

## Handout 2: Revision in mixture of probability distributions

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### Aim

To practice on probability calculations on compound distribution functions. To become familiar with distributions, Inverted Gamma, multivariate Normal, and multivariate Student T distributions.

### References:

- DeGroot, M. H. (1970, or 2005). Optimal statistical decisions (Vol. 82). John Wiley & Sons.
  - Part one: Survey of probability theory. Chapters 1-5 ; However the treatment of the Normal and Student T distributions is different than ours.
- Raiffa, H., & Schlaifer, R. (1961). Applied statistical decision theory.
  - Chapters 8.2, 8.3 ; However the treatment of the Normal and Student T distributions is different than ours.

### Web-applets

- [https://georgios-stats-3.shinyapps.io/demo\\_multivariatenormaldistribution/](https://georgios-stats-3.shinyapps.io/demo_multivariatenormaldistribution/)
- [https://github.com/georgios-stats/Shiny\\_applets/tree/master/demo\\_MultivariateNormalDistribution](https://github.com/georgios-stats/Shiny_applets/tree/master/demo_MultivariateNormalDistribution)

## 1 Mixture probability distribution

**Definition 1.** Consider a random variable  $x$  distributed according to  $F(x|z)$  with a pdf/pmf  $f(x|z)$  labeled by an unknown parameter  $z$ . Consider that  $z$  is distributed according to distribution  $\Pi(z)$  with pdf/pmf  $\pi(z)$ . Then:

1. This dependency can be represented as

$$\begin{aligned} x|z &\sim F(x|z) \\ z &\sim \Pi(z) \end{aligned}$$

where  $F(x|z)$  is called conditional (or parametrized) distribution,  $\Pi(z)$  is called mixing (or latent) distribution; and  $z$  is called mixing (or latent) variable.

2. Distribution  $G(x)$  that results by integrating the conditional distribution  $F(x|z)$  with respect to  $\Pi(z)$  as

$$G(x) = \int F(x|z) d\Pi(z)$$

is called the mixture (or compound) distribution of  $x$ . The mixture (or compound) distribution  $G(x)$  has PDF/PMF

$$g(x) = \int f(x|z) d\Pi(z) = \begin{cases} \int f(x|z) \pi(z) dz & , z \text{ cont.} \\ \sum_{\forall z} f(x|z) \pi(z) & , z \text{ discr.} \end{cases}$$

**Definition 2.**  $G(x)$  is also called continuous mixture distribution when the mixing/latent variable  $z$  is continuous.

$G(x)$  is also called finite mixture when  $z$  is discrete.

**Remark 3.** Recall from Handout 1 that

$$\begin{aligned} E_G(x) &= E_{\Pi}(E_F(x|z)) \\ \text{Var}_G(x) &= E_{\Pi}(\text{Var}(x|z)) + \text{Var}_{\Pi}(E_F(x|z)) \end{aligned}$$

## 2 Inverted Gamma distribution $x|a, b \sim \text{IG}(a, b)$

**Definition 4.** The random variable  $x \in (0, +\infty)$  follows an Inverted Gamma distribution  $x \sim \text{IG}(a, b)$ , if and only if  $x = \frac{1}{y}$  follows a Gamma distribution,  $y \sim \text{Ga}(a, b)$ , with  $a > 0$  and  $b > 0$ .

**Example 5.** Let  $x \sim \text{IG}(a, b)$ , then the PDF of  $x$  is

$$f_{\text{IG}(a,b)}(x) = \frac{b^a}{\Gamma(a)} x^{-a-1} \exp\left(-\frac{b}{x}\right) 1_{(0,+\infty)}(x) \quad (1)$$

**Solution.** It is

$$f_{\text{IG}(a,b)}(x) = f_{\text{Ga}(a,b)}\left(\frac{1}{x}\right) \left| \frac{d}{dx} \left(\frac{1}{x}\right) \right| = \frac{b^a}{\Gamma(a)} \left(\frac{1}{x}\right)^{a-1} \exp\left(-\frac{b}{x}\right) 1_{(0,+\infty)}\left(\frac{1}{x}\right) \left| -\frac{1}{x^2} \right|$$

