Bayesian Statistics III/IV (MATH3341/4031) Michaelmas term, 2021

Handout 12: Credible sets

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Aim: To explain and produce credible regions in the Bayesian framework.

References:

- Berger, J. O. (2013; Section 4.3.2). Statistical decision theory and Bayesian analysis. Springer Science & Business Media.
- Robert, C. (2007; Section 5.5). The Bayesian choice: from decision-theoretic foundations to computational implementation. Springer Science & Business Media.

Web applets:

- `https://georgios-stats-1.shinyapps.io/demo_CredibleSets/`

1 Set-up and aim

Notation 1. Consider a Bayesian model

$$\begin{cases} y|\theta & \sim F(y|\theta) \\ \theta & \sim \Pi(\cdot) \end{cases}$$

where $y := (y_1, ..., y_n) \in \mathcal{Y}$ is a sequence of observables, assumed to be generated from the parametric sampling distribution $F(y|\theta)$ with pdf/pmf $f(y|\theta)$ and labeled by an unknown parameter $\theta \in \Theta$ with a prior distribution $\Pi(\theta)$ with pdf/pmf $\pi(\theta)$. Also assume a sequence of m future outcomes $z = (y_{n+1}, ..., y_{n+m})$.

AIM: Instead of just reporting a point value for θ (or z) and the associated standard error, it is often desirable and clearer to report sets of values $C_a \subseteq \Theta$ (or $C_a \subseteq \mathcal{Z}$) with a specified probability a reflecting Your believe that $\theta \in C_a$ (or $z \in C_a$).

Note 2. Recall that

- Posterior degree of believe about uncertain parameter $\theta \in \Theta \subseteq \mathbb{R}^d$ is quantified via the posterior distribution $\Pi(\theta|y)$;
$$\mathrm{d}\Pi(\theta|y) = \pi(\theta|y)\mathrm{d}\theta$$
with cdf $\Pi(\theta|y)$ and pdf/pmf $\pi(\theta|y)$.
- Degree of believe about a future sequence of outcomes $z = (y_{n+1}, ..., y_{n+m}) \in \mathcal{Z}$ is quantified via the predictive distribution $G(z|y)$;
$$\mathrm{d}G(z|y) = g(z|y)\mathrm{d}z$$
with cdf $G(z|y)$ and pdf/pmf $g(z|y)$.

Notation 3. We present the parametric and predictive credible intervals in a unified framework. Consider unknown random quantity $x \in \mathcal{X} \subseteq \mathbb{R}^k$ following a distribution $Q(x|y)$;

$$\mathrm{d}Q(x|y) = q(x|y)\mathrm{d}x$$

with cdf $Q(x|y)$ and pdf/pmf $q(x|y)$. These are dummies for the following:

- In parametric inference, we have $x \equiv \theta$, $Q \equiv \Pi$, $q \equiv \pi$, and $k = d$.
- In predictive inference, we have $x \equiv z$, $Q \equiv G$, $q \equiv g$, and $k = m$.
- Note that x can also be any function of θ or z.

2 Credible Sets

Definition 4. A set $C_a \subseteq \mathcal{X}$ is called '$100(1-a)\%$' posterior credible set for x, with respect to the posterior distribution $Q(x|y)$ if

$$1 - a \leq \mathsf{P}_Q(x \in C_a|y) = \int 1\,(x \in C_a)\,\mathrm{d}Q(x|y)$$

Note 5. In Bayesian stats (unlike frequetist stats) we can correctly say that the $(1-a)100\%$ credible set C_a of unknown parameter θ means that the probability that θ is in C_a is $(1-a)100\%$. This is theoretically correct as everything unknown/uncertain is a random quantity following a distribution reflecting Your degree of believe.

Note 6. Note that different sets may satisfy Definition 4 and hence we are interested in using the most useful credible set for our application. This is addressed by imposing additional restrictions.

3 Highest probability density Credible sets

Note 7. Often it is useful to consider credible sets C_a which contain values of x that correspond to the highest pdf/pmf $q(x|y)$ (aka the most likely values of x). Then we can impose the restriction $q(x|y) \geq q(x'|y)$ for all $x \in C_a,\ x' \in C_a^{\complement}$, in Definition 4 which leads to Definition 8, the definition of the highest probability density (HPD) set.

