Bayesian Statistics III/IV (MATH3341/4031) Michaelmas term, 2019

Handout 16: Hierarchical Bayesian model

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Aim

To be able to specify and analyze a Hierarchical Bayesian, as well as to extend previously introduces concepts in the Hierarchical Bayes framework.

Basic reading list:

- Berger, J. O. (2013; Section 4.6). Statistical decision theory and Bayesian analysis. Springer.
- Robert, C. (2007, Sections 10.1-10.3). The Bayesian choice: from decision-theoretic foundations to computational implementation. Springer Science & Business Media.
- Robert, C. P., & Reber, A. (1998). Bayesian modelling of a pharmaceutical experiment with heterogeneous responses. Sankhyā: The Indian Journal of Statistics, Series B, 145-160.

R-scripts:

Bayesian Normal Mixture model (Appendix A): `http://htmlpreview.github.io/?https://github.com/georgios-stats/Bayesian_Statistics/blob/master/ComputerPracticals/Normal_Mixture_model/BayesianNormalMixtureModel.nb.html`

Bayesian Variable Selection in Linear model (Appendix B): `http://htmlpreview.github.io/?https://github.com/georgios-stats/Bayesian_Statistics/blob/master/ComputerPracticals/Bernoulli_regression_model_variable_selection/BernoulliRegressionModelVS_full.nb.html`

Random effect model (Appendix C): `https://github.com/georgios-stats/Bayesian_Statistics/blob/master/LectureHandouts/Rscripts/HierarchicalBayes/HierarchicalBayesPharmaceutical.R`

1 Hierarchical Bayesian Model

A Bayesian model can be hierarchical due to the sampling distribution modeling the observations or due to the decomposition of the prior information. A hierarchical Bayesian model involves several levels of conditional distributions.

Definition 1. A hierarchical Bayes model is a Bayesian statistical model with sampling distribution $y \sim f(y|\theta)$ and prior $\theta \sim \pi(\theta)$, where the prior distribution $\pi(\theta)$ is decomposed in conditional distributions. The Bayesian model is

$$\begin{cases} y|\theta & \sim f(y|\theta), \text{ sampling distribution} \\ \\ \theta & \sim \pi(\theta) \text{ marginal prior specified} \end{cases} \overset{\text{extend space}}{\Longrightarrow} \begin{array}{l} \begin{cases} y|\theta \sim f(y|\theta) \end{cases} \\ \begin{cases} \theta \sim \pi_1(\theta|\phi_1) & \text{1st level prior} \\ \phi_1|\phi_2 \sim \pi_2(\phi_1|\phi_2) & \text{2nd level hyper-prior} \\ \vdots & \\ \phi_j|\phi_{j+1} \sim \pi_{j+1}(\phi_j|\phi_{j+1}) & j\text{th level hyper-prior} \\ \vdots & \\ \phi_{m-1}|\phi_m \sim \pi_m(\phi_{m-1}|\phi_m) & m\text{th level hyper-prior} \end{cases} \end{array} \quad (1)$$

The joint distribution $p(y, \theta, \phi_1, ..., \phi_j, ...\phi_{m-1})$ has pdf

$$p(y, \theta, \phi_1, ..., \phi_j, ...\phi_{m-1}) = f(y|\theta)\pi_1(\theta|\phi_1)\pi_2(\phi_1|\phi_2)\pi_3(\phi_2|\phi_3)...\pi(\phi_{m-1}|\phi_m)$$

The marginal prior distribution $\pi(\theta)$ has pdf

$$\pi(\theta) = \int_{\Phi_1 \times \Phi_{m-1}} \pi_1(\theta|\phi_1)\pi_2(\phi_1|\phi_2)\mathrm{d}\phi_1\pi_3(\phi_2|\phi_3)\mathrm{d}\phi_2 ... \pi(\phi_{m-1}|\phi_m)\mathrm{d}\phi_{m-1}.$$

The parameters $\phi_j \in \Phi_j$ are called random hyper-parameters of level j for $1 \leq j \leq m-1$.

Remark 2. Hierarchical Bayesian model is simply a special type of Bayesian model, where

$$\begin{cases} y|\theta & \sim f(y|\theta) \\ \theta|\phi & \sim \pi(\theta|\phi) \\ \phi|\phi_m & \sim \pi(\phi|\phi_m) \end{cases} \tag{2}$$

for $\phi = (\phi_1, ..., \phi_{m-1})$, and ϕ_m fixed hyper-parameter.

Note 3. A hierarchical Bayesian model can be used as a mean to specify more diverse priors. This is achieved by setting ϕ to be a random hyper-parameter with $\phi|\phi_m \sim \pi_2(\phi|\phi_m)$ instead of setting ϕ to have a fixed value.

Note 4. A hierarchical Bayesian model can be used when the sampling distribution or the prior distributions justify a certain structure.

Example 5. Regarding the fully hierarchical model (1), the full conditionals distributions of each element of $\vartheta = (\theta, \phi_1, ..., \phi_{m-1}) \in \Theta \times \Phi$ are given as:

$$\pi(\vartheta_j|y, \vartheta_{-j}) = \pi(\vartheta_j|y, \vartheta_{j-1}, \vartheta_{j+1})$$

with the convention

$$\vartheta_j = \begin{cases} \theta & , j = 1 \\ \phi_{j-1} & , j = 2, ..., m \\ \phi_m & , j = m \end{cases}$$

and $\vartheta_{-j} = (\vartheta_1, ..., \vartheta_{j-1}, \vartheta_{j+1}, ...\vartheta_m)$.

Proof. Straightforward by using the Bayesian theorem. □

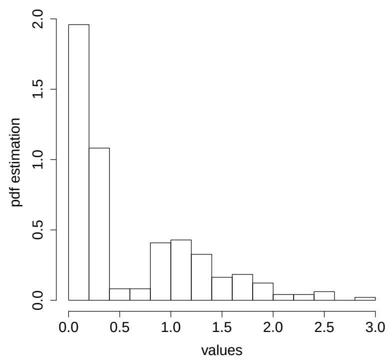
Example 6. Consider the following application where our concern is the distribution of enzymatic activity in the blood, for an enzyme involved in the metabolism of carcinogenic substances, among a group of $n = 245$ unrelated individuals; aka cluster analysis.

