Bayesian Statistics III/IV (MATH3361/4071) — Michaelmas term 2021

Exercise Sheet: Bayesian Statistics

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Part I

Matrix & vector calculus

The exercises about Matrix & vector calculus are optional and can be skipped.

Exercise 1. ($\star$)Let A, B be $K \times K$ invertible matrices . Show that

$$(A+B)^{-1} = A^{-1}(A^{-1}+B^{-1})^{-1}B^{-1}$$

Exercise 2. ($\star\star$)[Woodbury matrix identity] Verify that

$$(A+UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1}+VA^{-1}U\right)^{-1}VA^{-1}$$

if A and C are non-singular.

Exercise 3. ($\star\star$)[Sherman–Morrison formula] Let A be a $K \times K$ invertible matrix and u and v two $K \times 1$ column vectors. Verify that

$$\left(A+uv^{\top}\right)^{-1} = A^{-1} - \frac{1}{1+v^{\top}A^{-1}u}A^{-1}uv^{\top}A^{-1}$$

if $1+v^{\top}A^{-1}u \neq 0$, and if A is non-singular.

Exercise 4. ($\star\star\star$)[Block partition matrix inversion] Let A be $K \times K$ invertible matrix, and let $B = A^{-1}$ its inverse. Consider Partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix};\ B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Namely, $B_{11} = \left[A^{-1}\right]_{11}$ is the upper corner of the A^{-1}, etc...
Show that

$$A_{11}^{-1} = B_{11} = B_{12}B_{22}^{-1}B_{21}$$
$$A_{11}^{-1}A_{12} = -B_{12}B_{22}^{-1}$$

Hint: Start by noticing that

$$AB = I \iff \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \iff \begin{cases} A_{11}B_{11} + A_{12}B_{21} & = I \\ A_{11}B_{12} + A_{12}B_{22} & = 0 \end{cases}$$

Part II

Random variables

Exercise 5. ($\star$)Let $y \in \mathcal{Y} \subseteq \mathbb{R}$ be a univariate random variable with CDF $F_y(\cdot)$. Consider a bijective function $h : \mathcal{Y} \to \mathcal{Z}$ with $z = h(y)$, and h^{-1} its inverse. The PDF of z is

$$F_z(z) = \begin{cases} F_Y(h^{-1}(z)) & \text{if } h \nearrow \\ \\ 1 - F_Y(h^{-1}(z)) & \text{if } h \searrow \end{cases}$$

Exercise 6. ($\star$)Let $y \in \mathcal{Y} \subseteq \mathbb{R}$ be a univariate random variable with PDF $f_y(\cdot)$. Consider a bijective function $h : \mathcal{Y} \to \mathcal{Z} \subseteq \mathbb{R}$ and let h^{-1} be the inverse function of h. Consider a univariate random variable such that $z = h(y)$. The PDF of z is

$$f_z(z) \ = f_y(y)|\det(\frac{\mathrm{d}y}{\mathrm{d}z})| \ = f_y(h^{-1}(z))|\det(\frac{\mathrm{d}}{\mathrm{d}z}h^{-1}(z))|$$

Exercise 7. ($\star$)Let $y \sim \mathrm{Ex}(\lambda)$ r.v. with Exponential distribution with rate parameter $\lambda > 0$, and $f_{\mathrm{Ex}(\lambda)}(y) = \lambda \exp(-\lambda y) 1(y \geq 0)$. Let $z = 1 - \exp(-\lambda y)$. Calculate the PDF of z, and recognize its distribution.

Exercise 8. ($\star$)Prove the following properties

1. Let matrix $A \in \mathbb{R}^{q \times d}$, $c \in \mathbb{R}^q$, and $z = c + Ay$ then

$$\mathrm{E}(z) = \mathrm{E}(c + Ay) = c + A\mathrm{E}(y)$$

2. Let random variables $z \in \mathcal{Z}$ and $y \in \mathcal{Y}$, and let functions ψ_1 and ψ_2 defined on $\mathcal{Z}$ and $\mathcal{Y}$, then

$$\mathrm{E}(\psi_1(z) + \psi_2(y)) = \mathrm{E}(\psi_1(z)) + \mathrm{E}(\psi_2(y))$$

3. If random variables $z \in \mathcal{Z}$ and $y \in \mathcal{Y}$ are independent then

$$\mathrm{E}(\psi_1(z)\psi_2(y)) = \mathrm{E}(\psi_1(z))\mathrm{E}(\psi_2(y))$$

   for any functions ψ_1 and ψ_2 defined on $\mathcal{Z}$ and $\mathcal{Y}$.