**Example 6.** Let a random variable  $x \sim \text{IG}(a, b)$ , then

$$E_{\text{IG}(a,b)}(x) = \frac{b}{a-1}; \quad a > 1 \quad \text{and} \quad \text{Var}_{\text{IG}(a,b)}(x) = \frac{b^2}{(a-1)^2(a-2)}; \quad a > 2$$

**Solution.** It is

$$E_{\text{IG}(a,b)}(x) = \int x f_{\text{IG}(a,b)}(x) dx = \int_{(0,+\infty)} x \frac{b^a}{\Gamma(a)} x^{-a-1} \exp\left(-\frac{b}{x}\right) dx$$

Assume that  $a > 1$ . Then

$$\begin{aligned} E_{\text{IG}(a,b)}(x) &= \int_{(0,+\infty)} \frac{b^{a-1}}{\Gamma(a)} b \frac{\Gamma(a-1)}{\Gamma(a-1)} x^{-a+1-1} \exp\left(-\frac{b}{x}\right) dx = b \frac{\Gamma(a-1)}{\Gamma(a)} \int_{(0,+\infty)} \frac{b^{a-1}}{\Gamma(a-1)} x^{-a+1-1} \exp\left(-\frac{b}{x}\right) dx \\ &= b \frac{\Gamma(a-1)}{\Gamma(a)} \int_{(0,+\infty)} \frac{b^{a-1}}{\Gamma(a-1)} x^{-a+1-1} \exp\left(-\frac{b}{x}\right) dx = b \frac{\Gamma(a-1)}{(a-1)\Gamma(a-1)} \int f_{\text{IG}(a-1,b)}(x) dx = \frac{b}{a-1} \end{aligned}$$

Similarly

$$\begin{aligned} E_{\text{IG}(a,b)}(x^2) &= \int_{(0,+\infty)} x^2 \frac{b^a}{\Gamma(a)} x^{-a-1} \exp\left(-\frac{b}{x}\right) dx = \dots = b \frac{\Gamma(a-1)}{(a-1)\Gamma(a-1)} \int x f_{\text{IG}(a-1,b)}(x) dx \\ &= \frac{b}{a-1} \frac{b}{a-2}; \quad a > 2 \end{aligned}$$

So

$$\text{Var}_{\text{IG}(a,b)}(x) = E_{\text{IG}(a,b)}(x^2) - (E_{\text{IG}(a,b)}(x))^2 = \frac{b^2}{(a-1)^2(a-2)}$$

## 3 Multivariate Normal distribution $x|\mu, \Sigma \sim \mathbf{N}_d(\mu, \Sigma)$

**Definition 7.** A  $d$ -dimensional random variable  $x \in \mathbb{R}^d$  is said to have a multivariate Normal (Gaussian) distribution, if for every  $d$ -dimensional fixed vector  $\alpha \in \mathbb{R}^d$ , the random variable  $\alpha^\top x$  has a univariate Normal (Gaussian) distribution.

**Proposition 8.** A random vector  $x \in \mathbb{R}^d$  has a  $d$ -dimensional Normal distribution with mean  $\mu = E(x)$  and covariance matrix  $\Sigma = \text{Var}(x)$  if and only if random vector  $x \in \mathbb{R}^d$  has a characteristic function

$$\varphi_x(t) = \exp\left(it^\top \mu - \frac{1}{2}t^\top \Sigma t\right) \quad (2)$$

Hence: the  $d$ -dimensional Normal distribution is uniquely defined by the mean and the covariance matrix.

*Proof.* ( $\implies$ ) If  $x$  has a  $d$ -dimensional distribution then the characteristic function is  $\varphi_x(t) = \varphi_{t^\top x}(1)$ . Since  $x$  has a  $d$ -dimensional Normal distribution with mean  $\mu = E(x)$  and covariance matrix  $\Sigma = \text{Var}(x)$ ,  $t^\top x$  has a Normal distribution with mean  $E(t^\top x) = t^\top \mu$  and variance  $\text{Var}(t^\top x) = t^\top \Sigma t$ . Then (Handout 1, Example 31)

$$\varphi_x(t) = \varphi_{t^\top x}(1) = \exp\left(iE(t^\top x) - \frac{1}{2}\text{Var}(t^\top x)\right) = \exp\left(it^\top E(x) - \frac{1}{2}t^\top \text{Var}(x)t\right) = \exp\left(it^\top \mu - \frac{1}{2}t^\top \Sigma t\right)$$