Our observables are $y = (y_1, ..., y_n)$ with $n = 245$. In the Boxplot on the right, we can clear see that the distribution is multimodal, suggenting the existence of subpopulations/groups.

Interest lies on identifying subgroups of slow or fast metabolizers as a marker of genetic polymorphism in the general population. As we are interested in learning/identifying the sub-populations as the identity/label of the group from which each observation is drawn is unknown.

Questions of interest:

- How many groups exist?
- To which group each observartion belongs?.



Histogram of Enzyme dataset which is available from: `https://people.maths.bris.ac.uk/~mapjg/mixdata`

As for the sampling model, we can assume that the i-th observation y_i is randomly drawn from the j-th group which has proportion ϖ_j in the population and which is distributed according to the sampling distribution $y_i|\theta_j \sim f_j(y_i|\theta_j)$.

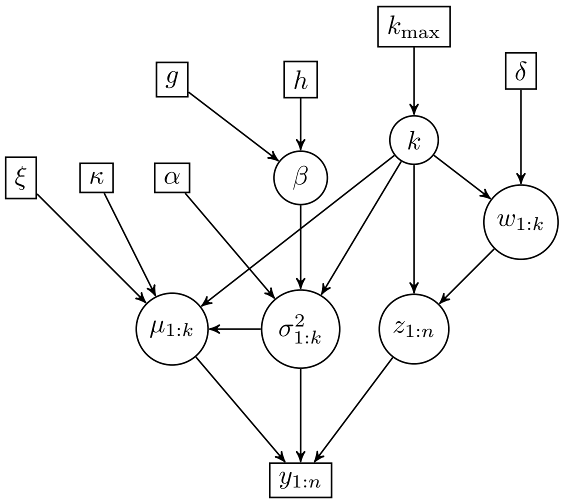
For simplicity, let's assume that all groups are Normally distributed but with different parameter values $\{\theta_j\}$; hence j-th group is $y_i|\mu_j, \sigma_j^2 \sim \mathrm{N}(y_i|\mu_j, \sigma_j^2)$ with $\theta_j = (\mu_j, \sigma_j^2)$.

It is natural to regard the group label z_i for the ith observation as a latent allocation variable: then z_i is supposed to be distributed as $z_i \sim f(z_i) = \varpi_{z_i}$ for $z_i \in \{1, ..., k\}$, and y_i is supposed to be distributed as $y_i|z_i, \theta_{z_i} \sim f_{z_i}(y_i|z_i, \theta_{z_i}) := \mathrm{N}(y_i|\mu_{z_i}, \sigma_{z_i}^2)$, for $i = 1, ..., n$; i.e.

$$\begin{cases} y_i|z_i, \mu_{z_i}, \sigma_{z_i}^2 & \sim f_j(y_i|\mu_{z_i}, \sigma_{z_i}^2) \\ z_i & \sim f(z_i) \end{cases} \Longrightarrow \begin{cases} y_i|z_i, \mu_{z_i}, \sigma_{z_i}^2 & \sim \mathrm{N}(y_i|\mu_{z_i}, \sigma_{z_i}^2) \\ z_i & \sim f(z_i) := \varpi_{z_i} \end{cases} \tag{3}$$

To complete the Bayesian model, we specify priors on the unknown quantities: Given there are k groups, $\varpi_{1:k} \sim \mathrm{Di}(\delta)$ for the group proportions, $\mu_j \sim \mathrm{N}(\xi_j, \sigma_j^2/\kappa)$ for the mean, and $\sigma_j^2 \sim \mathrm{Ga}(\alpha, \beta)$ for the variances. Assume we wish a more spread prior for σ_j^2 (for some reason), and hence we specify a hyper-prior on β as $\beta \sim \mathrm{Ga}(g, h)$. As the number of the groups is unknown, we assign prior $k \sim \pi(k) \in \mathrm{U_{discr}}(1, k_{\max})$.

$$\begin{cases} y_i|z_i, k, \varpi, \mu, \sigma^2 & \sim f_{z_i}(y_i|\varpi_{z_i}, \mu_{z_i}, \sigma_{z_i}^2) \\ & \qquad \text{for } i = 1, ..., n \\ z_i|k & \sim f(z_i) := \varpi_{z_i} \\ & \qquad \text{for } i = 1, ..., n \\ \varpi|k & \sim \mathrm{Di}(\delta) \\ \mu_j|\sigma_j^2, k & \sim \mathrm{N}(\xi, \sigma_j^2/\kappa) \\ & \qquad \text{for } j = 1, ..., k \\ \sigma_j^2|\beta, k & \sim \mathrm{Ga}(\alpha, \beta) \\ & \qquad \text{for } j = 1, ..., k \\ \beta & \sim \mathrm{Ga}(g, h) \\ k & \sim \mathrm{U_{discr}}(1, k_{\max}) \end{cases} \tag{4}$$



A Directed Acyclic Graph representation of the hierarchical model (4). Circles correspond to random quantities. Squares correspond to fixed quantities.

The joint distribution has pdf

Appendix A

$$p(y_{1:n}, z_{1:n}, k, \varpi_{1:k}, \mu_{1:k}, \sigma_{1:k}^2) = \overbrace{\prod_{i=1}^{n} \mathrm{N}(y_i|\mu_{z_i}, \sigma_{z_i}^2)}^{f(y_{1:n}|z_{1:n}, \mu_{1:k}, \sigma_{1:k}^2)} \overbrace{\prod_{i=1}^{n} \varpi_{z_i}}^{f(z_{1:n}|k)} \overbrace{\prod_{j=1}^{k} \mathrm{N}(\mu_k|\xi_k, \sigma_k^2/\kappa)}^{\pi(\mu_{1:k}|\sigma_{1:k}^2, k)} \overbrace{\prod_{j=1}^{k} \mathrm{Ga}(\sigma_k^2|\alpha, \beta)}^{\pi(\sigma_{1:k}^2|\beta, k)} \overbrace{\mathrm{Ga}(\beta|g, h)}^{\pi(\beta)} \overbrace{\frac{1}{|k_{\max}|}}^{\pi(k)}$$

The posterior $\pi(k, \varpi, \mu, \sigma^2, z|y)$ can be computed with the Bayesian theorem, and factorized as

$$\pi(k, \varpi, \mu, \sigma^2, \beta, z|y) \quad = \frac{p(y, z, k, \varpi, \mu, \sigma^2, \beta)}{\int p(y, z, k, \varpi, \mu, \sigma^2, \beta) \mathrm{d}(z, k, \varpi, \mu, \sigma^2, \beta)} \tag{5}$$

where to infer the number of groups from $\pi(k|y)$, the proportions in each group from $\pi(\varpi_{1:k}|y, k)$, the moments of each group from $\pi(\mu_{1:k}, \sigma_{1:k}^2|y, k)$, and the allocation of each observation to each group with $\pi(z|y, k, \varpi, \mu, \sigma^2)$.