Exercise 9. ($\star$)Prove the following properties of the covariance matrix

1. $\mathrm{Cov}(z, y) = \mathrm{E}(zy^\top) - \mathrm{E}(z)\left(\mathrm{E}(y)\right)^\top$
2. $\mathrm{Cov}(z, y) = \left(\mathrm{Cov}(y, z)\right)^\top$
3. $\mathrm{Cov}_\pi(c_1 + A_1 z, c_2 + A_2 y) = A_1 \mathrm{Cov}_\pi(x, y) A_2^\top$, for fixed matrices A_1, A_2, and vectors c_1, c_2 with suitable dimensions.

4. If z and y are independent random vectors then $\text{Cov}(z, y) = 0$

Exercise 10. ($\star$)Prove that the (i, j)-th element of the covariance matrix between vector z and y is the covariance between their elements z_i and y_j:

$$[\text{Cov}(z, y)]_{i,j} = \text{Cov}(z_i, y_j)$$

Exercise 11. ($\star$)Prove the following properties of $\text{Var}(Y)$ for a random vector $y \in \mathcal{Y} \subseteq \mathbb{R}^d$

1. $\text{Var}(y) = \text{E}(yy^\top) - \text{E}(y)\,(\text{E}(y))^\top$
2. $\text{Var}(c + Ay) = A\text{Var}(y)A^\top$, for fixed matrix A, and vectors c with suitable dimensions.
3. $\text{Var}(y) \geq 0$; (semi-positive definite)

Exercise 12. ($\star$)Prove the following properties of characteristic functions

1. $\varphi_{A+Bx}(t) = e^{it^T A}\varphi_x(B^T t)$ if $A \in \mathbb{R}^d$ and $B \in \mathbb{R}^{k\times d}$ are constants
2. $\varphi_{x+y}(t) = \varphi_x(t)\varphi_y(t)$ if and only if x and y are independent
3. if $M_x(t) = \text{E}(e^{t^T x})$ is the moment generating function, then $M_x(t) = \varphi_x(-it)$

Exercise 13. ($\star$)Show that if $X \sim \text{Ex}(\lambda)$ then $\varphi_X(t) = \frac{\lambda}{\lambda - it}$.

Exercise 14. ($\star$)

1. Find $\varphi_X(t)$ if $X \sim \text{Br}(p)$.
2. Find $\varphi_Y(t)$ if $Y \sim \text{Bin}(n, p)$

Exercise 15. ($\star\star\star$)Prove the following statement related to the Bayesian theorem:
Assume a probability space $(\Omega, \mathscr{F}, P)$. Let a random variable $y : \Omega \to \mathcal{Y}$ with distribution $F(\cdot)$. Consider a partition $y = (x, \theta)$ with $x \in \mathcal{X}$ and $\theta \in \Theta$. Then the probability density function (PDF), or the probability mass function (PMF) of $\theta|x$ is

$$f(\theta|x) = \frac{f(x|\theta)f(\theta)}{\int f(x|\theta)\text{d}F(\theta)} \tag{1}$$

Hint Consider cases where x is discrete and continuous. In the later case use the mean value theorem :

$$\int_A f(x)g(x)\text{d}x = f(\xi)\int_A g(x)\text{d}x$$

where $\xi \in A$ if A is connected, and $g(x) \geq 0$ for $x \in A$.

