Exercise Sheet: Bayesian Statistics

Lecturer & author: Georgios P. Karagiannis georgios.karagiannis@durham.ac.uk

Part I

Matrix & vector calculus

The exercises about Matrix & vector calculus are optional and can be skipped.

Exercise 1. ($\star$)Let A, B be $K \times K$ invertible matrices . Show that

$$(A+B)^{-1} = A^{-1}(A^{-1}+B^{-1})^{-1}B^{-1}$$

Exercise 2. ($\star\star$)[Woodbury matrix identity] Verify that

$$(A+UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1}+VA^{-1}U\right)^{-1}VA^{-1}$$

if A and C are non-singular.

Exercise 3. ($\star\star$)[Sherman–Morrison formula] Let A be a $K \times K$ invertible matrix and u and v two $K \times 1$ column vectors. Verify that

$$\left(A+uv^{\top}\right)^{-1} = A^{-1} - \frac{1}{1+v^{\top}A^{-1}u}A^{-1}uv^{\top}A^{-1}$$

if $1+v^{\top}A^{-1}u \neq 0$, and if A is non-singular.

Exercise 4. ($\star\star\star$)[Block partition matrix inversion] Let A be $K \times K$ invertible matrix, and let $B = A^{-1}$ its inverse. Consider Partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix};\ B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Namely, $B_{11} = \left[A^{-1}\right]_{11}$ is the upper corner of the A^{-1}, etc...
Show that

$$A_{11}^{-1} = B_{11} = B_{12}B_{22}^{-1}B_{21}$$
$$A_{11}^{-1}A_{12} = -B_{12}B_{22}^{-1}$$

Hint: Start by noticing that

$$AB = I \Longleftrightarrow \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Longleftrightarrow \begin{cases} A_{11}B_{11} + A_{12}B_{21} & = I \\ A_{11}B_{12} + A_{12}B_{22} & = 0 \end{cases}$$

Part II

Random variables

Exercise 5. ($\star$)Let $y \in \mathcal{Y} \subseteq \mathbb{R}$ be a univariate random variable with CDF $F_y(\cdot)$. Consider a bijective function $h : \mathcal{Y} \to \mathcal{Z}$ with $z = h(y)$, and h^{-1} its inverse. The PDF of z is

$$F_z(z) = \begin{cases} F_Y(h^{-1}(z)) & \text{if } h \nearrow \\ \\ 1 - F_Y(h^{-1}(z)) & \text{if } h \searrow \end{cases}$$

Exercise 6. ($\star$)Let $y \in \mathcal{Y} \subseteq \mathbb{R}$ be a univariate random variable with PDF $f_y(\cdot)$. Consider a bijective function $h : \mathcal{Y} \to \mathcal{Z} \subseteq \mathbb{R}$ and let h^{-1} be the inverse function of h. Consider a univariate random variable such that $z = h(y)$. The PDF of z is

$$f_z(z) \ = f_y(y)|\det(\frac{\mathrm{d}y}{\mathrm{d}z})| \ = f_y(h^{-1}(z))|\det(\frac{\mathrm{d}}{\mathrm{d}z}h^{-1}(z))|$$

Exercise 7. ($\star$)Let $y \sim \text{Ex}(\lambda)$ r.v. with Exponential distribution with rate parameter $\lambda > 0$, and $f_{\text{Ex}(\lambda)}(y) = \lambda \exp(-\lambda y) 1(y \geq 0)$. Let $z = 1 - \exp(-\lambda y)$. Calculate the PDF of z, and recognize its distribution.

Exercise 8. ($\star$)Prove the following properties

1. Let matrix $A \in \mathbb{R}^{q \times d}$, $c \in \mathbb{R}^q$, and $z = c + Ay$ then

$$\text{E}(z) = \text{E}(c + Ay) = c + A\text{E}(y)$$

2. Let random variables $z \in \mathcal{Z}$ and $y \in \mathcal{Y}$, and let functions ψ_1 and ψ_2 defined on $\mathcal{Z}$ and $\mathcal{Y}$, then

$$\text{E}(\psi_1(z) + \psi_2(y)) = \text{E}(\psi_1(z)) + \text{E}(\psi_2(y))$$

3. If random variables $z \in \mathcal{Z}$ and $y \in \mathcal{Y}$ are independent then

$$\text{E}(\psi_1(z)\psi_2(y)) = \text{E}(\psi_1(z))\text{E}(\psi_2(y))$$

   for any functions ψ_1 and ψ_2 defined on $\mathcal{Z}$ and $\mathcal{Y}$.