As the required integrals are intractable, we can resolve to numerical methods, etc... Monte Carlo e.g. via JAGS....

Remark 7. The Bayesian model with sampling distribution $y \sim f(y|\theta)$ and prior $\theta \sim \pi(\theta)$, can be recovered from 2 by marginalizing the prior as

$$\pi(\theta|\phi_m) = \int_{\Phi} \pi(\theta|\phi)\pi(\phi|\phi_m)\mathrm{d}\phi = \int_{\Phi_1 \times \Phi_{m-1}} \pi(\theta|\phi_1)\pi(\phi_1|\phi_2)\mathrm{d}\phi_1 ... \pi(\phi_{m-1}|\phi_m)\mathrm{d}\phi_{m-1}, \tag{6}$$

where ϕ_m is just a fixed hyper-parameter. This reduction shows that hierarchical modelings are indeed included in the Bayesian paradigm.

Note 8. A particularly appealing aspect of hierarchical models is that they allow for conditioning on all levels, and this easy decomposition of the posterior. Consider the Bayesian hierarchical model (2) a parametric model $f(y|\theta)$ with a hierarchical prior $\theta \sim \pi_1(\theta|\phi)$, and $\phi \sim \pi(\phi)$. The posterior distribution of θ is

$$\pi(\theta|y) = \int_{\Phi} \pi(\theta|y,\phi)\pi(\phi|y)\mathrm{d}\phi \tag{7}$$

where

$$\pi(\theta|y,\phi) = \frac{f(y|\theta)\pi_1(\theta|\phi)}{f_1(y|\phi)}; \qquad \pi(\phi|y) = \frac{f_1(y|\phi)\pi_2(\phi)}{f(y)};$$

$$f_1(y|\phi) = \int_{\Theta} f(y|\theta)\pi_1(\theta|\phi)\mathrm{d}\theta; \qquad f(y) = \int_{\Theta} f_1(y|\phi)\pi_2(\phi)\mathrm{d}\phi$$

Remark 9. Note 8 has important consequences in terms of the computation of Bayes estimators, since it shows that $\pi(\theta|y)$ can be simulated by generating, first, ϕ from $\pi(\phi|y)$ and then θ from $\pi(\theta|y,\phi)$, if these two conditional distributions are easier to work with. (Snapshot from Term 2).

Note 10. Hierarchical decomposition (2) may facilitate the computation of intractable posterior moments. Let h be a function $h : \Theta \rightarrow \mathbb{R}$, then

$$\mathrm{E}_{\pi}(h(\theta)|y) = \mathrm{E}_{\pi}\left(\mathrm{E}_{\pi}\left(h(\theta)|y,\phi\right)\, |y\right).$$

If $\mathrm{E}_{\pi}(h(\theta)|y) = \int h(\theta)\pi(\theta|y)\mathrm{d}\theta$ is intractable and θ has high dimensionality, one could possibly try to specify the prior decomposition $\pi(\theta) = \int_{\Phi} \pi_1(\theta|\phi)\pi_2(\phi|\phi_m)\mathrm{d}\phi$ in (6) such that $\mathrm{E}_{\pi}\left(h(\theta)|y,\phi\right)$ can be computed analytically, and ϕ has low dimensionality. In that case one would have to compute the equivalent but lower dimensional (and hence easier) integral $\mathrm{E}_{\pi}\left(\mathrm{E}_{\pi}\left(h(\theta)|y,\phi\right)\, |y\right) = \int \mathrm{E}_{\pi}\left(h(\theta)|y,\phi\right)\pi(\phi|y)\mathrm{d}\phi$.

Example 11. Consider the 'Challenger O-ring' example from the Computer practicals. Let y_i denote the presence of a defective O-ring in the ith flight (0 for absence, and 1 for presence).

Appendix B

Assume that y_i can be modeled as observations generated independently from a Bernoulli distribution with parameter p_i. Here, p_i denotes the relative frequency of defective O-rings at flight i. We study if 'presence of a defective O-ring' (y) depends on the 'temperature' (t), or the 'pressure' (s).

Let t_i denote the temperature (in F) in the platform, and let s_i denote the Leak check pressure (in PSI) before the ith flight. Here are some possible models of interest:

$$\mathscr{M}^{I}: \; p(t;\beta_{\mathscr{M}^{I}},\mathscr{M}^{I}) = \frac{\exp(\beta_0)}{1+\exp(\beta_0)} \quad ; \mathscr{M}^{IV}: \; p(t;\beta_{\mathscr{M}^{IV}},\mathscr{M}^{IV}) = \frac{\exp(\beta_0+\beta_1 t+\beta_2 s)}{1+\exp(\beta_0+\beta_1 t+\beta_2 s)}$$

$$\mathscr{M}^{II}: \; p(t;\beta_{\mathscr{M}^{II}},\mathscr{M}^{II}) = \frac{\exp(\beta_0+\beta_1 t)}{1+\exp(\beta_0+\beta_1 t)} \quad ; \mathscr{M}^{V}: \; p(t;\beta_{\mathscr{M}^{V}},\mathscr{M}^{V}) = \frac{\exp(\beta_0+\beta_1 t+\beta_2 s+\beta_3 ts)}{1+\exp(\beta_0+\beta_1 t+\beta_2 s+\beta_3 ts)}$$

$$\mathscr{M}^{III}: \; p(t;\beta_{\mathscr{M}^{III}},\mathscr{M}^{III}) = \frac{\exp(\beta_0+\beta_2 s)}{1+\exp(\beta_0+\beta_2 s)} \quad \text{etc...}$$