Exercise 16. ($\star$)Prove that:

1. if $Z \sim \text{N}(0, I)$ then $\varphi_Z(t) = \exp(-\frac{1}{2}t^T t)$, where $Z \in \mathbb{R}^d$

2. if $X \sim \mathrm{N}(\mu, \Sigma)$ then $\varphi_X(t) = \exp(it^T\mu - \frac{1}{2}t^T\Sigma t)$, where $X \in \mathbb{R}^d$

Hint: Assume as known that if $Z \sim \mathrm{N}(0, 1)$ then $\varphi_Z(t) = \exp(-\frac{1}{2}t^2)$, where $Z \in \mathbb{R}$

Exercise 17. ($\star$)Show the following properties of the Characteristic Function

1. $\varphi_x(0) = 1$ and $|\varphi_x(t)| \leq 1$ for all $t \in \mathbb{R}^d$
2. $\varphi_{A+Bx}(t) = e^{it^T A}\varphi_x(B^T t)$ if $A \in \mathbb{R}^d$ and $B \in \mathbb{R}^{k\times d}$ are constants
3. x and y are independent then $\varphi_{x+y}(t) = \varphi_x(t)\varphi_y(t)$ (we do not proov the other way around)
4. if $M_x(t) = \mathrm{E}(e^{t^T x})$ is the moment generating function, then $M_x(t) = \varphi_x(-it)$

Part III

Probability calculus

Exercise 18. ($\star$)Let a random variable $x \sim \text{IG}(a, b)$, a fixed value $c > 0$, and $y = cx$ then $y \sim \text{IG}(a, cb)$.

Exercise 19. ($\star\star\star$)Consider that x given z is distributed according to $\text{Ga}(\frac{n}{2}, \frac{nz}{2})$, and that z is distributed according to $\text{Ga}(\frac{m}{2}, \frac{m}{2})$; i.e.

$$\begin{cases} x|z & \sim \text{Ga}(\frac{n}{2}, \frac{nz}{2}) \\ z & \sim \text{Ga}(\frac{m}{2}, \frac{m}{2}) \end{cases}$$

Here, $\text{Ga}(\alpha, \beta)$ is the Gamma distribution with shape and rate parameters α and β, and PDF

$$f_{\text{Ga}(\alpha,\beta)}(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} 1(x > 0)$$

1. Show that the compound distribution of x is F $x \sim \text{F}(n, m)$, where $\text{F}(n, m)$ is F distribution with numerator and denumerator degrees of freedom n and m, and PDF

$$f_{\text{F}(n,m)}(x) = \frac{1}{x\,\text{B}\left(\frac{n}{2}, \frac{m}{2}\right)} \sqrt{\frac{(n\,x)^n\ m^m}{(n\,x + m)^{n+m}}} 1(x > 0)$$

2. Show that

$$\text{E}_{\text{F}(n,m)}(x) = \frac{m}{m-2}$$

3. Show that

$$\text{Var}_{\text{F}(n,m)}(x) = \frac{2m^2(n + m - 2)}{n(m-2)^2(m-4)}$$

Hint: If $\xi \sim \text{IG}(a, b)$ then $\text{E}_{\xi\sim\text{IG}(a,b)}(\xi) = \frac{b}{a-1}$, and $\text{Var}_{\xi\sim\text{IG}(a,b)}(\xi) = \frac{b^2}{(a-1)^2(a-2)}$

Exercise 20. ($\star\star$)Prove the following statement:
Let $x \sim \text{N}_d(\mu, \Sigma)$, $x \in \mathbb{R}^d$, and $y = (x - \mu)^{\top}\Sigma^{-1}(x - \mu)$. Then

$$y \sim \chi^2_d$$

Exercise 21. ($\star\star$)Let

$$\begin{cases} x|\xi & \sim \text{N}_d(\mu, \Sigma\xi) \\ \xi & \sim \text{IG}(a, b) \end{cases}$$

with PDF

$$f_{\text{N}_d(\mu,\Sigma\xi)}(x|\xi) = (2\pi)^{-\frac{d}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x - \mu)^{\top}\Sigma^{-1}(x - \mu)\right)$$

$$f_{\text{IG}(a,b)}(\xi) = \frac{b^a}{\Gamma(a)} \xi^{-a-1} \exp\left(-\frac{b}{\xi}\right) 1_{(0,\infty)}(\xi)$$