( $\impliedby$ ) If random vector  $x \in \mathbb{R}^d$  has a characteristic function  $\varphi_x(t) = \exp(it^\top \mu - \frac{1}{2}t^\top \Sigma t)$ , then for every  $d$ -dimensional fixed vector  $\alpha \in \mathbb{R}^d$  the characteristic function of  $\alpha^\top x$  is

$$\varphi_{\alpha^\top x}(t) = \varphi_x(t\alpha) = \exp\left(it\alpha^\top \mu - \frac{1}{2}t\alpha^\top \Sigma \alpha t\right) = \exp\left(it(\alpha^\top \mu) - \frac{1}{2}(\alpha^\top \Sigma \alpha)t^2\right)$$

which defines that  $\alpha^\top x$  has a univariate Normal distribution with mean  $\alpha^\top \mu$  and variance  $\alpha^\top \Sigma \alpha$ .  $\square$

**Notation 9.** We denote the  $d$ -dimensional Normal distribution with mean  $\mu$  and covariance matrix  $\Sigma \geq 0$  as  $N_d(\mu, \Sigma)$ .

**Notation 10.** The  $d$ -dimensional standardized Normal distribution is  $N_d(0, I)$ .

**Proposition 11.** Let random variable  $x \sim N_d(\mu, \Sigma)$ , fixed vector  $c \in \mathbb{R}^q$  and fixed matrix  $A \in \mathbb{R}^q \times \mathbb{R}^d$ . The random vector  $y = c + Ax$  has distribution  $y \sim N_q(c + A\mu, A\Sigma A^\top)$ .

*Proof.* First I show that  $y$  is Normally distributed. Let  $\alpha \in \mathbb{R}^q$  any fixed vector. Then  $\alpha^\top y = \tilde{\alpha}^\top x + \alpha^\top c$  where  $\tilde{\alpha} = A^\top \alpha$ . Because  $x$  is multivariate Normal, then  $\tilde{\alpha}^\top x$  is univariate Normal (by Definition 7), then  $\alpha^\top y$  is univariate Normal. So  $y$  is  $q$ -variate Normal. Also,  $E(y) = E(c + Ax) = c + AE(x)$ , and  $\text{Var}(y) = \text{Var}(c + Ax) = A\text{Var}(x)A^\top$ .  $\square$

**Proposition 12.** Let a  $d$ -dimensional random vector  $x \sim N_{(any)}(\mu, \Sigma)$ .

1. Let  $y = Ax$  and  $z = Bx$ , where  $A \in \mathbb{R}^{q \times d}$  and  $B \in \mathbb{R}^{k \times d}$ : The vectors  $y = Ax$  and  $z = Bx$  are independent if and only if  $A\Sigma B^\top = 0$ .
2. Let  $x = (x_1, \dots, x_d)^\top$ : The  $x_1, \dots, x_d$  are mutually independent if and only if the corresponding off diagonal parts of the  $\Sigma$  are zero.

*Proof.* In both cases, the CF (2) factorizes as  $\varphi_x(t) = \prod_j \varphi_{x_j}(t_j)$  only when the corresponding of diagonal parts of  $\Sigma$  are zero.  $\square$

**Proposition 13.** Any sub-vector of a vector with multivariate Normal distribution has a multivariate Normal distribution.

*Proof.* Let  $x \sim N_d(\mu, \Sigma)$ . Any sub-vector  $y$  of  $x$  can be expressed as  $y = 0 + Px$ , where  $P \in \mathbb{R}^{q \times d}$  is a suitable projection matrix. Then  $y \sim N_d(P\mu, P\Sigma P^\top)$ .  $\square$

**Proposition 14.** [Marginalization & conditioning] Let  $x \sim N_d(\mu, \Sigma)$ . Consider partition such that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix},$$

where  $x_1 \in \mathbb{R}^{d_1}$ , and  $x_2 \in \mathbb{R}^{d_2}$  Then:

1. For the marginal, it is  $x_1 \sim N_{d_1}(\mu_1, \Sigma_1)$ .

2. For  $x_{2.1} = x_2 - \Sigma_{21}\Sigma_1^{-1}x_1$ , with  $\Sigma_1 > 0$ , it is  $x_{2.1} \sim N_{d_2}(\mu_{2.1}, \Sigma_{2.1})$  where

$$\mu_{2.1} = \mu_2 - \Sigma_{21}\Sigma_1^{-1}\mu_1 \text{ and } \Sigma_{2.1} = \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top \quad (3)$$

3. Random variables  $x_1$  and  $x_{2.1}$  are independent.

4. For the conditional, if  $\Sigma_1 > 0$ , it is

$$x_2|x_1 \sim N_{d_2}(\mu_{2|1}, \Sigma_{2|1})$$

where

$$\mu_{2|1} = \mu_2 - \Sigma_{21}\Sigma_1^{-1}(x_1 - \mu_1) \text{ and } \Sigma_{2|1} = \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top \quad (4)$$

**Hint:** If that was a Homework it will be given as a hint to use, in (1.):  $x_1 = Ax$  with  $A = [I, 0]$ , and in (2.):  $x_{2.1} = Bx$  with  $[-\Sigma_{21}\Sigma_1^{-1}, I]$ .

**Solution.**

1. It is  $x_1 = Ax$  with  $A = [I, 0]$ . Then  $x_1 \sim N(A\mu, A\Sigma A^\top)$  where

$$A\mu = [I, 0] \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \mu_1; \quad A\Sigma A^\top = [I, 0] \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \Sigma_1$$

2. It is  $x_{2.1} = Bx$  with  $B = [-\Sigma_{21}\Sigma_1^{-1}, I]$ . Then  $x_{2.1} \sim N(B\mu, B\Sigma B^\top)$  where

$$\begin{aligned} B\mu &= [-\Sigma_{21}\Sigma_1^{-1}, I] \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = -\Sigma_{21}\Sigma_1^{-1}\mu_1 + \mu_2; \\ B\Sigma B^\top &= [-\Sigma_{21}\Sigma_1^{-1}, I] \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix} \begin{bmatrix} -\Sigma_1^{-1}\Sigma_{21}^\top \\ I \end{bmatrix} = [0, -\Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top + \Sigma_2] \begin{bmatrix} -\Sigma_{21}\Sigma_1^{-1} \\ I \end{bmatrix} \\ &= -\Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top + \Sigma_2 \end{aligned}$$

3.  $x_1$  and  $x_{2.1}$  are independent, because (i.)  $x_1$  and  $x_2$  are Normally distributed and (ii.) for  $x_1 = Ax$  with  $A = [I, 0]$  and  $x_{2.1} = Bx$  with  $[-\Sigma_{21}\Sigma_1^{-1}, 0]$  are

$$\begin{aligned} \text{Cov}(x_1, x_{2.1}) &= \text{Cov}(Ax, Bx) = A\Sigma B^\top = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix} \begin{bmatrix} -\Sigma_1^{-1}\Sigma_{21}^\top \\ I \end{bmatrix} = \\ &= \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \end{bmatrix} \begin{bmatrix} -\Sigma_1^{-1}\Sigma_{21}^\top \\ I \end{bmatrix} = -\Sigma_{21}^\top + \Sigma_{21}^\top = 0 \end{aligned}$$

4. From part ??, I observe that it is

$$x_{2.1} = x_2 - \Sigma_{21}\Sigma_1^{-1}x_1 \iff x_2 = x_{2.1} + \Sigma_{21}\Sigma_1^{-1}x_1. \quad (5)$$

In (5), if I condition  $x_2$  on a given value for  $x_1$ , the term  $\Sigma_{21}\Sigma_1^{-1}x_1$  is a constant, namely I have  $x_2|x_1 = x_{2.1}|x_1 + \text{const}$  where variation comes from  $x_{2.1}|x_1$ . From part ??,  $x_{2.1}$  is independent on  $x_1$ , and hence  $F(x_{2.1}|x_1) = F(x_{2.1})$ . Furthermore,  $x_{2.1}|x_1 \in \text{Normal}$  because  $x_{2.1} \in \text{Normal}$ . Consequently,  $x_2|x_1 \in \text{Normal}$  as a linear transformation of a Normal variate. Now, about the moments