The Bayesian hierarchical model under consideration is:

$$\begin{cases} y_i|\theta \sim f(y_i|\theta) \ :: & \left\{ y_i|\mathscr{M}, \beta_{\mathscr{M}} \sim \text{Br}\left(y_i| \frac{\exp(x_i^\top \beta_{\mathscr{M}})}{1+\exp(x_i^\top \beta_{\mathscr{M}})}\right), \quad \text{for, } i=1,...,n \right. \\ \theta|\phi_1 \sim \pi(\theta|\phi_1) \ :: & \begin{cases} \beta_j|\mathscr{M} \sim (1-\gamma_j)1_0(\beta_j) + \gamma_j \text{N}(\beta_j|\mu_0, \sigma_0^2) \ \ j=1,...,d \\ \begin{cases} \mathscr{M} & = (\gamma_1, ..., \gamma_d) \\ \gamma_j|\varpi & \sim \text{Br}(\varpi), \ \ j=1,...,d \end{cases} \end{cases} \\ \phi_1|\phi_2 \sim \pi(\phi_1|\phi_2) :: & \left\{ \varpi \quad \sim \text{Be}(a_0, b_0) \right. \end{cases}$$

where $\theta = (\mathscr{M}, \beta_{\mathscr{M}})$, $\phi_1 = \varpi$, and $\phi_2 = (a_0, b_0)$. Above, in the prior we considered an extra level of uncertainty by considering $\varpi \sim \text{Be}(a_0, b_0)$.

Here we added an additional level of uncertainty, and set $\varpi \sim \text{Be}(a_0, b_0)$ which creates a more diverse prior model, compared to the computer practical handout example where we had set $\varpi = 0.5$.

Now the joint probability distribution has pdf

$$p(y, \beta_{\mathscr{M}}, \mathscr{M}, \varpi) = \underbrace{\prod_{i=1}^n \text{Br}\left(y_i| \frac{\exp(x_i^\top \beta_{\mathscr{M}})}{1+\exp(x_i^\top \beta_{\mathscr{M}})}\right)}_{f(y|\theta)} \underbrace{\prod_{i=1}^n \left((1-\gamma_j)1_0(\beta_j) + \gamma_j \text{N}(\beta_j|\mu_0, \sigma_0^2)\right) \prod_{i=1}^n \text{Br}(\gamma_i|\varpi)}_{\pi(\theta|\phi_1)} \underbrace{\text{Be}(\varpi|a_0, b_0)}_{\pi(\phi_1|\phi_2)}$$

Example 12. Robert and Reber (1998) considers an experiment under which rats are intoxicated by a substance, then treated by either a placebo or a drug. (See: `https://www.jstor.org/stable/pdf/25053027.pdf`) Appendix C

Statistical model ($f(y|\theta)$)**:** The model associated with this experiment is a linear additive model effect: given x_{ij} , y_{ij} and z_{ij}, jth responses of the ith rat at the control, intoxication and treatment stages, respectively. The statistical model was specified such as that $(1 \le i \le I)$

$$\begin{aligned} x_{i,j} &\sim \text{N}(\theta_i, \sigma_c^2) && , 1 \le j \le J_i^c \\ y_{i,j} &\sim \text{N}(\theta_i + \delta_i, \sigma_a^2) && , 1 \le j \le J_i^a, \\ z_{i,j} &\sim \text{N}(\theta_i + \delta_i + \xi_i, \sigma_t^2) && , 1 \le j \le J_i^t, \end{aligned}$$

where θ_i is the average control measurement, δ_i the average intoxication effect and ξ_i the average treatment effect for the ith rat, the variances of these measurements being constant for the control, the intoxication and the treatment effects. An additional (observed) variable is w_i, which is equal to 1 if the rat is treated with the drug, and 0 otherwise.

Prior model $\pi(\theta|\phi)$**:** The different individual averages are related through a common (conjugate) prior distribution,

$$\begin{aligned} &\theta_i \sim \text{N}(\mu_\theta, \sigma_\theta^2), && \delta_i \sim \text{N}(\mu_\delta, \sigma_\delta^2), && \xi_i|w_i \sim \begin{cases} \text{N}(\mu_P, \sigma_P^2) & , w_i = 0 \\ \text{N}(\mu_D, \sigma_D^2) & , w_i = 1 \end{cases} \\ &\sigma_c \sim \pi(\sigma_c) \propto \frac{1}{\sigma_c}, && \sigma_a \sim \pi(\sigma_a) \propto \frac{1}{\sigma_a}, && \sigma_t \sim \pi(\sigma_t) \propto \frac{1}{\sigma_t}, \end{aligned} \tag{8}$$

This modeling seems to describe the natural phenomenon realistically enough, in the sense the responses x_{ij} , y_{ij} and z_{ij}

Hyper-priors $\pi(\phi|\phi_m)$: For the higher levels of prior ($\pi(\phi|\phi_m)$ in Eq 2), they considered improper (Jeffrey's) hyper-priors.

$$(\mu_\theta, \sigma_\theta) \sim \pi(\mu_\theta, \sigma_\theta) \propto \frac{1}{\sigma_\theta}, \qquad (\mu_\delta, \sigma_\delta) \sim \pi(\mu_\delta, \sigma_\delta) \propto \frac{1}{\sigma_\delta}, \qquad (\mu_P, \sigma_P) \sim \pi(\mu_P, \sigma_P) \propto \frac{1}{\sigma_P}, \tag{9}$$

$$(\mu_D, \sigma_D) \sim \pi(\mu_D, \sigma_D) \propto \frac{1}{\sigma_D}. \tag{10}$$

The priors in lines (8), (9) and (10) are improper non-informative priors. One could have have specify proper priors, like Normal-Inverse Gamma which are conjugate, however in that case he/she should have to specify the values for the fixed hyper-parameters.

As improper priors are specified, one need to study under what conditions the above improper priors lead to a proper (well defined) posterior –we omit this step here...