Definition 8. The $100(1-a)\%$ highest probability density (HPD) set for $x \in \mathcal{X}$ with respect to the posterior distribution $Q(x|y)$ is the subset C_a of Θ such that

1. $\mathsf{P}_Q(x \in C_a|y) \geq 1 - a$, and
2. $q(x|y) \geq q(x'|y)$ for all $x \in C_a,\ x' \in C_a^{\complement}$.

Note 9. Credible sets are considered as 'set estimators', and hence, they can be produced as Bayes decision rules under a specified loss function. See Examples 10 and 19.

Proposition 10. *[Minimal size region property] Let random quantity x follows $Q(x|y)$, let $\mathcal{D} = \{C; \mathsf{P}_Q(x \in C|y) \geq 1 - a\}$ be the decision space containing all possible $(1-a)$ credible sets of x, and let the loss function be*

$$\ell(x, C) = \kappa\,\|C\| - 1(x \in C), \quad \forall C \in \mathcal{D},\ \forall x \in \mathcal{X},\ \forall \kappa > 0, \tag{1}$$

No need to memorize Eq. 1

where $\|\cdot\|$ denotes a size of an area. Then:

1. *The Bayes rule (estimator) $\hat{C}$ has the minimum size among credible sets in $\mathcal{D}$.*
2. *$\hat{C}$ is the Bayes rule if and only if it is the $100(1-a)\%$ highest probability density (HPD) set as defined in Definition 8.*

Solution. The proof is omitted as too technical. (1.) is straightforward; while (2.) is just tricky calculus.

Note. HPD credible sets are credible sets with the minimum size (by Example 10). Clearly, loss (1) considers a trade off between two components: $\|C\|$ measuring the size of the credible set (the smaller the better), and $1(x \in C)$ indicating coverage of the credible set.

Remark 11. HPD credible sets are not, in general, invariant to transformations. If one has computed the HPD set for $x \sim Q(x|y)$, the HPD set for $\varphi = g(x)$ does not necessarily result by converting HPD set for x. To compute the HPD set for φ, one has to compute the posterior distribution

$$\mathrm{d}Q(\varphi|y) = \underbrace{q(g^{-1}(\varphi)|y) \left| \frac{\mathrm{d}}{\mathrm{d}\varphi} g^{-1}(\varphi) \right|}_{=\pi(\varphi|y)} \mathrm{d}\varphi,$$

and then compute the HPD set by implementing Definition 8.

3.1 General discussions

Definition 8 can be re-written equivalently as in Corollary 12, which provides a easier manner to compute credible regions in practice.

Corollary 12. *The* $100(1-a)\%$ *highest probability density (HPD) set for* $x \in \mathcal{X}$ *with respect to the posterior distribution* $Q(x|y)$ *is the subset* C_a *of* Θ *of the form*

$$C_a = \{x \in \mathcal{X} : q(x|y) \geq k_a\} \tag{2}$$

where k_a *is the largest constant such that*

$$1 - a \leq \mathsf{P}_Q(x \in C_a|y)$$

Proof. It is straightforward to show equivalence of (2) and Definition 8(2). ☐

Algorithm 13. *Based on Corollary 12, a (not-that-efficient) algorithm to compute HPD credible sets with a computer*[1]

- *Create a routine which computes all the solutions* $\{x^*\}$ *to the equation*

