Homework 1: Manipulation of multivariate probability distributions, and the Posterior distribution

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For Formative assessment, submit the solutions of the parts 1 and 2 from the Exercise 1, and the solution of the Exercise 2.

Exercise 1. ($\star\star$)

Let $x \sim \mathrm{T}_d(\mu, \Sigma, \nu)$. Recall that $x \sim \mathrm{T}_d(\mu, \Sigma, \nu)$ is the marginal distribution $f_x(x) = \int f_{x|\xi}(x|\xi) f_\xi(\xi) \mathrm{d}\xi$ of (x, ξ) where

$$x|\xi \sim \mathrm{N}_d(\mu, \Sigma\xi v)$$
$$\xi \sim \mathrm{IG}(\frac{v}{2}, \frac{1}{2})$$

Consider partition such that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}; \qquad \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix},$$

where $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$.

Address the following:

1. Show that the marginal distribution of x_1 is such that

$$x_1 \sim \mathrm{T}_{d_1}(\mu_1, \Sigma_1, \nu)$$

   Hint: Try to use the form $f_x(x) = \int f_{x|\xi}(x|\xi) f_\xi(\xi) \mathrm{d}\xi$.

2. Show that

$$\xi|x_1 \sim \mathrm{IG}(\frac{1}{2}(d_1 + v), \frac{1}{2}\frac{Q + v}{v})$$

   where $Q = (\mu_1 - x_1)^\top \Sigma_1^{-1} (\mu_1 - x_1)$.

   Hint: The PDF of $y \sim \mathrm{N}_d(\mu, \Sigma)$ is

$$f(y) = (2\pi)^{-\frac{d}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(y - \mu)^\top \Sigma^{-1} (y - \mu)\right)$$

   Hint: The PDF of $y \sim \mathrm{IG}(a, b)$ is

$$f_{\mathrm{IG}(a,b)}(y) = \frac{b^a}{\Gamma(a)} y^{-a-1} \exp(-\frac{b}{y}) 1_{(0,+\infty)}(y)$$

3. Let $\xi' = \xi \frac{v}{Q+v}$, with $Q = (\mu_1 - x_1)^\top \Sigma_1^{-1} (\mu_1 - x_1)$, show that

$$\xi'|x_1 \sim \mathrm{IG}(\frac{v + d_1}{2}, \frac{1}{2})$$

4. Show that the conditional distribution of $x_2|x_1$ is such that

$$x_2|x_1 \sim \mathrm{T}_{d_2}(\mu_{2|1}, \dot{\Sigma}_{2|1}, \nu_{2|1})$$

where

$$\begin{aligned}
\mu_{2|1} =&\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1) \\
\dot{\Sigma}_{2|1} =&\frac{\nu + (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)}{\nu + d_1}\Sigma_{2|1} \\
\Sigma_{2|1} =&\Sigma_{22} - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top \\
\nu_{2|1} =&\nu + d_1
\end{aligned}$$

Hint: You can use the Example [Marginalization & conditioning] from the Lecture Handout

Solution.

1. From what is given, it is $x|\xi \sim \mathrm{N}_d(\mu, \Sigma\xi v)$ and $\xi \sim \mathrm{IG}(\frac{v}{2}, \frac{1}{2})$ namely,

$$f_x(x) \;= \int f_{x_1,x_2|\xi}(x_1, x_2|\xi)f_\xi(\xi)\mathrm{d}\xi \;= \int f_{x_2|\xi,x_1}(x_2|\xi, x_1)f_{x_1|\xi}(x_1|\xi)f_\xi(\xi)\mathrm{d}\xi$$

It is

$$\begin{aligned}
f_{x_1}(x_1) &= \int\int f_{x_1,x_2|\xi}(x_1, x_2|\xi)f_\xi(\xi)\mathrm{d}\xi\mathrm{d}x_2 = \int\int f_{x_2|\xi,x_1}(x_2|\xi, x_1)f_{x_1|\xi}(x_1|\xi)f_\xi(\xi)\mathrm{d}\xi\mathrm{d}x_2 \\
&= \int \left(\int f_{x_2|\xi,x_1}(x_2|\xi, x_1)\mathrm{d}x_2\right) f_{x_1|\xi}(x_1|\xi)f_\xi(\xi)\mathrm{d}\xi = \int f_{x_1|\xi}(x_1|\xi)f_\xi(\xi)\mathrm{d}\xi
\end{aligned}$$

Because $x_1|\xi \sim \mathrm{N}_{d_1}(\mu_1, \Sigma_1\xi v)$, and $\xi \sim \mathrm{IG}(\frac{v}{2}, \frac{1}{2})$, it is $x_1 \sim \mathrm{T}_{d_1}(\mu_1, \Sigma_1, \nu)$ from the statement of the question.

