Bayesian Statistics III/IV (MATH3341/4031) Michaelmas term, 2021

Handout 7: Mixture priors

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Aim: Explain the mixture distribution. Explain, theorize, and construct conjugate mixture prior distribution.

References:

- Berger, J. O. (2013; Sections 4.2.2). Statistical decision theory and Bayesian analysis. Springer Science & Business Media.
- Robert, C. (2007; Sections 3, pp. 105-123, & pp. 127-141). The Bayesian choice: from decision-theoretic foundations to computational implementation. Springer Science & Business Media.

Web applets: `https://georgios-stats-1.shinyapps.io/demo_mixturepriors/`

1 Finite mixture distributions

Definition 1. Let $\mathcal{P}_m = \{\Pi_l(\theta|\chi_l);\, l = 1, ..., m\}$ be a collection of probability distributions where $\{\chi_l\}_{l=1}^m$ are parameters of the l-th component $\Pi_l(\theta|\chi_l)$. Let $\{\varpi_l\}_{l=1}^m$ be a set of weights where $\varpi_l > 0$ and $\sum_{l=1}^m \varpi_l = 1$. The mixture distribution derived from the aforementioned collections is

$$\Pi(\theta|\varpi, \chi) = \sum_{l=1}^m \varpi_l \Pi_l(\theta|\chi_l), \qquad \theta \in \Theta \tag{1}$$

where $\chi := (\chi_l,\, l = 1 : m)$ and $\varpi := (\varpi_l,\, l = 1, ..., m)$. $\Pi_l(\theta|\chi_l)$ is called l-th mixture component with mixture weight ϖ_l.

Example 2. Let $h(\cdot)$ be a function defined on Θ. The expectation of $h(\cdot)$ with respect to (1) is

$$\mathrm{E}_{\Pi}(h(\theta)|\varpi, \chi) = \int \sum_{l=1}^m \varpi_l h(\theta) \mathrm{d}\Pi_l(\theta|\chi_l) = \sum_{l=1}^m \varpi_l \int h(\theta)\mathrm{d}\Pi_l(\theta|\chi_l) = \sum_{l=1}^m \varpi_l \mathrm{E}_{\Pi_l}(h(\theta)|\chi_l)$$

where $\mathrm{E}_{\Pi_l}(h(\theta)|\chi_l) = \int h(\theta)\mathrm{d}\Pi_l(\theta|\chi_l)$.

Example 3. A mixture of probability distributions (1) is a probability distribution; i.e.

$$\int_\Theta \pi(\theta|\varpi, \chi)\mathrm{d}\theta = \sum_{l=1}^m \varpi_l \underbrace{\int_\Theta \mathrm{d}\pi_l(\theta|\chi_l)\mathrm{d}\theta}_{=1} = \sum_{l=1}^m \varpi_l = 1$$

Definition 4. The mixture is called finite mixture if $m < \infty$, and countably infinite mixture if $m \to \infty$. Here, we focus on finite mixtures.

Definition 5. A mixture model is called is called parametric mixture model if its components are members of the same parametric family of distributions eg. $\mathcal{P}_m = \{\Pi(\theta|\chi_l);\, l = 1, ..., m\}$, and hence

$$\Pi(\theta|\varpi, \chi) = \sum_{l=1}^{m} \varpi_l \Pi(\theta|\chi_l)$$

Example 6. An example of a parametric mixture model is the Normal mixture model

$$\pi(\theta|\varpi, \mu, \sigma^2) = \sum_{l=1}^{m} \varpi_l \mathrm{N}(\theta|\mu_l, \sigma_l^2),$$

where all components belong to the Normal distribution family.