$$q(x^*|y) = k_a \tag{3}$$

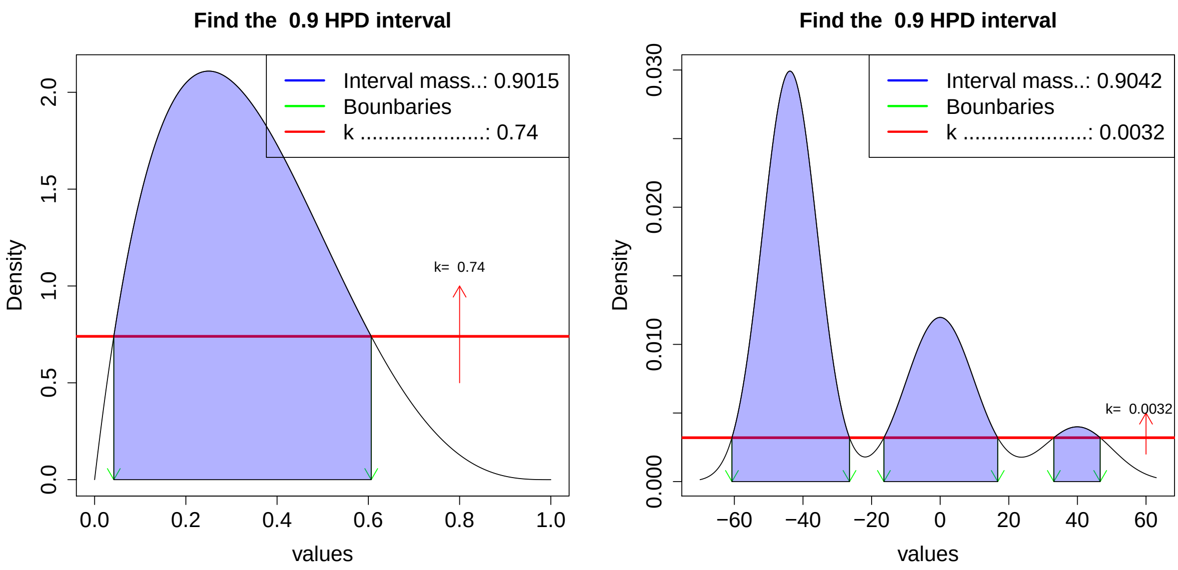
  for a given k_a. *Typically, these solutions* $\{x^*\}$ *are the boundaries of the set* $C_a = \{x \in \mathcal{X} : q(x|y) \geq k_a\}$.

- *Create a routine which computes the probability*

$$\mathsf{P}_Q(x \in C_a|y) = \int 1(x \in C_a)\, dQ(x|y) \tag{4}$$

- *Sequentially solve Equation 3 and obtain all the solutions* $\{x^*\}$, *by incrementally increasing* $k_a = \{\epsilon, \epsilon + \tau, \epsilon + 2\tau, \epsilon + 3\tau....\}$ *(such as starting from a tiny value* $\epsilon > 0$ *close to zero and recursively adding a tiny increments* $\tau > 0$*). Stop just before the probability in Equation 4 drops below* $1 - a$.

[1] Web-applet `https://georgios-stats-1.shinyapps.io/demo_CredibleSets/`



(a) Unimodal case; $C_a = [L, U]$

(b) Multimodal ; $C_a = [L_1, U_1] \cup [L_2, U_2]$

Figure 1: Schematic of Theorem 15 (in Fig. 1(1a)) and Algorithm 13 (in Fig. 1(1a) & Fig. 1(1b))

Note 14. For the simple 1D case, $x \in \mathcal{X}$ with $\dim(\mathcal{X}) = 1$, the following theorem can be used to compute HPD credible sets.

Theorem 15. *Let $x \in \mathbb{R}$ be a continuous random variable following distribution $Q(x|y)$ with unimodal density $q(x|y)$. If the interval $C_a = [L, U]$ satisfies*

1. $\int_L^U q(x|y)dx = 1 - a$,
2. $q(U) = q(L) > 0$, *and*
3. $x_{mode} \in (L, U)$, *where* x_{mode} is the mode of $q(x|y)$,

then it is the HPD interval of x with respect to $Q(x|y)$.

Proof. Use of the mean values theorem to prove. See, Casella, G., & Berger, R. L. (2002; pp. 441-443). Statistical inference (Vol. 2). Pacific Grove, CA: Duxbury. □

Remark 16. Theorem 15 suggests a procedure to find the boundaries of C_a in 1D cases. As is Figure 1a, we can imagine a horizontal bar which moves from the maximum of the density to zero, and intersects the density at locations which are the potential boundaries of C_a. The limits of the credible set are where the density above the two points the intersection take place (shaded area) is equal to $1 - a$. This mechanism is also described in the algorithm in suggested in Algorithm 13 and hence can also be used in multimodal densities (Figure 1b) or multivariate ones.