2. From what is given, it is $x|\xi \sim \mathrm{N}_d(\mu, \Sigma\xi v)$, and hence $x_1|\xi \sim \mathrm{N}_d(\mu_1, \Sigma_1\xi v)$ as marginal of a Normal distribution. From the Bayes Theorem, it is

$$\begin{aligned}
f_{\xi|x_1}(\xi|x_1) &\propto f_{x_1|\xi}(x_1|\xi)f(\xi) \\
&\propto \xi^{-\frac{d_1}{2}} \exp\left(-\frac{1}{2}(x_1 - \mu_1)^\top \left(\Sigma_1\xi v\right)^{-1}(x_1 - \mu_1)\right) \times \xi^{-\frac{d_1+v}{2}-1} \exp\left(-\frac{1}{\xi}\frac{1}{2}\right) \\
&\propto \xi^{-\frac{d_1+v}{2}-1} \exp\left(-\frac{1}{\xi}\frac{1}{2}\left[(x_1 - \mu_1)^\top\Sigma_1^{-1}(x_1 - \mu_1)\frac{1}{v} + 1\right]\right) \\
&\propto \xi^{-\frac{d_1+v}{2}-1} \exp\left(-\frac{1}{\xi}\frac{1}{2}\frac{Q + v}{v}\right)
\end{aligned}$$

This is the kernel of the Inverse Gamma distribution, and hence I can recognize that

$$\xi|x_1 \sim \mathrm{IG}(\frac{1}{2}(d_1 + v), \frac{1}{2}\frac{Q + v}{v}).$$

3. Let $\xi' = \xi\frac{v}{Q+v}$, with $Q = (\mu_1 - x_1)^\top\Sigma_1^{-1}(\mu_1 - x_1)$. Then it is

$$\begin{aligned}
f(\xi'|x_1) = f_{\mathrm{IG}(\frac{1}{2}(d_1+v),\frac{1}{2}\frac{Q+v}{v})}(\xi|x_1)\left|\frac{\mathrm{d}\xi}{\mathrm{d}\xi'}\right| \quad &\propto (Q\xi')^{-\frac{d_1+v}{2}-1}\exp(-\frac{1}{2}\frac{Q+v}{v}\frac{1}{\frac{Q+v}{v}\xi'})1_{(0,+\infty)}(\frac{Q+v}{v}\xi')\frac{Q+v}{v} \\
\propto (\xi')^{-\frac{d_1+v}{2}-1}\exp(-\frac{1}{2}\frac{1}{\xi'})1_{(0,+\infty)}(\xi') = f_{\mathrm{IG}(\frac{v+d_1}{2},\frac{1}{2})}(\xi')
\end{aligned}$$

So

$$\xi'|x_1 \sim \text{IG}(\frac{v+d_1}{2}, \frac{1}{2})$$

4. I will try to show that

$$x_2|\xi', x_1 \sim \text{N}_{d_2}\left(\mu_{2|1}, (v+d_1)\dot{\Sigma}_{2|1}\xi'\right)$$
$$\xi'|x_1 \sim \text{IG}(\frac{v+d_1}{2}, \frac{1}{2})$$

which leads to

$$x_2|x_1 \sim \text{T}_{d_2}(\mu_{2|1}, \dot{\Sigma}_{2|1}, \nu_{2|1})$$

since because

$$f_{x_2|x_1}(x_2|x_1) = \int f_{x_2|\xi,x_1}(x_2|\xi, x_1) f_\xi(\xi|x_1)\text{d}\xi$$

- I have calculated that
$$\xi'|x_1 \sim \text{IG}(\frac{v+d_1}{2}, \frac{1}{2})$$
where $\xi' = \xi\frac{v}{Q+v}$ with $Q = (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)$.
- It is (from multivariate Normal properties of the Example in the Hint)
$$x_2|\xi, x_1 \sim \text{N}_{d_2}\left(\mu_{2|1}, (\underbrace{\Sigma_{22} - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top}_{=\Sigma_{2|1}})\xi v\right) \equiv \text{N}_{d_2}\left(\mu_{2|1}, \Sigma_{2|1} v\xi\right)$$
where $\mu_{2|1} = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1)$. If I rearrange the parameters in order to appear $\xi' = \xi\frac{v}{Q+v}$ in the covariance I get
$$x_2|\xi, x_1 \sim \text{N}_{d_2}\left(\mu_{2|1}, \Sigma_{2|1} v\xi' \frac{v+Q}{v}\frac{v+d_1}{v+d_1}\right)$$
By setting
$$\dot{\Sigma}_{2|1} = \Sigma_{2|1}\frac{v+Q}{v+d_1}$$
I get
$$x_2|\xi', x_1 \sim \text{N}_{d_2}\left(\mu_{2|1}, (v+d_1)\dot{\Sigma}_{2|1}\xi'\right)$$

So I have

$$x_2|\xi', x_1 \sim \text{N}_{d_2}\left(\mu_{2|1}, (v+d_1)\dot{\Sigma}_{2|1}\xi'\right)$$
$$\xi'|x_1 \sim \text{IG}(\frac{v+d_1}{2}, \frac{1}{2})$$

which gives that $x_2|x_1 \sim \text{T}_{d_2}(\mu_{2|1}, \dot{\Sigma}_{2|1}, \nu_{2|1})$ with $\nu_{2|1} = v + d_1$. So the distribution of $x_2|x_1$ is $x_2|x_1 \sim \text{T}_{d_2}(\mu_{2|1}, \dot{\Sigma}_{2|1}, \nu_{2|1})$.