$$\begin{aligned} E(x_2|x_1) &= E(x_{2.1} + \Sigma_{21}\Sigma_1^{-1}x_1|x_1) = E(x_{2.1}|x_1) + E(\Sigma_{21}\Sigma_1^{-1}x_1|x_1) = [\mu_2 - \Sigma_{21}\Sigma_1^{-1}\mu_1] + [\Sigma_{21}\Sigma_1^{-1}x_1] \\ \text{Var}(x_2|x_1) &= \text{Var}(x_{2.1} + \Sigma_{21}\Sigma_1^{-1}x_1|x_1) = \text{Var}(x_{2.1}|x_1) = \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top \end{aligned}$$

**Proposition 15.** The density function of the  $d$ -dimensional Normal distribution with mean  $\mu$  and covariance matrix  $\Sigma$ , when  $\Sigma$  is symmetric positive definite matrix ( $\Sigma > 0$ ), exists and it is equal to

$$f(x) = (2\pi)^{-\frac{d}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x - \mu)^\top \Sigma^{-1}(x - \mu)\right) \quad (6)$$

*Proof.* Let  $x \sim N(\mu, \Sigma)$ . Because  $\Sigma > 0$ , we use Cholesky decomposition to define  $L$  such that  $\Sigma = LL^\top$ . Let  $z = L^{-1}(x - \mu)$ . It is  $E(z) = 0$ ,  $\text{Var}(z) = I$ ,  $z \sim N_d(0, I)$ , and hence  $z_1, \dots, z_d$  are mutually independent So

$$f_z(z) = \prod_{i=1}^d (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}z_i^2\right) = (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{1}{2}z^\top z\right)$$

Then

$$\begin{aligned} f_x(x) &= f_z(z) \left| \frac{dz}{dx} \right| = f_z\left(L^{-1}(x - \mu)\right) \left| \det\left(\frac{d}{dx} L^{-1}(x - \mu)\right) \right| \\ &= (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{1}{2}(x - \mu)^\top (L^{-1})^\top L^{-1}(x - \mu)\right) \det(L^{-1}) \\ &= (2\pi)^{-\frac{d}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x - \mu)^\top \Sigma^{-1}(x - \mu)\right) \det(\Sigma)^{-\frac{1}{2}} \end{aligned}$$

□

#### 4 Multivariate Student's T distribution $x \sim T_d(\mu, \Sigma, v)$

**Definition 16.** A  $d$ -dimensional random variable  $x \in \mathbb{R}^d$  is said to have a multivariate Student's T distribution with location parameter  $\mu$ , scale matrix  $\Sigma$ , and degrees of freedom  $v$ , and it is denoted as  $x \sim T_d(\mu, \Sigma, v)$ , if and only if

$$x = \mu + y\sqrt{v}\xi$$

where  $y \sim N_d(0, \Sigma)$  and  $\xi \sim \text{IG}(\frac{v}{2}, \frac{1}{2})$  are independent random variables.

**Example 17.** Definition 16 can be equivalently represented in terms of a mixture probability distribution (Section 1) as the compound (or mixture) distribution

$$x \sim T_d(\mu, \Sigma, v)$$

of

$$\begin{aligned} x|\xi &\sim N_d(\mu, \Sigma v\xi) \\ \xi &\sim \text{IG}\left(\frac{v}{2}, \frac{1}{2}\right) \end{aligned}$$

**Solution.** Straightforward derivation from Definition 16. We have that  $y \sim N_d(0, \Sigma)$ ,  $x = \mu + y\sqrt{v}\xi$ , and consider the rest  $\Sigma$ ,  $\mu$ ,  $v$ , and  $\xi$  as constant/known. So  $x|\mu, \Sigma, v, \xi$  follows a Normal distribution with mean  $E(x|\xi) = \mu$  and covariance matrix  $\text{Var}(x|\xi) = \Sigma v\xi$ . Hence, we can just write down  $x|\xi \sim N_d(\mu, \Sigma v\xi)$  by suppress the uninteresting constants  $\Sigma$ ,  $\mu$ , and  $v$  from the conditioning.