4 Examples

Example 17. Consider a Bayesian model

$$\begin{cases} y_i|\mu & \overset{\text{iid}}{\sim} \mathrm{N}_d(\mu,\Sigma), \qquad i=1,...,n \\ \mu & \sim \mathrm{N}_d(\mu_0,\Sigma_0) \end{cases}$$

where uncertain $\mu \in \mathbb{R}^d$, $d \geq 1$, and known $\Sigma > 0$, μ_0, $\Sigma_0 > 0$. Find the C_a parametric HPD credible set for μ.

Hint-1: If $z = (z_1, ..., z_d)^\top$ such as $z_j \overset{\text{iid}}{\sim} \mathrm{N}(0,1)$ for $j = 1, ..., d$, and $\xi = z^\top z = \sum_{j=1}^d z_j^2$, then $\xi \sim \chi^2_d$

Hint-2: It is

$$-\frac{1}{2}\sum_{i=1}^n (x-\mu_i)^\top \Sigma_i^{-1}(x-\mu_i)) = -\frac{1}{2}(x-\hat{\mu})^\top \hat{\Sigma}^{-1}(x-\hat{\mu})) + C(\hat{\mu},\hat{\Sigma}) \quad ;$$

$$\hat{\Sigma} = (\sum_{i=1}^n \Sigma_i^{-1})^{-1}; \quad \hat{\mu} = \hat{\Sigma}(\sum_{i=1}^n \Sigma_i^{-1}\mu_i);$$

$$C(\hat{\mu},\hat{\Sigma}) = \underbrace{\frac{1}{2}(\sum_{i=1}^n \Sigma_i^{-1}\mu_i)^\top (\sum_{i=1}^n \Sigma_i^{-1})^{-1}(\sum_{i=1}^n \Sigma_i^{-1}\mu_i) - \frac{1}{2}\sum_{i=1}^n \mu_i^\top \Sigma_i^{-1}\mu_i}_{=\text{independent of } x}$$

Solution. I will use the Definition 8.

- First, I compute the posterior of μ. It is

$$\begin{aligned}\pi(\mu|y) &\propto f(y|\mu)\pi(\mu) = \prod_{i=1}^n \mathrm{N}_d(y_i|\mu,\Sigma)\mathrm{N}_d(\mu|\mu_0,\Sigma_0) \\ &\propto \exp\left(-\frac{1}{2}\sum_{i=1}^n (y_i-\mu)^\top \Sigma^{-1}(y_i-\mu) - \frac{1}{2}(\mu-\mu_0)^\top \Sigma_0^{-1}(\mu-\mu_0)\right) \\ &\propto \exp\left(-\frac{1}{2}(\mu-\hat{\mu}_n)^\top \hat{\Sigma}_n^{-1}(\mu-\hat{\mu}_n)\right)\end{aligned}$$

  where

$$\hat{\Sigma}_n = (n\Sigma^{-1} + \Sigma_0^{-1})^{-1}; \qquad \hat{\mu}_n = \hat{\Sigma}_n(n\Sigma^{-1}\bar{y} + \Sigma_0^{-1}\mu_0)$$

  I recognize that $\pi(\mu|y) = \mathrm{N}_d(\mu|\hat{\mu}_n,\hat{\Sigma}_n)$, and hence $\mu|y \sim \mathrm{N}_d(\hat{\mu}_n,\hat{\Sigma}_n)$

- Now let's implement Definition 8. So,

$$\begin{aligned}C_a &= \left\{\mu \in \mathbb{R}^d : \pi(\mu|y) \geq k_a\right\} \\ &= \left\{\mu \in \mathbb{R}^d : \mathrm{N}_q(\mu|\hat{\mu}_n,\hat{\Sigma}_n) \geq k_a\right\} \\ &= \left\{\mu \in \mathbb{R}^d : (\mu-\hat{\mu}_n)^\top \hat{\Sigma}_n^{-1}(\mu-\hat{\mu}_n) \leq \underbrace{-2\log\left((2\pi)^{\frac{d}{2}}\det\left(\hat{\Sigma}_n\right)k_a\right)}_{=\tilde{k}_a}\right\}\end{aligned} \tag{5}$$

and I want the smallest constant $\tilde{k}_a$ (aka the largest constant k_a) such that

$$\mathsf{P}_{\Pi}\left(\mu \in C_a|y\right) \geq 1-a \Longleftrightarrow$$

$$\mathsf{P}_{\Pi}\left(\underbrace{(\mu-\hat{\mu}_n)^\top \hat{\Sigma}_n^{-1}(\mu-\hat{\mu}_n)}_{=\xi} \leq \tilde{k}_a\right) \geq 1-a \tag{6}$$