Note: Alternatively, one could prove sub-questions (2) and (4) by performing several pages of Matrix calculations to show that

$$\begin{aligned} f_X(x|\mu,\Sigma) =& \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})\nu^{\frac{d}{2}}\pi^{\frac{d}{2}}\det(\Sigma)^{\frac{1}{2}}}\left(1+\frac{1}{v}(x-\mu)^{\mathrm{T}}\Sigma^{-1}(x-\mu)\right)^{-\frac{\nu+d}{2}} \\ =& ... \\ =& \frac{\Gamma(\frac{\nu+d_1}{2})}{\Gamma(\frac{\nu}{2})\nu^{\frac{d_1}{2}}\pi^{\frac{d_1}{2}}\det(\Sigma_1)^{\frac{1}{2}}}\left(1+\frac{1}{v}(x_1-\mu_1)^{\mathrm{T}}\Sigma_1^{-1}(x_1-\mu_1)\right)^{-\frac{\nu+d_1}{2}} \\ & \times \frac{\Gamma(\frac{\nu_{2|1}+d_2}{2})}{\Gamma(\frac{\nu_{2|1}}{2})\nu_{2|1}^{\frac{d_2}{2}}\pi^{\frac{d_2}{2}}\det(\dot{\Sigma}_{2|1})^{\frac{1}{2}}}\left(1+\frac{1}{v_{2|1}}(x_2-\mu_{2|1})^{\mathrm{T}}\dot{\Sigma}_{2|1}^{-1}(x_2-\mu_{2|1})\right)^{-\frac{\nu_{2|1}+d_2}{2}} \end{aligned}$$

see Raiffa, H., & Schlaifer, R. (1961; Section 8.3). Applied statistical decision theory. This requires a lot of vector and matrix calculus.

Exercise 2. ($\star\star$)Let x be an observation. Consider the Bayesian model

$$\begin{cases} x|\theta & \sim \text{Pn}(\theta) \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $\text{Pn}(\theta)$ is the Poisson distribution with expected value θ. Consider a prior $\Pi(\theta)$ with density such as $\pi(\theta) \propto \frac{1}{\theta}$. Show that the posterior distribution is not always defined.

Hint-1: It suffices to show that the posterior is not defined in the case that you collect only one observation $x = 0$.

Hint-2: Poisson distribution: $x \sim \text{Pn}(\theta)$ has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

Solution. The prior with $\pi(\theta) \propto \frac{1}{\theta}$ is improper because

$$\int \pi(\theta)\mathrm{d}\theta \propto \int \frac{1}{\theta}\mathrm{d}\theta = \infty$$

So I need to check the proneness condition,

$$\underbrace{\int_{\mathbb{R}_+} \text{Pn}(x|\theta)\pi(\theta)\theta}_{\propto f(x)} \quad \begin{cases} < \infty & \text{posterior distribution is defined} \\ = \infty & \text{posterior distribution is not defined} \end{cases}$$

I will show that the posterior distribution is not defined given that I have collected a single observation $x = 0$. So I need to show that

$$\underbrace{\int_{\mathbb{R}_+} \text{Pn}(x=0|\theta)\pi(\theta)\theta}_{\propto f(x=0)} = \infty$$

It is

$$f(x) \propto \int_{\mathbb{R}_+} \text{Pn}(x|\theta)\frac{1}{\theta}\theta = \int_0^\infty \exp(-\theta)\frac{\theta^x}{x!}\frac{1}{\theta}\theta$$
$$f(x=0) \propto \int_{\mathbb{R}_+} \exp(-\theta)\frac{\theta^0}{0!}\frac{1}{\theta}\theta = \int_0^\infty \exp(-\theta)\frac{1}{\theta}\theta$$

We will use a convergence criteria in order to check if $\int_0^\infty \exp(-\theta)\frac{1}{\theta}\theta = \infty$.

Consider $h(\theta) = \exp(-\theta)\frac{1}{\theta}$. The function $h(\theta)$ has an improper behavior at 0, as it is not bounded there. Let $g(\theta) = \frac{1}{\theta}$. According to the Limit Comparison Test, it is

$$\lim_{\theta \to 0^+} \frac{h(\theta)}{g(\theta)} = \lim_{\theta \to 0^+} \frac{\frac{1}{\theta}\exp(-\theta)}{\frac{1}{\theta}} = 1 \neq 0$$

and

$$\int_0^\infty g(\theta)\theta = \int_0^\infty \frac{1}{\theta}\theta = \infty.$$

Therefore, it will be

$$\underbrace{\int_0^\infty h(\theta)\theta}_{=f(x=0)} = \infty$$

as well.