**Proposition 18.** If  $x \sim T_d(\mu, \Sigma, v)$  and  $\Sigma > 0$  then

1. The expected value is

$$E_{T_d(\mu, \Sigma, v)}(X) = \mu$$

2. The covariance matrix is

$$\text{Var}_{T_d(\mu, \Sigma, \nu)}(X) = \begin{cases} \frac{\nu}{\nu-2} \Sigma & , \text{ if } \nu > 2 \\ \text{undefined} & , \text{ else} \end{cases} \quad (7)$$

*Proof.* Given Definition 16,  $x \sim T_d(\mu, \Sigma, \nu)$  results as the marginal distribution of  $(x, \xi)$  where  $x|\xi \sim N_d(\mu, \Sigma\xi v)$  and  $\xi \sim \text{IG}(\frac{\nu}{2}, \frac{1}{2})$ .

1. It is

$$\mathbb{E}_{T_d(\mu, \Sigma, \nu)}(x) = \mathbb{E}_{\text{IG}(\frac{\nu}{2}, \frac{1}{2})}(\mathbb{E}_{N_d(\mu, \Sigma\xi v)}(x|\xi)) = \mathbb{E}_{\text{IG}(\frac{\nu}{2}, \frac{1}{2})}(\mu) = \mu \quad (8)$$

2. It is

$$\text{Var}_{T_d(\mu, \Sigma, \nu)}(x) = \mathbb{E}_{\text{IG}(\frac{\nu}{2}, \frac{1}{2})}(\text{Var}_{N_d(\mu, \Sigma\xi v)}(x|\xi)) + \text{Var}_{\text{IG}(\frac{\nu}{2}, \frac{1}{2})}(\mathbb{E}_{N_d(\mu, \Sigma\xi v)}(x|\xi)) \quad (9)$$

$$= \mathbb{E}_{\text{IG}(\frac{\nu}{2}, \frac{1}{2})}(\Sigma\xi v) + \cancel{\text{Var}_{\text{IG}(\frac{\nu}{2}, \frac{1}{2})}(\mu)} \stackrel{=0}{=} \Sigma v \mathbb{E}_{\text{IG}(\frac{\nu}{2}, \frac{1}{2})}(\xi) + 0$$

$$= \begin{cases} \Sigma v \frac{\frac{1}{2}}{\frac{\nu}{2}-1} & , \text{ if } \frac{\nu}{2} > 1 \\ \text{undefined} & , \text{ else} \end{cases}$$

□

**Example 19.** If  $x \sim T_d(\mu, \Sigma, \nu)$  and  $\Sigma > 0$  the PDF of  $x$  is

$$f_X(x|\mu, \Sigma, \nu) = \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2}) \nu^{\frac{d}{2}} \pi^{\frac{d}{2}} \det(\Sigma)^{\frac{1}{2}}} \left( 1 + \frac{1}{\nu} (x - \mu)^\top \Sigma^{-1} (x - \mu) \right)^{-\frac{\nu+d}{2}} \quad (10)$$

**Solution.** Given Definition 16,  $x \sim T_d(\mu, \Sigma, \nu)$  results as the marginal distribution of  $(x, \xi)$  where  $x|\xi \sim N_d(\mu, \Sigma\xi v)$  and  $\xi \sim \text{IG}(\frac{\nu}{2}, \frac{1}{2})$ . So it is