- I need to find quantile $\tilde{k}_a$. This requires to find the distribution of ξ. I know that

$$\xi = (\mu-\hat{\mu}_n)^\top \hat{\Sigma}_n^{-1}(\mu-\hat{\mu}_n) \sim \chi^2_d \tag{7}$$

because $\xi = z^\top z = \sum_{j=1}^n z_j$ with $z = L^{-1}(\mu-\hat{\mu}_n) \sim \mathrm{N}_d(0, I_d)$ where L is the lower matrix of the Cholesky decomposition of $\hat{\Sigma}_n = L^\top L$.

Hence Eq. 6, (due to Eqs. 5, 7) becomes

$$\mathsf{P}_{\chi^2_d}((\mu-\hat{\mu}_n)^\top \hat{\Sigma}_n^{-1}(\mu-\hat{\mu}_n) \leq \tilde{k}_a) = 1-a \tag{8}$$

which means that, $\tilde{k}_a$ is the $1-a$ quantile of the χ^2_d distribution, aka $\tilde{k}_a = \chi^2_{d,1-a}$

- Hence, the C_a parametric HPD credible set for μ is

$$C_a = \{\mu \in \mathbb{R}^d : (\mu-\hat{\mu}_n)^\top \hat{\Sigma}_n^{-1}(\mu-\hat{\mu}_n) \leq \chi^2_{d,1-a}\}$$

Example 18. Consider an exchangeable sequence of observables $y := (y_1, ...y_n) \in \mathbb{R}^n$ from model

$$\begin{cases} y_i|\theta & \stackrel{\text{iid}}{\sim} \text{Br}(\theta), \qquad i = 1, ..., n \\ \theta & \sim \text{Be}(a, b) \end{cases}$$

where $a = b = 2$, $n = 30$, and $\sum_{i=1}^{30} y_i = 15$. Find the 2-sides C_a parametric HPD credible interval for θ. Consider $a = 0.05$.

Solution.

- The posterior distribution of θ is $\text{Be}(a + n\bar{y}, b + n - n\bar{y})$, because

$$\pi(\theta|y) \propto \prod_{i=1}^n \text{Br}(y_i|\theta)\text{Be}(\theta|a, b) \propto \prod_{i=1}^n \theta^{y_i}(1-\theta)^{y_i}\theta^{a-1}(1-\theta)^{b-1} \propto \theta^{n\bar{y}+a-1}(1-\theta)^{n-n\bar{y}+b-1}$$

After substituting the values of the fixed parameters, I get $\pi(\theta|y) = \text{Be}(\theta|a_n = 17, b_n = 17)$.

  - To find the 2-sides C_a parametric HPD credible interval for θ, I use Theorem 15.

$$1 - a = \int_L^U \text{Be}(\theta|17, 17)\text{d}\theta = \mathsf{P}_{\text{Be}(17,17)}(\theta < U) - \mathsf{P}_{\text{Be}(17,17)}(\theta < L)$$

I note that the posterior is symmetric around 0.5 because $a_n = b_n$. Then,

$$1 - a = \mathsf{P}_{\text{Be}(17,17)}(\theta < U) - \left(1 - \mathsf{P}_{\text{Be}(17,17)}(\theta < U)\right) = 2\mathsf{P}_{\text{Be}(17,17)}(\theta < U) - 1$$

so $\mathsf{P}_{\text{Be}(17,17)}(\theta < U) = 1 - a/2$, and hence $U = \theta^*_{1-\frac{\alpha}{2}}$. Also,

$$\frac{1}{2} - L = U - \frac{1}{2} \implies L = 1 - U \implies L = 1 - \theta^*_{1-\frac{\alpha}{2}}$$

Putting these together, for $a = 0.05$, the 95% posterior credible interval for θ is

$$[L, U] = [0.36, 0.64].$$

- Note that, if we follow the same procedure, the compute the 95% prior credible interval for θ is

$$[L, U] = [0.14, 0.85].$$

As expected, the posterior 95 credible interval is narrower than the corresponding posterior one. (Try to check it in R).

```
> install.packages('HDInterval')
> library('HDInterval')
> hdi(qbeta, 0.95, shape1=17, shape2=17)
lower upper
0.3354445 0.6645555
```