I have an R script with a demo in `https://github.com/georgios-stats/Bayesian_Statistics/blob/master/LectureHandouts/Rscripts/HierarchicalBayes/HierarchicalBayesPharmaceutical.R`

Example 13. (Cont. Example 6) From another point of view, recall that the compound distribution $f(y_i|k, \varpi_{1:k}\theta_{1:k})$ of (3) is mixture model of distribution

$$y_i|k, \theta_{1:k} \sim f(y_i|k, \varpi_{1:k}, \theta_{1:k}) = \sum_{j=1}^{k} \varpi_j f_j(y_i|\theta_j) = \sum_{j=1}^{k} \varpi_j \mathrm{N}(y_i|\mu_j, \sigma_j^2), \text{ for } i = 1, ..., n \tag{11}$$

is a suitable sampling distribution for modeling heterogeneous populations. Then by marginalizing, we can get the equivalent model

$$\begin{cases} y_i|k, \varpi, \mu, \sigma^2 & \sim f(y_i|k, \varpi, \mu, \sigma^2) := \sum_{j=1}^{k} \varpi_j \mathrm{N}(y_i|\mu_j, \sigma_j^2), \text{ for } i = 1, ..., n \\ \varpi|k & \sim \mathrm{Di}(\delta) \\ \mu_j|\sigma_j^2, k & \sim \mathrm{N}(\mu_j|\xi, \sigma_j^2), \text{ for } j = 1, ..., k \\ \sigma_j^2|k & \sim \mathrm{Ga}(a, \beta), \text{ for } j = 1, ..., k \\ \beta & \sim \mathrm{Ga}(g, h) \\ k & \sim \mathrm{U}_{\text{discr}}(1, k_{\max}) \end{cases} \tag{12}$$

The joint distribution that admits pdf

$$p(y, k, \varpi, \mu, \sigma^2, \beta) = f(y|k, \varpi, \mu, \sigma^2)\pi(\varpi|k)\pi(\mu|\sigma^2, k)\pi(\sigma^2|k, \beta)\pi(\beta)\pi(k).$$

The posterior $\pi(k, \varpi, \mu, \sigma^2|y)$ can be computed with the Bayesian theorem, and factorized as

$$\pi(k, \varpi, \mu, \sigma^2, \beta|y) = \frac{p(y, \varpi, \mu, \sigma^2, \beta)}{\int p(y, k, \varpi, \mu, \sigma^2, \beta) \mathrm{d}(k, \varpi, \mu, \sigma^2, \beta)} \tag{13}$$

Models (4) and (12) in the sense that posterior (13) is the marginal of the posterior (5).

2 Non-identifiability issue

A parametric model for which an element of the parametrisation is redundant is said to be non-identified. Let Bayesian model $(f(y|\theta), \pi(\theta))$, where $\theta = (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$, and assume that the parametric model does not depend on θ_1; i.e. $f(y|\theta_1, \theta_2) = f(y|\theta_2)$. The fact that the likelihood does not depend on θ_1 suggests that y does not provide information about θ_1 directly.

Bayesian analysis of a non-identified model is always possible if a suitable prior $\Pi(\theta_1, \theta_2)$ on all the parameters is specified. For instance, if one specifies a priori that learning the value of θ_2 may change his belief about θ_1, via $\pi(\theta_1|\theta_2) \neq \pi(\theta_1)$.

Factorize the prior distribution as $\pi(\theta_1, \theta_2) = \pi(\theta_1|\theta_2)\pi(\theta_2)$. Then, we have the following PDF/PMF

$$\pi(\theta_1, \theta_2|y) \propto f(y|\theta_1, \theta_2)\pi(\theta_1, \theta_2) = f(y|\theta_2)\pi(\theta_1|\theta_2)\pi(\theta_2) \implies$$
$$\pi(\theta_1, \theta_2|y) = \pi(\theta_2|y)\pi(\theta_1|\theta_2) \implies$$
$$\pi(\theta_1|y, \theta_2) = \pi(\theta_1|\theta_2) \tag{14}$$
$$\pi(\theta_2|y) = \frac{f(y|\theta_2)\pi(\theta_2)}{\int_{\Theta_2} f(y|\theta_2)\pi(\theta_2)\mathrm{d}\theta_2} \quad .$$
$$\pi(\theta_1|y) = \int_{\Theta_2} \pi(\theta_1|\theta_2)\pi(\theta_2)\mathrm{d}\theta_2 \tag{15}$$

Here, θ_1 is said to be non-identifiable parameter from the data y, because y provides <u>no direct information</u> about θ_1. Inference about θ_1 based on marginal posterior $\pi(\theta_1|y)$ depends on y but the information provided about θ_1 comes indirectly through the marginal posterior of θ_2, see (15). Equivalently, (15) implies that y provides no information about θ_1 given θ_2.

If we <u>a priori</u> specify that learning the value of θ_2 does not change our belief about θ_1 $\pi(\theta_1|\theta_2) = \pi(\theta_1)$, then (15) becomes $\pi(\theta_1|y) = \pi(\theta_1)$ and hence data y provide no information about θ_1 at all.

Example 14. (Cont Example 13) It is not difficult to understand that the Bayesian model as defined in Example 6 is non-identifiable. For simplicity we focus on the Bayesian mixture of $k = 2$ components with Appendix A

$$y|\varpi, \mu, \sigma^2 \sim f(y|\varpi, \mu, \sigma^2) := \varpi_1 \mathrm{N}(y|\mu_1, \sigma_1^2) + \varpi_2 \mathrm{N}(y|\mu_2, \sigma_2^2)$$
$$\pi(\varpi, \mu, \sigma^2) = \underbrace{\mathrm{N}(\mu_1|\xi, \sigma_1^2)\mathrm{N}(\mu_2|\xi, \sigma_2^2)}_{\pi(\mu|\sigma^2)}\underbrace{\mathrm{Ga}(\sigma_1^2|\alpha, \beta)\mathrm{Ga}(\sigma_1^2|\alpha, \beta)}_{\pi(\sigma^2)}\mathrm{Di}(\varpi|\delta) \tag{16}$$

which leads to a posterior such as

$$\pi(\varpi, \mu, \sigma^2|y) \propto \left[\varpi_1 \mathrm{N}(y|\mu_1, \sigma_1^2) + \varpi_2 \mathrm{N}(y|\mu_2, \sigma_2^2)\right] \mathrm{N}(\mu_1|\xi, \sigma_1^2)\mathrm{N}(\mu_2|\xi, \sigma_2^2)\mathrm{Ga}(\sigma_1^2|\alpha, \beta)\mathrm{Ga}(\sigma_1^2|\alpha, \beta)\mathrm{Di}(\varpi|\delta)$$