$$\begin{aligned} f_x(x) &= \int f_{x|\xi}(x|\xi) f_\xi(\xi) d\xi = \int f_{N_d(\mu, \Sigma v \xi)}(x|\xi) f_{\text{IG}(\frac{\nu}{2}, \frac{1}{2})}(\xi) d\xi \\ &= \underbrace{\int \left( \frac{1}{2\pi} \right)^{\frac{d}{2}} \frac{1}{\sqrt{\det(\Sigma v \xi)}} \exp \left( -\frac{1}{2} (x - \mu)^\top \frac{\Sigma^{-1}}{v \xi} (x - \mu) \right) \frac{1}{\Gamma(\frac{\nu}{2})} \xi^{-\frac{\nu}{2}-1} \exp \left( -\frac{1}{\xi} \frac{1}{2} \right) 1_{(0, \infty)}(\xi) d\xi}_{=N_d(x|\mu, \Sigma v \xi)} \underbrace{\int \xi^{-\frac{\nu}{2}-1} \exp \left( -\frac{1}{\xi} \frac{1}{2} \right) d\xi}_{= \text{IG}(\xi|\frac{\nu}{2}, \frac{1}{2})} \\ &= \left( \frac{1}{2\pi} \right)^{\frac{d}{2}} \frac{1}{\sqrt{\det(\Sigma v)}} \frac{1}{\Gamma(\frac{\nu}{2})} \int \xi^{-\frac{\nu}{2}-\frac{d}{2}-1} \exp \left( -\frac{1}{\xi} \left[ \frac{1}{2v} (x - \mu)^\top \Sigma^{-1} (x - \mu) + \frac{1}{2} \right] \right) d\xi \quad (11) \\ &\quad \quad \quad = \Gamma\left(\frac{\nu}{2} + \frac{d}{2}\right) \left[ \frac{1}{2v} (x - \mu)^\top \Sigma^{-1} (x - \mu) + \frac{1}{2} \right]^{-\left(\frac{\nu}{2} + \frac{d}{2}\right)} \\ &= \left( \frac{1}{2\pi} \right)^{\frac{d}{2}} \frac{1}{\sqrt{\det(\Sigma v)}} \frac{1}{\Gamma(\frac{\nu}{2})} \Gamma\left(\frac{\nu}{2} + \frac{d}{2}\right) \left[ \frac{1}{2v} (x - \mu)^\top \Sigma^{-1} (x - \mu) + \frac{1}{2} \right]^{-\left(\frac{\nu}{2} + \frac{d}{2}\right)} \\ &= \left( \frac{1}{2\pi} \right)^{\frac{d}{2}} \frac{1}{\sqrt{\det(\Sigma v)}} \frac{1}{\Gamma(\frac{\nu}{2})} \Gamma\left(\frac{\nu}{2} + \frac{d}{2}\right) \left( \frac{1}{2} \right)^{-\frac{(\nu+d)}{2}} \left[ \frac{1}{v} (x - \mu)^\top \Sigma^{-1} (x - \mu) + 1 \right]^{-\frac{\nu+d}{2}} \\ &= \left( \frac{1}{\pi} \right)^{\frac{d}{2}} \frac{1}{\sqrt{\det(\Sigma)}} \left( \frac{1}{v} \right)^{\frac{d}{2}} \frac{1}{\Gamma(\frac{\nu}{2})} \Gamma\left(\frac{\nu+d}{2}\right) \left[ \frac{1}{v} (x - \mu)^\top \Sigma^{-1} (x - \mu) + 1 \right]^{-\frac{\nu+d}{2}} \end{aligned}$$

158 where the integral in (11) was calculated by recognizing the IG density from (1).

## 159 **5 Practice**

160 **Question 20.** *For practice try the Exercises 18, 19, and, 21, from the Exercise Sheet.*

## Supplementary material

### A The useful formulas regarding Normal PDF

The following formulas will be given as hints in the exercises and there is no need to be memorized. I present them for your information –no need to memorize.

**Fact 21.** If  $\Sigma_1 > 0$  and  $\Sigma_2 > 0$  symmetric

$$-\frac{1}{2}(x - \mu_1) \Sigma_1^{-1} (x - \mu_1)^\top - \frac{1}{2}(x - \mu_2) \Sigma_2^{-1} (x - \mu_2)^\top = -\frac{1}{2}(x - m) V^{-1} (x - m)^\top + C$$

where

$$V^{-1} = \Sigma_1^{-1} + \Sigma_2^{-1}; \quad m = V (\Sigma_1^{-1} \mu_1 + \Sigma_2^{-1} \mu_2); \quad C = \frac{1}{2} m^\top V^{-1} m - \frac{1}{2} (\mu_1^\top \Sigma_1^{-1} \mu_1 + \mu_2^\top \Sigma_2^{-1} \mu_2)$$