Show that the marginal PDF of x is

$$f(x) = \int f_{\mathrm{N}_d(\mu,\Sigma\xi)}(x|\xi) f_{\mathrm{IG}(a,b)}(\xi)\mathrm{d}\xi$$

$$= \frac{2a^{-\frac{d}{2}}}{\pi^{\frac{n}{2}}\sqrt{\det(\frac{b}{a}\Sigma)}} \frac{\Gamma\left(a+\frac{d}{2}\right)}{\Gamma(a)} \left[1 + \frac{1}{2a}(x-\mu)^\top \left(\frac{b}{a}\Sigma\right)^{-1}(x-\mu)\right]^{-\frac{(2a+d)}{2}} \quad (2)$$

FYI: For $a = b = \frac{v}{2}$, the marginal PDF is the PDF of the d-dimensional Student T distribution.

Exercise 22. $(\star\star\star)$

Let $x \sim \mathrm{T}_d(\mu,\Sigma,\nu)$. Recall that $x \sim \mathrm{T}_d(\mu,\Sigma,\nu)$ is the marginal distribution $f_x(x) = \int f_{x|\xi}(x|\xi)f_\xi(\xi)\mathrm{d}\xi$ of (x,ξ) where

$$x|\xi \sim \mathrm{N}_d(\mu, \Sigma\xi v)$$
$$\xi \sim \mathrm{IG}(\frac{v}{2},\frac{1}{2})$$

Consider partition such that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}; \qquad \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix},$$

where $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$.

Address the following:

1. Show that the marginal distribution of x_1 is such that

$$x_1 \sim \mathrm{T}_{d_1}(\mu_1,\Sigma_1,\nu)$$

   Hint: Try to use the form $f_x(x) = \int f_{x|\xi}(x|\xi)f_\xi(\xi)\mathrm{d}\xi$.

2. Show that

$$\xi|x_1 \sim \mathrm{IG}(\frac{1}{2}(d_1+v), \frac{1}{2}\frac{Q+v}{v})$$

   where $Q = (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)$.

   Hint: The PDF of $y \sim \mathrm{N}_d(\mu,\Sigma)$ is

$$f(y) = (2\pi)^{-\frac{d}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(y-\mu)^\top\Sigma^{-1}(y-\mu)\right)$$

   Hint: The PDF of $y \sim \mathrm{IG}(a,b)$ is

$$f_{\mathrm{IG}(a,b)}(y) = \frac{b^a}{\Gamma(a)} y^{-a-1}\exp(-\frac{b}{y})1_{(0,+\infty)}(y)$$

3. Let $\xi' = \xi\frac{v}{Q+v}$, with $Q = (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)$, show that

$$\xi'|x_1 \sim \mathrm{IG}(\frac{v+d_1}{2}, \frac{1}{2})$$

4. Show that the conditional distribution of $x_2|x_1$ is such that

$$x_2|x_1 \sim \mathrm{T}_{d_2}(\mu_{2|1}, \dot{\Sigma}_{2|1}, \nu_{2|1})$$

where

$$\begin{aligned}
\mu_{2|1} =& \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1) \\
\dot{\Sigma}_{2|1} =& \frac{\nu + (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)}{\nu + d_1}\Sigma_{2|1} \\
\Sigma_{2|1} =& \Sigma_{22} - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top \\
\nu_{2|1} =& \nu + d_1
\end{aligned}$$

Hint: You can use the Example [Marginalization & conditioning] from the Lecture Handout

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Exercise 23. ($\star\star\star$)Show that

1. If $x_i \sim \mathrm{N}_d(\mu_i, \Sigma_i)$ for $i = 1, .., n$ and $y = c + \sum_{i=1}^n B_i x_i$, then

$$y \sim \mathrm{N}_d(c + \sum_{i=1}^n \mu_i, \sum_{i=1}^n B_i \Sigma_i B_i^\top)$$

2. If $x_i \sim \mathrm{T}_d(\mu_i, \Sigma_i, v)$ for $i = 1, .., n$ and $z = c + \sum_{i=1}^n B_i x_i$, then

$$z \sim \mathrm{T}_d(c + \sum_{i=1}^n \mu_i, \sum_{i=1}^n B_i \Sigma_i B_i^\top, v)$$