Here the parametrization is non-identifiable because the symmetry in the sampling distribution

$$\varpi_1 \mathrm{N}(y|\mu_1, \sigma_1^2) + \varpi_2 \mathrm{N}(y|\mu_2, \sigma_2^2) = \varpi_2 \mathrm{N}(y|\mu_2, \sigma_2^2) + \varpi_1 \mathrm{N}(y|\mu_1, \sigma_1^2)$$

and the naive prior in (16) produce a posterior such that

$$\pi\left(\varpi = \begin{pmatrix}\varpi_1\\ \varpi_2\end{pmatrix}, \mu = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \sigma^2 = \begin{pmatrix}\sigma_1^2\\ \sigma_2^2\end{pmatrix}|y\right) = \pi\left(\varpi = \begin{pmatrix}\varpi_2\\ \varpi_1\end{pmatrix}, \mu = \begin{pmatrix}\mu_2\\ \mu_1\end{pmatrix}, \sigma^2 = \begin{pmatrix}\sigma_2^2\\ \sigma_1^2\end{pmatrix}|y\right)$$

This parametrization is not meaningful for parametric inference. In Bayesian stats this can be resolved for instance by changing the prior and imposing an identifiability constrain as

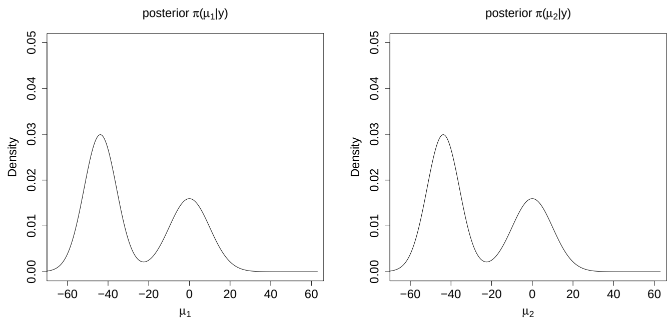
$$\pi^*(\mu|\sigma^2) = \frac{\mathrm{N}(\mu_1|\xi, \sigma_1^2)\mathrm{N}(\mu_2|\xi, \sigma_2^2)1\,(\mu_1 \leq \mu_2)}{\int \mathrm{N}(\mu_1|\xi, \sigma_1^2)\mathrm{N}(\mu_2|\xi, \sigma_2^2)1\,(\mu_1 \leq \mu_2)\,\mathrm{d}(\mu_1, \mu_2)} \propto \pi(\mu|\sigma^2)1\,(\mu_1 \leq \mu_2)$$

A schematic of the non-identifiability issue:

The non-identifiable model

$$\begin{cases} y_i|\varpi, \mu, \sigma^2 & \sim \varpi_1 \mathrm{N}(y|\mu_1, \sigma_1^2) + \varpi_2 \mathrm{N}(y|\mu_2, \sigma_2^2) \\ \varpi & \sim \mathrm{Di}(\delta) \\ \mu|\sigma^2 & \sim \mathrm{N}(\mu_1|\xi, \sigma_1^2)\mathrm{N}(\mu_2|\xi, \sigma_2^2) \\ \sigma^2 & \sim \mathrm{Ga}(\sigma_1^2|\alpha, \beta)\mathrm{Ga}(\sigma_1^2|\alpha, \beta) \end{cases} \tag{17}$$

produces marginal posteriors



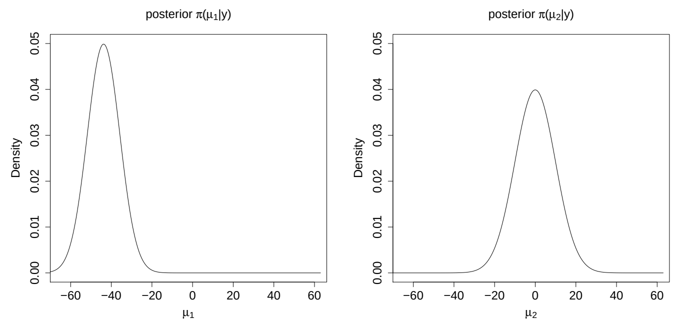
(a) Marginal posterior $\pi(\mu_1|y)$ (b) Marginal posterior $\pi(\mu_2|y)$

Figure 1: Some marginal posteriors of the non-identifiable model (17)

After non-identifiability is resolved, the identifiable model

$$\begin{cases} y_i|\varpi, \mu, \sigma^2 & \sim \varpi_1 \mathrm{N}(y|\mu_1, \sigma_1^2) + \varpi_2 \mathrm{N}(y|\mu_2, \sigma_2^2) \\ \varpi & \sim \mathrm{Di}(\delta) \\ \mu|\sigma^2 & \sim \mathrm{N}(\mu_1|\xi, \sigma_1^2)\mathrm{N}(\mu_2|\xi, \sigma_2^2)1\left(\mu_1 \leq \mu_2\right) \\ \sigma^2 & \sim \mathrm{Ga}(\sigma_1^2|\alpha, \beta)\mathrm{Ga}(\sigma_1^2|\alpha, \beta) \end{cases} \tag{18}$$

produces marginal posteriors



(a) Marginal posterior $\pi(\mu_1|y)$ (b) Marginal posterior $\pi(\mu_2|y)$

Figure 2: Some marginal posteriors of the non-identifiable model (18)

A Appendix: Bayesian Normal Mixture model

The complete R / RJAGS code is available from

- http://htmlpreview.github.io/?https://github.com/georgios-stats/Bayesian_Statistics/blob/master/ComputerPracticals/Normal_Mixture_model/BayesianNormalMixtureModel.nb.html
- https://raw.githubusercontent.com/georgios-stats/Bayesian_Statistics/master/ComputerPracticals/Normal_Mixture_model/enz.dat

Note 15. The RJAGS code for the analysis of the data set in the Example 6 is provided in Algorithm 1.

Algorithm 1 [Bayesian Variable Selection] RJAGS script for the analysis of the data set in the Example 6 .

```
rm(list=ls())
# Load rjags
library("rjags")
# define the Bayesian hierarchical model in JAGS syntax
jags_model <- "
  model{
    for (i in 1:n) {
    y[i] ~ dnorm(mu[zeta[i]], tau[zeta[i]])
    zeta[i] ~ dcat(varpi[])
  }
  for (i in 1:k) {
    mu[i] ~ dnorm(xi,kappa)
    tau[i] ~ dgamma(alpha,beta)
    sigma[i] <- 1/sqrt(tau[i])
  }
  beta ~ dgamma(g,h)
  varpi ~ ddirich(delta)
}
"
```

Example 16. (Cont. Example 6) As seen later, although equivalent in the sense that they produce the same inference, the expended hierarchical model (5) is computational convenient compared to hierarchical model (13) in the sense that its priors are conditional conjugate.