Note 7. Mixture distributions are useful because (among others):

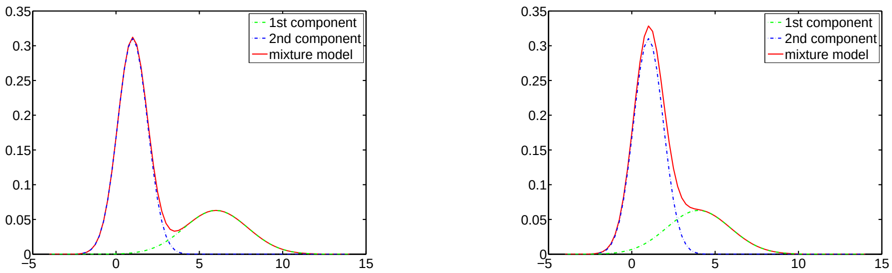
- they can approximate complicate other distributions by using a combination of simpler distributions $\{\Pi_l(\theta|\chi_l)\}$ which lead to more convenient computations.

- they can naturally model heterogeneity. E.g. consider a population which is heterogeneous in the sense that there are multiple sub-groups labeled by $\ell \in \{1, ..., m\}$, each group represented in the population with proportion ϖ_ℓ, and distributed as $\Pi_l(\theta|\chi_l)$. Then $y \sim \Pi(\theta|\chi)$ can be realized by drawing ℓ with probability $\mathsf{P}(\ell = l) = \varpi_l$ and drawing θ from $\Pi_\ell(\theta|\chi_\ell)$ given ℓ, aka

$$\begin{cases} \theta|\ell & \sim \Pi_\ell(\theta|\chi_\ell) \\ \ell & \sim \mathsf{P}(\ell) \end{cases} \quad \text{which implies } \theta \sim \Pi(\theta|\chi) \quad \text{since} \quad \Pi(\theta|\chi) = \sum_{\ell=1}^{m} \Pi(\theta|\chi_\ell)\mathsf{P}(\ell) = \sum_{l=1}^{m} \varpi_l \Pi(\theta|\chi_l)$$

  See Section 1 in Handout 2.

Example 8. A bimodal, or a right skewed (non-symmetric) distribution can be approximated by a Mixture of (unimodal and symmetric) Normal distributions with different parameter values. Also it describes a population with two groups Normally distributed with different parameters.



(a) Bimodal PDF:
$\pi(\theta) = 0.7\mathrm{N}(\theta|1, 0.9^2) + 0.3\mathrm{N}(\theta|6, 1.9^2)$

(b) Right skewed PDF:
$\pi(\theta) = 0.7\mathrm{N}(\theta|1, 0.9^2) + 0.3\mathrm{N}(\theta|4, 1.9^2)$

Figure 1: Normal mixture models. $\mathrm{N}(x|\mu, \sigma^2)$ denotes the Normal distribution density at value x with mean μ and variance σ^2.

2 Mixture prior distributions

Note 9. Mixture models can be used to specify priors distributions, either as a mean to approximate Your actual prior distribution with simpler & tractable distributions, or as a mean to represent heterogeneous prior believes.

Theorem 10. *Let $y := (y_1, ..., y_n)$ be observables generated from the sampling distribution $F(y|\theta)$.* Prior mixture distribution $\Pi(\theta|\varpi)$ is called the prior with pdf/pmf

$$\pi(\theta|\varpi) = \sum_{l=1}^{m} \varpi_l \pi_l(\theta), \tag{2}$$

where, $\mathcal{P}_m = \{\pi_l(\theta),\, \theta \in \Theta\}_{l=1}^{m}$ is a collection of distributions, and $\{\varpi_l\}$ are weights such that $\sum_{l=1}^{m} \varpi_l = 1$ and $\varpi_l > 0$. Then:

1. the posterior distribution $\Pi(\theta|y, \varpi)$ has pdf/pmf

$$\pi(\theta|y, \varpi) = \sum_{l=1}^{m} \varpi_l^* \pi_l(\theta|y)\,, \tag{3}$$

2. the predictive distribution of a future outcome z has pdf/pmf

$$g(z|y, \varpi) = \sum_{l=1}^{m} \varpi_l^* g_l(z|y)$$

where

$$\begin{aligned}
\varpi_l^* &= \frac{\varpi_l f_l(y)}{\sum_{l=1}^{m} \varpi_l f_l(y)} \;\propto\; \varpi_l f_l(y) \\
f_l(y) &= \int_\Theta f(y|\theta) \mathrm{d}\Pi_l(\theta) \\
\pi_l(\theta|y) &= \frac{f(y|\theta)\pi_l(\theta)}{\int_\Theta f(y|\vartheta)\mathrm{d}\Pi_l(\vartheta)} \;\propto\; f(y|\theta)\pi_l(\theta) \\
g_l(z|y) &= \int_\Theta f(z|\theta)\pi_l(\theta|y)\mathrm{d}\theta
\end{aligned}$$

Proof. From Bayes theorem, we have:

$$\begin{aligned}
\mathrm{d}\Pi(\theta|y) &= \frac{f(y|\theta)\mathrm{d}\Pi(\theta)}{\int_\Theta f(y|\vartheta)\mathrm{d}\Pi(\theta)} = \frac{f(y|\theta)\sum_{l=1}^{m}\varpi_l \mathrm{d}\Pi_l(\theta)}{\int_\Theta f(y|\vartheta)\sum_{l'=1}^{m}\varpi_{l'}\mathrm{d}\Pi_{l'}(\vartheta)} = \frac{\sum_{l=1}^{m}\varpi_l f(y|\theta)\mathrm{d}\Pi_l(\theta)}{\sum_{l'=1}^{m}\int_\Theta \varpi_{l'} f(y|\vartheta)\mathrm{d}\Pi_{l'}(\vartheta)} \\
&= \frac{\sum_{l=1}^{m}\varpi_l f_l(y)\overbrace{\frac{f(y|\theta)\mathrm{d}\Pi_l(\theta)}{f_l(y)}}^{=\mathrm{d}\Pi_l(\theta|y)}}{\sum_{l'=1}^{m}\int_\Theta \varpi_{l'} f_l'(y)\underbrace{\frac{f(y|\vartheta)\mathrm{d}\Pi_{l'}(\vartheta)}{f_{l'}(y)}}_{=\mathrm{d}\Pi_{l'}(\vartheta|y)}} = \frac{\sum_{l=1}^{m}\varpi_l f_l(y)\mathrm{d}\Pi_l(\theta|y)}{\sum_{l'=1}^{m}\varpi_{l'} f_{l'}(y)\underbrace{\int_\Theta \mathrm{d}\Pi_{l'}(\vartheta|y)}_{=1}} \\
&= \sum_{l=1}^{m}\underbrace{\frac{\varpi_l f_l(y)}{\sum_{l'=1}^{m}\varpi_{l'} f_{l'}(y)}}_{=\varpi_l^*}\mathrm{d}\Pi_l(\theta|y) = \sum_{l=1}^{m}\varpi_l^*\mathrm{d}\Pi_l(\theta|y).
\end{aligned}$$

Also

$$g(z|y, \varpi) = \int_\Theta f(z|\theta)\mathrm{d}\Pi_l(\theta|y) = \int_\Theta f(y|\theta)\sum_{l=1}^{m}\varpi_l^*\mathrm{d}\Pi_l(\theta|y) \quad = \sum_{l=1}^{m}\varpi_l^* g_l(z|y)$$

$\square$

Remark 11. Theorem 10 shows that the posterior distribution $\Pi(\theta|y)$ (derived by a mixture prior) is a mixture of 'individual posterior distributions' $\Pi_l(\theta|y)$ weighted by ϖ_l^*. It is determined not only by the observables but also by the weights of the individual distributions. Note that, for $l = 1, ..., m$, the prior weights ϖ_l and posterior weights ϖ_l^* my differ a lot, however they can be close each other.

