

Problem class 1^a**Nuisance parameters, conjugate priors, Jeffreys priors**

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^aAuthor: Georgios P. Karagiannis.**Nuisance parameters**

Exercise 1. (★★) Assume observable quantities $y = (y_1, \dots, y_n)$ forming the available data set of size n . Assume that the observations are drawn i.i.d. from a sampling distribution which is judged to be in the Normal parametric family of distributions $N(\mu, \sigma^2)$ with unknown mean μ and variance σ^2 . We are interested in learning μ and the next outcome $z = y_{n+1}$. We do not care about σ^2 . <-story

Assume You specify a Bayesian model

<-set-up

$$\begin{cases} y_i | \mu, \sigma^2 \sim N(\mu, \sigma^2), \text{ for all } i = 1, \dots, n & , \text{Statistical model} \\ \mu | \sigma^2 \sim N(\mu_0, \sigma^2 \frac{1}{\tau_0}) & , \text{prior} \\ \sigma^2 \sim \text{IG}(a_0, k_0) & , \text{prior} \end{cases}$$

1. Show that

$$\sum_{i=1}^n (y_i - \theta)^2 = n(\bar{y} - \theta)^2 + ns^2,$$

$$\text{where } s^2 = \frac{1}{2} \sum_{i=1}^n (y_i - \bar{y})^2.$$

2. Show that the joint posterior distribution $\Pi(\mu, \sigma^2 | y)$ is such as

$$\begin{aligned} \mu | y, \sigma^2 &\sim N(\mu_n, \sigma^2 \frac{1}{\tau_n}) \\ \sigma^2 | y &\sim \text{IG}(a_n, k_n) \end{aligned}$$

with

$$\mu_n = \frac{n\bar{y} + \tau_0\mu_0}{n + \tau_0}; \quad \tau_n = n + \tau_0; \quad a_n = a_0 + n$$

$$k_n = k_0 + \frac{1}{2}ns_n^2 + \frac{1}{2}\frac{\tau_0n(\mu_0 - \bar{y})^2}{n + \tau_0}$$

Hint: It is

$$-\frac{1}{2} \frac{(\mu - \mu_1)^2}{v_1} - \frac{1}{2} \frac{(\mu - \mu_2)^2}{v_2} \dots - \frac{1}{2} \frac{(\mu - \mu_n)^2}{v_n} = -\frac{1}{2} \frac{(\mu - \hat{\mu})^2}{\hat{v}} + C$$

where

$$\hat{v} = \left(\sum_{i=1}^n \frac{1}{v_i} \right)^{-1}; \quad \hat{\mu} = \hat{v} \left(\sum_{i=1}^n \frac{\mu_i}{v_i} \right); \quad C = \frac{1}{2} \frac{\hat{\mu}^2}{\hat{v}} - \frac{1}{2} \sum_{i=1}^n \frac{\mu_i^2}{v_i}$$

3. Show that the marginal posterior distribution $\Pi(\mu | y)$ is such as

$$\mu | y \sim T_1 \left(\mu_n, \frac{k_n}{a_n} \frac{1}{\tau_n}, 2a_n \right)$$

Hint-1: If $x \sim \text{IG}(a, b)$, $y = cx$, then $y \sim \text{IG}(a, cb)$.

Hint-2: The definition of Student T is considered as known

4. Show that the predictive distribution $\Pi(z|y)$ is Student T such as

$$z|y \sim T_1 \left(\mu_n, \frac{k_n}{a_n} \left(\frac{1}{\tau_n} + 1 \right), 2a_n \right)$$

Hint-1: Consider that

$$N(x|\mu_1, \sigma_1^2) N(x|\mu_2, \sigma_2^2) = N(x|m, v^2) N(\mu_1|\mu_2, \sigma_1^2 + \sigma_2^2)$$

where

$$v^2 = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{-1}; \quad m = v^2 \left(\frac{\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2} \right)$$

Hint-2: The definition of Student T is considered as known

Proper/improper priors

Exercise 2. (**) Consider the Bayesian model

$$\begin{cases} x|\sigma & \sim N(0, \sigma^2) \\ \sigma & \sim \text{Ex}(\lambda) \end{cases}$$

where $\text{Ex}(\lambda)$ is the exponential distribution with mean $1/\lambda$. Show that the posterior distribution is not defined always.

- HINT: Precisely, show that the posterior is not defined in the case that you collect only one observation $x = 0$.

Conjugate priors

Exercise 3. (**) Consider the Bayesian model

$$\begin{cases} x_i|\theta & \stackrel{\text{iid}}{\sim} \text{Mu}_k(\theta), \quad i = 1, \dots, n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $\theta \in \Theta$, with $\Theta = \{\theta \in (0, 1)^k \mid \sum_{j=1}^k \theta_j = 1\}$ and $\mathcal{X}_k = \{x \in \{0, 1\}^k \mid \sum_{j=1}^k x_j = 1\}$.