For simplicity assume that the number of groups is known and fixed to k. The full conditional posteriors in model (5) are:

$$w|y... \sim \text{Di}\left(\delta+n_1,...,\delta+n_k\right); \text{ where } n_j=\sum_{i=1}^{n} 1\left(z_i=j\right)$$

$$\mu_j|y... \sim \text{N}\left(\frac{\sum_{i:z_i=j} y_i - \xi\kappa}{n_j+\kappa}, \frac{\sigma_j^2}{n_j+\kappa}\right), \text{ for } j=1,...,k$$

$$\sigma_j^2|y... \sim \text{IG}\left(a+\frac{n_j}{2}, \beta+\frac{1}{2}\sum_{i:z_i=j}(y_i-\mu_j)^2\right), \text{ for } j=1,...,k$$

$$z_i|y... \sim \pi(z_i=j|y...)=\frac{\frac{w_j}{\sigma_j}\exp\left(-\frac{1}{2}\frac{(y_i-\mu_j)^2}{\sigma_j^2}\right)}{\sum_{j'=1}^{k}\frac{w_{j'}}{\sigma_{j'}}\exp\left(-\frac{1}{2}\frac{(y_i-\mu_{j'})^2}{\sigma_{j'}^2}\right)}; \text{ for } i=1,...,n$$

$$\beta|y... \sim \text{Ga}\left(g+k\alpha, h+\sum_{j=1}^{k}\sigma_j^2\right)$$

which can be used in Monte Carlo integration.

Model (13) does not produce full conditional posteriors of standard form, due to the summation in the likelihood.

B Appendix: Bayesian Variable Selection in Linear model

The complete R / RJAGS code is available from

- `http://htmlpreview.github.io/?https://github.com/georgios-stats/Bayesian_Statistics/blob/master/ComputerPracticals/Bernoulli_regression_model_variable_selection/BernoulliRegressionModelVS_full.nb.html`

Note 17. The RJAGS code for the analysis of the data set in the Example 11 is provided in Algorithm 2.

Algorithm 2 [Bayesian Mixed Effect Model] RJAGS script for the analysis of the data set in the Example 6 .

```
rm(list=ls())
# Load rjags
library("rjags")
# define the Bayesian hierarchical model in JAGS syntax
hierarhicalmodel<-"
model {
  for (i in 1:n) {
    mean[i] <- exp(inprod(X[i,],beta)) / (1+exp(inprod(X[i,],beta)))
    y[i] ~ dbern(mean[i])
  }
  betaT[1] ~ dnorm( 0 , 0.1 )
  beta[1] <- betaT[1]
  pp ~ dbeta(1.0,1.0)
  for (j in 1:(dmax-1)) {
    ind[j] ~ dbern( pp )
    betaT[j+1] ~ dnorm( 0 , 0.1 )
    beta[j+1] <- ind[j] * betaT[j+1]
  }
}
"
```

C Appendix: Random effect model

The complete R / RJAGS code is available from

- `https://github.com/georgios-stats/Bayesian_Statistics/blob/master/LectureHandouts/Rscripts/HierarchicalBayes/HierarchicalBayesPharmaceutical.R`

Note 18. The RJAGS code for the analysis of the data set in the Example 12 is provided in Algorithm 3.

Algorithm 3 [Bayesian Normal Mixture Model] RJAGS script for the analysis of the data set in the Example 12.

```
rm(list=ls())
# Load rjags
library("rjags")
# define the Bayesian hierarchical model in JAGS syntax
hierarhicalmodel <- "
  model {
     for ( i in 1 :  I ) {
      for ( j in 1 :  J ) {
        x[i,j] ~ dnorm( theta[i] , tau_c )
        y[i,j] ~ dnorm( theta[i] + delta[i] , tau_a )
        z[i,j] ~ dnorm( theta[i] + delta[i] + xi[i] , tau_t )
      }
      theta[i] ~ dnorm( mu_theta , tau_theta )
      delta[i] ~ dnorm( mu_delta , tau_delta )
      w_ind[i] <- ifelse(w[i] == 0, 0, 1)
      xi[i] <- w_ind[i]*x_d +(1-w_ind[i])*x_p
    }
    sig2_c <- 1/tau_c tau_c ~ dgamma(0.01 , 0.01)
    sig2_a = 1/tau_a tau_a ~ dgamma(0.01 , 0.01)
    sig2_t <- 1/tau_t tau_t ~ dgamma(0.01 , 0.01)
    mu_theta ~ dnorm( 0 , 0.001 )
    sig2_theta = 1/tau_theta
    tau_theta ~ dgamma(0.01 , 0.01)
    mu_delta ~ dnorm( 0 , 0.001 )
    sig2_delta <- 1/tau_delta
    tau_delta ~ dgamma(0.01 , 0.01)
    x_p ~ dnorm( mu_p , tau_p )
    x_d ~ dnorm( mu_d , tau_d )
    mu_p ~ dnorm( 0 , 0.001 )
    sig2_p <- 1/tau_p
    tau_p ~ dgamma(0.01 , 0.01)
    mu_d ~ dnorm( 0 , 0.001 )
    sig2_d <- 1/tau_d
    tau_d ~ dgamma(0.01 , 0.01)
  }
"
```

Example 19. (Cont...) You may use

$$-\frac{1}{2}\sum_{i=1}^{n}\frac{(x-\mu_i)^2}{\sigma_i^2}=-\frac{1}{2}\frac{(x-\hat{\mu})^2}{\hat{\sigma}^2}+C;\ \hat{\sigma}^2=(\sum_{i=1}^{n}\frac{1}{\sigma_i^2})^{-1};\ \hat{\mu}=\hat{\sigma}^2(\sum_{i=1}^{n}\frac{\mu_i}{\sigma_i^2});\quad C=\frac{1}{2}\frac{(\sum_{i=1}^{n}\frac{\mu_i}{\sigma_i^2})^2}{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}-\frac{1}{2}\sum_{i=1}^{n}\frac{\mu_i^2}{\sigma_i^2}$$