Example 19. Assume a 1-dimensional random quantity $x \sim Q(x|y)$, with unimodal density $q(x|y)$. Show that the $(1-a)$-credible interval $C_a = [L, U]$ for x as a Bayesian rule C_a under the loss function

$$\ell(x, C_a; L, U) = k(U - L) - 1(x \in [L, U]), \quad \text{with} \quad k \in (0, \max_{\forall x \in \mathbb{R}}(q(x|y)))$$

is given by $q(L) = q(U) = k$, and $\mathsf{P}_Q(x \in [L, U]|y) = 1 - a$.

Discuss known properties of the derived credible interval.

Solution. The decision space is $\mathcal{D} = \{C_a = [L, U] \,:\, \mathsf{P}_Q(x \in C_a|y) = 1 - a\}$. It is

$$\begin{aligned}
\mathrm{E}_Q\left(\ell(x, C_a; L)|y\right) &= \int \left(k(U - L) - 1(x \in [L, U])\right) \mathrm{d}Q(x|y) \\
&= \int k(U - L) q(x|y) \mathrm{d}x - \int_L^U q(x|y) \mathrm{d}x = k(U - L) - \int_{-\infty}^U q(x|y) \mathrm{d}x + \int_{-\infty}^L q(x|y) \mathrm{d}x
\end{aligned}$$

To find the critical values $\hat{L}$, and $\hat{U}$ for L and U, it is

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}L} \mathrm{E}_Q\left(\ell(x, C_a; L)|y\right)\Big|_{C_a = [\hat{L}, \hat{U}]} = \frac{\mathrm{d}}{\mathrm{d}L}\left(k(U - L) - \int_{-\infty}^U q(x|y) \mathrm{d}x + \int_{-\infty}^L q(x|y) \mathrm{d}x\right)\Big|_{C_a = [\hat{L}, \hat{U}]} \\
&= -k + q(\hat{L}|y) \implies q(\hat{L}|y) = k \\
0 &= \frac{\mathrm{d}}{\mathrm{d}U} \mathrm{E}_Q\left(\ell(x, C_a; U)|y\right)\Big|_{C_a = [\hat{L}, \hat{U}]} = \ldots = k - q(\hat{U}|y) \implies q(\hat{U}|y) = k
\end{aligned}$$

which are minimizers because

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}L^2} \mathrm{E}_Q\left(\ell(x, C_a; L)|y\right)\Big|_{C_a = [\hat{L}, \hat{U}]} = \frac{\mathrm{d}}{\mathrm{d}L} q(L|y)\Big|_{\hat{L}} > 0 \; ; \qquad \frac{\mathrm{d}^2}{\mathrm{d}L \mathrm{d}U} \mathrm{E}_Q\left(\ell(x, C_a; L)|y\right)\Big|_{C_a = [\hat{L}, \hat{U}]} = 0 \\
&\frac{\mathrm{d}^2}{\mathrm{d}U^2} \mathrm{E}_Q\left(\ell(x, C_a; U)|y\right)\Big|_{C_a = [\hat{L}, \hat{U}]} = -\frac{\mathrm{d}}{\mathrm{d}U} q(U|y)\Big|_{\hat{U}} > 0
\end{aligned}$$

So it is $C_a = [\hat{L}, \hat{U}]$ such that $q(\hat{L}|y) = q(\hat{U}|y) = k$, and $\mathsf{P}_Q(x \in [\hat{L}, \hat{U}]|y) = 1 - a$.

Based on Theorem 15, it is the HPD credible interval and in fact the shorter length credible interval.

Example 20. Assume an 1- dimensional random quantity $x \sim Q(x|y)$. In the Lecture Handout (Handout 11: Bayesian point estimation), discussed the follwoing Hint:

Hint: The Bayes estimate $\hat{\delta}$ of x under the linear loss function

$$\ell(x, \delta; \varpi) = (1 - \varpi)(\delta - x)1_{x \le \delta}(\delta) + \varpi(x - \delta)1_{x > \delta}(\delta),$$

where $\varpi \in [0, 1]$, is the ϖ-th quantile of distribution Q, let's denote it as x_{ϖ}.