Hint-1: Mu_k denotes the Multinomial probability distribution with PMF

$$\text{Mu}_k(x|\theta) = \begin{cases} \prod_{j=1}^k \theta_j^{x_j} & , \text{ if } x \in \mathcal{X}_k \\ 0 & , \text{ otherwise} \end{cases}$$

Hint-2: $\text{Di}_k(a)$ denotes the Dirichlet distribution with PDF

$$\text{Di}_k(\theta|a) = \begin{cases} \frac{\Gamma(\sum_{j=1}^k a_j)}{\prod_{j=1}^k \Gamma(a_j)} \prod_{j=1}^k \theta_j^{a_j-1} & , \text{ if } \theta \in \Theta \\ 0 & , \text{ otherwise} \end{cases}$$

1. Derive the conjugate prior distribution for θ , and recognize that it is a Dirichlet distribution family of distributions.
2. Verify that the prior distribution you derived above is indeed conjugate by using the definition.

Jeffreys priors

Exercise 4. (★★) Consider the trinomial distribution

$$p(x, y | \pi, \rho) = \frac{n!}{x! y! z!} \pi^x \rho^y \sigma^z, \quad (x + y + z = n) \\ \propto \pi^x \rho^y (1 - \pi - \rho)^{n-x-y}.$$

Specify a Jeffreys' prior for (π, ρ) .

HINT: It is $E(x) = n\pi$, $E(y) = n\rho$.

Exercise 5. (★★) Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \text{Ga}(\alpha, \beta), \quad \forall i = 1, \dots, n \\ (\alpha, \beta) & \sim \Pi(\alpha, \beta) \end{cases}$$

where $\text{Ga}(a, \beta)$ is the Gamma distribution with expected value α/β . Specify a Jeffrey's prior for $\theta = (\alpha, \beta)$.

Hint-1: Gamma distr.: $x \sim \text{Ga}(a, b)$ has pdf $f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) 1_{(0, +\infty)}(x)$, and Expected value $E_{\text{Ga}}(x|a, b) = \frac{a}{b}$

Hint-2: You may also need that the second derivative of the logarithm of a Gamma function is the 'polygamma function of order 1'. I.e.,

- $F^{(0)}(\alpha) = \frac{d}{d\alpha} \log(\Gamma(a))$
- $F^{(1)}(\alpha) = \frac{d^2}{d\alpha^2} \log(\Gamma(a))$

Hint-3: You may leave your answer in terms of function $F^{(1)}(\alpha)$.

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Exercise 6. (★★) Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \text{Ex}(\theta), \quad \forall i = 1, \dots, n \\ \theta & \sim \text{Ga}(a, b) \end{cases}$$

Hint-1: The PDF of $x \sim \text{G}(a, b)$ is $\text{Ga}(x|a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) 1_{(0, +\infty)}(x)$

Hint-2: The PDF of $x \sim \text{Ex}(\theta)$ is $\text{Ex}(x|\theta) = \text{Ga}(x|1, \theta)$

1. Show that the parametric model is member of the Exponential family, and the sufficient statistic for a sample of observables $x = (x_1, \dots, x_n)$.

2. Show that the posterior distribution θ given x is Gamma and compute its parameters.
3. Show that the predictive distribution $G(z|x)$ of a future z given $x = (x_1, \dots, x_n)$, has PDF

$$g(z|x) = \frac{a^*(b^*)^{a^*}}{(z + b^*)^{a^*+1}} 1(x \geq 0)$$

Further practice

From the exercise sheet, have a look at Exercises 26, 31, 47, 6, and 50.

A About Nuisance parameters

Assume observable quantities $y = (y_1, \dots, y_n)$. Assume that the sampling distribution is $dF(y|\theta)$ labeled by an unknown parameter $\theta \in \Theta$. Let $\theta = (\phi, \lambda)^\top$ with $\phi \in \Phi$ and $\lambda \in \Lambda$. Assume You are interested in learning parameter $\phi \in \Phi$, and You are not interested in learning the unknown parameter $\lambda \in \Lambda$; but both ϕ, λ are parts of the statistical model parameterisation. The unknown quantity $\lambda \in \Lambda$ is called nuisance parameter. We can call $\phi \in \Phi$ parameter of interest.

Note 7. In Bayesian Stats, learning (or quantifying uncertainty about) parameter of interest ϕ under the presence of a nuisance parameter $\lambda \in \Lambda$ is performed according to the Bayesian paradigm as usual: You specify a prior $d\Pi(\phi, \lambda)$ with PDF/PMF $\pi(\phi, \lambda) = \pi(\phi|\lambda)\pi(\lambda)$ on the joint space of ALL Your unknown parameters $\theta = (\phi, \lambda)^\top$; you compute the joint posterior distribution $d\Pi(\theta|y)$ of $\theta = (\phi, \lambda)^\top$ via the Bayesian theorem. Reasonably, Your posterior degree of believe about the parameter of interest ϕ given the data $y = (y_1, \dots, y_n)$ is given through the marginal posterior distribution $d\Pi(\phi|y)$.