The joint posterior pdf of $\vartheta = (\theta_{1:I}, \delta_{1:I}, \xi_{1:I}, \sigma_c^2, \sigma_a^2, \sigma_t^2, \sigma_\theta^2, \sigma_\delta^2, \sigma_P^2, \sigma_D^2, \mu_\theta, \mu_\delta, \mu_P, \mu_D)$ given obs. x, y, z is

$$
\begin{aligned}
\pi(\vartheta|x,y,z) \propto \prod_{i=1}^{I} \Bigg[& \exp\left(-\frac{(\theta_i-\mu_\theta)^2}{2\sigma_\theta^2} - \frac{(\delta_i-\mu_\delta)^2}{2\sigma_\delta^2}\right) \prod_{j=1}^{J_i^c} \exp\left(-\frac{(x_{i,j}-\theta_i)^2}{2\sigma_c^2}\right) \times \prod_{j=1}^{J_i^a} \exp\left(-\frac{(y_{i,j}-\theta_i-\delta_i)^2}{2\sigma_a^2}\right) \\
& \times \prod_{j=1}^{J_i^t} \exp\left(-\frac{(z_{i,j}-\theta_i-\delta_i-\xi_i)^2}{2\sigma_t^2}\right) \times \prod_{w_i=0} \exp\left(-\frac{(\xi_i-\mu_P)^2}{2\sigma_P^2}\right) \prod_{w_i=0} \exp\left(-\frac{(\xi_i-\mu_D)^2}{2\sigma_D^2}\right) \Bigg] \\
& \times \sigma_c^{-\sum_i J_i^c - 1} \sigma_a^{-\sum_i J_i^a - 1} \sigma_t^{-\sum_i J_i^t - 1} \sigma_\theta^{I-1} \sigma_\delta^{I-1} \sigma_D^{I_D - 1} \sigma_P^{I_P - 1}.
\end{aligned}
$$

The joint posterior distributions is not of standard form, and its pdf is intractable. However the full conditionals are of standard form. For instance, the full conditional posterior distribution density

$$
\begin{aligned}
& \pi(\delta_{1:I}|x_{\text{all}}, y_{\text{all}}, z_{\text{all}}, \theta_{1:I}, \xi_{1:I}, \sigma_c^2, \sigma_a^2, \sigma_t^2, \sigma_\theta^2, \sigma_\delta^2, \sigma_P^2, \sigma_D^2, \mu_\theta, \mu_\delta, \mu_P, \mu_D) \\
& \quad \propto \prod_{i=1}^{I} \left[\exp\left(-\frac{(\delta_i-\mu_\delta)^2}{2\sigma_\delta^2}\right) \times \prod_{j=1}^{J_i^a} \exp\left(-\frac{(y_{i,j}-\theta_i-\delta_i)^2}{2\sigma_a^2}\right) \times \prod_{j=1}^{J_i^t} \exp\left(-\frac{(z_{i,j}-\theta_i-\delta_i-\xi_i)^2}{2\sigma_t^2}\right) \right] \\
& \quad \propto \prod_{i=1}^{I} \left[\exp\left(-\frac{(\delta_i-\mu_\delta)^2}{2\sigma_\delta^2} - \sum_{j=1}^{J_i^a} \frac{(\delta_i-(y_{i,j}-\theta_i))^2}{2\sigma_a^2} - \sum_{j=1}^{J_i^t} \frac{(\delta_i-(z_{i,j}-\theta_i-\xi_i)}{2\sigma_t^2}\right) \right] \\
& \quad \propto \prod_{i=1}^{I} \left[\exp\left(-\frac{(\delta_i-\mu_{\delta,i}^*)^2}{2\left(\sigma_{\delta,i}^*\right)^2} + \text{const...}\right) \right] \propto \prod_{i=1}^{I} \left[\exp\left(-\frac{(\delta_i-\mu_{\delta,i}^*)^2}{2\left(\sigma_{\delta,i}^*\right)^2} + \text{const...}\right) \right] \\
& \quad \propto \prod_{i=1}^{I} \text{N}\left(\delta_i|\mu_{\delta,i}^*, \left(\sigma_{\delta,i}^*\right)^2\right)
\end{aligned}
$$

with

$$\delta_i|\text{rest}, ... \overset{\text{ind}}{\sim} \text{N}\left(\mu_{\delta,i}^*, \left(\sigma_{\delta,i}^*\right)^2\right), \ \forall i = 1, ..., n$$

where

$$\left(\sigma_{\delta,i}^*\right)^2 = \left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_a^2} J_i^a + \frac{1}{\sigma_t^2} J_i^t\right)^{-1}; \quad \mu_{\delta,i}^* = \left(\sigma_{\delta,i}^*\right)^2 \left(\frac{\mu_\delta}{\sigma_\delta^2} + \frac{\sum_{j=1}^{J_i^a} y_{i,j} - J_i^a \theta_i}{\sigma_a^2} + \frac{\sum_{j=1}^{J_i^a} y_{i,j} - J_i^t \theta_i - J_i^t \xi_i}{\sigma_t^2}\right)$$

Notice that δ_i are a postriori independent given all the resp unknown parameters $\left(\theta_{1:I}, \xi_{1:I}, \sigma_c^2, \sigma_a^2, \sigma_t^2, \sigma_\theta^2, \sigma_\delta^2, \sigma_P^2, \sigma_D^2, \mu_\theta, \mu_\delta, \mu_P, \mu_D\right)$. Notice that the prior $\delta_i \sim \text{N}(\mu_\delta, \sigma_\delta^2)$ in Example 12 is conditional conjugate prior of δ_i.

Try to compute the rest

$$
\begin{aligned}
& \pi(\theta_{1:I}|\text{rest}, ...) \sim ?\,; && \pi(\sigma_t^2|\text{rest}, ...) \sim ?\,; && \pi(\sigma_c^2|\text{rest}, ...) \sim ?\,; && \pi(\sigma_a^2|\text{rest}, ...) \sim ?\,; \\
& \pi(\xi_{1:I}|\text{rest}, ...) \sim ?\,; && \pi(\sigma_\theta^2|\text{rest}, ...) \sim ?\,; && \pi(\sigma_\delta^2|\text{rest}, ...) \sim ?\,; && \pi(\sigma_P^2|\text{rest}, ...) \sim ?, \ \text{etc...}
\end{aligned}
$$

See the solutions in: Robert, C. P., & Reber, A. (1998) from the link (`https://www.jstor.org/stable/pdf/25053027.pdf`).