1. Derive the $(1 - a)$-credible interval $C_a = [L, U]$ for x as a Bayesian rule C_a under the loss function

$$\ell(x, C_a; \varpi_L, \varpi_U) = \ell(x, L; \varpi_L) + \ell(x, U; \varpi_U) \tag{9}$$

   by computing L and U.

2. Your client is worried the same both for under-estimation and over-estimation; derive a suitable $(1 - a)$-credible interval $C_a = [L, U]$ based on (9) by computing L, and U.

3. Your client is worried only for over-estimation; derive a suitable $(1 - a)$-credible interval $C_a = [L, U]$ based on (9) by computing L and U.

Solution. It is given that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\delta} \mathrm{E}_Q \left(\ell(x, \delta; \varpi) | y \right) \Big|_{\delta = \hat{\delta}} = \frac{\mathrm{d}}{\mathrm{d}\delta} \int \ell(x, \delta; \varpi) \mathrm{d}Q(x|y) \Big|_{\delta = \hat{\delta}} \implies \hat{\delta} = x_{\varpi} \\ &= (1 - \varpi)\mathsf{P}_Q \left(\{x \le \hat{\delta}\} | y \right) - \varpi \mathsf{P}_Q \left(\{x \le \hat{\delta}\}^{\mathsf{C}} | y \right) \implies \hat{\delta} = x_{\varpi} \end{aligned}$$

1. The decision space is $\mathcal{D} = \{C_a = [L, U] \, : \, \mathsf{P}_Q(x \in C_a | y) = 1 - a\}$. Therefore, to find the Bayes rule (or Bayes estimate) of $C_a = [L, U]$ I need to minimize the expected posterior loss $\mathrm{E}_Q \left(\ell(x, C_a; \varpi_L, \varpi_U) | y \right)$ with respect to C_a or equivalently L, U, so

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}L} \mathrm{E}_Q \left(\ell(x, C_a; \varpi_L, \varpi_U) | y \right) \Big|_{C_a = [\hat{L}, \hat{U}]} = \mathrm{E}_Q \left(\ell(x, L; \varpi_L) | y \right) |_{L = \hat{L}} \implies \hat{L} = x_{\varpi_L} \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}U} \mathrm{E}_Q \left(\ell(x, C_a; \varpi_L, \varpi_U) | y \right) \Big|_{C_a = [\hat{L}, \hat{U}]} = \mathrm{E}_Q \left(\ell(x, U; \varpi_U) | y \right) |_{U = \hat{U}} \implies \hat{U} = x_{\varpi_U} \end{aligned}$$

   So $x \in [x_{\varpi_L}, x_{\varpi_U}]$ where $\varpi_U + \varpi_L = 1 - a$. It is the minimum because

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}U^2} \mathrm{E}_Q \left(\ell(x, C_a; \varpi_L, \varpi_U) | y \right) \Big|_{C_a = [\hat{L}, \hat{U}]} &= q(\hat{U}|y) > 0 \\ \frac{\mathrm{d}^2}{\mathrm{d}L^2} \mathrm{E}_Q \left(\ell(x, C_a; \varpi_L, \varpi_U) | y \right) \Big|_{C_a = [\hat{L}, \hat{U}]} &= q(\hat{L}|y) > 0 \\ \frac{\mathrm{d}}{\mathrm{d}U} \frac{\mathrm{d}}{\mathrm{d}L} \mathrm{E}_Q \left(\ell(x, C_a; \varpi_L, \varpi_U) | y \right) \Big|_{C_a = [\hat{L}, \hat{U}]} &= 0 \end{aligned}$$

   and hence the determinant of the Hessian in positive.

2. Then I can use the equi-tail interval: $x \in [x_{a/2}, x_{1-a/2}]$ with $\varpi_L = a/2$ and $\varpi_U = 1 - a/2$

3. Then I can use the lower-tail interval: $x \in (-\infty, x_{1-a}]$ with $\varpi_L = 0$ and $\varpi_U = 1 - a$.

Practice

Question 21. *To practice try to work on the Exercises 68, and 69 from the Exercise sheet.*