*Proof.* It is derived by  $\pm$ ing terms and doing matrix calculations. □

**Fact 22.** If  $f_{N_d(\mu, \Sigma)}(x)$  denotes the PDF of  $N_d(\mu, \Sigma)$ , then

$$f_{N_d(\mu_1, \Sigma_1)}(x) f_{N_d(\mu_2, \Sigma_2)}(x) = f_{N_d(m, V)}(x) f_{N_d(\mu_2, \Sigma_1 + \Sigma_2)}(\mu_1)$$

where

$$V^{-1} = \Sigma_1^{-1} + \Sigma_2^{-1}; \quad m = V (\Sigma_1^{-1} \mu_1 + \Sigma_2^{-1} \mu_2)$$

*Proof.* It is derived by exponentiation □

**Fact 23.** If  $\Sigma_i > 0$  symmetric for  $i = 1, \dots, n$

$$-\frac{1}{2} \sum_{i=1}^n (x - \mu_i) \Sigma_i^{-1} (x - \mu_i)^\top = -\frac{1}{2} (x - m) V^{-1} (x - m)^\top + C \quad (12)$$

where

$$V^{-1} = \sum_{i=1}^n \Sigma_i^{-1}; \quad m = V \left( \sum_{i=1}^n \Sigma_i^{-1} \mu_i \right); \quad C = \frac{1}{2} m^\top V^{-1} m - \frac{1}{2} \left( \sum_{i=1}^n \mu_i^\top \Sigma_i^{-1} \mu_i \right) \quad (13)$$

*Proof.* It is shown by induction from the (1.). □



## B Examples of mixture distributions

The following distributions below are given as examples of mixture distributions –no need to memorize.

### B.1 Fisher-Snedecor distribution $x \sim F(d_1, d_2)$

**Definition 24.** A random variable  $x \in \mathbb{R}$  is said to have a Fisher-Snedecor distribution with degrees of freedom  $n, m$ , and it is denoted as  $x \sim F(n, m)$ , if and only if

$$x = \frac{y_1/d_1}{y_2/d_2}$$

where  $y_1 \sim G(d_1/2, 1/2)$  and  $y_2 \sim G(d_2/2, 1/2)$  are independent random variables.

**Example 25.** The distribution function  $x \sim L(\mu, b)$  is such that

$$dF(x) = \left[ xB\left(\frac{d_1}{2}, \frac{d_2}{2}\right) \right]^{-1} \sqrt{\frac{(d_1 x)^{d_1} d_2^{d_2}}{(d_1 x + d_2)^{d_1 + d_2}}} dx$$

with mean  $E(x) = d_2/(d_2 - 1)$ .

**Solution.** See Statistical Concepts II (Exercises) which is proved by transformation of random variables and marginalization.

**Example 26.** The distribution function  $x \sim F(d_1, d_2)$  can be re-presented as a mixture distribution as

$$\begin{cases} x|v \sim \text{Ga}(d_1/2, zd_1/2) \\ z \sim \text{Ga}(d_2/2, d_2/2) \end{cases}$$

**Solution.** Show that  $f(x) = \int f(x|z)f(z)dz$ ; see Exercise 19 in the Exercise sheet.

### B.2 Laplace distribution $x \sim L(\mu, b)$

**Definition 27.** A random variable  $x \in \mathbb{R}$  is said to have a Laplace (or Double exponential) distribution with location parameter  $\mu$ , and scale parameter  $b$ , and it is denoted as  $x \sim L_d(\mu, b)$ , if and only if

$$x = \mu + y\sqrt{2b}$$

where  $y \sim N_d(0, b^2)$  and  $v \sim G(1, 1)$  are independent random variables.

**Example 28.** The distribution function  $x \sim L(\mu, b)$  can be re-presented as a mixture distribution as

$$\begin{cases} x|v \sim N(\mu, 2vb^2) \\ v \sim \text{Ga}(1, 1) \equiv \text{Ex}(1) \end{cases} \quad (14)$$

Show that the distribution function  $x \sim L(\mu, b)$  is such that

$$dF(x) = \frac{1}{2b} \exp\left(-\frac{|x - \mu|}{b}\right) dx$$

with mean  $E(x) = \mu$  and variance  $\text{Var}(x) = 2b^2$ .

**Solution.** By transformation of random variables you show (14). To show that  $f(x) = \int f(x|v)f(v)dv$ . Perform change of variable to compute the integral. See C. Robert's solution: <https://stats.stackexchange.com/questions/175458/show-that-a-scale-mixtures-of-normals-is-a-power-exponential>