Bayesian Statistics III/IV (MATH3361/4071) Michaelmas term 2021

Homework 4: Hypothesis test

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For Formative assessment, submit the solutions to the Exercise 1.

Exercise 1. $(\star\star)$

1. Consider the Single vs. General alternative Bayesian hypothesis test

$$\mathrm{H}_0 : \theta = \theta_0 \qquad \text{vs} \qquad \mathrm{H}_1 : \theta \neq \theta_0.$$

or more formally

$$\mathrm{H}_0 : \begin{cases} y|\theta_0 & \sim F(y|\theta_0) \\ \theta_0 \text{ is fixed} & \end{cases} \qquad \text{vs} \qquad \mathrm{H}_1 : \begin{cases} y|\theta & \sim F(y|\theta) \\ \theta & \sim \Pi_1(\theta),\ \theta \in \Theta_1 \end{cases} \tag{1}$$

Show that the Bayes factor B_{01} of H_0 and H_1 can be computed as

$$\mathrm{B}_{01} = \frac{\pi_1(\theta_0|y)}{\pi_1(\theta_0)}$$

where $\pi_1(\cdot)$ and $\pi_1(\cdot|\cdot)$, denote the PDF of $\Pi_1(\cdot)$ and $\Pi_1(\cdot|\cdot)$.

2. In the above hypothesis test, assume that $y|\theta \sim \mathrm{Bin}(n, \theta)$ has Binomial sampling distribution with unknown parameter θ.

   (a) Find an approximation of B_{01}, when n is large.

   Hint: [From Stats 2] How the aforesaid likelihoods or the posteriors do behave as $n$becomes big?

   (b) For large n, show that $\mathrm{B}_{01} > 1$ when $z_0 = \frac{p-\theta_0}{\sqrt{u}}$, with $p = \frac{y}{n}$ and $u = \frac{p(1-p)}{n}$, satisfies $|z| < \max\left(\sqrt{k}, 0\right)$ for some k that depends of n, p, and $\pi_1(\theta_0)$.

   (c) Let the conditional prior $\Pi_1(\theta)$ be a Uniform distribution with positive mass above the interval $[0, 1]$. Show that this choice of the conditional prior $\Pi_1(\theta)$ can create a "paradox" when compared with fixed size tests.

Solution.