Bayesian Statistics III/IV (MATH3361/4071)

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Homework 1: Manipulation of multivariate probability distributions, and the Posterior distribution

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For Formative assessment, submit the solutions of the parts 1 and 2 from the Exercise 1, and the solution of the Exercise 2.

Exercise 1. $(\star\star)$

Let $x \sim \mathrm{T}_d(\mu, \Sigma, \nu)$. Recall that $x \sim \mathrm{T}_d(\mu, \Sigma, \nu)$ is the marginal distribution $f_x(x) = \int f_{x|\xi}(x|\xi) f_{\xi}(\xi) \mathrm{d}\xi$ of (x, ξ) where

$$x|\xi \sim N_d(\mu, \Sigma \xi v)$$

 $\xi \sim IG(\frac{v}{2}, \frac{1}{2})$

Consider partition such that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}; \qquad \qquad \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix},$$

where $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$.

Address the following:

1. Show that the marginal distribution of x_1 is such that

$$x_1 \sim \mathsf{T}_{d_1}(\mu_1, \Sigma_1, \nu)$$

Hint: Try to use the form $f_x(x) = \int f_{x|\xi}(x|\xi) f_{\xi}(\xi) d\xi$.

2. Show that

$$\xi | x_1 \sim \text{IG}(\frac{1}{2}(d_1 + v), \frac{1}{2}\frac{Q + v}{v})$$

where $Q = (\mu_1 - x_1)^{\top} \Sigma_1^{-1} (\mu_1 - x_1)$.

Hint: The PDF of $y \sim N_d(\mu, \Sigma)$ is

$$f(y) = (2\pi)^{-\frac{d}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(y-\mu)^{\top}\Sigma^{-1}(y-\mu)\right)$$

Hint: The PDF of $y \sim IG(a, b)$ is

$$f_{\text{IG}(a,b)}(y) = \frac{b^a}{\Gamma(a)} y^{-a-1} \exp(-\frac{b}{y}) \mathbb{1}_{(0,+\infty)}(y)$$

3. Let $\xi' = \xi \frac{v}{Q+v}$, with $Q = (\mu_1 - x_1)^T \Sigma_1^{-1} (\mu_1 - x_1)$, show that

$$\xi'|x_1 \sim \text{IG}(\frac{v+d_1}{2}, \frac{1}{2})$$

4. Show that the conditional distribution of $x_2|x_1$ is such that

$$x_2|x_1 \sim \mathsf{T}_{d_2}(\mu_{2|1}, \dot{\Sigma}_{2|1}, \nu_{2|1})$$

where

$$\begin{split} \mu_{2|1} = & \mu_2 + \Sigma_{21} \Sigma_{11}^{-1} (x_1 - \mu_1) \\ \dot{\Sigma}_{2|1} = & \frac{\nu + (\mu_1 - x_1)^{\top} \Sigma_{1}^{-1} (\mu_1 - x_1)}{\nu + d_1} \Sigma_{2|1} \\ \Sigma_{2|1} = & \Sigma_{22} - \Sigma_{21} \Sigma_{1}^{-1} \Sigma_{21}^{\top} \\ \nu_{2|1} = & \nu + d_1 \end{split}$$

Hint: You can use the Example [Marginalization & conditioning] from the Lecture Handout

Exercise 2. $(\star\star)$ Let x be an observation. Consider the Bayesian model

$$\begin{cases} x | \theta & \sim \operatorname{Pn}(\theta) \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $Pn(\theta)$ is the Poisson distribution with expected value θ . Consider a prior $\Pi(\theta)$ with density such as $\pi(\theta) \propto \frac{1}{\theta}$. Show that the posterior distribution is not always defined.

Hint-1: It suffices to show that the posterior is not defined in the case that you collect only one observation x = 0.

Hint-2: Poisson distribution: $x \sim Pn(\theta)$ has PMF

$$Pn(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$