Note 8. To summarize; Specify the Bayesian model as:

<sum-up

$$\begin{cases} \overbrace{y|\phi, \lambda}^{=\theta} \sim F(\overbrace{y|\phi, \lambda}^{=\theta}) & , \text{ the statistical model} \\ \underbrace{(\phi, \lambda)}_{=\theta} \sim \Pi(\underbrace{\phi, \lambda}_{=\theta}) & , \text{ the prior model} \end{cases}$$

The joint posterior of θ given y is $d\Pi(\theta|y) = d\Pi(\lambda|y, \phi)d\Pi(\phi|y)$ is with PDF/PMF

$$\pi(\overbrace{\phi, \lambda}^{=\theta}|y) = \frac{f(y|\overbrace{\phi, \lambda}^{=\theta})\pi(\overbrace{\phi, \lambda}^{=\theta})}{f(y)} = \underbrace{\frac{f(y|\phi, \lambda)\pi(\lambda|\phi)}{f(y|\phi)}}_{=\pi(\lambda|y, \phi)} \underbrace{\frac{f(y|\phi)\pi(\phi)}{f(y)}}_{=\pi(\phi|y)} = \pi(\lambda|y, \phi)\pi(\phi|y)$$

The (marginal) likelihood $f(y|\phi)$ of y given ϕ is

$$f(y|\phi) = \underbrace{\int_{\Lambda} f(y|\overbrace{\phi, \lambda}^{=\theta})d\Pi(\lambda|\phi)}_{=E_{\Pi(\lambda|\phi)}(f(y|\phi, \lambda)|\phi)} = \begin{cases} \int_{\Lambda} f(y|\phi, \lambda)\pi(\lambda|\phi)d\lambda & , \text{ if } \lambda \text{ cont} \\ \sum_{\forall \lambda \in \Lambda} f(y|\phi, \lambda)\pi(\lambda|\phi) & , \text{ if } \lambda \text{ discr} \end{cases}$$

The PDF/PMF $\pi(\phi|y)$ of marginal posterior $d\Pi(\phi|y)$ of ϕ is

$$\pi(\phi|y) = \underbrace{\int_{\Lambda} \pi(\overbrace{\phi, \lambda}^{=\theta}|y)d\lambda}_{=E_{\Pi(\lambda|y)}(\pi(\phi|y, \lambda))} \quad \text{or equivalently} \quad \pi(\phi|y) = \frac{f(y|\phi)\pi(\phi)}{f(y)}$$

The predictive distribution $dG(z|y)$ of the next outcome $z = (y_{n+1}, \dots, y_{n+m})$ given y has pdf/pmf

$$g(z|y) = \int f(z|\overbrace{\phi, \lambda}^{=\theta})d\Pi(\overbrace{\phi, \lambda}^{=\theta}|y)$$

and the marginal likelihood $f(y)$ is

$$f(y) = \int f(y|\overbrace{\phi, \lambda}^{=\theta})\pi(\overbrace{\phi, \lambda}^{=\theta})d\phi d\lambda$$

B Criteria for integrals

General: Let integrable functions $f(x)$, and $g(x)$ for $x \geq a$.

Let

$$0 \leq f(x) \leq g(x), \quad \text{for } x \geq a$$

Then

$$\begin{aligned} \int_a^\infty g(x) dx < \infty &\implies \int_a^\infty f(x) dx < \infty \\ \int_a^\infty f(x) dx = \infty &\implies \int_a^\infty g(x) dx = \infty \end{aligned}$$

Type I: Let integrable functions $f(x)$, and $g(x)$ for $x \geq a$, and let $g(x)$ be positive.

Let

$$\lim_{n \rightarrow \infty} \frac{f(x)}{g(x)} = c$$

Then

- If $c \in (0, \infty)$:

$$\int_a^\infty g(x) dx < \infty \iff \int_a^\infty f(x) dx < \infty$$

- If $c = 0$:

$$\int_a^\infty g(x) dx < \infty \implies \int_a^\infty f(x) dx < \infty$$

- If $c = \infty$:

$$\int_a^\infty f(x) dx = \infty \implies \int_a^\infty g(x) dx = \infty$$

Type II: Let integrable functions $f(x)$, and $g(x)$ for $a < x \leq b$, and let $g(x)$ be positive.

Let

$$\lim_{n \rightarrow a^+} \frac{f(x)}{g(x)} = c$$

Then

- If $c \in (0, \infty)$:

$$\int_a^\infty g(x) dx < \infty \iff \int_a^\infty f(x) dx < \infty$$

- If $c = 0$:

$$\int_a^\infty g(x) dx < \infty \implies \int_a^\infty f(x) dx < \infty$$

- If $c = \infty$:

$$\int_a^\infty f(x) dx = \infty \implies \int_a^\infty g(x) dx = \infty$$

Note: A useful test function is

$$\int_0^\infty \left(\frac{1}{x}\right)^p dx \begin{cases} < \infty & , \text{ when } p > 1 \\ = \infty & , \text{ when } p \leq 1 \end{cases}$$