

Homework 3: Jeffreys' priors and Maximum entropy priors

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Exercise 1. (**) Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \text{Pn}(\theta), \forall i = 1, \dots, n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $\text{Pn}(\theta)$ is the Poisson distribution with expected value θ . Specify a Jeffreys' prior for θ .

Hint: Poisson distribution: $x \sim \text{Pn}(\theta)$ has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

Exercise 2. (**) Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \text{Pn}(\theta), \forall i = 1, \dots, n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $\text{Pn}(\theta)$ is the Poisson distribution with expected value θ . Specify a Maximum entropy prior under the constraint $E(\theta) = 2$ and reference measure such as $\pi_0(\theta) = \frac{1}{\sqrt{\theta}}$. In particular, you also have to state the name of the derived Maximum entropy prior distribution and report the values of its parameters.

Hint-1: Poisson distribution: $x \sim \text{Pn}(\theta)$ has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

Hint-2: Gamma distribution: $x \sim \text{Ga}(a, b)$ has PDF

$$\text{Ga}(x|a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-\beta x) 1(x > 0)$$

Exercise 3. (**) Let x be an observation. Consider the Bayesian model

$$\begin{cases} x | \theta & \sim \text{Pn}(\theta) \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $\text{Pn}(\theta)$ is the Poisson distribution with expected value θ . Consider a prior $\Pi(\theta)$ with density such as $\pi(\theta) \propto \frac{1}{\theta}$. Show that the posterior distribution is not always defined.

Hint-1: It suffices to show that the posterior is not defined in the case that you collect only one observation $x = 0$.**Hint-2:** Poisson distribution: $x \sim \text{Pn}(\theta)$ has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

The Limit Comparison Theorem for Improper Integrals

- Brand, L. (2006; Chapter 7). Advanced calculus: an introduction to classical analysis.

General: Let integrable functions $f(x)$, and $g(x)$ for $x \geq a$.

Let

$$0 \leq f(x) \leq g(x), \quad \text{for } x \geq a$$

Then

$$\begin{aligned} \int_a^\infty g(x) dx < \infty &\implies \int_a^\infty f(x) dx < \infty \\ \int_a^\infty f(x) dx = \infty &\implies \int_a^\infty g(x) dx = \infty \end{aligned}$$

Type I: Let integrable functions $f(x)$, and $g(x)$ for $x \geq a$, and let $g(x)$ be positive.

Let

$$\lim_{n \rightarrow \infty} \frac{f(x)}{g(x)} = c$$

Then

- If $c \in (0, \infty)$:

$$\int_a^\infty g(x) dx < \infty \iff \int_a^\infty f(x) dx < \infty$$

- If $c = 0$:

$$\int_a^\infty g(x) dx < \infty \implies \int_a^\infty f(x) dx < \infty$$

- If $c = \infty$:

$$\int_a^\infty f(x) dx = \infty \implies \int_a^\infty g(x) dx = \infty$$

Type II: Let integrable functions $f(x)$, and $g(x)$ for $a < x \leq b$, and let $g(x)$ be positive.

Let

$$\lim_{n \rightarrow a^+} \frac{f(x)}{g(x)} = c$$

Then

- If $c \in (0, \infty)$:

$$\int_a^\infty g(x) dx < \infty \iff \int_a^\infty f(x) dx < \infty$$

- If $c = 0$:

$$\int_a^\infty g(x) dx < \infty \implies \int_a^\infty f(x) dx < \infty$$

- If $c = \infty$:

$$\int_a^\infty f(x) dx = \infty \implies \int_a^\infty g(x) dx = \infty$$

Note: A useful test function is

$$\int_0^\infty \left(\frac{1}{x}\right)^p dx \begin{cases} < \infty & , \text{ when } p > 1 \\ = \infty & , \text{ when } p \leq 1 \end{cases}$$