Exercise 9. ($\star$)Prove the following properties of the covariance matrix

1. $\text{Cov}(z, y) = \text{E}(zy^\top) - \text{E}(z)\,(\text{E}(y))^\top$
2. $\text{Cov}(z, y) = (\text{Cov}(y, z))^\top$
3. $\text{Cov}_\pi(c_1 + A_1 z, c_2 + A_2 y) = A_1 \text{Cov}_\pi(x, y) A_2^\top$, for fixed matrices A_1, A_2, and vectors c_1, c_2 with suitable dimensions.

4. If z and y are independent random vectors then $\mathrm{Cov}(z, y) = 0$

Exercise 10. ($\star$)Prove that the (i, j)-th element of the covariance matrix between vector z and y is the covariance between their elements z_i and y_j:

$$[\mathrm{Cov}(z, y)]_{i,j} = \mathrm{Cov}(z_i, y_j)$$

Exercise 11. ($\star$)Prove the following properties of $\mathrm{Var}(Y)$ for a random vector $y \in \mathcal{Y} \subseteq \mathbb{R}^d$

1. $\mathrm{Var}(y) = \mathrm{E}(yy^\top) - \mathrm{E}(y)\,(\mathrm{E}(y))^\top$
2. $\mathrm{Var}(c + Ay) = A\mathrm{Var}(y)A^\top$, for fixed matrix A, and vectors c with suitable dimensions.
3. $\mathrm{Var}(y) \geq 0$; (semi-positive definite)

Exercise 12. ($\star$)Prove the following properties of characteristic functions

1. $\varphi_{A+Bx}(t) = e^{it^T A}\varphi_x(B^T t)$ if $A \in \mathbb{R}^d$ and $B \in \mathbb{R}^{k\times d}$ are constants
2. $\varphi_{x+y}(t) = \varphi_x(t)\varphi_y(t)$ if and only if x and y are independent
3. if $M_x(t) = \mathrm{E}(e^{t^T x})$ is the moment generating function, then $M_x(t) = \varphi_x(-it)$

Exercise 13. ($\star$)Show that if $X \sim \mathrm{Ex}(\lambda)$ then $\varphi_X(t) = \frac{\lambda}{\lambda - it}$.

Exercise 14. ($\star$)

1. Find $\varphi_X(t)$ if $X \sim \mathrm{Br}(p)$.
2. Find $\varphi_Y(t)$ if $Y \sim \mathrm{Bin}(n, p)$

Exercise 15. ($\star\star\star$)Prove the following statement related to the Bayesian theorem:
Assume a probability space $(\Omega, \mathscr{F}, P)$. Let a random variable $y : \Omega \to \mathcal{Y}$ with distribution $F(\cdot)$. Consider a partition $y = (x, \theta)$ with $x \in \mathcal{X}$ and $\theta \in \Theta$. Then the probability density function (PDF), or the probability mass function (PMF) of $\theta|x$ is

$$f(\theta|x) = \frac{f(x|\theta)f(\theta)}{\int f(x|\theta)\mathrm{d}F(\theta)} \tag{1}$$

Hint Consider cases where x is discrete and continuous. In the later case use the mean value theorem :

$$\int_A f(x)g(x)\mathrm{d}x = f(\xi)\int_A g(x)\mathrm{d}x$$

where $\xi \in A$ if A is connected, and $g(x) \geq 0$ for $x \in A$.

Exercise 16. ($\star$)Prove that:

1. if $Z \sim \mathrm{N}(0, I)$ then $\varphi_Z(t) = \exp(-\frac{1}{2}t^T t)$, where $Z \in \mathbb{R}^d$

2. if $X \sim \mathrm{N}(\mu, \Sigma)$ then $\varphi_X(t) = \exp(it^T\mu - \frac{1}{2}t^T\Sigma t)$, where $X \in \mathbb{R}^d$

Hint: Assume as known that if $Z \sim \mathrm{N}(0, 1)$ then $\varphi_Z(t) = \exp(-\frac{1}{2}t^2)$, where $Z \in \mathbb{R}$

Exercise 17. ($\star$)Show the following properties of the Characteristic Function

1. $\varphi_x(0) = 1$ and $|\varphi_x(t)| \leq 1$ for all $t \in \mathbb{R}^d$
2. $\varphi_{A+Bx}(t) = e^{it^TA}\varphi_x(B^Tt)$ if $A \in \mathbb{R}^d$ and $B \in \mathbb{R}^{k\times d}$ are constants
3. x and y are independent then $\varphi_{x+y}(t) = \varphi_x(t)\varphi_y(t)$ (we do not proov the other way around)
4. if $M_x(t) = \mathrm{E}(e^{t^Tx})$ is the moment generating function, then $M_x(t) = \varphi_x(-it)$