Remark 12. Mixture priors $\Pi(\theta)$ whose components $\{\Pi_l(\theta)\}$ are conjugate to the likelihood $f(y|\theta)$ can facilitate tractable Bayesian inference. In Theorem 10, if each $\Pi_l(\theta)$ is conjugate to the likelihood $f(y|\theta)$, then obviously each $\Pi_l(\theta|y)$ will belong to the same distribution family as the corresponding $\Pi_l(\theta)$ because $\pi_l(\theta|y) \propto f(y|\theta)\pi_l(\theta)$. Then, provided that components $\pi_l(\theta)$ are tractable, the components $\pi_l(\theta|y)$ will be tractable too (as they belong to the same distr. family). Likewise, the posterior weights can be calculated in closed form i.e., $\varpi_l^* \propto \varpi_l f_l(y)$ since the integral $f_l(y) = \int_\Theta f(y|\theta)\mathrm{d}\Pi_l(\theta)$ will be tractable. Hence, the produced posterior mixture pdf/pmf will be tractable.

Remark 13. Mixtures of conjugate priors are as easy to manipulate as regular conjugate distributions, while leading to a greater freedom in the modeling of the prior information.

Example 14. Let $y = (y_1, ..., y_n)$ observable quantities, generated iid from a Bernoulli sampling distribution with unknown parameter θ; aka $y_i|\theta \overset{\text{iid}}{\sim} \mathrm{Br}(\theta),\ i = 1, ..., n$.

1. Find the likelihood function
2. Find the PDF of the conjugate prior mixture prior $\pi(\theta) = \sum_{l=1}^{m} \varpi_l \pi_l(\theta)$, with m components $\{\pi_l(\theta)\}_{l=1}^m$.
3. Compute the pdf of the posterior distribution, and recognize it.
4. Compute the predictive pdf for the next outcome $z = (y_{n+1}, ..., y_{n+m})$ given we have observed y? What do you observe?

Hint: Beta distribution: $x \sim \mathrm{Be}(a, b)$ has pdf

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}1(x \in [0,1]); \quad \text{where} \qquad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, a > 0, b > 0$$

Solution.

1. The likelihood function is

$$f(y|\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \prod_{i=1}^{n} \mathrm{Br}(x_i|\theta) = \theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n y_i}$$

2. In previous examples, we found that

   - the sampling distribution $F(\cdot|\theta)$ is a Bernoulli distribution which is a member of the Exponential family.
   - the conjugate prior is Beta distribution, aka $\theta \sim \mathrm{Be}(a, b)$ $a > 0$ and $b > 0$.

   Therefore, the components $\{\pi_l(\theta)\}_{l=1}^m$ in the prior mixture distribution will be from the family of Beta distributions $\mathcal{P}_m = \{\mathrm{Be}(\theta|a_l, b_l); l = 1, ..., m\}$.

   Therefore, the conjugate mixture prior has pdf

$$\pi(\theta) = \sum_{l=1}^{m} \varpi_l \mathrm{Be}(\theta|a_l, b_l)$$

   where $\mathrm{Be}(\theta|a_l, b_l)$ is the pdf of $\mathrm{Be}(a_l, b_l)$, and $\{(a_l, b_l)\}$ are fixed prior hyper-parameters.

3. The posterior mixture posterior has PDF

$$\pi(\theta|y) = \sum_{l=1}^{m} \varpi_l^* \pi_l(\theta|y)$$

According to Theorem 10, the l-th components of the mixture posterior is

$$\begin{aligned}
\pi_l(\theta|y) &\propto f(y|\theta)\pi_l(\theta) = \prod_{i=1}^{n} \text{Br}(y_i|\theta) \times \text{Be}(\theta|a_l, b_l) \ \propto \prod_{i=1}^{n} [\theta^{y_i}(1-\theta)^{1-y_i}] \times \theta^{a_l-1}(1-\theta)^{b_l-1} \\
&\overset{r_n=\sum_{i=1}^n y_i}{\propto} \theta^{r_n+a_l-1}(1-\theta)^{n-r_n+b_l-1} \ \propto \ \ \text{Be}(\theta|a_l^*, b_l^*)
\end{aligned}$$

with $a_l^* = r_n + a_l$, $b_l^* = n - r_n + b_l$, and $r_n = \sum_{i=1}^n y_i$.

