta}_{\rm MLE}) = p(1-p)/n = u.$ 

So

$$\mathbf{B}_{01} = \frac{\pi_1 \left( \theta_0 | y \right)}{\pi_1 \left( \theta_0 \right)} \approx \frac{\left( 2\pi u \right)^{-1/2} \exp \left( -\frac{1}{2} \frac{\left( \theta_0 - p \right)^2}{u} \right)}{\pi_1 \left( \theta_0 \right)}$$

(b) For

$$B_{01} > 1 \iff -\log(B_{01}) < 0 \iff$$

$$\frac{(\theta_{0} - p)^{2}}{u} < \underbrace{-\log(2\pi u)}_{k^{*}} - 2\log(\pi_{1}(\theta_{0})) \iff$$

$$\frac{(\theta_{0} - p)^{2}}{u} < k^{*} - 2\log(\pi_{1}(\theta_{0}))$$

So re result follows for

$$k = k^* - 2\log\left(\pi_1\left(\theta_0\right)\right)$$

where

$$k^* = -\log(2\pi u) = -\log\left(2\pi \frac{p(1-p)}{n}\right)$$

- (c) I have  $\pi_1(\theta_0) = 1$ . Then  $B_{01} > 1$  when  $|z| < \sqrt{k^*}$ .
  - Assume I choose  $\sqrt{k^*} = 3$ , and I investigate what I get here:
  - In frequentist stasts, I suppose that the classical hypothesis test would reject  $H_0$  whenever  $|z| < \sqrt{k^*} = \sqrt{3}$  (this corresponds to in sig. level 1%). So if I get |z| = 3, I would reject  $H_0$ .
  - In Bayesian stats, |z| = 3 corresponds to

$$B_{01} \approx \frac{(2\pi u)^{-1/2} \exp\left(-\frac{1}{2} \frac{(\theta_0 - p)^2}{u}\right)}{\pi_1(\theta_0)} = \frac{\exp\left(\frac{1}{2}3\right) \exp\left(-\frac{1}{2}3\right)}{1} = 1$$

• For the record, I can get  $\sqrt{k^*}=2$  for a sample size  $n=2\pi p(1-p)\exp(9)\approx 16206\pi p(1-p)$