According to Theorem 10, the posterior weights can be calculated as

$$\begin{aligned}
\varpi_l^* \ &\propto \varpi_l f_l(y) \ \propto \ \varpi_l \int_{(0,\infty)} \prod_{i=1}^{n} f(x_i|\theta)\pi_l(\theta)\text{d}\theta = \varpi_l \int \prod_{i=1}^{n} \text{Br}(y_i|\theta)\text{Be}(\theta|a_l, b_l)\text{d}\theta \\
&= \varpi_l \int_{(0,\infty)} \theta^{r_n}(1-\theta)^{n-r_n} \frac{\Gamma(a_l+b_l)}{\Gamma(a_l)\Gamma(b_l)} \theta^{a_l-1}(1-\theta)^{b_l-1}\text{d}\theta \\
&= \varpi_l \frac{\Gamma(a_l+b_l)}{\Gamma(a_l)\Gamma(b_l)} \int_{(0,1)} \theta^{r_n+a_l-1}(1-\theta)^{n-r_n+b_l-1}\text{d}\theta \ = \varpi_l \frac{\Gamma(a_l+b_l)}{\Gamma(a_l)\Gamma(b_l)} \frac{\Gamma(a_l^*)\Gamma(b_l^*)}{\Gamma(a_l^*+b_l^*)}
\end{aligned} \tag{4}$$

namely,

$$\varpi_l^* = \frac{\varpi_l \frac{\Gamma(a_l+b_l)}{\Gamma(a_l)\Gamma(b_l)} \frac{\Gamma(a_l^*)\Gamma(b_l^*)}{\Gamma(a_l^*+b_l^*)}}{\sum_{l=1}^{m} \varpi_l \frac{\Gamma(a_l+b_l)}{\Gamma(a_l)\Gamma(b_l)} \frac{\Gamma(a_l^*)\Gamma(b_l^*)}{\Gamma(a_l^*+b_l^*)}}$$

4. It is ...

$$g(z|y) = \int_{\Theta} f(z|\theta)\text{d}\Pi(\theta|y) = \sum_{i=1}^{m} \varpi_l^*(y) \underbrace{\int_{\Theta} f(z|\theta)\text{d}\Pi_l(\theta|y)}_{=g_l(z|y)}$$

where the following is just copy-paste from Handout 3...

$$\begin{aligned}
g_l(z|y) &= \int_{\Theta} f(z|\theta)\pi_l(\theta|y)\text{d}\theta \ = \int_{\Theta} \prod_{i=1}^{m} f(z_i|\theta)\pi_l(\theta|y)\text{d}\theta \ = \int_{(0,\infty)} \prod_{i=1}^{m} \text{Br}(z_i|\theta)\text{Be}(\theta|a_l^*, b_l^*)\text{d}\theta \\
&= \int_0^1 \left[\theta^{\sum_{i=1}^m z_i}(1-\theta)^{m-\sum_{i=1}^m z_i}\right] \left[\frac{\theta^{a_l^*-1}(1-\theta)^{b_l^*-1}}{B(a_l^*, b_l^*)}\right] \text{d}\theta \ 1(z \in \{0,1\}^m) \\
&= \frac{1}{B(a_l^*, b_l^*)} \int_0^1 \theta^{\sum_{i=1}^m z_i + a_l^* - 1}(1-\theta)^{m-\sum_{i=1}^m z_i + b_l^* - 1}\text{d}\theta \ 1(z \in \{0,1\}^m) \\
&= \frac{B(\sum_{i=1}^m z_i + a_l^*, m - \sum_{i=1}^m z_i + b_l^*)}{B(a_l^*, b_l^*)} 1(z \in \{0,1\}^m)
\end{aligned}$$

3 Practice

Question 15. *Try the Exercise 51 from the Exercise sheet.*